MATH 6220 NOTES: TOPOLOGY 2

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Contents

Alas, I only have the last 3 weeks of lectures.

Lecture 1.

2019-04-12

"We'll have a reading course in the Fall to introduce stable/chromatic homotopy theory."

Today, Λ is a commutative unital ring. We'll aim to define the cross product and cup product on cohomology.

Definition 1.1 (Internal hom of chain complexes, Schreiber)**.** Let A and B be chain complexes. Define a chain complex $[A, B]$ to have components

$$[A,B]_n := \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{Mod}}(A_i, B_{i+n})$$

(the collection of *homological degree-n* maps between the underlying graded modules) and whose differential is defined on homogeneously graded elements $f \in [A, B]_n$ by

$$\delta f = \partial_B f - (-1)^n f \partial_A.$$

This complex is variously denoted

$$[A,B]_n = \mathrm{Hom}(A_\bullet, B_\bullet)_n = \mathrm{hom}(A,B) = \{f_i \colon A_i \to B_{i+n}\}_{i\in\mathbb{Z}}.$$

(Dually, maps of *cohomological degree-n* belong to the chain complex $[A, B]_{-n} = [A, B]^n$. Note the differential δf is well defined regardless of whether we're working with homological or cohomological degree.)

> *"One algebraic way to motivate this is to observe that the signs in the differential for the Hom are precisely what is needed for 0-cycles in the* $\mathrm{hom}(A, B)$ *complex to be the set of morphisms of complexes* $A \to B$ *(and also, that the 0th homology group* $H_0(\mathrm{hom}(A, B))$ *is the set of homotopy classes of morphisms* $A \to B$*). This is quite great. Once you decide you want this, all the other signs you mention follow because you need various things to hold. For example, you want the adjunction between* hom *and* $\otimes$ *to hold for the internal versions, so this forces you to add signs to the* $\otimes$*, and so on."*
> –*Mariano Suárez-Álvarez,* `https://math.stackexchange.com/questions/40468/`

Exercise 1.2 (Internal hom is a functor, Schreiber)**.**

(1) Verify $[-,-]\colon \mathsf{Ch}\times\mathsf{Ch}\to\mathsf{Ch}$ is the internal hom functor on Ch.
(2) Verify a map $f \in [A, B]$ of homological degree 0 is a cycle if and only if f is a chain map.
(3) Verify a map $h \in [A, B]$ of homological degree 1 such that $\delta h = f - g$ is a chain homotopy from g to f.
(4) Verify $H_0([A, B])$ is the group of homotopy-equivalence classes of chain maps in $[A, B]_0$.

> *"If one thinks of chain complexes as algebraic analogues of topological spaces and internal hom as an algebraic analogue of the internal hom in a nice category of topological spaces, then 0-cycles are analogues of points and 1-cycles are analogues of homotopies between points, so a 0-cycle in the internal hom is a continuous map and a 0-cycle up to boundaries is a homotopy class of continuous maps." –Qiaochu Yuan, math.stackexchange.com/questions/40468/*

Corollary 1.3. *Say G is a chain complex concentrated at degree 0. Then the cohomological degree of*

$$f\colon A_p \to G_0$$

is p. Since $\partial_G = 0$ (because it maps out of the concentration in degree 0), we see that the chain complex $[A, G]$ has differential $\delta f = (-1)^{\deg f + 1} f \partial_A$.

Note (Concentrated complexes and flat resolutions). For any PID R and finitely generated $M \in \mathsf{RMod}$, the finite flat resolution $F_\bullet(M)$ of M is quasi isomorphic to the complex M concentrated in degree 0:

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_0 \arrow[d] \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \end{tikzcd} \quad \text{is quasi-isomorphic if and only if} \quad 0 \to F_0 \to F_1 \to M \to 0 \quad \text{is exact.}$$

◀

To handle products (especially products of CW-complexes) in singular homology, we proved there exists a natural bilinear map $\times$ (the cross product) from the product of two singular chain complexes to the chain complex of the product of two spaces. With $\times$, we computed the boundary map of the chain complex $\Delta_*(X \times Y)$ using $\Delta_*(X) \times \Delta_*(Y)$ and the *degree of incidence* between n and $n-1$ cells. Then, taking the interval $I = Y$ as the second space gave proof that (CW homology, and thus) singular homology satisfied the homotopy axiom.

