MATH 6220 HOMEWORK 5

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We work in the category Ch, *chain complexes* of graded abelian groups.

1. For a given chain complex C, let $\mathrm{cyl}(C)$ denote the *mapping cylinder* of the identity map on $\mathrm{id}\colon C \to C$. The chain complex $\mathrm{cyl}(C)$ encodes the following data: If C is described by the level-wise groups and differentials

$$C = \left\{ C_{i+1} \xrightarrow{d_{i+1}} C_i : \text{for all } i \in \mathbb{Z} \right\},$$

then the mapping cylinder $\mathrm{cyl}(C)$ is described by level-wise groups and differentials

$$\mathrm{cyl}(C) = \left\{ \begin{tikzcd} C_i \arrow[rr, "d_i"] & & C_{i-1} \\ C_{i-1} \arrow[urr, "\mathrm{id}"] \arrow[rr, "-d_{i-1}"] \arrow[drr, "-\mathrm{id}"] & & C_{i-2} \\ C_i \arrow[rr, "d_i"] & & C_{i-1} \end{tikzcd} : \text{for all } i \in \mathbb{Z} \right\}.$$

Each level-wise group $(\mathrm{cyl}(C))_i$ is the direct sum of groups $C_i \oplus C_{i-1} \oplus C_i$, where $\mathrm{id}\colon C \to C$ is the identity, and where arrows indicate how a component group is mapped into a component group of lesser degree in the chain complex $\mathrm{cyl}(C)$.

Writing the level-wise groups as column vectors and the differential $[\partial]$ of $\mathrm{cyl}(C)$ as a matrix is elucidating, and will be helpful for computation:

$$[\partial] = \begin{bmatrix} d & \mathrm{id} & 0 \\ 0 & -d & 0 \\ 0 & -\mathrm{id} & d \end{bmatrix} : \begin{bmatrix} C_i \\ C_{i-1} \\ C_i \end{bmatrix} \longmapsto \begin{bmatrix} C_{i-1} \\ C_{i-2} \\ C_{i-1} \end{bmatrix}.$$

Now, say that D is another chain complex that's the target of two chain maps $f, g\colon C \to D$.

Claim (Extending chain maps to a mapping cylinder)**.** Two chain maps $f, g\colon C \to D$ are chain homotopic if and only if they extend to a map

$$\begin{bmatrix} f & \mathcal{S} & g \end{bmatrix} : \mathrm{cyl}(C) \to D,$$

where $\mathcal{S}\colon C \to D$ is a sequence[1] of level-wise sections of homological degree $+1$.

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ C_i \arrow[d] \arrow[ur, "s_i"] & D_{i+1} \arrow[d] \\ C_{i-1} \arrow[d] \arrow[ur, "s_{i-1}"] & D_i \arrow[d] \\ C_{i-r} \arrow[d] \arrow[ur, "s_{i-2}"] & D_i \arrow[d] \\ \vdots \arrow[ur, "s_{i-3}"] & \vdots \end{tikzcd}.$$

[1]Note that $\mathcal{S}$ is *not* a chain map. For example, if $f = g$, then the maps in $\mathcal{S}$ satisfy $ds + sd = 0$, and *not* $ds = sd$.

Proof. By definition, the chain maps f and g are chain homotopic if and only if there exists a sequence $\{H_i\}_{i\in\mathbb{Z}}$ of group homomorphisms

$$\left\{C_i \xrightarrow{H_i} D_{i+1}\right\}_{i\in\mathbb{Z}}$$

such that, for each $i \in \mathbb{Z}$, and each chain $c \in C_i$, the homomorphisms H_{i+1} and H_i satisfy the homotopy condition

$$(f_i - g_i)(c) = (d_i H_{i-1} - H_i d_{i-1})(c). \tag{1.1}$$

On the other hand, $\begin{bmatrix} f & \mathcal{S} & g \end{bmatrix}$ is a chain map if and only if for each $i \in \mathbb{Z}$, the following diagram commutes:

$$\begin{tikzcd}
(\mathrm{cyl}(C))_i \arrow[r, "{[\partial]}"] \arrow[d, "{\begin{bmatrix} f & \mathcal{S} & g \end{bmatrix}}"'] & (\mathrm{cyl}(C))_{i-1} \arrow[d, "{\begin{bmatrix} f & \mathcal{S} & g \end{bmatrix}}"] \\
D_i \arrow[r, "d"] & D_{i-1}
\end{tikzcd} \tag{1.2}$$

