# Chapter 1

# Homology Theory (IV in Bredon)

## Introduction

Algebraic topology is the study of algebraic invariants of topological spaces.

**Example 1.0.1.** 1. $\pi_0(X)$ is the set of path components of $X$

$$\pi_0 \colon \text{Top} \to \text{Sets}$$

Easy to compute for spaces like manifold.

2. $\pi_1(X, x)$ the fundamental group, is $\text{Map}_*(S^1, X)/\simeq$, where $\text{Map}_*$ denotes continuous, based point preserving functions, $\simeq$ is the relation of based point preserving homotopy, and $*$-composition of path is the group operation:

$$\pi_1 \colon \text{Top}_* \to \text{Groups}$$

Moderately computable using Van Kampen.

3. For $n > 1$, $\pi_n(X, x)$, the $n$th homotopy group $\text{Map}_*(S^n, X)/\simeq$

$$\pi_n \colon \text{Top}_* \to \text{AbGroups}$$

Very hard.

4. Homology (and cohomology) $H_n(X)$ are also invariants. For $n \geqslant 0$,

$$H_n \colon \text{Top} \to \text{AbGroups}$$

Very computable compared to the $\pi_n$s.

There are many applications. In mathematics it has become a fundamental tool, but it now also is used in industry (Topological Data Analysis).

## 1.1 Homology Groups

The fundamental group and homotopy groups are defined in terms of maps $S^n \to X$. Similarly, homology is defined based on maps from $\Delta^n \to X$ where $\Delta^n$ is an $n$-dimensional tetrahedron.

Throughout, a *map* between topological spaces will always mean a continuous function unless otherwise specified. Similarly, a map between groups is a group homomorphism.

### 1.1.1 Simplices

**Definition 1.1.1.** Let $\{e_n\}_{n\geqslant 0}$ be the standard basis for $\mathbb{R}^\infty$. The *standard p-simplex* is

$$\Delta_p = \{x \in \mathbb{R}^\infty \mid x = \sum_{i=0}^{p} \lambda_i e_i,\ 0 \leqslant \lambda_i \leqslant 1,\ \sum_{i=0}^{p} \lambda_i = 1\}.$$

The $\lambda_i$ are the *barycentric coordinates* of $x \in \Delta_p$.

**Example 1.1.2.** $\Delta_0 = \{e_0\}$, $\Delta_1$ is the line segment between $e_0$ and $e_1$, $\Delta_2$ is the triangle formed by the convex hull of $e_0, e_1, e_2$, etc.

**Definition 1.1.3.** For $v_0, \ldots, v_p \in \mathbb{R}^N$, we let $[v_0, \ldots, v_p]$ (called an *affine p-simplex*) be the map

$$\begin{aligned} \Delta_p &\to \mathbb{R}^N \\ \sum_{i=0}^{p} \lambda_i e_i &\mapsto \sum_{i=0}^{p} \lambda_i v_i. \end{aligned}$$

It is called *degenerate* if the $v_i$ are not affine independent (i.e., $v_1 - v_0, \ldots, v_p - v_0$ are not linearly independent).

**Example 1.1.4.** $[e_0, e_0, e_1]$ is degenerate.

**Definition 1.1.5.** The $i$th face of $\Delta_p$ is the map

$$F_i^p \colon [e_0, \ldots, \hat{e}_i, \ldots, e_{p-1}] \colon \Delta_{p-1} \to \Delta_p$$

**Example 1.1.6.** $[e_1, e_2]$ is $F_0^2$, $[e_0, e_2]$ is $F_1^2$ and $[e_0, e_1]$ is $F_2^2$

**Remark 1.1.7.** The faces satisfy

$$F_j^{p+1} \circ F_i^p = \begin{cases} [e_0, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_p] & i < j \\ [e_0, \ldots, \hat{e}_j, \ldots, \hat{e}_{i+1}, \ldots, e_p] & i \geqslant j. \end{cases}$$

In particular, $F_j^{p+1} \circ F_i^p = F_i^{p+1} \circ F_{j-1}^p$ if $i < j$. This is called a *simplicial identity.*

### 1.1.2 Singular Chains and Graded Abelian Groups

**Definition 1.1.8.** If $X$ is a topological space, a *(singular) $p$-simplex* of $X$ is a map

$$\sigma\colon \Delta_p \to X.$$

We write $\sigma(i) := \sigma(e_i)$ for $0 \leqslant i \leqslant p$. Let $\mathrm{Sing}_p(X)$ denote the set of all singular $p$-simplices. Let $\Delta_p(X)$ be the free abelian group on $\mathrm{Sing}_p(X)$:

$$\Delta_p(X) = \mathbb{Z}\{\mathrm{Sing}_p(X)\}$$

An element of $\Delta_p(X)$ is called a (singular) $p$-chain of $X$, and is of the form

$$\sum_\sigma n_\sigma \sigma$$

for $\sigma \in \mathrm{Sing}_p(X)$, $n_\sigma \in \mathbb{Z}$ with all but finitely many $n_\sigma = 0$.

**Remark 1.1.9.** Note that $\Delta_p(\varnothing) = 0$.

**Example 1.1.10.** 1. 0-simplices of $X$ are maps $\sigma\colon \Delta_0 = \{e_0\} \to X$. So the map

$$\mathrm{ev}\colon \mathrm{Sing}_0(X) \to X, \quad \sigma \mapsto \sigma(0)$$

is a bijection, which induces an isomorphism

$$\mathrm{ev}\colon \Delta_0(X) \xrightarrow{\cong} \mathbb{Z}\{X\}.$$

Under this identification, a 0-chain looks like

$$\sum_{x\in X} n_x x,\ \ n_x \in \mathbb{Z}, x \in X$$

and $n_x = 0$ for all but finitely many $x \in X$. Here, "$x$" represents the simplex $\sigma_x\colon \Delta_0 \to X$ which sends $e_0$ to $x$.

2. 1-simplices of $X$ are maps $\sigma\colon \Delta_1 \to X$. Since $\Delta_1 \cong [0,1]$, these are just paths in $X$.

**Definition 1.1.11.** A *graded abelian group* is a collection of abelian groups $C_p$ indexed by the integers $p \in \mathbb{Z}$. A map $f\colon \{C_p\} \to \{D_p\}$ of graded abelian groups is a sequence of homomorphisms $f_p\colon C_p \to D_p$.

**Example 1.1.12.** Let $\Delta_p(X) = 0$ for $p < 0$. Then $\{\Delta_p(X)\}$ is a graded abelian group. Further, for $f\colon X \to Y$ be a map of topological spaces. Then

$$f_\Delta\colon \Delta_*(X) \to \Delta_*(Y)$$

given by

$$f_\Delta\left(\sum_\sigma n_\sigma \sigma\right) = \sum_\sigma n_\sigma f\sigma$$

is homomorphism of graded abelian groups.

**Remark 1.1.13.** Its clear that $f_\Delta \circ g_\Delta = (f \circ g)_\Delta$ and that $1_\Delta = 1$. Letting $\Delta_p(f) = f_\Delta$, we get a *functor*

$$\Delta_p(-)\colon \mathrm{Top} \to \mathrm{Ab}_*.$$

### 1.1.3 Boundary Map and Chain Complexes

**Recall 1.1.14.** If $S$ is a set and $A$ is an abelian group, then there is a bijection

$$\{\text{Functions } S \to A\} \Leftrightarrow \{\text{Homomorphisms } \mathbb{Z}\{S\} \to A\}$$

which sends $g\colon S \to A$ to

$$g\left(\sum_{s\in S} n_s s\right) := \sum_{s\in S} n_s g(s).$$

I'll call this *extending linearly*.

**Definition 1.1.15.** If $\sigma \in \mathrm{Sing}_p(X)$ is a $p$-simplex, then

$$\sigma^{(i)} = \sigma \circ F_i^p$$

is the *ith face* of $\sigma$. The *boundary* of a $\sigma$ is the $p-1$-chain

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i \sigma^{(i)}.$$

Extending linearly gives a homomorphism

$$\partial_p\colon \Delta_p(X) \cong \mathbb{Z}\{\mathrm{Sing}_p(X)\} \to \Delta_{p-1}(X)$$

given by

$$\partial_p\left(\sum_\sigma n_\sigma \sigma\right) = \sum_\sigma n_\sigma \partial_p \sigma$$

**Remark 1.1.16.** Let $\sigma \in \Delta_1(X)$. Under the identification $\mathrm{ev} : \Delta_0(X) \cong \mathbb{Z}\{X\}$ of Example 1.1.10,

$$\partial_1(\sigma) = \sigma(1) - \sigma(0).$$

**Proposition 1.1.17.** $\partial_p \circ \partial_{p+1} = 0$

*Proof.* It's enough to check on the generators $\sigma\colon \Delta^{p+1} \to \Delta_p$. We have

$$\begin{aligned}
\partial_p \circ \partial_{p+1}(\sigma) &= \partial_p \sum_{j=0}^{p+1} (-1)^j \sigma \circ F_j \\
&= \sum_{j=0}^{p+1} (-1)^j \sum_{i=0}^{p} (-1)^i \sigma \circ F_j \circ F_i \\
&= \sum_{0\leqslant i<j\leqslant p+1} (-1)^{i+j} \sigma \circ F_j \circ F_i + \sum_{0\leqslant j\leqslant i\leqslant p} (-1)^{i+j} \sigma \circ F_j \circ F_i \\
&= \sum_{0\leqslant i<j\leqslant p+1} (-1)^{i+j} \sigma \circ F_i \circ F_{j-1} + \sum_{0\leqslant j\leqslant i\leqslant p} (-1)^{i+j} \sigma \circ F_j \circ F_i \\
&= \sum_{0\leqslant i\leqslant k\leqslant p} (-1)^{i+k+1} \sigma \circ F_i \circ F_k + \sum_{0\leqslant j\leqslant i\leqslant p} (-1)^{i+j} \sigma \circ F_j \circ F_i \qquad (k = j-1) \\
&= 0.
\end{aligned}$$

$\square$

We have the following structure:

$$
\begin{tikzcd}
\dots \arrow[r] & \Delta_{p+1}(X) \arrow[r, "\partial_{p+1}"] \arrow[rr, bend right, "0"'] & \Delta_p(X) \arrow[r, "\partial_p"] & \Delta_{p-1}(X) \arrow[r] & \dots \arrow[r] & \Delta_2(X) \arrow[r, "\partial_2"] \arrow[rr, bend right, "0"'] & \Delta_1(X) \arrow[r, "\partial_1"] & \Delta_0(X) \arrow[r] & 0
\end{tikzcd}
$$

**Definition 1.1.18.** A *chain complex* is a graded abelian group $\{C_p\}$ with a sequence of homomorphisms $\partial_p\colon C_p \to C_{p-1}$ such that $\partial_p \circ \partial_{p+1} = 0$.

$$
\begin{tikzcd}
\dots \arrow[r] & C_{p+1} \arrow[r, "\partial_{p+1}"] \arrow[rr, bend right, "0"'] & C_p \arrow[r, "\partial_p"] & C_{p-1} \arrow[r] & \dots
\end{tikzcd}
$$

The maps $\partial_i$ are called the *differentials* or *boundary operators*. We use the notation $C_* = (\{C_p\}, \{\partial_p\})$, and often drop the indices and write $\partial = \partial_p$.

A map $f\colon C_* \to D_*$ of chain complexes is a map of graded abelian groups such that the diagram

$$
\begin{tikzcd}
\dots \arrow[r] & C_{p+1} \arrow[r, "\partial_{p+1}"] \arrow[d, "f_{p+1}"] & C_i \arrow[r, "\partial_p"] \arrow[d, "f_p"] & C_{p-1} \arrow[r] \arrow[d, "f_{p-1}"] & \dots \\
\dots \arrow[r] & D_{p+1} \arrow[r, "\partial_{p+1}"] & D_p \arrow[r, "\partial_i"] & D_{p-1} \arrow[r] & \dots
\end{tikzcd}
$$

commutes, abbreviated:

$$f \circ \partial = \partial \circ f.$$

These are called *chain maps*.

**Example 1.1.19.** $\Delta_*(X) = (\{\Delta_p(X)\}, \{\partial_p\})$ is the *singular chain complex of* $X$.

**Proposition 1.1.20.** *For $f\colon X \to Y$ a map of topological spaces, $f_\Delta$ is a chain map, that is*

$$f_\Delta \circ \partial = \partial \circ f_\Delta.$$

*Proof.* We show that $f_\Delta \circ \partial(\sigma) = \partial \circ f_\Delta(\sigma)$ on the generators $\sigma \in \mathrm{Sing}_p(X)$, which implies that they agree on all of $\Delta_p(X)$:

$$
\begin{aligned}
f_\Delta(\partial\sigma) &= \sum_i (-1)^i f \circ \sigma^{(i)} \\
&= \sum_i (-1)^i f \circ \sigma \circ F_i^p \\
&= \sum_i (-1)^i (f \circ \sigma)^{(i)} \\
&= \partial(f_\Delta(\sigma)).
\end{aligned}
$$

$\square$

**Remark 1.1.21.** In fact, this is a functor

$$\Delta_*(-)\colon \mathrm{Top} \to \mathrm{Ch}\,.$$

### 1.1.4 Homology and Functorial Properties

**Definition 1.1.22.** Let $C_*$ be a chain complex. Let $c \in C_p$.

- If $c \in \ker(\partial_p)$, we call $c$ a *p-cycle* and let
$$Z_p(C_*) = \ker(\partial_p) \subseteq C_p.$$
- If $c \in \operatorname{im}(\partial_{p-1})$, we call $c$ a *p-boundary* and let
$$B_p(C_*) = \operatorname{im}(\partial_{p-1}) \subseteq Z_p(C_*).$$
- The $p$ *homology group* of $C_*$ is the group
$$H_p(C_*) = \frac{Z_p(C_*)}{B_p(C_*)} = \frac{\ker(\partial_p)}{\operatorname{im}(\partial_{p-1})}.$$
The book denotes by $[\![c]\!] \in H_p(C_*)$ the residue class of $c \in Z_p(C_*)$. It is called the *homology class* of $c$.
- If $c - c' \in B_p(C_*)$, $c$ and $c'$ are called *homologous*, denoted by $c \sim c'$. In this case, $[\![c]\!] = [\![c']\!]$.

We write $H_*(C_*)$ for the *graded abelian group* formed by the collection $\{H_p(C_*)\}$. We can also view this as a chain complex with zero differentials.

**Definition 1.1.23.** Let $X$ be a topological space. The $p$ *singular homology group* of $X$ is
$$H_p(X) = H_p(\Delta_*(X)).$$
We also let
$$Z_p(X) = Z_p(\Delta_*(X)) \qquad\qquad B_p(X) = B_p(\Delta_*(X)).$$

**Example 1.1.24.** Let $X = *$ be the one point space. There is a unique map
$$\sigma_p \colon \Delta_p \to *$$
$\Delta_p(X) \cong \mathbb{Z}\{\sigma_p\}$ for every $p$. Further,
$$\partial_p(\sigma_p) = \sum_{i=0}^{p}(-1)^i\sigma_p^{(i)} = \sum_{i=0}^{p}(-1)^i\sigma_{p-1} = \begin{cases} 0 & p \text{ odd} \\ \sigma_{p-1} & p \text{ even.}\end{cases}$$
So the singular chain complex looks like
$$\dots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$
Therefore,
$$H_p(X) = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p > 0.\end{cases}$$

**Exercise 1.1.25.** Let $X = \bigcup_{i \in I} X_i$ where the $X_i$ are the (disjoint) path components of $X$. Prove that

$$H_n(X) \cong \bigoplus_{i \in I} H_n(X_i).$$

**Proposition 1.1.26.** *A chain map $f\colon C_* \to D_*$ induces a map*

$$f_*\colon H_*(C_*) \to H_*(D_*)$$

*given by $f_*(\llbracket c \rrbracket) = \llbracket f(c) \rrbracket$.*

$$\begin{tikzcd} c \arrow[r, mapsto, "f"] \arrow[d, "\partial"'] \arrow[dr, phantom, "\circ"] & f(c) \arrow[d, "\partial"] \\ 0 \arrow[r, mapsto, "f"'] & 0 \end{tikzcd}$$

*Proof.* Let $c$ be a cycle in $C_p$. Then

$$\partial(f(c)) = f(\partial(c)) = f(0) = 0.$$

So $f$ induces a map

$$f\colon Z_p(C_*) \to Z_p(D_*).$$

$c \in Z_p(C_*)$
$f(c) \in Z_p(D_*)$

Since $f(\partial(c)) = \partial(f(c))$, ✓ $f$ also induces a map

$$f\colon B_p(C_*) \to B_p(D_*).$$

Say $k \in B_p(C_*)$.
Lift $k$ to $z$ in $C_p$
Then $f(k)$

$$\begin{tikzcd} z \arrow[r, mapsto, "f"] \arrow[d, "\partial"'] & (f(z)) \arrow[d, "\partial"] \\ k \arrow[r, mapsto, "f"'] & f(k) \end{tikzcd}$$

It follows that there is an induced map

$$f_*\colon H_p(C) = Z_p(C_*)/B_*(C_*) \to H_p(D) = Z_p(D_*)/B_p(D_*).$$

$\square$

**Exercise 1.1.27.** Check that $f_* \circ g_* = (f \circ g)_*$ and $1_* = 1$.

**Remark 1.1.28.** In fact, we have a homology is a *functor*:

$$H_*(-)\colon \text{Ch} \to \text{Ab}_*$$

where Ch is the category of chain complexes and $\text{Ab}_*$ of graded abelian groups.

**Remark 1.1.29.** Since $f_\Delta$ is a chain map

$$f_\Delta\colon \Delta_*(X) \to \Delta_*(Y)$$

it induce a homomorphisms

$$f_*\colon H_*(X) \to H_*(Y).$$

Singular homology is a thus functor

$$H_*(-)\colon \text{Top} \to \text{Ab}_*$$

It is the composition of the functors

$$\text{Top} \xrightarrow{\Delta_*(-)} \text{Ch} \xrightarrow{H_*(-)} \text{Ab}_*$$

**Corollary 1.1.30.** *Since functors preserve isomorphisms, $f\colon X \to Y$ is a homeomorphism, then $H_*(f)\colon H_*(X) \to H_*(Y)$ is an isomorphism.*

Later, we will discuss the following axiom, but use it in the meantime.

**Theorem 1.1.31** (Homotopy Axiom)**.** *Let $f \simeq g\colon X \to Y$ be homotopic maps. Then $f_* = g_*\colon H_*(X) \to H_*(Y)$.*

**Corollary 1.1.32.** *If $X \simeq \mathrm{pt}$ is contractible, then $H_*(X) \cong H_*(\mathrm{pt})$.*

*Proof.* Since $X$ is contractible, $\mathrm{id}\colon X \to X$ is homotopic to a map of the form

$$f\colon X \to \{x\} \hookrightarrow X.$$

So, $H_*(f) = \mathrm{id}_{H_*(X)}$. Since $H_*(f)$ factors through $H_*(\mathrm{pt})$, $\mathrm{id}_{H_p(X)} = 0$ for all $p > 0$, so $H_p(X) = 0$ for $p > 0$. Since a contractible space is path connected, $H_*(X) \cong H_*(\mathrm{pt})$. $\square$

## 1.2 Zeroth Homology Group

**Definition 1.2.1.** The homomorphism $\epsilon\colon \Delta_0(X) \cong_{\mathrm{ev}} \mathbb{Z}\{X\} \to \mathbb{Z}$ defined by

$$\epsilon\left(\sum_{x\in X} n_x x\right) = \sum_{x\in X} n_x$$

is called the *augmentation*.

**Remark 1.2.2.** If $X \neq \varnothing$, the map $\epsilon$ is surjective.

For $\sigma \in \mathrm{Sing}_1(X)$,

$$\epsilon(\partial_1(\sigma)) = \epsilon(\sigma(1) - \sigma(0)) = 1 - 1 = 0$$

so the composite

$$\Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

is zero, (as it is zero on all the generators of $\Delta_1(X)$). So we get an induced homomorphism

$$\epsilon_*\colon H_0(X) = \Delta_0(X)/\mathrm{im}(\partial_1) \to \mathbb{Z}$$

which is also called the augmentation.

**Notation.** For $X \neq \varnothing$ a path connected space, fix $x_0 \in X$ and let $\lambda_x\colon \Delta_1 \to X$ a path from $x_0$ to $x$, with the convention that $\lambda_{x_0}$ is the constant path. Note that this gives a group homomorphism

$$\lambda\colon \Delta_0(X) \cong \mathbb{Z}\{X\} \to \Delta_1(X)$$

determined by $\lambda(x) = \lambda_x$.

**Theorem 1.2.3.** *Let $X \neq \varnothing$ be a path connected space. Then $\epsilon_*$ is an isomorphism.*

*Proof.* Since $\epsilon_*$ is clearly surjective, it's enough to show that it is injective.

Suppose $\epsilon_*(\llbracket c \rrbracket) = 0$ for $c = \sum n_x x$, so that $\sum n_x = 0$. Then

$$c - \partial_1 \left( \sum n_x \lambda_x \right) = \sum n_x x - \sum n_x (x - x_0) = \left( \sum n_x \right) x_0 = 0.$$

So $c$ is a boundary, i.e., $\llbracket c \rrbracket = 0$. □

**Corollary 1.2.4.** *Let $X \neq \varnothing$ be a path connected space. Then $\llbracket x \rrbracket = \llbracket y \rrbracket$ for any $x, y \in X$, and $H_0(X) \cong \mathbb{Z}$, generated by $\llbracket x \rrbracket$ for any point $x \in X$.*

**Corollary 1.2.5.** *Let $X = \bigcup_{i \in I} X_i$ for $X_i$ path connected and non-empty. Let $x_i \in X_i$. Then*

$$H_0(X) \cong \bigoplus_{i \in I} \mathbb{Z}\{\llbracket x_i \rrbracket\}.$$

**Exercise 1.2.6.** Any map $f \colon X \to Y$ between path connected topological spaces induces an isomorphism $f_* \colon H_0(X) \xrightarrow{\cong} H_0(Y)$.

**Definition 1.2.7.** The *reduced* homology of $X$, denoted $\widetilde{H}_*(X)$ is the homology of the chain complex

$$\dots \longrightarrow \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \longrightarrow \dots$$

In particular, for $X = \bigcup_{i \in I} X_i$ for $X_i$ path connected and non-empty and $x_{i_0} \in X_{i_0}$ for any fixed $i_0 \in I$, then

Given a map
$f : X \longrightarrow$ point
$\Delta_*(X) \xrightarrow{\partial_0} \Delta_*(\text{point})$
$H_*(X) \xrightarrow{H_*(f)} \mathbb{Z}$
(localized in degree 0)
$\ker H_*(f) = \widetilde{H}_*(X)$

$$\widetilde{H}_p(X) = \begin{cases} H_p(X) & p > 0 \\ \bigoplus_{i \in I \setminus \{i_0\}} \mathbb{Z}\{\llbracket x_i \rrbracket - \llbracket x_{i_0} \rrbracket\} & p = 0 \\ 0 & p < 0. \end{cases}$$

**5.9. Example** (Reduced Homology). Consider the chain complex $C_*$ where $C_i = \Delta_i(X)$ for $i \geq 0$, $C_{-1} = \mathbf{Z}$, and $C_i = 0$ for $i < -1$, and where the differential $C_0 \to C_{-1}$ is the augmentation $\epsilon \colon \Delta_0(X) \to \mathbf{Z}$. The homology of this complex is called the "reduced homology" of $X$ and is denoted by $\widetilde{H}_*(X)$. This differs from $H_*(X)$ only in degree zero where $\widetilde{H}_0(X)$ can easily be seen to be the kernel of the map $H_0(X) \to H_0(\mathit{point})$ induced by the map of $X$ to the one-point space. (For $X = \varnothing$ it also differs in degree $-1$ since $\widetilde{H}_{-1}(\varnothing) \approx \mathbf{Z}$. However, one usually does not talk of reduced homology in this case.)

Note that if $X$ is path connected, $\widetilde{H}_0(X) = 0$.

**Relation of relative homology to reduced homology**

**Proposition 2.10.** *Let $X$ be a inhabited topological space and let $x : * \hookrightarrow X$ any point. Then the relative singular homology $H_n(X, *)$ is isomorphic to the absolute reduced singular homology $\tilde{H}_n(X)$ of $X$*

$$H_n(X, *) \simeq \tilde{H}_n(X).$$

## 1.3 First Homology Group

In this section, $X \neq \varnothing$ is a path connected space and $x_0 \in X$. Let $\pi_1 = \pi_1(X, x_0)$ and

$$\widetilde{\pi}_1 = \widetilde{\pi}_1(X, x_0) = \pi_1(X, x_0)/[\pi_1, \pi_1]$$

be its abelianization. The main theorem we prove here is

**Theorem 1.3.1** (Hurewicz)**.** *The map*

$$\phi = \phi_X \colon \pi_1(X, x_0) \to H_1(X)$$

*which sends the homotopy class $[\gamma] \in \pi_1(X, x_0)$ of a loop $\gamma \colon \Delta_1 \to X$ to the homology class $\llbracket \gamma \rrbracket$ induces an homomorphism*

$$\phi_* \colon \widetilde{\pi}_1(X, x_0) \to H_1(X).$$

*Furthermore, $\phi_*$ is an isomorphism if $X$ is path connected.*

**Corollary 1.3.2.** *There are isomorphisms*

$$H_1(S^1) \cong \mathbb{Z}, \qquad H_1(S^n) \cong 0, \quad n \geqslant 2 \qquad H_1(\mathbb{R}P^n) \cong \mathbb{Z}/2, \quad n \geqslant 2.$$

**Exercise 1.3.3.** Use the Hurewicz theorem to solve the following problems:

(a) Compute $H_1(K)$ for the Klein bottle $K$.

(b) Compute $H_1$ of $X = \prod_{j \in J} X_j$ for topological spaces $X_j$ in terms of $H_1(X_j)$.

(c) Let $X_i$ with base points $x_i \in X_i$. Suppose that there are open sets $x_i \in U_i \subseteq X_i$ such that $x_i$ is a deformation retract of $U_i$. Show that

$$H_1\left(\bigvee_{i \in I} X_i\right) \cong \bigoplus_{i \in I} H_1(X_i).$$

The fact that loops $\gamma\colon \Delta^1 \to X$ are 1-cycles implies that $[\![\gamma]\!] \in H_1(X)$. Since $\gamma$ is a loop, it factors

$$\begin{tikzcd} \Delta_1 \arrow[r, "\gamma"] \arrow[d, "\sigma_q"] & X \\ \Delta_1/(e_0 \sim e_1) \arrow[ur, dashed, "\overline{\gamma}"'] & \end{tikzcd}$$

where $\sigma_q$ is the quotient map. If $\gamma_1 \simeq \gamma_2$ through based point preserving homotopies, then $\overline{\gamma}_1 \simeq \overline{\gamma}_2$. By the homotopy axiom,

$$H_*(\overline{\gamma}_1) = H_*(\overline{\gamma}_2).$$

In particular,

$$[\![\gamma_1]\!] = H_*(\overline{\gamma}_1)([\![\sigma_q]\!]) = H_*(\overline{\gamma}_2)([\![\sigma_q]\!]) = [\![\gamma_2]\!].$$

So, $\phi$ is well defined.

Next, we show that $\phi$ respects composition.

**Lemma 1.3.4.** *If $f$ and $g$ are paths with $f(1) = g(0)$, then*

$$f + g - f * g$$

*is a boundary.*

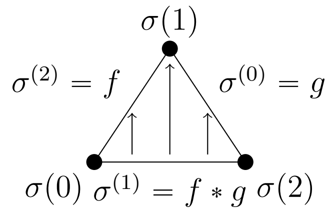
*Proof.* Define $\sigma\colon \Delta_2 \to X$ so that

- $f = \sigma^{(2)}$
- $g = \sigma^{(0)}$
- $f * g = \sigma^{(1)}$

- $\sigma$ is constant on the line segments perpendicular to the image of $[e_0, e_2]$

Then

$$\partial_2(\sigma) = g - f * g + f$$




$\square$

**Corollary 1.3.5.** *We have*

$$\phi([f][g]) = \phi([f * g]) = [\![f * g]\!] = [\![f + g]\!] = [\![f]\!] + [\![g]\!] = \phi([f]) + \phi([g]).$$

*That is, $\phi$ is a group homomorphism to an abelian group, and thus factors through $\widetilde{\pi}_1$.*

Next, we prove that $\phi_*$ is an isomorphism when $X$ is path connected. As before, let $\lambda\colon \Delta_0(X) \to \Delta_1(X)$ be the homomorphism

$$\sum n_x x \mapsto \lambda_{\sum n_x x} := \sum n_x \lambda_x$$

where $\lambda_x\colon \Delta_1 \to X$ is a path from $x_0$ to $x$ and $\lambda_{x_0}$ is the constant path. For $\sigma \in \mathrm{Sing}_1(X)$, let

$$\psi(\sigma) = [\lambda_{\sigma(0)} * \sigma * \lambda_{\sigma(1)}^{-1}] \in \widetilde{\pi}_1(X, x_0).$$

This extends to a map

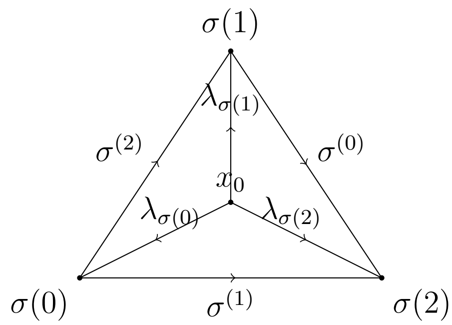
$$\psi\colon \Delta_1(X) \to \widetilde{\pi}_1(X, x_0)$$

Further, $B_1(X) \subseteq \ker(\psi)$ since, for $\sigma \in \mathrm{Sing}_2(X)$,

$$\begin{aligned}
\psi(\partial\sigma) &= \psi(\sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}) \\
&= [\lambda_{\sigma(1)} * \sigma^{(0)} * \lambda_{\sigma(2)}^{-1} * (\lambda_{\sigma(0)} * \sigma^{(1)} * \lambda_{\sigma(2)}^{-1})^{-1} * \lambda_{\sigma(0)} * \sigma^{(2)} * \lambda_{\sigma(1)}^{-1}] \\
&= [\lambda_{\sigma(1)} * \sigma^{(0)} * (\sigma^{(1)})^{-1} * \sigma^{(2)} * \lambda_{\sigma(1)}^{-1}] \\
&= 0.
\end{aligned}$$

So we get a homomorphism

$$\psi_*\colon H_1(X) \to \widetilde{\pi}_1(X, x_0)$$




**Lemma 1.3.6.** *$\psi_*$ is the two sided inverse of $\phi_*$.*

*Proof.* Let $[f] \in \widetilde{\pi}_1(X, x_0)$. Then

$$\psi_*\phi_*([f]) = \psi_*([\![f]\!]) = [\lambda_{x_0} * f * \lambda_{x_0}^{-1}] = [f]$$

since $\lambda_{x_0}$ is constant.

On the other hand, if $\sigma \in \mathrm{Sing}_1(X)$, then

$$\begin{aligned}\phi_*(\psi(\sigma)) &= \phi_*([\lambda_{\sigma(0)} * \sigma * \lambda_{\sigma(1)}^{-1}]) \\ &= [\![\lambda_{\sigma(0)} * \sigma * \lambda_{\sigma(1)}^{-1}]\!] \\ &= [\![\sigma - (\lambda_{\sigma(1)} - \lambda_{\sigma(0)})]\!] \\ &= [\![\sigma - \lambda_{\partial\sigma}]\!].\end{aligned}$$

Therefore, if $\partial c = 0$, then

$$\phi_*(\psi_*([\![c]\!])) = \phi_*(\psi(c)) = [\![c - \lambda_{\partial c}]\!] = [\![c]\!]. \qquad \square$$

**Theorem 1.3.7.** *The Hurewicz map is natural, i.e., if $f\colon X \to Y$ is a map of topological spaces such that $f(x_0) = y_0$, then the following diagram commutes:*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_\#"] \arrow[d, "\phi_X"] & \pi_1(Y, y_0) \arrow[d, "\phi_Y"] \\ H_1(X) \arrow[r, "f_*"] & H_1(Y). \end{tikzcd}$$

2019-01-28

## 1.4 Homological Algebra

### 1.4.1 Long Exact Sequence on Homology

**Definition 1.4.1.** A diagram of abelian groups

$$A \xrightarrow{i} B \xrightarrow{j} C \tag{1.1}$$

is *exact (at $B$)* if
$$\ker(j) = \mathrm{im}(i).$$
A diagram
$$\dots \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow A_{n-2} \longrightarrow A_{n-3} \longrightarrow \dots$$
which is exact at every term is called an *exact sequence* (or *long exact sequence*).

If the diagram
$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$
is exact, it is called a *short exact sequence.*

**Example 1.4.2.** The sequence
$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$$
is exact.

**Remark 1.4.3.** The map $i$ is injective if and only if
$$0 \longrightarrow A \xrightarrow{i} B$$
is exact (at $A$) and $j$ is surjective if and only if
$$B \xrightarrow{j} C \longrightarrow 0$$
is exact (at $C$). Finally, $i\colon A \to B$ is an isomorphisms if and only if
$$0 \longrightarrow A \xrightarrow{i} B \longrightarrow 0$$
is an exact sequence.

**Definition 1.4.4.** A sequence
$$A_* \xrightarrow{i} B_* \xrightarrow{j} C_*$$
of chain complexes is exact (at $B_*$) if it is exact *levelwise*. That is,
$$A_p \xrightarrow{i_p} B_p \xrightarrow{j_p} C_p$$
is exact at $B_p$ for each $p \in \mathbb{Z}$. The same holds for the definition of (long) exact sequences and short exact sequences.

**Theorem 1.4.5.** *Let*
$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \longrightarrow 0$$
*be a short exact sequence of chain complexes. There is a homomorphism*
$$\delta_*\colon H_p(C_*) \to H_{p-1}(A_*)$$

*called the* connecting homomorphism, *given by*

$$\delta_*([\![c]\!]) = [\![i^{-1} \circ \partial^B \circ j^{-1}(c)]\!],$$

*so that*

$$\dots \xrightarrow{\delta_*} H_p(A_*) \xrightarrow{i_*} H_p(B_*) \xrightarrow{j_*} H_p(C_*) \xrightarrow{\delta_*} H_{p-1}(A_*) \xrightarrow{i_*} \dots$$

*is a* long exact sequence.

This theorem has a long and tedious proof. The most important part for applications is to remember how the map $\delta_*$ is defined. So I'll focus mostly on that.

*Proof.* The proof of this theorem is called a *diagram chase*, and the relevant diagram is

$$\begin{tikzcd}
0 \arrow[r] & A_{p+1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{p+1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{p+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_p \arrow[r, "i"] \arrow[d, "\partial"] & B_p \arrow[r, "j"] \arrow[d, "\partial"] & C_p \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{p-1} \arrow[r, "i"] & B_{p-1} \arrow[r, "j"] & C_{p-1} \arrow[r] & 0
\end{tikzcd}$$

$b \xrightarrow{j(b)} c$

$a \xrightarrow{i(a)} \partial(b)$

pull it back!

First, fix $[\![c]\!] \in H_p(C)$. Then $\partial c = 0$. Since $j$ is surjective, we choose $b$ so that $j(b) = c$. But then

$$j\partial(b) = \partial j(b) = \partial c = 0,$$

so $\partial(b) = i(a)$ for $a \in A_{p-1}$. Since

$$i(\partial(a)) = \partial(i(a)) = \partial^2 b = 0$$

and $i$ is injective, $\partial(a) = 0$. We let

$$\delta_*([\![c]\!]) := [\![a]\!].$$

**Exercise 1.4.6.** Check that it doesn't depend on the choice of lift $b$ such that $j(b) = c$.

**Solution.** If $j(b') = c$, then $b' = b + i(a')$. The procedure

$$\partial(b') = \partial(b + i(a')) = \partial(b) + i(\partial(a')) = i(a + \partial(a'))$$

so that

$$\delta_*([\![c]\!]) := [\![a + \partial(a')]\!] = [\![a]\!].$$

It also doesn't depend on the choice of representative cycle $c$. If $c' = c + \partial(c'')$ so that $[\![c']\!] = [\![c]\!]$, then for $b''$ such that $j(b'') = c''$,

$$j(b + \partial(b'')) = c + \partial(c''),$$

so that the next step gives

$$\partial(b + \partial(b'')) = \partial(b) = i(a).$$

So, $\delta_*$ is well defined.

Bredon gives exactness rather by first showing each map is "order-two" then arguing that each group maps onto the next kernel.

**Exercise 1.4.7.** Check that $\delta_*$ is a homomorphism.

Finally, we need to show exactness. We prove exactness at $H_p(C_*)$. That is, we prove that

$$H_p(B_*) \xrightarrow{j_*} H_p(C_*) \xrightarrow{\delta_*} H_{p-1}(A_*)$$

is exact.

$$\begin{tikzcd}
0 \arrow[r] & A_{p+1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{p+1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{p+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_p \arrow[r, "i"] \arrow[d, "\partial"] & B_p \arrow[r, "j"] \arrow[d, "\partial"] & C_p \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{p-1} \arrow[r, "i"] & B_{p-1} \arrow[r, "j"] & C_{p-1} \arrow[r] & 0
\end{tikzcd}$$

$a'$, $a$, $i(a)$, $b$, $j(b)$, $c$, $i(a')$, $\partial(b)$, $j(i(a'))=0$ why?

Suppose that $[\![c]\!] \in \ker(\delta_*)$. Then $a = \partial(a')$, so that

$$\partial(b) = i(a) = i(\partial(a')) = \partial(i(a')).$$

So $b' = b - i(a')$ is a cycle, that is, $b' \in Z_p(B) = \ker(\partial)$. Therefore,

$$c = j(b) = j(i(a') + b') \overset{?}{=} j(b')$$

assuming $j \circ i = 0$? DUH

and $j_*([\![b']\!]) = [\![c]\!]$.

Third, for the exactness at $H_*(C_*)$, suppose that $\partial_*[\![c]\!] = 0$. Then for an element $b \in B_*$ for which $j(b) = c$, there is an $a \in A_*$ such that $i(a) = \partial b$, by the construction of $\partial_*$, and $a$ must be a boundary since it represents $\partial_*[\![c]\!] = 0$. Thus let $a = \partial a'$. Then $\partial i(a') = i(\partial a') = i(a) = \partial b$. Accordingly, $\partial(b - i(a')) = 0$ and $j(b - i(a')) = c - 0 = c$. Therefore $j_*[\![b - i(a')]\!] = [\![c]\!]$ as required. □

Finally, we must show that the indicated "long" sequence is exact. First, we show it is of "order two" (i.e., the composition of adjoining homomorphisms is zero). There are three cases. First, $j_* i_* = (j \circ i)_* = 0_* = 0$.

not just that $A \hookrightarrow B$ and $B \twoheadrightarrow C$, but that $A$ is collapsed.

Suppose that $[\![c]\!] = j_*([\![b']\!])$ for some $[\![b']\!] \in H_p(B)$. Then we can choose

$$b = b' + \partial(b'')$$

for some $b'' \in B_{p+1}$ and $b' \in Z_p(B)$. So,

$$\partial(b) = \partial(b' + \partial(b'')) = 0$$

so $i(0) = \partial(b)$. Hence,

$$\delta_*([\![c]\!]) = 0.$$

**Exercise 1.4.8.** Prove exactness at $H_p(B_*)$ and $H_p(A_*)$.

□

**Exercise 1.4.9.** Prove that the morphism $\delta_*$ is natural. That is, if

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "i"] \arrow[d, "f"] & B_* \arrow[r, "j"] \arrow[d, "g"] & C_* \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A'_* \arrow[r, "i"] & B'_* \arrow[r, "j"] & C'_* \arrow[r] & 0
\end{tikzcd}$$

a natural transformation from the cat. of ses of chain complexes to the cat of l.e.s...

is a commutative diagram with exact rows and $\delta'_* \colon H_*(C'_*) \to H_{*-1}(A'_*)$ the connecting homomorphism for the bottom row, then

$$\begin{tikzcd}
H_*(C_*) \arrow[r, "\delta_*"] \arrow[d, "h_*"] & H_{*-1}(A_*) \arrow[d, "f_*"] \\
H_*(C'_*) \arrow[r, "\delta'_*"] & H_{*-1}(A'_*)
\end{tikzcd}$$

commutes.

# INTRODUCTION TO HOMOLOGY

COLTON GRAINGER (MATH 6220 TOPOLOGY 2)

## Problems due 2019-01-30

**[1, p. 171].** *Given.* Let $X = \bigcup_{i \in I} X_i$ where the $X_i$ are the path components of $X$.

*To prove.* The homology $H_*(X)$ is isomorphic (as a graded abelian group) to the direct sum $\oplus_{i \in I} H_*(X_i)$.

*Proof.* For $p \in \mathbf{Z}_{\geq 0}$, we'll consider this portion of the chain complex:

$$\Delta_{p+1}(X) \xrightarrow{\partial_{p+1}} \Delta_p(X) \xrightarrow{\partial_p} \Delta_{p-1}(X) \text{ which is 0 when } p = 0.$$

We will will argue

$$\begin{aligned} \bigoplus_i H_p(X_i) &:= \bigoplus_i \frac{Z_p(X_i)}{B_p(X_i)} \\ &\cong \frac{\bigoplus_i Z_p(X_i)}{\bigoplus_i B_p(X_i)} && \text{by module theory} \\ &\cong \frac{Z_p(X)}{B_p(X)} && \text{TODO} \\ &=: H_p(X). \end{aligned}$$

generalize

First, suppose $p = 0$ (this should feel similar to augmentation [1, No. IV.2.3]).

| group isomorphism | justification |
|---|---|
| the cycle group $Z_0(X) \cong \oplus_i Z_0(X_i)$ | By convention, the 0-chain group $\Delta_0(X) \to 0$. Since $Z_0(X)$ is defined as the kernel, we have $\mathbf{Z}_0(X) := \Delta_0(X)$. Manipulating $\Delta_0(X)$ as the free abelian group (the direct sum of $\lvert X \rvert$ copies of the integers), $\mathbf{Z}\{X\} = \oplus_i \mathbf{Z}\{x \in X_i\} \cong \oplus_i Z_0(X_i)$. |
| the boundary group $B_0(X) \cong \oplus_i B_0(X_i)$ | Note $\mathrm{Sing}_1(X) = \sqcup_i \mathrm{Sing}_1(X_i)$, as the image of a 1-simplex must lie in entirely in a path component. Now $\partial$ is defined on generators, so consider $\partial(\mathrm{Sing}_1(X)) = \partial(\sqcup_i \mathrm{Sing}_1(X_i)) = \sqcup_i \partial(\mathrm{Sing}_1(X_i)) = \sqcup_i \{x - y : x, y \in X_i\}$. Extending linearly, $B_0(X) = \mathbf{Z}\{\sqcup_i \{x - y : x, y \in X_i\}\} \cong \oplus_i \mathbf{Z}\{x - y : x, y \in X_i\} \cong \oplus_i B_0(X_i)$. |

Now say $p > 0$. As before, any image of a simplex lies entirely in a path component:

$$\mathrm{Sing}_p(X) = \sqcup_i \mathrm{Sing}_p(X_i).$$

*Date*: 2019-01-14.

| group isomorphism | justification |
|---|---|
| cycles $Z_p(X) \cong \oplus_i Z_p(X_i)$ | Consider $\ker \partial_p$. $$0 = \partial\left(\sum_\sigma n_\sigma \sigma\right) = \sum_\sigma n_\sigma \partial\sigma$$ iff, (if $\sigma \in \mathrm{Sing}_p(X_i)$, then $\partial\sigma \in \mathrm{Sing}_{p-1}(X_i)$), $$0 = \sum_i \left(\sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma \delta\sigma\right)$$ which occurs iff for each $i$, $$0 = \sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma \partial\|_{\mathrm{Sing}_p(X_i)}\sigma.$$ So $$\ker\partial = \bigoplus_i \ker\partial\|_{\Delta_p(X_i)}.$$ |
| boundaries $B_p(X) \cong \oplus_i B_p(X_i)$ | For each path component $i$, does the restriction $\partial\|_{\Delta_{p+1}(X_i)}$ map into $\Delta_p(X_i)$ alone? Consider $\partial$ on generators $\partial\|_{\mathrm{Sing}\,p+1(X_i)}$. Does $\delta\sigma_{p+1}$ map into $\Delta_p(X_i)$? Yes! (Consider the face maps.) |

In conclusion, for all $p \in \mathbf{Z}_{\geq 0}$:

$$\bigoplus_i H_p(X_i) \cong \frac{\bigoplus_i Z_p(X_i)}{\bigoplus_i B_p(X_i)} \cong \frac{Z_p(X)}{B_p(X)} =: H_p(X).$$

□

**[1, p. 176].** *Given.* Let $X$ and $Y$ be path connected spaces, and let $f\colon X \to Y$ be a continuous map.

*To prove.* $f$ induces $f_*\colon H_0(X) \xrightarrow{\cong} H_0(Y)$.

*Proof sketch.* By [1], that $X$ and $Y$ are path connected implies for any $x \in X$, $y \in Y$, the homology groups $H_0(X)$ and $H_0(Y)$ are infinite cyclic, and generated by $[[x]]$ and $[[y]]$ respectively. Because $f_*\colon [[x]] \mapsto [[f(x)]]$ sends one generator to another, it's an isomorphism. □

*To prove.* Any map $f\colon X \to X$ induces the identity on $H_0(X)$.

*Proof sketch.* In this case, $f_*\colon [[x]] \mapsto [[f(x)]]$. Path connectedness implies $[[x]] = [[f(x)]]$. □

1. Let $X = \bigcup_{i\in I} X_i$ where $X_i$ are the path components of $X$. We have
$$H_*(X) \cong \bigoplus_{i\in I} H_*(X_i).$$

*Proof.* Since $\Delta_p$ is path connected, its image under a map is path connected and we have
$$\mathrm{Sing}_p(X) = \bigsqcup_{i\in I} \mathrm{Sing}_p(X_i).$$
Hence each $p$-simplex of $X$ may be realized as a $p$-simplex of some path component $X_i$ followed by the inclusion $f_i : X_i \hookrightarrow X$. This lets us define a map on chain groups
$$\phi : \bigoplus_{i\in I} \Delta_p(X_i) \to \Delta_p(X)$$
given by
$$\bigoplus_{i\in I} \sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma \sigma \mapsto \sum_{i\in I} \sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma (f_i\circ\sigma).$$
It's clear that $\phi$ is a group homomorphism. In fact $\phi$ is a group isomorphism with inverse $\phi^{-1}$ simply given by
$$\sum_{i\in I} \sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma (f_i\circ\sigma) \mapsto \bigoplus_{i\in I} \sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma \sigma.$$
We are justified in writing an arbitrary element of $\Delta_p(X)$ as a double sum as above because $\Delta_p(X)$ is abelian and $\mathrm{Sing}_p(X) = \bigsqcup_{i\in I} \mathrm{Sing}_p(X_i)$. Now we show that $\phi$ is in fact an isomorphism of chain complexes by checking $\partial\circ\phi = \phi\circ\partial$ and $\partial\circ\phi^{-1} = \phi^{-1}\circ\partial$. We have
$$\begin{aligned}
(\partial\circ\phi)\left(\bigoplus_{i\in I} \sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma\sigma\right) &= \partial\left(\sum_{i\in I} \sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma(f_i\circ\sigma)\right) && \text{(by definition of } \phi)\\
&= \sum_{i\in I} \partial\left(\sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma(f_i\circ\sigma)\right) && \text{(as } \partial \text{ is a homomorphism)}\\
&= \sum_{i\in I} (\partial\circ(f_i)_\Delta)\left(\sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma\sigma\right) && \text{(by definition of } (f_i)_\Delta)\\
&= \sum_{i\in I} ((f_i)_\Delta\circ\partial)\left(\sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma\sigma\right) && \text{(as } \Delta_* : \mathrm{Top}\to\mathrm{Ch} \text{ is a functor)}\\
&= \sum_{i\in I} (f_i)_\Delta\left(\sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma(\partial\circ\sigma)\right) && \text{(as } \partial \text{ is a homomorphism)}\\
&= \sum_{i\in I} \sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma(f_i\circ\partial\circ\sigma) && \text{(by definition of } (f_i)_\Delta)\\
&= \phi\left(\bigoplus_{i\in I} \sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma(\partial\circ\sigma)\right) && \text{(by definition of } \phi)\\
&= (\phi\circ\partial)\left(\bigoplus_{i\in I} \sum_{\sigma\in\mathrm{Sing}_p(X_i)} n_\sigma\sigma\right). && \text{(as } \partial \text{ is a homomorphism)}
\end{aligned}$$
A similar computation shows $\partial\circ\phi^{-1} = \phi^{-1}\circ\partial$. Since functors preserve isomorphism
$$\phi_* : H_*\left(\bigoplus_{i\in I} X_i\right)) \to H_*(X)$$

is an isomorphism.

Finally, since homology commutes with direct sum,

$$H_*\left(\bigoplus_{i\in I}\Delta_p(X_i)\right) \cong \bigoplus_{i\in I} H_*(\Delta_p(X_i)).$$

This implies that

$$\bigoplus_{i\in I} H_*(X_i) \cong H_*(X).$$

□

**[1, No. IV.3.7].**

(a) Compute $H_1(K)$ for the Klein bottle $K$.

- Apply Seifert Van Kampen [1, Figs. III–11] to find $\pi_1(K) = \langle a, b | bab^{-1}a^{-1} \rangle$.
- Quotient by the commutator, $\langle aba^{-1}b^{-1} \rangle$.
- By transitivity of factorization, $H_1(K) \cong \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$.

(b) Compute $H_1$ of $X = \prod_{j \in J} X_j$ for topological spaces $X_j$ in terms of $H_1(X_j)$.

- The fundamental group of a product is the product of fundamental groups.
- So $\pi_1(X) \cong \prod \pi_1(X_j)$.
- NO! It suffices to abelianize each component group.
  - Elements from $X_i$ and $X_j$ commute whenever $i \neq j$.
- We conclude $H_1(X) \cong \prod H_1(X_j)$.

(c) Let $X_i$ with base points $x_i \in X_i$. Suppose that there are open sets $x_i \in U_i \subseteq X_i$ such that $x_i$ is a deformation retract of $U_i$. Show that

$$H_1\left(\bigvee_{i \in I} X_i\right) \cong \bigoplus_{i \in I} H_1(X_i).$$

- Apply Seifert Van Kampen [1, Sec. III.9] to find $\pi_1(\vee_i X_i) = \bigstar_i \pi_1(X_i)$.

  The free product is the coproduct in the category of groups. That is, the free product plays the same role in group theory that disjoint union plays in set theory, or that the direct sum plays in module theory.

- Abelianize $\bigstar_i \pi_1(X_i)$ as $\bigoplus_i H_1(X_i)$.

**[1, p. 177].** *Given.* Let $f\colon X \to Y$ be a covering space of path connected spaces with $f(x_0) = y_0$. By the fundamental theorem of covering spaces, $f_\#\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a monomorphism. Is $f_*\colon H_*(X) \to H_*(Y)$ also a monomorphism?

*To demonstrate.* Not necessarily. We'll consider a cover $X$ of $Y = \mathbf{C} \setminus \{0, 1\}$ the plane with two points removed.

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[rr, bend right, "id"'] & Y \arrow[r] & X \end{tikzcd}$$

$$\begin{tikzcd} H_*(X) \arrow[r, "i_*"] \arrow[rr, bend right, "H_*(id) = id"'] & H_*(Y) \arrow[r] & H_*(X) \end{tikzcd}$$

*Demo.*

- Let $Y = \mathbf{C} \setminus \{0, 1\}$ and fix a point $p \in Y$.
- Deform retracting and applying Seifert van-Kampen,
$$\pi_1(Y, p) \cong \langle a, b \rangle.$$
- Because $aba^{-1}b^{-1}$ is nontrivial, $\langle aba^{-1}b^{-1} \rangle \cong \mathbf{Z}$ is an infinite cyclic subgroup of the fundamental group of the plane with two points removed.
  - Sketch $\mathbf{C}\{0, 1\}$ and the path $abab^{-1}$.
- By the Galois correspondence between covering spaces and subgroups of the fundamental group [2, No. 1.36], there's a cover $f\colon X \to Y$ such that
$$f_\#(\pi_1(X, q)) = \langle aba^{-1}b^{-1} \rangle \subset \pi_1(Y, p).$$
  - It's a cover $X$ of $Y$ for which the lift of $aba^{-1}b^{-1}$ is a loop without self-intersections.
  - Sketch Cayley diagram of universal cover. Sketch cyclic generator.
- Now $H_1(X) \cong \mathbf{Z}$, $H_1(Y) \cong \mathbf{Z}^2$, yet
$$f_*([[\underbrace{aba^{-1}b^{-1}}_{\text{in } \Delta_1(X)}]]) = [[a]] + [[b]] - [[a]] - [[b]],$$
  which is *not* an induced monomorphism. $\square$

$$\begin{tikzcd} (\prod G_i)^{ab} \arrow[r, dotted, "\exists \Phi"] & \prod G_i^{ab} \\ \prod G_i \arrow[u, "\pi"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

so $\Phi$ ain't injective

$$\begin{tikzcd} \prod G_i \arrow[r, two heads, "\varphi"] \arrow[d, "\pi"] & \prod G_i^{ab} \\ (\prod G_i)^{ab} \arrow[ur, dotted, "\exists! \Phi"'] & \end{tikzcd}$$

WTS $\ker \varphi \ngeq [\prod G_i, \prod G_i]$.

i.e., $\prod [G_i, G_i] \ngeq [\prod G_i, \prod G_i]$

We know $\ker\varphi \geq [\prod G_i, \prod G_i]$.

WTS that $\ker \varphi \ngeq [\prod G_i, \prod G_i]$.

Eg $\prod [G_i, G_i] \ngeq [\prod G_i, \prod G_i]$

Eg. we want $\exists (g_i) \in \prod [G_i, G_i]$ s.th. $(g_i) \notin [\prod G_i, \prod G_i]$.

$$\begin{tikzcd} \prod G_i \arrow[r] \arrow[dr, "\varphi"'] & (\prod G_i)^{ab} \arrow[d, dotted, "\exists!"] \\ & \prod G_i^{ab} \end{tikzcd}$$

ker

$$\begin{tikzcd} \mathbb{Z}/14 \arrow[r, "\times 7"] \arrow[d] & \mathbb{Z}/14 \\ \mathbb{Z}/7\mathbb{Z} \arrow[ur, dotted, "?"'] & \end{tikzcd}$$

no!

TODO

Quotient of product is the product of quotients...

?

$$\begin{tikzcd} \mathbb{Z}/14 \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z}/14 \\ \mathbb{Z}/7\mathbb{Z} \arrow[ur, dotted, "?"'] & \end{tikzcd}$$

yes!

