# FIRST LOOK AT THE UNIVERSAL COEFFICIENT THEOREM

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This problem is set from Bredon [1, No. IV.5.1]; it works out techniques that might later play into a proof of the universal coefficient theorem.

*Given.* Let $p\colon \mathbf{Z} \to \mathbf{Z}$ be multiplication by the prime $p$, and consider the short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \xrightarrow{\pi} \mathbf{Z}_p \to 0. \tag{1}$$

Say $X$ is a topological space. We assume $\Delta_*(-)\colon \mathsf{Top} \to \mathsf{Comp}$ is the singular chain functor from topological spaces to chain complexes of abelian groups, with $H_*(-)\colon \mathsf{Comp} \to \mathsf{GradedAb}$ the homology functor on chain complexes. For an abelian group $G$, denote by $H_*(C_*; G)$ the *homology of $C_*$ with coefficients in $G$*, which is the homology of the chain complex $G \otimes C_*$ with boundary operator $\mathrm{id}_G \otimes \partial$.

*To prove.* There is an exact sequence

$$0 \to \mathbf{Z}_p \otimes H_n(X) \to H_n(X; \mathbf{Z}_p) \to \mathrm{Tor}(H_{n-1}(X), \mathbf{Z}_p) \to 0, \tag{2}$$

with a homomorphism $H_n(X; \mathbf{Z}_p) \to \mathbf{Z}_p \otimes H_n(X)$ such that (2) splits.

*Proof.* We'll construct the exact sequence (2), then the splitting homomorphism.

Each group in $\Delta_*(X)$ is free abelian, and, in particular, flat as a $\mathbf{Z}$-module. Thus tensoring the short exact sequence (1) on the right by the $n$th singular chain $\Delta_n(X)$ group preserves exactness, and produces:

$$0 \longrightarrow \mathbf{Z} \otimes \Delta_n(X) \xrightarrow{p \otimes \mathrm{id}_\Delta} \mathbf{Z} \otimes \Delta_n(X) \xrightarrow{\pi \otimes \mathrm{id}_\Delta} \mathbf{Z}_p \otimes \Delta_n(X) \longrightarrow 0 \tag{3}$$

Lining up copies of (3) levelwise with the appropriate differentials, we exhibit the short exact sequence of chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathbf{Z} \otimes \Delta_n(X) \arrow[r, "p\otimes \mathrm{id}_\Delta"] \arrow[d, "\mathrm{id}_{\mathbf{Z}} \otimes \partial_n"] & \mathbf{Z} \otimes \Delta_n(X) \arrow[r, "\pi\otimes \mathrm{id}_\Delta"] \arrow[d, "\mathrm{id}_{\mathbf{Z}} \otimes \partial_n"] & \mathbf{Z}_p \otimes \Delta_n(X) \arrow[r] \arrow[d, "\mathrm{id}_{\mathbf{Z}_p} \otimes \partial_n"] & 0 \\
0 \arrow[r] & \mathbf{Z} \otimes \Delta_{n-1}(X) \arrow[r, "p\otimes \mathrm{id}_\Delta"] \arrow[d] & \mathbf{Z} \otimes \Delta_{n-1}(X) \arrow[r, "\pi\otimes \mathrm{id}_\Delta"] \arrow[d] & \mathbf{Z}_p \otimes \Delta_{n-1}(X) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{4}$$

As abelian groups, we'll identify $\mathbf{Z} \otimes \Delta_n(X) = \Delta_n(X)$ and $\mathbf{Z}_p \otimes \Delta_n(X) = \Delta_n(X)/p$ (the cokernel of the $p$th multiple map). Accordingly, we'll write the homomorphisms $p \otimes \mathrm{id}_\Delta = p$, $\pi \otimes \mathrm{id}_\Delta = \pi$, with all the differentials $\mathrm{id} \otimes \partial_n$ as $\partial$. Given these abbreviations, the short exact sequence (4) becomes:

$$0 \longrightarrow \Delta_*(X) \xrightarrow{p} \Delta_*(X) \xrightarrow{\pi} \Delta_*(X)/p \longrightarrow 0 \tag{5}$$

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By inspection of the definition of homology with coefficients along with the diagram (4), the long exact sequence of homology groups induced by (5) is:

$$\cdots \xrightarrow{\delta_*} H_n(X) \xrightarrow{p_*} H_n(X) \xrightarrow{\pi_*} H_n(X;\mathbf{Z}_p) \xrightarrow{\delta_*} H_{n-1}(X) \xrightarrow{p_*} \cdots \tag{6}$$