We compute the composition $(\mathrm{cyl}(C))_i \to (\mathrm{cyl}(C))_{i-1} \to D_{i-1}$ along the upper right corner of (1.2),

$$\begin{bmatrix} f & \mathcal{S} & g \end{bmatrix} \begin{bmatrix} d & \mathrm{id} & 0 \\ 0 & -d & 0 \\ 0 & -\mathrm{id} & d \end{bmatrix} = \begin{bmatrix} fd & f - sd - g & gd \end{bmatrix}. \tag{1.3}$$

We also compute the composition $(\mathrm{cyl}(C))_i \to D_i \to D_{i-1}$ along the lower left corner of (1.2),

$$\begin{bmatrix} d \end{bmatrix} \begin{bmatrix} f & \mathcal{S} & g \end{bmatrix} = \begin{bmatrix} df & ds & dg \end{bmatrix}. \tag{1.4}$$

Now $df = fd$ and $dg = gd$ by the hypotheses that $f, g \colon C \to D$ are chain maps. Comparing entries in (1.4) and (1.3), the following are equivalent:

- $f, g \colon C \to D$ extends to a chain map $\begin{bmatrix} f & \mathcal{S} & g \end{bmatrix} \colon \mathrm{cyl}(C) \to D$.
- The diagram (1.2) commutes.
- For each i, the maps $(\mathrm{cyl}(C))_i$ to D_{i-1} on the upper right and lower left of (1.2) are equal.
- For each i, the maps[2] $f_{i-1} - s_{i-2}d_{i-1} - g_{i-1} = d_i s_{i-1}$ from the center component of $(\mathrm{cyl}(C))_i$ to D_{i-1} are equal.
- For each i, the levelwise maps satisfy $f - g = ds + sd$.
- $\mathcal{S}$ is a chain homotopy between $f, g \colon C \to D$.

We have proven that $f, g \colon C \to D$ are chain homotopic if and only if they extend to a chain map $\mathrm{cyl}(C) \to D$. $\square$

2. Let A and B be chain complexes. Define the *tensor chain complex* $A \otimes B$ as the graded abelian group

$$(A \otimes B)_n := \bigoplus_{i+j=n} A_i \otimes B_j,$$

with differential, for each i-chain a and each j-chain b in A_i and B_j,

$$\partial(a \otimes b) := d_i^A(a) \otimes b + (-1)^i a \otimes d_j^B(b).$$

Claim (Tensor product of complexes are complexes)**.** $A \otimes B$ with differential ∂ is a chain complex.

[2]Note the indices: the center component of $(\mathrm{cyl}(C))_i$ is C_{i-1}. Maps out of the center component are thus shifted -1 degree from what one might expect.

Proof. It suffices to show that $\partial^2 = 0$, which follows. Consider $a \otimes b$, where $a \in A_i$ and $b \in B_j$

$$\begin{aligned}
\partial^2(a \otimes b) &= \partial\big(d_i^A(a) \otimes b \quad + \quad (-1)^i a \otimes d_j^B(b)\big) \\
&= \partial\big(d_i^A(a) \otimes b\big) \quad + \quad (-1)^i \partial\big(a \otimes d_j^B(b)\big) \\
&= d_{i-1}^A d_i^A(a) \otimes b + (-1)^{i-1} d_i^A(a) \otimes d_j^B(b) \\
&\qquad\qquad\qquad + \quad (-1)^i \big(d_i^A(a) \otimes d_j^B(b) + (-1)^i a \otimes d_{j-1}^B d_j^B(b)\big) \\
&= (-1)^{i-1} d_i^A(a) \otimes d_j^B(b) + (-1)^i \big(d_i^A(a) \otimes d_j^B(b)\big) \\
&= 0.
\end{aligned}$$

We have shown that ∂ is order 2, because $\partial^2(a \otimes b) = 0$, and each chain in any levelwise group $(A \otimes B)_i$ is a linear combination of tensors of the form $a \otimes b$. $\square$

Claim (Mapping cylinders are realized as tensor products)**.** Let I be the chain complex defined

- as the graded abelian group I such that $I_0 = \mathbb{Z}\{\ell_0, \ell_1\}$, $I_1 = \mathbb{Z}\{\ell\}$, and $I_i = 0$ if $i \neq 0, 1$,
- with differential d such that $d_1(\ell) = \ell_1 - \ell_0$ and $d_i = 0$ for all $i \neq 1$.

Then $\operatorname{cyl}(C) \cong I \otimes C$.