Also $\pi_1(\prod X) = \prod \pi_1(X)$

$$\begin{tikzcd} 0 \arrow[d] & & \\ B_p(X) \arrow[d] & & \\ Z_p(X) \arrow[r, "\alpha"] \arrow[d] & {} & \\ H_p(X) \arrow[ur, dotted] \arrow[d] & & \\ 0 & & \end{tikzcd}$$

if $\ker\alpha$

$$\begin{tikzcd} \mathbb{Z}/\cancel{\mathbb{Z}}(n) \arrow[r, "f:\, p^{th} \text{ multiple}"] \arrow[d, "\pi"] & \mathbb{Z}/(n) \\ \mathbb{Z}/(n,m) \arrow[ur, dotted, "?"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z}/\cancel{\mathbb{Z}}14 \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z}/14 \\ \mathbb{Z}/7 \arrow[ur, "\exists"'] & \end{tikzcd}$$

$0 \quad \overset{\varphi}{\prod} \mathbb{F}_2$

$(a_i b_i a_i^{-1} b_i^{-1}) \in \prod_1^\infty [\mathbb{F}_2, \mathbb{F}_2]$

$(a_i)(b_i)(a_i)^{-1}(b_i)^{-1} \in [\prod_1^\infty \mathbb{F}_2, \prod_1^\infty \mathbb{F}_2]$

$(a_i)(b_i)(a_i^{-1})(b_i^{-1}) \ldots$

$= (a_i b_i a_i^{-1} b_i^{-1})$

$\prod_1^\infty [\mathbb{F}_\infty, \mathbb{F}_\infty] \ni ([a_1, b_1], [a_2, b_2][a_2, c_2], [a_3, b_3][a_3, c_3][a_3, d_3]\ldots)$

yet not in $[\prod \mathbb{F}_\infty, \prod \mathbb{F}_\infty]$

Eg. for say $[(a_1, a_2, \ldots)(b_1, b_2, \ldots)]$

$[(0, c_2, \ldots)(0, \ldots)]$

$[(\cancel{a}0, a_2, \ldots)(0, 0, d_3)]$

$[(0, 0, a_3, \ldots)(0, 0, d_3)]$

$[a_2, b_2][a_2, c_2] = a_2 b_2 a_2^{-1} b_2^{-1} a_2 c_2 a_2^{-1} c_2^{-1}$

$\neq [a_2 b_2, a_2 c_2]$

If $f$ is "constant on fibers" then $f$ "descends to the quotient".

if $\ker f \geq \mathbb{Z}(m)$ then $f$ descends

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | 44 | 48 | 52 |
| | | | | 16 | 20 | 24 | 28 | 32 | 36 | 40 | 44 | 48 | 52 |
| 0 | 4 | 8 | 12 | 2 | 6 | 10 | 0 | 4 | 8 | 12 | 2 | 6 | 10 |

$4^{th}$ multiple map const. on fibers

$\ker p = \{ a \in \mathbb{Z}/(n) : pa = 0 \}$

counter example

the map descends if $\ker p \geq 14\mathbb{Z}$ (left)

$\pi_1^\infty \mathbb{F}_k = \pi_1(\prod_1^\infty \mathbb{C} \setminus \{k \text{ distinct points}\})$

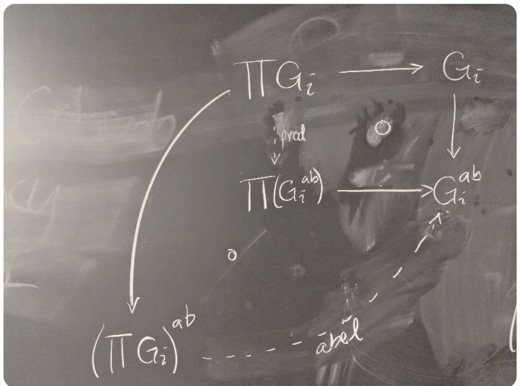
yet $(\prod_1^\infty \mathbb{F}_k)^{ab} \ncong \prod_1^\infty \mathbb{F}_k^{ab} \cong \prod_1^\infty \mathbb{Z}^k$

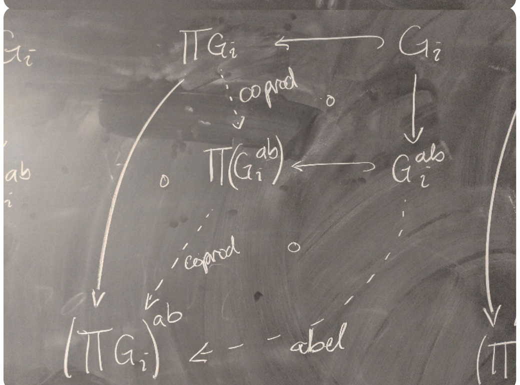
so it's not the case that $H_1(\prod X_i) = \prod H_1(X_i)$ such.

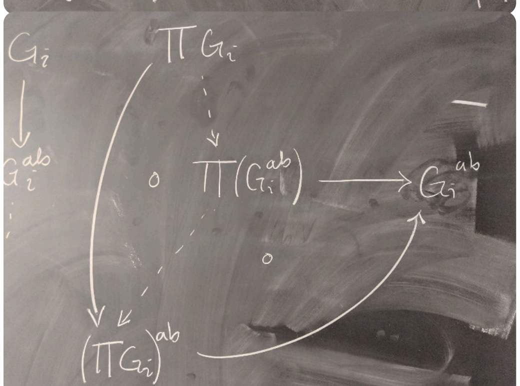
Because $(0, [a, b], [a, b][a, c], [a, b][a, c][a, d], \ldots)$ is in $\prod_1^\infty [\mathbb{F}_k, \mathbb{F}_2]$ yet not in $[\prod_1^\infty \mathbb{F}_k, \prod \mathbb{F}_k]$

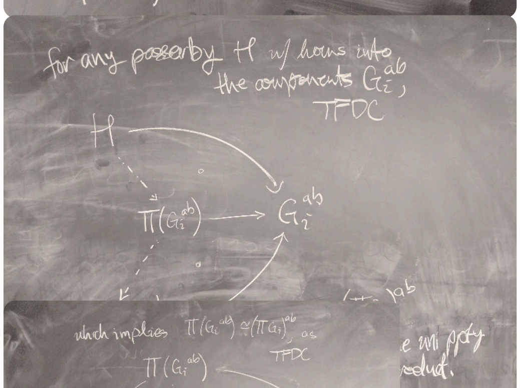
abelianizing

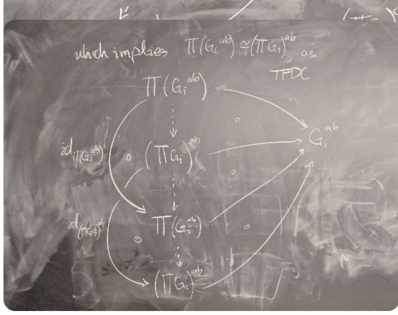
a <u>finite</u> product of groups
with universal properties
('s write up.)





# Homework 1 Problem 3

Jonathan Quartin

## 1

### 1.1 Part (a)

Compute $H_1(K)$ for the Klein bottle $K$.

### 1.2 Solution

First, we use Seifert Van Kampen's Theorem to determine the fundamental group. Let $U$ be a disk, fully contained within the interior of the square depicted above. Consider another disk contained within $U$, and let $V$ be the complement of this smaller disk. Then $\{U, V\}$ forms a cover of $K$, so by Seifert Van Kampen's Theorem,

$$\pi_1(K) = \frac{\langle a,b\rangle * \{e\}}{\langle abab^{-1}=e\rangle} = \langle a, b | abab^{-1} = e\rangle$$

By Hurewicz's Theorem, $H_1(K) = \pi_1(K)^{ab}$, so

$$H_1(K) = \frac{\langle a,b|abab^{-1}=e\rangle}{\langle aba^{-1}b^{-1}=e\rangle} = \frac{\langle a,b|aba^{-1}b^{-1}=e\rangle}{\langle abab^{-1}=e\rangle} \cong \frac{\mathbb{Z}\times\mathbb{Z}}{\langle a^2=e\rangle} \cong \mathbb{Z}_2 \times \mathbb{Z}$$

### 1.3 Part (b)

Compute $H_1$ of $X = \prod_{i=1}^{n} X_i$, for topological spaces $X_i$, in terms of $H_1(X_i)$.

### 1.4 Solution

Here is a solution that only uses universal properties. For a less categorical proof, skip to the alternate solution.

I will construct maps between $(G_1 \times G_2)^{ab}$ and $G_1^{ab} \times G_2^{ab}$, which are inverses of each other.

First, we have a map from $G_1 \times G_2$ to $G_1^{ab} \times G_2^{ab}$ by projecting onto factors, abelianizing, and then inclusions. Since $G_1^{ab} \times G_2^{ab}$ is abelian, there exists a unique map $\phi : (G_1 \times G_2)^{ab} \longrightarrow G_1^{ab} \times G_2^{ab}$ by the universal property of abelianization.

Now to get a map in the reverse direction. First, we have maps from $G_1$ and $G_2$ to $(G_1 \times G_2)^{ab}$ by inclusion into the direct product, and then abelianization. Hence there are unique maps from $G_1^{ab}$ and $G_2^{ab}$ mapping into $(G_1 \times G_2)^{ab}$ by the universal property of abelianization. So by the universal property of coproducts, $\exists! \ \psi : G_1^{ab} \times G_2^{ab} \longrightarrow (G_1 \times G_2)^{ab}$ making all compositions commute.

It suffices to check that $\phi$ and $\psi$ are inverses.

This can be shown by composing the two maps in either order alongside identity maps. Both will make the above constructed diagrams commute, so by uniqueness of $\phi, \psi$, we have that $\phi \circ \psi = Id_{G_1^{ab} \times G_2^{ab}}$, and $\psi \circ \phi = Id_{(G_1 \times G_2)^{ab}}$

The argument is sufficiently general to apply to arbitrary finite products of groups. Hence by Hurewicz's Theorem, we get the result for homology.

ALTERNATE SOLUTION:

<u>Induction</u>

**Base case:** Let $X = X_1 \bigcup X_2$.

For convenience, we label $\pi_1(X) = G, \pi_1(X_1) = H, \pi_1(X_2) = K$,

so $G = H \times K$

We can get a homomorphism $\phi : H \times K \longrightarrow H^{ab} \times K^{ab}$ as follows:

$(\iota_{H^{ab}}, \iota_{K^{ab}}) \circ (ab_H, ab_K) \circ (\pi_H \ \pi_K)$

Above, we first project, then abelianize, then include into the target group.

For $(h_1 h_2 h_1^{-1} h_2^{-1}, k_1 k_2 k_1^{-1} k_2^{-1}) \in [G, G]$, the image under the above map is $([e], [e])$.

$\implies [G, G] \in ker(\phi)$

The following proves the reverse inclusion:

$x \in ker(\phi) \Longrightarrow x = (h_1 \cdot ... \cdot h_m, k_1 \cdot ... \cdot k_n)$

where each $h_i$ is a generator in $[H, H]$, and each $k_i$ is a generator in $[K, K]$

Without loss of generality, assume $n > m$.

Then $x = (h_1, k_1) \cdot ... \cdot (h_m, k_m) \cdot (1, k_{m+1}) \cdot ... \cdot (1, k_n)$

Each of these tuples is in $[G, G]$, as was shown for the previous direction.

So $[G, G] = ker(\phi)$, and hence $G^{ab} = \frac{G}{[G,G]} = \frac{H \times K}{ker(\phi)} \cong H^{ab} \times K^{ab}$.

By Hurewicz, the result on homology follows.

**Inductive Step:** Assume for $G = \prod_{i=1}^{n} G_i$ that $G^{ab} = \prod_{i=1}^{n} G_i^{ab}$
(Here the $G_i$ are fundamental groups associated to topological spaces $X_i$)

Let $H = \prod_{i=1}^{n+1} G_i = G \times G_{n+1}$.

Then $H^{ab} = G^{ab} \times G_{n+1}^{ab}$ by the base case, so by the inductive assumption,

$H^{ab} = \prod_{i=1}^{n+1} G_i^{ab}$.

The result then follows by Hurewicz.

## 1.5 Part (c)

Let $X_i$ with base points $x_i \in X_i$. Suppose that there are open sets $x_i \in U_i \subset X_i$ such that $x_i$ is a deformation retract of $U_i$. Show that

$$H_1(\bigvee_{i \in I} X_i) \cong \bigoplus_{i \in I} H_1(X_i).$$

## 1.6 Solution

Applying Seifert Van Kampen yields $\pi_1(\bigvee_i X_i) = *_i \pi_1(X_i)$.

For the following chain of equalities, we use these universal properties in order: abelianization, free product (in category of groups), abelianization, direct sum (in category of abelian groups).

$\hom((*\pi_1(X_i))^{ab}, C)$

$= \hom((*\pi_1(X_i)), C)$

$= \bigoplus \hom(\pi_1(X_i), C)$

$= \bigoplus \hom(\pi_1(X_i)^{ab}, C)$

$= \hom(\bigoplus \pi_1(X_i)^{ab}, C)$,

so $(*\pi_1(X_i))^{ab} \cong \bigoplus \pi_1(X_i)^{ab}$, i.e. $H_1(\bigvee_{i\in I} X_i) \cong \bigoplus_{i\in I} H_1(X_i)$.

2019-02-01

### 1.4.2 Homological Algebra Continued

First, some remarks from last class

also the co-product bull-shittery with abelianization...

**Remark 1.4.10.** (a) Let $f\colon A_* \to B_*$ be a chain map. Then $f$ is an isomorphism (i.e. has a two sided inverse chain map $g$) if and only if $f_p\colon A_p \to B_p$ is an isomorphism for all $p \in \mathbb{Z}$. Note that an isomorphism of chain complexes induces an isomorphism on homology since homology is a functor.

(b) If $A_* = \bigoplus_i (A_i)_*$, then $H_*(A) \cong \bigoplus_i H_*((A_i)_*)$.

An *additive category* is an **Ab**-category $\mathcal{A}$ with a zero object (i.e., an object that is initial and terminal) and a product $A \times B$ for every pair $A, B$ of objects in $\mathcal{A}$. This structure is enough to make finite products the same as finite coproducts. The zero object in **Ch** is the complex "0" of zero modules and maps. Given a family $\{A_\alpha\}$ of complexes of $R$-modules, the product $\Pi A_\alpha$ and coproduct (direct sum) $\oplus A_\alpha$ exist in **Ch** and are defined degreewise: the differentials are the maps

$$\prod d_\alpha : \prod_\alpha A_{\alpha,n} \to \prod_\alpha A_{\alpha,n-1} \quad \text{and} \quad \oplus d_\alpha : \oplus_\alpha A_{\alpha,n} \to \oplus_\alpha A_{\alpha,n-1},$$

respectively. These suffice to make **Ch** into an additive category.

[Wei94, ch1

**Exercise 1.2.1** Show that direct sum and direct product commute with homology, that is, that $\oplus H_n(A_\alpha) \cong H_n(\oplus A_\alpha)$ and $\Pi H_n(A_\alpha) \cong H_n(\Pi A_\alpha)$ for all $n$.

### 1.4.3 Relative Homology Groups

**Definition 1.4.11.** A pair of spaces $(X, A)$ is a topological space $X$ together with a subspace $A \subseteq X$. A map of pairs $f\colon (X, A) \to (Y, B)$ is a continuous function $f\colon X \to Y$ such that $f(A) \subseteq B$.

**Definition 1.4.12.** A subcomplex $B_*$ of $C_*$ is a sequence $B_p \subseteq C_p$ of subgroups such that $\partial_p(B_p) \subseteq B_{p-1}$. If $B_*$ is a subcomplex, then the $\partial_p$ induces a map $C_p/B_p \to C_{p-1}/B_{p-1}$. The quotient complex is $C_*/B_*$ with the induced boundary map.

For example if $A$ is a subspace of $X$, then $\Delta_*(A)$ is a subcomplex of $\Delta_*(X)$.

**Definition 1.4.13.** Let $(X, A)$ be a pair of spaces. The *relative chains complex* is

$$\Delta_*(X, A) = \Delta_*(X)/\Delta_*(A).$$

The relative homology is

$$H_p(X, A) := H_p(\Delta_*(X, A)).$$

**Theorem 1.4.14.** *If $(X, A)$ is a pair, there is a long exact sequence of homology groups*

$$\dots \xrightarrow{\delta_*} H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{j_*} H_p(X, A) \xrightarrow{\delta_*} H_{p-1}(A) \xrightarrow{i_*} \dots$$

*Proof.* This is the long exact sequence associated to

$$0 \to \Delta_*(A) \to \Delta_*(X) \to \Delta_*(X, A) \to 0.$$

□

### 1.4.4 Homology with coefficients and Cohomology

There is a functor

$$\mathrm{Hom}_{\mathrm{Ab}}(-,-)\colon \mathrm{Ab}^{\mathrm{op}} \times \mathrm{Ab} \to \mathrm{Ab},$$

which maps $(A, B)$ to the abelian group $\mathrm{Hom}_{\mathrm{Ab}}(A, B)$ (group structure is given by adding values). If $\phi\colon A \to B$, $f\colon C \to A$ and $g\colon B \to D$, then

$$\mathrm{Hom}_{\mathrm{Ab}}(f, g)(\phi) = g \circ \phi \circ f.$$

[EJ99]

**Definition 1.4.9.** If $\mathsf{C}$ is any category, we define $\mathsf{C}^0$ (the *category opposite* $\mathsf{C}$) to be the category such that $\mathrm{Ob}(\mathsf{C}^0) = \mathrm{Ob}(\mathsf{C})$, and $\mathrm{Hom}_{\mathsf{C}^0}(B, A) = \mathrm{Hom}_{\mathsf{C}}(A, B)$ where for $g : C \to B$, $f : B \to A$ in $\mathsf{C}^0$, $f \circ g$ is defined to be the morphism $g \circ f$ of $\mathsf{C}$. So then a contravariant functor $T : \mathsf{C} \to \mathsf{D}$ is simply a functor $T : \mathsf{C}^0 \to \mathsf{D}$ or a functor $T : \mathsf{C} \to \mathsf{D}^0$.

There is also functor

$$(-) \otimes (-) \colon \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$$

which takes $(A, B)$ to

$$A \otimes B := \{a \otimes b : a \in A,\ b \in B\}/\sim$$

where the equivalence relation is

$$(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$$
$$a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2.$$

For $f \colon A \to C$ and $g \colon B \to D$, we have a map

$$f \otimes g \colon A \otimes B \to C \otimes D$$

given by

$$f \otimes g(a \otimes b) = f(a) \otimes f(b).$$

This has the universal property that there are isomorphisms of abelian groups

$$\mathrm{Hom}_{\mathrm{Ab}}(A \otimes B, C) \cong \mathrm{Bilin}_{\mathrm{Ab}}(A \times B, C) \cong \mathrm{Hom}_{\mathrm{Ab}}(A, \mathrm{Hom}_{\mathrm{Ab}}(B, C)).$$

**Definition 1.4.15.** Let $G$ be an abelian group and $C_*$ be a chain complex:

(a) The *homology of $C_*$ with coefficients in $G$*, denoted $H_*(C_*; G)$, is the homology of the chain complex $G \otimes C_*$, whose $p$th group is $G \otimes C_p$ and boundary $\partial_p^{C_*, G} = \mathrm{id}_G \otimes \partial_p$.

(b) The *cohomology of $C_*$ with coefficients in $G$*, denoted $H^*(C_*; G)$, is the homology of the cochain complex $\mathrm{Hom}_{\mathrm{Ab}}(C_*, G)$, whose $p$th group is $\mathrm{Hom}_{\mathrm{Ab}}(C_p, G)$ and boundary $\partial^p_{C_*, G} = \mathrm{Hom}_{\mathrm{Ab}}(\partial_p, \mathrm{id}_G)$.

(c) The *homology of $X$ with coefficients in $G$* is $H_*(X; G) := H_*(\Delta_*(X); G)$.

(d) The *cohomology of $X$ with coefficients in $G$* is $H^*(X; G) = H^*(\Delta_*(X); G)$. We also write $H^*(X) = H^*(X; \mathbb{Z})$ and call it the cohomology of $X$.

We define homology groups with coefficients in $G$ by

$$\boxed{H_p(X;G) = H_p(\Delta_*(X) \otimes G).}$$

For $A \subset X$, the sequence

$$(1) \qquad 0 \to \Delta_*(A) \otimes G \to \Delta_*(X) \otimes G \to \Delta_*(X, A) \otimes G \to 0$$

is exact because of the splitting map $\Delta_*(X, A) \to \Delta_*(X)$. Define

$$\boxed{H_p(X, A; G) = H_p(\Delta_*(X, A) \otimes G).}$$

Then (1) induces the long exact sequence

$$\boxed{\cdots \to H_p(A; G) \to H_p(X; G) \to H_p(X, A; G) \to H_{p-1}(A; G) \to \cdots.}$$

If $0 \to G' \to G \to G'' \to 0$ is an exact sequence of abelian groups then

$$0 \to \Delta_*(X) \otimes G' \to \Delta_*(X) \otimes G \to \Delta_*(X) \otimes G'' \to 0$$

is exact, since $\Delta_*(X)$ is free abelian. Consequently, there is the long exact sequence

$$\boxed{\cdots \to H_p(X; G') \to H_p(X; G) \to H_p(X; G'') \to H_{p-1}(X; G') \to \cdots}$$

and similarly for the relative groups $H_*(X, A; \cdot)$. The connecting homomorphism $H_p(X; G'') \to H_{p-1}(X; G')$ is sometimes called the "Bockstein" homomorphism in this case, although that appellation is usually reserved for the special cases of the coefficient sequences $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_p \to 0$ and $0 \to \mathbf{Z}_p \to \mathbf{Z}_{p^2} \to \mathbf{Z}_p \to 0$.

Especially note the case of coefficients in a field $F$. In this case all the groups such as $H_p(X, A; F)$ are vector spaces over $F$. More generally, if $R$ is a commutative ring, then $H_p(X, A; R)$ is an $R$-module.

**Remark 1.4.16.** Recall that

$$\Delta_*(X, A) = \Delta_*(X)/\Delta_*(A).$$

Note that

$$\mathrm{Sing}_p(X) = \mathrm{Sing}_p(A) \sqcup \{\sigma \in \mathrm{Sing}_p(X) : \sigma(\Delta_p) \nsubseteq A\}.$$

Furthermore,

$$\mathbb{Z}\{\sigma \in \mathrm{Sing}_p(X) : \sigma(\Delta_p) \nsubseteq A\} \cancel{\to \Delta_p(X)} \to \Delta_p(X, A)$$

is an isomorphism.

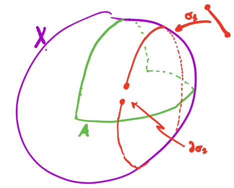
Observe

$$\begin{array}{c} \mathbb{Z}\{\sigma \in \mathrm{Sing}_p(X) : \sigma(\Delta_p) \subseteq A\} \\ \parallel \\ 0 \to \Delta_p(A) \to \Delta_p(X) \to \Delta_p(X, A) \to 0 \end{array}$$

splits levelwise

for each $p \in \mathbb{Z}_{\geq 0}$

$$\begin{tikzcd}
 & \Delta_p(X) \arrow[dr] \arrow[d, "\cong"] & & \\
0 \arrow[r] & \Delta_p(A) \arrow[u] \arrow[d] & \Delta_p(X,A) \arrow[r] & 0 \\
 & \mathbb{Z}\{\sigma_p : \Delta_p \to X \mid \sigma_p(\Delta_p) \not\subset A\} \oplus \Delta_p(A) \arrow[ur] & &
\end{tikzcd}$$




So, the exact sequence

E.g., not split as a whole chain complex.

$$0 \to \Delta_*(A) \xrightarrow{i} \Delta_*(X) \xrightarrow{j} \Delta_*(X,A) = \Delta_*(X)/\Delta_*(A) \to 0$$

is "level-wise split."

It follows that

$$0 \to G \otimes \Delta_*(A) \to G \otimes \Delta_*(X) \to G \otimes \Delta_*(X,A) \to 0$$

and

$$0 \to \mathrm{Hom}_{\mathrm{Ab}}(\Delta_*(X,A),G) \to \mathrm{Hom}_{\mathrm{Ab}}(\Delta_*(X),G) \to \mathrm{Hom}_{\mathrm{Ab}}(\Delta_*(A),G) \to 0$$

are exact sequences (since they are also levelwise split), so there are long exact sequences

$$\dots \xrightarrow{\delta_*} H_p(A;G) \xrightarrow{i_*} H_p(X;G) \xrightarrow{j_*} H_p(X,A;G) \xrightarrow{\delta_*} H_{p-1}(A;G) \xrightarrow{i_*} \dots$$

and

$$\dots \xrightarrow{\delta^*} H^p(X,A;G) \xrightarrow{i^*} H^p(X;G) \xrightarrow{j^*} H^p(A;G) \xrightarrow{\delta^*} H^{p+1}(X,A;G) \xrightarrow{i^*} \dots$$

Similarly, if

$$0 \to G \to G' \to G'' \to 0$$

Suppose we want

$0 \to G \xrightarrow{i} G' \xrightarrow{j} G'' \to 0$ tensored on the right by $\mathbb{F}_2$.

Compute $G \otimes \mathbb{F}_2 \cong (G \otimes_{\mathbb{Z}} \mathbb{Z}) \oplus (G \otimes_{\mathbb{Z}} \mathbb{Z})$
$\cong (G \oplus G)$. $\quad gn \otimes (x,y) = g \otimes (nx, ny)$

is an exact sequence of abelian groups, since $\Delta_p(X)$ is free, then

$$0 \to G \otimes \Delta_p(X) \to G' \otimes \Delta_p(X) \to G'' \otimes \Delta_p(X) \to 0$$

whence

$0 \to G^{\oplus 2} \to G'^{\oplus 2} \to G''^{\oplus 2} \to 0$

$i \otimes \mathrm{id}_{\mathbb{F}_2} : g \otimes (x,y) \mapsto g' \otimes (x,y)$ $\quad$ not.

and

$$0 \to \mathrm{Hom}(\Delta_p(X),G) \to \mathrm{Hom}(\Delta_p(X),G') \to \mathrm{Hom}(\Delta_p(X),G'') \to 0$$

are exact (check it!). So, there are long exact sequences

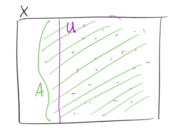
$$\dots \xrightarrow{\delta_*} H_p(X;G) \xrightarrow{i_*} H_p(X;G') \xrightarrow{j_*} H_p(X;G'') \xrightarrow{\delta_*} H_{p-1}(X;G) \xrightarrow{i_*} \dots$$

and

$$\dots \xrightarrow{\delta^*} H^p(X;G) \xrightarrow{i^*} H^p(X;G') \xrightarrow{j^*} H^p(X;G'') \xrightarrow{\delta^*} H^{p+1}(X;G) \xrightarrow{i^*} \dots$$

2019-02-04

## 1.5 Axioms for Homology




Now we consider homology as a functor of pairs. Note that $H_*(A, \varnothing) \cong H_*(A)$. So we can think of Top as being embedded in $\text{Top}_{\text{pairs}}$

**Definition 1.5.1.** $(X; A, B)$ is an excisive triad if $X$ is the union of the interiors of $A$ and $B$. An excisive triple is a triple $(X, A, U)$ with $\overline{U} \subseteq \text{int}(A)$.

$U = X \cap B^c$
$U^c = X^c \cup B$
$B = U^c \cap X$
$B = X \setminus U$

**Remark 1.5.2.** One can go from an excisive triad to an excisive triple by letting $U = X - B$ and vice versa.

**Definition 1.5.3.** A *generalized* homology theory on the category of pairs of topological spaces is a covariant functor $h_*$ to graded abelian groups together with a natural transformation

$$\delta_*\colon h_p(X, A) \to h_{p-1}(A)$$

that satisfies

**Remark 1.3.5** The data of the long exact sequence is sometimes organized into the mnemonic shape

$$\begin{tikzcd} H_*(A) \arrow[rr] & & H_*(B) \arrow[dl] \\ & H_*(C) \arrow[ul, "\partial"] & \end{tikzcd}$$

This is called an *exact triangle* for obvious reasons. This mnemonic shape is responsible for the term "triangulated category," which we will discuss in Chapter 10. The category **K** of chain equivalence classes of complexes and maps (see exercise 1.4.5 in the next section) is an example of a triangulated category.

1. (Homotopy Axiom) If $f \simeq g\colon (X, A) \to (Y, B)$, then

$$h_*(f) = h_*(g)\colon h_*(X, A) \to h_*(Y, B)$$

2. (Exactness axiom) For any pair $(X, A)$, there is a long exact sequence

$$\ldots \xrightarrow{\delta_*} h_p(A) \xrightarrow{i_*} h_p(X) \xrightarrow{j_*} h_p(X, A) \xrightarrow{\delta_*} h_{p-1}(A) \xrightarrow{i_*} \ldots$$

where $i\colon A \to X$ and $j\colon X \to (X, A)$ are the inclusions.

3. (Excision Axiom) If $(X, A, U)$ is an excisive triple, then $(X - U, A - U) \to (X, A)$ induces an isomorphism

$$h_*(X - U, A - U) \xrightarrow{e_*} h_*(X, A)\,.$$

Equivalently, given an excisive triad $(X; A, B)$, the inclusion $(B, A \cap B) \to (X, A)$ induces an isomorphism

$$h_*(B, A \cap B) \xrightarrow{e_*} h_*(X, A)\,.$$

4. (Additivity Axiom) If $(X, A) = \coprod (X_i, A_i)$, then

$$h_*(X, A) \cong \bigoplus h_*(X_i, A_i)$$

and the isomorphism is induced by the inclusions.

for pairs!

If $h_*$ satisfies

6. (Dimension axiom) $h_*(\text{pt}) = 0$ if $* > 0$.

then we say that $h_*$ is *ordinary* and call the group $h_0(\text{pt})$ the coefficients of $h_*$.

If $X \neq \varnothing$, then the *reduced homology* $\widetilde{h}_*(X)$ is defined by the exact sequence

$$0 \to \widetilde{h}_*(X) \to h_*(X) \to h_*(\text{pt}) \to 0.$$

If $A \neq \varnothing$, we also let $\widetilde{h}_*(X, A) = h_*(X, A)$.

$0 \to h_*(A) \to h_*(X) \to h_*(X,A) \to 0$

We define $\tilde{h}_*(X,A) = h_*(X,A)$, if $A \neq \emptyset$.

We define $\tilde{h}_*(X)$ in the s.e.s.

$0 \to \tilde{h}_*(X) \to h_*(X) \to h_*(pt) \to 0$, which has apparent comm. diagram

$$\ldots \xrightarrow{\delta_*} \tilde{h}_p(X) \xrightarrow{i_*} h_p(X) \xrightarrow{j_*} \cancel{h_p(pt)}^{\,0} \xrightarrow{\delta_*} \tilde{h}_{p-1}(X) \xrightarrow{i_*} h_p(X) \xrightarrow{j_*} \cancel{h_p(pt)}^{\,0} \xrightarrow{\delta_*} \ldots$$

$$\tilde{h}_p(X) = \begin{cases} h_p(X) & p \geq 1 \\ & p = 0 \end{cases}$$

# EX 1.5.4

It will be convenient to introduce the *reduced homology* of a space, which is defined to be $\widetilde{H}_n(X) := \ker(H_n(X) \to H_n(pt))$. Note that $\widetilde{H}_n(X) = H_n(X)$ for $n > 0$. Furthermore, a continuous map $f : X \to Y$ defines a map of reduced homology groups.

**Corollary 5.5.** *There is a long exact sequence in reduced homology*

$$\cdots \longrightarrow \widetilde{H}_1(A) \longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_1(X,A) \xrightarrow{\delta} \widetilde{H}_0(A) \longrightarrow \widetilde{H}_0(X) \longrightarrow H_0(X,A) \longrightarrow 0.$$

*Proof.* By the Snake Lemma

$$\cdots \longrightarrow H_1(X,A) \xrightarrow{\delta} H_0(A) \longrightarrow H_0(X) \longrightarrow H_0(X,A) \longrightarrow 0$$

is exact. It is an exercise to show that $\widetilde{H}_0(A) \to \widetilde{H}_0(X)$, and that $A \hookrightarrow X$ maps $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$ in $H_0(A) = \widetilde{H}_0(A) \oplus \mathbb{Z}$ and $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$. So the sequence in reduced homology is exact. ☐

$(X,A) \xrightarrow{f} (P,P)$ induces

$H_*(X,A) \xrightarrow{f_*} H_*(P,P)$, surjective because

$(P,P) \xrightarrow{x} (X,A) \xrightarrow{f} (P,P)$ is the identity on $(P,P)$

Defined by lifting the image of $\widetilde{H}_0(A)$ in $H_0(A)$.

Defined by pushing from $\widetilde{H}_0(A)$ to $H_0(A)$, then to $H_0(X)$, then lifting the image of $\widetilde{H}_0(X)$ in $H_0(X)$.

Defined as the composition.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \widetilde{H}_0(A) \arrow[r] \arrow[d] & \widetilde{H}_0(X) \arrow[d] \arrow[dr] & \widetilde{H}_0(X,A) \arrow[d, equal] & \\
H_1(X,A) \arrow[ur] \arrow[r] & \widetilde{H}_0(A)\oplus H_0(P) \arrow[r] \arrow[d, "g+c"] & \widetilde{H}_0(X)\oplus H_0(P) \arrow[r] \arrow[d, "\tilde{g}+\tilde{c}"] & H_0(X,A) \arrow[r] \arrow[d] & H_{-1}(A) \\
0 \arrow[r] & H_0(P) \arrow[r, "\sim"] \arrow[d] & H_0(P) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

Say $g \in H_0(P)$, $c \in \widetilde{H}_0(A)$.

$g$ $\quad$ $g = \tilde{g}$

$P \hookrightarrow A \to P$ (id)

$H_*(P) \hookrightarrow H_*(A) \twoheadrightarrow H_*(P)$ splits leftwise.

$A \hookrightarrow X \to A$ (id)

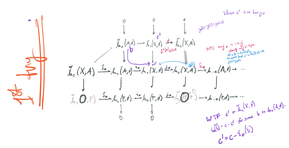
$H_*(A) \hookrightarrow H_*(X) \to H_*(A)$

$g+c \longmapsto \tilde{g}+\tilde{c} \longmapsto \tilde{g}$

☐

also true for a general homology theory

1st try



5. ⟡ Prove from the axioms that $H_i(\varnothing) = 0$ for all $i$, and that $H_i(X,X) = 0$ for all $i$ and all spaces $X$.

$\varnothing \hookrightarrow P \twoheadrightarrow \varnothing$

$H_*(\varnothing) \hookrightarrow H_*(P) \twoheadrightarrow H_*(\varnothing)$

So $H_i(\varnothing) = 0$ for all $i \neq 0$.

$(\varnothing,\varnothing) \hookrightarrow (P,\varnothing) \hookrightarrow (P,\varnothing)$ has l.e.s.

$h_1(P) \xrightarrow{\delta_*} h_0(\varnothing) \to h_0(P) \to h_0(P) \xrightarrow{\delta_*} h_{-1}(\varnothing) \cdots$

$0 \to h_0(\varnothing) \to h_0(P) \xrightarrow{\sim} h_0(P) \to 0$ s.e.s.

So $h_i(\varnothing) = 0$ for all $i$.

Now consider the les for $(X,X)$.

$\cdots \xrightarrow{0} h_{n+1}(X,X) \xrightarrow{\delta_*} h_n(X) \xrightarrow{\sim} h_n(X) \xrightarrow{0} h_n(X,X) \xrightarrow{\delta_*} h_{n-1}(X) \xrightarrow{\sim} h_{n-1}(X) \xrightarrow{0} h_{n-1}(X,X) \xrightarrow{\delta_*} h_{n-2}(X) \to \cdots$

hence $h_i(X,X) = 0$ for all $i$.

$\underline{\boxed{\text{PROP}}}$ Let $h$ be an ordinary homology theory. Let $X \neq \varnothing$ with $x \in X$, say $P = \{x\}$. Then $\tilde{h}_*(X) \cong h_*(X,P)$. Given. $h$, an ordinary homology theory. $\tilde{h}_*(X)$ def'd $\tilde{h}_*(X) \hookrightarrow h_*(X) \twoheadrightarrow h_*(P)$. $\tilde{h}_n(X,A) = h_n(X,A)$ for all $n$.

*Proof.* The inclusions $(P,\varnothing) \hookrightarrow (X,\varnothing) \hookrightarrow (X,P)$ induce a l.es. in reduced homology

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{h}_3(X,P) \arrow[r, "\delta_*"] \arrow[d, equal] & \cancel{\tilde{h}_2(P)} \arrow[r] & \tilde{h}_2(X) \arrow[r] & \tilde{h}_2(X,P) \arrow[r, "\delta_*"] \arrow[d, equal] & \cancel{\tilde{h}_1(P)} \arrow[r] & \tilde{h}_1(X) \arrow[r] & \tilde{h}_1(X,P) \arrow[r, "\delta_*"] \arrow[d, equal] & \cancel{\tilde{h}_0(P)} \arrow[r] & \tilde{h}_0(X) \arrow[r] & \tilde{h}_0(X,P) \arrow[r, "\delta_*"] \arrow[d, equal] & \cancel{\tilde{h}_{-1}(P)} \arrow[r] & \cdots \\
 & h_3(X,P) & & & h_2(X,P) & & & h_1(X,P) & & & h_0(X,P) & &
\end{tikzcd}$$

☐

**Proposition 2.9**. *If $A \hookrightarrow X$ is a topological subspace inclusion which is good in the sense of def. 2.7, then the $A$-relative singular homology of $X$ coincides with the reduced singular homology of the quotient space $X/A$:*

$$H_n(X,A) \simeq \tilde{H}_n(X/A).$$

For instance (Hatcher, prop. 2.22).

***Proof***. By assumption we can find a neighbourhood $A \xrightarrow{j} U \hookrightarrow X$ such that $A \hookrightarrow U$ has a deformation retract and hence in particular is a homotopy equivalence and so induces also isomorphisms on all singular homology groups.

It follows in particular that for all $n \in \mathbb{N}$ the canonical morphism $H_n(X,A) \xrightarrow{H_n(\mathrm{id},j)} H_n(X,U)$ is an isomorphism, by prop. .

Given such $U$ we have an evident commuting diagram of pairs of topological spaces

$$\begin{array}{ccccc} (X,A) & \overset{(\mathrm{id},j)}{\longrightarrow} & (X,U) & \leftarrow & (X-A, U-A) \\ \downarrow & & \downarrow & & \downarrow \\ (X/A, A/A) & \overset{(\mathrm{id},j/A)}{\longrightarrow} & (X/A, U/A) & \leftarrow & (X/A - A/A, U/A - A/A) \end{array}.$$

Here the right vertical morphism is in fact a homeomorphism.

Applying relative singular homology to this diagram yields for each $n \in \mathbb{N}$ the commuting diagram of abelian groups

$$\begin{array}{ccccc} H_n(X,A) & \underset{\simeq}{\overset{H_n(\mathrm{id},j)}{\longrightarrow}} & H_n(X,U) & \overset{\simeq}{\leftarrow} & H_n(X-A, U-A) \\ \downarrow & & \downarrow & & \downarrow \\ H_n(X/A, A/A) & \underset{\simeq}{\overset{H_n(\mathrm{id},j/A)}{\longrightarrow}} & H_n(X/A, U/A) & \overset{\simeq}{\leftarrow} & H_n(X/A - A/A, U/A - A/A) \end{array}.$$

Here the left horizontal morphisms are the above isomorphims induced from the deformation retract. The right horizontal morphisms are isomorphisms by prop. 2.3 and the right vertical morphism is an isomorphism since it is induced by a homeomorphism. Hence the left vertical morphism is an isomorphism (2-out-of-3 for isomorphisms). ∎

### Relation to reduced homology

Singular!

**Proposition 2.10**. *Let $X$ be a inhabited topological space and let $x : * \hookrightarrow X$ any point. Then the relative singular homology $H_n(X,*)$ is isomorphic to the absolute reduced singular homology $\tilde{H}_n(X)$ of $X$*

$$H_n(X,*) \simeq \tilde{H}_n(X).$$

***Proof***. This is the special case of prop. 2.9 for $A$ a point. ∎

## 3. Examples

### Basic examples

**Example 3.1**. The reduced singular homology of the $n$-sphere $S^n$ equals the $S^{n-1}$-relative homology of the $n$-disk with respect to the canonical boundary inclusion $S^{n-1} \hookrightarrow D^n$: for all $n \in \mathbb{N}$

$$\tilde{H}_\bullet(S^n) \simeq H_\bullet(D^n, S^{n-1}).$$

***Proof***. The $n$-sphere is homeomorphic to the $n$-disk with its entire boundary identified with a point:

$$S^n \simeq D^n/S^{n-1}.$$

**Proposition 3.3**. *For $X$ an inhabited topological space, its reduced singular homology, def. 2.2, coincides with its relative singular homology relative to any base point $x : * \to X$:*

$$\tilde{H}_\bullet(X) \simeq H_\bullet(X,*).$$

***Proof***. Consider the sequence of topological subspace inclusions

$$\varnothing \hookrightarrow * \overset{x}{\hookrightarrow} X.$$

By the discussion at *Relative homology - long exact sequences* this induces a long exact sequence of the form

$$\cdots \to \cancel{H_{n+1}(*)} \to H_{n+1}(X) \to H_{n+1}(X,*) \to \cancel{H_n(*)} \to H_n(X) \to H_n(X,*) \to \cdots \to H_1(X) \to H_1(X,*) \to \cancel{H_0(*)} \xrightarrow{H_0(x)} H_0(X) \to H_n(X,*) \to 0.$$

Here in positive degrees we have $H_n(*) \simeq 0$ and therefore exactness gives isomorphisms

$$H_n(X) \xrightarrow{\simeq} H_n(X,*) \quad \forall_{n \geq 1}$$

and hence with prop. 3.1 isomorphisms

$$\tilde{H}_n(X) \xrightarrow{\simeq} H_n(X,*) \quad \forall_{n \geq 1}.$$

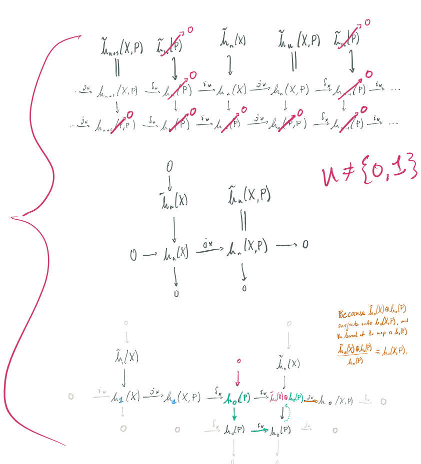
It remains to deal with the case in degree 0. To that end, observe that $H_0(x) : H_0(*) \to H_0(X)$ is a monomorphism: for this notice that we have a commuting diagram

$$\begin{array}{ccc} H_0(*) & \overset{\mathrm{id}}{\to} & H_0(*) \\ \downarrow & \nearrow & \downarrow \simeq \\ H_0(X) & \underset{}{\overset{H_0(\epsilon)}{\longrightarrow}} & \mathbb{Z} \end{array},$$

where $f : X \to *$ is the terminal map. That the outer square commutes means that $H_0(\epsilon) \circ H_0(x) = H_0(\epsilon)$ and hence the composite on the left is an isomorphism. This implies that $H_0(x)$ is an injection.

wrong idea

$n \neq \{0,1\}$



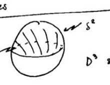
# § IV.6 Homology of Spheres

2017-12-06 #5

Homology of spheres

$S^n \subset \mathbb{R}^{n+1}$

$D^{n+1} \subset \mathbb{R}^{n+1}$



$D^2_+$ $S^2$ $D^3$ solid

$\partial D^{n+1} = S^n$

$D^n_+ \subset S^n$ s.th. $D^n_+$ has $x_n \geq 0$.

Include $S^{n-1} \subset S^n$ so that $S^{n-1} \subset D^n_\pm$.

PROP) Let $n \geq 0$ and there are iso $\tilde{h}_*(S^n) \cong h_*(D^n, S^{n-1}) \cong h_p(S^n, D^n_+)$

$$\cong \begin{cases} G & p = n \\ 0 & p \neq n. \end{cases}$$

Q) what's $h_*(S^0)$? $\bigoplus_1^2 h_*(pt)$?

$h_0(S^0) = G^{\oplus 2}$ ← kernel of the maps to

$\tilde{h}_0(S^0) = G$ ← reduced homology

$$\boxed{\begin{array}{c} 0 \to \tilde{h}(S^0) \longrightarrow G \oplus G \xrightarrow{\text{add } \oplus \text{ vals}} G \longrightarrow 0 \\ \uparrow \text{ what's the kernel?} \\ G\{[+1] - [-1]\} \end{array}}$$

$) \to \cdots$

$h_0(\underset{pt}{D^0_-}, \phi) = G$, $h_0(S^0, D^0_+) = G$.

WANT) les for pairs $(D^n_-, S^{n-1})$

$\cdots \to \tilde{h}_p(S^{n-1}) \to \underset{0}{\tilde{h}_p(D^n_-)} \to h_p(D^n_-, S^{n-1}) \xrightarrow{\delta^*} \tilde{h}_{p-1}(S^{n-1}) \to \underset{0}{\tilde{h}_p(D^n_-)}$

hence $h_p(D^n_-, S^{n-1}) \xrightarrow{\delta_*} \tilde{h}_{p-1}(S^{n-1})$

$$h_*(S^0) = \begin{cases} G^{\oplus 2} & \text{in } * = 0, \\ 0 & \text{else.} \end{cases}$$

Hence $0 \longrightarrow \underset{\therefore G}{\tilde{h}_0(S^0)} \longrightarrow \underset{G^{\oplus 2}}{h_0(S^0)} \longrightarrow \underset{G}{h_0(pt)} \longrightarrow 0$. splits

So $\tilde{h}_*(S^0) = \begin{cases} G & \text{in } * = 0, \\ 0 & \text{else}. \end{cases}$

Also $h_*(D^0_-) \underset{h.a.}{\cong} h_*(pt) = \begin{cases} G & \text{in } * = 0 \\ 0 & \text{else} \end{cases}$

Lastly $h_*(S^0, D^0_+) = \begin{cases} G & \text{in } * = 0 \\ 0 & \text{else} \end{cases}$

Consider the reduced l.e.s. for pairs $(D^n_-, S^{n-1})$

$\cdots \to \tilde{h}_p(S^{n-1}) \longrightarrow \cancel{\tilde{h}_p(D^n_-)} \longrightarrow h_p(D^n_-, S^{n-1}) \xrightarrow{\delta_*} \tilde{h}_{p-1}(S^{n-1}) \to \cancel{\tilde{h}_{p-1}(D^n_-)} \to \cdots$ for all $p$. By dimension axiom & defn of reduced homology...

Thus $h_p(D^n_-, S^{n-1}) \cong \tilde{h}_{p-1}(S^{n-1})$.

Next, consider the reduced l.e.s. for $(S^n, D^n_+)$, we see $h_p(S^n, D^n_+) \cong \tilde{h}_{p-1}(S^n)$.

$\cdots \to \tilde{h}_{p+1}(S^n) \to \cancel{\tilde{h}_{p+1}(D^n_+)} \longrightarrow h_{p+1}(S^n, D^n_+) \xrightarrow{\delta_*} \tilde{h}_p(S^n) \to \cancel{\tilde{h}_p(D^n_+)} \to h_p(S^n, D^n_+) \xrightarrow{\delta_*} \cdots$

Lastly, we argue for $(S^n, D^n_+, N)$ where $N$ denotes the north pole... $(S^n \setminus N, D^n_+ \setminus N) \simeq (D^n_-, S^{n-1})$

$$h_*(S^n, D^n_+) \xrightarrow[e^*]{\cong} h_*(S^n \setminus N, D^n_+ \setminus N) \xrightarrow[h.a.]{\cong} h_*(D^n_-, S^{n-1})$$

We conclude

$\tilde{h}_{p-1}(S^{n-1}) \xrightarrow{\sim} h_p(D^n_-, S^{n-1})$

$\tilde{h}_{p-1}(S^n) \longleftarrow h_p(S^n, D^n_+)$

Now, because $\tilde{h}_*(S^0) = \begin{cases} G & * = 0 \\ 0 & \text{else} \end{cases}$

we have that

# Exercises

**Corollary 1.6.2.** *$S^{n-1}$ is not a retract of $D^n$*

**Corollary 1.6.3** (Brouwer's Fixed Point Theorem)**.** *Any map $f \colon D^n \to D^n$ has a fixed point.*

*Proof.* Let $r(x) \in S^{n-1}$ be the point where the ray from $f(x)$ to $x$ passes through $S^{n-1}$. This is well defined since $f$ has no fixed point and one can check it's continuous (see alternate approach in Chapter 2, Corollary 11.12 or Bredon). The map $r$ is a retract of $D^n$ onto $S^{n-1}$, which is a contradiction. □

(a)

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \longrightarrow 0$$

a short exact sequence of chain complexes, prove the exactness of

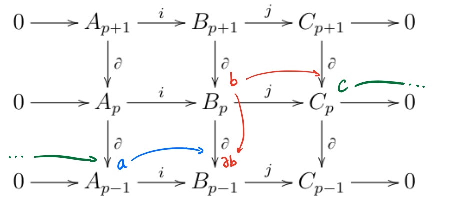
$$\ldots \xrightarrow{\delta_*} H_p(A_*) \xrightarrow{i_*} H_p(B_*) \xrightarrow{j_*} H_p(C_*) \xrightarrow{\delta_*} H_{p-1}(A_*) \xrightarrow{i_*} \ldots$$

at $H_p(B_*)$ and at $H_p(A_*)$.

"Order two" WTS $j_* \circ i_* \equiv 0$, also $i_* \circ \delta_* \equiv 0$. Levelwise

Pf. Because, by assumption, $0 \to A_* \xrightarrow{i_*} B_* \xrightarrow{\partial_*} C_* \to 0$ is exact, $j_* \circ i_* \equiv 0$.

Now consider $i_* \circ \delta_*$.



Let $c + B_p(C) \in H_p(C)$.

$\delta_* : c + B_p(C) \longmapsto a + B_{p-1}(A)$ s.th. $a \in i^{-1}(\partial^B(j^{-1}(c)))$.

Then $i_* : a + B_{p-1}(A) \longmapsto 0 + B_{p-1}(B)$, b/c $i(a) = \partial b$ for some $b \in j^{-1}(c)$. □

"Each group maps onto the next kernel"

WTS For $a + B_{p-1}(A) \in \ker i_*$ there exists $c + B_p(C)$ in $H_p(C)$ sth $\delta_*(c + B_p(C)) = a + B_{p-1}(A)$. Likewise for $H_p(B)$.

Pf. Let $a + B_{p-1}(A) \in \ker i_*$. Then $i(a) \in B_p(B)$ for our rep $a$. There exists $b \in Z_p(B)$ sth $\partial b = i(a)$. Let $c = j(b)$. Is $c$ a cycle? $\partial c = \partial j(b) = j\partial(b) = (j \circ i)(a) = 0$ by exactness. So $c \in Z_p(C)$.

Then $c + B_p(C) \xmapsto{\delta_*} a + B_{p-1}(A)$, by construction.

Second, for the exactness at $H_*(A_*)$, suppose that $i_*[\![a]\!] = 0$. Then $i(a) = \partial b$ for some $b \in B_*$. Then put $c = j(b)$. We have $\partial c = \partial j(b) = j(\partial b) = j(i(a)) = 0$. Thus $c$ represents a homology class, and by construction of $\partial_*$, $\partial_*[\![c]\!] = [\![a]\!]$.

Third, for the exactness at $H_*(C_*)$, suppose that $\partial_*[\![c]\!] = 0$. Then for an element $b \in B_*$ for which $j(b) = c$, there is an $a \in A_*$ such that $i(a) = \partial b$, by the construction of $\partial_*$, and $a$ must be a boundary since it represents $\partial_*[\![c]\!] = 0$. Thus let $a = \partial a'$. Then $\partial i(a') = i(\partial a') = i(a) = \partial b$. Accordingly, $\partial(b - i(a')) = 0$ and $j(b - i(a')) = c - 0 = c$. Therefore $j_*[\![b - i(a')]\!] = [\![c]\!]$ as required. □

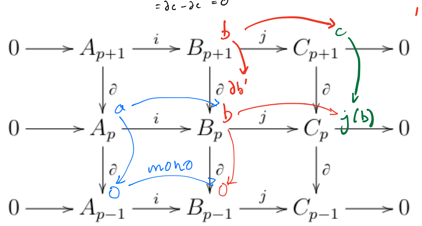
First, we show the exactness at $H_*(B_*)$. Suppose that $j_*[\![b]\!] = 0$. This means that $j(b) = \partial c$ for some $c \in C_*$. Let $b' \in B_*$ be such that $j(b') = c$. Then

$$j(b - \partial b') = j(b) - j(\partial b') = \partial c - \partial(j(b')) = \partial c - \partial c = 0.$$

||

This shows that we could have taken the representative $b$ of its homology class to be such that $j(b) = 0$. For this choice, then, $b = i(a)$ for some $a \in A_*$ (and $\partial a = 0$ since it maps, by the monomorphism $i$, into $\partial b = 0$). Thus $[\![b]\!] = i_*[\![a]\!]$ as claimed.

$$\begin{aligned} j(b - \partial b') &= j(b) - j(\partial b') \\ &= \partial c - \partial(j(b')) \\ &= \partial c - \partial c = 0 \end{aligned}$$

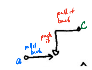


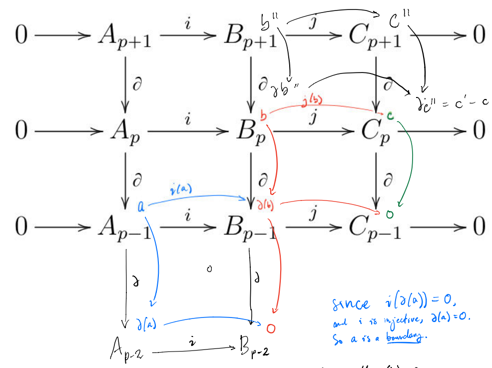
so $\partial_*[\![a]\!] = [\![b]\!]$, because $j(b) = 0$ implies $b \in i(A_p)$ and $\partial b = 0$ implies $\partial a = 0$.

Recall if $0 \to A_* \to B_* \to C_* \to 0$ is an exact sequence of chain complexes there's a l.e.s. $\to H_p(A_*) \xrightarrow{i} H_p(B_*) \xrightarrow{j} H_p(C_*) \xrightarrow{\delta_*}$

$\delta_*([\![c]\!]) = [\![i^{-1}\partial^B j^{-1}(c)]\!]$

$= [\![a]\!]$ as below.



since $i(\partial(a)) = 0$, and $i$ is injective, $\partial(a) = 0$. So $a$ is a boundary.

