INTRODUCTION TO HOMOLOGY

COLTON GRAINGER (MATH 6220 TOPOLOGY 2)

Problems due 2019-01-30

[1, p. 171]. *Given.* Let $X = \bigcup_{i \in I} X_i$ where the X_i are the path components of X.

To prove. The homology $H_*(X)$ is isomorphic (as a graded abelian group) to the direct sum $\oplus_{i \in I} H_*(X_i)$.

Proof. For $p \in \mathbf{Z}_{\geq 0}$, we'll consider this portion of the chain complex:

$$\Delta_{p+1}(X) \xrightarrow{\partial_{p+1}} \Delta_p(X) \xrightarrow{\partial_p} \Delta_{p-1}(X) \text{ which is 0 when } p = 0.$$

We will will argue

$$\begin{aligned} \bigoplus_i H_p(X_i) &:= \bigoplus_i \frac{Z_p(X_i)}{B_p(X_i)} \\ &\cong \frac{\bigoplus_i Z_p(X_i)}{\bigoplus_i B_p(X_i)} && \text{by module theory} \\ &\cong \frac{Z_p(X)}{B_p(X)} && \text{TODO} \\ &=: H_p(X). \end{aligned}$$

First, suppose $p = 0$ (this should feel similar to augmentation [1, No. IV.2.3]).

group isomorphism	justification
the cycle group $Z_0(X) \cong \oplus_i Z_0(X_i)$	By convention, the 0-chain group $\Delta_0(X) \to 0$. Since $Z_0(X)$ is defined as the kernel, we have $\mathbf{Z}_0(X) := \Delta_0(X)$. Manipulating $\Delta_0(X)$ as the free abelian group (the direct sum of $\lvert X \rvert$ copies of the integers), $\mathbf{Z}\{X\} = \oplus_i \mathbf{Z}\{x \in X_i\} \cong \oplus_i Z_0(X_i)$.
the boundary group $B_0(X) \cong \oplus_i B_0(X_i)$	Note $\operatorname{Sing}_1(X) = \sqcup_i \operatorname{Sing}_1(X_i)$, as the image of a 1-simplex must lie in entirely in a path component. Now ∂ is defined on generators, so consider $\partial(\operatorname{Sing}_1(X)) = \partial(\sqcup_i \operatorname{Sing}_1(X_i)) = \sqcup_i \partial(\operatorname{Sing}_1(X_i)) = \sqcup_i \{x - y : x, y \in X_i\}$. Extending linearly, $B_0(X) = \mathbf{Z}\{\sqcup_i \{x - y : x, y \in X_i\}\} \cong \oplus_i \mathbf{Z}\{x - y : x, y \in X_i\} \cong \oplus_i B_0(X_i)$.

Now say $p > 0$. As before, any image of a simplex lies entirely in a path component:

$$\operatorname{Sing}_p(X) = \sqcup_i \operatorname{Sing}_p(X_i).$$

Date: 2019-01-14.

group isomorphism	justification
cycles $Z_p(X) \cong \oplus_i Z_p(X_i)$	Consider $\ker \partial_p$. $$0 = \partial\left(\sum_\sigma n_\sigma \sigma\right) = \sum_\sigma n_\sigma \partial\sigma$$ iff, (if $\sigma \in \mathrm{Sing}_p(X_i)$, then $\partial\sigma \in \mathrm{Sing}_{p-1}(X_i)$), $$0 = \sum_i \left(\sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma \delta\sigma\right)$$ which occurs iff for each i, $$0 = \sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma \partial\|_{\mathrm{Sing}_p(X_i)}\sigma.$$ So $$\ker\partial = \bigoplus_i \ker\partial\|_{\Delta_p(X_i)}.$$
boundaries $B_p(X) \cong \oplus_i B_p(X_i)$	For each path component i, does the restriction $\partial\|_{\Delta_{p+1}(X_i)}$ map into $\Delta_p(X_i)$ alone? Consider ∂ on generators $\partial\|_{\mathrm{Sing}\,p+1(X_i)}$. Does $\delta\sigma_{p+1}$ map into $\Delta_p(X_i)$? Yes! (Consider the face maps.)

In conclusion, for all $p \in \mathbf{Z}_{\geq 0}$:

$$\bigoplus_i H_p(X_i) \cong \frac{\bigoplus_i Z_p(X_i)}{\bigoplus_i B_p(X_i)} \cong \frac{Z_p(X)}{B_p(X)} =: H_p(X).$$

□

[1, p. 176]. *Given.* Let X and Y be path connected spaces, and let $f\colon X \to Y$ be a continuous map.