Now we seek to define the cross product $\times$ on cochain complexes. (TODO Motivate.) Consider the cochains $f\colon \Delta_p(X) \to \Lambda$ and $g\colon \Delta_q(X) \to \Lambda$. There's a map induced by the ring structure

$$f \otimes g\colon \Delta_p(X) \otimes \Delta_q(Y) \to \Lambda \otimes \Lambda.$$

The tensor product is over Λ, so multiplication $\Lambda \otimes \Lambda \xrightarrow{m} \Lambda$ gives an isomorphism. Composing the previous two maps, for any p- and q-simplices σ and τ,

$$\sigma \otimes \tau \longmapsto f(\sigma) \otimes g(\tau) \longmapsto f(\sigma) g(\tau).$$

Recall also, from the Eilenberg-Zilber theorem, there's a natural chain equivalence $\theta\colon \Delta_*(X \times Y) \to \Delta_*(X) \otimes \Delta_*(Y)$. (The content of this theorem was in constructing a chain homotopies $\theta \circ \times \simeq \mathrm{id}$).

Definition 1.4 (Cross product on cochain complexes)**.** Define the cross product

$$\times\colon \Delta^p(X; \Lambda) \otimes \Delta^q(Y; \Lambda) \to \Delta^{p+q}(X \times Y; \Lambda),$$

by the rule

$$f \times g = f \otimes g \circ \theta.$$

Remark 1.5. Yu asked how

$$\theta\colon \Delta_p(X \times Y; \Lambda) \to (\Delta_*(X) \otimes \Delta_*(Y))_p$$

could be well defined. TODO Write $f\colon \Delta_p(X) \to \Lambda$, then extend to all of $\Delta_*(X)$ (by varying p?). ◀

Lemma 1.6. *$\delta(f \times g) = \delta f \times g + (-1)^{\deg f} f \times \delta g$.*

Proof. Acyclic models. □

Proposition 1.7. *There's a natural (linear map out of the) product of homology groups*

$$\times\colon H^p(X; \Lambda) \otimes H^q(Y; \Lambda) \to H^{p+q}(X \times Y; \Lambda)$$

such that

$$[f] \otimes [g] \longmapsto [f] \times [g],$$

which is induced by

$$\Delta_*(X \times Y) \xrightarrow{\theta} \Delta_*(X) \otimes \Delta_*(Y) \xrightarrow{f \otimes g} \Lambda\ .$$

Proof. Let the unit $1 \in H^0(X;\mathbb{Z})$ be the class of the augmentation $\varepsilon\colon \Delta_0(X) \to \mathbb{Z}$. We appeal to the universal property of the product space $X \times Y$. The component projections induce chain maps on the cochain complexes.

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_X"] \arrow[dr, "\pi_Y"'] & X \\ & Y \end{tikzcd} \quad \overset{\Delta^*(-)}{\rightsquigarrow} \quad \begin{tikzcd} \Delta^*(X \times Y) & \Delta^*(X) \arrow[l, "\pi_X^*"'] \\ & \Delta^*(Y) \arrow[ul, "\pi_Y^*"] \end{tikzcd}$$

I claim that the maps π_X^* and π_Y^* descend to cohomology, i.e., that there're well-defined homomorphisms $H^*(X) \to H^*(X \times Y)$ sending α under π_X^* to $\alpha \times 1$ and symmetrically $H^*(Y) \to H_*(X \times Y)$ sending $\beta \mapsto 1 \times \beta$.