EX| To check $\delta_*$ doesn't depend on choice of lift $b$ sth $j(b) = c$.

Demo. Let $b' \in B_p$ map $j(b') = c$. Again, $\partial(b')$ is in the kernel of $j : B_{p-1} \to C_{p-1}$, so also in the (isom) image of $A_{p-1}$. We need to show that $\underbrace{i^{-1}(\partial(b'))}_{a'}$ and $\underbrace{i^{-1}(\partial(b))}_{a}$ differ by a boundary.

We must show that this does not depend on the choices of $b$ and of $c$ within its homology class. First suppose $c = j(b')$, so that $j(b - b') = 0$. Then $b - b' = i(a_0)$ for some $a_0 \in A_p$. Thus $\partial b - \partial b' = \partial(i(a_0)) = i(\partial a_0)$. But the left-hand side of this equation is $\partial b - \partial b' = i(a) - i(a') = i(a - a')$. It follows that $a - a' = \partial a_0$ and so $a \sim a'$ as desired.

$b - b' = i(a_0)$, so $\partial b - \partial b' = \partial(i(a_0)) = i(\partial(a_0))$.

Simultaneously, $\partial b = i(a)$, $\partial b' = i(a')$,

and $i$ is a monomorphism, so $i(\partial a_0) = \partial b - \partial b' = i(a) - i(a') = i(a - a')$,

Thus $\partial a_0 = a - a'$. So $a \sim a'$. □

EX| $\delta^*([\![c]\!]) = \delta^*([\![c']\!])$ regardless of choice of cycle $c \sim c'$.

So let $c = c' + \partial c''$.

It also doesn't depend on the choice of representative cycle $c$. If $c' = c + \partial(c'')$ so that $[\![c']\!] = [\![c]\!]$, then for $b''$ such that $j(b'') = c''$,

$$j(b + \partial(b'')) = c + \partial(c''),$$

so that the next step gives

$$\partial(b + \partial(b'')) = \partial(b) = i(a).$$

So, $\delta_*$ is well defined.

$j(b + \partial b'') = c + \partial c'' = c + c' - c = c'$

$\partial(b + \partial b'') = \partial b = i(a)$

calculate $j(b') = j(b) + j(\partial b'') = c + \partial c'' = c'$. But $\partial b' = \partial b + \partial\partial b'' = \partial b$ and so $\partial b$ and $\partial b'$, being equal, pull back to the same thing under $i^{-1}$.

EX| We show $\delta_*$ is a homomorphism from $H_p(C) \to H_{p+1}(A)$.

So let $c + B_p(C), c' + B_p(C) \in H_p(C)$. Consider $\delta_*(c + B_p(C)) + \delta_*(c' + B_p(C))$

$= i^{-1}(\partial b) + B_{p+1}(A) + i^{-1}(\partial b') + B_{p+1}(A) = i^{-1}(\partial b) + i^{-1}(\partial b') + B_{p+1}(A)$

$= i^{-1}(\partial(b + b')) + B_{p+1}(A)$, as $i^{-1}$ is an isom onto $\ker j$ and $\partial b, \partial b' \in \ker j$, and also $\partial$ is a gp hom, and this is just $\delta_*(c + c' + B_p(C))$. □

Bredon gives exactness rather by first showing each map is "order two" then arguing that each group maps onto the next kernel

**Exercise 1.4.7.** Check that $\delta_*$ is a homomorphism.

Finally, we need to show exactness. We prove exactness at $H_p(C_*)$. That is, we prove that

$$H_p(B_*) \xrightarrow{j_*} H_p(C_*) \xrightarrow{\delta_*} H_{p-1}(A_*)$$

is exact.

Suppose that $[\![c]\!] \in \ker(\delta_*)$. Then $a = \partial(a')$, so that

$$\partial(b) = i(a) = i(\partial(a')) = \partial(i(a')).$$

So $b' = b - i(a')$ is a cycle, that is, $b' \in Z_p(B) = \ker(\partial)$. Therefore,

$$c = j(b) = j(i(a') + b') \overset{?}{=} j(b')$$

assuming $ji = 0$? DUH

and $j_*([\![b']\!]) = [\![c]\!]$.

Suppose that $[\![c]\!] = j_*([\![b']\!])$ for some $[\![b']\!] \in H_p(B)$. Then we can choose

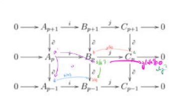
$$b = b' + \partial(b'')$$

for some $b'' \in B_{p+1}$ and $b' \in Z_p(B)$. So,

$$\partial(b) = \partial(b' + \partial(b'')) = 0$$

so $i(0) = \partial(b)$. Hence,

$$\delta_*([\![c]\!]) = 0$$



Third, for the exactness at $H_*(C_*)$, suppose that $\partial_*[\![c]\!] = 0$. Then for an element $b \in B_*$ for which $j(b) = c$, there is an $a \in A_*$ such that $i(a) = \partial b$, by the construction of $\partial_*$, and $a$ must be a boundary since it represents $\partial_*[\![c]\!] = 0$. Thus let $a = \partial a'$. Then $\partial i(a') = i(\partial a') = i(a) = \partial b$. Accordingly, $\partial(b - i(a')) = 0$ and $j(b - i(a')) = c - 0 = c$. Therefore $j_*[\![b - i(a')]\!] = [\![c]\!]$ as required. □

Finally, we must show that the indicated "long" sequence is exact. First, show it is of "order two" (i.e., the composition of adjoining homomorphisms is zero). There are three cases. First, $j_* i_* = (j \circ i)_* = 0_* = 0$.

→ not just that $A \hookrightarrow B$ and $B \twoheadrightarrow C$, but that $A$ is collapsed.

the "connecting homomorphism"

(b) Prove that the morphism $\delta_*$ is natural. That is, if

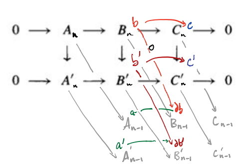
$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{j} & C_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A'_* & \xrightarrow{i} & B'_* & \xrightarrow{j} & C'_* & \longrightarrow & 0
\end{array}$$

is a commutative diagram with exact rows and $\delta'_*: H_*(C'_*) \to H_{*-1}(A'_*)$ the connecting homomorphism for the bottom row, then TFDC

$$\begin{array}{ccccc}
 & [\![c]\!] & & [\![a]\!] & \\
\longrightarrow & H_n(C) & \xrightarrow{\partial_*} & H_{n-1}(A) & \longrightarrow \\
 & \downarrow & & \downarrow & \\
 & [\![c']\!] & & [\![a']\!] & \\
\longrightarrow & H_n(C') & \xrightarrow{\partial_*} & H_{n-1}(A') & \longrightarrow
\end{array}$$

Proof. We assume that the chain complexes are between $R$-modules for a comm. unital ring

Let $c$ be a cycle in $Z_n(C)$ and $[\![c]\!]$ in $H_n(C)$ its homology class. Let $c'$ in $H_n(C')$ be the image of $c$ in $H_n(C')$ by the morphism in $S$. Then $[\![c']\!]$ is represented by $c'$ the image of $c$ in $Z_n(C')$.



Say that $b$ in $B_n$ lifts $c$, then by commutativity of the map between short exact sequences, $b'$ lifts $c'$.

Push $b$ and $b'$ to find $a$ in $A_{n-1}$ and $a'$ in $A'_{n-1}$ s.th. $i(a) = \partial b$, $i'(a) = \partial b'$.

The chain complex map $B$ to $B'$ forces $\partial b \longmapsto \partial b'$. The map of short exact sequences forces $a \longmapsto a'$ (say $a \mapsto a''$ for $\times$, then $a \mapsto a'' \mapsto b'' \neq \partial' \leftarrow a'$.)

We conclude $\delta_*([\![c]\!])$ is represented by $a$, $\delta_*([\![c']\!])$ is represented by $a'$, and $a \longmapsto a'$ is induced by the morphism in $S$.

We shall now explain what we mean by the naturality of $\partial$. There is a category $\mathcal{S}$ whose objects are short exact sequences of chain complexes (say, in an abelian category $\mathcal{C}$). Commutative diagrams [Wei94, ch. 1]

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

$(*)$ the "morphism in $S$"

give the morphisms in $\mathcal{S}$ (from the top row to the bottom row). Similarly, there is a category $\mathcal{L}$ of long exact sequences in $\mathcal{C}$.

**Proposition 1.3.4** *The long exact sequence is a functor from $\mathcal{S}$ to $\mathcal{L}$. That is, for every short exact sequence there is a long exact sequence, and for every map $(*)$ of short exact sequences there is a commutative ladder diagram*

we call the functor from $S$ to $\mathcal{L}$ the "evident commutative diagram"

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial} & H_n(A) & \longrightarrow & H_n(B) & \longrightarrow & H_n(C) & \xrightarrow{\partial} & H_{n-1}(A) \longrightarrow & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \xrightarrow{\partial} & H_n(A') & \longrightarrow & H_n(B') & \longrightarrow & H_n(C') & \xrightarrow{\partial} & H_{n-1}(A') \longrightarrow & \cdots .
\end{array}$$

*Proof* All we have to do is establish the ladder diagram. Since each $H_n$ is a functor, the left two squares commute. Using the Embedding Theorem 1.6.1, we may assume $\mathcal{C} = \mathbf{mod}\text{–}R$ in order to prove that the right square commutes. Given $z \in H_n(C)$, represented by $c \in C_n$, its image $z' \in H_n(C')$ is represented by the image of $c$. If $b \in B_n$ lifts $c$, its image in $B'_n$ lifts $c'$. Therefore by 1.3.3 $\partial(z') \in H_{n-1}(A')$ is represented by the image of $db$, that is, by the image of a representative of $\partial(z)$, so $\partial(z')$ is the image of $\partial(z)$. ◇

- There is a functor assigning to a pair of spaces $(X, A)$ the associated long exact sequence of homology groups. Morphisms in the domain category are maps of pairs, and in the target category morphisms are maps between exact sequences forming commutative diagrams. This functor is the composition of two functors, the first assigning to $(X, A)$ a short exact sequence of chain complexes, the second assigning to such a short exact sequence the associated long exact sequence of homology groups. Morphisms in the intermediate category are the evident commutative diagrams.

Another sort of process we have encountered is the transformation of one functor into another, for example:

- Boundary maps $H_n(X, A) \to H_{n-1}(A)$ in singular homology, or indeed in any homology theory.

[Hat02]



- Change-of-coefficient homomorphisms $H_n(X; G_1) \to H_n(X; G_2)$ induced by a homomorphism $G_1 \to G_2$, as in the proof of Lemma 2.49.

In general, if one has two functors $F, G : \mathcal{C} \to \mathcal{D}$ then a **natural transformation** $T$ from $F$ to $G$ assigns a morphism $T_X : F(X) \to G(X)$ to each object $X \in \mathcal{C}$, in such a way that for each morphism $f : X \to Y$ in $\mathcal{C}$ the square at the right commutes. The case that $F$ and $G$ are contravariant rather than covariant is similar.

$$\begin{array}{ccc}
F(X) & \xrightarrow{F(f)} & F(Y) \\
\downarrow{\scriptstyle T_X} & & \downarrow{\scriptstyle T_Y} \\
G(X) & \xrightarrow{G(f)} & G(Y)
\end{array}$$

**Exercise 1.3.3** (5–Lemma) In any commutative diagram

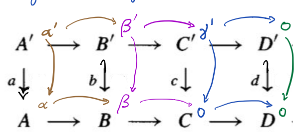
[Wei94, ch 1]

$$
\begin{tikzcd}
A' \arrow[r] \arrow[d, "a", "\cong"'] & B' \arrow[r] \arrow[d, "b", "\cong"'] & C' \arrow[r] \arrow[d, "c"] & D' \arrow[r] \arrow[d, "d", "\cong"'] & E' \arrow[d, "e", "\cong"'] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E
\end{tikzcd}
$$

with exact rows in any abelian category, show that if $a$, $b$, $d$, and $e$ are isomorphisms, then $c$ is also an isomorphism. More precisely, show that if $b$ and $d$ are monic and $a$ is an epi, then $c$ is monic. Dually, show that if $b$ and $d$ are epis and $e$ is monic, then $c$ is an epi.

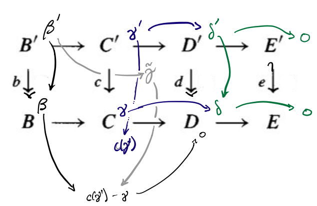
WTS. (i) $a$ epi, $b$ and $d$ monic $\implies$ $c$ monic
(ii) $e$ monic, $b$ and $d$ epi $\implies$ $c$ epi.

Proof. (i) Let $\gamma' \in C'$ s.th. $c(\gamma') = 0$.



Say $\delta'$ is the image of $\gamma'$ in $D'$. Since the image of $0$ in $D$ is $0$, and $d$ is monic, $\delta' = 0$.
By exactness at $C'$, $\gamma'$ lifts to $\beta'$ in $B'$. Pushing $\beta'$ down to $\beta$, we see $\beta \mapsto 0$.
So $\beta$ lifts to $\alpha$ in $A$, and $\alpha$ lifts to (perhaps a non-unique) $\alpha'$ in $A'$.
Because $b$ is monic, and $\beta' \overset{b}{\longmapsto} \beta$ commutativity implies $\alpha' \longmapsto \beta'$.
Since $\gamma'$ is in the image of $A'$, and exactness of the top row implies the composition of two successive maps is $0$, we see $\gamma' = 0$.
So $c$ is monic.

(ii) Say $\gamma \in C$.



Let $\delta$ be the image of $\gamma$. Because $d$ is epi, lift $\delta$ to $\delta'$ in $D'$.
By exactness at $D$, $\delta \longmapsto 0$ in $E$. With $e$ monic and the last square commuting, we must have $\delta' \longmapsto 0$ in $E'$. By exactness at $D'$, there's a lift of $\delta'$ to, call it $\gamma'$, in $C'$. Now $c(\gamma') \longmapsto \delta$ and $\gamma \longmapsto \delta$.
So the image in $D$ of the difference of $c(\gamma')$ and $\gamma$ is $0$. We lift $c(\gamma') - \gamma$ to $\beta$ in $B$. Because $b$ is epi, lift $\beta$ to $\beta'$. Say $\beta' \longmapsto \tilde{\gamma}$.
Then $c(\tilde{\gamma}) = c(\gamma') - \gamma$ so $\gamma = c(\gamma' - \tilde{\gamma})$. $\square$

CORO I

(c) Consider a commutative diagram with exact rows

$$
\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1", "\cong"'] & A_2 \arrow[r] \arrow[d, "f_2", "\cong"'] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4", "\cong"'] & A_5 \arrow[d, "f_5", "\cong"'] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5.
\end{tikzcd}
$$

Prove that if $f_1, f_2, f_4, f_5$ are isomorphisms, so is $f_3$.

See also

12. (The Five Lemma). Let

[@Hun74, number IV.1.12]

$$
\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\alpha_1"] & A_2 \arrow[r] \arrow[d, "\alpha_2"] & A_3 \arrow[r] \arrow[d, "\alpha_3"] & A_4 \arrow[r] \arrow[d, "\alpha_4"] & A_5 \arrow[d, "\alpha_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}
$$

be a commutative diagram of $R$-modules and $R$-module homomorphisms, with exact rows. Prove that:
(a) $\alpha_1$ an epimorphism and $\alpha_2, \alpha_4$ monomorphisms $\Rightarrow \alpha_3$ is a monomorphism;
(b) $\alpha_5$ a monomorphism and $\alpha_2, \alpha_4$ epimorphisms $\Rightarrow \alpha_3$ is an epimorphism.

Tensoring an exact sequence of R-modules. [DF04, ch 10.4]

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\psi"] \arrow[dr] & M \arrow[r, "\varphi"] \arrow[d, "\cong"] & N \arrow[r] \arrow[l, dashed, bend right] & 0 \\ & & L \oplus N \arrow[ur] & & \end{tikzcd}$$

splits, then

$$\begin{tikzcd} 0 \arrow[r] & D \otimes_R L \arrow[r] \arrow[dr] & D \otimes_R (L \oplus N) \arrow[r] \arrow[d, "\cong"] & D \otimes_R N \arrow[r] & 0 \\ & & (D \otimes_R L) \oplus (D \otimes_R N) \arrow[ur] & & \end{tikzcd}$$

is also split exact!

We now consider the behavior of extensions $0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0$ of $R$-modules with respect to tensor products.

Suppose that $D$ is a *right* $R$-module. For any homomorphism $f : X \to Y$ of left $R$-modules we obtain a homomorphism $1 \otimes f : D \otimes_R X \to D \otimes_R Y$ of abelian groups (Theorem 13). If in addition $D$ is an $(S, R)$-bimodule (for example, when $S = R$ is commutative and $D$ is given the standard $(R, R)$-bimodule structure as in Section 4), then $1 \otimes f$ is a homomorphism of left $S$-modules. Put another way,

$$D \otimes_R \_\_ : X \longrightarrow D \otimes_R X$$

is a *covariant functor* from the category of left $R$-modules to the category of abelian groups (respectively, to the category of left $S$-modules when $D$ is an $(S, R)$-bimodule), cf. Appendix II. In a similar way, if $D$ is a left $R$-module then $\_\_ \otimes_R D$ is a covariant functor from the category of right $R$-modules to the category of abelian groups (respectively, to the category of right $S$-modules when $D$ is an $(R, S)$-bimodule). Note that, unlike Hom, the tensor product is covariant in both variables, and we shall therefore concentrate on $D \otimes_R \_\_$, leaving as an exercise the minor alterations necessary for $\_\_ \otimes_R D$.

We have already seen examples where the map $1 \otimes \psi : D \otimes_R L \to D \otimes_R M$ induced by an injective map $\psi : L \hookrightarrow M$ is no longer injective (for example the injection $\mathbb{Z} \hookrightarrow \mathbb{Q}$ of $\mathbb{Z}$-modules induces the zero map from $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$). On the other hand, suppose that $\varphi : M \to N$ is a surjective $R$-module homomorphism. The tensor product $D \otimes_R N$ is generated as an abelian group by the simple tensors $d \otimes n$ for $d \in D$ and $n \in N$. The surjectivity of $\varphi$ implies that $n = \varphi(m)$ for some $m \in M$, and then $\boxed{1 \otimes \varphi(d \otimes m)} = d \otimes \varphi(m) = d \otimes n$ shows that $1 \otimes \varphi$ is a surjective homomorphism of abelian groups from $D \otimes_R M$ to $D \otimes_R N$. This proves most of the following theorem.

**Theorem 39.** Suppose that $D$ is a right $R$-module and that $L$, $M$ and $N$ are left $R$-modules. If

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0 \quad \text{is exact,}$$

then the associated sequence of abelian groups

$$D \otimes_R L \xrightarrow{1 \otimes \psi} D \otimes_R M \xrightarrow{1 \otimes \varphi} D \otimes_R N \longrightarrow 0 \quad \text{is exact.} \tag{10.13}$$

If $D$ is an $(S, R)$-bimodule then (13) is an exact sequence of left $S$-modules. In particular, if $S = R$ is a commutative ring, then (13) is an exact sequence of $R$-modules with respect to the standard $R$-module structures. The map $1 \otimes \varphi$ is not in general injective, i.e., the sequence (13) cannot in general be extended to a short exact sequence.

The sequence (13) is exact for *all* right $R$-modules $D$ if and only if

$$L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0 \quad \text{is exact.}$$

$D \otimes_R L \xrightarrow{1 \otimes \psi} D \otimes_R M \xrightarrow{1 \otimes \varphi} D \otimes_R N$

An elem of the image of $1 \otimes \psi$ is of the form $\sum d_i \otimes \psi(l_i)$, and we map such through $1 \otimes \varphi$.

*Proof:* For the first statement it remains to prove the exactness of (13) at $D \otimes_R M$. Since $\varphi \circ \psi = 0$, we have

$$(1 \otimes \varphi)\left(\sum d_i \otimes \psi(l_i)\right) = \sum d_i \otimes (\varphi \circ \psi(l_i)) = 0 ✓$$

and it follows that image$(1 \otimes \psi) \subseteq \ker(1 \otimes \varphi)$. In particular, there is a natural projection $\pi : (D \otimes_R M)/\text{ image}(1 \otimes \psi) \to (D \otimes_R M)/\ker(1 \otimes \varphi) = D \otimes_R N$. The composite of the two projection homomorphisms

$$D \otimes_R M \to (D \otimes_R M)/\text{ image}(1 \otimes \psi) \xrightarrow{\pi} D \otimes_R N$$

So we take out more of the quotient...

$$\begin{tikzcd} D \otimes_R M \arrow[r] \arrow[dr, "1 \otimes \varphi"'] & D \otimes_R M/\mathrm{im}(1 \otimes \psi) \arrow[d, dashed, "\pi"] \\ & D \otimes_R M/\ker(1 \otimes \varphi) = D \otimes_R N \end{tikzcd}$$

is the quotient of $D \otimes_R M$ by $\ker(1 \otimes \varphi)$, so is just the map $1 \otimes \varphi$. We shall show that $\pi$ is an isomorphism, which will show that the kernel of $1 \otimes \varphi$ is just the kernel of the first projection above, i.e., image$(1 \otimes \psi)$, giving the exactness of (13) at $D \otimes_R M$. To see that $\pi$ is an isomorphism we define an inverse map. First define $\pi' : D \times N \to (D \otimes_R M)/\text{ image}(1 \otimes \psi)$ by $\pi'((d, n)) = d \otimes m$ for any $m \in M$ with $\varphi(m) = n$. Note that this is well defined: any other element $m' \in M$ mapping to $n$ differs from $m$ by an element in $\ker \varphi = \text{image } \psi$, i.e., $\boxed{m' = m + \psi(l)}$ for some $l \in L$, and $d \otimes \psi(l) \in \text{image}(1 \otimes \psi)$. It is easy to check that $\pi'$ is a balanced map, so induces a homomorphism $\tilde{\pi} : D \otimes N \to (D \otimes_R M)/\text{ image}(1 \otimes \psi)$ with $\tilde{\pi}(d \otimes n) = d \otimes m$. Then $\tilde{\pi} \circ \pi(d \otimes m) = \tilde{\pi}(d \otimes \varphi(m)) = d \otimes m$ shows that $\tilde{\pi} \circ \pi = 1$. Similarly, $\pi \circ \tilde{\pi} = 1$, so that $\pi$ and $\tilde{\pi}$ are inverse isomorphisms, completing the proof that (13) is exact. Note also that the injectivity of $\psi$ was not required for the proof.

$\pi' : D \times N \longrightarrow D \otimes_R M / \mathrm{im}(1 \otimes \psi)$

$(d, n) \longmapsto \overline{d \otimes m}$ for any $m \in M$ s.th. $\varphi(m) = n$

Well def'd? Choose $m'$ s.th. $\varphi(m') = n$. Then $\varphi(m' - m) = 0$.

So $\exists l \in L$ s.th. $m' = \psi(l) + m$. $d \otimes m + \mathrm{im}(1 \otimes \psi) \overset{?}{=} d \otimes (\psi(l) + m) + \mathrm{im}(1 \otimes \psi)$

Yes, because $d \otimes (\psi(l) + m) = \underbrace{d \otimes \psi(l)}_{\text{in image}(1 \otimes \psi)} + d \otimes m$. ✓

Balanced?

**Definition.** Let $M$ be a right $R$-module, let $N$ be a left $R$-module and let $L$ be an abelian group (written additively). A map $\varphi : M \times N \to L$ is called *R-balanced* or *middle linear with respect to R* if $D \times N \longrightarrow D \otimes_R N$

$$\varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n)$$
$$\varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2)$$
$$\varphi(m, rn) = \varphi(mr, n)$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, and $r \in R$.

Let $d, d_1, d_2 \in D$, $n, n_1, n_2 \in N$, and $r \in R$.

$\pi(d_1 + d_2, n) = (d_1 + d_2) \otimes m + \mathrm{im}(1 \otimes \psi)$

$= (d_1 \otimes m + d_2 \otimes m) + \mathrm{im}(1 \otimes \psi)$

$= \pi(d_1, n) + \pi(d_2, n)$

$\pi(d, n_1 + n_2) = d \otimes (m_1 + m_2) + \mathrm{im}(1 \otimes \psi)$

$= (d \otimes m_1 + d \otimes m_2) + \mathrm{im}(1 \otimes \psi)$

$= \pi(d, n_1) + \pi(d, n_2)$

$\pi(dr, n) = dr \otimes m + \mathrm{im}(1 \otimes \psi)$

$= d \otimes rm + \mathrm{im}(1 \otimes \psi)$

$= \pi(d, rn)$ yes.

Finally, suppose (13) is exact for every right $R$-module $D$. In general, $R \otimes_R X \cong X$ for any left $R$-module $X$ (Example 1 following Corollary 9). Taking $D = R$ the exactness of the sequence $L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ follows.

**Theorem 17.** *(Tensor Products of Direct Sums)* Let $M, M'$ be right $R$-modules and let $N, N'$ be left $R$-modules. Then there are unique group isomorphisms

$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$$
$$M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N')$$

such that $(m, m') \otimes n \mapsto (m \otimes n, m' \otimes n)$ and $m \otimes (n, n') \mapsto (m \otimes n, m \otimes n')$ respectively. If $M, M'$ are also $(S, R)$-bimodules, then these are isomorphisms of left $S$-modules. In particular, if $R$ is commutative, these are isomorphisms of $R$-modules.

*Proof:* The map $(M \oplus M') \times N \to (M \otimes_R N) \oplus (M' \otimes_R N)$ defined by $((m, m'), n) \mapsto (m \otimes n, m' \otimes n)$ is well defined since $m$ and $m'$ in $M \oplus M'$ are uniquely defined in the direct sum. The map is clearly $R$-balanced, so induces a homomorphism $f$ from $(M \oplus M') \otimes N$ to $(M \otimes_R N) \oplus (M' \otimes_R N)$ with

$$f((m, m') \otimes n) = (m \otimes n, m' \otimes n).$$

In the other direction, the $R$-balanced maps $M \times N \to (M \oplus M') \otimes_R N$ and $M' \times N \to (M \oplus M') \otimes_R N$ given by $(m, n) \mapsto (m, 0) \otimes n$ and $(m', n) \mapsto (0, m') \otimes n$, respectively, define homomorphisms from $M \otimes_R N$ and $M' \otimes_R N$ to $(M \oplus M') \otimes_R N$. These in turn give a homomorphism $g$ from the direct sum $(M \otimes_R N) \oplus (M' \otimes_R N)$ to $(M \oplus M') \otimes_R N$ with

$$g((m \otimes n_1, m' \otimes n_2)) = (m, 0) \otimes n_1 + (0, m') \otimes n_2.$$

An easy check shows that $f$ and $g$ are inverse homomorphisms and are $S$-module isomorphisms when $M$ and $M'$ are $(S, R)$-bimodules. This completes the proof.

The previous theorem clearly extends by induction to any finite direct sum of $R$-modules. The corresponding result is also true for arbitrary direct sums. For example

$$M \otimes (\oplus_{i \in I} N_i) \cong \oplus_{i \in I} (M \otimes N_i),$$

where $I$ is any index set (cf. the exercises). This result is referred to by saying that *tensor products commute with direct sums.*

(2) Let $\mathrm{Tor}(\mathbb{Z}/p, H_{n-1}(X)) = \ker(H_{n-1}(X) \xrightarrow{p} H_{n-1}(X))$. Prove that there is a split exact sequence

$$0 \to H_n(X;\mathbb{Z}) \otimes \mathbb{Z}/p \to H_n(X;\mathbb{Z}/p) \to \mathrm{Tor}(\mathbb{Z}/p, H_{n-1}(X)) \to 0.$$

$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/(p) \longrightarrow 0$ exact... $\Delta_*$ is flat.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \otimes \Delta_n(X) \arrow[r, "p \otimes id_\Delta"] \arrow[d, "id_{\mathbb{Z}} \otimes \partial_n"] & \mathbb{Z} \otimes \Delta_n(X) \arrow[r, "\pi \otimes id_\Delta"] \arrow[d, "id_{\mathbb{Z}} \otimes \partial_n"] & \mathbb{Z}_p \otimes \Delta_n(X) \arrow[r] \arrow[d, "id_{\mathbb{Z}_p} \otimes \partial_n"] & 0 \\
0 \arrow[r] & \mathbb{Z} \otimes \Delta_{n-1}(X) \arrow[r, "p \otimes id_\Delta"] & \mathbb{Z} \otimes \Delta_{n-1}(X) \arrow[r, "\pi \otimes id_\Delta"] & \mathbb{Z}_p \otimes \Delta_{n-1}(X) \arrow[r] & 0
\end{tikzcd}$$

carrying this ses to a les $\cdots \xrightarrow{\partial_*} H_p(X;G) \xrightarrow{i_*} H_p(X;G') \xrightarrow{j_*} H_p(X;G'') \xrightarrow{\partial_*} H_{p-1}(X;G) \xrightarrow{i_*}$

The *homology of $C_*$ with coefficients in $G$*, denoted $H_*(C_*;G)$, is the homology of the chain complex $G \otimes C_*$, whose $p$th group is $G \otimes C_p$ and boundary $\partial_p^{C_*,G} = \mathrm{id}_G \otimes \partial_p$.

$$\begin{tikzcd}[column sep=small]
 & & \operatorname{coker}\{H_n(X;\mathbb{Z}) \xrightarrow{(p\otimes id)_*} H_n(X;\mathbb{Z})\} \arrow[dr] & & C \arrow[dr] & & \\
\cdots \arrow[r, "\delta_*"] & H_n(X;\mathbb{Z}) \arrow[r, "(p \otimes id)_*"] \arrow[dr] & H_n(X;\mathbb{Z}) \arrow[r, "(\pi \otimes id)_*"] \arrow[u] & H_n(X;\mathbb{Z}_p) \arrow[r, "\delta_*"] \arrow[dr] & H_{n-1}(X;\mathbb{Z}) \arrow[r, "(p\otimes id)_*"] \arrow[u] & H_{n-1}(X;\mathbb{Z}) \arrow[r] \arrow[dr] & \cdots \\
 & & C \arrow[u] & & \ker\{H_{n-1}(X;\mathbb{Z}) \xrightarrow{(p\otimes id)_*} H_{n-1}(X;\mathbb{Z})\} \arrow[u] & & C
\end{tikzcd}$$

splits?

splitting? Consider the ses.

$$0 \longrightarrow Z_n \xrightarrow{f} \Delta_n(X) \longrightarrow B_{n-1} \longrightarrow 0$$

which splits. $g \circ f = id_{Z_n}$ (they're all free abelian)

Extend $g$ to the quotient $H_n(X)$.

$$\begin{tikzcd}
\Delta_n(X) \arrow[d, "g"'] \arrow[dr, dashed, "g'"] & \\
Z_n \arrow[r, "\text{quotient}"'] & H_n(X)
\end{tikzcd}$$

Consider $\cdots \xrightarrow{0} H_{n+1}(X) \xrightarrow{0} H_n(X) \xrightarrow{0} H_{n-1}(X) \xrightarrow{0} \cdots$ as the chain complex $H_*(X)$

We induce a chain map $G$ from $\Delta_*(X)$ to $H_*(X)$ given $g'$

Observe that $\partial\Delta_{n+1}(X) = B_n(X) = \ker(Z_n \to H_n(X))$ so that each square commutes

$$\begin{tikzcd}
\vdots \arrow[d, "\partial"] & \vdots \arrow[d, "0"] \\
\Delta_{n+1}(X) \arrow[r, "g'"] \arrow[d, "\partial"] \arrow[dr, phantom, "\circ"] & H_{n+1}(X) \arrow[d, "0"] \\
\Delta_n(X) \arrow[r, "g'"] \arrow[d, "\partial"] \arrow[dr, phantom, "\circ"] & H_n(X) \arrow[d, "0"] \\
\Delta_{n-1}(X) \arrow[r, "g'"] \arrow[d, "\partial"] & H_{n-1}(X) \arrow[d, "0"] \\
\vdots & \vdots
\end{tikzcd}$$

$$\Delta_*(X) \xrightarrow{G} H_*(X)$$

Tensor both chains by $\mathbb{Z}_p$ to obtain

When we take the homology on $\mathbb{Z}_p \otimes \Delta_*(X)$, by definition, we have $H_*(X, \mathbb{Z}_p)$. For the RHS, because the image of each map in the complex is trivial, we obtain $\mathbb{Z}_p \otimes H_*(X)$. The induced map on homology gives the desired chain map

$$H_*(X, \mathbb{Z}_p) \xrightarrow{(id \otimes G)_*} \mathbb{Z}_p \otimes H_*(X)$$

$$\begin{tikzcd}
\vdots \arrow[d, "id \otimes \partial"] & \vdots \arrow[d, "0"] \\
\mathbb{Z}_p \otimes \Delta_{n+1}(X) \arrow[r, "id \otimes g'"] \arrow[d, "id \otimes \partial"] & \mathbb{Z}_p \otimes H_{n+1}(X) \arrow[d, "0"] \\
\mathbb{Z}_p \otimes \Delta_n(X) \arrow[r, "id \otimes g'"] \arrow[d, "id \otimes \partial"] & \mathbb{Z}_p \otimes H_n(X) \arrow[d, "0"] \\
\mathbb{Z}_p \otimes \Delta_{n-1}(X) \arrow[r, "id \otimes g'"] \arrow[d, "id \otimes \partial"] & \mathbb{Z}_p \otimes H_{n-1}(X) \arrow[d, "0"] \\
\vdots & \vdots
\end{tikzcd}$$

$$\mathbb{Z}_p \otimes \Delta_*(X) \xrightarrow{id \otimes G} \mathbb{Z}_p \otimes H_*(X).$$

In particular, at the $n^{th}$ level, we have

$$H_n(X, \mathbb{Z}_p) \xrightarrow{id \otimes g'} \mathbb{Z}_p \otimes H_n(X) \cong \operatorname{coker}\{H_n(X;\mathbb{Z}) \xrightarrow{(p\otimes id)_*} H_n(X;\mathbb{Z})\}$$

for $A \in Ab$, $\mathbb{Z}_p \otimes_{\mathbb{Z}} A \simeq A/p$

1. Multiplication by the prime $p: \mathbf{Z} \to \mathbf{Z}$ fits in a short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \to \mathbf{Z}_p \to 0.$$

Use this to derive the natural split exact sequence

$$0 \to \frac{H_n(X)}{pH_n(X)} \to H_n(X;\mathbf{Z}_p) \to \ker\{p: H_{n-1}(X) \to H_{n-1}(X)\} \to 0.$$

(The splitting is not natural.)

We move from a l.e.s. to a s.e.s again...

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d] & C \arrow[r, "h"] & D \arrow[r, "k"] & E \\
 & B/\ker g \arrow[r, "\simeq"] & C' \arrow[u] & &
\end{tikzcd}$$

exact so $\operatorname{im} f = \ker g$

Let $c \in \ker h'$. Then $c \in \ker h$. So $c \in \operatorname{im} g$. Passing to the quotient, $c \in \operatorname{im} g'$.

$$0 \longrightarrow B/\operatorname{im} f \xrightarrow{g'} C \xrightarrow{h'} \ker k \longrightarrow 0$$

why exact?

$\operatorname{im} g' \subset \ker h'$

$\ker h' \subset \operatorname{im} g'$

The significance of short exact sequence shows up when we try to break down a long exact sequence into short exact sequences. Consider the exact sequence of $R$-modules $\cdots \to A_{n+2} \to A_{n+1} \to A_n \to A_{n-1} \to A_{n-2} \to \cdots$. Let

$$C_n \cong \ker\,(A_n \to A_{n-1}) \cong \operatorname{im}\,(A_{n+1} \to A_n).$$

[Che09]

As the algebraic structure underlying $R$-module is abelian group, the cokernel of each homomorphism exists such that $C_n \cong \operatorname{coker}(A_{n+2} \to A_{n+1})$. Then we obtain the following commutative diagram, in which all the diagonal sequences are short exact:

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & \\
 & & C_{n+1} \arrow[ur] \arrow[dr] & & & & C_{n-1} \arrow[ur] \arrow[dr] & & \\
\cdots \arrow[r] & A_{n+2} \arrow[rr] \arrow[ur] & & A_{n+1} \arrow[r] \arrow[dr] & A_n \arrow[rr] \arrow[urr] & & A_{n-1} \arrow[rr] \arrow[drr] & & \cdots \\
 & C_{n+2} \arrow[u] & & & C_n \arrow[u] \arrow[dr] & & & & C_{n-2} \arrow[dr] \\
0 \arrow[ur] & & & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd} \tag{2.5}$$

Conversely, given any short exact sequences overlapped in this way, their middle terms form an exact sequence.

Marvin on splitting:

$$[\![c]\!] \xleftarrow{\text{forget } \mathbb{Z}_p} [\![c]\!]$$

$H_n(X, \mathbb{Z}_p)$

$\Delta_n(X) \otimes \mathbb{Z}_p$

$\overset{\text{lift}}{\longleftarrow}$ $\ni c$

not nec. well defined...

$\Delta_n(X) \otimes \mathbb{Z}_p \cong \Delta_n(X)/(p)$

now observe $H_n(X, \mathbb{Z}_p)$ is larger!

Another idea: a ses of vector spaces always splits.

Beaudry $\Delta_n(X)/p \cong Z_n(X)/p \oplus B_{n-1}(X)/p$

$\downarrow$

$\Delta_{n-1}(X)/p$

**Theorem 3.21.** *If $C$ is a chain complex of free abelian groups, then there are natural short exact sequences*

$$0 \to H_n(C) \otimes G \to H_n(C;G) \to Tor_1(H_{n-1}(C), G) \to 0 \tag{3.22}$$

*for all $n$ and all $G$, and these sequences split, though not naturally.* [Che09]

To prove the splitting, we go back to the split short exact sequence $0 \to Z_n \xrightarrow{f} C_n \xrightarrow{g} B_{n-1} \to 0$. Splitting implies that there is $p : C_n \to Z_n$ such that $p \circ f = 1_{Z_n}$. Further $p$ can be extended to $p'$, making the following diagram commutes:

$$\begin{tikzcd}
C_n \arrow[d, "p"'] \arrow[dr, dashed, "p'"] & \\
Z_n \arrow[r, "q"'] & H_n(C).
\end{tikzcd} \tag{3.23}$$

To get a chain map $F : C. \to H.(C)$, we make $H.$ a chain complex by adding trivial boundary maps between them. Tensor with $G$, which yields $F \otimes 1 : C. \otimes G \to H.C \otimes G$. When we take the homology of $C. \otimes G$, we get the usual $H_n(C;G)$. When we take the homology of $H.C \otimes G$, however, it gives us $H_n(C) \otimes G$, due to the zero homomorphisms. Thus we have the induced homomorphism on homology $F_* : H_n(C;G) \to H_n(C) \otimes G$, which proves the desired splitting.

**Splitting Lemma.** *For a short exact sequence $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ of abelian groups the following statements are equivalent*:

(a) *There is a homomorphism $p: B \to A$ such that $pi = \mathbb{1}: A \to A$.* [Hat02]

(b) *There is a homomorphism $s: C \to B$ such that $js = \mathbb{1}: C \to C$.*

(c) *There is an isomorphism $B \approx A \oplus C$ making a commutative diagram as at the right, where the maps in the lower row are the obvious ones, $a \mapsto (a, 0)$ and $(a, c) \mapsto c$.*

$$\begin{tikzcd}[column sep=small]
 & & B \arrow[d, "\approx"] \arrow[dr, "j"] & & \\
0 \arrow[r] & A \arrow[ur, "i"] \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

If these conditions are satisfied, the exact sequence is said to **split**. Note that (c) is symmetric: There is no essential difference between the roles of $A$ and $C$.

**Sketch of Proof**: For the implication (a) $\Rightarrow$ (c) one checks that the map $B \to A \oplus C$, $b \mapsto (p(b), j(b))$, is an isomorphism with the desired properties. For (b) $\Rightarrow$ (c) one uses instead the map $A \oplus C \to B$, $(a, c) \mapsto i(a) + s(c)$. The opposite implications (c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) are fairly obvious. If one wants to show (b) $\Rightarrow$ (a) directly, one can define $p(b) = i^{-1}(b - sj(b))$. Further details are left to the reader. $\square$

# FIRST LOOK AT THE UNIVERSAL COEFFICIENT THEOREM

COLTON GRAINGER (SCRIBE) AND MARVIN QI (PRESENTER)

This problem is set from Bredon [1, No. IV.5.1]; it works out techniques that might later play into a proof of the universal coefficient theorem.

*Given.* Let $p\colon \mathbf{Z} \to \mathbf{Z}$ be multiplication by the prime $p$, and consider the short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{p} \mathbf{Z} \xrightarrow{\pi} \mathbf{Z}_p \to 0. \tag{1}$$

Say $X$ is a topological space. We assume $\Delta_*(-)\colon \mathsf{Top} \to \mathsf{Comp}$ is the singular chain functor from topological spaces to chain complexes of abelian groups, with $H_*(-)\colon \mathsf{Comp} \to \mathsf{GradedAb}$ the homology functor on chain complexes. For an abelian group $G$, denote by $H_*(C_*; G)$ the *homology of $C_*$ with coefficients in $G$*, which is the homology of the chain complex $G \otimes C_*$ with boundary operator $\mathrm{id}_G \otimes \partial$.

*To prove.* There is an exact sequence

$$0 \to \mathbf{Z}_p \otimes H_n(X) \to H_n(X; \mathbf{Z}_p) \to \mathrm{Tor}(H_{n-1}(X), \mathbf{Z}_p) \to 0, \tag{2}$$

with a homomorphism $H_n(X; \mathbf{Z}_p) \to \mathbf{Z}_p \otimes H_n(X)$ that splits (2).

*Proof.* We'll construct the exact sequence (2), then the splitting homomorphism.

Each group in $\Delta_*(X)$ is free abelian, and, in particular, flat as a $\mathbf{Z}$-module. Thus tensoring the short exact sequence (1) on the right by the $n$th singular chain $\Delta_n(X)$ group preserves exactness, and produces:

$$0 \longrightarrow \mathbf{Z} \otimes \Delta_n(X) \xrightarrow{p \otimes \mathrm{id}_\Delta} \mathbf{Z} \otimes \Delta_n(X) \xrightarrow{\pi \otimes \mathrm{id}_\Delta} \mathbf{Z}_p \otimes \Delta_n(X) \longrightarrow 0 \tag{3}$$

Lining up copies of (3) levelwise with the appropriate differentials, we exhibit the short exact sequence of chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \mathbf{Z} \otimes \Delta_n(X) \arrow[r, "p\otimes \mathrm{id}_\Delta"] \arrow[d, "\mathrm{id}_{\mathbf{Z}} \otimes \partial_n"] & \mathbf{Z} \otimes \Delta_n(X) \arrow[r, "\pi\otimes \mathrm{id}_\Delta"] \arrow[d, "\mathrm{id}_{\mathbf{Z}} \otimes \partial_n"] & \mathbf{Z}_p \otimes \Delta_n(X) \arrow[r] \arrow[d, "\mathrm{id}_{\mathbf{Z}_p} \otimes \partial_n"] & 0 \\
0 \arrow[r] & \mathbf{Z} \otimes \Delta_{n-1}(X) \arrow[r, "p\otimes \mathrm{id}_\Delta"] \arrow[d] & \mathbf{Z} \otimes \Delta_{n-1}(X) \arrow[r, "\pi\otimes \mathrm{id}_\Delta"] \arrow[d] & \mathbf{Z}_p \otimes \Delta_{n-1}(X) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{4}$$

As abelian groups, we'll identify $\mathbf{Z} \otimes \Delta_n(X) = \Delta_n(X)$ and $\mathbf{Z}_p \otimes \Delta_n(X) = \Delta_n(X)/p$ (the cokernel of the $p$th multiple map). Accordingly, we'll write the homomorphisms $p \otimes \mathrm{id}_\Delta = p$, $\pi \otimes \mathrm{id}_\Delta = \pi$, with all the differentials $\mathrm{id} \otimes \partial_n$ as $\partial$. Given these abbreviations, the short exact sequence (4) becomes:

$$0 \longrightarrow \Delta_*(X) \xrightarrow{p} \Delta_*(X) \xrightarrow{\pi} \Delta_*(X)/p \longrightarrow 0 \tag{5}$$

*Date*: 2019-02-13.

By inspection of the definition of homology with coefficients along with the diagram (4), the long exact sequence of homology groups induced by (5) is:

$$\cdots \xrightarrow{\delta_*} H_n(X) \xrightarrow{p_*} H_n(X) \xrightarrow{\pi_*} H_n(X;\mathbf{Z}_p) \xrightarrow{\delta_*} H_{n-1}(X) \xrightarrow{p_*} \cdots \tag{6}$$

We now make an aside to argue the algebraic structure of a long exact sequence is encoded in a series of intertwined short exact sequences [2, No. 2.5]. Start by considering an exact sequence of abelian groups

$$\cdots \to A_{n+2} \to A_{n+1} \to A_n \to A_{n-1} \to A_{n-2} \to \cdots .$$

By definition of exactness at $A_n$,

$$\operatorname{im}(A_{n+1} \to A_n) = \ker(A_n \to A_{n-1}). \tag{7}$$

Quotienting by the kernel of the map out of $A_n$, we have an isomorphism onto the image in $A_{n-1}$

$$A_n/\ker(A_n \to A_{n-1}) \cong \operatorname{im}(A_n \to A_{n-1}) \tag{8}$$

Now, the cokernel of an additive homomorphism is the target group quotiented by the incoming image. Substituting (7) into (8) gives an example of such

$$\operatorname{coker}(A_{n+1} \to A_n) := A_n/\operatorname{im}(A_{n+1} \to A_n) \cong \operatorname{im}(A_n \to A_{n-1}).$$

Putting it all together, there's a group $C_n$ that is

- the kernel of the map $A_n \to A_{n-1}$,
- isomorphic to the image of the map $A_{n+1} \to A_n$, and
- isomorphic to the cokernel of the map $A_{n+2} \to A_{n+2}$.

Moreover, there are groups $C_{n+k}$ for $k \in \mathbf{Z}$ satisfying analogously conditions. We may now decorate our long exact sequence with inclusions and natural projections so that the diagram

(9)

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & & & 0 \\
 & & & C_{n+1} \arrow[ur] \arrow[dr] & & & & C_{n-1} \arrow[ur] \arrow[dr] & & & & C_{n-3} \arrow[ur] & \\
\cdots \arrow[rr] & & A_{n+2} \arrow[ur] \arrow[rr] & & A_{n+1} \arrow[rr] \arrow[dr] & & A_n \arrow[ur] \arrow[rr] & & A_{n-1} \arrow[rr] \arrow[dr] & & A_{n-2} \arrow[ur] \arrow[rr] & & \cdots \\
 & C_{n-2} \arrow[ur] & & & & C_n \arrow[ur] \arrow[dr] & & & & C_{n-2} \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

is exact. We emphasize that the crossing maps are either *inclusions* or *natural projections*. The crossing sequences are short exact because, e.g., for $0 \to C_n \to A_n \to C_{n-1} \to 0$,

- $C_n$ is the kernel, so injects and passes through $A_n$ to 0 in $C_{n-1}$,
- $A_n$ surjects onto $C_{n-1}$ and passes to 0,
- any element in $A_n$ that passes to 0 in $C_{n-1}$ must be in the image of $C_n$ in $A_n$, because $A_n \to C_{n-1}$ is the natural projection onto $A_n/\ker(A_n \to A_{n-1}) = A_n/C_n$.

We now construct the desired short exact sequence (2). Placing the long exact sequence of homology groups (6) into the position of the long exact sequence in the cross stitched diagram (9), one visibly obtains the short exact sequence:

$$0 \longrightarrow \operatorname{coker}\left(H_n(X) \xrightarrow{p_*} H_n(X)\right) \longrightarrow H_n(X;\mathbf{Z}_p) \longrightarrow \ker\left(H_{n-1}(X) \xrightarrow{p_*} H_{n-1}(X)\right) \longrightarrow 0 \tag{10}$$

This sequence (10) is isomorphic to the one we wanted (2), as

$$\operatorname{coker}\left(H_n(X) \xrightarrow{p_*} H_n(X)\right) \cong \mathbf{Z}_p \otimes H_n(X)$$

and

$$\ker\left(H_{n-1}(X) \xrightarrow{p_*} H_{n-1}(X)\right) =: \mathrm{Tor}(H_{n-1}(X), \mathbf{Z}_p).$$

What remains to be proven is that there is a homomorphism from $H_n(X, \mathbf{Z}_p)$ down to $\mathbf{Z}_p \otimes H_n(X)$ that (left) inverts the inclusion of $\mathbf{Z}_p \otimes H_n(X)$ up into $H_n(X, \mathbf{Z}_p)$. My argument here closely follows [2, No. 3.21].

We'll work levelwise for a while in the singular chain complex $\Delta_*(X)$. Let $Z_n$ be the cycle subgroup of $\Delta_n(X)$ and $B_{n-1}$ the boundary subgroup $\delta\Delta_n(X)$ in $\Delta_{n-1}(X)$. Observe that $Z_n$ includes into $\Delta_n(X)$ which projects onto $B_{n-1}$. So we have the short exact sequence of free abelian groups

(11)
$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, "f"] & \Delta_n(X) \arrow[r] \arrow[l, bend left, dashed, "g"] & B_{n-1} \arrow[r] & 0, \end{tikzcd}$$

that splits with $g \circ f = \mathrm{id}_{Z_n}$. Extend $g$ to the quotient $H_n(X)$ so that the following diagram commutes:

(12)
$$\begin{tikzcd} \Delta_n(X) \arrow[d, "g"] \arrow[dr, dashed, "g'"] & \\ Z_n \arrow[r] & H_n(X) \end{tikzcd}$$

Consider the graded abelian group $\{H_k(X)\}_{k\in\mathbf{Z}}$ in a chain with trivial differentials

$$\ldots \xrightarrow{0} H_{n+1}(X) \xrightarrow{0} H_n(X) \xrightarrow{0} H_{n-1}(X) \xrightarrow{0} \ldots.$$

Call this chain complex $H_*(X)$. We induce a chain map $g_*$ from $\Delta_*(X)$ to $H_*(X)$ with $g'$:

(13)
$$\begin{tikzcd} \vdots \arrow[d, "\partial"] & \vdots \arrow[d, "0"] \\ \Delta_{n+1}(X) \arrow[r, "g'"] \arrow[d, "\partial"] & H_{n+1}(X) \arrow[d, "0"] \\ \Delta_n(X) \arrow[r, "g'"] \arrow[d, "\partial"] & H_n(X) \arrow[d, "0"] \\ \Delta_{n-1}(X) \arrow[r, "g'"] \arrow[d, "\partial"] & H_{n-1}(X) \arrow[d, "0"] \\ \vdots & \vdots \end{tikzcd}$$

(14)
$$\begin{tikzcd} \vdots \arrow[d, "\mathrm{id}\otimes\partial"] & \vdots \arrow[d, "0"] \\ \mathbf{Z}_p \otimes \Delta_{n+1}(X) \arrow[r, "\mathrm{id}\otimes g'"] \arrow[d, "\mathrm{id}\otimes\partial"] & \mathbf{Z}_p \otimes H_{n+1}(X) \arrow[d, "0"] \\ \mathbf{Z}_p \otimes \Delta_n(X) \arrow[r, "\mathrm{id}\otimes g'"] \arrow[d, "\mathrm{id}\otimes\partial"] & \mathbf{Z}_p \otimes H_n(X) \arrow[d, "0"] \\ \mathbf{Z}_p \otimes \Delta_{n-1}(X) \arrow[r, "\mathrm{id}\otimes g'"] \arrow[d, "\mathrm{id}\otimes\partial"] & \mathbf{Z}_p \otimes H_{n-1}(X) \arrow[d, "0"] \\ \vdots & \vdots \end{tikzcd}$$

Observe that $\partial(\Delta_{n+1}(X)) = B_n(X) = \ker(Z_n \to H_n(X))$ so that each square in (13) commutes. We may then tensor both chains by $\mathbf{Z}_p$ on the right, obtaining

$$\mathbf{Z}_p \otimes \Delta_*(X) \xrightarrow{\mathrm{id}\otimes g_*} \mathbf{Z}_p \otimes H_n(X).$$

When we take homology on $\mathbf{Z}_p \otimes \Delta_*(X)$ by definition we have $H_*(X, \mathbf{Z}_p)$. However, when computing homology in the chain complex $\mathbf{Z}_p \otimes H_n(X)$, the image of each map in the complex is *trivial*, so the homology of the complex is $\mathbf{Z}_p \otimes H_*(X)$.

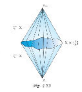
The map $\mathrm{id}\otimes g_*$ of chain complexes induces a homomorphism $H_*(\mathrm{id}\otimes g')$ on homology from $H_*(X, \mathbf{Z}_p) \to \mathbf{Z}_p \otimes H_*(X)$. In particular, at the $n$th level,

(15)
$$H_n(X, \mathbf{Z}_p) \xrightarrow{H_*(\mathrm{id}\otimes g')} \mathbf{Z}_p \otimes H_n(X).$$

By construction in (12) of $g'$ as the extension of the splitting map $g$ (where $g \circ f = \mathrm{id}_{Z_n}$), the induced map $H_*(\mathrm{id}\otimes g')$ in (15) is a left inverse to the inclusion of $\mathbf{Z}_p \otimes H_n(X)$ into $H_n(X, \mathbf{Z}_p)$. □

[Bre 93, p. 181]

(3) 1. ✧ Define the "unreduced suspension" $\Sigma X$ of a space $X$ to be the quotient space of $I \times X$ obtained by identifying $\{0\} \times X$ and $\{1\} \times X$ to points. (This is the union of two cones on $X$.) For any homology theory (satisfying the axioms) show that there is a natural isomorphism $\tilde{H}_i(X) \xrightarrow{\approx} \tilde{H}_{i+1}(\Sigma X)$. Here "natural" means that for a map $f: X \to Y$, and its suspension $\Sigma f: \Sigma X \to \Sigma Y$, the following diagram commutes:



$$\begin{array}{ccc} \tilde{H}_i(X) & \xrightarrow{\approx} & \tilde{H}_{i+1}(\Sigma X) \\ \downarrow f_* & & \downarrow (\Sigma f)_* \\ \tilde{H}_i(Y) & \xrightarrow{\approx} & \tilde{H}_{i+1}(\Sigma Y). \end{array}$$

idea: land the diagram in the l.e.s.

not using open sets

$(X,\emptyset) \xrightarrow{i} (\Sigma X, \emptyset) \xrightarrow{j} (\Sigma X, X)$ induces the les of reduced homology

$$\cdots \xrightarrow{\delta_*} \tilde{h}_p(X) \xrightarrow{i_*} \tilde{h}_p(\Sigma X) \xrightarrow{j_*} h_p(\Sigma X, X) \xrightarrow{\delta_*} \tilde{h}_{p-1}(X) \xrightarrow{i_*} \tilde{h}_{p-1}(\Sigma X) \xrightarrow{j_*} \cdots$$

Let $V_+$ and $V_-$ be defined by $[0, 1/3) \times X$ and $(2/3, 1] \times X$ in $\Sigma X$. Then $\overline{V_+ \cup V_-} \subset \mathrm{Int}(\Sigma X \setminus X)$; Observe that $(\Sigma X \setminus V, X \setminus V$

Let $V = V_+ \sqcup V_-$. So $(\Sigma X, X, V)$ is an excise triad.