To prove. f induces $f_*\colon H_0(X) \xrightarrow{\cong} H_0(Y)$.

Proof sketch. By [1], that X and Y are path connected implies for any $x \in X$, $y \in Y$, the homology groups $H_0(X)$ and $H_0(Y)$ are infinite cyclic, and generated by $[[x]]$ and $[[y]]$ respectively. Because $f_*\colon [[x]] \mapsto [[f(x)]]$ sends one generator to another, it's an isomorphism. □

To prove. Any map $f\colon X \to X$ induces the identity on $H_0(X)$.

Proof sketch. In this case, $f_*\colon [[x]] \mapsto [[f(x)]]$. Path connectedness implies $[[x]] = [[f(x)]]$. □

[1, No. IV.3.7].

(a) Compute $H_1(K)$ for the Klein bottle K.

- Apply Seifert Van Kampen [1, Figs. III–11] to find $\pi_1(K) = \langle a, b | bab^{-1}a^{-1} \rangle$.
- Quotient by the commutator, $\langle aba^{-1}b^{-1} \rangle$.
- By transitivity of factorization, $H_1(K) \cong \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$.

(b) Compute H_1 of $X = \prod_{j \in J} X_j$ for topological spaces X_j in terms of $H_1(X_j)$.

- The fundamental group of a product is the product of fundamental groups.
- So $\pi_1(X) \cong \prod \pi_1(X_j)$.
- It suffices to abelianize each component group.
  - Elements from X_i and X_j commute whenever $i \neq j$.
- We conclude $H_1(X) \cong \prod H_1(X_j)$.

(c) Let X_i with base points $x_i \in X_i$. Suppose that there are open sets $x_i \in U_i \subseteq X_i$ such that x_i is a deformation retract of U_i. Show that

$$H_1\left(\bigvee_{i \in I} X_i\right) \cong \bigoplus_{i \in I} H_1(X_i).$$

- Apply Seifert Van Kampen [1, Sec. III.9] to find $\pi_1(\vee_i X_i) = \bigstar_i \pi_1(X_i)$.

  The free product is the coproduct in the category of groups. That is, the free product plays the same role in group theory that disjoint union plays in set theory, or that the direct sum plays in module theory.

- Abelianize $\bigstar_i \pi_1(X_i)$ as $\bigoplus_i H_1(X_i)$.

[1, p. 177]. *Given.* Let $f\colon X \to Y$ be a covering space of path connected spaces with $f(x_0) = y_0$. By the fundamental theorem of covering spaces, $f_\#\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a monomorphism. Is $f_*\colon H_*(X) \to H_*(Y)$ also a monomorphism?

To demonstrate. Not necessarily. We'll consider a cover X of $Y = \mathbf{C} \setminus \{0, 1\}$ the plane with two points removed.

Demo.

- Let $Y = \mathbf{C} \setminus \{0, 1\}$ and fix a point $p \in Y$.
- Deform retracting and applying Seifert van-Kampen,
$$\pi_1(Y, p) \cong \langle a, b \rangle.$$
- Because $aba^{-1}b^{-1}$ is nontrivial, $\langle aba^{-1}b^{-1} \rangle \cong \mathbf{Z}$ is an infinite cyclic subgroup of the fundamental group of the plane with two points removed.
  - Sketch $\mathbf{C}\{0, 1\}$ and the path $abab^{-1}$.
- By the Galois correspondence between covering spaces and subgroups of the fundamental group [2, No. 1.36], there's a cover $f\colon X \to Y$ such that
$$f_\#(\pi_1(X, q)) = \langle aba^{-1}b^{-1} \rangle \subset \pi_1(Y, p).$$
  - It's a cover X of Y for which the lift of $aba^{-1}b^{-1}$ is a loop without self-intersections.
  - Sketch Cayley diagram of universal cover. Sketch cyclic generator.
- Now $H_1(X) \cong \mathbf{Z}$, $H_1(Y) \cong \mathbf{Z}^2$, yet
$$f_*([[\underbrace{aba^{-1}b^{-1}}_{\text{in } \Delta_1(X)}]]) = [[a]] + [[b]] - [[a]] - [[b]],$$
which is *not* an induced monomorphism. □

References

[1] G. E. Bredon, *Topology and Geometry.* New York: Springer-Verlag, 1993.

[2] A. Hatcher, *Algebraic Topology.* Cambridge University Press, 2002.