Math Binder - AMC/AIME

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Contents

1 § Polynomials

1.1 Definition and Basics

A polynomial is defined as $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ with names corresponding to their degree (constant, linear, quadratic, cubic, quartic).

The factored form is written as $P(x) = a(x-r)(x-p)\cdots(x-q)$. The simplest and most useful polynomial is the quadratic. It can be written as $ax^2 + bx + c$ and factored respectively. The formula to solve for x is $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. The most important formula for polynomials is the Vieta Formulas.

Formula 1.1 (Vieta Formulas) *Sum of roots* $(r_1+r_2+r_3+\cdots+r_n)$*:* $-\frac{a_{n-1}}{a_n}$
Product of roots $(r_1 r_2 r_3 \cdots r_n)$*:* $(-1)^n \cdot \frac{a_0}{a_n}$
Pairwise sums of p *(*$p = 2$*:* $r_1r_2 + r_1r_3 + r_1r_4 + \cdots + r_{n-1}r_n$*):* $(-1)^p \cdot \frac{a_{n-p}}{a_n}$

Theorem 1.1 (Fundamental theorem of algebra) *It states that a single variable polynomial with degree* n *has exactly* n *complex roots.*

Problem 1.1 *Let* $r, s,$ *and* t *be the roots of* $3x^3 - 4x^2 + 5x + 7 = 0$*. (IA 8.20 p.249)*

1. *Find* $r + s + t$ $(\frac{4}{3})$*.*
2. *Find* $r^2 + s^2 + t^2$ $(\frac{-14}{9})$*.*
3. *Find* $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$ $(\frac{-5}{7})$*.*

1.2 Synthetic Division

A simplification of traditional polynomial division. Note this only works when the coefficients of the linear term in the divisor is 1. It is also know as the **Ruffini's Rule**.

Example:

$$\begin{array}{c|ccccc} 3 & 1 & -3 & 7 & -1 & 5 \\ & & 3 & 0 & 21 & 60 \\ \hline & 1 & 0 & 7 & 20 & \multicolumn{1}{|c}{65} \end{array}$$

Which is the same as $(x^4 - 3x^3 + 7x^2 - x + 5) \div (x - 3) =$. Notice you work from left to right, and multiply to get the next number in the second row. If your divisor doesn't have 1 as its coefficient in the linear term, you can divide it by $1/n$ and in the end also multiply the quotient and remainder by $1/n$.

Usually, you write the result of polynomial division as $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$.

1.3 Rational Root Theorem

Theorem 1.2 (Rational Root Theorem) *A rational root of a polynomial in the form* $\pm\frac{p}{q}$ *where* p *and* q *are relatively prime must follow the condition* $p|a_0$ *and* $q|a_n$.

1.4 Remainder Theorem

Theorem 1.3 (Remainder Theorem) *When a polynomial* $f(x)$ *is divided by* $x - a$, *the remainder is determined by* $f(a)$.

Theorem 1.3 can be proven using the form $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$ and synthetic division.

Proof 1.1 (Remainder Theorem)

$$
\begin{aligned}
f(x) &= (x-a)q(x) + r(x) \\
&= (x-a)q(x) + c. \\
f(a) &= (a-a)q(a) + c \\
&= 0 \cdot q(a) + c = c
\end{aligned}
$$

Thus, $f(a)$ always returns the remainder of $f(x) \div (x-a)$.

1.5 Practice Problems

Problems from Intermediate Algebra, AoPS.

1. 6.11, pg.181
2. 6.21 pg. 191
3. 6.17, pg. 187
4. 6.22, pg. 191
5. 6.27, pg. 191
6. 6.29, pg. 192
7. $\star$ Challenge Problems, pg. 192