Lecture 4: Black-Scholes Formulas and Implied Volatility

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Outline

Asset Model and Black-Scholes PDE

In Lecture 3, we first introduced the following asset model

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z}. \tag{1}$$

Then we derived the famous *Black – Scholes* partial differential equation (PDE).

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{2}$$

Black-Scholes PDE

Remarks

- If some asset's price $S(t)$ follows the process (1), then the PDE (2) must be satisfied for any option whose value depends on S and t and is paid up-front.
- When deriving (2), we didn't make use of any specific information from the option type (e.g. a call option or a put option).
- The drift parameter μ in the asset model does not appear in the PDE (2).
- This idea of continuously fine-tuning the portfolio in order to reduce or remove risk is known as *dynamic hedging*.

Nobel Prize in Economics

A Bit of History

- This equation is named after its inventors, Fisher Black and Myron Scholes.
- Robert Merton also made significant contributions here.
- Merton and Scholes received the 1997 Nobel Prize in Economics for this work. Unfortunately, Black passed away in 1995.
- Though ineligible for the prize because of his death in 1995, Black was mentioned as a contributor by the Swedish Academy.

Black-Scholes PDE

- The PDE (2) can have many solutions.
- $V(S,t) = S$ is one solution.

$$\frac{\partial V}{\partial t} = 0, \qquad \frac{\partial V}{\partial S} = 1, \qquad \frac{\partial^2 V}{\partial S^2} = 0$$

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} - rV = 0 + rS - rV = 0.$$

- $V(S,t) = e^{rt}$ is another one.

$$\frac{\partial V}{\partial t} = re^{rt} = rV, \qquad \frac{\partial V}{\partial S} = 0, \qquad \frac{\partial^2 V}{\partial S^2} = 0$$

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} - rV = rV - rV = 0.$$

Black-Scholes PDE

- The function $V(S,t) = S$ satisfies the PDE (2). But it is definitely not the right solution for a call option. Why?
- To uniquely determine $V(S,t)$, we have to specify other conditions that involve information about the particular option.

- Consider a European call option $C(S,t)$. Let's see what we know about $C(S,t)$:
  - *Terminal condition* of the PDE (2):

$$C(S,T) = \max(S(T) - K, 0). \tag{3}$$

  - *Lower boundary condition* of (2):

$$\lim_{S \to 0} C(S,t) = 0, \text{ for any } 0 \leq t \leq T. \tag{4}$$

  - *Upper boundary condition* of (2):

$$\lim_{S \to \infty} C(S,t) = S - Ke^{-r(T-t)}, \text{ for any } 0 \leq t \leq T. \tag{5}$$

Black-Scholes PDE

- Let's look at the conditions (3)-(5) a bit closer.
- (3) is just the payoff definition of a European call option.
- When $S = 0$, from our asset model, the asset price would stay at 0 for any time t. So the option value is 0. The lower boundary condition (4) is justified.
- When $S \to \infty$, it becomes ever more likely that the option will be exercised and the magnitude of the exercise price becomes less and less important.
- Thus, as $S \to \infty$ the call option becomes a forward contract with K as the delivery price. It can be proved that the value of the option in this case is

$$S - Ke^{-r(T-t)}.$$

which is the upper boundary condition (5).

Forward Contracts

- Recall the payoff at maturity for a long forward contract position is

$$S(T) - K.$$

- Then we can prove that the value function $V(S(t), t)$ satisfies $V(S, t) = S(t) - Ke^{-r(T-t)}$.
- Let's see whether the above $V(S, t)$ satisfies (2).

$$\frac{\partial V}{\partial t} = -rKe^{-r(T-t)}, \quad \frac{\partial V}{\partial S} = 1, \quad \frac{\partial^2 V}{\partial S^2} = 0$$

Hence,

$$\begin{aligned} &\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} - rV \\ &= -rKe^{-r(T-t)} + rS - rV = r(S - Ke^{-r(T-t)}) - rV = 0. \end{aligned}$$