Proof. Recognize the free abelian group $\mathbb{Z}\{\ell_0, \ell_1\} \cong \mathbb{Z} \oplus \mathbb{Z}$. Because the tensor product commutes with direct sums, for an arbitrary abelian group $\mathcal{A}$, there's an natural isomorphism

$$\mathbb{Z}\{\ell_0, \ell_1\} \otimes \mathcal{A} \xrightarrow{\cong} (\mathbb{Z} \otimes \mathcal{A})^{\oplus 2}.$$

Moreover, this tensor product is over $\mathbb{Z}$, so $(\mathbb{Z} \otimes \mathcal{A})^{\oplus 2} \cong \mathcal{A} \oplus \mathcal{A}$. Accounting for $\mathbb{Z}\{\ell\}$ in a similar fashion, it follows that, for any degree $i \in \mathbb{Z}$, the abelian group $(I \otimes C)_i$ is naturally isomorphic to the direct sum

$$(\mathbb{Z}\ell_1 \otimes C_i) \oplus (\mathbb{Z}\ell \otimes C_{i-1}) \oplus (\mathbb{Z}\ell_0 \otimes C_i) \xrightarrow{\cong} C_i \oplus C_{i-1} \oplus C_i. \tag{2.1}$$

Therefore, as graded abelian groups, $I \otimes C \cong_{\mathsf{GrAb}} \operatorname{cyl}(C)$.

Considering the RHS and LHS of (2.1), we deduce that d of I induces the differential ∂ on $I \otimes C$ as follows:

$$\left\{ \begin{tikzcd}
\mathbb{Z}\ell_1 \otimes C_i \arrow[r, "\mathrm{id}\otimes d_i"] & \mathbb{Z}\ell_1 \otimes C_{i-1} \\
\mathbb{Z}\ell \otimes C_{i-1} \arrow[r, "-\mathrm{id}\otimes d_{i-1}"] \arrow[ur, "{[\ell\to\ell_0]\otimes\mathrm{id}}"] \arrow[dr, "{-[\ell\to\ell_1]\otimes\mathrm{id}}"] & \mathbb{Z}\ell \otimes C_{i-2} \\
\mathbb{Z}\ell_0 \otimes C_i \arrow[r, "\mathrm{id}\otimes d_i"] & \mathbb{Z}\ell_0 \otimes C_{i-1}
\end{tikzcd} : i \in \mathbb{Z} \right\} \rightsquigarrow \left\{ \begin{tikzcd}
C_i \arrow[r, "d_i"] & C_{i-1} \\
C_{i-1} \arrow[r, "-d_{i-1}"] \arrow[ur, "\mathrm{id}"] \arrow[dr, "-\mathrm{id}"] & C_{i-2} \\
C_i \arrow[r, "d_i"] & C_{i-1}
\end{tikzcd} : i \in \mathbb{Z} \right\}.$$

Hence, if $\varphi \colon I \otimes C \to \operatorname{cyl}(C)$ is the natural isomorphism of graded abelian groups in 2.1, then $\varphi \circ \partial = d \circ \varphi$. So φ is an invertible chain map, thus $I \otimes C \cong_{\mathsf{Ch}} \operatorname{cyl}(C)$ as chain complexes. $\square$

Note. Say that Δ_1 and Δ_2 are abstract ordered simplices. The product $\Delta_1 \times \Delta_2$ contains 6 vertices, so is *not* a 3-simplex. However, there's an operator, call it $\times$, that takes $\Delta_1 \times \Delta_2$ and makes an ordered decomposition into 3 adjacent 3-simplices, each pairwise sharing 3 vertices. How does the rule $\times$ for the decomposition

$$\Delta_1 \times \Delta_2 \xmapsto{\times} \Delta_3 \sqcup \Delta_3 \sqcup \Delta_3 / \sim$$

correspond to the rule on signs for the differential $[\partial]$ on the mapping cylinder? I really don't know. ◀

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\begin{tikzcd}
 & C_{i-1} \arrow[dd, "-d_{i-1}"] \arrow[dddl, "-\mathrm{id}"'] \arrow[dddr, "\mathrm{id}"] & \\
C_i \arrow[ur, dotted] \arrow[dd, "d_i"'] \arrow[ddrr, dotted] & & C_i \arrow[ul, dotted] \arrow[ll, dotted, bend left] \arrow[dd, "d_i"] \\
 & C_{i-2} & \\
C_{i-1} \arrow[ur, dotted] & & C_{i-1} \arrow[ul, dotted] \arrow[ll, dotted, bend left]
\end{tikzcd}
$$