Math Binder - AMC/AIME

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Contents

1 Polynomials

1.1 Definition and Basics

A polynomial is defined as $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ with names corresponding to their degree (constant, linear, quadratic, cubic, quartic).

The factored form is written as $P(x) = a(x-r)(x-p)\cdots(x-q)$. The simplest and most useful polynomial is the quadratic. It can be written as $ax^2 + bx + c$ and factored respectively. The formula to solve for x is $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. The most important formula for polynomials is the Vieta Formulas.

Formula 1.1 (Vieta Formulas) *Sum of roots* $(r_1+r_2+r_3+\cdots+r_n)$*:* $-\frac{a_{n-1}}{a_n}$
Product of roots $(r_1r_2r_3\cdots r_n)$*:* $(-1)^n \cdot \frac{a_0}{a_n}$
Pairwise sums of p *(*$p=2$*:* $r_1r_2 + r_1r_3 + r_1r_4 + \cdots + r_{n-1}r_n$*):* $(-1)^p \cdot \frac{a_{n-p}}{a_n}$

Theorem 1.1 (Fundamental theorem of algebra) *It states that a single variable polynomial with degree* n *has exactly* n *complex roots.*

Problem 1.1 *Let* $r, s,$ *and* t *be the roots of* $3x^3 - 4x^2 + 5x + 7 = 0$*. (Intermediate Algebra, AoPS, 8.20 pg.249)*

1. *Find* $r + s + t$ $(\frac{4}{3})$.
2. *Find* $r^2 + s^2 + t^2$ $(\frac{-14}{9})$.
3. *Find* $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$ $(\frac{-5}{7})$.

1.2 Synthetic Division

A simplification of traditional polynomial division. Note this only works when the coefficients of the linear term in the divisor is 1. It is also know as the **Ruffini's Rule**.

Example:

$$\begin{array}{c|ccccc} 3 & 1 & -3 & 7 & -1 & 5 \\ & & 3 & 0 & 21 & 60 \\ \hline & 1 & 0 & 7 & 20 & \multicolumn{1}{|c}{65} \end{array}$$

Which is the same as $(x^4 - 3x^3 + 7x^2 - x + 5) \div (x - 3) = x^3 + 7x + 20 + \frac{65}{x-3}$. Notice you work from left to right, and multiply to get the next number in the second row. If your divisor doesn't have 1 as its coefficient in the linear term, you can divide it by $1/n$ and in the end also multiply the quotient and remainder by $1/n$.

Usually, you write the result of polynomial division as $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$.

1.3 Rational Root Theorem

Theorem 1.2 (Rational Root Theorem) *A rational root of a polynomial in the form* $\pm\frac{p}{q}$ *where* p *and* q *are relatively prime must follow the condition* $p|a_0$ *and* $q|a_n$.

1.4 Remainder Theorem

Theorem 1.3 (Remainder Theorem) *When a polynomial $f(x)$ is divided by $x - a$, the remainder is determined by $f(a)$.*

Theorem 1.3 can be proven using the form $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$ and synthetic division.

Proof 1.1 (Remainder Theorem)

$$\begin{aligned} f(x) &= (x-a)q(x) + r(x) \\ &= (x-a)q(x) + c. \\ f(a) &= (a-a)q(a) + c \\ &= 0 \cdot q(a) + c = c \end{aligned}$$

Thus, $f(a)$ always returns the remainder of $f(x) \div (x - a)$.

1.5 Factor Theorem

Theorem 1.4 (Factor Theorem) *given the expression $x - a$, it is a divisor of $p(x)$ if and only if $p(a) = 0$. This can be proven with the remainder theorem.*

1.6 Miscellaneous

To find the sum of the coefficients of a polynomial $P(x)$, plug $x = 1$! Example problem: Practice problem #2

1.7 Practice Problems

Problems from *Intermediate Algebra, AoPS*.