Let $\varepsilon > 0$ and let $\mathcal{U} = (\frac{1}{3} - \varepsilon, \frac{2}{3} + \varepsilon) \times X$. Then $X = \mathrm{int}\,\mathcal{U} \cup \mathrm{int}\,V$. So $(X; \mathcal{U}, V)$ is an excisive triad. Moreover, $\mathcal{U}$ is homotopic to $X$. So the l.e.s. in reduced homology

Then $h_*(\Sigma X, \mathcal{U}) \xrightarrow[h.a.]{\cong} h_*(\Sigma X, X)$

sloppy

$(CX, X)$

$X, X \setminus V)$ excisive triad

$h_p(CX \setminus V, \mathcal{U} \setminus V)$

$\downarrow h.a.$

$\tilde{h}_p(CX \setminus V) \to h_p(CX \setminus V, X) \to \tilde{h}_{p-1}(X) \to \tilde{h}_{p-1}(CX \setminus V)$

$\tilde{h}_p(X) \to \tilde{h}_p(X) \to h_p(CX \setminus V, X) \to \tilde{h}_{p-1}(X) \to \tilde{h}_{p-1}(X) \to$

$q : I \times X \to \Sigma X$

$\mathcal{U} = q([0, 1/2] \times X)$, contractible.

$V = q([0, 1/3) \times X)$

woah! $\mathcal{U} \neq \bar{V}$

Better thicken up, then we $\underbrace{\mathcal{U} \setminus V = X \times I \simeq X}_{\text{homotopy}}$

Consider the pair $(\Sigma X, \mathcal{U})$, with l.e.s.

$$h_{i+1}(\mathcal{U}) \longrightarrow h_{i+1}(\Sigma X) \xrightarrow{\sim} h_{i+1}(\Sigma X, \mathcal{U}) \longrightarrow h_i(\mathcal{U}) = 0$$

$\downarrow$ by excision

$h_{i+1}(\Sigma X \setminus V, \mathcal{U} \setminus V)$

$\downarrow$ identity

$$0 = h_{i+1}(\Sigma X \setminus V) \longrightarrow h_{i+1}(\Sigma X \setminus V, X) \longrightarrow h_i(X) \longrightarrow h_i(\Sigma X \setminus V) = 0$$

these should all be reduced!

Discussion:

We ought to think of $\Sigma$ as a functor from Top to ...

Why is it true that $\Sigma f$ should be defined? Because the $I \times -$ functor descends to the quotient.

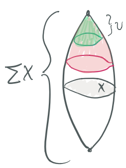
What maps in the homology axioms are natural?

Excision is a functor on triples, a map from

$H_*(X \setminus \mathcal{U}, A \setminus \mathcal{U}) \longrightarrow H_*(X, A)$

what's the easiest way? $\Sigma$ then is a functor

Justin



Consider the l.e.s. in reduced homology for the pair $(\Sigma X, \mathcal{U})$

$$\cdots \longrightarrow \underbrace{\cancel{\tilde{h}_{p+1}(\mathcal{U})}}_{h.a.}^{0} \longrightarrow \tilde{h}_{p+1}(\Sigma X) \xrightarrow[\therefore \cong]{} h_{p+1}(\Sigma X, \mathcal{U}) \longrightarrow \underbrace{\cancel{\tilde{h}_p(\mathcal{U})}}_{h.a.}^{0} \longrightarrow \cdots$$

Notice that $(\Sigma X, \mathcal{U}, V)$ is an excisive triple, so that $(\Sigma X \setminus V, \mathcal{U} \setminus V) \hookrightarrow (\Sigma X, \mathcal{U})$ induces $h_*(\Sigma X \setminus V, \mathcal{U} \setminus V) \cong h_*(\Sigma X, \mathcal{U})$. Lastly, because $\mathcal{U} \setminus V \simeq X$, we have a l.e.s. for $(\Sigma X \setminus V, X)$

$$\cdots \longrightarrow \tilde{h}_p(X) \longrightarrow \underbrace{\tilde{h}_p(\cancel{\Sigma X} \setminus V)}_{h.a.}^{0} \longrightarrow h_p(\Sigma X \setminus V, X) \xrightarrow[\therefore \cong]{\delta_*} \tilde{h}_{p-1}(X) \longrightarrow \underbrace{\tilde{h}_{p-1}(\cancel{\Sigma X} \setminus V)}_{h.a.}^{0} \longrightarrow \tilde{h}_{p-1}(\Sigma X \setminus V, X) \longrightarrow \cdots$$

We've shown that $\tilde{h}_{p+1}(\Sigma X) \cong h_{p+1}(\Sigma X, \mathcal{U}) \cong h_{p+1}(\Sigma X \setminus V, \mathcal{U} \setminus V) \cong h_{p+1}(\Sigma X \setminus V, X) \cong \tilde{h}_p(X)$.

Say that $f: X \to Y$ is a cts. map. Let $\Sigma f: \Sigma X \to \Sigma Y$ be defined by $\Sigma f(\{t\} \times \{x\}) = \{t\} \times \{f(x)\}$.

Observe that

$$\begin{array}{ccc} \tilde{h}_p(X) & \xrightarrow{T_X} & \tilde{h}_{p+1}(\Sigma X) \\ h_*(f) \downarrow & & \downarrow h_*(\Sigma f) \\ \tilde{h}_p(Y) & \xrightarrow{T_Y} & \tilde{h}_{p+1}(\Sigma Y) \end{array}$$

$$\begin{tikzcd} \mathrm{Top}_* \arrow[r, "\Sigma"] \arrow[dr, "h_* \circ \Sigma"'] & \mathrm{Top}_* \arrow[d, "h_*"] \\ & \mathrm{GradedAb} \end{tikzcd}$$

natural transf. $T$

For each $X \in \mathrm{Top}_*$, let $T_X : \tilde{h}_p(X) \longmapsto \tilde{h}_{p+1}(\Sigma X)$.

**Proposition 2.22.** *For good pairs $(X, A)$, the quotient map $q: (X, A) \to (X/A, A/A)$ induces isomorphisms $q_*: H_n(X, A) \to H_n(X/A, A/A) \approx \tilde{H}_n(X/A)$ for all $n$.*

**Proof**: Let $V$ be a neighborhood of $A$ in $X$ that deformation retracts onto $A$. We have a commutative diagram

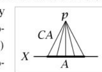
$$\begin{array}{ccccc} H_n(X,A) & \longrightarrow & H_n(X,V) & \longleftarrow & H_n(X-A, V-A) \\ \downarrow q_* & & \downarrow q_* & & \downarrow q_* \\ H_n(X/A, A/A) & \longrightarrow & H_n(X/A, V/A) & \longleftarrow & H_n(X/A - A/A, V/A - A/A) \end{array}$$

The upper left horizontal map is an isomorphism since in the long exact sequence of the triple $(X, V, A)$ the groups $H_n(V, A)$ are zero for all $n$, because a deformation retraction of $V$ onto $A$ gives a homotopy equivalence of pairs $(V, A) \simeq (A, A)$, and $H_n(A, A) = 0$. The deformation retraction of $V$ onto $A$ induces a deformation retraction of $V/A$ onto $A/A$, so the same argument shows that the lower left horizontal map is an isomorphism as well. The other two horizontal maps are isomorphisms directly from excision. The right-hand vertical map $q_*$ is an isomorphism since $q$ restricts to a homeomorphism on the complement of $A$. From the commutativity of the diagram it follows that the left-hand $q_*$ is an isomorphism. ◻

This proposition shows that relative homology can be expressed as reduced absolute homology in the case of good pairs $(X, A)$, but in fact there is a way of doing this



for arbitrary pairs. Consider the space $X \cup CA$ where $CA$ is the cone $(A \times I)/(A \times \{0\})$ whose base $A \times \{1\}$ we identify with $A \subset X$. Using terminology introduced in Chapter 0, $X \cup CA$ can also be described as the mapping cone of the inclusion $A \hookrightarrow X$. The assertion is that $H_n(X, A)$ is isomorphic to $\tilde{H}_n(X \cup CA)$ for all $n$ via the sequence of isomorphisms




$$\tilde{H}_n(X \cup CA) \approx H_n(X \cup CA, CA) \approx H_n(X \cup CA - \{p\}, CA - \{p\}) \approx H_n(X, A)$$

where $p \in CA$ is the tip of the cone. The first isomorphism comes from the exact sequence of the pair, using the fact that $CA$ is contractible. The second isomorphism is excision, and the third comes from a deformation retraction of $CA - \{p\}$ onto $A$.

In general, if one has two functors $F, G: \mathcal{C} \to \mathcal{D}$ then a **natural transformation** $T$ from $F$ to $G$ assigns a morphism $T_X : F(X) \to G(X)$ to each object $X \in \mathcal{C}$, in such a way that for each morphism $f: X \to Y$ in $\mathcal{C}$ the square at the right commutes. The case that $F$ and $G$ are contravariant rather than covariant is similar.

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow T_X & & \downarrow T_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

**20.** Show that $\tilde{H}_n(X) \approx \tilde{H}_{n+1}(SX)$ for all $n$, where $SX$ is the suspension of $X$. More generally, thinking of $SX$ as the union of two cones $CX$ with their bases identified, compute the reduced homology groups of the union of any finite number of cones $CX$ with their bases identified.

Let $\gamma_k X$ denote the pushout or the union of $k$ many cones on $X$ with base points identified.

$$\begin{array}{ccc} \coprod_{i=1}^{k} X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod_{i=1}^{k} CX & \longrightarrow & \gamma_k X \end{array}$$

(4) Let $h$ be an ordinary homology theory. Let $X_i$ with base points $x_i \in X_i$. Suppose that there are open sets $x_i \in U_i \subseteq X_i$ such that $x_i$ is a strong deformation retract of $U_i$. Show that

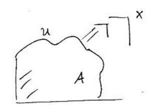
$$\widetilde{h}_n\left(\bigvee_{i\in I} X_i\right) \cong \bigoplus_{i\in I} \widetilde{h}_n(X_i).$$

(Hint: try to prove first for just two spaces

3. ◇ Let $X$ be a Hausdorff space and let $x_0 \in X$ be a point having a closed neighborhood $N$ in $X$, of which $\{x_0\}$ is a strong deformation retract. Let $Y$ be a Hausdorff space and let $y_0 \in Y$. Define $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$, the "one-point union" of $X$ and $Y$. Show that the inclusion maps induce isomorphisms $\widetilde{H}_i(X) \oplus \widetilde{H}_i(Y) \xrightarrow{\approx} \widetilde{H}_i(X \vee Y)$, for any homology theory, whose inverse is induced by the projections of $X \vee Y$ to $X$ and $Y$. )

Also use excision.

Q] $H_*(A_1 \sqcup A_2, B_1 \sqcup B_2) \overset{?}{=} H_*(A_1, B_1) \oplus H_*(A_2, B_2)$




"locality condition"
"homology is local"
↓
esp. relative homology

[Hat02]

**Corollary 2.25.** *For a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism $\bigoplus_\alpha i_{\alpha*} : \bigoplus_\alpha \widetilde{H}_n(X_\alpha) \to \widetilde{H}_n(\bigvee_\alpha X_\alpha)$, provided that the wedge sum is formed at basepoints $x_\alpha \in X_\alpha$ such that the pairs $(X_\alpha, x_\alpha)$ are good.*

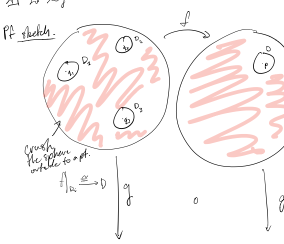
**Proof**: Since reduced homology is the same as homology relative to a basepoint, this follows from the proposition by taking $(X, A) = (\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\})$. □

CORO] Let $f : S^n \to S^n$ be cts map. Let $p \in S^n$ and spose for open nbhd $U \ni p$, $f : f^{-1}(U) \to U$ is smooth and $p$ is a regular value. Say $f^{-1}(p) = \{q_i\}_1^k$. By rotations, we compute the deg of the derivative(s) to find that $\deg(f) = \sum_{i=1}^k \operatorname{sgn}(\det(D_{q_i} f))$
?

Q] We can choose any regular value?

A] So long as the nbhd is one on which $f$ is smooth.

Pf sketch.




$f|_{D_i} \xrightarrow{\cong} D$

$g$ ○ $g'$

$D_i/\partial D_i \hookrightarrow D_1/\partial D_1 \vee D_2/\partial D_2 \vee D_3/\partial D_3 \longrightarrow D/\partial D$

compute locally

then show $\tilde{h}(\bigvee_i D_i/\partial D_i) \cong \bigoplus_i \tilde{h}(D_i/\partial D_i)$.

WTS $\deg(f) = \deg(g' \circ f)$.

Hint: avoid over geometrizing ...

The Brouwer Fixed Point Theorem (Theorem 11.12 of Chapter II), follows from this by the same simple geometric argument given in Chapter II.

The reader should note the form of the proof of Corollary 6.7. The geometric assumption that there is a retraction can be stated in terms of maps and their compositions (the identity map of the sphere to itself factors through the disk). This statement translates, by applying the homology functor, to an analogous statement about groups and homomorphisms and their compositions. Since the resulting statement about groups is obviously false, the original one about spaces must also be false. This type of argument is typical of applications of homology theory.

[Hat02, p. 160]

**Axioms for Homology**

For simplicity let us restrict attention to CW complexes and focus on reduced homology to avoid mentioning relative homology. A (reduced) **homology theory** assigns to each nonempty CW complex $X$ a sequence of abelian groups $\widetilde{h}_n(X)$ and to each map $f : X \to Y$ between CW complexes a sequence of homomorphisms $f_* : \widetilde{h}_n(X) \to \widetilde{h}_n(Y)$ such that $(fg)_* = f_* g_*$ and $\mathbb{1}_* = \mathbb{1}$, and so that the following three axioms are satisfied.

(1) If $f \simeq g : X \to Y$, then $f_* = g_* : \widetilde{h}_n(X) \to \widetilde{h}_n(Y)$.

(2) There are boundary homomorphisms $\partial : \widetilde{h}_n(X/A) \to \widetilde{h}_{n-1}(A)$ defined for each CW pair $(X, A)$, fitting into an exact sequence

$$\cdots \xrightarrow{\partial} \widetilde{h}_n(A) \xrightarrow{i_*} \widetilde{h}_n(X) \xrightarrow{q_*} \widetilde{h}_n(X/A) \xrightarrow{\partial} \widetilde{h}_{n-1}(A) \xrightarrow{i_*} \cdots$$

where $i$ is the inclusion and $q$ is the quotient map. Furthermore the boundary maps are natural: For $f : (X, A) \to (Y, B)$ inducing a quotient map $\bar{f} : X/A \to Y/B$, there are commutative diagrams

$$\begin{array}{ccc} \widetilde{h}_n(X/A) & \xrightarrow{\partial} & \widetilde{h}_{n-1}(A) \\ \downarrow \bar{f}_* & & \downarrow f_* \\ \widetilde{h}_n(Y/B) & \xrightarrow{\partial} & \widetilde{h}_{n-1}(B) \end{array}$$

(3) For a wedge sum $X = \bigvee_\alpha X_\alpha$ with inclusions $i_\alpha : X_\alpha \hookrightarrow X$, the direct sum map $\bigoplus_\alpha i_{\alpha*} : \bigoplus_\alpha \widetilde{h}_n(X_\alpha) \to \widetilde{h}_n(X)$ is an isomorphism for each $n$.

Negative values for the subscripts $n$ are permitted. Ordinary singular homology is zero in negative dimensions by definition, but interesting homology theories with nontrivial groups in negative dimensions do exist.

The third axiom may seem less substantial than the first two, and indeed for finite wedge sums it can be deduced from the first two axioms, though not in general for infinite wedge sums, as an example in the Exercises shows.

## Relation to relative homology

**Proposition 3.3.** *For $X$ an inhabited topological space, its reduced singular homology, def. 2.2, coincides with its relative singular homology relative to any base point $x : * \to X$:*

$$\tilde{H}_\bullet(X) \simeq H_\bullet(X, *) .$$

***Proof***. Consider the sequence of topological subspace inclusions

$$\emptyset \hookrightarrow * \overset{x}{\hookrightarrow} X .$$

By the discussion at *Relative homology - long exact sequences* this induces a long exact sequence of the form

$$\cdots \to H_{n+1}(*) \to H_{n+1}(X) \to H_{n+1}(X, *) \to H_n(*) \to H_n(X) \to H_n(X, *) \to \cdots \to H_1(X) \to H_1(X, *) \to H_0(*) \xrightarrow{H_0(x)} H_0(X) \to H_n(X, *) \to 0 .$$

Here in positive degrees we have $H_n(*) \simeq 0$ and therefore exactness gives isomorphisms

$$H_n(X) \overset{\simeq}{\to} H_n(X, *) \ \ \forall_{n \geq 1}$$

and hence with prop. 3.1 isomorphisms

$$\tilde{H}_n(X) \overset{\simeq}{\to} H_n(X, *) \ \ \forall_{n \geq 1} .$$

It remains to deal with the case in degree 0. To that end, observe that $H_0(x) : H_0(*) \to H_0(X)$ is a monomorphism: for this notice that we have a commuting diagram

$$\begin{array}{ccc} H_0(*) & \overset{\mathrm{id}}{\to} & H_0(*) \\ \downarrow & \nearrow_{H_0(x)} & \downarrow_{\simeq} \\ H_0(X) & \longrightarrow & \mathbb{Z} \end{array} ,$$

# Homework 2, Problem 4

Marvin Qi

**Theorem 0.1.** *Let $h$ be an ordinary homology theory. Let $X_i$ with base points $x_i \in X_i$. Suppose that there are open sets $x_i \in U_i \subset X_i$ such that $x_i$ is a strong deformation retract of $U_i$. Show that*

$$\tilde{h}_n\left(\bigvee_i X_i\right) \cong \bigoplus_i \tilde{h}_n(X_i) \tag{0.1}$$

*Proof.* We show this directly for the case where $i = 1, 2$. We will take advantage of the fact that $X_1 \vee X_2 \setminus \{x_1 = x_2\}$ is a disjoint union $X_1 \setminus \{x_1\} \coprod X_2 \setminus \{x_2\}$.

By excision,

$$\begin{aligned} h_n(X_1 \vee X_2, U_1 \vee U_2) &\cong h_n(X_1 \vee X_2 \setminus \{x_1 = x_2\}, U_1 \cup U_2 \setminus \{x_1 = x_2\}) \\ &\cong h_n(X_1 \setminus \{x_1\} \sqcup X_2 \setminus \{x_2\}, U_1 \setminus \{x_1\} \sqcup U_2 \setminus \{x_2\}) \\ &\cong h_n(X_1 \setminus \{x_1\}, U_1 \setminus \{x_1\}) \oplus h_n(X_2 \setminus \{x_2\}, U_2 \setminus \{x_2\}) \\ &\cong h_n(X_1, U_1) \oplus h_n(X_2, U_2) \\ &\cong h_n(X_1, x_1) \oplus h_n(X_2, x_2) \\ &\cong \tilde{h}_n(X_1) \oplus \tilde{h}_n(X_2) \end{aligned} \tag{0.2}$$

The first equality follows from excision. The second line follows from the fact that because the total space is a wedge sum, removal of the identified point leads to a disjoint union of the original spaces without the identified points. The third equality follows from the sum axiom of homology theories. The fourth equality is excision but applied backwards. The fifth equality is via the application of the homotopy axiom, which is applicable since $x_i$ is a deformation retract of $U_i$ and therefore $U_i \simeq x_i$. The last equality is a result of the fact that reduced homology is equivalent to homology of pairs where the subspace is a point.

Moreover, we also have

$$h_n(X_1 \vee X_2, U_1 \vee U_2) \cong h_n(X_1 \vee X_2, x_1 = x_2) \cong \tilde{h}_n(X_1 \vee X_2) \tag{0.3}$$

where the first equality follows from the homotopy axiom and the second is the property of reduced homology. The space $U_1 \vee U_2$ is homotopy equivlent to the identified point because the point is a strong deformation of both $U_1$ and $U_2$. Given a strong deformation retract $F_1 : U_1 \times [0,1] \to U_1$ and $F_2 : U_2 \times [0,1] \to U_2$ such that $F_i(x_i, t) = x_i$, we can construct a strong deformation retract $F : U_1 \vee U_2 \times [0,1] \to U_1 \vee U_2$ as

$$F(x,t) = \begin{cases} F_1(x,t), & \text{if } x \in U_1 \setminus \{x_1 = x_2\} \\ F_2(x,t), & \text{if } x \in U_2 \setminus \{x_1 = x_2\} \\ x, & \text{if } x = x_1 = x_2 \end{cases} \tag{0.4}$$

This strong deformation retract is continuous and well defined. Therefore, the identified point is a strong deformation retract of $U_1 \vee U_2$ and therefore is homotopy equivlent to it. Combining (0.3) and (0.2), we establish the result that

$$\tilde{h}_n(X_1 \vee X_2) \cong \tilde{h}_n(X_1) \bigoplus \tilde{h}_n(X_2) \tag{0.5}$$

The argument extends to arbitrary wedge sums because the only assumptions used in (0.2) were that removal of the identified point leads to a disjoint union, and that disjoint unions of spaces give direct sums of homologies, neither of which depends on the cardinality of the indexing set. In (0.3) and (0.4), the only change would be to introduce a strong deformation retract $F_i$ for each element of the indexing set, and construct $F$ via the same procedure. $\square$

Optional!

2. If $f: \mathbf{S}^n \to \mathbf{S}^n$ is a map without fixed points, show that $\deg(f) = (-1)^{n+1}$.

Optional!

4. If $n$ is even, show that any map $f: \mathbf{P}^n \to \mathbf{P}^n$ (real projective $n$-space) has a fixed point. (*Hint*: Use Corollary 6.13.)

# § IV.7 Degree of maps

Which general linear maps preserve the sphere?

exact: $0 \longrightarrow SO(n+1) \longrightarrow O(n+1) \xrightarrow{\det} \{1,-1\} \longrightarrow 0$

Moreover, $SO(n+1)$ is <u>path connected</u>. (Rotations)

OTOH, $\pi_0\, O(n+1) = \{1,-1\}$. (Reflections)

Now both $O(n+1)$ and $SO(n+1)$ are groups of maps.

$\mathrm{Map}(I, \mathrm{Map}(X,Y))$ is the collection of homotopies between maps $f,g : X \longrightarrow Y$.

**Definition 1.6.4.** Let $h_*$ have coefficient $\mathbb{Z}$ and let $n \geqslant 0$. Let $f\colon S^n \to S^n$. The *degree* of $f$, denoted $\deg(f)$, is the unique integer $d$ such that $f_*\colon \widetilde{h}_n(S^n) \to \widetilde{h}_n(S^n)$ is multiplication by $d$.

Note that $\deg(f \circ g) = \deg(f)\deg(g)$, and that if $f$ is null-homotopic, then $\deg(f) = 0$. Furthermore, $\deg(\mathrm{id}) = 1$. Finally, if $f$ is a homeomorphism, then $\deg(f) = \pm 1$, and $\deg(f) = \deg(f^{-1})$.

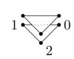
Let $f : S^n \to S^n$ be a continuous map. Recall $\widetilde{H}_n(S^n) \simeq \mathbb{Z}$ for $n \geq 0$.

**Definition 7.1.** $f_*$ in dimension $n$ is a homomorphism $\mathbb{Z} \to \mathbb{Z}$, i.e. $f_*$ in dimension $n$ is multiplication by an integer $\deg(f)$, the *degree* of $f$.

*Note* 7.2. (i) $\deg(\mathrm{id}_{S^n}) = 1$ as $(\mathrm{id}_{S^n})_* = \mathrm{id}_{H_n}$
(ii) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$ as $(f \circ g)_* = f_* \circ g_*$
(iii) $f \simeq g \implies \deg(f) = \deg(g)$ as $f_* = g_*$
(iv) $f \simeq * \implies \deg(f) = 0$ as $f_* = 0$.

**Example 7.3.**

(1) $f$ is induced by the reflection in $\mathbb{R}^n i \times \{0\} \subset \mathbb{R}^{n+1}$, then $\deg(f) = -1$:
$H_n(S^n) = \langle \Delta_1 - \Delta_2 \rangle$, $f$ interchanges $\Delta_1$ and $\Delta_2$. So $f_*(\Delta_1 - \Delta_2) = (\Delta_2 - \Delta_1) = -(\Delta_1 - \Delta_2)$.




(2) $f$ is the antipodal map $S^n \to S^n$, $x \mapsto -x$, then $\deg(f) = (-1)^{n+1}$:
$f$ is induced by $-I : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $-I = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & 1 & 0 \\ 0 & \cdots & 0 & -1 \end{pmatrix}$.
Each of the $n+1$ matrices on the right hand side is a reflection homotopic to the one in (1) hence, using (ii) and (iii), $\deg(f) = (-1)^{n+1}$.

to show (missed class)

**Proposition 1.6.5.** *Let $f_i\colon S^n \to S^n \subseteq \mathbb{R}^{n+1}$ be the map defined by*

$$f_i(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n).$$

*Then* $\deg(f_i) = -1$.

*Proof.* For $0 \leqslant i \leqslant n$, $f_i = s \circ f_0 \circ s^{-1}$ where

$$s(x_0, \ldots, x_n) = (x_i, x_1, \ldots, x_{i-1}, x_0, x_{i+1}, \ldots, x_n).$$

So, $\deg(f_i) = \deg(f_0)$ since $\deg(s) = \deg(s^{-1}) = \pm 1$. We show that $\deg(f_0) = -1$. We abbreviate $f = f_0$.

Note that

$$h_0(S^0) \cong h_0(\{(1)\}) \oplus h_0(\{(-1)\}).$$

Fix a generator of $h_0(\{1\})$ and call it $a_+$. Note that $f_0$ restricts to a map from $\{(1)\}$ to $\{(-1)\}$. Let $a_- = f_*(a_+)$. So,

$$h_0(S^0) \cong \mathbb{Z}\{a_+, a_-\}$$

and $f_*$ interchanges $a_-$ and $a_+$. There is an exact sequence

$$0 \to \widetilde{h}_0(S^0) \to h_0(S^0) \to h_0(\mathrm{pt}) \to 0.$$

Since $h_0(\{(\pm 1)\}) \to h_0(\mathrm{pt})$ is the identity, the kernel is

$$\widetilde{h}_0(S^0) \cong \mathbb{Z}\{a_+ - a_-\}.$$

So $f_*\colon \widetilde{h}_0(S^0) \to \widetilde{h}_0(S^0)$ maps $a_+ - a_-$ to $-(a_+ - a_-)$, and so $\deg(f) = -1$.

Assume that the claim holds for $S^{n-1}$ and let $S^{n-1} \subseteq S^n$ be embedded as the points for which $x_n = 0$. The hemispheres $D^n_+$ and $D^n_-$ are invariant under $f$. We have

$$\begin{tikzcd}
\widetilde{h}_n(S^n) \arrow[r, "\cong", "j_*"'] \arrow[d, "f_*"] & h_n(S^n, D^n_+) \arrow[d, "f_*"] & h_n(D^n_-, S^{n-1}) \arrow[l, "\cong"', "e_*"] \arrow[r, "\cong", "\delta_*"'] \arrow[d, "f_*"] & \widetilde{h}_{n-1}(S^{n-1}) \arrow[d, "f_*", "(-1)"'] \\
\widetilde{h}_n(S^n) \arrow[r, "\cong", "j_*"'] & h_n(S^n, D^n_+) & h_n(D^n_-, S^{n-1}) \arrow[l, "\cong"', "e_*"] \arrow[r, "\cong", "\delta_*"'] & \widetilde{h}_{n-1}(S^{n-1})
\end{tikzcd}$$

where the horizontal isomorphism are the ones we fixed in the boxes above. The claim follows. ☐

**Corollary 1.6.6.** *Any reflection across a hyperplane which intersects the sphere at a great circle has degree $-1$. If $g \in O(n+1)$, then $\deg(g) = 1$ if it is int he component of the identity and $-1$ otherwise. In particular, any rotation has degree $1$.*

*Proof.* Such a map of the sphere to itself differs from $f_0$ by conjugation by a rotation of $\mathbb{R}^{n+1}$.

The group $O(n+1)$ has two path components given by those matrices of determinant 1 and those matrices of determinant $-1$. The group $SO(n+1)$ is the component of the identity. If $g \in SO(n+1)$, then a path $\gamma\colon I \to SO(n+1)$ for $g$ to the identity gives a homotopy

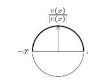
$$G\colon I \times S^n \to S^n$$

$G(t,x) = \gamma(t)(x)$ from $g$ to the identity of $S^n$. So $\deg(g) = \deg(\mathrm{id}) = 1$ for $g \in SO(n+1)$. Similarly, $\deg(g) = -1$ for $g$ in the path component of $-\mathrm{id}$. ☐

**Application 7.4.** *A continuous function $v : S^n \to \mathbb{R}^{n+1}$ defines a tangent vector field on $S^n$ if $v(x) \perp x$ for all $x \in S^n$. $S^n$ has a continuous nowhere zero tangent vector field if and only if $n$ is odd.*

*Proof.* Suppose $v : S^n \to \mathbb{R}^{n-1}$ is continuous with $v(x) \perp x$ for all $x$. Suppose further $v(x) \neq 0$ for all $x$. Consider $F : S^n \times [0,1] \to S^n$ given by $F(x,t) = (\cos \pi t)x + (\sin \pi t)\frac{v(x)}{|v(x)|}$.

Note $F(x,0) = x$ and $F(x,1) = -x$. Hence $F$ defines a homotopy from the identity map to the antipodal map. Hence they must have the same degree (by (iii)) and $1 = (-1)^{n+1}$. So $n$ must be odd.




Conversely, if $n$ is odd, define

$$\begin{aligned} v : S^n &\longrightarrow \mathbb{R}^{n+1} \\ (x_1, \ldots, x_{2k}) &\longmapsto (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1}). \end{aligned}$$

□

**Corollary 1.6.7.** *The antipodal map $-1\colon S^n \to S^n$ has degree $(-1)^{n+1}$. In particular, if $n$ is even, it is not homotopic to the identity map.*

*Proof.* It is the composite $f_0 \circ f_1 \circ \ldots \circ f_n$. □

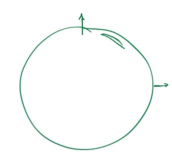
**Corollary 1.6.8.** *If $n$ is even, then for $f\colon S^n \to S^n$, there exists $x$ such that $f(x) = \pm x$.*

*Proof.* See Corollary IV.6.13. The idea is to use $f$ (assuming it doesn't send a point to itself or its antipode) to define homotopies that give $\mathrm{id} \simeq f \simeq -\mathrm{id}$. Namely,

$$F(x,t) = \frac{tf(x) + (1-t)x}{\|tf(x) + (1-t)x\|} \qquad G(x,t) = \frac{-tx + (1-t)f(x)}{\|-tx + (1-t)f(x)\|}$$

□

just a trick when the degree of an antipodal map is understood



**Corollary 1.6.9.** *If $n$ is even, the sphere $S^n$ does not have a non-vanishing continuous tangent vector field.*

*Proof.* Let $\xi\colon S^n \to TS^n$ be a continuous nowhere vanishing vector field, write $x \mapsto \xi_x \in T_xS^n$, where we identify $T_xS^n \subseteq \mathbb{R}^{n+1}$ as the $n$-dimensional subspace orthogonal to $x$. Let

$$f(x) = \xi_x / \|\xi_x\|.$$

Then $f\colon S^n \to S^n$. Since $\langle f(x), x\rangle = 0$, $f(x) \neq \pm x$ for any $x \in S^n$, a contradiction. □

## 1.7 Computing the Degree Map

TODO

**Proposition 1.7.1.** *Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a non-singular linear map. We can define a map*

$$A^*\colon S^n \to S^n$$

*by viewing $S^n$ as the one point compactification of $\mathbb{R}^n$ and setting $A^*(\infty) = \infty$. Then,*

$$\deg(A^*) = \mathrm{sgn}(\det(A)).$$

*Proof.* The matrix representing $A$ is a product of elementary matrices. If the claim holds for elementary matrices, it will hold for $A$.

- Let $E$ be an elementary matrix with one diagonal entry a real number $\alpha \neq 0$. Then, $E^*$ is homotopic to the identity if $\alpha > 0$. If $\alpha < 0$, $E^*$ is homotopic to a diagonal matrix with one diagonal $-1$ and the others all 1. This is one of the reflections $f_i$ discussed before whose degree is $-1$.

- If $E$ is the identity with one non-zero off diagonal entry, then $E$ is homotopic to the identity, so $\deg(E^*) = 1$.

- If $E$ is the identity with two columns swapped, then $E$ is a reflection accross an $n-1$-dimensional hyperplane, so $\deg(E^*) = -1$.

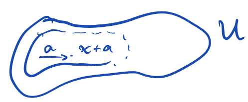
**Last time** Let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be in $GL_n(\mathbb{R})$.

Say $A^+ : (\mathbb{R}^n)^+ \longrightarrow (\mathbb{R}^n)^+$ $\quad \deg(A) := \operatorname{sgn}(\det(A))$

$$\begin{array}{ccc} (\mathbb{R}^n)^+ & \longrightarrow & (\mathbb{R}^n)^+ \\ \| & & \| \\ S^n & & S^n \end{array}$$

by 1-pt compactification.

**THM** (Taylor) Let $U \subset \mathbb{R}^n$ be open. Let $f : U \longrightarrow \mathbb{R}^m$ be $C^2$.
Let $a \in U$. Consider $r(x) := f(a+x) - (f(a) + D_a f(x))$ for $x$ s.th $a+x \in U$.
Then $\lim\limits_{x \to 0} \dfrac{r(x)}{\|x\|} = 0$. In particular, there exists $\varepsilon > 0$ s.th. $\|r(x)\| < \frac{1}{2}\|x\|$ for all $\|x\| < \varepsilon$.



review proof (hard)

**Proposition 1.7.3.** *Identify $S^n = \mathbb{R}^n \cup \{\infty\}$. Let $f\colon S^n \to S^n$ be a map which is smooth on $V = f^{-1}(U)$ for $U$ an open neighborhood of 0 and such that*

- $f^{-1}(\{0\}) = \{0\}$, so that $Df_0$ is surjective
- *0 is a regular value for $f|_V$ and*
- *$Df_0\colon \mathbb{R}^n \to \mathbb{R}^n$ is the identity.*

*Then* $\deg(f) = 1$.

*Proof.* The trick is to prove that $f \simeq \mathrm{id}$. First, we show that $f \simeq f_1$ for a map $f_1$ such that, for some small disk $D$ around 0,

- $f_1|_D = \mathrm{id}$
- $f_1(D') \subseteq D'$ where $D' = S^n \setminus D$, and $f_1|_{\partial D'} = f_1|_{\partial D} = \mathrm{id}$.

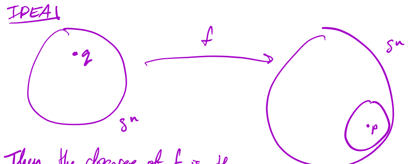
For the disk $D^n$ and a function $g\colon D^n \to D^n$ such that $g|_{\partial D^n} = \mathrm{id}$, the map $G\colon D^n \times I \to D^n$ given by

$$G(x,t) = tx + (1-t)g(x)$$

is a homotopy from $g$ to id relative to the boundary $\partial D^n$. So both $f_1|_D$ and $f_1|_{D'}$ are homotopic to the identity relative the boundary $\partial D = \partial D'$. So, $f_1 \simeq \mathrm{id}$ and we get

$$\deg(f) = \deg(f_1) = \deg(\mathrm{id}) = 1.$$

IDEA!




$p$ a regular value eg $f^{-1}(p) = q$

Then the degree of $f$ is the sign of the Jacobian determinant computed in coordinates in $p$ and $q$ that "differ by a rotation"

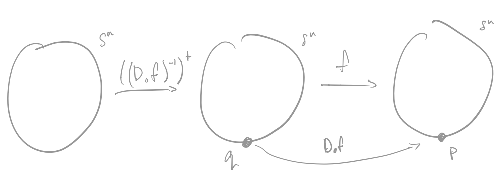
If $p$ is a reg value, then the deriv of $f$ at $q$ is surjective, by dim count.

Then rotate so that both $p$ and $q$ are at the origin. Knowing $(D_q f)^+$ is an iso, we precompose:

$$\deg\left(f \circ \left(\underbrace{(D_q f)^+}_{}\right)^{-1}\right) = 1$$

$D_q f \in GL_n(\mathbb{R})$, yet extended to the one pt. compactification $S^n = \mathbb{R}^n \cup \{\infty\}$.

We really only need that
(i) $p$ is a regular value
(ii) $f^{-1}(p) = q$ is the only pt in the preimage.

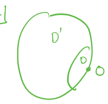



We conclude that $f_1|_D = \mathrm{id}$

let $D' = S^n \setminus D$. Then $f_1(D') \subset D'$

and $\boxed{f_1|_{\partial D'} = f_1|_{\partial D} = \mathrm{id}.}$

IDEA!




we'll homotope $D'$.

Say $g: D^n \to D^n$ s.th $g|_{\partial D^n} = \mathrm{id}$.

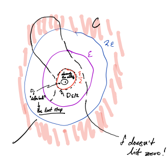
Then say $G: I \times D^n \to D^n$, by the rule

$G(x,t) = xt + (1-t)g(x)$, a homotopy rel $\partial D^n$.

$f_1 \simeq f \simeq \mathrm{id}$, hence $\deg(f) = 1$. □

Pf. Let $\varepsilon > 0$ s.th $f(x) = x + r(x)$ for $\|x\| < 2\varepsilon$ with also $\|r(x)\| < \frac{1}{2}\|x\|$ for $\|x\| < 2\varepsilon$ (by Taylor's thm about 0 with $D_0 f = \mathrm{id}$). Now $F(x,t)$ is a homotopy defined by

$$F(x,t) = \begin{cases} f(x) & \|x\| \geq 2\varepsilon \\ f(x) - t\left(2 - \frac{\|x\|}{\varepsilon}\right)r(x) & \varepsilon \leq \|x\| \leq 2\varepsilon \\ f(x) - t\,r(x) & \|x\| \leq \varepsilon \end{cases}$$




$$F(x,1) = \begin{cases} f(x) & \|x\| \geq 2\varepsilon \\ x - \underbrace{\left(1 - \frac{\|x\|}{\varepsilon}\right)}_{\lambda_x} r(x) & \varepsilon \leq \|x\| \leq 2\varepsilon \\ x & \|x\| \leq \varepsilon \end{cases}$$

Observe $|\lambda_x| \leq 1$. Consider the ball at $\frac{\varepsilon}{2}$ radius.

$$\begin{aligned} \|x - \lambda_x r(x)\| &\geq \big|\, \|x\| - |\lambda_x|\,\|r(x)\| \,\big| \\ &\geq \|x\| - |\lambda_x|\,\|r(x)\| \\ &> 1/2\,\varepsilon \quad \text{by Taylor's thm.} \end{aligned}$$

Let $C = \underbrace{\overline{S^n - D_{\varepsilon/2}}}_{\text{cmpt}} \subset S^n$, so $f_1(C) \subset S^n$ is cmpt

thus $S^n \setminus f_1(C)$ is open

and $0 \in S^n \setminus f_1(C)$.

**Remark 1.7.4.** Let $f\colon S^n \to S^n$ be a smooth map. Suppose that $p$ is a regular value. Suppose that $f^{-1}(p) = q$. Up to precomposing $f$ by a rotation, we can assume that $p = q$. Again, by composing by rotations, we can assume that $p$ is the origin if we model $S^n$ by $\mathbb{R}^n \cup \{\infty\}$. We identify $T_0 S^n = \mathbb{R}^n \subseteq S^n$.

Let $Df_0$ be the derivative of $f$ at 0. Since 0 is a regular value, $Df_0\colon \mathbb{R}^n \to \mathbb{R}^n$ is surjective, and so is an isomorphism. Then, $f \circ (Df_0^{-1})^*$ satisfies the condition of the previous theorem, so has degree 1. Therefore,

$$\deg(f) = \deg((Df_0)^*) = \operatorname{sgn}(\det(Df_0)).$$

In terms of the original function, this is the sign of the Jacobian determinant at $q$, computed coordinate systems at $p$ and $q$ which differ from a rotation. We write let $d(f,q)$ be this sign.

**Corollary 1.7.5.** *Let $f\colon S^n \to S^n$ be a continuous map which is smooth on $f^{-1}(U)$ for $U$ a neighborhood of $p$. Suppose that $p$ is a regular value and let*

$$f^{-1}(p) = \{q_1, \ldots, q_k\}.$$

*Then,*

$$\deg(f) = \sum_{i=1}^{k} d(f, q_i).$$

*Proof.* By the inverse function theorem, there is a disk $D$ containing $p$ such that $f^{-1}(D)$ is a disjoint union of disks $D_i$ around $q_i$ and each $D_i$ maps diffeomorphically onto $D$ under $f$. Consider the composite

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "g"'] & S^n \arrow[d, "g'"] \\ \bigvee_i D_i/\partial D_i \arrow[r, "h"] & S^n \end{tikzcd}$$

where $g$ sends every point outside of the interior of $f^{-1}(D)$ to a point, and $g'$ maps every point outside of the interior of $D$ to a point. The map $h$ is the induced map.

**Exercise 1.7.6.** Use additivity and the fact that

$$\widetilde{h}\left(\bigvee_i D_i/\partial D_i\right) \cong \bigoplus_i \widetilde{h}(D_i/\partial D)$$

to prove that

$$\deg(f) = \deg(g' \circ f) = \deg(h \circ g) = \sum_{i=1}^{k} d(f, q_i).$$

□

**Corollary 1.7.7.** *The degree of the $k$-fold cover $S^1 \to S^1$ is $k$.*

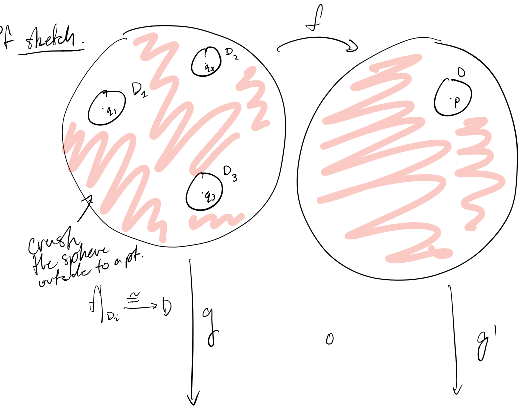
<u>CORO</u> Let $f: S^n \to S^n$ be cts map. Let $p \in S^n$ and spose for open nbhd $\mathcal{U} \ni p$, $f: f^{-1}(\mathcal{U}) \to \mathcal{U}$ is smooth and $p$ is a regular value. Say $f^{-1}(p) = \{q_i\}_1^k$. By rotations, we compute the deg of the derivative(s) to find that $\deg(f) = \sum_{i=1}^k \operatorname{sgn}(\det(D_{q_i} f))$

?

<u>Q</u> We can choose any regular value?

<u>A</u> So long as the nbhd is one on which $f$ is smooth.

Pf <u>sketch</u>.




$$D_i/\partial D_i \hookrightarrow D_1/\partial D_1 \vee D_2/\partial D_2 \vee D_3/\partial D_3 \longrightarrow D/\partial D$$

compute locally

then show $\tilde{h}(\bigvee_i D_i/\partial D_i) \cong \bigoplus_i \tilde{h}(D_i/\partial D_i)$.

<u>WTS</u> $\deg(f) = \deg(g' \circ f)$.

# EXERCISES (degree maps)

For $A \in GL(\mathbb{R}^n)$, verify $A^*$ is continuous at $\infty$; take an open set $\mathcal{U} \ni \infty$ and pull it back to the complement of a cmpt set.

Relate homogeneous manifolds to degrees of maps?

We want a properly "discontinuous action" of $\mathbb{Z}_p$ on $S^{2n-1}$. So consider the map $(z_1, \dots, z_n) \longmapsto (e^{2\pi i p} z_1, \dots, e^{2\pi i p} z_n)$ [Bre93, IV.14.2]

We have a "p-fold" covering $S^{2n-1} \to L$ of its "orbit space".

We could also have a "free and proper" action of $S^1$ on $S^{2n-1}$ via the map $(z_1, \dots, z_n) \longmapsto (e^{2\pi i t} z_1, \dots, e^{2\pi i t} z_n)$. [Lee03, Prob 21-3]

- What's the analogue of a p-fold covering for a free action?
- Main differences between the orbit spaces?

# READ

‖ **Proposition 2.30.** $\deg f = \sum_i \deg f \,|\, x_i$.

Let $\mathbb{RP}^2$ be the space obtained by either identifying antipodal points on the boundary $\partial D^2 = S^1$ of $D^2$ or by attaching a 2-cell to the cellular complex $K^{(0)} = \{point\}$ and $K^{(1)} = \{ \circlearrowleft \}$ with a degree 2 map. That is, $K^{(2)} = K^{(2+n)}$ is the pushout of the diagram where $f_{\partial\sigma}$ has degree 2.

$$\begin{tikzcd} S^1 \arrow[r, "f_{\partial\sigma}"] \arrow[d] & K^{(1)} \arrow[d] \\ D^2 \arrow[r, "f_\sigma"] & K^{(2)} \end{tikzcd}$$

We'll compute reduced homology with Mayer-Vietoris.

Let $A = \mathrm{Int}\, D^2$ be the open 2-cell in $\mathbb{RP}^2$, find a point $p$ in $A$, and let $B = D^2 \setminus \{p\}$ be the punctured 2-cell. Observe $\mathbb{RP}^2 = \mathrm{Int}\, A \cup \mathrm{Int}\, B$.

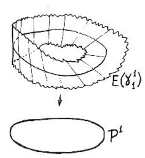
By Mayer-Vietoris, there's a l.e.s. in reduced homology

$$\cdots \longrightarrow \tilde H_n(A) \oplus \tilde H_n(B) \longrightarrow \tilde H_n(\mathbb{RP}^2) \xrightarrow{\delta} \tilde H_{n-1}(A\cap B) \longrightarrow \tilde H_{n-1}(A) \oplus \tilde H_{n-1}(B) \longrightarrow \cdots$$

First for homotopy equivalences. $A$ is an open ball so deform retracts to $\{p\}$.

$B$ is an open Möbius band, and deforms to $S^1$.

As well, $A \cap B$ deform retracts to $S^1$.




$$\tilde H_*(A) = 0$$
$$\tilde H_*(B) = \tilde H_*(A\cap B) = \{\mathbb{Z} \text{ if } * = 1, \text{ else } 0\}.$$

Our l.e.s. becomes trivial above degree 2, so we consider

$$\begin{tikzcd} \tilde H_2(A) \oplus \tilde H_2(B) \arrow[r] \arrow[d, "\cong"] & \tilde H_2(\mathbb{RP}^2) \arrow[r, "\delta"] \arrow[d, "\cong"] & \tilde H_1(A\cap B) \arrow[r] \arrow[d, "\cong"] & \tilde H_1(A) \oplus \tilde H_1(B) \arrow[r] \arrow[d, "\cong"] & \tilde H_1(\mathbb{RP}^2) \arrow[r, "\delta"] \arrow[d, "\cong"] & \tilde H_0(A\cap B) \arrow[d, "\cong"] \\ 0 \arrow[r] & \tilde H_2(\mathbb{RP}^2) \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r] & \tilde H_1(\mathbb{RP}^2) \arrow[r] & 0 \end{tikzcd}$$

Fix generators in $\tilde H_1(A\cap B)$ and $\tilde H_1(B)$ by letting $\gamma$ be a loop in $A \cap B$ winding once around $\{p\}$ in the mathematically positive direction and letting $b$ in $B$ be the path starting at $K^{(0)}$ and traversing half way around $K^{(1)}$ in the math. pos. direction. The inclusion $i_{A\cap B} : A \cap B \longrightarrow B$ induces $H_1(i_{A\cap B})[\gamma] = 2[b]$.

Or, follow Bredon to determine the degree map of the attaching $f_\sigma$.

We conclude
$$\tilde H_2(\mathbb{RP}^2) \cong \ker(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}) = 0,$$
$$\tilde H_1(\mathbb{RP}^2) \cong \mathrm{coker}(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}) = \mathbb{Z}/(2),$$
$$\tilde H_*(\mathbb{RP}^2) \cong 0 \text{ for } * \neq 1.$$

**Proposition 1.7.8.** *There is an isomorphism*

$$\tilde h_p(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}/2 & p = 1 \\ 0 & \textit{otherwise} \end{cases}$$

*Proof.* Let $\mathbb{R}P^2$ be constructed by attaching a disk $D^2$ to $S^1$ along the degree 2 map. Let $D$ be the disk of radius 1/2 in $D^2$ and let $U = \mathbb{R}P^2 - \{0\}$ where 0 is the center of $D^2$.

There's a long exact sequence

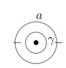
$$\begin{tikzcd} h_{p+1}(\mathbb{R}P^2, U) \arrow[r] & \tilde h_p(U) \arrow[r] & \tilde h_p(\mathbb{R}P^2) \arrow[r] & h_p(\mathbb{R}P^2, U) \\ h_{p+1}(\mathbb{R}P^2 - S^1, U - S^1) \arrow[r] \arrow[u, "\cong"] & \tilde h_p(U - S^1) \arrow[r] \arrow[u] & \tilde h_p(\mathbb{R}P^2 - S^1) \arrow[r] \arrow[u] & h_p(\mathbb{R}P^2 - S^1, U - S^1) \arrow[u, "\cong"] \\ h_{p+1}(D, \partial D) \arrow[r] \arrow[u, "\cong"] & \tilde h_p(\partial D) \arrow[r] \arrow[u] & \tilde h_p(D) \arrow[r] \arrow[u] & h_p(D, \partial D) \arrow[u, "\cong"] \end{tikzcd}$$

Note that
$$U \simeq U - S^1 \simeq \partial D \simeq S^1.$$
If $p \geqslant 3$, then this gives $\tilde h_p(\mathbb{R}P^2) = 0$ and similarly if $p = 0$. So it remains to study the cases $p = 1, 2$.

$$\begin{tikzcd} & & h_2(D, \partial D) \arrow[r, "\cong"] \arrow[d, "\cong"] & \tilde h_1(\partial D) \arrow[d] & & & \\ 0 \arrow[r] & \tilde h_2(\mathbb{R}P^2) \arrow[r] \arrow[d, "="] & h_2(\mathbb{R}P^2, U) \arrow[r] \arrow[d, "\cong"] & \tilde h_1(U) \arrow[r] \arrow[d, "\cong"] & \tilde h_1(\mathbb{R}P^2) \arrow[r] \arrow[d, "="] & h_1(\mathbb{R}P^2, U) \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & \tilde h_2(\mathbb{R}P^2) \arrow[r] \arrow[d, "="] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "="] & \mathbb{Z} \arrow[r] \arrow[d, "="] & \tilde h_1(\mathbb{R}P^2) \arrow[r] \arrow[d, "\cong"] & h_1(\mathbb{R}P^2, U) \arrow[r] \arrow[d, "="] & 0 \\ & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 & \end{tikzcd}$$

Since $S^1 \cong \partial D \to U \simeq S^1$ is the multiplication by 2 map, we get the result. $\square$

1. Recall that we can express the real projective plane $\mathbb{R}P^2$ as the quotient space of $\mathbb{S}^2$ modulo antipodal points or as a quotient of $\mathbb{D}^2$:
$$\mathbb{R}P^2 \cong \mathbb{S}^2/\pm\mathrm{id} \cong \mathbb{D}^2/z \sim -z \text{ for } z \in \mathbb{S}^1.$$
We use the latter definition and set $X = \mathbb{R}P^2$, $A = X \setminus \{[0,0]\}$ (which is an open Möbius strip and hence homotopically equivalent to $\mathbb{S}^1$) and $B = \mathring{\mathbb{D}}^2$. Then
$$A \cap B = \mathring{\mathbb{D}}^2 \setminus \{[0,0]\} \simeq \mathbb{S}^1.$$
Thus we know that $H_1(A) \cong \mathbb{Z}$, $H_1(B) \cong 0$ and $H_2(A) = H_2(B) = 0$. We choose generators for $H_1(A)$ and $H_1(A \cap B)$ as follows:




Let $a$ be the path that runs along the outer circle in mathematical positive direction half around starting from the point $(1, 0)$. This is the generator for $H_1(A)$. Let $\gamma$ be the loop that runs along the inner circle in mathematical positive direction. This is the generator for $H_1(A \cap B)$; note that $A \cap B \simeq \mathbb{D} \setminus \{0\}$. Then the inclusion $i_{A\cap B} \colon A \cap B \to A$ induces
$$H_1(i_{A\cap B})[\gamma] = 2[a].$$
This suffices to compute $H_*(\mathbb{R}P^2)$ up to degree two because the long exact sequence is
$$\tilde H_2(A) \oplus \tilde H_2(B) = 0 \to \tilde H_2(X) \to \tilde H_1(A\cap B) \cong \mathbb{Z} \xrightarrow{\cdot 2} \tilde H_1(A) \cong \mathbb{Z} \to \tilde H_1(X) \to \tilde H_0(A\cap B) = 0.$$
On the two copies of the integers, the map is given by multiplication by two and thus we obtain:
$$H_2(\mathbb{R}P^2) \cong \ker(2\cdot\colon \mathbb{Z} \to \mathbb{Z}) = 0,$$
$$H_1(\mathbb{R}P^2) \cong \mathrm{coker}(2\cdot\colon \mathbb{Z} \to \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},$$
$$H_0(\mathbb{R}P^2) \cong \mathbb{Z}.$$
The higher homology groups are trivial, because there $H_n(\mathbb{R}P^2)$ is located in a long exact sequence between trivial groups.

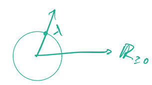
Beardsley wk 6

20190218 Top2 1

Complex projective space $\mathbb{C}P^n$

lines in $\mathbb{C}^{n+1}$ through the origin

$[z_0 : \dots : \underline{\underline{z_n}}]$ line through $(z_0, \dots, z_n) \in S^{2n+1} \subseteq \mathbb{C}^{n+1}$




Say $z_n \in \mathbb{C}$. There's $\lambda \in S^1$ s.th. $\lambda z_n \in \mathbb{R}_{\geq 0}$

Choose canonical reps $z_n \in \mathbb{R}_{\geq 0}$ if $z_n > 0$, then its unique.