Black-Scholes Formulas

- With three conditions imposed on (2), we can derive a unique solution for the call option value.
- But the derivation is beyond the scope of this course.
- Instead, we give the solution directly, and then verify that the solution $C(S,t)$ satisfies (2) and the terminal and boundary conditions.
- The solution function $C(S,t)$ is

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{6}$$

where $N(x)$ is the cumulative density function of $\mathbf{N}(0,1)$, and

$$d_1 = \frac{\ln(S/K) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}, \tag{7}$$

$$d_2 = \frac{\ln(S/K) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}. \tag{8}$$

Black-Scholes Formulas

- Given $C(S, t)$, we can derive the value $P(S, t)$ of a European put option using the call-put parity equation:

$$C(S, t) + Ke^{-r(T-t)} = P(S, t) + S$$

- The solution function $P(S, t)$ is

$$P(S, t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1). \tag{9}$$

- Alternatively, let's see what we know about $P(S, t)$:
  - *Terminal condition* of the PDE (2):

$$P(S, T) = \max(K - S(T), 0). \tag{10}$$

  - *Lower boundary condition* of (2):

$$\lim_{S \to 0} P(S, t) = Ke^{-r(T-t)}, \text{ for any } 0 \leq t \leq T. \tag{11}$$

  - *Upper boundary condition* of (2):

$$\lim_{S \to \infty} P(S, t) = 0, \text{ for any } 0 \leq t \leq T. \tag{12}$$

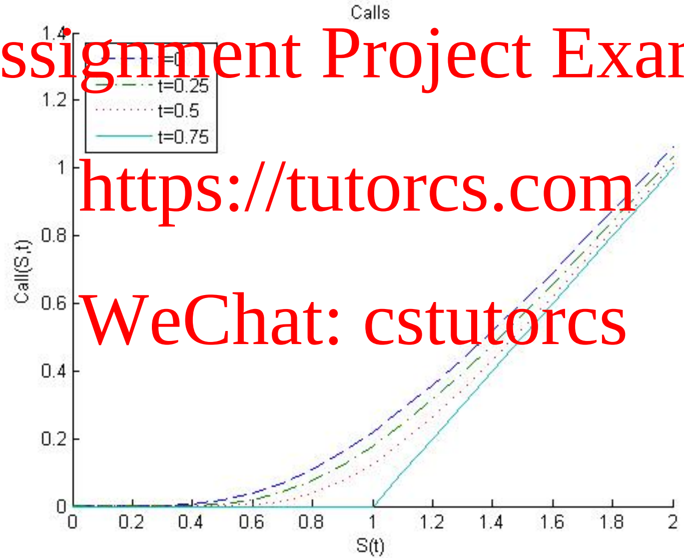
Black-Scholes PDE

- In summary, to valuate a financial derivative using (2), we need to specify the terminal and boundary conditions.
- Some financial insight should be utilized to choose suitable conditions for the derivative at hand.

- Most of the time, there is no closed-form solution from (2).
- It has to be solved using numerical techniques, for example, the finite difference method, the finite element method, etc.

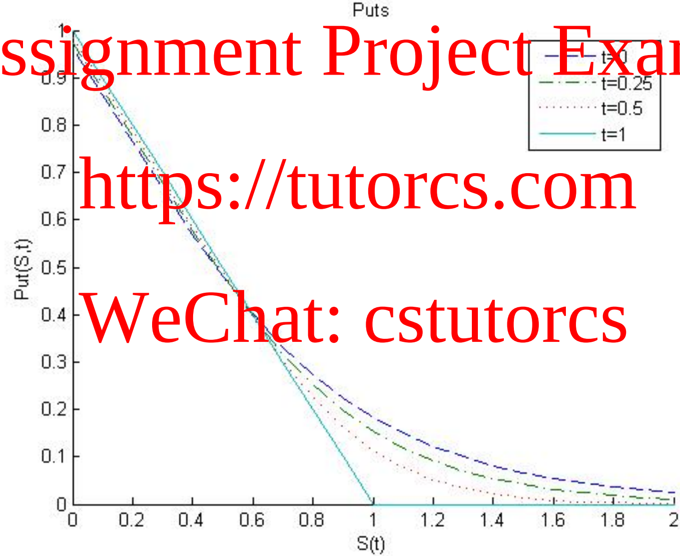
European Call Plot

$K = 1$; $r = 0.05$; $\sigma = 0.6$; $T = 0.75$; The call option value decreases when getting closer to maturity.