To see this, consider the "slice map"

$$\begin{tikzcd} \Delta_p(X) \otimes \Delta_0(Y) \arrow[r, "\times"] & \Delta_p(X \times Y) \arrow[r, "\pi_{X\Delta}"] & \Delta_p(X) \\ \tau \otimes y \arrow[r, mapsto] & (\tau \times y)(x) = (\tau(z), y) \arrow[r, mapsto] & \tau \end{tikzcd}$$

Extend the slice map linearly to a map on chains, then obtain an induced map on cochains. $\square$

Example 1.8 (The twist map)**.** How can we compare $\alpha \times \beta \in H^*(X \times Y;\Lambda)$ to $\beta \times \alpha \in H^*(Y \times X;\Lambda)$?

Consider the twist map $T\colon X \times Y \to Y \times X$ taking $(\alpha,\beta) \mapsto (\beta,\alpha)$, which induces a map on cohomology.

$$H^*(X \times Y;\Lambda) \xleftarrow[T^*]{} H^*(Y \times X;\Lambda)\ .$$

◂

If $X = Y$, we apply proposition 1.7 and pull back the diagonal map $d\colon X \to X \times X$ (where $x \mapsto (x,x)$) to obtain the cup product.

Definition 1.9 (Cup product)**.** The *cup product* is the bilinear map

$$\begin{tikzcd} H^*(X;\Lambda) \otimes H^*(X;\Lambda) \arrow[r, "\times"] & H^*(X \times X;\Lambda) \arrow[r, "d^*"] & H^*(X;\Lambda)\ , \\ \alpha \otimes \beta \arrow[rr, mapsto] & & \alpha \smile \beta \end{tikzcd}$$

given by the pullback of the diagonal map

$$d^*\colon H^*(X \times X) \to H^*(X)$$

precomposed with the cross product $\times$. (To compute the cup product on cochains f^p and g^q in $\Delta^p(X)$ and $\Delta^q(X)$, see definition 2.7.)

Exercise 1.10. From example 1.8, prove that the cup product is a graded-commutative operation.

Lecture 2.

2019-04-15

"Does the torus $S^1 \times S^1$ have a cohomology ring[1] *that's not a polynomial ring? Does any space?"*

Recall that last week, we proved the Eilenberg-Zilber chain map θ was a natural homotopy inverse of the cross product $\times$. And last lecture, knowing θ and $\times$ are (chain-)homotopy equivalences, we exploited the chain equivalence $\theta\colon \Delta_*(X \times Y) \to \Delta_*(X) \otimes \Delta_*(Y)$ to define the cross product and the cup product on cohomology groups.

Note. It's not at all apparent, however, what map $\times$ induces on cohomology:

$$\times\colon H^*(X;\Lambda) \otimes H^*(Y;\Lambda) \overset{?}{\to} H^*(X \times Y;\Lambda). \tag{2.1}$$

In general, there's no Künneth theorem for the cohomology cross product. But, if the ring of coefficients Λ is a field and either $H_*(X;\Lambda)$ or $H_*(Y;\Lambda)$ is of finite type, then the map in (2.1) is an isomorphism. (E.g., if $\Lambda = \mathbb{R}$ and X is a compact n-manifold.) ◂

This lecture we'll give details for making computations with the cup product. Let X and Y be topological spaces and Λ a commutative unital ring.

[1] Hint. The cohomology ring for S^1 is to $\mathbb{RP}^2$ as – is to – (?) with spheres adjoined (!?). Consider the homogeneous elements of the ring. Just describe the ring. For example, when algebraic topologists write $\mathbb{Z}[x]/\langle x^4 \rangle$, it is understood to be *a homogeneous ring*. Addition is *strictly levelwise*. Products are defined between two homogeneous elements of non-homogeneous degrees.

Exercise 2.2. Say $f\colon A_* \to \Lambda$ and $g\colon B_* \to \Lambda$ are cochains in the complexes $A^*(\Lambda)$ and $B^*(\Lambda)$. Check that the ring structure of $\Lambda \otimes \Lambda = \Lambda$ forces the definition

$$f \otimes g\colon A_* \otimes B_* \to \Lambda$$
$$(f \otimes g)(a \otimes b) = (-1)^{\deg a \deg g} f(a)g(b),$$

for chains $a \in A_*$ and $b \in B_*$.

