2019-04-12 LECTURE

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There's an announcement for the reading course in homotopy theory, which will probably will start with the "homotopy category" and fibrations.

topic	when
pairings	today
Poincaré duality	soon

Pairings

We have a sign convention[1] for the *maps of cohomological degree p*:

$$\mathrm{Hom}(A_*, B_*)^p = \{f_i : A_i \to B_{i-p}\}.$$

That is, if f has cohomological degree p, then the chain map looks like

$$\begin{tikzcd} A_{i+1} \arrow[r] \arrow[d] & B_{i+1-p} \\ A_i \arrow[r] \arrow[d] & B_{i-p} \\ A_{i-1} \arrow[r] & B_{i-1-p} \end{tikzcd}.$$

Claim. $\mathrm{Hom}(A_*, B_*)^p$ is a chain complex with differential

$$(\delta f)(a) = \partial(f(a)) - (-1)^{\deg f} f(\partial(a)).$$

Why? Consider the (chain map) element $f \in \mathrm{Hom}(A_*, B_*)$. Then f is a cocycle if and only if f has cohomological degree 0.

Corollary. Consider the chain complex *concentrated at degree* 0:

$$G_* = \{0 \quad \text{not in degree 0 else } G \quad \}.$$

With our sign conventions, the degree of each (chain map) element $f \in\in \mathrm{Hom}(A_p, G)$ is $(-1)^{\deg f+1}$. Therefore (by the previous claim)

$$\mathrm{Hom}(A_*, G) \quad \text{is a chain complex with differential} \quad \delta f = (-1)^{\deg f+1} f\partial.$$

Keep in mind that the resolution $F_*(G) \to G$ (for G a finitely generated abelian group) induces a chain map.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & G \arrow[r] & 0 \end{tikzcd}.$$

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[1]This is also (homologically) $\mathrm{Hom}(A_*, B_*)_{-p}$.

Cross products. Let Λ be a commutative unital ring. Define

$$\times\colon \Delta^p(X;\Lambda)\otimes\Delta^q(Y;\Lambda)\to\Delta^{p+q}(X\times Y;\Lambda)$$

such that for (the cochains) $f\colon \Delta_p(X)\to\Lambda$ and $g\colon \Delta_q(X)\to\Lambda$ we have

$$f\otimes g\colon \Delta_p(X)\otimes\Delta_q(Y)\to\Lambda\otimes\Lambda.$$

But we can also map

$$\Lambda\otimes\Lambda\xrightarrow{m}\Lambda$$

so that $\sigma\otimes\tau\mapsto f(\sigma)\otimes g(\tau)\mapsto f(\sigma)g(\tau)$.

Recall from the Eilenberg-Zilber theorem (which is only *hard* for singular chain complexes generated on *spaces*), there's a map $\Theta\colon \Delta_*(X\times Y)\to\Delta_*(X)\otimes\Delta_*(Y)$ for which we have the chain homotopies $\Theta\circ\times\sim\mathrm{id}$ and $\times\circ\theta\sim\mathrm{id}$. Define

$$f\times g=f\otimes g\circ\Theta.$$

Yu asked how

$$\Theta\colon \Delta_p(X\times Y;\Lambda)\to(\Delta_*(x)\otimes\Delta_*(Y))_p$$

could be well defined.

- Fix $f\colon \Delta_p(X)\to\Lambda$,
- then extend to *all of* $\Delta_*(X)$ by varying p.
- **TODO**

Lemma. $\delta(f\times g)=\delta f\times g+(-1)^{\deg f}f\times\delta g$.

Proof. **TODO** (via acyclic models.)

Theorem. There's a natural product

$$\times\colon H^p(X;\Lambda)\otimes H^q(Y;\Lambda)\to H^{p+q}(X\times Y;\Lambda)$$

such that

$$f\otimes g\mapsto f\times g.$$

Proof sketch.

$$\Delta_*(X\times Y)\xrightarrow{\Theta}\Delta_*(X)\otimes\Delta_*(Y)\xrightarrow{f\otimes g}\Lambda\ .$$

Cup products. Define $1\in H^0(X;\mathbf{Z})$ to be the class[2] of $\epsilon\colon \Delta_0(X)\to\mathbf{Z}$.

From the product

$$\begin{tikzcd} X\times Y \arrow[r, "\pi_X"] \arrow[dr, "\pi_Y"] & X \\ & Y \end{tikzcd}$$

we have the induced map on $H^*(X)\to H^*(X\times Y)$ taking α under π_X^* to $\alpha\times 1$ (similarly $\beta\mapsto 1\times\beta$).

In particular, consider the "slice map"

$$\begin{tikzcd} \Delta_p(X)\otimes\Delta_0(X) \arrow[r, "\times"] & \Delta_p(X\times Y) \arrow[r, "\pi_{X\Delta}"] & \Delta_p(X) \\ \tau\otimes y \arrow[r, mapsto] & (\tau\times y)(x)=(\tau(z),y) \arrow[r, mapsto] & \tau \end{tikzcd}$$

TODO Once extended linearly, we have a map on chains, therefore an induced map.

[2]What?

How can we compare $\alpha \times \beta \in H^*(X \times Y; \Lambda)$ to $\beta \times \alpha \in H^*(Y \times X; \Lambda)$? There's a twist map $T\colon X \times Y \to Y \times X$ taking $(\alpha, \beta) \mapsto (\beta, \alpha)$, which induces a map on cohomology

$$H^*(X \times Y; \Lambda) \xleftarrow[T^*]{} H^*(Y \times X; \Lambda)\ .$$

Note. From this, one shows that the cup product is anti-commutative.

To finally define the cup product, we need the diagonal map $d\colon X \to X \times X$ such that $d(x) = (x, x)$. The induced map d^* back on cohomology along with the map from the tensor product into the product (given by Eilenberg-Zilber) allow one to string together (this magic of topological spaces):

$$\begin{array}{ccccc} H^*(X;\Lambda) \otimes H^*(X;\Lambda) & \xrightarrow{\times} & H^*(X \times X;\Lambda) & \xrightarrow{d^*} & H^*(X;\Lambda)\ . \\ \\ \alpha \otimes \beta & \longmapsto & & \longrightarrow & \alpha\text{"cup"}\beta \end{array}$$