We now make an aside to argue the algebraic structure of a long exact sequence is encoded in a series of intertwined short exact sequences [2, No. 2.5]. Start by considering an exact sequence of abelian groups

$$\cdots \to A_{n+2} \to A_{n+1} \to A_n \to A_{n-1} \to A_{n-2} \to \cdots .$$

By definition of exactness at $A_n$,

$$\operatorname{im}(A_{n+1} \to A_n) = \ker(A_n \to A_{n-1}). \tag{7}$$

Quotienting by the kernel of the map out of $A_n$, we have an isomorphism onto the image in $A_{n-1}$

$$A_n/\ker(A_n \to A_{n-1}) \cong \operatorname{im}(A_n \to A_{n-1}) \tag{8}$$

Now, the cokernel of an additive homomorphism is the target group quotiented by the incoming image. Substituting (7) into (8) gives an example of such

$$\operatorname{coker}(A_{n+1} \to A_n) := A_n/\operatorname{im}(A_{n+1} \to A_n) \cong \operatorname{im}(A_n \to A_{n-1}).$$

Putting it all together, there's a group $C_n$ that is

- the kernel of the map $A_n \to A_{n-1}$,
- isomorphic to the image of the map $A_{n+1} \to A_n$, and
- isomorphic to the cokernel of the map $A_{n+2} \to A_{n+2}$.

Moreover, there are groups $C_{n+k}$ for $k \in \mathbf{Z}$ satisfying analogously conditions. We may now decorate our long exact sequence with inclusions and natural projections so that the diagram

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & & & 0 \\
 & & & C_{n+1} \arrow[ur] \arrow[dr] & & & & C_{n-1} \arrow[ur] \arrow[dr] & & & & C_{n-3} \arrow[ur] & \\
\cdots \arrow[rr] & & A_{n+2} \arrow[ur] \arrow[rr] & & A_{n+1} \arrow[rr] \arrow[dr] & & A_n \arrow[ur] \arrow[rr] & & A_{n-1} \arrow[rr] \arrow[dr] & & A_{n-2} \arrow[ur] \arrow[rr] & & \cdots \\
 & C_{n-2} \arrow[ur] & & & & C_n \arrow[ur] \arrow[dr] & & & & C_{n-2} \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd} \tag{9}$$

is exact. We emphasize that the crossing maps are either *inclusions* or *natural projections*. The crossing sequences are short exact because, e.g., for $0 \to C_n \to A_n \to C_{n-1} \to 0$,

- $C_n$ is the kernel, so injects and passes through $A_n$ to 0 in $C_{n-1}$,
- $A_n$ surjects onto $C_{n-1}$ and passes to 0,
- any element in $A_n$ that passes to 0 in $C_{n-1}$ must be in the image of $C_n$ in $A_n$, because $A_n \to C_{n-1}$ is the natural projection onto $A_n/\ker(A_n \to A_{n-1}) = A_n/C_n$.

We now construct the desired short exact sequence (2). Placing the long exact sequence of homology groups (6) into the position of the long exact sequence in the cross stitched diagram (9), one visibly obtains the short exact sequence:

$$0 \longrightarrow \operatorname{coker}\left(H_n(X) \xrightarrow{p_*} H_n(X)\right) \longrightarrow H_n(X;\mathbf{Z}_p) \longrightarrow \ker\left(H_{n-1}(X) \xrightarrow{p_*} H_{n-1}(X)\right) \longrightarrow 0 \tag{10}$$

This sequence (10) is isomorphic to the one we wanted (2), as

$$\operatorname{coker}\left(H_n(X) \xrightarrow{p_*} H_n(X)\right) \cong \mathbf{Z}_p \otimes H_n(X)$$

and

$$\ker\left(H_{n-1}(X) \xrightarrow{p_*} H_{n-1}(X)\right) =: \mathrm{Tor}(H_{n-1}(X), \mathbf{Z}_p).$$

What remains to be proven is that there is a homomorphism from $H_n(X, \mathbf{Z}_p)$ down to $\mathbf{Z}_p \otimes H_n(X)$ that (left) inverts the inclusion of $\mathbf{Z}_p \otimes H_n(X)$ up into $H_n(X, \mathbf{Z}_p)$. My argument here closely follows [2, No. 3.21].

