Fundamentals of theory of computation I

Sample for Test 1 – Sample Solutions

Exercise 1: Let $L_1 = \{a^n b^n \,|\, n \in \mathbb{N}\}$, $L_2 = \{a^n b^k \,|\, n \geq 0, k \equiv 1 \pmod 3\}$, Determine the following languages.

$$L_1 L_2 = \{a^n b^n a^m b^k \,|\, n, m \in \mathbb{N}, k \equiv 1 \pmod 3\}$$
$$L_1 \setminus L_2 = \{a^n b^n \,|\, n \equiv 0 \text{ or } 2 \pmod 3\}$$
$$\mathrm{Pre}(L_2) = \{a^n b^k \,|\, n, k \in \mathbb{N}\}$$

Exercise 2: Let $N_t(u)$ denote the number of t's occuring in the word u. Make a type 3 grammar generating the language $L = \{u \in \{a, b, c\}^* \,|\, bb \not\subseteq u, N_a(u) \leq 1\}$. (I.e., there are at most 1 a and no consecutive b's in the words of L.)

Idea: is the last letter of the already generated word equal to b or not? How many a's were already generated? We can preserve both pieces of information throughout the derivation.

X_0: the last letter is not a b, 0 a's were generated so far.
X_1: the last letter is not a b, exactly 1 a's were generated so far.
B_0: the last letter is a b, 0 a's were generated so far.
B_1: the last letter is a b, exactly 1 a's were generated so far.

$$X_0 \to aX_1 \,|\, bB_0 \,|\, cX_0 \,|\, \varepsilon$$
$$X_1 \to bB_1 \,|\, cX_1 \,|\, \varepsilon$$
$$B_0 \to aX_1 \,|\, cX_0 \,|\, \varepsilon$$
$$B_1 \to cX_1 \,|\, \varepsilon$$

X_0 is the start symbol.

Exercise 3: Determine $L(G)$ for the grammar $G = \langle \{S, X, Y\}, \{a, b\}, P, S \rangle$, where P is as follows.

$$S \to aXa \,|\, bSb \,|\, a \,|\, b \,|\, \varepsilon$$
$$X \to aYa \,|\, bSb \,|\, b \,|\, \varepsilon$$
$$Y \to bSb \,|\, b$$

List all the Chomsky grammar classes having G as a member.

Observations:

- G is linear, so there will be at most 1 nonterminal in the sentential forms.
- Terminal symbols are generated in a symmetric way to the nonterminal, so all generated words are palindromes.
- If a pair of b's is generated, the nonterminal becomes an S.
- If a pair of a's is generated, S becomes X, X becomes Y. It is not allowed to generate a pair of a's, if the nonterminal is a Y.
- So having S in the sentential form means, that there's no a's right before (and right after) S in the sentential form. X means, that there's exactly one a's right before (and right after) X in the sentential form. Y means, that there's exactly two a's right before (and right after) Y in the sentential form.

Checking the terminating rules as well, one can conlude:

$L(G) = \{u \in \{a,b\}^* \,|\, u = u^R \wedge aaa \not\subseteq u\}$, i.e., $L(G)$ is the language of palindromes over $\{a,b\}$ without 3 consecutive a's.

G is a type 2 and type 0 grammar.

Exercise 4: Give a regular expression describing the following languages.

$$L_1 = \{u \in \{a,b\}^* \,|\, u \text{ contains an even number of } a\text{'s, and if } u \text{ contains a } b, \text{ then the number of } a\text{'s after the last } b \text{ is odd}\}.$$

$$L_2 = \{u \in \{a,b,c\}^* \,|\, u \text{ does not contain the subword } ab\}.$$

$R_1 = (b^*ab^*a)^*b^*ab^*ba(aa)^* + (aa)^*$ $\qquad L(R_1) = L_1.$

$R_2 = (b + a^*c)^*a^*$ $\qquad L(R_2) = L_2.$

Exercise 5: Construct a type 3 grammar in normal form, according to the algorithm we learnt, generating the same language as the following type 3 grammar G. $G = \langle \{S,A,B,C\}, \{a,b\}, P, S\rangle$, where the set of production rules P consists of the following rules.

$S \to aabA \,|\, bB$
$A \to C \,|\, aS$
$B \to bC \,|\, \varepsilon$
$C \to B \,|\, a$

$S \to aD \,\|\, bB$	$H_0(S) = H_1(S) = \{S\} = H(S)$	$S \to aD \,\|\, bB$
$A \to C \,\|\, aS$	$H_0(A)=\{A\}, H_1(A)=\{A,C\}, H_2(A)=H_3(A)=\{A,B,C\}=H(A)$	$A \to bC \,\|\, \varepsilon \,\|\, aF \,\|\, aS$
$B \to bC \,\|\, \varepsilon$	$H_0(B) = H_1(B) = \{B\} = H(B)$	$B \to bC \,\|\, \varepsilon$
$C \to B \,\|\, aF$	$H_0(C) = \{C\}, H_1(C) = H_2(C) = \{C,B\} = H(C)$	$C \to bC \,\|\, \varepsilon \,\|\, aF$
$D \to aE$	$H_0(D) = H_1(D) = H(D) = \{D\}$	$D \to aE$
$E \to bA$	$H_0(E) = H_1(E) = H(E) = \{E\}$	$E \to bA$
$F \to \varepsilon$	$H_0(F) = H_1(F) = H(F) = \{F\}$	$F \to \varepsilon$