$\{[z_0 : \dots : z_{n-1} : 0] \in \mathbb{C}P^n\} = \mathbb{C}P^{n-1}$

Say $w_0 = (z_0, \dots, z_{n-1})$. Let $p = (w, \underbrace{\sqrt{1-|w|^2}}_{= z_n})$

if $|w| = 1$, then $[p] \in \mathbb{C}P^{n-1}$

else $|w| < 1$ and ...

$D^{2n} - S^{2n-1} \cong \{(w, \sqrt{1-|w|^2}) : |w| < 1\}$ is an open $2n$ cell in $\mathbb{C}P^{n-1}$

1 cell in each even dimension

$$\begin{tikzcd} S^{2n-1} \arrow[r, "q"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{C}P^{n-1} \arrow[d] \\ D^{2n} \arrow[r] & \mathbb{C}P^n \end{tikzcd}$$

$q$: quotients by action of $S^1$ on a line

EX] $\mathbb{C}P^0 = \{*\}$, $\mathbb{C}P^1 = S^2$ TODO pushout

EX] "killing a generator in $\pi_1(X)$"

Say $X$ is a top space.

Can I kill generators in $X$?

No, but I can pushout to $X \cup_{S^1} D^2$, a CW complex s.th. I've space to do homotopy.

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, hook] & X \arrow[d] \\ D^2 \arrow[r] & X \cup_{S^1} D^2 \end{tikzcd}$$

**Complex Projective Space**

Similarly, $\mathbb{C}P^n$, which is the space of complex lines in $\mathbb{C}^{n+1}$. It is the quotient space of $\mathbb{C}^{n+1} - \{0\}$ by the action of $\mathbb{C}^\times$, or equivalently, the quotient of $S^{2n+1} \subseteq \mathbb{C}^n$ by the action of $S^1$,

$$(z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n), \quad \lambda \in S^1 = \{x \in \mathbb{C} : |x| = 1\}.$$

Points of $\mathbb{C}P^n$ are denoted by $[z_0 : \dots : z_n]$. Note that $\mathbb{C}P^0$ is a point. The points of the form $[z_0 : \dots : z_{n-1} : 0]$ form a copy of $\mathbb{C}P^{n-1} \subseteq \mathbb{C}P^n$. Any other point of $\mathbb{C}P^n$ is of the form $[z_0 : \dots : z_{n-1} : z_n]$ for $z_n \neq 0$ and $(z_0, \dots, z_n) \in S^{2n+1}$.

We show that, as a CW-complex, $\mathbb{C}P^n$ has one cell in each even dimension $0 \leqslant k \leqslant n$ and is obtained from $\mathbb{C}P^{n-1}$ by attaching a $2n$-cell along the quotient map $f_{\partial\sigma_{2n}} : S^{2n-1} \to \mathbb{C}P^{n-1}$.



For each complex number $z$, there is a complex number $\lambda \in S^1$ such that $\lambda z \in \mathbb{R}_{\geqslant 0} \subseteq \mathbb{C}$ and this number is unique if $z \neq 0$.

So, each point of $\mathbb{C}P^n$ has a representative $(z_0, \dots, z_n) \in S^{2n+1}$ such that $z_n \in \mathbb{R}_{\geqslant 0} \subseteq \mathbb{C}$, and the representative is unique if $z_n \neq 0$.

Let $w = (z_0, \dots, z_{n-1})$. Then, $(w, z_n) \in S^{2n+1}$, so $|w| \leqslant 1$ and

$$z_n = \sqrt{|z_n|^2} = \sqrt{1 - |w|^2}.$$

So, each point in $\mathbb{C}P^n$ has a representative of the form

$$(w, \sqrt{1 - |w|^2}), \quad w \in \mathbb{C}^n, \quad |w| \leqslant 1$$

The subspace of points for which $|w| = 1$ is precisely $\mathbb{C}P^{n-1}$. The subspace where the representatives are unique is

$$D^{2n} - S^{2n-1} \cong \{(w, \sqrt{1 - |w|}) : w \in \mathbb{C}^n, |w| < 1\} \subseteq \mathbb{C}P^n$$

That's the $2n$-cell.

Let $f_{\partial\sigma_{2n}} : S^{2n-1} \to \mathbb{C}P^{n-1}$ be the projection. Then $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by the following pushout:

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f_{\partial\sigma_{2n}}"] \arrow[d] & \mathbb{C}P^{n-1} \arrow[d] \\ D^{2n} \arrow[r] & \mathbb{C}P^n. \end{tikzcd}$$

In fact, $\mathbb{C}P^k \subseteq \mathbb{C}P^n$, given by the points of the form $[z_0 : \dots : z_k : 0 \dots : 0]$, is the $k$-skeleton.

**Definition 1.8.6.** The space $\mathbb{C}P^\infty$ is defined to be $\bigcup_{n \in \mathbb{N}} \mathbb{C}P^n$ with the weak topology.

2019-02-15

# CW Complexes (see Hatcher appendices)

**Recall** Say $A, X, Y$ are top spaces. For $f: A \to X$, $g: A \to Y$ (continuous maps), we have the "gluing" (pushout) $X \cup_A Y$ which is the space on the set $X \sqcup Y$

$$X \cup_A Y := X \sqcup Y / g(a) \sim f(a) \qquad i_X : X \longrightarrow X \cup_A Y$$
$$i_Y : Y \longrightarrow X \cup_A Y$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i_X"] \arrow[ddr, bend left, "h"] & \\
Y \arrow[r, "i_Y"'] \arrow[drr, bend right, "k"'] & X \cup_A Y \arrow[dr, dashed, "\exists! F"] & \\
& & Z
\end{tikzcd}$$

pushouts

univ propty

$$\begin{tikzcd}
A \arrow[r, bend left, "h \circ f"] \arrow[r, bend right, "k \circ g"'] & Z
\end{tikzcd}$$

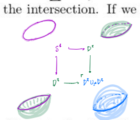
implies $\exists! F$

topology: $U \subseteq X \cup_A Y$ open iff $i_X^{-1}(U)$ open in $X$ and $i_Y^{-1}(U)$ open in $Y$.

[Rie15]

**Gluings.** Suppose $X$ and $Y$ are two sets that are not disjoint but share a common subset $A = X \cap Y$. We can form a new set by *gluing* $X$ to $Y$ along $A$. We denote this set by $X \cup_A Y$. Formally, $X \cup_A Y$ is the quotient of $X \coprod Y$ by the relation that identifies those points in $X$ and in $Y$ that lie in the intersection. If we label the inclusions, as in the diagram

(1.1)
$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "i"'] & Y \arrow[d] \\
X \arrow[r] & X \cup_A Y
\end{tikzcd}$$



we say that points in $X \cup_A Y$ are points in $X \coprod Y$ subject to the identifications $i(a) \sim j(a)$ for each $a \in A$.[2] We arrange inclusions into a square (1.1) to represent that the inclusions *commute*: a point $a \in A$ has the same image in $X \cup_A Y$ regardless of whether it is mapped first into $X$ and then into $X \cup_A Y$ or mapped first into $Y$ and then into $X \cup_A Y$.

Gluings are also called *pushouts*. The symbol "⌜" is included to remind the reader that the set in the lower right-hand corner is constructed as a gluing.

**Theorem** (the universal property of the gluing topology II). *Let $h: X \to Z$ and $k: Y \to Z$ be any continuous functions so that $hf = kg$. Then there exists a unique continuous function $j: X \cup_A Y \to Z$ so that its restriction to $X$ is the function $h$ and its restriction to $Y$ is the function $k$, i.e., so that the diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\bar{g}"] \arrow[ddr, bend left, "h"] & \\
Y \arrow[r, "\bar{f}"'] \arrow[drr, bend right, "k"'] & X \cup_A Y \arrow[dr, dashed, "\exists! j"] & \\
& & Z
\end{tikzcd}$$

*commutes.*

**Theorem 3.10.** *Let $X$ be any set equipped with functions $f_\alpha: X_\alpha \to X$ whose domains are topological spaces $X_\alpha$, and suppose the topology on $X$ is defined to be the finest topology so that these maps are continuous. Then $U \subset X$ is open if and only if $f_\alpha^{-1}(U) \subset X_\alpha$ is open for each $\alpha$.*

**Definition 3.3.** Given spaces and continuous functions $X \xleftarrow{i} A \xrightarrow{j} Y$, the *gluing topology* on $X \cup_A Y$ is the finest topology so that the maps $X \to X \cup_A Y \leftarrow Y$ are continuous.

**DEF** A C.W. complex $K$ is the following data:

- $K^{(0)}$ is a discrete set of points.
- Suppose $K^{(n-1)}$ has been constructed.
- we have a collection of continuous fns

$$f_{\partial\sigma} : S_\sigma^{n-1} \longrightarrow K^{(n-1)}$$

for $\sigma$ in some index set $I_n$.

$K^{(n)}$ is the pushout

the n-skeleton

$$\begin{tikzcd}
\coprod_{\sigma \in I_n} S_\sigma^{n-1} \arrow[r, "\sqcup f_{\partial\sigma}"] \arrow[d] & K^{(n-1)} \arrow[d] \\
\coprod_{\sigma \in I_n} D_\sigma^n \arrow[r, "\sqcup f_\sigma"'] & K^{(n)}
\end{tikzcd}$$

attaching maps

characteristic maps

$K = \bigcup_{n \geq 0} K^{(n)}$ with the weak topology

$U \subseteq K$ is open iff $U \cap K^{(n)}$ is open for all n

open with respect to the inclusions... eg.

**Theorem 3.7** (theorem-schema F[4]). *Consider a family of spaces $X_\alpha$ and a set $X$ given the finest topology so that certain maps $X_\alpha \to X$ are continuous. Then if $Z$ is any space, a function $X \to Z$ is continuous if and only if the composite maps $X_\alpha \to X \to Z$ are continuous.*

**DATA** $K^{(0)}$ the zero skeleton and the later attaching maps

$\{f_{\partial\sigma} : \sigma \in I_n\}_{n \geq 0}$

[EJ11]

**Definition 1.4.19.** Let C be an Abelian category. Then a *pushout* of the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \\
C &
\end{tikzcd}$$

in C is an object $D$ together with morphisms $h : B \to D$ and $k : C \to D$ such that $kg = hf$ and if

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\
C \arrow[r, "f'"'] & D'
\end{tikzcd}$$

is any commutative diagram in C, then there is a unique morphism $D \to D'$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \arrow[ddr, "g'"] & \\
C \arrow[r, "k"'] \arrow[drr, "f'"'] & D \arrow[dr] & \\
& & D'
\end{tikzcd}$$

is commutative.

The diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
C \arrow[r, "k"'] & D
\end{tikzcd}$$

in the above is called a *pushout diagram*.

**EX** $X \vee Y$ is the pushout

$$\begin{tikzcd}
* \arrow[r, "*"] \arrow[d, "\partial"'] & X \arrow[d] \\
Y \arrow[r] & X \cup_* Y
\end{tikzcd}$$

If $K$ has a finite number of cells, then we say $K$ is a finite CW-complex.

A subcomplex is a CW complex built from a subset of "the cells" of $K$.

**Q** what are the characterizations of the topology on $K$?

**A** $U \subseteq K$ is open iff $U \cap K_\sigma$ is open, for all $\sigma \in I_n$.
$f_\sigma(D_\sigma^n)$

**Example 1.4.20.**

1. In the above, if we set $D = (C \oplus B)/K$ where $K = \{(g(a), -f(a)) : a \in A\}$, and let $h(b) = (0, b) + K$, $k(c) = (c, 0) + K$, then $D$ with morphisms $h, k$ is a pushout.

# § REAL PROJECTIVE SPACE

20190301 (reordered)

DEMO] $C_\bullet^{CW}(\mathbb{R}P^n)$ is computed.

$\partial_k = (1+(-1)^k)$ for all $k$, no free groups above $n$.

Let $T : S^{k-1} \longrightarrow S^{k-1}$. Define $f_{T\sigma_k} = Tf_{\sigma_k}$. For the reflection.

$S^n$ the CW comp 2 cells in each dim $\sigma_k$ and $T\sigma_k$.

$$\begin{tikzcd}
\partial I^k_{\sigma_k} \cup \partial I^k_{T\sigma_k} \arrow[r, "f_{\partial\sigma_k} \cup f_{\partial T\sigma_k}"] \arrow[d] & S^k_{\sigma_k} \arrow[d] \\
I^k_{\sigma_k} \cup I^k_{T\sigma_k} \arrow[r, "f_{\sigma_k} \cup f_{T\sigma_k}"] & S^{k+1}_{\sigma_k}
\end{tikzcd}$$

Having $p_{\sigma_k} f_{\sigma_k} = \gamma_k : I^k \xrightarrow{\gamma_1 \wedge \cdots \wedge \gamma_1} S^k$ defining $p_{\sigma_k}$,

and also $p_{T\sigma_k} f_{T\sigma_k} = \gamma_k$, we conclude $p_{\sigma_k} = p_{T\sigma_k} \circ T$.

So also $p_{\sigma_k} \circ T = p_{T\sigma_k}$. We need the boundary map for

the caps $\partial\sigma_k$ and $\partial T\sigma_k$. So go on...

$$\partial\sigma_k = \deg(p_{\sigma_{k-1}} f_{\partial\sigma_k})\,\sigma_{k-1} + \deg(p_{T\sigma_k} f_{\partial\sigma_k})\,T\sigma_{k-1}$$

$$\partial T\sigma_k = \deg(p_{\sigma_{k-1}} f_{\partial T\sigma_k})\,\sigma_{k-1} + \deg(p_{T\sigma_k} f_{\partial T\sigma_k})\,T\sigma_{k-1}.$$

We find by symmetry

$p_{T\sigma_{k-1}} \circ f_{\partial T\sigma_k} = p_{\sigma_{k-1}} \circ f_{\partial\sigma_k}$ Likewise for

$p_{\sigma_{k-1}} \circ f_{\partial T\sigma_k} = p_{T\sigma_{k-1}} \circ f_{\partial\sigma_k}$. Fix $\deg(p_{\sigma_{k-1}} \circ f_{\partial\sigma_k}) = 1$.

TFDC
$$\begin{tikzcd}
\partial I^k_{\sigma_k} \arrow[r, "f_{\partial\sigma_k}"] & S^{k-1} \arrow[r, "p_{\sigma_{k-1}}"] \arrow[d, "T"'] & S^{k-1} \\
& S^{k-1} \arrow[ur, "p_{T\sigma_{k-1}}"'] &
\end{tikzcd}$$

Because $1 = \deg(p_{\sigma_{k-1}} f_{\partial\sigma_k}) = \deg(p_{T\sigma_{k-1}} \circ T \circ f_{\partial\sigma_k}) = \deg(T)\deg(p_{T\sigma_{k-1}} f_{\partial\sigma_k})$

We have $\deg(T\sigma_{k-1}, f_{\partial\sigma_k}) = \deg(T) = (-1)^k$. $\square$

IDEA] This CW structure for $\mathbb{R}P^n$ is equivariant under the antipodal action.

Define $\pi_\Delta : S^n \to \mathbb{R}P^n$ s.th. $\pi_\Delta(\sigma_k) = \pi_\Delta(T\sigma_k) = \tau_k$.

Then $\pi_\Delta$ is a cellular map, so a chain map, thence ... (?)

# Spheres as CW complexes 20190215

EX| $S^n = D^n / \partial D^n$

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * = K^{(0)} = \cdots = K^{(n-1)} \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd}$$

EX| $S^n = \{(x_0, \dots, x_n) : |x| = 1\} \subset \mathbb{R}^{n+1}$.

$\cup$

$S^k = \{(x_0, \dots, x_k, 0, 0, \dots, 0) \in S^n\}$

$k$-cell $\{\sigma_k, T_{\sigma_k}\}$ $T : S^n \to S^n$ antipodal map

$K_{\sigma_k} = D^k_+$ $K_{T_{\sigma_k}} = D^k_-$.

Say we have $S^{k-1}$.

$$\begin{tikzcd} S^{k-1}_{\sigma_k} \cup S^{k-1}_{T\sigma_k} \arrow[r, "\mathrm{id} \sqcup T"] \arrow[d] & S^{k-1} = (S^n)^{(k-1)} \arrow[d] \\ D^k_{\sigma_k} \sqcup D^k_{T\sigma_k} \arrow[r] & S^k = (S^n)^{(k)} \end{tikzcd}$$

$S^\infty = \bigcup_{k \geq 0} S^k$

TODO, verify closure finiteness in $S^\infty$

This procedure allows the quotient by the action of the antipodal map gives a cell structure for $\mathbb{R}P^n$ s.th. there's only 1 cell in each dim.

EX| Say we've built $\mathbb{R}P^k$. Let $S^k \xrightarrow{q} \mathbb{R}P^k$. Attach another cell...

$$\begin{tikzcd} S^k \arrow[r] \arrow[d] & \mathbb{R}P^k \arrow[d] \\ D^{k+1} \arrow[r] & \mathbb{R}P^{k+1} \end{tikzcd}$$

$\mathbb{R}P^{k+1}$

DEF| Let $K, L$ be CW complexes. A cellular map is a continuous map $f : K \to L$ s.th. $f(K^{(n)}) \subseteq L^{(n)}$.

THM| Any map between CW complexes

$$\begin{tikzcd} S^k \arrow[rr] \arrow[dr] & & S^n \\ & * \arrow[ur] & \end{tikzcd}$$

$k < n$

homotopic to a cellular map

thence $\pi_k S^n = 0$.

pick your cw structure!

### 1.8.1 Some important examples of CW complexes

**The spheres**

The sphere $S^n$ has many different decompositions as a CW-complex. One example is to have $(S^n)^{(k)} = \{*\}$ for $0 \leqslant k < n$ and have one $n$-cell $\sigma_n$ attached by the only map $f_{\partial\sigma_n} : S^{n-1} \to *$:

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & (S^n)^{(n-1)} \cong * \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd}$$

Another CW-structure has two cells in each dimensions $0 \leqslant k \leqslant n$. The $k$-skeleton of $S^n$ can be taken to be

$$S^k = \{(x_0, \dots, x_k, 0, \dots, 0) \in S^n\}.$$

and the cells are

$$K_{\sigma_k} = D^k_+ = \{(x_0, \dots, x_k, 0, \dots, 0) \in S^n : x_k \geqslant 0\}$$
$$K_{T\sigma_k} = D^k_- = \{(x_0, \dots, x_k, 0, \dots, 0) \in S^n : x_k \leqslant 0\}.$$

Let the two cells be denoted by $\sigma_k$ and $T\sigma_k$ where $T$ is the antipodal map. Then the attaching maps are chosen so that $f_{T\sigma_k} = T \circ f_{\sigma_k}$.

With this model,

$$S^0 \subseteq S^1 \subseteq S^2 \dots$$

and

$$S^\infty = \bigcup_{n \geqslant 0} S^n$$

with the weak topology.

**Real Projective Space**

Recall that $\mathbb{R}P^n$ is the space of real lines in $\mathbb{R}^{n+1}$. It can be described as a quotient space of $\mathbb{R}^{n+1} - \{0\}$ by the action of $\mathbb{R}^\times$. Equivalently, it's the quotient of $S^n$ by the relation

$$(x_0, \dots, x_n) \sim T(x_0, \dots, x_n) = (-x_0, \dots, -x_n).$$

We denote the corresponding point of $\mathbb{R}P^n$ by $[x_0 : \dots : x_n]$. Note that $\mathbb{R}P^0$ is a point.

The space $\mathbb{R}P^n$ can be realized as the following CW-complex with one $k$-cell for each $0 \leqslant k \leqslant n$, obtained from $\mathbb{R}P^{n-1}$ by attaching one $n$-cell.

Note that $\mathbb{R}P^{n-1} \subseteq \mathbb{R}P^n$ as the subset of points of the form $[x_0 : \dots : x_{n-1} : 0]$. Each point $p \in \mathbb{R}P^n$ has a representative $(x_0, \dots, x_n) \in S^n$ with $x_n \geqslant 0$ and the representative is unique if the last coordinate is non zero.

For such a representative, letting $y = (x_0, \dots, x_{n-1})$, we have that $|y| \leqslant 1$ and

$$x_n = \sqrt{|x_n|^2} = \sqrt{1 - |y|^2}.$$



So, each point of $\mathbb{R}P^n$ can be represented as

$$(y, \sqrt{1 - |y|^2}), \quad y \in \mathbb{R}^n, \quad |y| \leqslant 1.$$

Those points for which $|y| = 1$ are precisely the points of $\mathbb{R}P^{n-1}$. There is a homeomorphism

$$D^n - S^{n-1} \cong \{(y, \sqrt{1 - |y|^2}) : y \in \mathbb{R}^n, |y| < 1\} \subseteq \mathbb{R}P^n.)$$

This is the open $n$-cell.

Let

$$f_{\partial\sigma_n} : S^{n-1} \to \mathbb{R}P^{n-1}$$

be the quotient map. The image of this map is those points represented by $(x_0, \dots, x_n)$ such that $x_n = 0$. Then $\mathbb{R}P^n$ is obtained as the pushout:

$$\begin{tikzcd} S^{n-1} \arrow[r, "f_{\partial\sigma_n}"] \arrow[d] & \mathbb{R}P^{(n-1)} \arrow[d] \\ D^n \arrow[r] & \mathbb{R}P^n \end{tikzcd}$$

The $k$-skeleton of $\mathbb{R}P^n$ is then $\mathbb{R}P^k \subseteq \mathbb{R}P^n$ realized as those points of the form

$$[x_0 : \dots : x_k : 0 : \dots : 0].$$

In fact, for this structure and the structure of $S^n$ with two $k$-cells for each $k$ described above, the quotient map $S^n \to \mathbb{R}P^n$ is a cellular map and we have a commutative diagram

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d, "\varphi^1"] & S^1 \arrow[r] \arrow[d, "\varphi^2"] & S^2 \arrow[r] \arrow[d, "\varphi^3"] & \dots \\ \mathbb{R}P^0 \arrow[r] & \mathbb{R}P^1 \arrow[r] & \mathbb{R}P^2 \arrow[r] & \dots \end{tikzcd}$$

**Definition 1.8.5.** The space $\mathbb{R}P^\infty$ is defined to be $\bigcup_{n \in \mathbb{N}} \mathbb{R}P^n$ with the weak topology.

DEF "Closure finiteness": Let $K$ be a CW complex. Each closed cell of $K$ is contained in a finite subcomplex of $K$. Any compact subset of $K$ is thus contained in a finite subcomplex. Proof: TODO (see Top1)

$S^{n-1}$ is compact, so $f_{\partial\sigma}(S^{n-1})$ is in a finite subcomplex.

### 1.8.2 Closure finiteness

We won't prove the following statement (See Prop 8.1, Thm 8.2 of Bredon):

**Proposition 1.8.7.** *Let $K$ be a CW-complex. Then*

- *Each cell of $K$ is contained in a finite sub-complex of $K$. (This is* **closure finite***.)*
- *Any compact subset of $K$ is contained in a finite subcomplex. In particular, for any $\sigma$, the image of $f_{\partial\sigma}$ is contained in a finite subcomplex.*

EX "Local homology of $M$ in $\mathrm{Diff}^n$ at $p$"
(Sam asked of choosing a generator for orientation was as in diff geo 1)

Say $M \in \mathrm{Diff}^n$. Fix $p \in M$. Let $U_p$ be a convex ball about $p$.

Then $H_n(M, M-p) \xleftarrow[ex]{\sim} H_n(D^n, D^n - 0) \overset{h.a.}{\simeq} H_n(D^n, S^{n-1}) \cong \mathbb{Z}$

"local homology of $M$ at $p$" excision homotopy axiom

Orientation is a means of choosing a generator that's consistent for all points in the nbhd of $U_p$

provisional DEF

An orientation is a fn which assigns to each $p \in M$ a generator $\xi_p \in H_n(M, M \setminus p)$ sth ... $\forall p \in M$, $\exists B_p \ni p$ open sth $\forall q \in B_p$

$$\begin{array}{c} \xi \in H_n(M, M \setminus B_p) \cong \mathbb{Z} \\ \downarrow \qquad \downarrow \\ \xi_q \in H_n(M, M \setminus q) \end{array}$$

Here is an application of the machinery we have developed, a classical result of Brouwer from around 1910 known as 'invariance of dimension', which says in particular that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ if $m \neq n$.

**Theorem 2.26.** *If nonempty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.*

**Proof**: For $x \in U$ we have $H_k(U, U - \{x\}) \approx H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ by excision. From the long exact sequence for the pair $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ we get $H_k(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \approx \widetilde{H}_{k-1}(\mathbb{R}^m - \{x\})$. Since $\mathbb{R}^m - \{x\}$ deformation retracts onto a sphere $S^{m-1}$, we conclude that $H_k(U, U - \{x\})$ is $\mathbb{Z}$ for $k = m$ and $0$ otherwise. By the same reasoning, $H_k(V, V - \{y\})$ is $\mathbb{Z}$ for $k = n$ and $0$ otherwise. Since a homeomorphism $h: U \to V$ induces isomorphisms $H_k(U, U - \{x\}) \to H_k(V, V - \{h(x)\})$ for all $k$, we must have $m = n$. ◻

Generalizing the idea of this proof, the **local homology groups** of a space $X$ at a point $x \in X$ are defined to be the groups $H_n(X, X - \{x\})$. For any open neighborhood $U$ of $x$, excision gives isomorphisms $H_n(X, X - \{x\}) \approx H_n(U, U - \{x\})$ assuming points are closed in $X$, and thus the groups $H_n(X, X - \{x\})$ depend only on the local topology of $X$ near $x$. A homeomorphism $f: X \to Y$ must induce isomorphisms $H_n(X, X - \{x\}) \approx H_n(Y, Y - \{f(x)\})$ for all $x$ and $n$, so the local homology groups can be used to tell when spaces are not locally homeomorphic at certain points, as in the preceding proof. The exercises give some further examples of this. [Hat02]

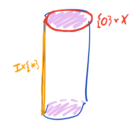
CW complex conventions $\quad * \in X \quad$ basepoint

DEF| For $X, Y \in \mathrm{Top}_*$,

"wedge" $\quad X \vee Y := X \cup_* Y \cong \underbrace{X \times \{*\} \cup \{*\} \times Y} \subseteq X \times Y$

"smash" $\quad X \wedge Y := X \times Y / X \vee Y$

EX| $I = [0,1] = I^1$ basept $0 \in I$. Then $I \wedge X$ is ...




EX| $\underbrace{S^1 \wedge X}$ but forget to crush the basepoint of $X$ is $\underbrace{\Sigma X}$

"reduced" suspension of $X$ $\quad$ "unreduced"

If the basept has a deform retractible basept nbhd, then $\Sigma X \cong S^1 \wedge X$.

PROP| $S^p = \underbrace{S^1 \wedge S^1 \wedge \dots \wedge S^1}_{p \text{ times.}} \xleftarrow[\gamma_p]{\gamma^1 \wedge \dots \wedge \gamma^1} \underbrace{I^1 \times \dots \times I^1}_{p \text{ times}} := I^p.$

assoc operation up to homeo

Q| What's $\gamma_p = \gamma_1 \wedge \dots \wedge \gamma_1$ as a set theoretic function?

in Bre93 for 2pgs

A| Clearly $I^1 \xrightarrow{\gamma_1} S^1 = I^1/\partial I^1$

By induction $I^1 \times I^1 \xrightarrow{\gamma_1 \times \gamma_1} S^1 \times S^1$

then quotient $S^1 \times S^1$ by $S^1 \vee S^1$

$$\begin{tikzcd} I^1 \times I^1 \arrow[r, "\gamma_1 \times \gamma_1"] \arrow[dr, "\gamma_1 \wedge \gamma_1"'] & S^1 \times S^1 \arrow[d] \\ & S^1 \wedge S^1 \end{tikzcd}$$

What was meant to be shown?

notice $\partial I^{p+1} \cong S^p$

$$\begin{tikzcd} I^p \arrow[r, "\gamma_p"] \arrow[d] & S^p \\ I^p/\partial I^p \arrow[ur, "\tilde{\gamma}_p", "\cong"'] \end{tikzcd}$$

induced how?

Observe $I^p \times I^q = I^{p+q}$

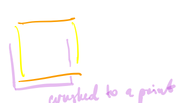
$\quad \big\downarrow \gamma_{p+q} = \gamma_p \wedge \gamma_q \quad$ induces $\partial \gamma_{p+1}$ (how?)

$S^p \wedge S^q = S^{p+q}$

TODO| Define $\partial \gamma_{p+1}$

$$\begin{tikzcd} \partial I^{p+1} = \partial I^1 \times I^p \cup I^1 \times \partial I^p \arrow[r] \arrow[rr, phantom, bend left, "\partial\gamma_{p+1} \text{ homotopy equiv.}"] & S^0 \wedge S^p \cong S^p \\ I^1 \times I^p \arrow[u, phantom, "\cap"] \arrow[r, "1 \wedge \gamma_p"] & I^1 \wedge S^p \arrow[u, phantom, "\cap"] \end{tikzcd}$$

$(1, x) \longmapsto x$




Now we need to pick generators.

Fix $[I^0] \in H_0(I^0) \cong \mathbb{Z}$

Spose we have fixed a generator $[I_p] \in H_p(I_p, \partial I_p) \overset{?}{\cong} \underbrace{H_p(S_p, \partial S_p)}_{\text{by}} \cong \mathbb{Z}$

$[S^p] = (\gamma_p)_*([I_p]) \in H_p(S^p, *)$

prove with some excision

Having constructed $\partial\gamma_{p+1}$ as a h.e. $\partial I^{p+1} \to S^p$, $(\partial\gamma_{p+1})^* : H_p(\partial I^{p+1}, *) \xrightarrow{\sim} H_p(S^p, *)$, with $[\partial I^{p+1}]$ def'd so that $(\partial\gamma_{p+1})_*([\partial I^{p+1}]) = [S^p]$. **Define** $[I^{p+1}]$ so that $\delta_*([I^{p+1}]) = [\partial I^{p+1}]$.

There's a Bockstein connecting from here so that we've a inductively defined orientation by ...

### 1.8.3 Conventions for CW Complexes

First, let's fix some things. First, we work with singular homology $H_*$ with coefficients $\mathbb{Z}$, but some of this works more generally.

TODO! By 20190220 || 9. Conventions for CW-Complexes

**Definition 1.8.8.** If $X$ and $Y$ are pointed spaces, we let their base point be $*$. Then wedge product is

$$X \vee Y = X \cup_* Y \cong X \times \{*\} \cup \{*\} \times Y.$$

The smash product is

$$X \wedge Y := X \times Y/(X \vee Y)$$

If $f\colon X \to Z$ and $g\colon Y \to W$ are maps of spaces which preserve the base points, then there is an induced map

$$f \wedge g\colon X \wedge Y \to Z \wedge W.$$

Let $I^1 = [0,1]$ with base point $\{0\}$ and $S^1 = I^1/\partial I^1 = I^1/\{0,1\}$ with base point $\partial I^1$. Let $\gamma_1\colon I^1 \to S^1$ be the quotient. Let

$$I^p = \underbrace{I^1 \times \ldots \times I^1}_{p}$$

and

$$S^p = \underbrace{S^1 \wedge \ldots \wedge S^1}_{p}$$

and

$$\gamma_p = \gamma_1 \wedge \ldots \wedge \gamma_1\colon I^p \to S^p.$$

Then,

$$\gamma_{p+q} = \gamma_p \wedge \gamma_q\colon I^p \times I^q \to S^p \wedge S^q = S^{p+q}.$$

So we have $I^p \times I^q = I^{p+q}$ and $\partial I^p \cong S^{p-1}$. In fact, we fix a homotopy equivalence

$$\begin{tikzcd} \partial I^{p+1} \arrow[r, equal] \arrow[rrr, bend left, ":=\partial\gamma_{p+1}"] & \partial I^1 \times I^p \cup I^1 \times \partial I^p \arrow[r, dotted] \arrow[d, "\subseteq"] & S^0 \wedge S^p \arrow[r, "\cong"'] \arrow[d, "\subseteq"] & S^p \\ I^{p+1} \arrow[r, equal] & I^1 \times I^p \arrow[r, "1 \wedge \gamma_p"] & I^1 \wedge S^p \end{tikzcd}$$

We also fix a homeomorphism

$$\begin{tikzcd} I^p \arrow[r, "\gamma_p"] \arrow[d] & S^p \\ I_p/\partial I^p \arrow[ur, "\tilde{\gamma}_p"'] \end{tikzcd}$$

We choose generators of some relevant homology groups.

- Fix $[I^0] \in H_0(I^0) = H_0(I^0, \partial I^0) \cong \mathbb{Z}$ a generator and suppose that $[I^p] \in H_p(I^p, \partial I^p)$ has been determined.
- Let $[S^p] \in H_p(S^p, *)$ be given by $[S^p] = (\gamma_p)_*([I^p])$ where
$$(\gamma_p)_*\colon H_p(I^p, \partial I^p) \to H_p(S^p, *).$$
- Fix a generator $[\partial I^{p+1}] \in H_p(\partial I^{p+1}, *)$ such that
$$[S^p] = (\partial\gamma_{p+1})_*([\partial I^{p+1}]).$$
- Let $[I^{p+1}] \in H_{p+1}(I^{p+1}, \partial I^{p+1})$ be chosen so that
$$\delta_*([I^{p+1}]) = [\partial I^{p+1}].$$

Now that we have fixed all these generators, we can speak of the degree of a map between the relevant pairs. For example, a map $f\colon \partial I^p \to S^p$ has a degree defined by

$$f_*([I^p]) = \deg(f)[S^p].$$

Recap we've generators

$[I^p]$, $[S^p]$, $[\partial I^p]$

$H_p(I^p, \partial I^p)$

$H_p(S^p, *)$

$H_p(\partial I^p, *)$

consistent with orientation.

Suppose $K$ is a CW complex, then  20190220

$$I^n_\sigma \xrightarrow{f_\sigma} K^{(n)} \longrightarrow K^{(n)}/K^{(n-1)} = \bigvee_\mu I^n_\mu / \partial I^n_\mu \xrightarrow{g_\mu} S^n_\sigma$$

define $p$ for each $n$

Say $\sigma$ is an $n$-cell, $\tau$ is an $n-1$ cell

$$\partial I^n_\sigma \xrightarrow{f_{\partial\sigma}} K^{(n-1)}/K^{(n-2)} \xrightarrow{p_\tau} S^{n-1}_\tau$$

$\deg(p_\tau f_{\partial\sigma}) =: [\tau:\sigma]$

$(p_\tau f_{\partial\sigma})_*([\partial I^n_\sigma]) = [\tau:\sigma][S^{n-1}_\tau]$

## Rel Homology

Let $K$ be a CW complex, let $A \subseteq K$ be a subcomplex. computable version

Denote $K^{(n)}_A = K^{(n)} \cup A$ for $n \geq 0$. TODO define $C_*(K,A)$ "cellular chain complex"

$C^n(K,A) := \mathbb{Z}\{n\text{-cells of } K \text{ that are } \underline{\text{not}} \text{ in } A\}$

Eg  $C_1(K,A) \cong \mathbb{Z}$

We define on each of the generators $\partial_n(\sigma) = \sum_\tau [\tau:\sigma]\tau$ ($\sigma$: $n$-cell, $\tau$: $n-1$ cells)

$C_n(K,A) \longrightarrow C_{n-1}(K,A)$

$$\partial I^n_\sigma \xrightarrow{f_{\partial\sigma}} K^{(n-1)}/K^{(n-2)} \xrightarrow{p_\tau} S^{n-1}_\tau$$

$$K^{(n-1)}/K^{(n-2)} = \bigvee_\tau I^{n-1}_\tau / \partial I^{n-1}_\tau$$

**EXERCISE** Over $i \in I$, TFDC

$$\begin{tikzcd} S^m \arrow[r] \arrow[dr, "f_i"'] & \bigvee_i S^m \arrow[d, "p_i \text{ projection}"] \\ & S^n_i \end{tikzcd}$$

?

Show that $[S^m] \longmapsto \sum_i \deg(f_i)[S^n_i]$.

**EXERCISE** $C_1(K,A) \xrightarrow{\partial_1} C_0(K,A)$  $\partial_1$ acts on $\sigma$ how? Prove $y - x = \partial(\sigma)$. ✓

$\sigma \in C_1(K,A)$

 $C_1(K,A) \cong \mathbb{Z}$

then $\partial I^1_\sigma \longrightarrow \underbrace{K^{(0)}/K^{(-1)}}_{= K^{(0)} \cup \{*\}}$, $K^{(-1)} = \emptyset$, $\longrightarrow S^0$

$f_{\partial\sigma}(0) = x$, $f_{\partial\sigma}(1) = y$

then $\partial I^1_\sigma \longrightarrow \underbrace{K^{(0)}/K^{(-1)}}_{= K^{(0)} \cup \{*\}}$, $K^{(-1)} = \emptyset$, $\longrightarrow S^0_\tau$, with $\partial I^1_\sigma = \{0,1\}$

Boiled down $\partial I^1_\sigma \longrightarrow \{x, y, *\} \xrightarrow{p_x} \{0,1\}$

reminiscent of singular homology

"attaching 1-cells"

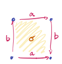
$p_x$: $x \mapsto 1$, $y \mapsto 0$, $* \mapsto 0$

$p_y$: $x \mapsto 0$, $y \mapsto 1$, $* \mapsto 0$

deg zero most confusing...

**DEFN** Cellular homology of $(K,A)$ is the homology of $C_*(K,A)$.
We'll show that cellular homology and singular hom are equiv.

**EX** We compute for a Klein bottle, $K$.



$K^{(0)} = \{*\}$
$K^{(1)} = \{a, b\}$
$K^{(2)} = \{\sigma\}$

$$\begin{tikzcd} S^1_\sigma \arrow[r, "a*b*a^{-1}*b"] \arrow[d] & S^1_a \vee S^1_b \arrow[d] \\ D^2_\sigma \arrow[r] & K^{(2)} \end{tikzcd}$$

We compute with $A = \emptyset$ and $K$ as given,

$$\longrightarrow C_3(K) \longrightarrow C_2(K) \longrightarrow C_1(K) \longrightarrow C_0(K) \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}\{\sigma\} \xrightarrow{\partial_2} \mathbb{Z}\{a,b\} \xrightarrow{0} \mathbb{Z}\{*\} \longrightarrow 0$$

why?

$\sigma \overset{\partial_2}{\longmapsto} 0 \cdot a + 2b$

worth less

Note it  $S^1_a$, $S^1_b$, crush $b$

$S^1_\sigma \xrightarrow{f_{\partial\sigma}} S^1_a \vee S^1_b \xrightarrow{p_a} S^1_a$, $g_{1a} = 0$, $\deg = 0$

$S^1_\sigma \xrightarrow{f_{\partial\sigma}} S^1_a \vee S^1_b \xrightarrow{p_b} S^1_b$, $g_{1b} = 0$, $\deg = 0$

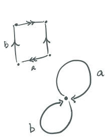
$\longrightarrow C_3(K) \longrightarrow C_2(K) \longrightarrow C_1(K) \longrightarrow C_0(K) \longrightarrow 0$

$0 \longrightarrow \mathbb{Z}\{\sigma\} \xrightarrow{\partial_2} \mathbb{Z}\{a,b\} \xrightarrow{0} \mathbb{Z}\{*\} \longrightarrow 0$ why?

$\sigma \overset{\partial_2}{\longmapsto} 0 \cdot a + 2b$

compute the kernels!

Torus $\mathbb{T}^2 \cong S^1 \times S^1$, $\mathbb{RP}^2$ at home.



$a * b * a^{-1} * b$

$K^{(2)} = \{e^2\}$, $K^{(1)} = \{a^1, b^1\}$, $K^{(0)} = \{e^0\}$.

Suppose $f_{\partial a^1}, f_{\partial b^1} : \partial I^1 \to K^{(0)}$, the same unique map.

Let $f_{\partial e^2} : \partial I^2 \longrightarrow K^{(1)}$ trace the path $a^1 * b^1 * a^1 * (b^1)^{-1}$.

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} \mathbb{Z}^{\oplus 2} \xrightarrow{[0\ 0]} \mathbb{Z} \longrightarrow 0$$

$\begin{bmatrix} 0 \\ 2 \end{bmatrix}[e^2] = \begin{bmatrix} 0 \\ 2e^2 \end{bmatrix}$,  $[0\ \ 0]\begin{bmatrix} a^1 \\ b^1 \end{bmatrix} = 0$

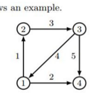
$k \geq 3$: $H_k(KB) = 0$, $H_2(KB) = 0$, $H_1(KB) = \mathbb{Z} \oplus \mathbb{Z}_2$, $H_0(KB) = \mathbb{Z}$

---

The node-arc incidence matrix of a directed graph with $n$ nodes and $m$ arcs is the $n \times m$ matrix $A$ with elements

$$A_{ij} = \begin{cases} -1 & \text{if arc } j \text{ leaves node } i \\ 1 & \text{if arc } j \text{ enters node } i \\ 0 & \text{otherwise.} \end{cases}$$

The figure shows an example.



$$A = \begin{bmatrix} -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

**10.5. Example.** Consider the torus $K = \mathbf{T}^2$ as the space arising from the unit square by identifying opposite sides. The obvious structure as a CW-complex is to let the corners be the unique 0-cell $x$, the edges of the square giving two 1-cells $a, b$ and the interior giving the single 2-cell $\sigma$. Thus $C_0(\mathbf{T}^2) \approx \mathbf{Z}$, generated by $x$, $C_1(\mathbf{T}^2) \approx \mathbf{Z} \oplus \mathbf{Z}$, generated by $a$ and $b$, and $C_2(\mathbf{T}^2) \approx \mathbf{Z}$, generated by $\sigma$. By the above remarks $\partial a = 0$ and $\partial b = 0$. The attaching map for $\sigma$ is the loop in the figure eight (formed by $a$ and $b$) running around $a$ then $b$ then $a^{-1}$ then $b^{-1}$. The composition $p_a f_{\partial\sigma}$ is this map followed by mapping the $a$ part of the figure eight around the circle by degree $\pm 1$, and the $b$ part to the base point. This is just the loop $a * \text{constant} * a^{-1} * \text{constant} \simeq \text{constant}$. Hence it has degree 0, and similarly for $p_b f_{\partial\sigma}$. Consequently, $[a:\sigma] = 0$ and $[b:\sigma] = 0$ so that $\partial = 0$ in degree 2, and thus in all degrees. Therefore $H_*(\mathbf{T}^2) \approx C_*(\mathbf{T}^2) \approx \mathbf{Z}, \mathbf{Z} \oplus \mathbf{Z}, \mathbf{Z}$ in degrees 0, 1, 2, respectively.

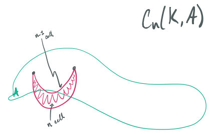
**10.6. Example.** The Klein bottle $\mathbf{K}^2$ can be constructed similarly to the torus but with the "$b$" edges identified with a flip. Thus the attaching map, up to orientation of cells, would be the loop $a * b * a^{-1} * b$. In this case the composition $p_b f_{\partial\sigma}$ becomes the loop, in $\mathbf{S}^1$, constant$*b*$constant$*b$ which has degree $\pm 2$. Thus the boundary map computation will give $[b:\sigma] = \pm 2$, where the sign depends on how one defines the attaching maps in detail. Thus $\partial\sigma = [a:\sigma]a + [b:\sigma]b = 0a \pm 2b = \pm 2b$. It then follows that $H_2(\mathbf{K}^2) = 0$, $H_1(\mathbf{K}^2) \approx \mathbf{Z} \oplus \mathbf{Z}_2$, and $H_0(\mathbf{K}^2) \approx \mathbf{Z}$.

Let $K$ be a CW complex, let $A \subseteq K$ be a subcomplex.
Denote $K_A^{(n)} = K^{(n)} \cup A$ for $n \geq 0$. TODO define $C_*(K, A)$
$C^n(K,A) := \mathbb{Z}\{$n-cells of $K$ that are not in $A\}$

(Theorem)

**ALT DEF** Say $H_*(-)$ is the singular homology functor. $C_n(K,A) \cong H_n(K_A^{(n)}, K_A^{(n-1)})$ with $\partial_n$ identified with

$$\begin{tikzcd} C_n(K,A) \cong H_n(K_A^{(n)}, K_A^{(n-1)}) \arrow[r, "\delta_*"] \arrow[rr, bend right, "B_n"'] & H_{n-1}(K_A^{(n-1)}) \arrow[r, "i_*"] & H_{n-1}(K_A^{(n-1)}) \arrow[r] & H_{n-1}(K_A^{(n-1)}, K_A^{(n-2)}) \cong C_{n-1}(K,A). \end{tikzcd}$$




$B_n$, the connecting hom for the l.e.s. of a triple $(K_A^{(n)}, K_A^{(n-1)}, K_A^{(n-2)})$ TODO

We want to argue that $H_*(K,A) \cong H_*(C_*(K,A))$.

20190222

free abelian gp of n-cells in $K$ not in $A$

**THM** $C_n(K,A) \cong H_n(K_A^{(n)}, K_A^{(n-1)})$

with boundary operator given by the blockstein conn. homo for triples...

**Lemma.** There's an isom $\bigoplus_{\sigma \notin A} H_*(I_\sigma^n, \partial I_\sigma^n) \xrightarrow{\cong} H_*(K_A^{(n)}, K_A^{(n-1)})$.

**Pf sketch.**

$$\begin{tikzcd}
H_*\left(\coprod_{\sigma \notin A} (I_\sigma^n, \partial I_\sigma^n)\right) \arrow[r, "\coprod_{\sigma \notin A} f_\sigma"', "\therefore \cong"] \arrow[d, "\cong \text{ by homotopy axiom (h.a.)}"] & H_*(K_A^{(n)}, K_A^{(n-1)}) \arrow[d, "\cong \text{ ``h.a.''}"] \\
H_*\left(\coprod_{\sigma \notin A} (I_\sigma^n, \partial I_\sigma^n \setminus \{p_0\})\right) \arrow[r, "\coprod_{\sigma \notin A} f_\sigma"] & H_*(K_A^{(n)}, K_A^{(n)} - \{f_\sigma(p_0)\}) \\
H_*\left(\coprod_{\sigma \notin A} (\mathrm{Int}(I_\sigma^n), \mathrm{Int}(I_\sigma^n) \setminus \{p_0\})\right) \arrow[u, "\cong \text{ excision (ex) (rel } \partial I_\sigma^n)"] \arrow[r, "\coprod_{\sigma \notin A} f_\sigma"] & H_*(K_A^{(n)} - K_A^{(n-1)}, K_A^{(n)} \setminus K_A^{(n-1)} \setminus \{f_\sigma(p_0)\}) \arrow[u, "e_*"]
\end{tikzcd}$$

$p_0 \in \mathrm{Int}(I_\sigma^n)$ take out the range of $p_0$

obviously an isom since the pairs are homeomorphic

try to quotient in your head

Wherefore $\underbrace{\bigoplus_{\sigma \notin A} H_*(I_\sigma^n, \partial I_\sigma^n)}_{\text{just } H_*(D^n, S^{n-1})} \xrightarrow{\cong} H_*(K_A^{(n)}, K_A^{(n-1)}) = \begin{cases} C_n(K,A) & * = n \\ 0 & * \neq n \end{cases}$

which means TFDC

$$\begin{tikzcd}
H_*(K_A^{(n)}, K_A^{(n-1)}) \arrow[r, "B_n"] & H_{*-1}(K_A^{(n-1)}, K_A^{(n-2)}) \\
\bigoplus_{\sigma \notin A} H_*(I_\sigma^n, \partial I_\sigma^n) \arrow[u, "\bigoplus (f_\sigma)_*"] \arrow[r, "\delta_*"] & \bigoplus_{\sigma \notin A} H_{*-1}(\partial I_\sigma^n) \arrow[u, "\bigoplus (f_{\partial\sigma})_*"']
\end{tikzcd}$$

(b/c $\beta_n$ the connecting homomorphisms are nat'l by assumption)

**Remark** | Above dimension $n$, $K^{(n)}$ has no homology, and the intuition is that only the $n+1$ skeleton matters. $\Big\}$ unpack "cellular maps" for singular homology

**Remark** | If $(X,A) = \bigcup_{n\ge 0}(X_n, A_n)$ with the weak topology and $(X_n, A_n) \xrightarrow[\text{inclusion}]{\subseteq} (X_{n+1}, A_{n+1})$, thus

$$H_*(X,A) \cong \operatorname{colim} H_*(X_n, A_n)$$
$$\cong \bigoplus H_*(X_n, A_n)/\sim \text{ generated by } [\![c]\!] \sim [\![i_*(c)]\!]$$

homology over the skeleton stops and stabilizes...

Pf in
- Bredon appendix
- May "Consice"

**Lemma** | If $i > n$ or $i < 0$, then $H_i(K_A^{(n)}, A) \cong 0$.
Also if $k \ge 1$, then $\boxed{H_n(K_A^{(n+1)}, A)} \xrightarrow{\cong} H_n(K^{(n+k)}, A)$.
(say $K$ is a finite CW complex, then $\boxed{H_n(K,A)} \cong H_n(K_A^{(n+1)}, A)$.) apparent.

**Pf.** Consider the LES for the triple $(K_A^{(n)}, K_A^{(n-1)}, A)$. Induct on $n$.

$H_i(K_A^{(0)}, A)$ is clear? $H(A,A) = 0$ $i > 0$ ? TODO

$H_{i+1}(K_A^{(n)}, K_A^{(n-1)}) \to H_i(K_A^{(n-1)}, A) \to \cancel{H_i(K_A^{(n}}$ ... TODO

We've shown $H_i(K_A^{(n)}, A) = 0$.

Moreover, shifting up indices, we find $\cancel{H_n(K_A^{(n+2)}, K_A^{(n+1)})}$ forces
$$H_n(K_A^{(n+1)}, A) \xrightarrow{\cong} H_n(K^{(n+2)}, A). \qquad \square$$

**Recall, our GOAL** | $H_*(K,A)$ is computed by cellular homology
$$H_n(K_A^{(n)}, K_A^{(n-1)}) \xrightarrow{\beta_n} H_{n-1}(K_A^{(n-1)}, K_A^{(n-2)}).$$

**THM** | $H_n(K,A) \cong \ker\beta_n / \operatorname{im}\beta_{n+1}$.

**Pf.** We've just demonstrated $H_n(K,A) \cong H_n(K_A^{(n+1)}, A)$.
**WTS** $H_n(K_A^{(n+1)}, A) \cong H_n(K_A^{(n+1)}, K_A^{(n-2)})$. Consider LES for the triple $(K_A^{(n+1)}, K_A^{(n-2)}, A)$. So $H_n(K_A^{(n+1)}, K_A^{(n-2)}) \cong H_n(K,A)$.
Lastly we aim to show $H_n(K_A^{(n+1)}, K_A^{(n-2)}) \cong \ker\beta_n/\operatorname{im}\beta_{n+1}$.

Vertically, the LES for $(K_A^{(n)}, K_A^{(n-1)}, K_A^{(n-2)})$
Horizontally, $(K_A^{n+1}, K_A^{n}, K_A^{n-2})$

and we have the exact sequence

$$0 \to H_n(K^{(n)}, A) \to H_n(K^{(n)}, K^{(n-1)}) \xrightarrow{\partial_*} H_{n-1}(K^{(n-1)}, A) \to H_{n-1}(K^{(n)}, A) \to 0.$$

Consider the following diagram with exact row and column, defining $\beta_{n+1}$ by commutativity:

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
H_{n+1}(K^{(n+1)}, A) \arrow[r, "j_{n+1}"] & H_{n+1}(K^{(n+1)}, K^{(n)}) \arrow[r, "\partial_{n+1}"] \arrow[dr, "\beta_{n+1}"'] & H_n(K^{(n)}, A) \arrow[r, "i_n"] \arrow[d, "j_n"] & H_n(K^{(n+1)}, A) \to 0 \\
 & & H_n(K^{(n)}, K^{(n-1)}) \arrow[d, "\partial_n"] & \\
 & & \tilde{H}_{n-1}(K^{(n-1)}, A). &
\end{tikzcd}$$

Accordingly, $\beta_{n+1} = j_n \circ \partial_{n+1}$. Since $\beta_n\beta_{n+1} = j_{n-1}\partial_n j_n \partial_{n+1} = j_{n-1} \circ 0 \circ \partial_{n+1} = 0$, this gives a boundary operator $\beta$. Note that

$$\ker \beta_{n+1} = \ker \partial_{n+1} = \operatorname{im} j_{n+1}.$$

Hence

$$\ker \beta_n = \ker \partial_n = \operatorname{im} j_n,$$
$$\operatorname{im} \beta_{n+1} = j_n(\operatorname{im} \partial_{n+1})$$

and $j_n$ is a monomorphism on $H_n(K^{(n)})$. Consequently, $j_n$ induces isomorphisms

$$\begin{array}{ccc}
H_n(K^{(n)}, A) & \xrightarrow{\approx} & \ker \beta_n \\
\bigcup & & \bigcup \\
\operatorname{im} \partial_{n+1} & \xrightarrow{\approx} & \operatorname{im} \beta_{n+1}
\end{array}$$

and hence induces

$$H_n(K^{(n+1)}, A) \approx \operatorname{coker} \partial_{n+1} \xrightarrow{\approx} \ker \beta_n/\operatorname{im} \beta_{n+1}.$$

But

$$H_n(K^{(n+1)}, A) \xrightarrow{\approx} H_n(K^{(n+2)}, A) \xrightarrow{\approx} H_n(K^{(n+3)}, A) \xrightarrow{\approx} \cdots$$

since $H_n(K^{(i)}, K^{(i-1)}) = 0$ for $i > n$, and by the exact sequences. Thus, if $\dim(K) < \infty$, we get

$$H_n(K, A) \approx H_n(K^{(n+1)}, A) \approx \ker \beta_n/\operatorname{im} \beta_{n+1}.$$

20190227

CW complexes

$\mathbb{RP}^n$, $\mathbb{CP}^n$ via antipodal maps

cellular homology

showing that cellular homology and singular homology compute the same groups...

} summary.

Often the degree doesn't matter up to signs... (the kernel and the cokernel just don't depend on it).

# CELLULAR MAPS

**DEF |** A cellular map on a CW complex is stable on the $n$-skeleton.

**DEF |** Let $(K,A)$ and $(L,B)$ be pairs with $A,B$ subcomplexes. A cellular map of pairs is a cell map $\varphi : K \to L$ s.th. $\varphi(A) \subset B$. The induced map $\varphi_* : H_*(K,A) \longrightarrow H_*(L,B)$ is a chain map and well defined.