1. ~~6.11, pg. 181~~
2. 6.21 pg. 191
3. 6.17, pg. 187
4. 6.22, pg. 191
5. 6.27, pg. 191
6. 6.29, pg. 192
7. $\star$ Challenge Problems, pg. 192
8. `https://numbertheoryguydotcom.files.wordpress.com/2016/03/polynomials.pdf`

2 Algebraic Manipulations

2.1 Definitions

Sophie Germain:

$$x^4 + 4y^4 = (x^2 - 2xy + y^2)(x^2 + 2xy + y^2)$$

$(x+y)^3 = x^3 + 3xy(x+y) + y^3 = x^3 + 3x^2y + 3xy^2 + y^3.$

$(x-y)^3 = x^3 - 3xy(x-y) + y^3 = x^3 - 3x^2y + 3xy^2 - y^3.$

$x^3 + y^3 = (x+y)(x^2 - xy + y^2).$

$x^3 - y^3 = (x-y)(x^2 + xy + y^2).$

$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx).$

2.2 Practice Problems

1. Given that x and y are distinct nonzero real numbers such that $x+2/x = y+2/y$, what is xy?

   Solution:

$$\begin{aligned} x - y &= 2/y - 2/x \\ x - y &= \frac{2x - 2y}{xy} \\ &= \frac{2(x-y)}{xy} \\ &= \frac{2}{xy} = 1 \\ xy &= \boxed{2}. \end{aligned}$$

2. Suppose that real number x satisfies $\sqrt{49-x^2}-\sqrt{25-x^2}=3$. What is the value of $\sqrt{49-x^2}+\sqrt{25-x^2}$

Solution:

$$\begin{aligned}
&(\sqrt{49-x^2}-\sqrt{25-x^2})(\sqrt{49-x^2}+\sqrt{25-x^2})\\
&=(49-x^2)-(25-x^2)\\
&=24=3(\sqrt{49-x^2}+\sqrt{25-x^2})(\sqrt{49-x^2}+\sqrt{25-x^2})=\boxed{8}.
\end{aligned}$$

3. What is the minimum value of the expression $x^2+8x+13$ for any real x?

Solution:

$$\begin{aligned}
x^2+8x+13&=(x+4)^2-3\\
\min((x+4)^2-3)&=\boxed{-3}.
\end{aligned}$$

4. Real numbers x and y satisfy the equation $x^2+y^2=10x-6y-34$. What is $x+y$?

Solution:

$$\begin{aligned}
(x^2-10x)+(y^2+6y)&=-34\\
(x-5)^2+(y+3)^2&=0\\
x=5,y&=-3\\
x+y&=\boxed{2}.
\end{aligned}$$

5. There is a positive integer n such that $(n+1)!+(n+2)!=n!\cdot 440$. What is the sum of the digits of n?

Solution:

$$\begin{aligned}
n!(n+1)^2(n+2) &= n!(440)\\
(n+1)(n+3) &= 440\\
((n+2)+1)((n+2)-1) &= 440\\
(n+2)^2 - 1^2 &= 440\\
(n+2)^2 &= 441\\
n+2 &= 21\\
n &= \boxed{19}.
\end{aligned}$$

6. For all integers $n \geq 9$, the value of $\frac{(n+2)!-(n+1)!}{n!}$ is always which of the following?

   Solution:

   $$\frac{n!((n+2)(n+1)-(n+1))}{n!} = (n+1)^2.$$

   The expression must always be a perfect square.

7. Let $f(x) = x^2(1-x)^2$. What is the value of the sum $f\left(\frac{1}{2019}\right) - f\left(\frac{2}{2019}\right) + f\left(\frac{3}{2019}\right) - f\left(\frac{4}{2019}\right) + \cdots + f\left(\frac{2017}{2019}\right) - f\left(\frac{2018}{2019}\right)$?

   (A) 0 **(B)** $\frac{1}{2019^4}$ **(C)** $\frac{2018^2}{2019^4}$ **(D)** $\frac{2020^2}{2019^4}$ **(E)** 1

   Solution:

   $$\left(f\left(\frac{1}{2019}\right) - f\left(\frac{1}{2019}\right)\right) + \left(f\left(\frac{2}{2019}\right) - f\left(\frac{2}{2019}\right)\right) + \cdots + \left(f\left(\frac{1009}{2019}\right) - f\left(\frac{1009}{2019}\right)\right)$$

   The answer is $\boxed{\textbf{(A) } 0}$.

8. Given that $x + \frac{1}{x} = 3$, find

   (a) $x^2 + \frac{1}{x^2} = (x + \frac{1}{x})^2 = x^2 + 2 + \frac{1}{x^2} = 3^2 - 2 = \boxed{7}$.

   (b) $x^3 + \frac{1}{x^3} = (x + \frac{1}{x})(x^2 - 1 + \frac{1}{x^2}) = 3(7 - 1) = \boxed{18}$.

   (c) $x^4 + \frac{1}{x^4} = 7^2 + 2 = 49 + 2 = \boxed{51}$.

