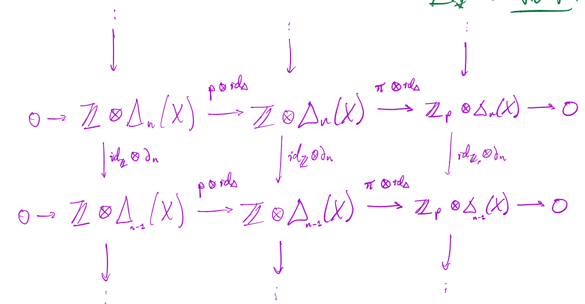
(2) Let $\mathrm{Tor}(\mathbb{Z}/p, H_{n-1}(X)) = \ker(H_{n-1}(X) \xrightarrow{p} H_{n-1}(X))$. Prove that there is a split exact sequence

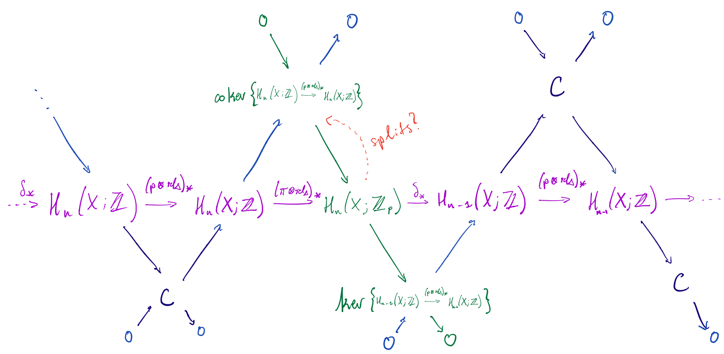
$$0 \to H_n(X;\mathbb{Z}) \otimes \mathbb{Z}/p \to H_n(X;\mathbb{Z}/p) \to \mathrm{Tor}(\mathbb{Z}/p, H_{n-1}(X)) \to 0.$$

$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/(p) \longrightarrow 0$ exact ... Δ_* is flat.



carry this ses to a les $\cdots \xrightarrow{\delta_*} H_p(X;G) \xrightarrow{i_*} H_p(X;G') \xrightarrow{j_*} H_p(X;G'') \xrightarrow{\delta_*} H_{p-1}(X;G) \xrightarrow{i_*}$

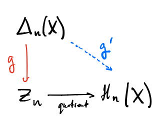
The *homology of C_* with coefficients in G*, denoted $H_*(C_*;G)$, is the homology of the chain complex $G \otimes C_*$, whose pth group is $G \otimes C_p$ and boundary $\partial_p^{C_*,G} = \mathrm{id}_G \otimes \partial_p$.



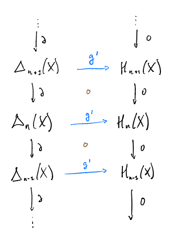
splitting? Consider the ses. $0 \longrightarrow Z_n \xrightarrow{f} \Delta_n(X) \longrightarrow B_{n-1} \longrightarrow 0$

which splits. $g \circ f = \mathrm{id}_{Z_n}$ (they're all free abelian)

Extend g to the quotient $H_n(X)$.

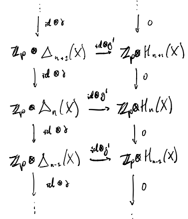


Consider $\cdots \xrightarrow{\partial} H_{n+1}(X) \xrightarrow{0} H_n(X) \xrightarrow{0} H_{n-1}(X) \xrightarrow{0} \cdots$ as the chain complex $H_*(X)$. We induce a chain map G from $\Delta_*(X)$ to $H_*(X)$ given g'. Observe that $\partial\Delta_{n+1}(X) = B_n(X) = \ker(Z_n \to H_n(X))$ so that each square commutes



$\Delta_*(X) \xrightarrow{G} H_*(X)$

Tensor both chains by $\mathbb{Z}_p$ to obtain



$\mathbb{Z}_p \otimes \Delta_*(X) \xrightarrow{\mathrm{id}\otimes G} \mathbb{Z}_p \otimes H_*(X)$

When we take the homology on $\mathbb{Z}_p \otimes \Delta_*(X)$, by definition, we have $H_*(X,\mathbb{Z}_p)$. For the RHS, because the image of each map in the complex is trivial, we obtain $\mathbb{Z}_p \otimes H_*(X)$. The induced map on homology gives the desired chain map

$$H_*(X,\mathbb{Z}_p) \xrightarrow{(\mathrm{id}\otimes G)_*} \mathbb{Z}_p \otimes H_*(X)$$

In particular, at the nth level, we have

$$H_n(X,\mathbb{Z}_p) \xrightarrow{\mathrm{id}\otimes g'} \mathbb{Z}_p \otimes H_n(X) \cong \operatorname{coker}\{H_n(X;\mathbb{Z}) \xrightarrow{(p\otimes \mathrm{id})_*} H_n(X;\mathbb{Z})\}$$

for $A \in \mathrm{Ab}$, $\mathbb{Z}_p \otimes_{\mathbb{Z}} A \simeq A/p$

1. Multiplication by the prime $p: \mathbf{Z} \to \mathbf{Z}$ fits in a short exact sequence

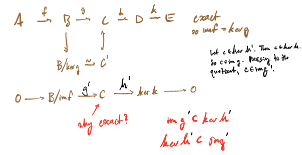
$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}_p \to 0.$$

Use this to derive the natural split exact sequence

$$0 \to \frac{H_n(X)}{pH_n(X)} \to H_n(X;\mathbf{Z}_p) \to \ker\{p: H_{n-1}(X) \to H_{n-1}(X)\} \to 0.$$

(The splitting is not natural.)