TODO, show with large diagram

$$\begin{array}{ccc} H_n(K_A^{(n)}, K_A^{(n-1)}) & \xrightarrow{\varphi_*} & H_{n+1}(L_B^{(n)}, L_B^{(n-1)}) \\ \parallel & & \parallel \\ C_n(K,A) & \xrightarrow{\text{def } \varphi_\Delta} & C_n(L,B) \end{array}$$

cooking up a bunch of spaces...

$$\begin{tikzcd} L_\mu^{(n)} \arrow[r, "p_\mu"] \arrow[d] & S \\ L^{(n)}/L^{(n-1)} \arrow[ur, dashed] & \end{tikzcd}$$

Recall that we fixed the notion of degree

Say $\sigma$ and $\mu$ are $n$-cells

$$\begin{tikzcd} I_\sigma^n \arrow[r, "f_\sigma"] \arrow[d, "\partial_n"] & K^{(n)} \arrow[r, "\varphi"] & L^{(n)} \arrow[r, "p_\mu"] & S_\mu^n \arrow[d, equal] \\ S_\sigma^n \arrow[r, "\bar{f}_\sigma"] & K^{(n)}/K^{(n-1)} \arrow[r, "\bar{\varphi}"] & L^{(n)}/L^{(n-1)} \arrow[r, "\bar{p}_\mu"'] & S_\mu^n \end{tikzcd}$$

$$\varphi_\Delta[I_\sigma^n] = \sum_{\mu \notin B} \deg(p_\mu \circ \varphi \circ f_\sigma)\mu.$$

__Cellular approximation__ We may imagine taking $I^q$ and $I^p$ as the product $I^{q+p}$. In some cases the product of two CW complexes has a product topology and the structure of a CW complex. For big spaces, the product top & the weak top fail to coincide.

**THM |** Say $K$ or $L$ is locally compact, or that both $K$ and $L$ have the structure of a CW complex, with characteristic maps

$$\underset{\uparrow\, p\text{-cell}}{f_\sigma} \times \underset{\uparrow\, q\text{-cell}}{f_\mu} : I_\sigma^p \times I_\mu^q \longrightarrow K^{(p)} \times L^{(q)} = (K \times L)^{(p+q)}$$

where $f_\sigma$ and $f_\mu$ are char. maps of $K$ and $L$ respectively.

**EX |** In homotopy theory, we need to take products with the interval $I$. Because $I$ is locally compact, we have for any CW complex $K$ a well defined product CW complex $K \times I$. $I^{(0)} = \{0,1\}$, $I^{(1)} = \{\{0,1\}\}$, $I^{(k)} = I^{(1)}$.

Proof with homotopy extension property.

**THM |** Cellular approximation. Let $(K,A)$ and $(L,B)$ be CW-pairs. Let $\varphi : (K,A)$ to $(L,B)$ be a map of pairs. Then $\varphi$ is homotopic to a cellular map of pairs. Moreover if $\varphi, \psi : (K,A) \longrightarrow (L,B)$ is a cellular map of pairs s.th. $\psi \simeq \varphi$. Then

# § PRODUCT CW COMPLEXES

20190304

Say $K$ and $L$ are CW complexes. When do we have a CW-structure on $K \times L$?

**THM** Say $K$ or $L$ is locally compact and that both $K$ and $L$ have the structure of a CW complex, with characteristic maps

$$\begin{tikzcd}
\underset{\uparrow\, p\text{-cell}}{f_\sigma} \times \underset{\uparrow\, q\text{-cell}}{f_\mu} \arrow[d, equal] & : & I^p_\sigma \times I^q_\mu \arrow[d, equal] \arrow[r] & K^{(p)} \times L^{(q)} \arrow[d, equal] & \\
f_{\sigma\times\mu} & : & I^{p+q}_{\sigma\times\mu} \arrow[r] \arrow[rr, bend right, "g^{p+q}"'] & (K\times L)^{(p+q)} \arrow[r, "\rho_{\sigma\times\mu}"] & S^{p+q}_{\sigma\times\mu}
\end{tikzcd}$$

**DEF** "differential operator" Let $\sigma$ be a $p$-cell of $K$, and $\mu$ a $q$-cell of $L$. Let $p+q=n$. The boundary in $C_n(K\times L) \longrightarrow C_{n-1}(K\times L)$ is found to be

$$\sigma\times\mu \overset{\partial}{\longmapsto} (\partial\sigma \times \mu) + (-1)^p(\sigma \times \partial\mu)$$

Pf sketch.

$$\begin{tikzcd}
\partial(I^p \times I^q) \arrow[r, "f_{\partial(\sigma\times\mu)}"] \arrow[d, equal] & (K\times L)^{(p+q-1)} \arrow[r, "\underset{\rho_\tau \wedge \rho_\varphi}{\rho_{\tau\times\varphi} \,=}"'] & S^{p+q-1} \\
\partial(\partial I^p \times I^q \cup I^p \times \partial I^q) \arrow[ur, "f_{\partial\sigma}\times f_\mu \,\cup\, f_\sigma \times f_{\partial\mu}"'] & &
\end{tikzcd}$$

Then we know $\displaystyle \partial(\sigma\times\mu) = \sum_{\substack{\tau\times\varphi \\ \text{for all } \tau\times\varphi \\ \text{in } (K\times L)^{(n-1)}}} [\tau\times\varphi : \sigma\times\mu]\, \tau\times\varphi$

I **claim** $[\tau\times\varphi : \sigma : \mu]$ is $0$ unless

i. $\deg \tau = p-1$ and $\deg \varphi = q$, or

ii. $\deg \tau = p$ and $\deg \varphi = q-1$.

As well, in these cases

i. $\varphi = \mu$ and ii. $\tau = \sigma$.

To justify i, say that $\varphi \neq \mu$, then $\underbrace{p_\varphi \circ f_\mu}_{\text{same dim}} = *$ (unless $\varphi = \mu$).

Always, the null map has degree 0.

Thus $\partial(\sigma\times\mu) = \sum_\tau [\tau\times\mu : \sigma\times\mu]\, \tau\times\mu + \sum_\varphi [\sigma\times\varphi : \sigma\times\mu]\, \sigma\times\varphi$

Why? What do we want from $\partial(\sigma\times\mu)$?

**LEMMA** For a p-cell $\sigma$, a p-1 cell $\tau$, a q cell $\mu$, a q-1 cell $\varphi$, the incidence of cells $\sigma\times\mu$ to $\tau\times\mu$ in $(K\times L)^{(n)}$ to $(K\times L)^{(n-1)}$ is $[\tau\times\mu : \sigma\times\mu] = [\tau:\sigma]$ and the incidence of $\sigma\times\mu$ to $\sigma\times\varphi$ is $-[\sigma\times\varphi : \sigma\times\mu] = (-1)^p[\varphi:\mu]$.

We have attaching maps

$$\partial(I^p\times I^q) \xrightarrow[f_{\partial(\sigma\times\mu)}]{} (K\times L)^{(p+q-1)} \xrightarrow[p_{\tau\times\mu}]{} S^{p+q-1},$$ which gives

$$\begin{aligned}(p_{\tau\times\mu}\circ f_{\partial(\sigma\times\mu)})(x,y) &= p_{\tau\times\mu}(f_\sigma(x), f_\mu(y)) \\ &= (p_\tau f_\sigma(x), p_\mu f_\mu(y)) \\ &= (p_\tau f_\sigma(x), \gamma_q(y)) \\ &= (p_\tau f_\sigma\wedge 1_q)\circ(1\wedge\gamma_q)(x,y)\end{aligned}$$

$$\begin{tikzcd} (x,y) \in \partial(I^p\times I^q) \arrow[d, "1\wedge\gamma_q"'] \arrow[rd, dashed] & \\ {[(x,\gamma_q(y))]} \in \partial I^p\wedge S^q \arrow[r, "p_\tau f_\sigma\wedge 1_q"'] & S^{p-1}\wedge S^q \end{tikzcd}$$

Similarly, taking the other sphere to play.

$$\partial(I^p\times I^q) \xrightarrow[\gamma_p\wedge 1]{} S^p\wedge\partial I^q \xrightarrow[1\wedge p_\varphi f_{\partial\mu}]{} S^p\wedge S^{q-1}$$

NTS $g:\partial I^p \longrightarrow S^{p-1}$, $\deg((g\wedge 1)\circ(1\wedge\gamma_q)) = \deg(g)$
$g:\partial I \longrightarrow S$, $\deg((g\wedge 1)\circ(1\wedge\quad)) = \deg(g)$ } REVISE

there's a natural suspension isom

$$\begin{tikzcd} H_*(X) \arrow[r, "\cong"] \arrow[rd, "\cong"'] & H_{*+1}(\Sigma X) \arrow[d] \\ & H_{*+1}(X\wedge S^1) \end{tikzcd}$$


so that $H_*(X) \xrightarrow[\cong]{\Psi} H_{*+q}(X\wedge S^q)$

$$\begin{tikzcd} H_{p-1}(\partial I^p) \arrow[r, "\Psi", "\cong"'] \arrow[d, "g_*"] & H_{p+q-1}(\partial I^p\wedge S^q) \arrow[d, "(g\wedge 1)_*"] \\ H_{p-1}(S^{p-1}) \arrow[r, "\Psi", "\cong"'] & H_{p+q-1}(S^{p+q-1}) \end{tikzcd}$$

Orient $H_{p+q-1}(\partial I^p\wedge S^q) \ni [\partial I^p\wedge S^q]$

$$\begin{matrix} H_{p+q-1}(\partial I^p\wedge S^q) & \ni & [\partial I^p\wedge S^q] \\ \downarrow & & \downarrow \\ H_{p+q-1}(S^{p+q-1}) & \ni & [S^{p+q-1}] \end{matrix}$$

as we had $\partial I^p \xrightarrow[1\wedge\gamma_{p-1}]{} S$

# PRODUCT CW-COMPLEXES (CORRECTIONS) 20190306

Write $(K\times L)^{(n)} = \bigcup_{i=0}^{n} K^{(i)} \times L^{(n-i)}$. Let $\sigma$ be a $p$-cell, $\mu$ a $q$-cell, then

$$\begin{tikzcd} \underset{\uparrow\,p\text{-cell}}{f_\sigma} \times \underset{\uparrow\,q\text{-cell}}{f_\mu} : I^p_\sigma \times I^q_\mu \arrow[r] \arrow[d, equal] & K^{(p)} \times L^{(q)} \arrow[r, "p_\sigma \wedge p_\mu"] \arrow[d, equal] & S^p \wedge S^q \arrow[d, equal] \\ f_{\sigma\times\mu} : I^{p+q}_{\sigma\times\mu} \arrow[r] \arrow[rr, bend right, "g^{p+q}"'] & (K\times L)^{(p+q)} \arrow[r, "p_{\sigma\times\mu}"] & S^{p+q}_{\sigma\times\mu} \end{tikzcd}$$

Last time, we fudged up the def$^{\underline{n}}$ of $p_{\sigma\times\mu}$. We'll revise.

Let $z=(x,y) \in \bigcup_{i=0}^{n} \left(K^{(i)} \times L^{(n-i)}\right)$

Then $p_{\sigma\times\mu}$ is the smash product on the cell of relevance, and crushes everything else to the basepoint.

**DEF** $p_{\sigma\times\mu}(z) = \{ p_\sigma(x) \wedge p_\mu(y)$ if $\sigma\times\mu \in K^{(p)}\times L^{(q)}$, else $* \}$

**EX** We appeal to the suspension functor to produce an isomorphism.

For __any__ $g : \partial I^p \to S^{p-1}$ and $h : \partial I^q \to S^{q-1}$

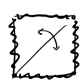
$$\begin{tikzcd} H_*(\partial I^p) \arrow[r, "\Sigma_{top}"] \arrow[d, "1\wedge\gamma_{p-1}"'] & H_*(\partial I^p \wedge S^q) \arrow[r, "s_{top}?"] \arrow[d, "1\wedge\gamma_{p-1}\wedge 1_q"'] & H_*(S^q \wedge \partial I^p) \arrow[d, "1_q\wedge 1\wedge\gamma_{p-1}"] \\ H_*(S^{p-1}) \arrow[r, "\Sigma_{bot}"] & H_*(S^{p-1}\wedge S^q) \arrow[r, "s_{bot}?"] & H_*(S^q \wedge S^{p-1}) \end{tikzcd}$$

oriented

$\therefore \deg \Sigma_{Top} = \deg \Sigma_{bot}$ & $\deg s_{top} = \deg s_{bot}$.

$$\begin{aligned} I^p \times I^q &\longrightarrow I^q \times I^p \\ (x,y) &\longmapsto (y,x) \end{aligned}$$
induces maps of deg $(-1)^{pq}$

__Work__ inductively: $\begin{aligned} I\times I &\to I\times I \\ (x,y) &\mapsto (y,x) \end{aligned}$ has deg $-1$ via action



crush to $S^2$

TODO

__Observe__: No matter the orientation, we __must__ have the equations

$$\deg(1_p \wedge \gamma_q) = \varepsilon$$
$$\deg(\gamma_q \wedge 1_p) = (-\varepsilon)^q$$

for $\varepsilon \in \mathbb{Z}^\times$.

Compute $H_*(S^1\times S^1)$

as a PRODUCT CW-COMPLEX.

**12.1. Lemma.** $[\tau\times\mu : \sigma\times\mu] = [\tau:\sigma]$.

PROOF. By the above remarks, we need to show that

$$\deg((p_\tau f_{\partial\sigma} \wedge 1_q)\circ(1_p \wedge \gamma_q)) = \deg(p_\tau f_{\partial\sigma}).$$

In general, for a map $g: \partial\mathbf{I}^p \to \mathbf{S}^{p-1}$ we will show that the composition

$$\partial(\mathbf{I}^p \times \mathbf{I}^q) \xrightarrow{1_p\wedge\gamma_q} \partial\mathbf{I}^p \wedge \mathbf{S}^q \xrightarrow{g\wedge 1_q} \mathbf{S}^{p-1} \wedge \mathbf{S}^q = \mathbf{S}^{p+q-1}$$

has the same degree as does $g$. That is, $\deg(g\wedge 1_q \circ 1_p \wedge \gamma_q) = \deg g$.

Recall from Problem 1 of Section 6 that there is a natural "suspension" isomorphism $\tilde{H}_k(X) \xrightarrow{\approx} \tilde{H}_{k+1}(\Sigma X)$. Also, for pointed $X$, there is a canonical map $\Sigma X \to X \wedge \mathbf{S}^1$ collapsing the arc between the poles through the base point. For most $X$ this is a homotopy equivalence, and that is true when $X \approx \mathbf{S}^k$. (We only need that $H_{k+1}(\Sigma X) \xrightarrow{\approx} H_{k+1}(X\wedge\mathbf{S}^1)$ when $X\approx\mathbf{S}^k$ and that is an easy consequence of Theorem 7.4 by comparing both spaces to the sphere obtained by collapsing the complement of a nice disk away from the "base point arc.") Therefore, the composition $\tilde{H}_k(X) \to \tilde{H}_{k+1}(X\wedge\mathbf{S}^1)$, of these two maps, is a natural homomorphism which is an isomorphism when $X\approx\mathbf{S}^k$. Iterating this $q$ times and using that $\mathbf{S}^q = \mathbf{S}^1\wedge\cdots\wedge\mathbf{S}^1$, $q$ times, gives the natural homomorphism $\phi: \tilde{H}_k(X) \to \tilde{H}_{k+q}(X\wedge\mathbf{S}^q)$ which is an isomorphism for $X\approx\mathbf{S}^k$. (Incidentally, a general condition on $X$ for $\phi$ to be an isomorphism is that $X$ be "well-pointed"; see Theorem 1.9 of Chapter VII.) Thus there is the commutative diagram

$$\begin{tikzcd} H_{p-1}(\partial\mathbf{I}^p) \arrow[r, "\phi", "\approx"'] \arrow[d, "g_*"] & H_{p+q-1}(\partial\mathbf{I}^p\wedge\mathbf{S}^q) \arrow[d, "(g\wedge 1_q)_*"] \\ H_{p-1}(\mathbf{S}^{p-1}) \arrow[r, "\phi", "\approx"'] & H_{p+q-1}(\mathbf{S}^{p-1}\wedge\mathbf{S}^q) = H_{p+q-1}(\mathbf{S}^{p+q-1}) \end{tikzcd}$$

for *any* $g$.

Define $\epsilon = \pm 1$ by $\phi[\mathbf{S}^{p-1}] = \epsilon[\mathbf{S}^{p+q-1}]$. By definition, $g_*[\partial\mathbf{I}^p] = (\deg g)[\mathbf{S}^{p-1}]$. Hence

$$(g\wedge 1_q)_*\phi[\partial\mathbf{I}^p] = \phi g_*[\partial\mathbf{I}^p] = \epsilon\deg(g)[\mathbf{S}^{p+q-1}].$$

In the special case where $g$ is $1_1\wedge\gamma_{p-1}: \partial\mathbf{I}^p \to \mathbf{S}^{p-1}$, our orientation conventions specify that $\deg(1_1\wedge\gamma_{p-1}) = 1$. Substituting $1_1\wedge\gamma_{p-1}$ for $g$ in the above equation gives

$$(1_1\wedge\gamma_{p-1}\wedge 1_q)_*\phi[\partial\mathbf{I}^p] = \epsilon[\mathbf{S}^{p+q-1}].$$

For general $g$ then, these two equations show that

$$\deg(g\wedge 1_q) = \deg(g)\cdot\deg(1_1\wedge\gamma_{p-1}\wedge 1_q).$$

(The degrees make sense only after orienting $\partial\mathbf{I}^p\wedge\mathbf{S}^q$, but the *equation* is not affected by that choice.) Composing this with $1_p\wedge\gamma_q$, and noting that $(1_1\wedge\gamma_{p-1}\wedge 1_q)\circ(1_p\wedge\gamma_q) = 1_1\wedge\gamma_{p+q-1}$ in the diagram

$$\begin{tikzcd} \partial(\mathbf{I}^p\times\mathbf{I}^q) \arrow[r, "1_p\wedge\gamma_q"] \arrow[d, "1_1\wedge\gamma_{p+q-1}"'] & \partial\mathbf{I}^p\wedge\mathbf{S}^q \arrow[d, "1_1\wedge\gamma_{p-1}\wedge 1_q"] \\ \mathbf{S}^{p+q-1} \arrow[r, equal] & \mathbf{S}^{p-1}\wedge\mathbf{S}^q \end{tikzcd}$$

we conclude that

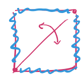
$$\begin{aligned} \deg(g\wedge 1_q\circ 1_p\wedge\gamma_q) &= \deg(g)\cdot\deg((1_1\wedge\gamma_{p-1}\wedge 1_q)\circ(1_p\wedge\gamma_q)) \\ &= \deg(g)\cdot\deg(1_1\wedge\gamma_{p+q-1}) = \deg(g), \end{aligned}$$

where the last equation is by our orientation conventions. $\square$

# Exercises (Product CW-Complexes, "orientation")

**TODO!** $I^p \times I^q \longrightarrow I^q \times I^p$ induces maps of deg $(-1)^{pq}$

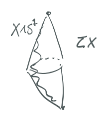
$(x,y) \longmapsto (y,x)$

Work inductively: $I \times I \longrightarrow I \times I$, $(x,y) \longmapsto (y,x)$ has deg $-1$ via action  crush to $S^2$

TODO, show $\deg g_* = \deg (g \wedge 1)_*$ and $\deg g_* = -\deg(1 \wedge g)_*$

**TODO!** Show there's a natural suspension isom

$$\begin{tikzcd} H_*(X) \arrow[r, "\cong"] \arrow[dr, "\cong"'] & H_{*+1}(\Sigma X) \arrow[d] \\ & H_{*+1}(X \wedge S^1) \end{tikzcd}$$




so that $H_*(X) \xrightarrow[\cong]{\psi} H_{*+q}(X \wedge S^q)$

(1) Use Excision to prove that if an open subset $U \subseteq \mathbb{R}^m$ is homeomorphic to an open subset $V \subseteq \mathbb{R}^n$, then $n = m$. (We generalize to topological manifolds)

Given. Say $M$ and $N$ are connected topological manifolds of dimension $m$ and $n$ resp.

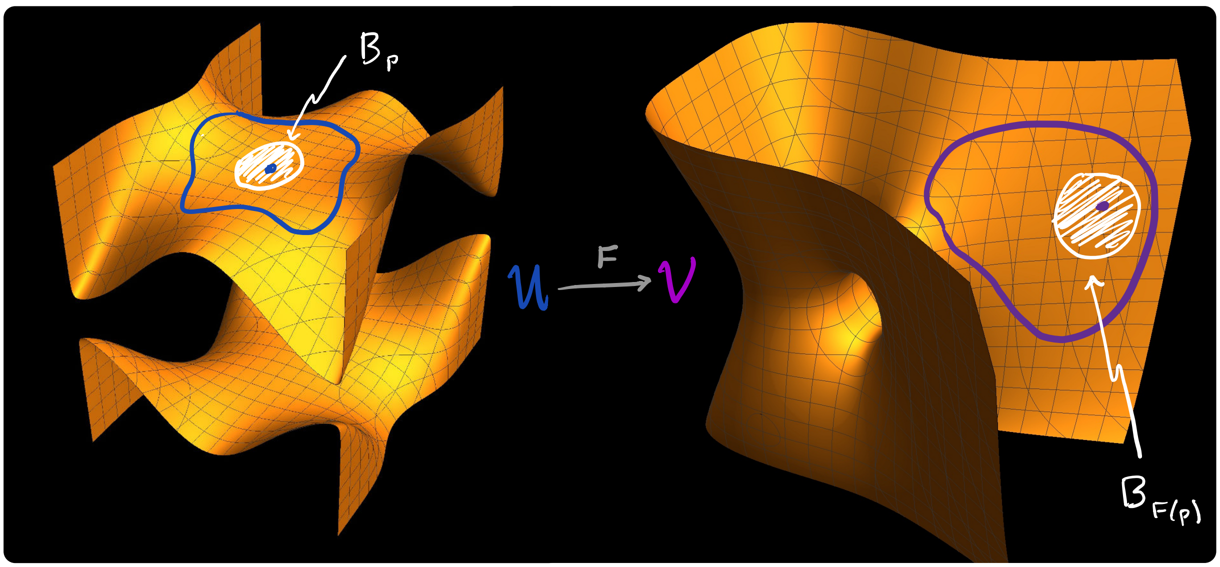
Suppose $F$ is a homeo from open $\mathcal{U} \subset M$ to open $\mathcal{V} \subset N$. (Suppose both $\mathcal{U}$ and $\mathcal{V}$ are nonempty.)

Let $h_*(-) : \mathrm{Top}_{\text{pairs}} \longrightarrow \mathrm{GradedAb}$ be an ordinary homology theory.

To prove. $M$ and $N$ have the same dimension; $m = n$.

Proof. WLOG, suppose $\mathcal{U}$ and $\mathcal{V}$ are chart domains with $(\mathcal{U}, \varphi)$ and $(\mathcal{V}, \psi)$ charts to $\mathbb{R}^m$, $\mathbb{R}^n$.

Let $p \in \mathcal{U}$ and find $F(p)$ in $\mathcal{V}$. Let $B_p$ be an open ball about $p$ in $\mathcal{U}$.




$(\mathcal{U}, \mathcal{U} \setminus \{p\}, \mathcal{U} \setminus B_p)$ is an excisive triple.

$(B_p, B_p \setminus \{p\})$ deform retracts to $(D^m, S^{m-1})$, where $M$ has dimension $m$.

$$h_*(\mathcal{U}, \mathcal{U} \setminus \{p\}) \xleftarrow[ex]{\sim} h_*(B_p, B_p \setminus \{p\}) \xrightarrow[h.a.]{\sim} h_*(D^m, S^{m-1}).$$

For a ball $B_{F(p)}$ about $F(p)$ in $\mathcal{V}$, excision and the homotopy axiom also imply

$$h_*(\mathcal{V}, \mathcal{V} \setminus \{F(p)\}) \xleftarrow[ex]{\sim} h_*(B_{F(p)}, B_{F(p)} \setminus \{F(p)\}) \xrightarrow[h.a.]{\sim} h_*(D^n, S^{n-1}) \quad \text{(for } N \text{ of dimension } n\text{)}.$$

Because $F$ is a homeo, $h_*(F) : h_*(\mathcal{U}, \mathcal{U} \setminus \{p\}) \to h_*(\mathcal{V}, \mathcal{V} \setminus \{F(p)\})$ is an isom. in GradedAb.

Thus $h_*(D^m, S^{m-1}) \cong h_*(D^n, S^{n-1})$. These groups must match in each degree, so $m = n$. $\square$

(2) Compute the homology of $\mathbb{R}P^2$ and $\mathbb{R}P^3$. Chapter 4, Section 7, Problem 3 (p.194) and Bredon, Problem 6, p.206 and Bredon.

Let $\mathbb{R}P^2$ be the space obtained by either identifying antipodal points on the boundary $\partial D^2 = S^1$ of $D^2$ or by attaching a 2-cell to the cellular complex $K^{(0)} = \{\text{point}\}$ and $K^{(1)} = \{ \circlearrowleft \}$ with a degree 2 map. That is, $K^{(2)} = K^{(2+n)}$ is the pushout of the diagram where $f_{\partial\sigma}$ has degree 2.

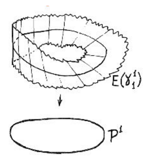
$$
\begin{tikzcd}
S^1 \arrow[r, "f_{\partial\sigma}"] \arrow[d] & K^{(1)} \arrow[d] \\
D^2 \arrow[r, "f_\sigma"] & K^{(2)}
\end{tikzcd}
$$

We'll compute reduced homology with Mayer-Vietoris.

Let $A = \operatorname{Int} D^2$ be the open 2-cell in $\mathbb{R}P^2$, find a point $p$ in $A$, and let $B = D^2 \setminus \{p\}$ be the punctured 2-cell. Observe $\mathbb{R}P^2 = \operatorname{Int} A \cup \operatorname{Int} B$. By Mayer-Vietoris, there's a l.e.s. in reduced homology

$$\cdots \longrightarrow \tilde{H}_n(A) \oplus \tilde{H}_n(B) \longrightarrow \tilde{H}_n(\mathbb{R}P^2) \xrightarrow{\delta} \tilde{H}_{n-1}(A \cap B) \longrightarrow \tilde{H}_{n-1}(A) \oplus \tilde{H}_{n-1}(B) \longrightarrow \cdots$$

First for homotopy equivalences. $A$ is an open ball so deform retracts to $\{p\}$. $B$ is an open Möbius band, and deforms to $S^1$. As well, $A \cap B$ deform retracts to $S^1$.




$$\tilde{H}_*(A) = 0$$

$$\tilde{H}_*(B) = \tilde{H}_*(A \cap B) = \{\mathbb{Z} \text{ if } * = 1, \text{ else } 0\}.$$

Our l.e.s. becomes trivial above degree 2, so we consider

$$
\begin{tikzcd}
\tilde{H}_2(A) \oplus \tilde{H}_2(B) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_2(\mathbb{R}P^2) \arrow[r, "\delta"] \arrow[d, "\cong"] & \tilde{H}_1(A \cap B) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_1(A) \oplus \tilde{H}_1(B) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_1(\mathbb{R}P^2) \arrow[r, "\delta"] \arrow[d, "\cong"] & \tilde{H}_0(A \cap B) \arrow[d, "\cong"] \\
0 \arrow[r] & \tilde{H}_2(\mathbb{R}P^2) \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r] & \tilde{H}_1(\mathbb{R}P^2) \arrow[r] & 0
\end{tikzcd}
$$

Fix generators in $\tilde{H}_1(A \cap B)$ and $\tilde{H}_1(B)$ by letting $\gamma$ be a loop in $A \cap B$ winding once around $\{p\}$ in the mathematically positive direction and letting $b$ in $B$ be the path starting at $K^{(0)}$ and traversing half way around $K^{(1)}$ in the math. pos. direction. The inclusion $i_{A \cap B} : A \cap B \longrightarrow B$ induces $H_1(i_{A\cap B})[\gamma] = 2[b]$.

Or, follow Bredon to determine the degree map of the attaching $f_\sigma$.

We conclude

$$\tilde{H}_2(\mathbb{R}P^2) \cong \ker(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}) = 0,$$

$$\tilde{H}_1(\mathbb{R}P^2) \cong \operatorname{coker}(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}) = \mathbb{Z}/(2),$$

$$\tilde{H}_*(\mathbb{R}P^2) \cong 0 \quad \text{for } * \neq 1.$$

**Proposition 1.7.8.** *There is an isomorphism*

$$\tilde{h}_p(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}/2 & p = 1 \\ 0 & \textit{otherwise} \end{cases}$$

*Proof.* Let $\mathbb{R}P^2$ be constructed by attaching a disk $D^2$ to $S^1$ along the degree 2 map. Let $D$ be the disk of radius 1/2 in $D^2$ and let $U = \mathbb{R}P^2 - \{0\}$ where 0 is the center of $D^2$.

There's a long exact sequence

$$
\begin{tikzcd}
h_{p+1}(\mathbb{R}P^2, U) \arrow[r] & \tilde{h}_p(U) \arrow[r] & \tilde{h}_p(\mathbb{R}P^2) \arrow[r] & h_p(\mathbb{R}P^2, U) \\
h_{p+1}(\mathbb{R}P^2 - S^1, U - S^1) \arrow[r] \arrow[u, "\cong"] & \tilde{h}_p(U - S^1) \arrow[r] \arrow[u] & \tilde{h}_p(\mathbb{R}P^2 - S^1) \arrow[r] \arrow[u] & h_p(\mathbb{R}P^2 - S^1, U - S^1) \arrow[u, "\cong"] \\
h_{p+1}(D, \partial D) \arrow[r] \arrow[u, "\cong"] & \tilde{h}_p(\partial D) \arrow[r] \arrow[u] & \tilde{h}_p(D) \arrow[r] \arrow[u] & h_p(D, \partial D) \arrow[u, "\cong"]
\end{tikzcd}
$$

Note that

$$U \simeq U - S^1 \simeq \partial D \simeq S^1.$$

If $p \geqslant 3$, then this gives $\tilde{h}_p(\mathbb{R}P^2) = 0$ and similarly if $p = 0$. So it remains to study the cases $p = 1, 2$.

$$
\begin{tikzcd}
 & & h_2(D, \partial D) \arrow[r, "\cong"] \arrow[d, "\cong"] & \tilde{h}_1(\partial D) \arrow[d] & & & \\
0 \arrow[r] & \tilde{h}_2(\mathbb{R}P^2) \arrow[r] \arrow[d, "="] & h_2(\mathbb{R}P^2, U) \arrow[r] \arrow[d, "\cong"] & \tilde{h}_1(U) \arrow[r] \arrow[d, "\cong"] & \tilde{h}_1(\mathbb{R}P^2) \arrow[r] \arrow[d, "="] & h_1(\mathbb{R}P^2, U) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \tilde{h}_2(\mathbb{R}P^2) \arrow[r] \arrow[d, "="] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "="] & \mathbb{Z} \arrow[r] \arrow[d, "="] & \tilde{h}_1(\mathbb{R}P^2) \arrow[r] \arrow[d, "\cong"] & h_1(\mathbb{R}P^2, U) \arrow[r] \arrow[d, "="] & 0 \\
 & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 &
\end{tikzcd}
$$

Since $S^1 \cong \partial D \to U \simeq S^1$ is the multiplication by 2 map, we get the result. □

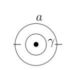
1. Recall that we can express the real projective plane $\mathbb{R}P^2$ as the quotient space of $\mathbb{S}^2$ modulo antipodal points or as a quotient of $\mathbb{D}^2$:

$$\mathbb{R}P^2 \cong \mathbb{S}^2 / \pm \mathrm{id} \cong \mathbb{D}^2 / z \sim -z \text{ for } z \in \mathbb{S}^1.$$

We use the latter definition and set $X = \mathbb{R}P^2$, $A = X \setminus \{[0,0]\}$ (which is an open Möbius strip and hence homotopically equivalent to $\mathbb{S}^1$) and $B = \mathbb{D}^2$. Then

$$A \cap B = \mathring{\mathbb{D}}^2 \setminus \{[0,0]\} \simeq \mathbb{S}^1.$$

Thus we know that $H_1(A) \cong \mathbb{Z}$, $H_1(B) \cong 0$ and $H_2(A) = H_2(B) = 0$. We choose generators for $H_1(A)$ and $H_1(A \cap B)$ as follows:




Let $a$ be the path that runs along the outer circle in mathematical positive direction half around starting from the point $(1,0)$. This is the generator for $H_1(A)$. Let $\gamma$ be the loop that runs along the inner circle in mathematical positive direction. This is the generator for $H_1(A \cap B)$; note that $A \cap B \simeq \mathbb{D} \setminus \{0\}$. Then the inclusion $i_{A\cap B} \colon A \cap B \to A$ induces

$$H_1(i_{A\cap B})[\gamma] = 2[a].$$

This suffices to compute $H_*(\mathbb{R}P^2)$ up to degree two because the long exact sequence is

$$\tilde{H}_2(A) \oplus \tilde{H}_2(B) = 0 \to \tilde{H}_2(X) \to \tilde{H}_1(A \cap B) \cong \mathbb{Z} \xrightarrow{\cdot 2} \tilde{H}_1(A) \cong \mathbb{Z} \to \tilde{H}_1(X) \to \tilde{H}_0(A \cap B) = 0.$$

On the two copies of the integers, the map is given by multiplication by two and thus we obtain:

$$H_2(\mathbb{R}P^2) \cong \ker(2\cdot \colon \mathbb{Z} \to \mathbb{Z}) = 0,$$
$$H_1(\mathbb{R}P^2) \cong \operatorname{coker}(2\cdot \colon \mathbb{Z} \to \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},$$
$$H_0(\mathbb{R}P^2) \cong \mathbb{Z}.$$

The higher homology groups are trivial, because there $H_n(\mathbb{R}P^2)$ is located in a long exact sequence between trivial groups.

3. If $X_k = \mathbf{S}^1 \cup_{\phi_k} \mathbf{D}^2$, where $\phi_k$ is the map of Example 7.7, compute $H_*(X_k)$.

6. Compute the homology of the space obtained from a circle by attaching a 2-cell by a map of degree 2, and another 2-cell by a map of degree 3. Generalize.

Hints (from Bredon p. 194, 206)

TODO 1/2

# $\mathbb{R}P^3$

$\mathbb{R}P^3$ has the one cell structure in each dimension and the skeleta $\mathbb{R}P^k = \mathbb{R}P^{k-1} \cup_{f_k} D^k$

where $f_{\partial k} : \partial D^k \longrightarrow \mathbb{R}P^{k-1}$

$S^{k-1} \longrightarrow S^{k-1}/\sim$ is the quotient

Having only one cell in each dim, the incidence degree of $\partial D^k$ attaching by $f_{\partial k}$ is

$$\deg\left(S^{k-1} \xrightarrow{f_{\partial k}} S^{k-1}/\sim \;:=\; \mathbb{R}P^{k-1} \xrightarrow{p} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \cong S^{k-1}\right)$$

is this map unique?

where $p \circ f|_{D_+^{k-1}}$ and $p \circ f|_{D_-^{k-1}}$ are related by the antipodal map. We compute

$$[k : \partial k] = \deg(p \circ f_{\partial k})$$

$$= \deg(id) + \deg(\text{antipodal})$$

$$= 1 + (-1)^k \text{ for attaching } D^k \text{ to } \mathbb{R}P^{k-1}.$$

So for $\mathbb{R}P^3$, the cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

3-cell 2 cell 1 cell pt

why? The antipodal map from eg, $S^1$ to $S^1$ is homotopic to id.

Thus $H_*(\mathbb{R}P^3) = \begin{cases} \mathbb{Z}, & * = 0 \\ \mathbb{Z}/(2) & * = 1 \\ 0 & * = 2 \\ \mathbb{Z} & * = 3 \\ 0 & \text{else} \end{cases}$

$\square$

(a) Compute the homology of the tori $T^2$ and $T^3$ ($T^n$ is the product of $n$ circles). Conclude that $T^2$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology, but show that they are not homotopy equivalent.

$$\tilde{h}_*\left(\bigvee_1^3 S^1\right) \cong \bigoplus_1^3 \tilde{h}_*(S^1) \cong \{\mathbb{Z}^{\oplus 3} \text{ if } *=1, \text{ else } 0\}$$

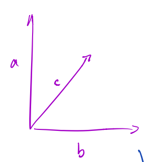
$$\tilde{h}_*(S^1 \vee S^1 \vee S^2) \cong \begin{cases} 0 & *=0 \\ \mathbb{Z}^{\oplus 2} & *=1 \\ \mathbb{Z} & *=2 \\ 0 & \text{else} \end{cases} \qquad \underline{\text{unreduced}} \quad \begin{matrix} \mathbb{Z} & *=0 \\ \mathbb{Z}^{\oplus 2} & *=1 \\ \mathbb{Z} & *=2 \\ 0 & \text{else} \end{matrix}$$

$$\begin{tikzcd} I \arrow[r, "f_{\partial\sigma}"] & K^{(1)} \arrow[r] \arrow[rr, bend right, "pa"] & K^{(1)}/K^{(0)} \arrow[r] & S^1 \end{tikzcd}$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z} \longrightarrow 0$$

$\mathbb{Z}\{[a,b,c]\}$ // $\qquad \mathbb{Z}^3 = \mathbb{Z}\{[a,c],[b,c],[a,b]\}$

$$\partial[a,b,c] = \deg[a,b] + \boxed{\overset{\text{focus}}{\deg[a,c]}} + \deg[b,c]$$




$$\longrightarrow (T^3)^{(2)}/(T^3)^{(1)} = S^2_{[a,b]} \vee S^2_{[b,c]} \vee S^2_{[a,c]}$$

$$\downarrow$$

$$\longrightarrow S^2_{[a,b]}$$

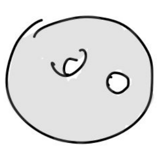
Whence

factors through

Recall $\partial I^1 \times I^2 \cup I^1 \times \partial I^2 = \partial(I^1 \times I^2) \xrightarrow{1 \wedge \gamma_2} S^2$ by convention $\deg(1 \wedge \gamma_2) = 1$

$$\| \\ \partial(I^3)$$

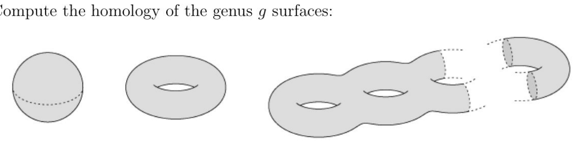
(b) Compute the homology of the punctured torus, i.e., the surface

$T^2 \setminus \{pt\} =$ 

There's a deform retract to $S^1 \vee S^1$. $\tilde{h}_*(T^2 \setminus \{pt\}) \underset{h.e.}{\cong} h_*(S^1 \vee S^1)$.

$\tilde{h}_*\left(\bigvee_1^2 S^1\right) \cong \bigoplus_1^2 \tilde{h}_*(S^1) \cong \{\mathbb{Z}^{\oplus 2} \text{ if } * = 1, \text{ else } 0\}$. □

(c) Compute the homology of the genus $g$ surfaces:



$K^{(0)} = \{p\}$

$$\begin{tikzcd} \coprod_{\sigma \in \{a,b\}} S^0_\sigma \arrow[r, "\sqcup f_{\partial\sigma}"] \arrow[d] & K^{(0)} \arrow[d] \\ \coprod_\sigma D^1_\sigma \arrow[r, "\sqcup f_\sigma"] & K^{(1)} \end{tikzcd}$$

$K^{(1)} =$ 

$K^{(2)} =$

$$\begin{tikzcd} S^1_\tau \arrow[r, "f_{\partial\tau}"] \arrow[d] & K^{(1)} \arrow[d] \\ D^2_\tau \arrow[r] & K^{(2)} \end{tikzcd}$$

Q| What's degree here?

boundary maps?

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{\oplus 2} \longrightarrow \mathbb{Z} \longrightarrow 0$$

deg(

# Homework 3 Problem 3

Justin Willson

February 27, 2019

(a) First, we compute the homology $T^2$ using the cellular structure that has

- One 0-cell $x$
- Two 1-cells, $a$, and $b$, that are attached by the constant map to $x$
- One 2-cell $\sigma$, that is attached by the loop $aba^{-1}b^{-1}$

This yields the chain complex

$$\dots \longrightarrow 0 \xrightarrow{0} \mathbb{Z}\{\sigma\} \xrightarrow{j} \mathbb{Z}\{a,b\} \xrightarrow{i} \mathbb{Z}\{x\} \xrightarrow{0} 0 \ .$$

Now, because $a$ and $b$ are attached by the constant map, we know that $i$ is the zero map, so $H_0(T^2) = \mathbb{Z}$. Next, in order to determine $H_1(T^2)$, we must find the kernel of $i$ and the image of $j$. Because both $a$ and $b$ are sent to 0, the kernel of $j$ is all of $\mathbb{Z}\{a,b\}$. Next, the image of $j$ is determined by $j(\sigma) = deg(p_a \circ f_\sigma)a + deg(p_b \circ f_\sigma)b$. Looking at $p_a \circ f_\sigma$, we see that the attaching map for $\sigma$ results in $aba^{-1}b^{-1}$ and $p_a(aba^{-1}b^{-1}) = aa^{-1}$, which is the constant map, so it has degree zero. We see the same calculation arise for $p_b \circ f_\sigma$. Hence the image of $j$ is 0, and its kernel is all of $\mathbb{Z}\{\sigma\}$. Thus $H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$, and $H_2(T^2) = \mathbb{Z}$.

Next, we compute the homology of $T^3$. The following diagram is helpful in understanding its cell structure.

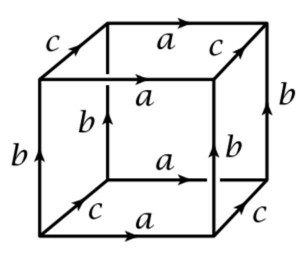


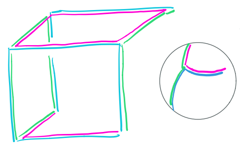

Figure 1: Diagram From Hatcher p142

From this we see the structure is given by

$$K^{(0)} = \{e^0\}, \quad K^{(1)} = \{a, b, c\}, \quad K^{(2)} = \{ab, bc, ac\}, \quad K^{(3)} = \{e^3\}.$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{\oplus 3} \xrightarrow[\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}]{} \mathbb{Z}^{\oplus 3} \xrightarrow[\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}]{} \mathbb{Z} \longrightarrow 0$$

$$\begin{bmatrix} [ab : e^3] \\ [bc : e^3] \\ [ac : e^3] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



a "tiling"

- One 0-cell $x$
- Three 1-cells, $a$, $b$, and $c$ that are attached by the constant map to $x$
- Three 2-cell $ab$, $ac$, and $bc$ that is attached by the maps $aba^{-1}b^{-1}$, $aca^{-1}c^{-1}$, and $bcb^{-1}c^{-1}$
- One 3-cell $abc$

This gives us the chain complex

$$\dots \longrightarrow 0 \xrightarrow{0} \mathbb{Z}\{abc\} \xrightarrow{k} \mathbb{Z}\{ab, ac, bc\} \xrightarrow{j} \mathbb{Z}\{a, b, c\} \xrightarrow{i} \mathbb{Z}\{x\} \xrightarrow{0} 0 \ .$$

Note that $i$ is just the zero map because the attaching maps are constant. Also, $j$ is the zero map by the same calculation as the one for $T^2$, just with a bouquet of 3 circles instead of 2. Finally, we claim that $k$ is the zero map as well. This is because when we project the image of the attaching map onto one of the 2-cells we end up with two copies of that 2-cell with opposite orientation. Because the degree is 0 when projected onto each 2-cell, we see the map is in fact the zero map. Thus the homology groups are

- $H_0(T^2) = \mathbb{Z}$
- $H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
- $H_2(T^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$
- $H_3(T^2) = \mathbb{Z}$,

with all other homology groups being 0.

Compared to these last two problems, computing the cellular homology of $X = S^1 \vee S^1 \vee S^2$ is fairly simple. Its cell structure is given by

- One 0-cell $x$
- Two 1-cells, $a$, and $b$, that are attached by the constant map to $x$
- One 2-cell $\sigma$, that is attached by the constant map to $x$.

Because the attaching maps are constant maps, the connecting homomorphism in the chain complex are all the 0 map. That is we have the chain complex

$$\dots \longrightarrow 0 \xrightarrow{0} \mathbb{Z}\{\sigma\} \xrightarrow{0} \mathbb{Z}\{a, b\} \xrightarrow{0} \mathbb{Z}\{x\} \xrightarrow{0} 0 \ .$$

This is the exact same chain complex as the one for $T^2$ (recall $i$ and $j$ turned out to be the 0 map). Thus, they have the same homology. However,the two spaces are not homotopy equivalent because if they have different fundamental groups (the fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$ while the fundamental group of $X$ is $\mathbb{Z} \star \mathbb{Z}$).

(b) Let $X$ represent the punctured torus. Looking at $X$ as a square with a hole punched out of it and the sides identified as usual, we see it deformation retracts to a bouquet of two circles. The cellular structure for the bouquet is given by

- One 0-cell $x$
- Two 1-cells, $a$, and $b$, that are attached by the constant map

$$\ldots \longrightarrow 0 \xrightarrow{0} \mathbb{Z}\{a,b\} \xrightarrow{i} \mathbb{Z}\{x\} \xrightarrow{0} 0 \ .$$

The computation of $H_0$ and $H_1$ for this chain is the same as the computation for $H_0(T^2)$ and $H_1(T^2)$ in part $a$. Thus

$$H_0(X) = \mathbb{Z} \quad \text{and} \quad H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$$

(c) We will determine the cellular structure for $X$, a genus $g$ surface, by looking at the associated regular $4g$ sided polygon with its boundary given by $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$. From this we see the cell structure on $X$ is given by

- One 0-cell $x$
- 2g 1-cells, $a_1, b_1, \ldots, a_g, b_g$ , that are attached by the constant map to $x$
- One 2-cell $\sigma$, that is attached by the loop $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$.

This yields the chain complex

$$\ldots \longrightarrow 0 \xrightarrow{0} \mathbb{Z}\{\sigma\} \xrightarrow{j} \mathbb{Z}\{a_1, b_1, \ldots, a_g, b_g\} \xrightarrow{i} \mathbb{Z}\{x\} \xrightarrow{0} 0 \ .$$

From here, computing the homology is very similar to computing the homology for the torus. Because the attaching maps for the 1-cells is the constant map, we see that the connecting homomorphism, $i$ is just the zero map. Further, like when we computed the second connecting homomorphism for the torus, we see that when we project the image of $\sigma$ onto any of the 1-cells, we are left with the constant map. Hence, $j$ is the zero map as well. Thus the homology groups of $X$ are given by

- $H_0(T^2) = \mathbb{Z}$
- $H_1(T^2) = \mathbb{Z}^{\oplus 2g}$
- $H_2(T^2) = \mathbb{Z}$

with all other homology groups 0.

12. For a CW-complex, show that $\sum_\tau [\omega:\tau][\tau:\sigma] = 0$ for all $(n+1)$-cells $\sigma$ and $(n-1)$-cells $\omega$, and with $\tau$ ranging over all $n$-cells.

maybe 1/2

We require $\partial^2 = 0$. Let $A = \emptyset$. $\sigma$ $n+1$ cell, $\omega$ $n-1$ cell fixed.

By definition, $\partial(\sigma) = \sum_\tau [\tau:\sigma]\tau$.

<u>Whence</u> $\partial^2(\sigma) = \sum_\tau [\tau:\sigma]\partial[\tau]$ (linearity)

$= \sum_\tau [\tau:\sigma] \sum_\omega [\omega:\tau]\omega$

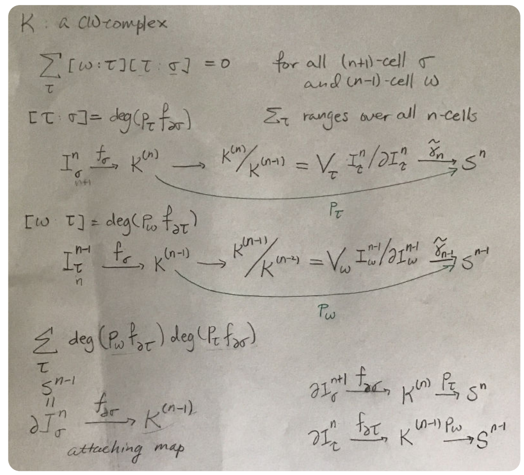
$= \sum_{\forall\tau,\omega} [\omega:\tau][\tau:\sigma]\omega.$

Knowing $\partial^2(\sigma) = 0$, we have $\sum_\tau [\omega:\tau][\tau:\sigma] = 0$

<u>Q</u> Why should it be that $\partial^2 = 0$?

<u>A</u> Consider the long exact sequence for a triple, write down the $\beta_n$ to see. Cf [Bre93, p.202].

As well, do consider that $\partial D^n$ is compact mapping into CW complex. TODO, show a compact subset of the CW complex meets only finitely many cells.


K : a CW-complex

$\sum_\tau [\omega:\tau][\tau:\sigma] = 0$ for all $(n+1)$-cell $\sigma$ and $(n-1)$-cell $\omega$

$[\tau:\sigma] = \deg(p_\tau f_{\partial\sigma})$ $\quad \Sigma_\tau$ ranges over all n-cells

$$I^n_\sigma \xrightarrow{f_\sigma} K^{(n)} \longrightarrow K^{(n)}/K^{(n-1)} = \vee_\tau I^n_\tau/\partial I^n_\tau \xrightarrow{\tilde{\gamma}_n} S^n, \quad p_\tau : K^{(n)} \to S^n$$

$[\omega:\tau] = \deg(p_\omega f_{\partial\tau})$

$$I^{n-1}_\tau \xrightarrow{f_\sigma} K^{(n-1)} \longrightarrow K^{(n-1)}/K^{(n-2)} = \vee_\omega I^{n-1}_\omega/\partial I^{n-1}_\omega \xrightarrow{\tilde{\gamma}_{n-1}} S^{n-1}, \quad p_\omega : K^{(n-1)} \to S^{n-1}$$

$\sum_\tau \deg(p_\omega f_{\partial\tau})\deg(p_\tau f_{\partial\sigma})$

$S^{n-1} = \partial I^n_\sigma \xrightarrow{f_{\partial\sigma}} K^{(n-1)}$ attaching map

$\partial I^{n+1}_\sigma \xrightarrow{f_{\partial\sigma}} K^{(n)} \xrightarrow{p_\tau} S^n$

$\partial I^n_\tau \xrightarrow{f_{\partial\tau}} K^{(n-1)} \xrightarrow{p_\omega} S^{n-1}$

# INCIDENCE NUMBERS IN CELLULAR HOMOLOGY

COLTON GRAINGER (SCRIBE) AND SHEN LU (PRESENTER)

This problem is set from Bredon [1, No. IV.11.12]. It demonstrates

- for a CW-complex $K$, the differential $\beta$ of the chain complex $C_*^{\mathrm{CW}}(K)$ satisfies $\beta^2 = 0$, and thus
- for an $n+1$ cell $\sigma$ and an $n-1$ cell $\omega$, we've $\sum_\tau [\omega : \tau][\tau : \sigma] = 0$ with $\tau$ ranging over all $n$-cells.

**Given.** Let $K$ be a CW-complex, with $n$-skeleton $K^{(n)}$ for $n \geq 0$. Because $K^{(n)}$ contains an open neighborhood around the closed subset $K^{(n-1)}$ that deform retracts onto $K^{(n-1)}$, we know:

- The relative homology $H_*(K^{(n)}, K^{(n-1)})$ is isomorphic to the reduced homology $\tilde{H}_*(K^{(n)}/K^{(n-1)})$.
- The quotient space $K^{(n)}/K^{(n-1)}$ is homeomorphic to the wedge $\vee(I^n/\partial I^{n-1}) \approx \vee S^n$, and thus

$$H_*(K^{(n)}, K^{(n-1)}) \xrightarrow[e_*]{\cong} \tilde{H}_*(K^{(n)}/K^{(n-1)}) \xrightarrow[\text{h.a.}]{\cong} \tilde{H}_*(\vee(I^n/\partial I^{n-1})) \xrightarrow{\cong} \bigoplus_{n\text{-cells of } K^{(n)}} \tilde{H}_*(S^n). \tag{1}$$

We may define a chain complex $C_*^{\mathrm{CW}}(K)$ associated to $K$ as follows:

- Let the chain group $C_n^{\mathrm{CW}}(K)$ be $H_n(K^{(n)}, K^{(n-1)})$. This is the free abelian group (in the $n$th degree of the graded group) at the end of (1) whose basis is the set of $n$-cells attached to $K^{(n-1)}$.
- Let the differential $\beta_n \colon C_n^{\mathrm{CW}}(K) \to C_{n-1}^{\mathrm{CW}}(K)$ be the composite

$$\begin{tikzcd} C_n^{\mathrm{CW}}(K) = H_n(K^{(n)}, K^{(n-1)}) \arrow[r, "\delta_n"] \arrow[rr, bend right, "\beta_n"'] & H_{n-1}(K^{(n-1)}) \arrow[r, "j_{n-1}"] & H_{n-1}(K^{(n-1)}, K^{(n-2)}) = C_{n-1}^{\mathrm{CW}}(K). \end{tikzcd} \tag{2}$$

In (2), the boundary map $\delta_n$ arises from the long exact sequence for the pair $(K^{(n)}, K^{(n-1)})$, and the map of relative homology groups $j_{n-1} \colon H_{n-1}(K^{(n-1)}) \to H_{n-1}(K^{(n-1)}, K^{(n-2)})$ is induced by the inclusion of skeleta $j \colon (K^{(n-1)}, \varnothing) \hookrightarrow (K^{(n-1)}, K^{(n-2)})$.

From lecture [2, No. 1.11.3], we know $\delta_n$ respects the attaching maps; for an $n$-cell $\sigma$ with attaching map $f_{\partial\sigma}$,

$$\delta_n[I^n_\sigma] = [f_{\partial\sigma}(\partial I^n_\sigma)].$$

And so, the differential $\beta_n$ can be described with "incidence numbers" [3, No. 8.5]. For an $n$-cell $\sigma$ and an $n-1$ cell $\tau$, define

$$[\tau : \sigma] := \deg\left( \begin{tikzcd} S^{n-1} \arrow[r, "f_{\partial\sigma}"] \arrow[d, phantom, "\wr"] & K^{(n-1)} \arrow[r] \arrow[rrr, bend right, "p_\tau"'] & K^{(n-1)}/K^{(n-2)} \arrow[r, "\approx"] & \vee S^{n-1} \arrow[r, "\text{find } \tau"] & S^{n-1} \arrow[d, phantom, "\wr"] \\ \partial I^n_\sigma & & & & I^{n-1}_\tau/\partial I^{n-1}_\tau \end{tikzcd} \right). \tag{3}$$

To make three comments. First, we take for granted the rule $\sigma \mapsto \sum_\tau [\tau : \sigma]\tau$ on generators $\sigma$ in $C_n^{\mathrm{CW}}(K)$ extends linearly and *is* the differential $\beta_n$ in (2). See [1, p. 203]. So write $\beta_n(\sigma) := \sum_\tau [\tau : \sigma]\tau$. Second, all but finitely many terms in the sum $\sum_\tau [\tau : \sigma]\tau$ must be zero. This is because the compact set $\partial I^n_\sigma$ is

*Date*: 2019-03-01.

attached by $f_{\partial\sigma}$ to a *compact* subset of $K^{(n-1)}$. Third, the projection $p_\tau$ that "finds" $\tau$ in (3) is the unique map $p_\tau \colon K^{(n-1)} \to S^{n-1}$ satisfying:

i. $p_\tau \circ f_\tau = \gamma_{n-1} =$ smash product $\gamma \wedge \cdots \wedge \gamma$ of $n-1$ copies of the quotient map $\gamma \colon I^1 \to S^1$
ii. $p_\tau \circ f_{\tau'} =$ constant map to base point, for $\tau' \neq \tau$.