Corollary 2.3. *For the singular chain complexes over a topological space X, let f, g be cochains in $\Delta^*(X)$ and α, β chains in $\Delta_*(X)$. Then, $(f \otimes g)(\alpha \times \beta) = (-1)^{\deg \alpha \deg g} f(\alpha)g(\beta)$.*

Now, we certainly have an *evaluation* ev from the group of p-cochains $f\colon \Delta_p(X) \to \Lambda$ tensored with the group $\Delta_p(X)$ of p-chains,

$$\Delta^p(X;\Lambda) \otimes \Delta_p(X) \xrightarrow{\text{ev}} \Lambda,$$

defined by

$$\text{ev}\colon f \otimes c \longmapsto f(c).$$

In fact, ev induces a map on cohomology, which is denoted by the angle brackets

$$\langle [f], [c] \rangle \in \Lambda.$$

Lemma 2.4 (Krönecker pairing). *The evaluation $\Delta^p(X;\Lambda) \otimes \Delta_p(X) \xrightarrow{\text{ev}} \Lambda$ induces a Λ-linear map*

$$H^p(X;\Lambda) \otimes H_p(X) \to \Lambda \tag{2.5}$$

such that $\langle |f|, |c| \rangle \mapsto f(c)$.

Proof. (TODO: revise) The *Kronecker pairing* is the argument that $f(c)$ does not depend on representatives f or c (from the cochain, resp, chain complexes). Consider that in the proof of the universal coefficient theorem, we found a map β from $H^*(X;\Lambda)$ to $\text{Hom}(H_p(X), \Lambda)$ such that $|f| \mapsto \{|c| \mapsto f(c)\}$ gave a group homomorphism. Use this. □

Remark 2.6. The *cap product* operation over a topological space X is the above (2.5) pairing, which is "given by combining the *Kronecker pairing* of the cohomology class with the image of the homology class under diagonal and using the Eilenberg-Zilber theorem." (See `https://ncatlab.org/nlab/show/cap+product`.) ⋖

Definition 2.7 (The cup product on cochains)**.** Let $X \in \mathsf{Top}$. The diagonal map $d\colon X \to X \times X$, induces $d_\Delta\colon \Delta_*(X) \to \Delta_*(X)$. Define a natural *diagonal approximation* $\Delta = d_\Delta \circ \theta$, where θ is the chain equivalence from the Eilenberg-Zilber theorem.

$$\begin{tikzcd} 0 \arrow[r] & \Delta_*(X) \arrow[r, "d_\Delta"] \arrow[rr, bend right, "\Delta"'] & \Delta_*(X \times X) \arrow[r, "\theta"] & \Delta_*(X) \otimes \Delta_*(X) \end{tikzcd}$$

The *cup product of homogeneous cochains* f and g is

$$f \smile g = (f \otimes g)\theta d_\Delta.$$

Note. The equation

$$\delta(f \smile g) = \delta f \smile g + (-1)^{\deg} f \smile \delta g$$

follows from the boundary formula for the cross product $\times$. ⋖

Proposition 2.8.

(1) *The cup product is natural for X in* Top *and Λ in* Ring. *Given a continuous map $\varphi\colon X \to Y$ in* Top, *the induced map on cohomology satisfies*

$$\varphi^*(\alpha \smile \beta) = \varphi^*(\alpha) \smile \varphi^*(\beta)$$

for all homogeneous cochains α, β in $\Delta^*(X)$.

