Math Binder - AMC/AIME

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Contents

1 Polynomials

1.1 Definition and Basics

A polynomial is defined as $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ with names corresponding to their degree (constant, linear, quadratic, cubic, quartic).

The factored form is written as $P(x) = a(x-r)(x-p)\cdots(x-q)$. The simplest and most useful polynomial is the quadratic. It can be written as $ax^2 + bx + c$ and factored respectively. The formula to solve for x is $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. The most important formula for polynomials is the Vieta Formulas.

Formula 1.1 (Vieta Formulas) *Sum of roots ($r_1 + r_2 + r_3 + \cdots + r_n$):* $-\frac{a_{n-1}}{a_n}$
Product of roots ($r_1 r_2 r_3 \cdots r_n$): $(-1)^n \cdot \frac{a_0}{a_n}$
Pairwise sums of p ($p = 2$: $r_1r_2 + r_1r_3 + r_1r_4 + \cdots + r_{n-1}r_n$): $(-1)^p \cdot \frac{a_{n-p}}{a_n}$

Theorem 1.1 (Fundamental theorem of algebra) *It states that a single variable polynomial with degree n has exactly n complex roots.*

Problem 1.1 *Let $r, s,$ and t be the roots of $3x^3 - 4x^2 + 5x + 7 = 0$. (IA 8.20 p.249)*

1. *Find $r + s + t$ ($\frac{4}{3}$).*

2. *Find* $r^2 + s^2 + t^2$ $(\frac{-14}{9})$.

3. *Find* $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$ $(\frac{-5}{7})$.

1.2 Synthetic Division

A simplification of traditional polynomial division. Note this only works when the coefficients of the linear term in the divisor is 1. It is also know as the **Ruffini's Rule**.

Example:
$$\begin{array}{c|ccccc} 3 & 1 & -3 & 7 & -1 & 5 \\ & & 3 & 0 & 21 & 60 \\ \hline & 1 & 0 & 7 & 20 & \multicolumn{1}{|c}{65} \end{array}$$

Which is the same as $(x^4 - 3x^3 + 7x^2 - x + 5) \div (x - 3) =$. Notice you work from left to right, and multiply to get the next number in the second row. If your divisor doesn't have 1 as its coefficient in the linear term, you can divide it by $1/n$ and in the end also multiply the quotient and remainder by $1/n$.

Usually, you write the result of polynomial division as $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$.

1.3 Rational Root Theorem

A rational root of a polynomial in the form $\pm\frac{p}{q}$ where p and q are relatively prime must follow the condition $p|a_0$ and $q|a_n$.