European Put Plot

The put option value can decrease or increase when getting closer to maturity, depending on the spot level.



European Option Values and Time to Maturity

- In Lecture 2, we have proved that the value of a European call option on a non-dividend-paying asset is non-decreasing as a function of the time to maturity.
- Actually it can be proved that the European call option value is a monotonically increasing function of time-to-maturity.
- But the value of a European put option is not a monotonic function of time to maturity.
- In most cases, the longer the time to maturity is, the more valuable is an European put option.
- But for a deep in-the-money put option, it could be the opposite if the risk-free interest rate is really high, e.g., two European puts options, strikes are both 10000, current spot is 0.01, interest rate is 100%, time to maturity: 1 day and 2 days, respectively. Which one is more valuable?

Main Takeaway

The key points:

- The drift term μ in the asset model (1) doesn't matter for option evaluation.
- Risk can be eliminated by holding a portfolio in which the random parts of two different sub-portfolios cancel each other
- No-arbitrage principle implies that a portfolio from which risk has been eliminated must grow at the risk-free rate.
- A European option's value can be replicated by a (self-financing) portfolio consisting of dynamically trading in stock and risk-free bond.
- Get familiar with the closed-form formulas for European Call/Put options.

Practicalities of Trading Options

- So far, we have learned quite a bit about options.
- You might start thinking to get your feet wet and trade some options. But before that you have to answer the following question first.
  - With the same underlying asset, there are many different options with different strikes, different maturities, and different payoff types (Call or Put).
  - How will you decide which one to buy or sell? Intuitively, you should buy cheaper ones and sell more expensive ones.
  - You need a systematic way to decide the relative cheapness/richness among different options.
- Can we directly look at the prices of options, just similarly to what we normally do with stocks?
- Not a good idea! Why?
- The rest of this lecture gives you a new tool that could help you on this.

What is Implied Volatility

- The Black-Scholes formula gives the value of an option as a function of several inputs: S_0, K, T, r, and σ.
- Of these only one is not specified in the contract or readily observable, the volatility σ.
- What we can observe from the market are option prices.
- Since the option price is a monotonic function with respect to σ, given the option price V, there exists a unique σ when substituted into Black-Scholes formula that gives the option price V, which is called *implied volatility*.
- In practice, we calculate the implied volatilities for different strikes and different maturities, and then generate an implied volatility surface using some numerical interpolations.

Why implied volatility

- A convenient quantity to measure the cheapness or dearness of an option.
- We can look at the option premium directly for cheapness/dearness. Not a good idea.
  - Options with different strikes, different maturities, and different underlying assets are essentially different contracts. Direct comparison is meaningless.
- With implied volatilities, there are two ways of judging the cheapness or dearness of options.
- The first is simply by comparing current implied volatility with past levels of implied volatility on the same underlying asset.
- The second is by comparing current implied volatility with the historical volatility of the underlying itself (will be discussed in next lecture).

Why implied volatility

- Implied volatility is relatively more stable than stock levels.
- Complex products need the volatilities implied by the prices of simple options observed from the market.
- Later we will introduce some much more complex options, e.g., Asian options, basket options, etc.
- All of these are OTC products. Not standardized products. Their prices cannot be observed directly from the market. You have to check with various banks for the price.
- Then how do banks get the volatility information to price them?
- They use the implied volatility surface obtained from the market prices of simpler products (European/American call/put options).

Nonlinear implicit Equation

- Suppose that on a certain date we observe an asset price of S_0 and an interest rate of r.
- If we also observe the value V of a call option with expiry time T and strike price K.