(2) $\alpha \smile 1 = \alpha = 1 \smile \alpha$, *where* 1 *is the class of the augmentation* ε *(TODO. Be specific.)*

(3) *The cup product $\smile$ is associative.*

(4) *The cup product is* skew-commutative*:*

$$\alpha \smile \beta = (-1)^{\deg \alpha \deg \beta}(\beta \smile \alpha).$$

Definition 2.9 (Alexander–Whitney diagonal approximation)**.** Let $\sigma\colon \Delta_n \to X$ be a singular n-simplex in X. The *Alexander–Whitney diagonal approximation* explicitly computes the image of σ under the chain map $\Delta\colon \Delta_*(X) \to \Delta_*(X) \otimes \Delta_*(X)$ from the *front and back faces* of σ.

$$\Delta\sigma = \sum_{p+q=n} \|\sigma\|^p_{\text{front}} \otimes \|\sigma\|^q_{\text{back}}.$$

Exercise 2.10. Any two chain maps $\Phi, \Psi\colon \Delta_*(X) \to \Delta_*(X \otimes X)$ that agree with the diagonal approximation

$$\Delta(x) = x \otimes x \quad \text{in the 0th degree}$$

are chain homotopic: $\Phi \simeq \Psi$.

Proposition 2.11 (Computing the cup product)**.** *Say f and g are in the cochains with degrees p and q respectively, such that $p + q = n$. Then*

$$\begin{aligned}
(f \smile g)(\sigma) &= (f \otimes g)(\Delta\sigma) \\
&= (f \otimes g)\left(\sum_{i+j=n} \|\sigma\|^i_{front} \otimes \|\sigma\|^j_{back}\right) \\
&= (f \otimes g)\Big(\|\sigma\|^p_{front} \otimes \|\sigma\|^q_{back}\Big) \\
&= (-1)^{\deg g \deg f} f(\|\sigma\|^p_{front}) g(\|\sigma\|^q_{back}) \quad (\textit{an element of } \Lambda).
\end{aligned}$$

Exercise 2.12 (A derivation from the cup product)**.** Let $A, B \subset X$ in Top be open in X. Verify the following:

(1) $\Delta_*(A) + \Delta_*(B) \twoheadrightarrow \Delta_*(A \smile B)$.
(2) $H^*(X, A; \Lambda) \otimes H^*(X, B; \Lambda) \to H^*(X, A; \Lambda)$. .
(3) From the snake lemma and (2), there's a long exact sequence

$$\cdots \longleftarrow H^{*+1}(X, A; \Lambda) \xleftarrow[\delta^*]{} H^*(A; \Lambda) \xleftarrow[i^*]{} H^*(X; \Lambda) \longleftarrow \cdots$$

that's natural in X, A, B and Λ.
(4) The connecting homomorphism δ is a derivation (TODO of what?) defined by

$$\delta^*(\alpha \smile i^*(\beta)) = \alpha \smile \delta^*(\beta).$$

Lecture 3.

2019-04-19

"À bas Euclide! Mort aux triangles![2] [Down with Euclid! Death to triangles!]"

Today, we'll define the cap product and sketch Poincaré duality for (nice) topological manifolds. Let $X \in \mathsf{Top}$. To reference the degrees of cochains in $\Delta^p(X)$ and chains in $\Delta_q(X)$ let p, n be nonnegative integers with $q = n - p$. We'll regard a p-cochain f to "be defined but equal to zero on i-simplices when $i \neq p$". (So for $c \in \Delta_i(X)$, $f(c) = 0$ if $i \neq p$.)

Definition 3.1 (The cap product on cochains and chains)**.** Suppose $f \in \Delta^p(X; \Lambda)$ and $c \in \Delta_n(X)$. The *cap product* on the chain–cochain level is the linear map

$$\begin{aligned}
\cap\colon \Delta^p(X) \otimes \Delta_n(X) &\to \Delta_{n-p=q}(X) \\
f \otimes c &\xmapsto{\cap} f \cap c := (1 \otimes f)\Delta c
\end{aligned}$$

where $\Delta\colon \Delta_*(X) \to \Delta_*(X) \otimes \Delta_*(X)$ is some diagonal approximation.