Now, we almost done setting up results and rehashing definitions needed to make $C_*^{\mathrm{CW}}(K)$ a chain complex. It remains to argue that the differential $\beta$ is of order 2, i.e., that $\beta^2 = 0$. So consider the following three long exact sequences in relative homology. (This is Ulrike Tillmann's argument [3, No. 8.6].)

(4)
$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(K^{(n+1)}, K^{(n)}) \arrow[r, "\delta_{n+1}"] & H_n(K^{(n)}) \arrow[r] \arrow[dl, equal] & H_n(K^{(n+1)}) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_n(K^{(n)}) \arrow[r, "j_n"'] & H_n(K^{(n)}, K^{(n-1)}) \arrow[r, "\delta_n"] & H_{n-1}(K^{(n-1)}) \arrow[r] \arrow[dl, equal] & \cdots \\
 & \cdots \arrow[r] & H_{n-1}(K^{(n-1)}) \arrow[r, "j_{n-1}"'] & H_{n-1}(K^{(n-1)}, K^{(n-2)}) \arrow[r] & \cdots
\end{tikzcd}$$

Notice $\beta_n\beta_{n+1} = (j_{n-1}\delta_n)(j_n\delta_{n+1}) = j_{n-1}(\delta_n j_n)\delta_{n+1} = 0$, as $\delta_n j_n = 0$ by exactness of the middle row.

**To prove.** Let $K$ be a CW-complex. For all $n+1$ cells $\sigma$ and $n-1$ cells $\omega$,

$$\sum_\tau [\omega : \tau][\tau : \sigma] = 0, \tag{5}$$

where $\tau$ ranges over all $n$-cells.

**Proof.** We require $\beta^2 = 0$, as in (4). We also require $\beta(\sigma) = \sum_\tau [\tau : \sigma]\tau$, as discussed after (3). Whence

$$\begin{aligned}
\beta^2(\sigma) &= \sum_\tau [\tau : \sigma]\beta(\tau) && (\beta \text{ is linear})\\
&= \sum_\tau [\tau : \sigma] \sum_\omega [\omega : \tau]\omega && (\text{evaluate})\\
&= \sum_{\forall \tau, \omega} [\omega : \tau][\tau : \sigma]\omega. && (\mathbf{Z} \text{ is a commutative ring})
\end{aligned}$$

$C_{n-1}^{\mathrm{CW}}(K)$ is the free abelian group whose basis is the set of $n-1$ cells in $K$. So if $\beta^2(\sigma) = 0$, then the coefficient $[\omega : \tau][\tau : \sigma]$ of each $n-1$ cell $\omega$ had better be zero. Thus, fixing $\omega$, we conclude $\sum_\tau [\omega : \tau][\tau : \sigma]\omega = 0$. $\square$

**Remarks.**

i. Here's another way to remember (5) in the case that $K$ is finite. Take the matrices $[\beta_{n+1}]$ and $[\beta_n]$ representing $\beta_{n+1}$ and $\beta_n$ with respect to the *finite* bases for $C_{n+1}^{\mathrm{CW}}(K)$ and $C_n^{\mathrm{CW}}(K)$. Because $\beta$ is order two, $[\beta_n][\beta_{n+1}] = 0$ and (5) follows from matrix multiplication.

ii. If the coefficient $[\tau : \sigma]$ is the "incidence" of $\sigma$ to $\tau$, then the matrix $[\beta_n]$ suggests itself as the "incidence matrix" of the differential $\beta_n$.

iii. But the term "incidence matrix" is typically reserved for the following situation: Take the differential $\beta_1 \colon C_1^{\mathrm{CW}}(K) \to C_0^{\mathrm{CW}}(K)$. How does the matrix $[\beta_1]$ describe the *directed graph* whose vertices are 0-cells in $K^{(0)}$ and whose directed edges are oriented 1-cells in $K^{(1)}$?

# § THE HOMOTOPY AXIOM (Need to settle that sing. homology satisfies the axioms)

<u>DEF</u>| Say $A$ and $B$ are chain complexes. Let $\varphi, \psi : A \to B$ be chain maps. We say $\varphi$ and $\psi$ are <u>chain homotopic</u> if there are maps for all $p \in \mathbb{Z}$ s.th. $D : A_p \to B_{p+1}$ such that $\partial D + D\partial = \varphi - \psi$. (Note if $\varphi$ and $\psi$ are chain maps then $\varphi - \psi$ is too.)

$$\begin{tikzcd}
A_{p+1} \arrow[r, "\varphi-\psi"] \arrow[d, "\partial"] & B_{p+1} \arrow[d, "\partial"] \\
A_p \arrow[r, "\varphi-\psi"] \arrow[d, "\partial"] & B_p \arrow[d, "\partial"] \\
A_{p-1} \arrow[r, "\varphi-\psi"] \arrow[d, "\partial"] & B_{p-1} \arrow[d, "\partial"] \\
A_{p-2} \arrow[r, "\varphi-\psi"] & B_{p-2}
\end{tikzcd}$$

Homotopy should give isomorphisms on homotopy groups, so $A_*$ and $B_*$ should satisfy the following:

<u>LEMMA</u>| If $\varphi \simeq \psi$, then $\varphi_* = \psi_*$ as maps $H_*(A) \to H_*(B)$.

<u>Pf.</u> Let $[\![a]\!] \in H_p(A)$, $a \in Z_p(A)$. Then
$$\begin{aligned}\varphi_*([\![a]\!]) &= [\![\varphi(a)]\!] \\ &= [\![\psi(a) + \partial D(a) + D\partial(a)]\!] \\ &= [\![\psi(a)]\!] \\ &= \psi_*([\![a]\!]).\end{aligned}$$
cycle

<u>IDEA</u>| Chain homotopy is an equiv. relation. (TODO)
Therefore we've an algebraic analogue $K(Ch)$ homotopy cat. of $Ch$
$Obj(K(Ch)) = Obj(Ch)$, yet $Mor(K(Ch)) = Mor(K)/\cong$ homotopic.

Chain homotopic Chain complexes  Assume singular homology.  20190208

<u>DEF</u>| $\varphi, \psi : A_* \to B_*$ are chain homotopic if there's a $D$ that raises index $D : A_p \to D_{p+1}$ for all $p$ s.th. $D\partial + \partial D = \varphi - \psi$.

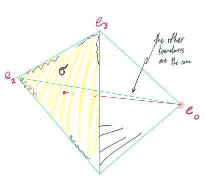
<u>DEF</u>| "chain homotopy equiv" "triangulated category"
if $\varphi \simeq \psi$ as chain maps, then $H_*(\varphi) = H_*(\psi)$.

<u>THM</u>| Say $X$ has $id : X \to X$ null homotopic. Then $H_*(X) \cong H_*(pt)$.
<u>Pf.</u> Let $F : X \times I \to X$ $F(0,x) = x$, $F(x,1) = x_0$, $\forall x \in X$. Recall the augmentation map $\varepsilon : \Delta_*(X) \to \Delta_*(X)$ sth

$$\varepsilon : \Delta_p(X) \xrightarrow{0} \Delta_p(X) \qquad p > 0$$
$$\varepsilon : \Delta_0(X) \longrightarrow \Delta_0(X) \text{ and } \varepsilon\left(\sum n_x x\right) = \left(\sum n_x\right) x_0.$$

<u>WTS</u>| $\Delta_*(id) \cong \varepsilon$ as chain maps so that $H_*(\varepsilon)$ induces the identity map?

Explicitly,
$$\begin{tikzcd}
\arrow[d] & \arrow[d] \\
\Delta_n(X) \arrow[r, "1"] \arrow[d] & \Delta_n(X) \arrow[d] \\
\Delta_{n-1}(X) \arrow[r, "1"] \arrow[d] \arrow[ur, "D"] & \Delta_{n-1}(X) \arrow[d] \\
\Delta_{n-2}(X) \arrow[r, "1"] \arrow[d] \arrow[ur, "D"] & \Delta_{n-2}(X) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$
sth $D\partial + \partial D = 1 - \varepsilon$.




We need a fn to contract the $n-1$ simplex as a face of the $n$-simplex.  "σ doesn't eat $e_n$'s for dinners"

$$(D\sigma)\left(\sum_{i=0}^{n} \lambda_i e_i\right) = \begin{cases} F\left(\sigma\left(\sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_0} e_{i-1}\right) \lambda_0\right), & \lambda_0 \neq 1 \\ x_0, & \lambda_0 = 1 \end{cases}$$

$$(D\sigma)^{(i)} = \begin{cases} D\sigma^{(i-1)} & 0 < i \leq n \quad \text{☆ easy} \\ \sigma & i = 0 \quad \text{☆ } \underline{\text{hard}} \text{ TODO, lots of indices} \end{cases}$$

If $n > 1$, then $\partial D\sigma = \sum_{i=0}^{n} (-1)^i (D\sigma)^{(i)}$
$$= \sigma - \sum_{i=1}^{n} (-1)^i (D\sigma)^{(i)} = \sigma - \sum_{j=0}^{n-1} (-1)^j (D(\sigma^{(j)})) = \sigma - D\partial\sigma. \text{ Thus } \partial D\sigma + D\partial\sigma = \sigma - \overset{0}{\overset{\parallel}{\varepsilon(\sigma)}}.$$

Else if $n = 1$, then $\partial D\sigma = (D\sigma)^{(0)} - (D\sigma)^{(1)} = \sigma - x_0 = \sigma - \varepsilon(\sigma)$.

IDEA $\left\{\begin{array}{l}\text{One chain map does the same as} \\ \text{the other up to the } n^{th} \text{ deg}\end{array}\right.$

Thus $\partial D\sigma + D\partial\sigma = \sigma - \varepsilon(\sigma)$ in all cases, for all $\sigma \in Sing_n(X)$ for all $n \geq 0$. □

**16.1. Theorem.** *There exist bilinear maps* $\times : \Delta_p(X) \times \Delta_q(Y) \to \Delta_{p+q}(X \times Y)$ *such that:*

(1) *for* $x \in X$, $y \in Y$, $\sigma : \Delta_q \to Y$, *and* $\tau : \Delta_p \to X$, $x \times \sigma$ *and* $\tau \times y$ *are as described above;*

(2) *(naturality) if* $f : X \to X'$ *and* $g : Y \to Y'$ *and if* $\langle f, g \rangle : X \times Y \to X' \times Y'$ *denotes the product map, then*

$$\langle f, g \rangle_\Delta(a \times b) = f_\Delta(a) \times g_\Delta(b); \quad \textit{and}$$

(3) *(boundary formula)* $\partial(a \times b) = \partial a \times b + (-1)^{\deg a} a \times \partial b$.

**THM** There's a bilinear map $\times : \Delta_p(X) \times \Delta_q(Y) \to \Delta_{p+q}(X \times Y)$ s.th. HOT!

(1) for $x \in X$, $y \in Y$ ...

(2) The cross product is natural.

(3) $\partial(\sigma \times \tau) = \partial\sigma \times \tau + (-1)^p(\sigma \times \partial\tau)$.

TODO Rotman for const. of prism operator.

Take $\Delta_p(X) \times \Delta_q(Y)$ and want to form a canonical

For pairs of spaces $(X,A), (Y,B)$, there's an induced map $\times : H_p(X,A) \times H_q(Y,B) \to H_{p+q}(X \times Y, \underbrace{(X \times B) \cup (A \times Y)}_{\shortparallel\ (X,A) \times (Y,B)})$

- coherence verification

$[\![a]\!] \times [\![b]\!] \longmapsto [\![a \times b]\!]$

Recall from last time: 20190311

**THM** Say $X$ has id $: X \to X$ null homotopic. Then $H_*(X) \cong H_*(pt)$.

**IDEA** To show $H_*(-)$ singular homology satisfies the homotopy axiom, we needed the "cross product"!

(1) We assume $\sigma_x : pt \to x \in X$. Then with $x \in Sing_0(X)$ and $\tau \in Sing_p(Y)$ $x \times \tau : \Delta_p \to X \times Y$ s.th. $(x \times \tau)(z) = (x, \tau(z)) \in X \times Y$. Moreover, for $\sigma \times y \in Sing_p(X)$ TODO

(2) Suppose $f : X \to X'$ and $g : Y \to Y'$ are morphisms in Top

$$\begin{tikzcd}
\Delta_p(X) \times \Delta_q(Y) \arrow[r, "f_\Delta \times g_\Delta"] \arrow[d, "\times"] & \Delta_p(X') \times \Delta_q(Y') \arrow[d, "\times"] \\
\Delta_{p+q}(X \times Y) \arrow[r, "(f \times g)_\Delta"] & \Delta_{p+q}(X' \times Y')
\end{tikzcd}$$

Pf sketch. Induct on $p+q$. Say $\Delta_i(X) \times \Delta_j(Y) \xrightarrow{\times} \Delta_{i+j}(X \times Y)$ for $i+j < p+q$ has been def'd and satisfies (1–3). Let $\iota_p : \Delta_p \to \Delta_p$ be the identity. Then $\iota_p \in \Delta_p(\Delta_p)$.

Construct $\iota_p \times \iota_q : \Delta_p(\Delta_p) \times \Delta_q(\Delta_q) \to \Delta_{p+q}(\Delta_p \times \Delta_q)$. It's sufficient to define $\iota_p \times \iota_q$ so that by (2) with $\sigma : \Delta_p \to X$ s.th. $\sigma = \sigma_\Delta(\iota_p)$ and $\tau : \Delta_q \to Y$ s.th. $\tau = \tau_\Delta(\iota_q)$.

**TFDC**

$$\begin{tikzcd}
\Delta_p(\Delta_p) \times \Delta_q(\Delta_q) \arrow[r, "\sigma_\Delta \times \tau_\Delta"] \arrow[d, "\times"] \arrow[dr, "\sigma \times \tau \ \circ"] \arrow[dr, bend left, "(\sigma \times \tau)_\Delta(\iota_p \times \iota_q)"] & \Delta_p(X) \times \Delta_q(Y) \arrow[d, "\times"] \\
\Delta_{p+q}(\Delta_p \times \Delta_q) \arrow[r, "(\sigma \times \tau)_\Delta"] & \Delta_{p+q}(X \times Y)
\end{tikzcd}$$

Yoneda lemma. TODO
$\Delta_p(-)$ is a representable functor, as a Hom. $\Delta_p$ is a rep'ble obj.
We get natural transformations.

If the bdry formula (3) holds, then we shall have $\partial(\iota_p \times \iota_q) = \underbrace{\partial\iota_p \times \iota_q + (-1)^p \iota_p \times \partial\iota_q}_{\text{inductively defined}} \in \Delta_{p+q-1}(\Delta_p \times \Delta_q)$

Lastly, should $\partial(\iota_p \times \iota_q) = 0$, then $\iota_p \times \iota_q$ exists in $\Delta_p \times \Delta_q$. But $\partial(\iota_p \times \iota_q) = 0$ as $\Delta_p \times \Delta_q \simeq \{pt\}$. (contractible)

Note $\partial^2$ should be $0$. So $\partial^2(\iota_p \times \iota_q) = \partial^2\iota_p \times \iota_q + (-1)^{p-1}(\partial\iota_p \times \partial\iota_q) + (-1)^p(\partial\iota_p \times \partial\iota_q) + (-1)^p(\iota_p \times \partial^2\iota_q)$ by (3) applied inductively. We let $\partial(\iota_p \times \iota_q) = \xi$. Since $\xi$ is a cycle, it lifts to a boundary. Let $\iota_p \times \iota_q$ be the map s.th. $\partial(\iota_p \times \iota_q) = \xi$.

logical ordering here ???

**Lemma** Proof. WTS $\exists D : \Delta_n(X) \to \Delta_{n+1}(\Delta^1 \times X)$ with $D\partial + \partial D = (\eta_1)_\Delta - (\eta_0)_\Delta$

Define $D = \iota_1 \times (-) : \Delta_n(X) \to \Delta_{n+1}(\Delta^1 \times X)$. Observe

$(\eta_i)_\Delta(\sigma)(x) = (\eta_i \circ \sigma)(x) = (e_i, \sigma(x)) = (e_i \times \sigma)(x)$. by defn (f)$_\Delta$ and (1).

For the boundary: $\partial(\iota_1 \times \sigma) = \partial\iota_1 \times \sigma - \underbrace{\iota_1 \times \partial\sigma}_{= D\partial\sigma}$

$= (e_1 - e_0) \times \sigma - D\partial\sigma$

$= (e_1 - \sigma) - e_0 \times \sigma - D\partial\sigma$

$= (\eta_1)_\Delta(\sigma) - (\eta_0)_\Delta(\sigma) - D\partial\sigma$. ∎

**CORO** Say $f, g : X \to Y$ with $f \simeq g$. Then $f_* = g_*$.

Pf. Let $h : \Delta^1 \times X \to Y$ be a homotopy. Then with

$\eta_i : X \to \Delta^1 \times X$ s.th. $\eta_i(x) = (e_i, x)$ for $i = 0, 1$,

Then $h\eta_0 = f$, $h\eta_1 = g$. Thus $f_* = (h\eta_0)_* = h_*\eta_{0*}$
$= h_*\eta_{1*}$
$= (h\eta_1)_*$
$= g_*$. ∎

# § Exercises (Homotopy Axiom)

## CHECK

Show that if $\varphi \simeq \psi$ as chain maps, then $H_*(\varphi) = H_*(\psi)$.

## VERIFY

For pairs of spaces $(X, A), (Y, B)$, there's an induced map $\times : H_p(X,A) \times H_q(Y,B) \longrightarrow H_{p+q}(\underbrace{X \times Y, (X \times B) \cup (A \times Y)}_{= (X,A) \times (Y,B)})$

- coherence verification

$[\![a]\!] \times [\![b]\!] \longmapsto [\![a \times b]\!]$

## LOOK-UP or ASK SEBASTIAN

$\Delta_p(-)$ is a representable functor, as a Hom. $\Delta_p$ is a rep'ble obj.

Yoneda lemma. We get natural transformations.

## COMPUTE

$$(D\sigma)^{(i)} = \begin{cases} D\sigma^{(i-1)} & 0 < i \leq n \quad \text{* easy} \\ \sigma & i = 0 \quad \text{* hard TODO, lots of indices} \end{cases}$$

**Exercise 1.4.5** In this exercise we shall show that the chain homotopy classes of maps form a quotient category **K** of the category **Ch** of all chain complexes. The homology functors $H_n$ on **Ch** will factor through the quotient functor $\mathbf{Ch} \to \mathbf{K}$.

1. Show that chain homotopy equivalence is an equivalence relation on the set of all chain maps from $C$ to $D$. Let $\mathrm{Hom}_{\mathbf{K}}(C, D)$ denote the equivalence classes of such maps. Show that $\mathrm{Hom}_{\mathbf{K}}(C, D)$ is an abelian group.
2. Let $f$ and $g$ be chain homotopic maps from $C$ to $D$. If $u: B \to C$ and $v: D \to E$ are chain maps, show that $vfu$ and $vgu$ are chain homotopic. Deduce that there is a category **K** whose objects are chain complexes and whose morphisms are given in (1).
3. Let $f_0$, $f_1$, $g_0$, and $g_1$ be chain maps from $C$ to $D$ such that $f_i$ is chain homotopic to $g_i$ ($i = 1, 2$). Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that **K** is an additive category, and that $\mathbf{Ch} \to \mathbf{K}$ is an additive functor.
4. Is **K** an abelian category? Explain.

# § Mayer-Vietoris

THM (Mayer-Vietoris). Suppose

$$X = \operatorname{int}(X_1) \cup \operatorname{int}(X_2).$$

There's a l.e.s.

$$\to H_n(X_1 \cap X_2) \xrightarrow{(i_1)_* \oplus (i_2)_*} H_n(X_1) \oplus H_n(X_2) \xrightarrow{(j_1)_* - (j_2)_*} H_n(X) \xrightarrow{D} H_{n-1}(X_1 \cap X_2) \to \cdot$$

which arises from

$$\begin{tikzcd} H_n(X) \arrow[r] \arrow[drr, "D"] & H_n(X, X_2) & & \text{les for } (X, X_2) \\ & H_n(X_1, X_1 \cap X_2) \arrow[u, "e_*", "\cong"'] \arrow[r, "\partial_*"] & H_{n-1}(X_1 \cap X_2) & \text{les for } (X_1, X_1 \cap X_2). \end{tikzcd}$$

IDEA

Equiv. to excision.

# § Exercises

# ARGUE

**CHAPTER 6 CLASSIFICATION OF SURFACES**

**Definition.** An $n$-manifold without boundary is a (Hausdorff and second countable) topological space $M$ with the property that for each $x \in M$ there is an open set $U$, with $x \in U \subset M$, such that $U$ is homeomorphic to $\mathbb{R}^n$.

**Definition.** Suppose $M_1$ and $M_2$ are connected $n$-manifolds and $B_1$ and $B_2$ are $n$-balls with $B_1 \subset M_1$ and $B_2 \subset M_2$. The manifold $(M_1 - \operatorname{int} B_1) \cup_h (M_2 - \operatorname{int} B_2)$, where $h$ is a homeomorphism from the boundary of $B_1$ to the boundary of $B_2$, is called the connected sum $M_1 \# M_2$.

**6.1 Theorem*.** *Let $M$ be a compact connected 2-manifold without boundary then $M$ is homeomorphic to one and only one of*

(a) $M_0 = S^2$

(b) $M_g = T \# T \# \ldots \# T$, *where $T$ is the torus $S^1 \times S^1$ and there are $g$ summands*

(c) $N_h = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$, *where $\mathbb{R}P^2$ is the real projective plane and there are $h$ summands.*

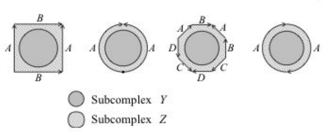
why?

**Note.** The Mayer-Vietoris Theorem implies that

$$H_1(M_g) \cong \bigoplus_{2g \text{ copies}} \mathbb{Z}, \qquad H_1(N_h) \cong \mathbb{Z}/2 \oplus \bigoplus_{(h-1) \text{ copies}} \mathbb{Z}.$$

# COMPUTE

3. For $X$ the torus, the sphere, the two-holed torus, and the projective plane, assume there is a triangulation $X = Y \cup Z$ as pictured (with $Y \cap Z$ homeomorphic to the circle), and use the Mayer-Vietoris sequence to compute $H_*(X)$.



# § Simplicial homology

**Algebraic Topology 2004** **Example Sheet 3s**

Supplement

$1\frac{1}{2}$. Here is a more careful articulation of the definition of triangulation. Recall that the standard $n$-simplex $\Delta[n]$ has vertices $\{0, \ldots, n\}$; a *subsimplex* of $\Delta[n]$ is a (affine) map $\Delta[i] \to \Delta[n]$ induced by an order preserving injection $\{0, \ldots, i\} \to \{0, \ldots, n\}$ – so $\Delta[n]$ has precisely $2^{n+1}$ subsimplices (including the non-proper subsimplex id: $\Delta[n] \to \Delta[n]$). If $\sigma \colon \Delta[n] \to X$ is a map, we will call the composite $\Delta[i] \to \Delta[n] \to X$ a subsimplex of $\sigma$. We write $|\sigma|$ for the image of $\sigma$ in $X$ and $|\mathring{\sigma}|$ for the image of $\Delta[n] - \partial\Delta[n]$ (note $\partial\Delta[0]$ is empty). With these clarifications, the definition from class (repeated below) is now precise.

**Definition** Let $X$ be a compact hausdorff space. A (finite) triangulation on $X$ consists of a finite set $\mathcal{T}$ of maps $\sigma \colon \Delta[n] \to X$ that are homeomorphisms onto their images, such that:

(i) If $\sigma \colon \Delta[n] \to X$ is in $\mathcal{T}$, then every subsimplex of $\sigma$ is in $\mathcal{T}$.
(ii) Every element of $X$ is in $|\mathring{\sigma}|$ for a unique $\sigma \in \mathcal{T}$.
(iii) If $|\sigma| \cap |\tau|$ is non-empty for $\sigma, \tau \in \mathcal{T}$, then there exists $\rho \in \mathcal{T}$ that is a subsimplex of $\sigma$ and a subsimplex of $\tau$ and that satisfies $|\rho| = |\sigma| \cap |\tau|$.

Show that the underlying combinatorial structure of $\mathcal{T}$ is a simplicial complex:

(a) Let $V = \{\sigma \colon \Delta[0] \to X \mid \sigma \in \mathcal{T}\}$, and let $S \subset \mathcal{P}V$ be

$$S = \{\{a_0, \ldots, a_n\} \subset V \mid \text{There exists } \sigma \in \mathcal{T} \text{ such that } a_0, \ldots, a_n \text{ are subsimplexes of } \sigma\}.$$

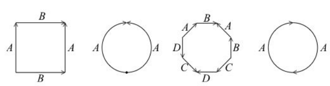
Show that $V, S$ defines a simplicial complex

(b) Show that the elements of $S$ are in one-to-one correspondence with the elements of $\Sigma$ (say $A \mapsto \sigma_A$) such that $A \subset B$ if and only if $\sigma_A$ is a subsimplex of $\sigma_B$.

# § Exercises

## TRIANGULATE

1. Polygon gluing diagrams are not triangulations of the spaces. Subdivide the gluing diagrams for the torus, the sphere, the two-holed torus[1], and the projective plane to get triangulations.




## COMPUTE

7. Use your triangulations from problem 1 to compute the homology groups of: the torus, the sphere, the two-holed torus, and the projective plane.

**Application 1.1.3** (Simplicial homology) Here is a topological application we shall discuss more in Chapter 8. Let $K$ be a geometric simplicial complex, such as a triangulated polyhedron, and let $K_k$ $(0 \le k \le n)$ denote the set of $k$-dimensional simplices of $K$. Each $k$-simplex has $k+1$ faces, which are ordered if the set $K_0$ of vertices is ordered (do so!), so we obtain $k+1$ set maps $\partial_i \colon K_k \to K_{k-1} (0 \le i \le k)$. The *simplicial chain complex* of $K$ with coefficients in $R$ is the chain complex $C_.$, formed as follows. Let $C_k$ be the free $R$-module on the set $K_k$; set $C_k = 0$ unless $0 \le k \le n$. The set maps $\partial_i$ yield $k+1$ module maps $C_k \to C_{k-1}$, which we also call $\partial_i$; their alternating sum $d = \sum(-1)^i \partial_i$ is the map $C_k \to C_{k-1}$ in the chain complex $C_.$. To see that $C_.$ is a chain complex, we need to prove the algebraic assertion that $d \circ d = 0$. This translates into the geometric fact that each $(k-2)$-dimensional simplex contained in a fixed $k$-simplex $\sigma$ of $K$ lies on exactly two faces of $\sigma$. The homology of the chain complex $C_.$ is called the *simplicial homology* of $K$ with coefficients in $R$. This simplicial approach to homology was used in the first part of this century, before the advent of singular homology.

## Wednesday, March 13

(1) Chapter IV, Section 12 of Bredon (p.214), problems 1, 2, 3

(2) Read Chapter IV, Section 13 of Bredon (p. 216). Do problems 1, 2 and 4 of that section.

(3) Chapter IV, Section 18 of Bredon (p.230), problems 1, 2, 3

(4) Read Chapter IV, Section 17 starting on p.223 of Bredon. There is no problem to solve here, but we will have a discussion about it in class.

$\frac{3}{3}$ 1 Compute $H_i(\mathbf{S}^p \times \mathbf{S}^q)$.

$\frac{2}{3}$ 2 Compute $H_i(\mathbf{P}^2 \times \mathbf{P}^2)$.

$\frac{2}{3}$ 3 Compute the homology of the product of a Klein bottle and a real projective plane.

**12.1** We compute $H_*(S^p \times S^q)$ for all $p, q \geq 0$.

Let both $S^p$ and $S^q$ have the cell structure of a point and a single $p$-cell, resp., $q$-cell attached. To wit:

$$\boxed{S^{p(0)} = S^{q(0)} = \{e\}, \quad S^{p(p)} = \{p\}, \quad S^{q(q)} = \{q\}}$$

Notice in any case $p \geq 0$, $\partial p = 0$, likewise for $q$.
Therefore on generators, $\partial(p \times q) = \cancel{\partial p}^{0} \times q + (-1)^p p \times \cancel{\partial q}^{0} = 0$
$\partial(p \times e) = \partial p \times e - (-1)^p p \times \partial e = 0$, $\partial(e \times q) = \partial e \times q + (-1)^0 e \times \partial q = 0$.
Since the boundary operator for $C_*(S^p \times S^q)$ is trivial at each level, $H_*(S^p \times S^q)$ is just $C_*(S^p \times S^q)$ with the differential forgotten.

If we define the direct sum in Graded Ab to be the levelwise direct sum, then $H_*(S^p \times S^q) \cong H_*(S^p) \oplus H_*(S^q)$. no! □

1) Let $S^p$ and $S^q$ be CW-complexes with $K^{(1)} = \{e\}$ and $K^{(p)} = p$ and $K^{(q)} = q$. Then $S^p \times S^q$ as a CW complex has cell. homology

CASE $p - q > 1$ and $q \geq 2$

$0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \to 0 \to \cdots \to \mathbb{Z} \to 0$

$\mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z}$

CASE $p - q > 1$ and $q = 1$

$0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \cdots 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$

$\mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z}$

$\partial(p \times q) = \cancel{\partial p}^{0} \times q + (-1)^p (p \times \cancel{\partial q})^{0} = 0$

CASE $p - q > 1$ and $q = 0$

$0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z}^{\oplus 2} \to 0$

$\mathbb{Z} \quad \mathbb{Z}^{\oplus 2}$

$\partial(p \times q) = \cancel{\partial p}^{0} \times q + (-1)^p p \times \cancel{\partial q}^{0} = 0$

CASE $p - q = 1$ and $q \geq 2$

$0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to \cdots \to \mathbb{Z} \to 0$

$\mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z}$

$\partial(p \times q) = \cancel{\partial p}^{0} \times q + (-1)^p p \times \cancel{\partial q}^{0}$ duh
$\partial(p \times e) = \cancel{\partial p}^{0} \times e + (-1)^p p \times \cancel{\partial e}^{0}$
$\partial(e \times q) = \cancel{\partial e}^{0} \times q + e \times \cancel{\partial q}^{0}$ duh

CASE $p - q = 1$ and $q = 1$

$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$

$\mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z}$

$\partial p = 0$
$\partial q = 0$
$\partial e = 0$

CASE $p - q = 1$ and $q = 0$

$0 \to \mathbb{Z}^{\oplus 2} \to \mathbb{Z}^{\oplus 2} \to 0$

$\mathbb{Z}^{\oplus 2} \quad \mathbb{Z}^{\oplus 2}$

$\partial p = 0$, $\partial q = 0$, $\partial e = 0$

CASE $p = q$, $q \geq 2$

$0 - \mathbb{Z} \cdots 0 \cdots \mathbb{Z}^{\oplus 2} \cdots 0 \cdots \mathbb{Z} - 0$

$\mathbb{Z} \quad \mathbb{Z}^{\oplus 2} \quad \mathbb{Z}$

CASE $p = q = 1$

$0 \to \mathbb{Z} \to \mathbb{Z}^{\oplus 2} \to \mathbb{Z} \to 0$

$\mathbb{Z} \quad \mathbb{Z}^{\oplus 2} \quad \mathbb{Z}$

$\partial p = 0$, $\partial q = 0$, $\partial e = 0$

CASE $p = q = 0$

$0 \to \mathbb{Z}^{\oplus 4} \to 0$

$\mathbb{Z}^{\oplus 4}$

**12.2** Let $\mathbb{P}^2$ have the CW-structure given by the quotient of $S^2$ by $T$, where $S^2$ has two $k$-cells in each dimension $k = 0, 1, 2$ given by the two hemispheres of $S^k \subset S^2$. Let these $k$-cells be denoted $\sigma_k$ and $T\sigma_k$. We identify points of $\sigma_k$ and $T\sigma_k$ with the antipodal map $T: S^2 \to S^2$, so that [Bre93, §IV.14]

- $\mathbb{P}^2$ has one $k$-cell $\sigma_k$ in each dimension $k = 0, 1, 2$
- $\partial \sigma_k = (1 + (-1)^k)\, \sigma_{k-1}$ is the boundary map.

Then $C_*^{CW}(\mathbb{P}^2)$ is $\cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$,

with homology $H_*(\mathbb{P}^2) = \{\mathbb{Z}$ if $* = 2$, $\mathbb{Z}/(2)$ if $* = 1$, $\mathbb{Z}$ if $* = 0$, $\{0\}$ else$\}$.

Construct a CW-complex structure on $\mathbf{S}^n$ with exactly two cells in each dimension $i$, $0 \leq i \leq n$, by letting the $i$-cells be two hemispheres of $\mathbf{S}^i \subset \mathbf{S}^n$ for each $i$. For $k \leq n$ denote the two $k$-cells by $\sigma_k$ and $T\sigma_k$. We can take the characteristic map of the latter to be

$$f_{T\sigma_k} = T \circ f_{\sigma_k}$$

where $T: \mathbf{S}^n \to \mathbf{S}^n$ is the antipodal map. Note that $p_{T\sigma} f_{T\sigma} = p_\sigma f_\sigma$ since both are equal to $\gamma_k$. Also note that the first of these equals $p_{T\sigma} T f_\sigma$. Since the equation $p_\sigma f_\sigma = \gamma_k$ characterizes the projection $p_\sigma$ we conclude that

$$p_{T\sigma} T = p_\sigma.$$

Now the composition $p_{\sigma_{k-1}} f_{\partial \sigma_k}: \mathbf{S}^{k-1} \to \mathbf{S}^{k-1}$ collapses a hemisphere to a point and is otherwise a homeomorphism. This is clearly homotopic to a homeomorphism and thus has degree $\pm 1$. (This also follows directly from Proposition 7.2 or Corollary 7.5.)

$C_*^{CW}(\mathbb{P}^2 \times \mathbb{P}^2)$ is therefore

$$\begin{array}{ccccccccccc} & & \text{4 cells} & & \text{3 cells} & & \text{2 cells} & & \text{1 cells} & & \text{0-cells} \\ 0 & \to & \mathbb{Z}\{\sigma_2 \times \mu_2\} & \to & \mathbb{Z}\{\sigma_2 \times \mu_1, \sigma_1 \times \mu_2\} & \to & \mathbb{Z}\{\sigma_2 \times \mu_0, \sigma_0 \times \mu_2, \sigma_1 \times \mu_1\} & \to & \mathbb{Z}\{\sigma_1 \times \mu_0, \sigma_0 \times \mu_1\} & \to & \mathbb{Z}\{\sigma_0 \times \mu_0\} \to 0 \end{array}$$

After brief computations with the boundary operator on a product of CW-complexes,

$\partial(\sigma_2 \times \mu_2) = \partial\sigma_2 \times \mu_2 + (-1)^2 \sigma_2 \times \partial\mu_2 = 2\sigma_1 \times \mu_2 + \sigma_2 + 2\mu_1$ $\quad [\partial_4] = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\quad$ $\ker \partial_4 = \{0\}$, so $H_4 = 0$

$\partial(\sigma_2 \times \mu_1) = 2\sigma_1 \times \mu_1 + (-1)^2 \sigma_2 \times 0 = 2\sigma_1 \times \mu_1$
$\partial(\sigma_1 \times \mu_2) = \cancel{\partial\sigma_1}^{0} \times \mu_2 + (-1)^1 \sigma_1 \times \partial\mu_2 = -\sigma_1 \times 2\mu_1$ $\quad [\partial_3] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & -2 \end{bmatrix}$ $\quad$ $\ker \partial_3 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$, $\operatorname{im} \partial_4 = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\}$, so $H_3 = \mathbb{Z}/(2)$

$\partial(\sigma_2 \times \mu_0) = \partial\sigma_2 \times \mu_0 + (-1)^2 \sigma_2 \times \cancel{\partial\mu_0}^{0} = 2\sigma_1 \times \mu_0$
$\partial(\sigma_0 \times \mu_2) = \cancel{\partial\sigma_0}^{0} \times \mu_2 + (-1)^2 \sigma_0 \times \partial\mu_2 = \sigma_0 \times 2\mu_1$
$\partial(\sigma_1 \times \mu_1) = \cancel{\partial\sigma_1}^{0} \times \mu_1 - \sigma_1 \times \cancel{\partial\mu_1}^{0} = 0$ $\quad [\partial_2] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ $\quad$ $\ker \partial_2 = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$, $\operatorname{im} \partial_3 = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}\right\}$, so $H_2 = \mathbb{Z}/(2)$

$\ker \partial_1 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$, $\operatorname{im} \partial_2 = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right\}$ so $H_1 = [\mathbb{Z}/(2)]^{\oplus 2}$

$\partial(\sigma_1 \times \mu_0) = \cancel{\partial\sigma_1}^{0} \times \mu_0 + 0$
$\partial(\sigma_0 \times \mu_1) = 0 + (-1)^0 \sigma_0 \times \cancel{\partial\mu_1}^{0}$ $\quad [\partial_1] = [0 \;\; 0]$

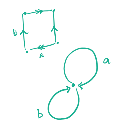
lastly, $H_0 = \mathbb{Z}$

$0 \quad * \geq 4$

We conclude $H_*(\mathbb{P}^2 \times \mathbb{P}^2) = \begin{cases} \mathbb{Z}/2 & * = 2,3 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & * = 1 \\ \mathbb{Z} & * = 0 \end{cases}$ $\square$

## 12.3 Recall that KB has CW structure

$KB^{(0)} = \{e_0\}$, $KB^{(1)} = \{a, b\}$, $KB^{(2)} = \{e_2\}$ with $f_{\partial e_2}$ tracing $a * b * a * b^{-1}$.



$a * b * a^{-1} * b$

$K^{(2)} = \{e^2\}$, $K^{(1)} = \{a^1, b^1\}$, $K^{(0)} = \{e^0\}$.

Suppose $f_{\partial a^1}, f_{\partial b^1} : \partial I^1 \to K^{(0)}$, the same unique map.

Let $f_{\partial e^2} : \partial I^2 \longrightarrow K^{(1)}$ trace the path $a^1 * b^1 * a^1 * (b^1)^{-1}$.

$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} \mathbb{Z}^{\oplus 2} \xrightarrow{[0\ 0]} \mathbb{Z} \to 0$$

$\begin{bmatrix} 0 \\ 2 \end{bmatrix}[e^2] = \begin{bmatrix} 0 \\ 2e^2 \end{bmatrix}$, $[0\ 0]\begin{bmatrix} a^1 \\ b^1 \end{bmatrix} = 0$

$$\begin{matrix} 0 & 0 & \mathbb{Z} \oplus \mathbb{Z}_2 & \mathbb{Z} \\ \parallel & \parallel & \parallel & \parallel \\ H_k(KB) & H_2(KB) & H_1(KB) & H_0(KB) \end{matrix}$$

$k \geq 3$

$$C_*^{CW}(KB) \text{ is } \quad 0 \longrightarrow \underset{2\text{ cells}}{\mathbb{Z}\{e_2\}} \xrightarrow[\begin{bmatrix} 0 \\ 2 \end{bmatrix}]{} \underset{1\text{ cells}}{\mathbb{Z}\{a, b\}} \xrightarrow[{[0\ 0]}]{} \underset{0\text{ cell}}{\mathbb{Z}\{e_0\}} \longrightarrow 0.$$

$$C_*^{CW}(\mathbb{P}^2) \text{ is } \quad 0 \longrightarrow \mathbb{Z}\{\sigma_2\} \xrightarrow[{[2]}]{} \mathbb{Z}\{\sigma_1\} \xrightarrow[{[0]}]{} \mathbb{Z}\{\sigma_0\} \longrightarrow 0$$

Thus $C_*^{CW}(\mathbb{P}^2 \times KB)$ is

$0 \longrightarrow \mathbb{Z}\{\sigma_2 \times e_2\} \cong \mathbb{Z}$

$\partial \sigma_2 \times e_2 + (-1)^2 \sigma_2 \times \partial e_2 = 2\sigma_1 \times e_2 + \sigma_2 \times 2a$

$\therefore [\partial_4] = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

$\xrightarrow{\partial_4} \mathbb{Z}\{\sigma_2 \times a, \sigma_2 \times b, \sigma_1 \times e_2\} \cong \mathbb{Z}^{\oplus 3}$

$\partial(\sigma_2 \times a) = \partial\sigma_2 \times a + (-1)^2 \sigma_2 \times \cancel{\partial a}^{\,0} = 2\sigma_1 \times a$

$\partial(\sigma_2 \times b) = 2\sigma_1 \times b$

$\partial(\sigma_1 \times e_2) = -\sigma_1 \times 2a$

$\therefore [\partial_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$

$\xrightarrow{\partial_3} \mathbb{Z}\{\sigma_2 \times e_0, \sigma_0 \times e_2, \sigma_1 \times a, \sigma_1 \times b\} \cong \mathbb{Z}^{\oplus 4}$

$\partial(\sigma_2 \times e_0) = 2\sigma_1 \times e_0$

$\partial(\sigma_0 \times e_2) = \sigma_0 \times 2a$

$\partial(\sigma_1 \times a) = 0$

$\partial(\sigma_1 \times b) = 0$

$\therefore [\partial_2] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{\partial_2} \mathbb{Z}\{\sigma_1 \times e_0, \sigma_0 \times a, \sigma_0 \times b\} \cong \mathbb{Z}^{\oplus 3}$

$\partial(\sigma_1 \times e_0) = 0$

$\partial(\sigma_0 \times a) = 0$

$\partial(\sigma_0 \times b) = 0$

$\therefore [\partial_1] = [0\ 0\ 0]$

$\xrightarrow{\partial_1} \mathbb{Z}\{\sigma_0 \times e_0\} \cong \mathbb{Z}$

$\longrightarrow 0$

Therefore, to compute homology:

$\ker \partial_4 = 0 \qquad \operatorname{im} \partial_5 = 0$

$\ker \partial_3 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} \qquad \operatorname{im} \partial_4 = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}\right\}$

$\ker \partial_2 = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\} \qquad \operatorname{im} \partial_3 = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}\right\}$

$\ker \partial_1 = \mathbb{Z}^{\oplus 3} \qquad \operatorname{im} \partial_2 = \operatorname{span}\left\{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\right\}$

$$H_*(\mathbb{P}^2 \times KB) = \begin{cases} 0 & * \geq 4 \\ \mathbb{Z}/2 & * = 3 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & * = 2 \\ \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & * = 1 \\ \mathbb{Z} & * = 0 \end{cases} \quad \square$$

# Problem 1: Products of CW Complexes

Marvin Qi

## 1 Compute $H_i(S^p \times S^q)$

For the sphere $S^p$, we'll use the cell structure with a single 0-cell $x$ and a single $p$-cell $\sigma$ where the attaching map sends the boundary of the $p$-disk to the point. We do the same for the sphere $S^q$; the cell structure contains a single 0-cell $y$ and a single $q$-cell $\tau$ where the attaching map sends the boundary of the $q$-disk to $y$. Note that in both $S^p$ and $S^q$, the boundary maps are simply the zero map.

Using this, we can straightforwardly compute the cellular homology of the product $S^p \times S^q$ using the formula $\partial(\alpha \times \beta) = \partial\alpha \times \beta + (-1)^p \alpha \times \partial\beta$ where $\alpha$ is a $p$-cell.

Consider the case where $p \neq q$. The cell complex of $S^p \times S^q$ consists of a single 0-cell $x \times y$, a single $p$-cell $\sigma \times y$, a single $q$-cell $x \times \tau$, and a $(p+q)$-cell $\sigma \times \tau$. Thus, the cellular chain complex is

$$\begin{aligned} C_0 &= \mathbb{Z}\{x \times y\} \\ C_p &= \mathbb{Z}\{\sigma \times y\} \\ C_q &= \mathbb{Z}\{x \times \tau\} \\ C_{p+q} &= \mathbb{Z}\{\sigma \times \tau\} \end{aligned} \tag{1.1}$$

Since the boundary map is simply the zero map, the kernels of the boundary maps are just the entire group, and the images of the boundary maps are zero. Therefore, we have

$$\begin{aligned} H_0 &= \mathbb{Z}\{x \times y\} \\ H_p &= \mathbb{Z}\{\sigma \times y\} \\ H_q &= \mathbb{Z}\{x \times \tau\} \\ H_{p+q} &= \mathbb{Z}\{\sigma \times \tau\} \end{aligned} \tag{1.2}$$

For the case $p = q$, then we instead have

$$C_p = C_q = \mathbb{Z}\{x \times \tau\} \oplus \mathbb{Z}\{\sigma \times y\} \tag{1.3}$$

which modifies the homology to

$$\begin{aligned} H_0 &= \mathbb{Z}\{x \times y\} \\ H_p &= \mathbb{Z}\{\sigma \times y\} \oplus \mathbb{Z}\{x \times \tau\} \\ H_{2p} &= \mathbb{Z}\{\sigma \times \tau\} \end{aligned} \tag{1.4}$$

## 2 Compute $P^2 \times P^2$

For $P^2$, we'll use the cell structure with a single $k$-cell in each dimension. Label the 0-cell $x$, the 1-cell $a$, and the 2-cell $\sigma$. The attaching map for the 2-cell identifies antipodal points. Therefore the cell structure for $P^2 \times P^2$ has a single 0-cell $x \times x$, two 1-cells $x \times a$ and $a \times x$, three 2-cells $x \times \sigma$, $a \times a$, and $\sigma \times x$, two 3-cells $a \times \sigma$ and $\sigma \times a$, and one 4-cell $\sigma \times \sigma$. The cellular chain complex in each dimension is the free abelian group generated by each cell.

The boundary map for $P^2$ acts on the chain complexes as follows:

$$\begin{aligned} \partial\sigma &= 2a \\ \partial a &= 0 \\ \partial x &= 0 \end{aligned} \tag{2.1}$$

Therefore, the boundary map for $P^2 \times P^2$ acts as

$$\begin{aligned} \partial_4(\sigma \times \sigma) &= 2a \times \sigma + 2\sigma \times a \\ \partial_3(a \times \sigma) &= -2a \times a \\ \partial_3(\sigma \times a) &= 2a \times a \\ \partial_2(x \times \sigma) &= 2x \times a \\ \partial_2(a \times a) &= 0 \\ \partial_2(\sigma \times a) &= 2a \times x \\ \partial_1(x \times a) = \partial_1(a \times x) &= 0 \end{aligned} \tag{2.2}$$

From here we see that

$$\begin{aligned} &\ker(\partial_{i\geq 4}) = 0 \rightarrow H_{i\geq 4}(P^2 \times P^2) = 0 \\ &\ker(\partial_3) = \mathbb{Z}\{a \times \sigma + \sigma \times a\}, \quad \operatorname{im}(\partial_4) = 2\mathbb{Z}\{a \times \sigma + \sigma \times a\} \rightarrow H_3(P^2 \times P^2) = \mathbb{Z}_2 \\ &\ker(\partial_2) = \mathbb{Z}\{a \times a\}, \quad \operatorname{im}(\partial_3) = 2\mathbb{Z}\{a \times a\} \rightarrow H_2(P^2 \times P^2) = \mathbb{Z}_2 \\ &\ker(\partial_1) = \mathbb{Z}\{a \times x\} \oplus \mathbb{Z}\{x \times a\}, \quad \operatorname{im}(\partial_2) = 2\mathbb{Z}\{a \times x\} \oplus 2\mathbb{Z}\{x \times a\} \rightarrow H_1(P^2 \times P^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \end{aligned}$$

Finally $P^2 \times P^2$ is connected so $H_0(P^2 \times P^2) = \mathbb{Z}$.

## 3 Compute $K \times P^2$

For $P^2$ we use the same structure as above, calling the 0-cell $y$, the 1-cell $c$, and the 2-cell $\tau$. For $K$, we use the "square with identified sides" cell structure, with a single 0-cell $x$, two 1-cells $a$ and $b$, and one 2-cell $\sigma$. For $K$, the $a$ cells are identified antiparallel while the $b$ cells are identified parallel. In $K$, the boundary map acts on the cells as

$$\begin{aligned} \partial\sigma &= 2a \\ \partial a &= 0 \\ \partial b &= 0 \\ \partial x &= 0 \end{aligned} \tag{3.1}$$

In $P^2$, the boundary map acts on the cells as in problem 2.

The cell structure of $K \times P^2$ consists of all possible products of the cells in $K$ and the cells in $P^2$. The boundary map acts on them in the following ways:

$$\begin{aligned}
\partial_4(\sigma \times \tau) &= 2a \times \tau + 2\sigma \times c \\
\partial_3(\sigma \times c) &= 2a \times c \\
\partial_3(a \times \tau) &= -2a \times c \\
\partial_3(b \times \tau) &= -2b \times c \\
\partial_2(\sigma \times y) &= 2a \times y \\
\partial_2(x \times \tau) &= 2x \times c \\
\partial_2(a \times c) &= 0 \\
\partial_2(b \times c) &= 0 \\
\partial_1(x \times c) &= 0 \\
\partial_1(a \times y) &= 0 \\
\partial_1(b \times y) &= 0
\end{aligned} \tag{3.2}$$

From here we see that

$$\begin{aligned}
\ker\partial_4 = 0 &\rightarrow H_4(K \times P^2) = 0 \\
\ker\partial_3 = \mathbb{Z}\{\sigma \times c + a \times \tau\}, \mathrm{im}\partial_4 = 2\mathbb{Z}\{\sigma \times c + a \times \tau\} &\rightarrow H_3(K \times P^2) = \mathbb{Z}_2 \\
\ker\partial_2 = \mathbb{Z}\{a \times c\} \oplus \mathbb{Z}\{b \times c\}, \mathrm{im}\partial_3 = 2\mathbb{Z}\{a \times c\} \oplus 2\mathbb{Z}\{b \times c\} &\rightarrow H_2(K \times P^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\
\ker\partial_1 = \mathbb{Z}\{x \times c\} \oplus \mathbb{Z}\{a \times y\} \oplus \mathbb{Z}\{b \times y\}&, \\
\mathrm{im}\partial_2 = 2\mathbb{Z}\{x \times c\} \oplus 2\mathbb{Z}\{a \times y\} \oplus 0 &\rightarrow H_1(K \times P^2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}
\end{aligned} \tag{3.3}$$

Finally, $H_i(K \times P^2)$ for $i \geq 5$ is zero since there are no cells in 5 or higher dimension, and $H_0(K \times P^2)$ is $\mathbb{Z}$ since the space is connected.

(1) Use the knowledge of the covering spaces of the torus, but do not use the knowledge of its homology groups, to show that its Euler characteristic is zero.

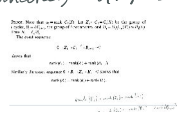
$\frac{2}{3}$ (2) If $X$ is a finite CW-complex of dimension two, and if $X$ is simply connected then show that $\chi(X)$ determines $H_2(X)$ completely. What are the possible values for $\chi(X)$ in this situation?

$\frac{1}{3}$ (4) If $X$ and $Y$ are finite CW-complexes, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

13.1| Let $\mathbb{T}^2 \approx \mathbb{R}^2/\mathbb{Z}^2$. Apparently $\mathbb{R}^2/\mathbb{Z}$ and $\mathbb{R}^2$ are infinitely sheeted covers of $\mathbb{T}^2$. Consider a finitely sheeted cover of the form $\mathbb{R}^2/_{n\mathbb{Z}\times m\mathbb{Z}}$, with nm sheets. Since $\chi(\mathbb{T}^2)$ is a topological invariant, if $\mathbb{T}^2 \approx \mathbb{R}^2/_{n\mathbb{Z}\times m\mathbb{Z}}$, then $\chi(\mathbb{R}^2/(n\mathbb{Z}\times m\mathbb{Z})) = \chi(\mathbb{T}^2)$. Indeed $\mathbb{T}^2 \xrightarrow[\cong]{\varphi} \mathbb{R}^2/_{n\mathbb{Z}\times m\mathbb{Z}}$ where $\varphi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} nx \\ my \end{bmatrix}$. But by prop 13.5, because $\mathbb{R}^2/_{n\mathbb{Z}\times m\mathbb{Z}}$ is an nm-sheeted cover, $\chi(\mathbb{T}^2) = mn\chi(\mathbb{R}^2/_{n\mathbb{Z}\times m\mathbb{Z}})$.
$= mn\chi(\mathbb{T}^2)$.

Therefore $\chi(\mathbb{T}^2) = 0$. (Note that a similar argument would hold for $\mathbb{T}^k$ $k \geq 0$.) $\square$

13.2| Given. Suppose $X$ is a simply connected CW-complex of dim. 2 with finitely many cells. To prove. $\chi(X)$ determines $H_2(X)$ up to isomorphism. Proof. As $X$ is simply connected, $H_0(X) = \mathbb{Z}$ and $H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)] \cong \{0\}$.


Proof. Note that ... Let ... be the group of ... cycles, ... the group of ... boundaries and ...
The exact sequence
shows that
Similarly the exact sequence ... shows that

Because $X$ has dim 2, the group of 2 boundaries $B_2 := \partial C_3 = \{0\}$.

Therefore $H_2(X) \cong C_2(X)$. Say $\chi(X)$ is given. Then $\chi(X) = \operatorname{rank}(H_2) + \operatorname{rank}(H_0)$.

So $\operatorname{rank}(H_2) = \chi(X) - 1$. Because $H_2(X)$ is free, any other CW-complex $Y$ as described above has the same homology as $X$ iff $X$ and $Y$ have the same number of 2-cells. In this case $\chi(X) = \chi(Y)$ iff $H_2(X) = H_2(Y)$.

Note $\chi(X) = \operatorname{rank}(H_2)$.

13.3| TODO! Say $X$ and $Y$ are finite CW complexes, with $n_i^X$ the number of $i$-cells in $X$, and $n_j^Y$ the number of $j$-cells in $Y$. Then $X \times Y$ has $i+j=k$ cells $\sum_{i+j=k} n_i^X n_j^Y$. The convolution product yields
$\sum_{k\geq 0} (-1)^k \left(\sum_{i+j=k} n_i^X n_j^Y\right) = \left(\sum_i (-1)^i n_i^X\right) \times \left(\sum_j (-1)^j n_j^Y\right)$. So $\chi(X\times Y) = \chi(X)\cdot\chi(Y)$. $\square$

# Homework 4 Problem 2

Jonathan Quartin

**Exercise 1.** Use the knowledge of the covering spaces of the torus, but do not use the knowledge of its homology groups, to show that its Euler characteristic is zero.