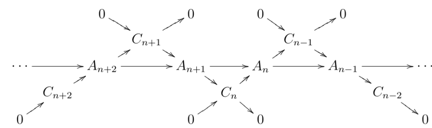
We move from a l.e.s. to a s.e.s again...



The significance of short exact sequence shows up when we try to break down a long exact sequence into short exact sequences. Consider the exact sequence of R-modules $\cdots \to A_{n+2} \to A_{n+1} \to A_n \to A_{n-1} \to A_{n-2} \to \cdots$. Let

$$C_n \cong \ker(A_n \to A_{n-1}) \cong \mathrm{im}(A_{n+1} \to A_n). \qquad \text{[Che09]}$$

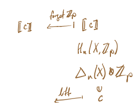
As the algebraic structure underlying R-module is abelian group, the cokernel of each homomorphism exists such that $C_n \cong \mathrm{coker}(A_{n+2} \to A_{n+1})$. Then we obtain the following commutative diagram, in which all the diagonal sequences are short exact:



(2.5)

Conversely, given any short exact sequences overlapped in this way, their middle terms form an exact sequence.

Mauvin on splitting:



not nec. well defined...

$\Delta_n(X) \otimes \mathbb{Z}_p \cong \Delta_n(X)/(p)$

now observe $H_n(X,\mathbb{Z}_p)$ is larger!

Another idea: a ses of vector spaces always splits.

$\Delta_n(X)/p \cong Z_n(X)/p \oplus B_{n-1}(X)/p$

Theorem 3.21. *If C is a chain complex of free abelian groups, then there are natural short exact sequences*

$$0 \to H_n(C) \otimes G \to H_n(C;G) \to Tor_1(H_{n-1}(C), G) \to 0 \qquad (3.22)$$

for all n and all G, and these sequences split, though not naturally. [Che09]

To prove the splitting, we go back to the split short exact sequence $0 \to Z_n \xrightarrow{f} C_n \xrightarrow{g} B_{n-1} \to 0$. Splitting implies that there is $p: C_n \to Z_n$ such that $p \circ f = 1_{Z_n}$. Further p can be extended to p', making the following diagram commutes:

$$\begin{tikzcd} C_n \arrow[d, "p"'] \arrow[dr, dashed, "p'"] & \\ Z_n \arrow[r, "q"] & H_n(C). \end{tikzcd} \qquad (3.23)$$

To get a chain map $F: C. \to H.(C)$, we make $H.$ a chain complex by adding trivial boundary maps between them. Tensor with G, which yields $F \otimes 1: C. \otimes G \to H.C \otimes G$. When we take the homology of $C. \otimes G$, we get the usual $H_n(C;G)$. When we take the homology of $H.C \otimes G$, however, it gives us $H_n(C) \otimes G$, due to the zero homomorphisms. Thus we have the induced homomorphism on homology $F_*: H_n(C;G) \to H_n(C) \otimes G$, which proves the desired splitting.

Splitting Lemma. *For a short exact sequence $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ of abelian groups the following statements are equivalent:*

(a) *There is a homomorphism $p: B \to A$ such that $pi = \mathbb{1}: A \to A$.* [Hat02]
(b) *There is a homomorphism $s: C \to B$ such that $js = \mathbb{1}: C \to C$.*
(c) *There is an isomorphism $B \approx A \oplus C$ making a commutative diagram as at the right, where the maps in the lower row are the obvious ones, $a \mapsto (a,0)$ and $(a,c) \mapsto c$.*

$$\begin{tikzcd} & B \arrow[dr, "j"] \arrow[d, "\approx"] & & \\ 0 \arrow[r] & A \arrow[u, "i"] \arrow[r] & A\oplus C \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

If these conditions are satisfied, the exact sequence is said to **split**. Note that (c) is symmetric: There is no essential difference between the roles of A and C.

Sketch of Proof: For the implication (a) ⇒ (c) one checks that the map $B \to A \oplus C$, $b \mapsto (p(b), j(b))$, is an isomorphism with the desired properties. For (b) ⇒ (c) one uses instead the map $A \oplus C \to B$, $(a,c) \mapsto i(a) + s(c)$. The opposite implications (c) ⇒ (a) and (c) ⇒ (b) are fairly obvious. If one wants to show (b) ⇒ (a) directly, one can define $p(b) = i^{-1}(b - sj(b))$. Further details are left to the reader. □