- Recall the Black-Scholes formula

$$C(\sigma) = S_0 N(d_1) - K e^{-rT} N(d_2) = V,$$

where

$$d_1 = d_2 + \sigma\sqrt{T} = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

- Since everything else is known, this is an equation for σ.
- But it is an implicit equation. We cannot just rearrange it to isolate σ and thus read off its value.
- We have to do some work to find the value of σ^* that makes $C(\sigma^*) - V = 0$.

Solving nonlinear implicit equations

Nonlinear implicit equation

- Consider the following general problem: given some *continuous* nonlinear monotonic function $f(x)$, find a real number x^* such that

$$f(x^*) = 0.$$

- There are a couple of options open to us.
  - Use guesswork.
  - Use inspired guesswork: use our previous guesses to make new, better ones.

Bisection Method

- If we have two guesses, x_a and x_b, with $f(x_a)f(x_b) < 0$, then we know that $f(x)$ must cross zero somewhere between x_a and x_b.
- We can use a divide-and-conquer approach to find x^*.

Bisection Method

- The bisection method goes as follows:

  Step 1: Find x_a and x_b with $x_a < x_b$ such that $f(x_a)f(x_b) < 0$.

  Step 2: Set $x_{\mathrm{mid}} := \frac{x_a + x_b}{2}$ and evaluate $f(x_{\mathrm{mid}})$.

  Step 3: If $f(x_{\mathrm{mid}}) = 0$ then stop. If $f(x_{\mathrm{mid}})f(x_a) < 0$, then reset $x_b = x_{\mathrm{mid}}$. Otherwise, reset $x_a = x_{\mathrm{mid}}$.

  Step 4: If $x_b - x_a < \epsilon$, then stop and use $\frac{x_a + x_b}{2}$ as the approximation to x^*. Otherwise, return to Step 2.

- Note that we must choose a value $\epsilon > 0$ for our stopping criterion $x_b - x_a < \epsilon$.
- The approximation error is guaranteed to be no more than $\epsilon/2$.

Bisection Method

- There is no foolproof procedure for finding suitable x_a and x_b in Step 1.
- Without specific knowledge of the function $f(x)$, we must resort to trial and error.
- The bisection method halves the length of the interval $[x_a, x_b]$ on each iteration, the error at the k-th iteration is bounded by $\frac{L}{2^{k+1}}$, where L is the length of the original interval, $x_b - x_a$.
- This is referred to as a *linear convergence bound* because the error bound decreases by a linear factor (in this case $\frac{1}{2}$).
- We consider next a faster method.

Newton-Raphson method

- Let's look at a faster method: Newton-Raphson method.
- Suppose that we have a current guess x_n. Then, assuming $f(x)$ is differentiable, and writing $\epsilon_n = x^* - x_n$, the Taylor series expansion gives

$$0 = f(x^*) = f(x_n + \epsilon_n) = f(x_n) + \epsilon_n f'(x_n) + \frac{\epsilon^2}{2} f''(x_n) + \cdots .$$

- If we ignore the terms of second order or higher, we get

$$\epsilon_n \approx -\frac{f(x_n)}{f'(x_n)} \Rightarrow x^* \approx x_n - \frac{f(x_n)}{f'(x_n)}.$$

- We set this estimate as our next guess:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- To see how quickly the error decreases, let's go back and look at the Taylor expansion:

$$\epsilon_{n+1} = x^* - x_{n+1} = x^* - x_n + \frac{f(x_n)}{f'(x_n)} = \epsilon_n + \frac{f(x_n)}{f'(x_n)}$$

$$\approx -\frac{\epsilon_n^2}{2}\frac{f''(x_n)}{f'(x_n)}$$

- If we could assume that $\frac{f''(x_n)}{f'(x_n)}$ is bounded by a constant C, then the new error is proportional to the square of the old.

- Second order convergence.
- Note that the result requires the starting value x_0 is sufficiently close to x^*. Otherwise, it may fail to converge.

