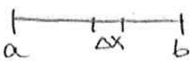
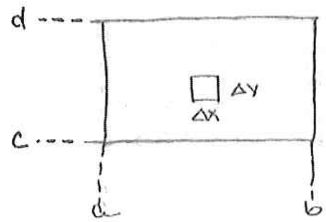
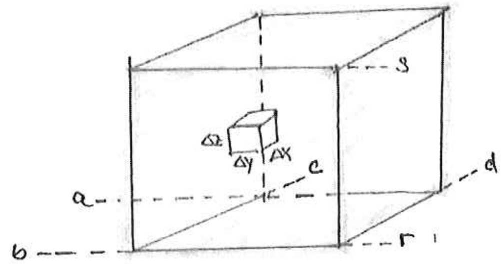
Discussion #16: Triple Integrals in rectangular & cylindrical coordinates.

$$\int_a^b f(x)\,dx \qquad \int_c^d\int_a^b f(x,y)\,dx\,dy \qquad \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$$

Region of integration



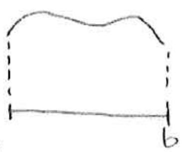




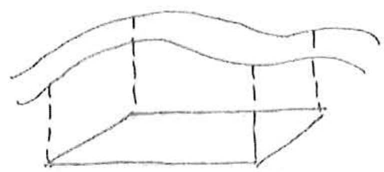
définition

$$\lim_{\Delta x\to 0}\sum f(x)\,dx \qquad \lim_{\Delta x,\Delta y\to 0}\sum\sum f(x,y)\,\Delta x\,\Delta y \qquad \lim_{\Delta x\,\Delta y\,\Delta z\to 0}\sum\sum\sum f(x,y,z)\,\Delta x\,\Delta y\,\Delta z$$

Area/Volume meaning



Area under f(x)

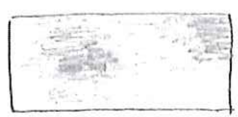


Volume under f(x,y)

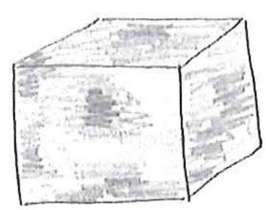
???

density meaning

density of rod



density of plate



density in box

Ex1: $$\int_0^3\int_0^1\int_0^2 (6y^2z - x)\,dx\,dy\,dz =$$

Ex2: $$\int_0^2\int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx\,dz =$$

* **Cylindrical Coordinates:**

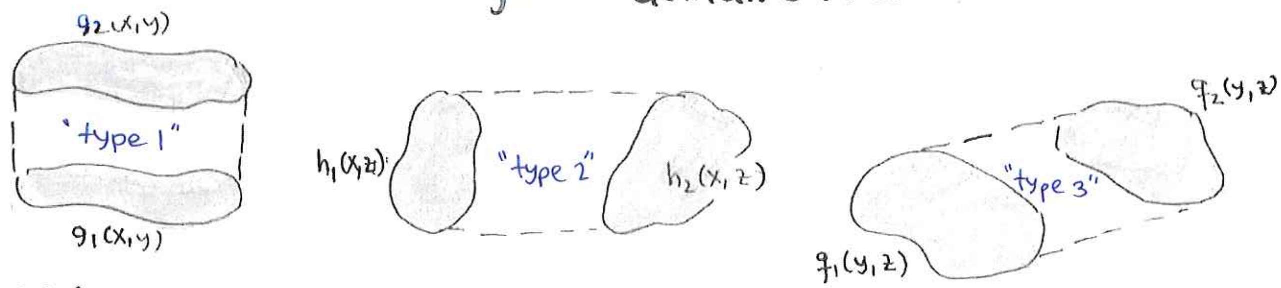
why this name?

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \text{ Polar in } x \,\&\, y$$
$$z = z$$

$\Rightarrow$ Volume form

$dV =$

Non-rectangular domains in 3D:



$$\iint\left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right)dy\,dx \qquad \iint\left(\int_{h_1(x,z)}^{h_2(x,z)} f(x,y,z)\,dy\right)dx\,dz \qquad \iint\left(\int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)\,dx\right)dy\,dz$$

Ex 3: W is the region in the first octant bounded by the plane $z=4$ and the surface $z=x^2+y^2$. Evaluate $\iiint_W x\,dV$

Ex 4: Evaluate $\iiint_W (1-z^2)\,dV$. for W is the pyramid with vertices $(0,0,0)$, $(1,0,0)$ $(0,1,0)$ $(1,1,0)$ & $(0,0,1)$

Triple Integrals:

Ex 1: $$\int_0^3\int_0^1\int_0^2 (6y^2z - x)\,dx\,dy\,dz = \int_0^3\int_0^1 \left(6y^2zx - \frac{x^2}{2}\right)\Big|_0^2 dy\,dz = \int_0^3\int_0^1 (12y^2z - 2)\,dy\,dz$$

$$= \int_0^3 (4y^3z - 2y)\Big|_0^1 dz = \int_0^3 (4z-2)\,dz = (2z^2 - 2z)\Big|_0^3 = 18 - 6 = 12$$

Ex 2: $$\int_0^2\int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx\,dz = \int_0^2\int_0^3 y\Big|_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx\,dz = \int_0^2\int_0^3 2\sqrt{9-x^2}\,dx\,dz$$

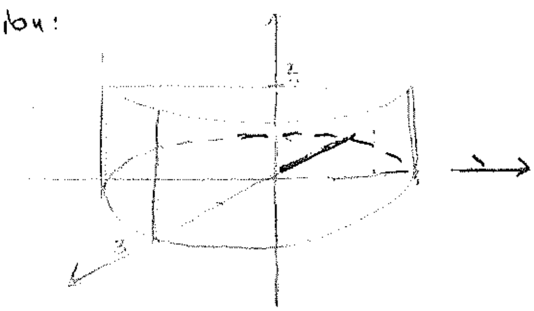
$$= \int_0^3 4\sqrt{9-x^2}\,dx =$$

need Trig substitution

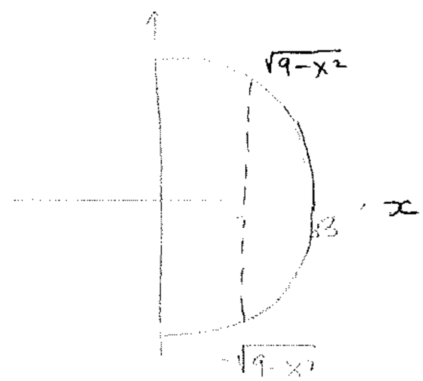
Alternative way: $\iiint_W dy\,dx\,dz$ = Volume of Region W

find Region:

in $\mathbb{R}^3$:



in $\mathbb{R}$:



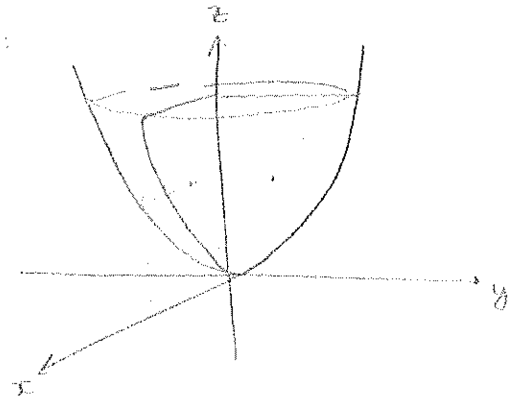
(half) cylinder of radius = 3 and height = 2

Area of circle = πr^2 ⟹ Area of base $\frac{1}{2}\pi 3^2 = \frac{9}{2}\pi$ ⟹ Volume = 9π

also, could use polar coordinates: (these are actually cylindrical coordinates in $\mathbb{R}^3$)

$$\iiint_W dV = \int_0^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^3 r\,dr\,d\theta\,dz = \int_0^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^2}{2}\Big|_0^3 d\theta\,dz = \int_0^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{9}{2}\,d\theta\,dz = \int_0^2 \frac{9}{2}\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz = 9\pi$$

Ex 3:



$0 \le x \le 2$

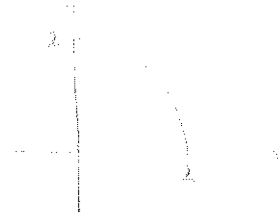
$0 \le y \le \sqrt{z - x^2}$

$0 \le z \le x^2+y^2$

① Region:

know: $0 \le x$, $0 \le y$ } need to set this

$x^2+y^2 \le z \le 4$



$0 \le x \le 2$

$0 \le y \le \sqrt{4-x^2}$

integrate:
$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} x\,dz\,dy\,dx = \int_0^2\int_0^{\sqrt{4-x^2}} xz\Big|_{x^2+y^2}^{4} dz\,dy\,dx = \int_0^2\int_0^{\sqrt{4-x^2}} [4x - x(x^2+y^2)]\,dy\,dx$$

$4x - x^3 - xy^2$

$$= \int_0^2 \left((4x-x^3)y - \frac{xy^3}{3}\right)\Bigg|_0^{\sqrt{4-x^2}} dx = \int_0^2 \left(x\sqrt{4-x^2}^{\,3} - \frac{x\sqrt{4-x^2}^{\,3}}{3}\right)dx = \int_0^2 \frac{2}{3}x\sqrt{4-x^2}^{\,3}\,dx$$

$$= -\frac{1}{3}\int_4^0 \sqrt{u}^{\,3}\,du = \frac{1}{3}\cdot\frac{2}{5}u^{\frac{5}{2}}\Big|_0^4 = \frac{2^6}{15} = \frac{64}{15}$$