As a consequence, if Δ is the Alexander–Whitney diagonal approximation and σ is an n-simplex in $\Delta_n(X)$, then

$$\begin{aligned}
f \cap \sigma &= (1 \otimes f)\Delta\sigma \\
&= (1 \otimes f) \sum_{p+q=n} (\sigma\rfloor_q \otimes {}_p\lfloor\sigma) \\
&= (-1)^{pq} f({}_p\lfloor\sigma) \cdot \sigma\rfloor_{n-p}.
\end{aligned}$$

Proposition 3.2 (Properties of the cap product with respect to other operations)**.**

(1) *(Augmentation) For the augmentation $\varepsilon\colon \Delta_0(X) \to \Lambda$ and any 0-chain $c \in \Delta_0(X)$,*

$$\varepsilon \cap c = c.$$

[2] Jean Dieudonné, keynote address at the Royaumont Seminar (1959)

(2) *(Krönecker pairing) For any cochain $f \in \Delta^p(X)$ and chain $c \in \Delta_p(X)$ (of the same degree),*

$$\begin{array}{ccccc} \Delta^p(X) \otimes \Delta_p(X) & \xrightarrow{\cap} & \Delta_0(X) & \xrightarrow{\varepsilon} & \Lambda \\ f \otimes c & \longmapsto & f \cap c & \longmapsto & \varepsilon(f \cap c) = f(c) \end{array}$$

That is, the cap product coincides with the Krönecker pairing (if interpreted correctly).

(3) *(Cup product) For any two cochains $f \in \Delta^p(X)$, $g \in \Delta^k(X)$, and any chain $c \in \Delta_{n+k}(X)$,*

$$(f \cup g) \cap c = f \cap (g \cap c) \quad \textit{(which is 0 if } p + k \geq n\textit{)}.$$

(4) *(Induced chain maps) For any map of spaces $X \xrightarrow{\varphi} Y$, any cochain $f \in \Delta^p(X)$, and any chain $c \in \Delta_n(X)$, the chain maps φ_Δ and φ^Δ satisfy*

$$\varphi_\Delta\big(\varphi^\Delta(f) \cap c\big) = f \cap \varphi_\Delta(c).$$

(5) *(Boundary maps) For any cochain f and chain c,*

$$\partial(f \cap c) = \delta(f) \cap c + (-1)^{\deg f} f \cap \partial c.$$

The cap product on chains and cochains descends to (co)homology by the boundary formula above (2.7).

$$\frown : H^p(X) \otimes H_n(X) \to H_{n-p}(X).$$

Proposition 3.3 (Relation of the cap product on homology classes to other operations)**.**

(1) *(Triviality) For $1 \in H^0(X)$ the class of the augmentation, and any $\gamma \in H_n(X)$,*

$$1 \frown \gamma = \gamma.$$

(2) *(Krönecker pairing) For ε_* induced from the augmentation, $\alpha \in H^p$, and $\gamma \in H_p$,*

$$\varepsilon_*(\alpha \frown \gamma) = \langle \alpha, \gamma \rangle .$$

(3) *(Cup product) $(\alpha \smile \beta) \frown \gamma = \alpha \frown (\beta \frown \gamma)$.*
(4) *(Naturality) $\varphi_*(\varphi^*\alpha \frown \gamma) = \alpha \frown \varphi_*\gamma$.*
(5) *(Annihilation) For $\alpha \smile \beta$ in H^p, $\gamma \in H_p$, TODO.*
(6) *(Cross product) We have $(\alpha \times \beta) \cap (a \times b) = (-1)^{\deg \alpha \deg \beta}(\alpha \cap a) \times (\beta \cap b)$.*

The cup and cap product have an *adjoint-ish* relationship with each other (but, on facebook, `it's complicated`).

Note. Manifolds will have a canonical class to cap with. The purpose of the development now is to reach Poincaré duality for manifolds. Suppose I have a symmetric monoidal category. Then I have a notion of "a ring object" in the category. The cap product then gives a *pairing*. ⋖

Remark 3.4 (Fundamental class of a manifold). Say $M \in \mathsf{Man}$ is a (closed, compact, orientable) n-manifold. Then with $[M] \in H_n(M)$ the fundamental class, the pairing

$$\begin{array}{c} H^p(M) \otimes H_n(M) \to H_{n-p}(M) \\ \alpha \longmapsto \alpha \cap [M] \end{array}$$

will induce an isomorphism. ⋖

Definition 3.5 (Topological manifolds)**.** An n-manifold is a (second-countable) Hausdorff topological space M such that every $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^n$.