Examples

Find the reciprocal

- Given a number $a > 0$, find $1/a$ without doing any division!
- Set $f(x) = \frac{1}{ax} - 1$. Then Newton-Raphson gives

$$x_{n+1} = x_n - \frac{\frac{1}{ax_n} - 1}{-\frac{1}{ax_n^2}} = x_n + x_n - ax_n^2 = x_n(2 - ax_n).$$

Find the square root

- Given $a > 0$, find $\sqrt{a}$.
- Set $f(x) = x^2 - a$. Then Newton-Raphson gives

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{x_n^2 + a}{2x_n}.$$

General Observations

- Newton-Raphson will normally converge very quickly, roughly doubling the number of decimal places of accuracy at each iteration.
- It will fail to produce this kind of convergence if $f'(x^*) = 0$, e.g., $f(x) = x^3$.
- A suitable initial guess has to be supplied, otherwise, it may produce disastrous results.
- It is always a good idea to do a sanity check on the results.
- Bisection can be used to generate a good initial guess.

Implied Volatility

- Recall our problem of solving implied volatility: given S_0, K, r, T, and the call option value V, find σ such that

$$C(\sigma) - V = 0.$$

- Write $f(\sigma) = C(\sigma) - V = S_0 N(d_1(\sigma)) - Ke^{-rT} N(d_2(\sigma)) - V$, where

$$d_1(\sigma) = d_2(\sigma) + \sigma\sqrt{T} = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

- Then $f'(\sigma) = S_0\sqrt{T}N'(d_1(\sigma))$.
- Note that $f'(\sigma) > 0$, so $f(\sigma)$ is monotonically increasing with respect to σ.
- $\lim_{\sigma \to 0^+} C(\sigma) = \max(S_0 - Ke^{-rT}, 0)$.
- $\lim_{\sigma \to \infty} C(\sigma) = S_0$.

Implied Volatility

- Thus if $V \in (\max(S_0 - Ke^{-rT}, 0), S_0)$, there will be a unique solution to the equation $f(\sigma) = 0$.
- From the Black-Scholes formula for a European call option,

$$\frac{\partial C}{\partial \sigma} = S_0 \sqrt{T} N'(d_1)$$

$$\frac{\partial^2 C}{\partial \sigma^2} = -\frac{S_0\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} d_1 \frac{\partial d_1}{\partial \sigma} = \frac{d_1 d_2}{\sigma} \frac{\partial C}{\partial \sigma}$$

- Thus $\partial C / \partial \sigma$ is maximum over $[0, \infty]$ at $\sigma = \hat{\sigma}$, where

$$\hat{\sigma} = \sqrt{2 \left| \frac{\ln S_0/K + rT}{T} \right|}$$

- For $\sigma < \hat{\sigma}$, $f'(\sigma)$ is increasing, and for $\sigma > \hat{\sigma}$ $f'(\sigma)$ is decreasing.
- We shall start our iteration with $\sigma_0 = \hat{\sigma}$, and since $f(\sigma)$ is decreasing away from $\hat{\sigma}$, this will guarantee that each ϵ_n has the same sign as ϵ_0 and is smaller.

Example


```
%%%%%%%%%%%%%%%%%%%%%%% parameters %%%%%%%%%%%%%%%%%%%%%%%
r = 0.03; S = 2; K = 2; T = 3; tau = T; sigma_true = 0.3;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%starting value
sigmahat = sqrt(2*abs( (log(S/K) + r*T)/T ) );
```

Example

```
%%%%%%%%%%%%%% Newton's Method %%%%%%%%%%%
tol = 1e-8;
sigma = sigmahat;
sigmadiff = 1;
n = 1;
nmax = 100;
while (sigmadiff >= tol & n < nmax)
    C = price_call(S,K,r,sigma,tau);
    Cvega = vega_call(S,K,r,sigma,tau);
    increment = (C-C_true)/Cvega;
    sigma = sigma - increment;
    n = n+1;
    sigmadiff = abs(increment);
end
```