$u = 4-x^2$

$du = -2x\,dx$

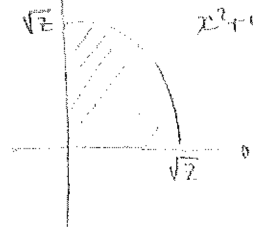
$x=0 \to u=4$

$x=2 \to u=0$

Alt method: (using cylindrical coordinate)

$$\int_0^4\left(\iint_D f\,dA\right)dz = \int_0^4\left(\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{z}} x\,r\,dr\,d\theta\right)dz = \int_0^4\left(\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{z}} r^2\cos\theta\,dr\,d\theta\right)dz$$

need to change $x = r\cos\theta$



$x^2+y^2=z$

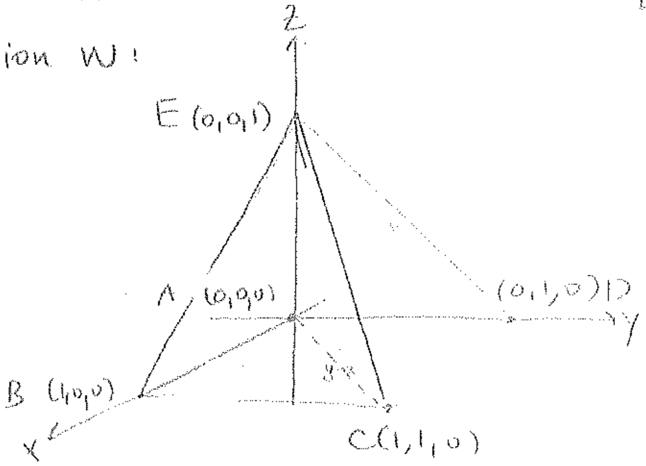
at level z

$$= \int_0^4\int_0^{\frac{\pi}{2}} \frac{r^3}{3}\cos\theta\Big|_0^{\sqrt{z}} d\theta\,dz = \int_0^4\int_0^{\frac{\pi}{2}} \frac{\sqrt{z}^{\,3}}{3}\cos\theta\,d\theta\,dz$$

$$= \int_0^4 \frac{\sqrt{z}^{\,3}}{3}\sin\theta\Big|_0^{\frac{\pi}{2}} dz = \int_0^4 \frac{z^{\frac{3}{2}}}{3}\,dz = \frac{1}{3}\cdot\frac{2}{5}\cdot z^{\frac{5}{2}}\Big|_0^4 = \frac{64}{15}$$

Ex 4: $\iiint_W (1-z^2)\,dV$

① Region W:



Equations of planes:

CDE: $z+y=1 \longrightarrow z=1-y$

EBC: $z+x=1 \Longrightarrow z=1-x$

$0 \le x \le 1$

$0 \le y$

$\left.\begin{array}{l} 0 \le z \le 1-x \\ \quad \text{or} \\ 0 \le z \le 1-y \end{array}\right\}$ split

Method 1

(can use symmetry $V_1 = V_2$)

$$V_1 = \int_0^1 \int_0^x \int_0^{1-x} (1-z^2)\,dz\,dy\,dx \qquad \{0 \le x \le 1;\ 0 \le y \le x;\ 0 \le z \le 1-x\}$$

$+$

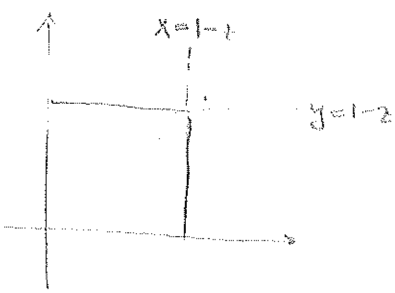
$$V_2 = \int_0^1 \int_x^1 \int_0^{1-y} (1-z^2)\,dz\,dy\,dx \qquad \{0 \le x \le 1;\ x \le y \le 1;\ 0 \le z \le 1-y\}$$

Method 2:

$$\int_0^1 \left(\begin{array}{c} (1-z^2) \times \\ \text{Area of square in} \\ xy \text{ plane} \end{array} \right) dz = \int_0^1 \Big[(1-z^2)\cdot \underbrace{(1-z)^2}_{1-2z+z^2}\Big]\,dz$$

$$= \int_0^1 (1+3z^2-z^4)\,dz = \left(z - \frac{3z^4}{4} - \frac{z^5}{5}\right)\Bigg|_0^1$$

$$= 1 - \frac{3}{4} - \frac{1}{5} = \frac{1}{20}$$



Area $= (1-z)^2$