Remark 3.6 ("Feeling" version of dual triangulations). Say that I triangulate M with simplices σ_i such that the alternating face maps are coherent with the triangulation. If it is possible to define coherently, the *fundamental class* of the manifold M is the boundary $\partial(\sum_i \sigma_i) = [M]$. We'll define this rigorously in a bit.

For example, $\mathbb{RP}^2$ can be triangulated, but fails to admit oriented 2-cells. ⋖

Example 3.7 (Dual cells as indicator functions)**.** Let $M \in \mathsf{Man}$ be sufficiently nice (compact, orientable) of dimension n (for example, 2). Then the 0-cells (say, u, w, and v) are *paired* to the *indicator functions* on the 0-cells.

$$D(u)^*, D(w)^*, D(v)^* : C_2^D(M) \to \Lambda. \tag{3.8}$$

The dual cell structure arises from assigning each k-cell to a $n-k$-cell by the rule

$$\langle u_0, u_1, \ldots, u_k \rangle \leftrightsquigarrow D(u_0) \cap D(u_1) \cap \ldots \cap D(u_k).$$

⋖

Exercise 3.9. Find a planar graph (V, E, F), and algorithmically compute the dual graph.

Lecture 4.

2019-04-22

In the last section of the course, for orientations and Poincaré duality, we'll make the lateral move to Peter May's Concise.

Fix an $n \in \mathbb{N}$ today for the dimension of our topological manifolds. E.g., we work in Man^n.

Remark 4.1 (Local excision). Let $x \in U \subset M$ be a point of the manifold $M \in \mathsf{Man}^n$, with U a chart domain. From an application of excision, the coordinate balls in M have trivial homology groups:

$$H_i(M, M - x; G) \cong H_i(U, U - x; G) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

That is, for any $x \in U$, there's an isomorphism $H_n(M, M \setminus U) \xrightarrow{\cong} H_n(M, M - x)$.

Fix coefficients over a commutative unital ring Λ.

Definition 4.2 (Fundamental class)**.** A *fundamental class* of M at $X \subset M$ is a class $z \in H_n(M, M \setminus X; \Lambda)$ such that the image of z in $H_n(M, M - x; \Lambda)$ is a generator for all $x \in X$.

$$j_{x,X} \colon H_n(M, M \setminus X) \to H_n(M, M - x) \cong \Lambda$$

$$z \xmapsto{j_{x,X}} j_{x,:}(z) \text{ which needs to be a unit in } \Lambda.$$

Note. Observe that the fundamental class at X doesn't necessarily exist, but if U is a "fundamental" open set, then this class *always* exists.

Definition 4.3 (Orientation (May))**.** An Λ-orientation is an open cover $\{U_i\}$ of M with fundamental classes z_i of M at U_i

$$z_i \in H_i(M, M \setminus U_i; \Lambda)$$

such that z_i and z_j map to the same class in the intersected "local homology"

$$H_i\left(M, M \setminus \underset{\text{nonempty}}{(U_i \cap U_j)}; \Lambda\right)$$

(Note that the small neighborhood $U_i \cap U_j$ is more susceptible to *receive* induced homology maps than are either of the larger open neighborhoods U_i or U_j.)

Here's a differential geometric interpretation. Pick a frame in some n-manifold over some point $a \in M$. Consider the set S_a of all frames over points b such that there's a continuous path from a to b in M. Either, for some homotopy class of paths in M that return to a, there exist frames in S_a whose determinant differs by -1, or no such homotopy class of paths exists. In the later case, M is orientable. *It's an n-finger rule!*

Example 4.4 (Orientation double cover)**.** Suppose now that the coefficient ring is $\mathbb{Z}$. Define the *orientation double cover of M* to be the set of pairs

$$\tilde{M}_{\mathbb{Z}} = \{(x, a) : x \in M, a \in H_n(M, M - x)\}$$

En masse, say we have an open cover $\mathcal{U}$ of M. For each open set $U \subset M$, we need to define a *topology basis* for the orientation bundle $\tilde{m}$. But for U in the open cover $\mathcal{U}$, there's a fundamental class $a \in H_n(M, M \setminus U)$. So take a open set $U_a \subset \tilde{M}$ (thought of as the open set $U \subset M$ "evaluated" at the class a) defined by

$$U_a = \{(x, b) \in \tilde{M} : x \in U, b = j_{x,U}(a)\}$$

as a basis element for the topology of the orientation bundle $\tilde{M}$.