We'll work levelwise for a while in the singular chain complex $\Delta_*(X)$. Let $Z_n$ be the cycle subgroup of $\Delta_n(X)$ and $B_{n-1}$ the boundary subgroup $\delta\Delta_n(X)$ in $\Delta_{n-1}(X)$. Observe that $Z_n$ includes into $\Delta_n(X)$ which projects onto $B_{n-1}$. So we have the short exact sequence of free abelian groups

(11)
$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, "f"] & \Delta_n(X) \arrow[r] \arrow[l, bend left, dashed, "g"] & B_{n-1} \arrow[r] & 0, \end{tikzcd}$$

that splits with $g \circ f = \mathrm{id}_{Z_n}$. Extend $g$ to the quotient $H_n(X)$ so that the following diagram commutes:

(12)
$$\begin{tikzcd} \Delta_n(X) \arrow[d, "g"] \arrow[dr, dashed, "g'"] & \\ Z_n \arrow[r] & H_n(X) \end{tikzcd}$$

Consider the graded abelian group $\{H_k(X)\}_{k\in\mathbf{Z}}$ in a chain with trivial differentials

$$\ldots \xrightarrow{0} H_{n+1}(X) \xrightarrow{0} H_n(X) \xrightarrow{0} H_{n-1}(X) \xrightarrow{0} \ldots.$$

Call this chain complex $H_*(X)$. We induce a chain map $g_*$ from $\Delta_*(X)$ to $H_*(X)$ with $g'$:

(13)
$$\begin{tikzcd} \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "0"] \\ \Delta_{n+1}(X) \arrow[r, "g'"] \arrow[d, "\partial"] & H_{n+1}(X) \arrow[d, "0"] \\ \Delta_n(X) \arrow[r, "g'"] \arrow[d, "\partial"] & H_n(X) \arrow[d, "0"] \\ \Delta_{n-1}(X) \arrow[r, "g'"] \arrow[d, "\partial"] & H_{n-1}(X) \arrow[d, "0"] \\ \vdots & \vdots \end{tikzcd}$$

(14)
$$\begin{tikzcd} \vdots \arrow[d, "\mathrm{id}\otimes\partial"] & \vdots \arrow[d, "0"] \\ \mathbf{Z}_p \otimes \Delta_{n+1}(X) \arrow[r, "\mathrm{id}\otimes g'"] \arrow[d, "\mathrm{id}\otimes\partial"] & \mathbf{Z}_p \otimes H_{n+1}(X) \arrow[d, "0"] \\ \mathbf{Z}_p \otimes \Delta_n(X) \arrow[r, "\mathrm{id}\otimes g'"] \arrow[d, "\mathrm{id}\otimes\partial"] & \mathbf{Z}_p \otimes H_n(X) \arrow[d, "0"] \\ \mathbf{Z}_p \otimes \Delta_{n-1}(X) \arrow[r, "\mathrm{id}\otimes g'"] \arrow[d, "\mathrm{id}\otimes\partial"] & \mathbf{Z}_p \otimes H_{n-1}(X) \arrow[d, "0"] \\ \vdots & \vdots \end{tikzcd}$$

Observe that $\partial(\Delta_{n+1}(X)) = B_n(X) = \ker(Z_n \to H_n(X))$ so that each square in (13) commutes. We may then tensor both chains by $\mathbf{Z}_p$ on the right, obtaining

$$\mathbf{Z}_p \otimes \Delta_*(X) \xrightarrow{\mathrm{id}\otimes g_*} \mathbf{Z}_p \otimes H_n(X).$$

When we take homology on $\mathbf{Z}_p \otimes \Delta_*(X)$ by definition we have $H_*(X, \mathbf{Z}_p)$. However, when computing homology in the chain complex $\mathbf{Z}_p \otimes H_n(X)$, the image of each map in the complex is *trivial*, so the homology of the complex is $\mathbf{Z}_p \otimes H_*(X)$.

The map $\mathrm{id}\otimes g_*$ of chain complexes induces a homomorphism $H_*(\mathrm{id}\otimes g')$ on homology from $H_*(X, \mathbf{Z}_p) \to \mathbf{Z}_p \otimes H_*(X)$. In particular, at the $n$th level,

(15)
$$H_n(X, \mathbf{Z}_p) \xrightarrow{H_*(\mathrm{id}\otimes g')} \mathbf{Z}_p \otimes H_n(X).$$

By construction in (12) of $g'$ as the extension of the splitting map $g$ (where $g \circ f = \mathrm{id}_{Z_n}$), the induced map $H_*(\mathrm{id}\otimes g')$ in (15) is a left inverse to the inclusion of $\mathbf{Z}_p \otimes H_n(X)$ into $H_n(X, \mathbf{Z}_p)$. □

## References

[1] G. E. Bredon, *Topology and Geometry.* New York: Springer-Verlag, 1993.

[2] J. L. Chen, "Universal Coefficient Theorem for Homology," 2009.