MATH 6220 HOMEWORK 6

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1. (a) Chapter V, Section 7, 284: 5. *If $H_*(X)$ is finitely generated, then*
$$\chi(X) = \sum(-1)^i \dim H_i(X; \Lambda) \quad \textit{for any field } \Lambda.$$

(b) Chapter VI, Section 1, 321: 3. *For spaces X, Y of bounded finite type,*
$$\chi(X \times Y) = \chi(X)\chi(Y).$$

Note (See Chapter VI, Section 4 and Example 4.12.). By a *graded commutative ring*, we will mean a graded abelian group R^* together with a homomorphism of graded abelian groups $\mu\colon R^* \otimes R^* \to R^*$ such that,

- There exists $1 \in R^0$ which is a two sided unit for μ, and
- $\mu(a \otimes b) = (-1)^{\deg(a)\deg(b)}\mu(b \otimes a)$.

The cup product gives a graded commutative ring structure on the cohomology of a space. ◁

2. Chapter VI, Section 1, 321: 2. *Let X_p be the space resulting from attaching an n-cell to S^{n-1} by a map of degree p. Use the Künneth Theorem to compute the homology of $X_p \times X_q$ for any p, q.*

3. (a) *Write down the ring structure of* $H^*(S^n)$ *and of* $H^*(S^n \times S^m)$.

(b) *We will see later that* $H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x]/\langle x^{n+1} \rangle$ *for* $x \in H^1(\mathbb{R}P^n; \mathbb{Z}/2)$. *Use this to prove that* $\mathbb{R}P^3$ *is not homotopy equivalent to* $\mathbb{R}P^2 \vee S^3$.

(c) Chapter VI, 334: 3. *Show that any map* $S^4 \to S^2 \times S^2$ *must induce the zero homomorphism on* $H_4(-)$.

4. Chapter VI, 334: 5. *Any two chain maps $\alpha, \beta \colon \Delta_*(X) \to \Delta_*(X \otimes X)$ that agree with the diagonal approximation*

$$\Delta(x) = x \otimes x \quad \textit{in the 0th degree}$$

are chain homotopic: $\alpha \simeq \beta$.

Proof. Let X be a topological space. We will manipulate the functor:

$$\begin{tikzcd} \mathsf{Top} \arrow[r, "\mathrm{Maps}(\Delta^n_{\mathsf{Top}}, -)"] \arrow[rrr, bend right, "\Delta_*(-)"'] & \mathsf{sSet} \arrow[r, "\mathbb{Z}\langle - \rangle"] & \mathsf{sAb} \arrow[r, "C(-)"] & \mathsf{Ch}_*^+ \end{tikzcd} \tag{4.1}$$

that eventually[1] sends X to its singular chain complex $\Delta_*(X)$.

So, let $\alpha, \beta \colon \Delta_*(X) \to \Delta_*(X) \otimes \Delta_*(X)$ be chain homomorphisms such that $\alpha = \beta$ on $\Delta_0(X)$:

$$\alpha(x) = \beta(x) = x \otimes x \quad \text{for } x \colon \Delta_0 \to X.$$

To see that $\alpha \simeq \beta$ are chain homotopic, we need to construct a sequence of homomorphisms

$$\left\{h_n \colon \Delta_n(X) \to (\Delta_*(X) \otimes \Delta_*(X))_{n+1}\right\}_{n \in \mathbb{Z}_{\geq 0}}$$

such that, for each $n \in \mathbb{Z}_{\geq 0}$ and each singular simplex $\sigma \in \Delta_n(X)$, the homomorphisms h_n and h_{n-1} satisfy the homotopy condition

$$(\alpha_n - \beta_n)(\sigma) = (\delta_{n+1} h_n - h_{n-1} \partial_n)(\sigma), \tag{4.2}$$

which is an equality of chains in $(\Delta_*(X) \otimes \Delta_*(X))_n$.

For the BASE CASE, let X be *any topological space* and α, β *any diagonal approximations.* Define

$$h_0 \colon \Delta_0(X) \to \Delta_0(X) \otimes \Delta_0(X)$$

by $h_0(x) = 0$. Then (trivially)

$$0 = \alpha_0 - \beta_0 = \delta_1 h_0 - h_{-1} \partial_0$$

as α and β are diagonal approximations that agree on 0-simplices.

For the INDUCTIVE STEP, let X, α, and β be as above, and say the first $p < n$ homomorphisms

$$\left\{h_p \colon \Delta_p(X) \to (\Delta_*(X) \otimes \Delta_*(X))_{p+1}\right\}_{p<n}$$

and satisfy the homotopy condition (4.2).

We will complete the inductive step finding a chain homotopy η on an acyclic model, Δ_n. Consider the identity map $\iota_n \colon \Delta_n \to \Delta_n$ from the topological n-simplex Δ_n to itself. Note ι_n is both a singular n-simplex

$$\iota_n \in \mathrm{Sing}_n := \mathrm{Maps}(\Delta^n_{\mathsf{Top}}, \Delta_n),$$

and a continuous map between topological spaces

$$\Delta_n \xrightarrow{\iota_n} \Delta_n.$$

In particular,

$$\delta \iota_n \in \Delta_{n-1}(\Delta_n)$$

is a $n-1$-chain. So say that A and B are diagonal approximations for the singular chain complex $\Delta_*(\Delta)_n$. By the inductive hypothesis, the homomorphisms η_p for $p < n$ are defined. Therefore

$$\begin{aligned} &(A_{n-1} - B_{n-1})(\partial \iota_n) = (\delta_n \eta_{n-1} - \eta_{n-2} \partial_{n-1})(\partial \iota_r) \\ \iff\ &(A_{n-1} - B_{n-1} - \eta_{n-2} \partial_{n-1})(\partial \iota_n) = (\delta_n \eta_{n-1})(\partial \iota_r) \\ \iff\ &(A_{n-1} - B_{n-1})(\partial \iota_n) = (\delta_n \eta_{n-1})(\partial \iota_r) \end{aligned}$$

is an equality of chains in $(\Delta_*(\Delta)_r \otimes \Delta_*(\Delta)_n)_{n-1}$.

[1]Here $\mathrm{Maps}(\Delta^n_{\mathsf{Top}}, -)$ takes the space X to the simplicial set $\mathrm{Sing}X$, which $\mathbb{Z}\langle - \rangle$ takes to the simplicial (free) abelian group $\mathbb{Z}\langle \mathrm{Sing}X \rangle$, which then $C(-)$ takes to the alternating face map complex $\Delta_*(X)$.

Computing the boundary of ... (we finished this proof in lecture.) $\square$