Proposition 4.5 (The orientation bundle is a double cover)**.** *As defined in 4.4, the projection from the bundle $\tilde{M} \twoheadrightarrow M$ such that $(x, a) \mapsto x$ is a double cover. (It's a trivial, path disconnected, double cover if M is orientable; it's* always *a double cover.)*

Proof. TODO. Here's the sketch: consider the lift of an open set $U \subset M$. Then $p^{-1}(U)$ should be isomorphic to the group of units $(\mathbb{Z}/(2))$ of the local homology. We want to establish the double cover on U.

$$\begin{tikzcd} p^{-1}(U) \arrow[dr] & U \times \mathbb{Z}^\times \arrow[l] \arrow[d] \\ & U \end{tikzcd}$$

□

Definition 4.6 (Orientation (Bredon))**.** Let $X \subset M$ be any subset of the manifold M. Let $\tilde{M}$ be the orientation bundle. Then *an orientation of M along X* is a continuous section $X \xrightarrow{s} \tilde{M} \xrightarrow{p} M$ with $p \circ s = \mathrm{id}$.

Exercise 4.7 ($SU(2)$ is a cover of $SO(3)$)**.** Show that either $SU(2)$ is or is not an orientation bundle of $S0(3)$.

Proposition 4.8 (Sufficient conditions for orientability)**.** *Let M be a connected n-manifold. The following are equivalent.*

(1) *M is orientable.*
(2) *M is orientable along any compact subset.*
(3) *$\tilde{M}$ is a trivial double cover, and $\tilde{M} \cong M \sqcup M$.*

Note the two definitions are almost equivalent, but the former definition was a bit too *raw*, as in, May gave us an actual bundle, not an equivalence class of bundles.

Proof. 1 implies 2. Consider the definition of continuous sections, then march along any compact subset (get a Lebesgue cover). 2 implies 3. Assume also that $\tilde{M}$ is connected. Let $x \in M$. Let $\sigma\colon [0,1] \to \tilde{M}$ be a path, starting at (x,a) and ending at $(x,-a)$. Then project $\sigma([0,1])$ down to M. Consider that $p\sigma([0,1]) \subset M$ is compact, so by (2) there's a section $s\colon p\sigma([0,1]) \to \tilde{M}$ along the image of the path.

$$\begin{tikzcd} & \tilde{M} \\ {[0,1]} \arrow[ur, "\sigma"] \arrow[r, "p\sigma"'] & M \arrow[u, "s"] \end{tikzcd}$$

3 implies 1. If there's a trivial double cover of M, then it's easy (since the projection is a local homeomorphism) to obtain a continuous section from M back into the orientation bundle $\tilde{M}$. ☐

Proposition 4.9 ($\mathbb{Z}/2$ orientation)**.** *Let the ring of coefficients $\Lambda = \mathbb{Z}/(2)$. (TODO, is this true for* any *2-torsion ring?) Running the same argument as in the proof of 4.8, the projection from $\tilde{M}_{\mathbb{Z}}$ to M induces a projection from the $\mathbb{Z}/(2)$ orientation bundle. TFDC:*

$$\begin{tikzcd} \tilde{M}_{\mathbb{Z}} \arrow[rr] \arrow[dr] & & \tilde{M}_{\mathbb{Z}/2} \arrow[dl] \\ & M & \end{tikzcd}$$

It follows that $\tilde{M}_{\mathbb{Z}/2} \cong M \sqcup M$, so that M is orientable $\mod 2$.