9. Real numbers x and y satisfy $x + y = 4$ and $x \cdot y = -2$. What is the value of

$$x + \frac{x^3}{y^2} + \frac{y^3}{x^2} + y?$$

(A) 360 **(B)** 400 **(C)** 420 **(D)** 440 **(E)** 480

Solution:

$$\begin{aligned} &\frac{x^3}{x^2} + \frac{x^3}{y^2} + \frac{y^3}{x^2} + \frac{y^3}{y^2} \\ &= \frac{x^3 + y^3}{x^3} + \frac{x^3 + y^3}{y^2} \\ &= \frac{88}{x^2} = \frac{88(x^2 + y^2)}{4} = \boxed{44}. \end{aligned}$$

10. Let r, s, and t be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find $(r + s)^3 + (s + t)^3 + (t + r)^3$.

Solution:

$$\begin{aligned} (r + s)^3 + (s + t)^3 + (t + r)^3 \\ = (0 - t)^3 + (0 - r)^3 + (0 - s)^3 = -(r^3 + s^3 + t^3) \\ r^3 + s^3 + t^3 - 3rst = (r + s + t)(r^2 + s^2 + t^2 - rs - st - tr) = 0 \\ r^3 + s^3 + t^3 = 3rst = -251, -251 \cdot -3 = \boxed{753}. \end{aligned}$$

11. Find $3x^2y^2$ is x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$.

Solution:

$$a = x^2, b = y^2$$

$$\begin{aligned}
b + 3ab &= 30a + 517 \\
ab + \frac{b}{3} - 10a &= \frac{517}{3} \\
(a + \frac{1}{3})(b - 10) &= \frac{507}{3} \\
(3x^2 + 1)(y^2 - 10) &= 507
\end{aligned}$$

$$(x, y) = (\pm 2, \pm 7)$$

$3x^2y^2 = 3 \cdot 4 \cdot 49 = \boxed{588}$.

12. What is the remainder when $s^{202} + 202$ is divided by $2^{101} + 2^{51} + 1$.

13. Two non-zero real numbers, a and b, satisfy $ab = a - b$. Which of the following is a possible value of $\frac{a}{b} + \frac{b}{a} - ab$?

Solution:

$$\begin{aligned}
\frac{a^2 + b^2}{ab} - ab &= \frac{(a - b)^2 + 2ab}{ab} - ab \\
&= \frac{(ab)^2 + 2ab}{ab} - ab \\
&= ab + 2 - ab = \boxed{2}
\end{aligned}$$

14. If $x + \frac{1}{x} = 4$, then what is the value of $x^8 + \frac{1}{x^8}$?

Solution:

$$\begin{aligned}
x^2 + \frac{1}{x^2} &= 4^2 - 2 = 14 \\
x^4 + \frac{1}{x^4} &= 14^2 - 2 = 194 \\
x^8 = \frac{1}{x^8} &= 194^2 - 2 = \boxed{37624}.
\end{aligned}$$

15. Let a and b be relatively prime positive integers with $a > b > 0$ and $\frac{a^3 - b^3}{(a-b)^3} = \frac{73}{3}$. What is $a - b$?
Solution:

$$\begin{aligned}
\frac{a^3 - b^3}{(a-b)^3} &= \frac{(a-b)(a^2 + ab + b^2)}{(a-b)^3} \\
&= \frac{a^2 + ab + b^2}{(a-b)^2} \\
&= \frac{(a-b)^2 - 3ab}{(a-b)^2} \\
&= 1 + \frac{3ab}{(a-b)^2} \\
\frac{3ab}{(a-b)^2} &= \frac{70}{3} \\
9ab &= 70(a-b)^2 \\
3\sqrt{\frac{ab}{70}} &= a - b \\
ab|70, 3&|a - b.
\end{aligned}$$

Thus $a = 10, b = 7$ so $a - b = \boxed{3}$.

16. If $(a - 1/a)^2 = 4$, what is the absolute value of $a^3 - 1/a^3$?

Solution:

$$
\begin{aligned}
a - \frac{1}{a} &= 2 \\
(a - \frac{1}{a})^3 &= a^3 - \frac{1}{a^3} - 3(x - \frac{1}{x}) \\
2^3 &= a^3 - \frac{1}{a^3} - 6 \\
a^3 - \frac{1}{a^3} &= \boxed{14}
\end{aligned}
$$

3 § Complex Numbers

3.1 Definition and Forms

Definition of i: $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

Rectangular form: $a + bi$.

Exponential form: $re^{i\theta}$ where r is the magnitude and θ is the angle around the polar plane.