Consider the double cover of the torus by itself:

$$f : T = S^1 \times S^1 \longrightarrow T = S^1 \times S^1$$
$$(x, y) \longrightarrow (x^2, y)$$

From Bredon, we have that if $X \to Y$ is a covering map with $k$ sheets ($k$ finite) and $Y$ is a finite CW-complex then $X$ is also a CW-complex and $\chi(X) = k\chi(Y)$.

So in our case, we get $\chi(T) = 2\chi(T)$ which implies that $\chi(T) = 0$

**Exercise 2.** If X is a finite CW-complex of dimension two, and if $X$ is simply connected then show that $\chi(X)$ determines $H_2(X)$ completely. What are the possible values for $\chi(X)$ in this situation?

That $X$ simply connected implies that $H_0(X) = \mathbb{Z}$ and that

$$H_1(X) = \pi_1(X)^{ab} = 0^{ab} = 0.$$

Therefore, $\chi(X) = 1 + \text{rank}(H_2(X))$.

Now, since $X$ is 2-dimensional, $H_2(X)$ is free.

Thus $H_2(X) = \mathbb{Z}^{(\chi(X)-1)}$, so $H_2(X)$ is completely determined by $\chi(X)$.

The Euler characteristic $\chi(X)$ can be any natural number, as follows: For $X = D^2$,

$$\text{rank}(H_2(X)) = 0,$$

and for $X$ a wedge of $n$ copies of $S^2$,

$$\text{rank}(H_2(X)) = n.$$

**Exercise 3.** If $X$ and $Y$ are finite CW-complexes, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

Let $a_n, b_n, c_n$ be the number of $n$-cells of $X \times Y, X, Y$ respectively. By Theorem 13.3 in Bredon, $\chi(X \times Y) = \sum_i (-1)^i a_i$. By definition of the product CW-structure, $a_i = \sum\limits_{p+q=i} b_p c_q$. Plugging the second sum into the first yields:

$$\begin{aligned}
\chi(X \times Y) &= \sum_i (-1)^i \sum_{p+q=i} b_p c_q \\
&= \sum_i \sum_{p+q=i} (-1)^{p+q} b_p c_q \\
&= \left(\sum_p (-1)^p b_p\right) \left(\sum_q (-1)^q c_q\right) \\
&= \chi(X)\chi(Y).
\end{aligned}$$

1. Use the Mayer–Vietoris sequence to give another derivation of the homology groups of spheres (of all dimensions).

2. Use the Mayer–Vietoris sequence to compute the homology of the space which is the union of three $n$-disks along their common boundaries.

3. Use the Mayer–Vietoris sequence to give another derivation of the homology groups of the projective plane.

1] We work with reduced homology. Let $S^n = \mathcal{U}_n^+ \cup \mathcal{U}_n^-$ for $\mathcal{U}_n^+$ and $\mathcal{U}_n^-$ open sets obtained by puncturing the sphere $S^n$ at opposite poles. Now $\mathcal{U}_n^+ \simeq \mathcal{U}_n^- \simeq pt$ and $\mathcal{U}_n^+ \cap \mathcal{U}_n^- \simeq S^{n-1}$. So we proceed inductively.

Recall $\tilde{H}_*(S^0) = \{\mathbb{Z} \ * = 0 \text{ else } 0\}$. Spose $\tilde{H}_*(S^{n-1}) = \{\mathbb{Z} \ * = n-1 \text{ else } 0\}$ has been proven.

Then the les for Mayer Vietoris is

$$\begin{tikzcd}
\cancel{\tilde{H}_{n+1}(\mathcal{U}_n^+)}^{0} \oplus \cancel{\tilde{H}_{n+1}(\mathcal{U}_n^-)}^{0} \arrow[r] & \tilde{H}_{n+1}(S^n) \arrow[r, "\partial"] & \tilde{H}_n(\mathcal{U}_n^+ \cap \mathcal{U}_n^-) \arrow[r, "(\iota_1)_* \oplus (\iota_2)_*"] \arrow[d, "\cong"', "h.a."] & \cancel{\tilde{H}_n(\mathcal{U}_n^+)}^{0} \oplus \cancel{H_n(\mathcal{U}_n^-)}^{0} \arrow[r, "(j_1)_* - (j_2)_*"] & \tilde{H}_n(S^n) \arrow[r, "\partial"] & \tilde{H}_{n-1}(\mathcal{U}_n^+ \cap \mathcal{U}_n^-) \arrow[r] \arrow[d, "\cong"', "h.a."] & \cancel{\tilde{H}_{n-1}(\mathcal{U}_n^+)}^{0} \oplus \cancel{\tilde{H}_{n-1}(\mathcal{U}_n^-)}^{0} \arrow[r] & \cdots \\
 & & \tilde{H}_n(S^{n-1}) & & & \tilde{H}_{n-1}(S^{n-1}) & &
\end{tikzcd}$$

So $\tilde{H}_k(S^n) \cong \tilde{H}_{k-1}(S^{n-1})$ for all $k \geq 1$.

Thus $\tilde{H}_*(S^n) = \{\mathbb{Z} \text{ if } * = n \text{ else } 0\}$. $\square$

2] Given. Let $X$ be the CW structure $X^{(0)}=\{e_0\}$, $X^{(n-1)}=\{e_{n-1}\}$, $X^{(n)}=\{e_n^1, e_n^2, e_n^3\}$.

**Let $\mathcal{U}$ be an open nbhd of $e_n^1 \vee e_n^2$ in $X$ that deform retracts to $e_n^1 \vee e_n^2 \approx S^n$.**

**Let $V$ be an open nbhd of $e_n^3$ in $X$ that deform retracts to $e_n^3 \simeq$ point.**

Then by Mayer-Victories, the inclusions
$$\begin{tikzcd} & \mathcal{U} \arrow[dr] & \\ \mathcal{U}\cap V \arrow[ur] \arrow[dr] & & X \\ & V \arrow[ur] & \end{tikzcd}$$
induce a l.e.s. of reduced homology:

$$\cdots \xrightarrow{D} \tilde{H}_k(\mathcal{U}\cap V) \to \tilde{H}_k(\mathcal{U}) \oplus \tilde{H}_k(V) \to \tilde{H}_k(X) \xrightarrow{D} \tilde{H}_{k-1}(\mathcal{U}\cap V) \to \tilde{H}_{k-1}(\mathcal{U}) \oplus \tilde{H}_{k-1}(V) \to \tilde{H}_{k-1}(X) \xrightarrow{D} \cdots$$

$$\cdots \to \tilde{H}_k(S^{n-1}) \to \tilde{H}_k(S^n) \oplus \{0\} \to \tilde{H}_k(X) \xrightarrow{D} \tilde{H}_{k-1}(S^{n-1}) \to \tilde{H}_{k-1}(S^n) \oplus \{0\} \to \tilde{H}_{k-1}(X) \xrightarrow{D} \tilde{H}_{k-2}(S^{n-1}) \cdots$$

let $k=n$.

$$0 \to \mathbb{Z} \to \tilde{H}_n(X) \xrightarrow{D} \mathbb{Z} \to 0 \to 0 \to \cancel{\tilde{H}_{k-1}(X)}^{\,0} \to 0 \to 0 \cdots$$

I claim $\tilde{H}_n(X) \cong \mathbb{Z}^{\oplus 2}$. Consider $\partial_* : H_n(\mathcal{U}, \mathcal{U}\cap V) \to H_{n-1}(\mathcal{U}\cap V)$.

One "sheaf" maps to $0$, the other to $1$.

<u>IDEA!</u>
$$\begin{tikzcd} H_n(X) \arrow[r] \arrow[drr, "D"] & H_n(X, V) & \\ & H_n(\mathcal{U}, \mathcal{U}\cap V) \arrow[u, "e_*", "\cong"'] \arrow[r, "\partial_*"] & H_{n-1}(\mathcal{U}\cap V) \end{tikzcd}$$
.

Let $X$ be the CW structure $X^{(0)}=\{e_0\}$, $X^{(n-1)}=\{e_{n-1}\}$, $X^{(n)}=\{e_n^1, e_n^2, e_n^3\}$. **We assume $n \geq 2$.**

Then the cellular chain complex is $0 \to \mathbb{Z}\{e_n^1, e_n^2, e_n^3\} \xrightarrow{\partial_n} \mathbb{Z}\{e_{n-1}\} \xrightarrow{\partial_{n-1}} \cdots \to \mathbb{Z}\{e_0\} \to 0$

$[\partial_n] = \big[[e_{n-1}:e_n^1] \ [e_{n-1}:e_n^2] \ [e_{n-1}:e_n^3]\big] = [1 \ \ 1 \ \ 1]$ and $[\partial_{n-1}] = [0]$ for all $n \geq 1$. So $H_n^{cw}(X) = \ker \partial_n = \mathbb{Z}^{\oplus 2}$

$H_{n-1}^{cw}(X) = \ker \partial_{n-1} / \operatorname{im} \partial_n = \mathbb{Z}/\mathbb{Z} = \{0\}$. For $n \geq 2$, $H_*^{cw}(X) = \{\mathbb{Z}^{\oplus 2}$ if $*=n$, $\mathbb{Z}$ if $*=0$, $0$ else$\}$.

Let $\mathbb{R}P^2$ be the space obtained by either identifying antipodal points on the boundary $\partial D^2 = S^1$ of $D^2$ or by attaching a 2-cell to the cellular complex $K^{(0)} = \{\text{point}\}$ and $K^{(1)} = \{ G \}$ with a degree 2 map. That is, $K^{(2)} = K^{(2+n)}$ is the pushout of the diagram where $f_{\partial\sigma}$ has degree 2.

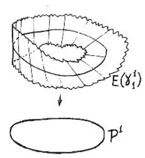
$$\begin{tikzcd} S^1 \arrow[r, "f_{\partial\sigma}"] \arrow[d] & K^{(1)} \arrow[d] \\ D^2 \arrow[r, "f_\sigma"] & K^{(2)} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

We'll compute reduced homology with Mayer-Vietoris.

Let $A = \operatorname{Int} D^2$ be the open 2-cell in $\mathbb{R}P^2$, find a point $p$ in $A$, and let $B = D^2 \setminus \{p\}$ be the punctured 2-cell. Observe $\mathbb{R}P^2 = \operatorname{Int} A \cup \operatorname{Int} B$. By Mayer-Vietoris, there's a l.e.s. in reduced homology

$$\cdots \longrightarrow \tilde{H}_n(A) \oplus \tilde{H}_n(B) \longrightarrow \tilde{H}_n(\mathbb{R}P^2) \xrightarrow{\delta} \tilde{H}_{n-1}(A\cap B) \longrightarrow \tilde{H}_{n-1}(A) \oplus \tilde{H}_{n-1}(B) \longrightarrow \cdots$$

First for homotopy equivalences. $A$ is an open ball so deform retracts to $\{p\}$. $B$ is an open Möbius band, and deforms to $S^1$. As well, $A\cap B$ deform retracts to $S^1$.




$$\tilde{H}_*(A) = 0$$
$$\tilde{H}_*(B) = \tilde{H}_*(A\cap B) = \{\mathbb{Z} \text{ if } * = 1, \text{ else } 0\}.$$

Our l.e.s. becomes trivial above degree 2, so we consider

$$\begin{tikzcd} \tilde{H}_2(A) \oplus \tilde{H}_2(B) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_2(\mathbb{R}P^2) \arrow[r, "\delta"] \arrow[d, "\cong"] & \tilde{H}_1(A\cap B) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_1(A) \oplus \tilde{H}_1(B) \arrow[r] \arrow[d, "\cong"] & \tilde{H}_1(\mathbb{R}P^2) \arrow[r, "\delta"] \arrow[d, "\cong"] & \tilde{H}_0(A\cap B) \arrow[d, "\cong"] \\ 0 \arrow[r] & \tilde{H}_2(\mathbb{R}P^2) \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r] & \tilde{H}_1(\mathbb{R}P^2) \arrow[r] & 0 \end{tikzcd}$$

Fix generators in $\tilde{H}_1(A\cap B)$ and $\tilde{H}_1(B)$ by letting $\gamma$ be a loop in $A\cap B$ winding once around $\{p\}$ in the mathematically positive direction and letting $b$ in $B$ be the path starting at $K^{(0)}$ and traversing half way around $K^{(1)}$ in the math. pos. direction. The inclusion $i_{A\cap B} : A\cap B \longrightarrow B$ induces $H_1(i_{A\cap B})[\gamma] = 2[b]$.

Or, follow Bredon to determine the degree map of the attaching $f_\sigma$.

We conclude
$$\tilde{H}_2(\mathbb{R}P^2) \cong \ker(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}) = 0,$$
$$\tilde{H}_1(\mathbb{R}P^2) \cong \operatorname{coker}(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}) = \mathbb{Z}/(2),$$
$$\tilde{H}_*(\mathbb{R}P^2) \cong 0 \text{ for } * \neq 1.$$

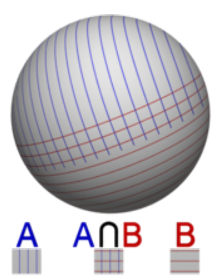
# Homework 4 Problem 3

James Cates
Math 6220

March 19, 2019

1. Use the Mayer-Vietoris sequence to give another derivation of the homology groups of spheres (of all dimensions).
2. Use the Mayer-Vietoris sequence to compute the homology of the space which is the union of three $n$-disks along their common boundaries.
3. Use the Mayer-Vietoris sequence to give another derivation of the homology groups of the projective plane.

**Solution** 1. *Proof.* We will prove the reduced homology case. For our sphere $S^n$, let $U_n^+$ be the northern hemisphere of $S^n$ with a little overlap in the southern hemisphere. Let $U_n^-$ be the southern hemisphere with some overlap in the northern hemisphere. Both $U_n^+$ and $U_n^-$ are homotopic to a point, and their intersection is homotopic to $S^{n-1}$. Also, $U_n^+ \cup U_n^- = S^n$.




[1]

This can visualized in the image above, where $A = U_n^+$ and $B = U_n^-$. We know that $\tilde{H}_*(S^0) = \{\mathbb{Z} : \text{if } * = 0 \text{ and } 0 \text{ else}\}$. Suppose that

$$\tilde{H}_*(S^{n-1}) = \{\mathbb{Z} \;\text{ if } * = n-1 \text{ and } 0 \text{ else}\}$$

has been proven. We will now use Mayer Vietoris to prove that this implies $\tilde{H}_*(S^n) = \{\mathbb{Z} \text{ if } * = n \text{ and } 0 \text{ else}\}$. Using Mayer Vietoris, we get the exact sequence

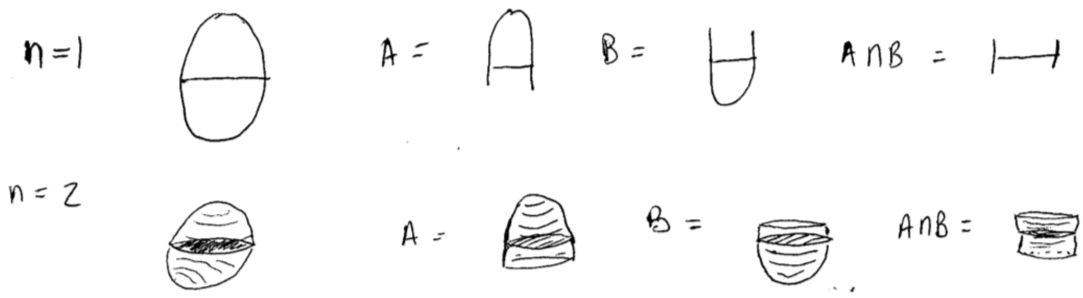
$$\dots \longrightarrow \tilde{H}_k(U_n^+) \oplus \tilde{H}_k(U_n^-) \longrightarrow \tilde{H}_k(S^n) \longrightarrow \tilde{H}_{k-1}(S^{n-1}) \longrightarrow \tilde{H}_{k-1}(U_n^+) \oplus \tilde{H}_{k-1}(U_n^-) \longrightarrow \dots$$

This reduces to

$$\ldots \longrightarrow 0 \oplus 0 \longrightarrow \tilde{H}_k(S^n) \longrightarrow \tilde{H}_{k-1}(S^{n-1}) \longrightarrow 0 \oplus 0 \longrightarrow \ldots$$

Now when $k = n$, we have that $\tilde{H}_*(S^n) = \{\mathbb{Z} \text{ if } * = n \text{ and } 0 \text{ else}\}$. $\square$

2. *Proof.* We will look at the cases $n = 1$ and $n = 2$ to visualize this problem.




Clearly in both cases, A and B are homotopic to spheres, and $A \cap B$ is homotopic to a point. In the general case, we let A be the union of the top n-disk, middle n-disk, and some of the bottom disk, and the opposite for B. We then have that A and B are homotopic to n-spheres, and $A \cap B$ is homotopic to a point. Let $X = A \cup B$. Now by Mayer Vietoris, we get the following exact sequence:

$$\ldots \longrightarrow \tilde{H}_k(A \cap B) \longrightarrow \tilde{H}_k(A) \oplus \tilde{H}_k(B) \longrightarrow \tilde{H}_k(X) \longrightarrow \tilde{H}_{k-1}(A \cap B) \longrightarrow \ldots$$

By the homotopy axiom, the sequence reduces to

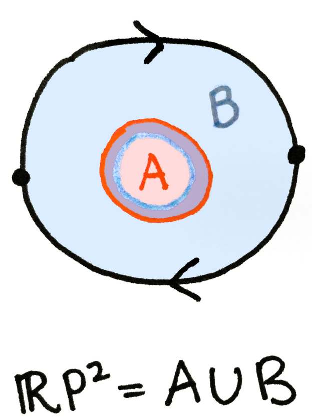
$$\ldots \longrightarrow \tilde{H}_k(*) \longrightarrow \tilde{H}_k(S^n) \oplus \tilde{H}_k(S^n) \longrightarrow \tilde{H}_k(X) \longrightarrow \tilde{H}_{k-1}(*) \longrightarrow \ldots$$

and for when $k = n$, we get

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \tilde{H}_k(X) \longrightarrow 0 \longrightarrow \ldots$$

So $\tilde{H}_k(X) = \{\mathbb{Z} \oplus \mathbb{Z} \text{ if } k = n, 0 \text{ else}\}$. $\square$

3. *Proof.* For $\mathbb{R}P^2$, we construct our open sets as follows. We let $A$ be an open disk from the CW structure, and we let $B$ be the complement of a smaller open disk that is resides in $A$. This can be visualized in the figure below.




[2]

We note that $A$ is homotopy equivalent to a point, and $B$ and $A \cap B$ are homotopy equivalent to $S^1$. Now by Mayer Vietoris, we get the following exact sequence, which allows us to calculate $H_2$:

$$\dots \longrightarrow \tilde{H}_2(\mathbb{R}P^2) \longrightarrow \tilde{H}_1(A \cap B) \longrightarrow \tilde{H}_1(A) \oplus \tilde{H}_1(B) \longrightarrow \dots$$

This is equivalent to:

$$\dots \longrightarrow 0 \oplus 0 \longrightarrow \tilde{H}_2(\mathbb{R}P^2) \xrightarrow{\partial_*} \tilde{H}_1(S^1) \xrightarrow{i_*^A \oplus i_*^B} \tilde{H}_1(*) \oplus \tilde{H}_1(S^1) \longrightarrow \dots$$

Clearly, $\ker(\partial_*) = 0$ because of exactness. We also have $im(\partial_*) = 0$, and this follows because $\ker(i_*^A \oplus i_*^B) = 0$. The map $i_*^A \oplus i_*^B = 0 \oplus \times 2$ because $i_*^B$ is the covering map and we already know that this map is of degree 2. Therefore, we have $\tilde{H}_2(\mathbb{R}P^2) = 0$.
Now for $\tilde{H}_1$, we look at the portion of the Mayer Vietoris sequence:

$$\dots \longrightarrow \tilde{H}_1(A) \oplus \tilde{H}_1(B) \longrightarrow \tilde{H}_1(\mathbb{R}P^2) \longrightarrow \tilde{H}_0(A \cap B) \longrightarrow \tilde{H}_0(A) \oplus \tilde{H}_0(B) \longrightarrow \dots$$

which is the same as

$$\dots \xrightarrow{i_*^A \oplus i_*^B} 0 \oplus \mathbb{Z} \xrightarrow{\phi} \tilde{H}_1(\mathbb{R}P^2) \xrightarrow{\psi} 0 \xrightarrow{\alpha} 0 \oplus 0 \longrightarrow \dots$$

We know that im($\psi$)=0 by exactness. We also know that $\ker(\phi) = 2\mathbb{Z}$ from the image of the double covering map $i_*^B$. This implies that $\ker(\psi) = im(\phi) = \mathbb{Z}/2\mathbb{Z}$.
Therefore, $\tilde{H}_1 = \mathbb{Z}/2\mathbb{Z}$. For $k > 2$ or $k = 0$, we have $\tilde{H}_k(\mathbb{R}P^2) = 0$, and this follows from all the homology groups around $\tilde{H}_k$ being zero. $\square$

# References


[1] https://en.wikipedia.org/wiki/Mayer?Vietoris_sequence
[2] https://www.math3ma.com/blog/the-fundamental-group-of-the-real-projective-plane

**Given.** Let $cyl(C)$ denote the mapping cylinder of the identity map $id_C$ of $C$. It has $C_n \oplus C_{n-1} \oplus C_n$ in degree $n$ and differential given by the matrix
$$\begin{bmatrix} d & id & 0 \\ 0 & -d & 0 \\ 0 & -id & d \end{bmatrix} : \begin{array}{c} C_n \\ \oplus \\ C_{n-1} \\ \oplus \\ C_n \end{array} \to \begin{array}{c} C_{n-1} \\ \\ C_{n-2} \\ \\ C_{n-1} \end{array}$$

**To prove.** Two chain maps $f, g : C \to D$ are homotopic iff they extend to a map $(f, s, g) : cyl(C) \to D$.

**Part A|** **Given.** Let $f, g : C \to D$ be chain homotopic. Suppose $cyl(C)$ is the chain complex $C_n \oplus C_{n-1} \oplus C_n$.

Let $d : \begin{bmatrix} c \\ c' \\ c'' \end{bmatrix} \longmapsto \begin{bmatrix} dc + c' \\ -dc' \\ dc'' - c' \end{bmatrix}$.

WTS| $(f,s,g)\, d = d_D (f,s,g)$.

Compute $(f,s,g)\, d \begin{bmatrix} c \\ c' \\ c'' \end{bmatrix} = [f \;\; s \;\; g] \begin{bmatrix} dc + c' \\ -dc' \\ dc'' - c' \end{bmatrix} = (f-g)(c') + df(c) + dg(c'') - sd(c')$.

$Hd + dH = f - g$

Compute $d_D(f,s,g) \begin{bmatrix} c \\ c' \\ c'' \end{bmatrix} = df(c) + ds(c') + dg(c'')$.

$$\begin{tikzcd} C_{n+1} \arrow[r] \arrow[d, "f-g"] & C_n \arrow[r] \arrow[d, "f-g"] \arrow[dl, "H"'] & C_{n-1} \arrow[d, "f-g"] \arrow[dl, "H"'] \\ D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \end{tikzcd}$$

$(f,s,g)\, d \begin{bmatrix} c \\ c' \\ c'' \end{bmatrix} = [f \;\; s \;\; g] \begin{bmatrix} dc + c' \\ -dc' \\ dc'' - c' \end{bmatrix} = (f-g)(c') + df(c) + dg(c'') - Hd(c')$.

$d_D(f,s,g) \begin{bmatrix} c \\ c' \\ c'' \end{bmatrix} = df(c) + dH(c') + dg(c'')$.

can replace iff $f \simeq g$ chain homotopic.

else not equal... ☹ $\square$

**Part B|** $A, B \in Ch$. $(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j$.

(c) Let $A$ and $B$ be chain complexes. Define the chain complex $A \otimes B$ by

$(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j$, $d^{\otimes}(a \otimes b) = d(a) \otimes b + (-1)^{i} a \otimes d(b)$, ...

(i) Check that $A \otimes B$ is a chain complex.

(ii) Define $I$ by $I_1 = \mathbb{Z}\{l\}$, $I_0 = \mathbb{Z}\{l_0, l_1\}$, $I_n = 0$ if $n \neq 0, 1$, with $d_1(l) = l_1 - l_0$. Prove that for a chain complex $C$ the cylinder $cyl(C) \cong I \otimes C$.

GOAL|
$(d^{\otimes})^2 = 0$ (thence $A \otimes B \in Ch$). ✓ nice!

GOAL| **Given.** $C \in Ch$. $cyl(C) = C_n \oplus C_{n-1} \oplus C_n$ with diff $d = \begin{bmatrix} & & \end{bmatrix}$

To prove.

Pf. Compute $(I_0 \otimes C_n) \oplus (I_1 \otimes C_{n-1})$ with also $\varphi(c_0, c, c_1)$ defined $\varphi(c_0, c, c_1) = (l_1 \otimes c_0 + l_0 \otimes c_1) \oplus (l \otimes c)$. "elementary tensors"

if $c_0$ and $c_1$ are not multiples of eachother, then $l_1 \otimes c_0 + l_0 \otimes c_1$ is not a generic element...

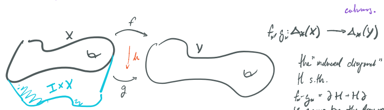
$d^{\otimes} \varphi \begin{bmatrix} c_0 \\ c \\ c_1 \end{bmatrix} = \begin{bmatrix} l_1 \otimes dc_0 + l_0 \otimes dc_1 \\ l_1 \otimes c - l_0 \otimes c - l \otimes dc \end{bmatrix}$

$l_1 \otimes C_{1i}$ ✓

$\varphi d^{\otimes} \begin{bmatrix} c_0 \\ c \\ c_1 \end{bmatrix} = l_1 \otimes dc_0 + \underset{\underset{l_1 \otimes c}{\vee}}{l_1} \otimes dc_1 - l_0 \otimes c - l \otimes dc$ ✓

The same? Yes. Be careful with the columns.

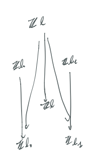
$\partial(l \otimes c_{i-1}) = (l_1 - l_0) \otimes c_{i-1} - l \otimes dc_{i-1}$

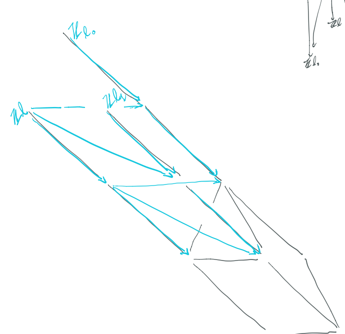



$f_*, g_* : \Delta_*(X) \longrightarrow \Delta_*(Y)$

the "induced diagonal" $H$ s.th. $f_* - g_* = \partial H + H \partial$ is given by the diagonal?

we need a "cylinder object" for the data of the DIAGONAL!





## ## mapping cones

Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a chain map for $\mathcal{C}, \mathcal{D} \in Ch$.

Construct $\mathcal{E} \in Ch$ by $E_n = C_{n-1} \oplus D_n$ with $\partial_{\mathcal{E}} = \begin{bmatrix} -\partial_{\mathcal{C}} & 0 \\ -\varphi & \partial_{\mathcal{D}} \end{bmatrix}$.

Then $\partial_{\mathcal{E}}^2 = \begin{bmatrix} -\partial_{\mathcal{C}} & 0 \\ -\varphi & \partial_{\mathcal{D}} \end{bmatrix}^2 = \begin{bmatrix} \partial_{\mathcal{C}}^2 & 0 \\ \varphi\partial_{\mathcal{C}} - \partial_{\mathcal{D}}\varphi & \partial_{\mathcal{D}}^2 \end{bmatrix} = 0$, as $\varphi\partial_{\mathcal{C}} = \partial_{\mathcal{D}}\varphi$.

So $\mathcal{E}$ is a chain complex. With $\mathcal{D} \overset{\iota}{\hookrightarrow} \mathcal{E}$ via $\iota = \begin{bmatrix} 0 \\ id \end{bmatrix}$, we see

$$\begin{tikzcd} D_{n+1} \arrow[r, "\partial_{\mathcal{D}}"] \arrow[d, "\iota"] & D_n \arrow[r, "\partial_{\mathcal{D}}"] \arrow[d, "\iota"] & D_{n-1} \arrow[d, "\iota"] \\ E_{n+1} \arrow[r, "\partial_{\mathcal{E}}"] & E_n \arrow[r, "\partial_{\mathcal{E}}"] & E_{n-1} \end{tikzcd}$$

$\iota \partial_{\mathcal{D}} = \begin{bmatrix} 0 \\ id \end{bmatrix} \partial_{\mathcal{D}} = \begin{bmatrix} 0 \\ \partial_{\mathcal{D}} \end{bmatrix}$ and

$\partial_{\mathcal{E}} \iota = \begin{bmatrix} -\partial_{\mathcal{C}} & 0 \\ \varphi & \partial_{\mathcal{D}} \end{bmatrix} \begin{bmatrix} 0 \\ id \end{bmatrix} = \begin{bmatrix} 0 \\ \partial_{\mathcal{D}} \end{bmatrix}$, so $\iota : \mathcal{D} \hookrightarrow \mathcal{E}$ is a chain map.

Now write $\mathcal{E} = cone(\varphi)$, the mapping cone of $\varphi$.

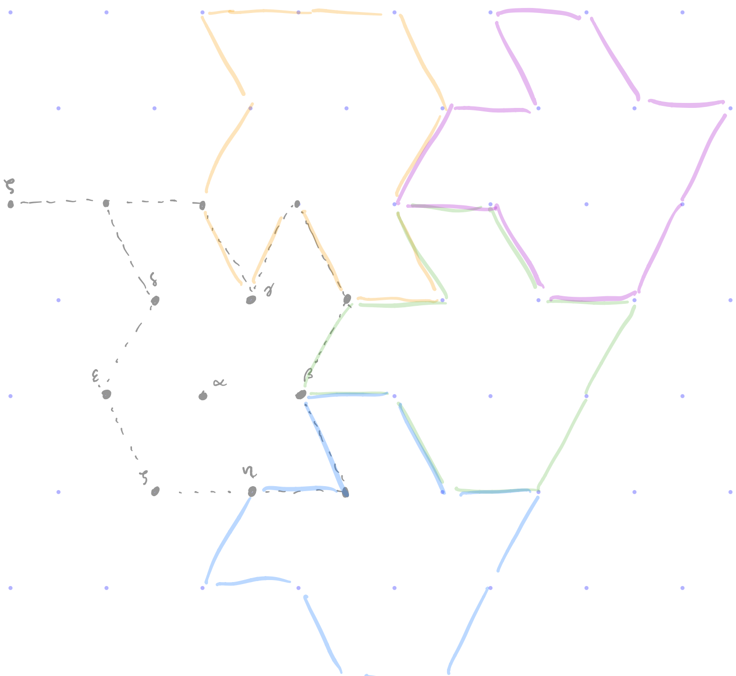
For $\mathcal{C}, \mathcal{D}, \mathcal{C}', \mathcal{D}' \in Ch$ with chain maps $\varphi, \varphi'$ s.th.

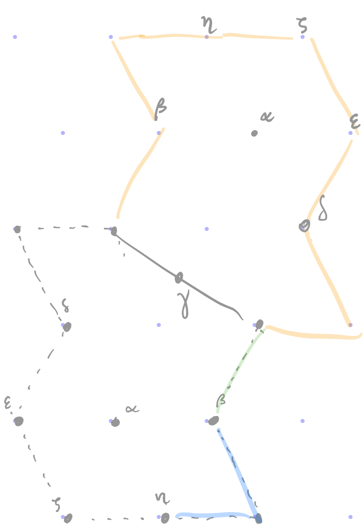
TFDC
$$\begin{tikzcd} \mathcal{C} \arrow[r, "\varphi"] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathcal{C}' \arrow[r, "\varphi'"] & \mathcal{D}' \end{tikzcd}.$$
consider $cone(\varphi)$ and $cone(\varphi')$.

We see $cone(\varphi) = \{C_n \oplus D_{n-1}\}$ has a map on comp.

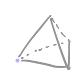
$C_n \oplus D_{n-1} \longrightarrow C_n' \oplus D_{n-1}'$

**2.1** Triangulate the torus; compute its simplicial homology.







2.2) Triangulate the Klein bottle and compute simplicial homology with coeffs $\mathbb{Z}/2$.

# § Cohomology chit-chat

**the machinery**

$\Delta^*(X,A;G) := \mathrm{Hom}_{Gp}(\Delta_*(X,A);G)$ $\quad \partial^p$ (boundary in cochain complex) is $\mathrm{Hom}_{Gp}(\partial_p;G)$

$\partial^p_{\mathrm{Hom}} = \mathrm{Hom}(\partial_{p+1}, G)$

$$\begin{tikzcd} C_{p+1} \arrow[d, "\partial_{p+1}"'] \arrow[dr, dotted] & \\ C_p \arrow[r, "\varphi"] \arrow[d, "\partial_p"'] & G \\ C_{p-1} \arrow[d] & \\ \vdots & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}(C_p, G) \arrow[d, "\partial^p"] \\ \mathrm{Hom}(C_{p+1}, G), \end{tikzcd}$$

recall $\left\{ \begin{array}{l} B \xleftarrow{f} A \\ \mathrm{Hom}(f;G) : \mathrm{Hom}(B,G) \longrightarrow \mathrm{Hom}(A,G) \end{array} \right.$

$H^*(X,A;G) := H^*(\Delta^*(X,A;G))$

convention

B/c indices inverse, $\Delta^0(X;G) \xrightarrow{\partial^1} \Delta^1(X;G) \xrightarrow{\partial^2} \Delta^2(X;G)$ } cochain complex

$$\begin{tikzcd} \Delta^0(X;G) \arrow[r, "\partial^1"] \arrow[rr, bend right, "0"'] & \Delta^1(X;G) \arrow[r, "\partial^2"] & \Delta^2(X;G) \end{tikzcd}$$

DREAM for the cohomology of $X$

just 0? No, cohomology is <u>not</u> just the dual of $H_*(X)$.

<u>Error</u> $\to H^*(X) \longrightarrow \mathrm{Hom}_{Gp}(H_*(X), \mathbb{Z}) \longrightarrow 0$

~ universal coefficient <u>thm</u>

The hom functor doesn't behave well with resp. to the homology functor.

$\boxed{\text{EX}}$ if $\mathbb{Z}/2$ is in the $H_*(X)$ then $\mathrm{Hom}(H_*(X), \mathbb{Z})$ kills the torsion

when $\mathbb{F}$ is the coeff gp. then many groups collapse (eg. torsion groups)
a field!
so that in fact $\mathrm{Hom}(H_*(X), \mathbb{Z}) \cong H^*(X)$.

# Exercises (chit-chat)

Show an example where $\mathrm{Hom}(-, \mathbb{Z})$ is <u>not</u> exact

# § User's guide to Ext & Tor in the category Ab.

20190320

**Exercise.** Say $G$ is an ab. gp and $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is ses.

Then $\mathrm{Hom}_{Ab}(-, G)$ is a left-exact functor: $0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G)$

also $\mathrm{Hom}_{Ab}(G, -)$ is too (left-exact) $0 \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, B) \to \mathrm{Hom}(G, C)$

then also $(- \otimes G)$ is right exact: $A \otimes G \xrightarrow{f \otimes 1} B \otimes G \xrightarrow{g \otimes 1} C \otimes G \to 0$

$$\sum b_i \otimes x_i \longmapsto \sum c_i \otimes x_i \quad \text{with}$$

**COUNTER EXAMPLE** $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ is a ses. Let $G = \mathbb{Z}/2$.

Then only $\underset{\mathbb{Z}/2}{\mathbb{Z}/2 \otimes G} \xrightarrow[0]{2 \otimes 1} \underset{\mathbb{Z}/2}{\mathbb{Z}/4 \otimes \mathbb{Z}/2} \to \underset{\mathbb{Z}/2}{\mathbb{Z}/2 \otimes \mathbb{Z}/2} \to 0$ fails to be exact...

**DEF** We call $G$

- injective if $\mathrm{Hom}(-, G)$ is an exact functor.
- projective if $\mathrm{Hom}(G, -)$ is " .
- flat if $- \otimes G$ " " .

**Recall** $Ab = \mathbb{Z}$-mod, but in $R$-mod, these defns matter.

**Some facts.** $G$ is injective iff $G$ is divisible. $\to$ For each $g \in G$ and each $n \in \mathbb{Z} \setminus \{0\}$, there exists $e \in G$ s.th $n.e = g$. Eg $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$

$G$ is projective iff $G$ is free.

$$\begin{tikzcd} & G \arrow[dl, dashed, "\exists"'] \arrow[d] \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

$G$ is flat iff $G$ is torsion free.

**Remark.** If $G$ is free, then $G$ is flat. Eg. $\mathbb{Q}$ is flat not free.

**DEF** $(\otimes)$ $\mathrm{Tor}(-,-) : Ab \times Ab \to Ab$ "torsion"

(Hom) $\mathrm{Ext}(-,-) : Ab^{op} \times Ab \to Ab$ "extension"

**Facts.** $\mathrm{Tor}(A, B) = \mathrm{Tor}(B, A)$ ($= A * B$ in Bredon)

↳ why? $A \otimes B \xrightarrow[\cong]{s} B \otimes A$, $s(a \otimes b) \longmapsto b \otimes a$

Let $G \in Ab$ and $0 \to A \to B \to C \to 0$. There's a les.

$$0 \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, B) \to \mathrm{Hom}(G, C) \to \mathrm{Ext}(G, A) \to \mathrm{Ext}(G, B) \to \mathrm{Ext}(G, C) \to 0$$

$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G) \to \mathrm{Ext}(C, G) \to \mathrm{Ext}(B, G) \to \mathrm{Ext}(A, G) \to 0$$

cohomological functors

} also called $\mathrm{Hom}(G, A) = \mathrm{Ext}^0(G, A)$ and $\mathrm{Ext}(G, A) = \mathrm{Ext}^1(G, A)$.

$$0 \to \mathrm{Tor}(G, A) \to \mathrm{Tor}(G, B) \to \mathrm{Tor}(G, C) \to G \otimes A \to G \otimes B \to G \otimes C \to 0$$

a homological functor

} Tor is right exact! remember!

} outside of Ab (eg in R-mod) these sequences are not nec. finished after 6 terms. They continue.

Say $G$ is an abelian gp. Then $G$ has a presentation

$$0 \to F_1(G) \xrightarrow{d_1} F_0(G) \xrightarrow{d_0} G \xrightarrow{\text{"-1"}} 0$$

free!

"free resolution of $G$"

↝ projective

$H_1(0 \to F_1 \otimes G' \to F_0 \otimes G' \to 0) = \mathrm{Tor}(G, G')$

so $\mathrm{Tor}(G, G') := \ker\left(d_1 \otimes \mathrm{id}_{G'} : F_1(G) \otimes G' \to F_0(G) \otimes G'\right)$

and $\mathrm{Ext}(G, G') := \mathrm{Hom}(F_1(G), G') / \mathrm{im}(\mathrm{Hom}(d_1, G'))$

with $H^1(0 \to \mathrm{Hom}(F_0, G') \to \mathrm{Hom}(F_1, G') \to 0) := \mathrm{Ext}(G, G')$

20190322 Yano

**THEME]** Cohomology (Univ. Coeff thm)

Turns out cohomology theories will give a contravariant functor

Existence or unexistence of vector fields on subsets of $\mathbb{R}^2$?

$H^\bullet_{DR}(X) \cong H^\bullet(X, \mathbb{R})$ (DeRham's thm) $\rightsquigarrow$ Poincaré duality gives a corresp.

Want a <u>derived</u> functor (compositions of homology & cohomology).

**TOPIC]** $\mathrm{Ext}^i_{\mathbb{Z}}$, $\mathrm{Tor}^{\mathbb{Z}}_i$. Let $M, N \in Ab$.

Then recall $\mathrm{Hom}_{\mathbb{Z}}(M,-)$, $\mathrm{Hom}_{\mathbb{Z}}(-,N)$, $M \otimes_{\mathbb{Z}} -$ and, $- \otimes_{\mathbb{Z}} N$ are functors $Ab \to Ab$.

Let
$$\begin{tikzcd} & & & M \arrow[d] \arrow[dl, dashed] & \\ 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[dl, dashed] & A'' \arrow[r] & 0 \\ & N & & & \end{tikzcd}$$

$\mathrm{Hom}_{\mathbb{Z}}(M,-)$ is a covariant left exact functor. (if $A''$ is projective, then $A \to A''$ lifts)

$\mathrm{Hom}_{\mathbb{Z}}(-,N)$ is a <u>contravariant</u> left exact functor (if $A'$ is injective, then $A' \to A$ extends)

Q] Why welldefined? Choice of proj. res $\Rightarrow$ chain homotopic.

<u>Defn</u> $\mathrm{Ext}(M,N)$

① $0 \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \to 0$ by FTFGM over PIDs

② 
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d] & P_0 \arrow[r] \arrow[d, "\pi"] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$
quasi-isomorphism on complexes

$H_0(B_\cdot) \cong H_0(A_\cdot)$

$A_\cdot \xrightarrow{f_\cdot} B_\cdot$ induces $H_\cdot(A) \xrightarrow[\text{isom}]{H_\cdot(f)} H_\cdot(B_\cdot)$

③ Apply the functor $\mathcal{F}$.

④ Take (co)homology.

EX] $0 \longrightarrow \mathrm{Hom}(P_0, N) \longrightarrow \mathrm{Hom}(P_1, N) \longrightarrow 0$

Then produces $H^1(\mathrm{Hom}(-,?)(P_0, N)) := \mathrm{Ext}^1(M,N)$

EX] Compute $\mathrm{Ext}^1(\mathbb{Z}/2, \mathbb{Z}/3)$.

① $0 \longrightarrow \overset{P_1}{\mathbb{Z}} \xrightarrow[\cdot 2]{d_1} \overset{P_0}{\mathbb{Z}} \longrightarrow \mathbb{Z}/_2 \longrightarrow 0$

"find a complex that's quasi-isom to the (co)homology

② $0 \longrightarrow \overset{P_1}{\mathbb{Z}} \xrightarrow[\cdot 2]{d_1} \mathbb{Z} \longrightarrow 0$ forget the end

③ $\mathrm{Hom}(-, \mathbb{Z}/3)$ applied, thus $0 \to \mathbb{Z}/_3 \xrightarrow{2} \mathbb{Z}/_3 \longrightarrow 0$

# § Exereises (User's Guide to Ext & Tor in Ab)

<u>Riddle</u>. Cycles mod boundaries gives kernel mod image. What's kernel mod cokernel?

G is injective iff G is <u>divisible</u>.

For each $g \in G$ and each $n \in \mathbb{Z} \setminus \{0\}$, there exists $e \in G$ sth $n.e = g$.

**PROVE** E.g. $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$

G is <u>projective</u> iff G is <u>free</u>.

$$\begin{tikzcd} & G \arrow[d] \arrow[dl, dashed, "\exists"'] & \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

$b$, $c$, $x$

**DEFINE** the def$^{\underline{n}}$ of injective relies on a dual diagram to the one here.

**RIDDLE** $\mathrm{Ext}^0_{\mathbb{Z}}(M, N) = \mathrm{Hom}_{\mathbb{Z}}(M, N)$. Apply $\mathrm{Hom}_{\mathbb{Z}}(-, N)$

① $$\begin{tikzcd} 0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \\ 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

② $$\begin{tikzcd} & 0 & & \\ 0 \arrow[r] & \mathrm{Hom}(P_0, N) \arrow[r] \arrow[u] & \mathrm{Hom}(P_1, N) \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(M, N) \arrow[r] \arrow[u] & 0 & \\ & 0 \arrow[u] & & \end{tikzcd}$$

### *A3.10.1 Exercises: Tor*

The name "Tor" comes from the following computation, which connects Tor with torsion.

**Exercise A3.16:*** Let $x \in R$ be a nonzerodivisor. Show that

$$\mathrm{Tor}_1(R/x, M) = \{m \in M | xm = 0.\}$$

**Exercise A3.17:** If $I$ and $J$ are any ideals of $R$, then $IJ \subset I \cap J$. Show that $\mathrm{Tor}_1(R/I, R/J) = (I \cap J)/(IJ)$. This usefully encapsulates several often-used cases (of course these can also be proven directly). For example, use it to show that $I \cap J = IJ$ in the following cases:

a. $I + J = R$.

b. $I$ is generated by a sequence of elements that form a regular sequence mod $J$.

**Exercise A3.18 ("Betti" numbers):** Let $(R, \mathfrak{m})$ be a local ring. We say that a free resolution

$$F : \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \xrightarrow{\varphi_1} F_0$$

of a module $M$ is **minimal** if each $\varphi_i$ has an image contained in $\mathfrak{m}F_{i-1}$. (If the $F_i$ are finitely generated modules, then Nakayama's lemma shows that this is equivalent to a more obviously natural formulation. See Chapter 20.) If $F$ as above is a minimal free resolution of $M$ and rank $F_i = b_i$, then show that $\mathrm{Tor}_i(R/\mathfrak{m}, M) = (R/\mathfrak{m})^{b_i}$. The $b_i$ are called Betti numbers of $M$, in loose analogy with the situation in topology, where $F$ is a chain complex.

COMPUTE Ext & Tor for $M$ and $N$...

(1) $M = \mathbf{Z}$ and $N = \mathbf{Z}$,

(2) $M = \mathbf{Z}/4\mathbf{Z}$ and $N = \mathbf{Z}/8\mathbf{Z}$,

(3) $M = \mathbf{Q}$ and $N = \mathbf{Z}/2\mathbf{Z}$, and

(4) $M = \mathbf{Z}/2\mathbf{Z}$ and $N = \mathbf{Q}/\mathbf{Z}$.

(5) $M = \mathbb{Z}^r$ and $N = \mathbb{Z}/3\mathbb{Z}$

(6) $M \cong \mathbb{Z}^r \oplus \mathbb{Z}/p_1^{n_1} \oplus \cdots \oplus \mathbb{Z}/p_l^{n_l}$

$N \cong \mathbb{Z}^q \oplus$ " $\lambda_1^{m_1}$ " $\oplus$ " $\lambda_j^{m_j}$

$\boxed{\text{EX}}$ $M = N = \mathbb{Z}$. Want $\text{Ext}^i(M,N)$.

① $0 \to \mathbb{Z} \xrightarrow{id} \mathbb{Z} \to 0$

② $0 \to \mathbb{Z} \to 0$

③ $\text{Hom}(0,\mathbb{Z}_3) \leftarrow \text{Hom}(\mathbb{Z},\mathbb{Z}) \leftarrow \text{Hom}(0,\mathbb{Z}_3)$

$0 \leftarrow \mathbb{Z} \leftarrow 0$

④ $\text{Ext}^1(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}$ in what degree?

Tor

① $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$

② $0 \to \mathbb{Z} \to 0$

③ $0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to 0$

$\downarrow \cong$

$0 \to \mathbb{Z} \to 0$

④ $\text{Tor}^1(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}$ ?

WRONG!

$\boxed{\text{EX}}$ $M = \mathbb{Z}_4$, $N = \mathbb{Z}_8$

Ext

① $0 \to \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \to \mathbb{Z}/4 \to 0$

② $0 \to \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \to 0$

③ $0 \to \text{Hom}(\mathbb{Z},\mathbb{Z}_8) \xrightarrow{\text{Hom}(\cdot 4,\mathbb{Z}_8)} \text{Hom}(\mathbb{Z},\mathbb{Z}_8) \to 0$

# § Projective Resolutions and more Ext

20190401

We're in $Ab = \mathbb{Z}$mod, where "projective" means "free."

**Remark** If $F$ is free, then for any exact seq. $M \xrightarrow{g} N \to 0$ with a map $f : F \to N$, there's a $\varphi$ lifting $f$ to $M$.

$$\begin{tikzcd} & F \arrow[d, "f"] \arrow[dl, dashed, "\exists\varphi"', "\text{non-unique}"] \arrow[dl, phantom, shift right=3, "\circ"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

**PROP** Let $F_\bullet(G) \to G$ and $F_\bullet(G') \to G'$ be presentations.

Let $f = f_{-1} : G \to G'$ be a homomorphism.

Then there exists a lift $f_\bullet : F_\bullet(G) \to F_\bullet(G')$, i.e.,

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & F_1(G) \arrow[r] \arrow[d, "f_1"] & F_0(G) \arrow[r] \arrow[d, "f_0"] & F_{-1}(G) \arrow[r] \arrow[d, "f_{-1}"] & 0 \arrow[d] \\ 0 \arrow[r] & F_1(G') \arrow[r] & F_0(G') \arrow[r] & F_{-1}(G) \arrow[r] & 0 \end{tikzcd}$$

} any two lifts are chain homotopic.

**Proof.** We'll show a lift exists and any two lifts are chain homotopic.

Existence of lift. then uniqueness up to quasi-isom? chain homotopy?

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & F_1(G) \arrow[r, "d_1^G"] \arrow[d, "f_1 - g_1"] \arrow[dl, "D"'] & F_0(G) \arrow[r, "d_0^G"] \arrow[d, "f_0 - g_0"] \arrow[dl, "D"'] & F_{-1}(G) \arrow[r] \arrow[d, "f_{-1} - g_{-1}"] \arrow[dl, "D"'] & 0 \arrow[d] \arrow[dl, "0"] \\ 0 \arrow[r] & F_1(G') \arrow[r, "d_1^{G'}"] & F_0(G') \arrow[r, "d_0^{G'}"] & F_{-1}(G) \arrow[r] & 0 \end{tikzcd}$$

$\therefore$ use surjectivity

The s.e.s. has $f_0$ and $f_{-1}$, so $f_1$ exists indeed.

use order 2

$$\begin{tikzcd} 0 \arrow[r] & F_1(G) \arrow[r] \arrow[d, "\therefore\exists"'] & F_0(G) \arrow[r] \arrow[d, "f_0"] \arrow[dr, phantom, "\circ"] & F_{-1}(G) \arrow[r] \arrow[d, "f_{-1}"] & 0 \\ 0 \arrow[r] & F_1(G') \arrow[r] & F_0(G') \arrow[r] & F_{-1}(G) \arrow[r] & 0 \end{tikzcd}$$

$$\ker\left(F_0(G) \to F_{-1}(G)\right) \xrightarrow[\text{induced?}]{\text{inclusion?}} \ker\left(F_0(G') \to F_{-1}(G')\right)$$

Let $g$ be another such lift

use comm. square.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & F_1(G) \arrow[r] \arrow[d, "f_1 - g_1"] & F_0(G) \arrow[r] \arrow[d, "f_0 - g_0"] \arrow[dl, "\exists D"'] & G \arrow[r] \arrow[d, "0"] \arrow[dl] & 0 \arrow[d] \\ 0 \arrow[r] & F_1(G') \arrow[r] & F_0(G') \arrow[r] & G' \arrow[r] & 0 \end{tikzcd}$$

TODO

$f \simeq g$ homotopic maps?

$\square$

Recall the structure theorem for F.G. Ab gps.

EX] Say $p, q$ primes in $\mathbb{Z}$. Let $n, m \in \mathbb{N}$. Then for a gp. $\mathbb{Z}/p^n$, we've the presentation

① $$0 \to \mathbb{Z} \xrightarrow{p^n} \mathbb{Z} \to \mathbb{Z}/p^n \to 0$$

② Apply $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/q^m)$ to the chain comp. $0 \to \mathbb{Z} \xrightarrow{p^n} \mathbb{Z} \to 0$.

$$\begin{tikzcd}
0 & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/q^m) \arrow[l] \arrow[d, "\cong", no head, double] & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/q^m) \arrow[l, "{\mathrm{Hom}(p^n, \mathbb{Z}/q^m)}"'] \arrow[d, "\cong", no head, double] & 0 \arrow[l] & \\
0 & \mathbb{Z}/q^m \arrow[l] & \mathbb{Z}/q^m \arrow[l, "p^n"'] & 0 \arrow[l] & (\star)
\end{tikzcd}$$

④ Compute $H^0(\star) = \begin{cases} 0, & p \neq q \\ \mathbb{Z}/p^k & \text{else} \end{cases}$, $H^1(\star) = \begin{cases} 0, & p \neq q \\ \mathbb{Z}/p^k & \text{else} \end{cases}$
with $k = \min\{p, q\}$.

$G$ is an extension of $\widetilde{\mathbb{Z}}/p$ by $\mathbb{Z}/p$

Q] $\mathrm{Ext}(\widetilde{\mathbb{Z}}/p, \mathbb{Z}/p)^{\cong \mathbb{Z}/p}$ classifies s.e.s. up to equivalences of ses.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/p \arrow[r] \arrow[d, no head, double] & G \arrow[r] \arrow[d, "\cong"] & \widetilde{\mathbb{Z}}/p \arrow[r] \arrow[d, no head, double] & 0 \\
0 \arrow[r] & \mathbb{Z}/p \arrow[r] & G' \arrow[r] & \widetilde{\mathbb{Z}}/p \arrow[r] & 0
\end{tikzcd}$$

EX] $0^{th}$ extension $0 \to \widetilde{\mathbb{Z}}/p \to \widetilde{\mathbb{Z}}/p \oplus \mathbb{Z}/p \to \mathbb{Z}/p \to 0$, $p-1$ other extensions for $\lambda \in (\mathbb{Z}/p)^\times$
$$0 \to \mathbb{Z}/p \xrightarrow{\lambda p} \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

**Exercise A3.26 (Yoneda's description of $\mathrm{Ext}^1$):** The ideas in this and the next exercise give a useful and appealing interpretation of the elements of Ext. See, for example, MacLane [1963, Chapter III] for more details.

a. If

$$\alpha: \quad 0 \to B \to X \to A \to 0$$
$$\alpha': \quad 0 \to B \to X' \to A \to 0$$

are short exact sequences, we say that $\alpha$ is **Yoneda equivalent** to $\alpha'$ if there exists a map $f: X \to X'$ making the diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, no head, double] & X \arrow[r] \arrow[d, "f"] & A \arrow[r] \arrow[d, no head, double] & 0 \\
0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

commute. Show that Yoneda equivalence is an equivalence relation (reflexive, symmetric, and transitive). Show that $\alpha$ is Yoneda equivalent to the "split" sequence

$$0: 0 \to A \to A \oplus B \to B \to 0$$

iff $\alpha$ is itself split.

We shall write $[\alpha]$ for the Yoneda equivalence class of a short exact sequence $\alpha$.

We now define $E^1(A, B)$ to be the set of equivalence classes of short exact sequences as above. We shall see that $E^1_R(A, B)$ is naturally isomorphic to $\mathrm{Ext}^1_R(A, B)$.

THM] "Horseshoe lemma." (Produces l.e.s.) Given $0 \to A \to B \to C \to 0$, and presentations $F_\cdot(A) \to A$, $F_\cdot(C) \to C$, we have a "horseshoe". Then there's a presentation $F_\cdot(B) \to B$ s.th. the rows and columns are exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_1(A) \arrow[r] \arrow[d] & F_1(B) \arrow[r] \arrow[d] & F_1(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_0(A) \arrow[r] \arrow[d] & F_0(B) \arrow[r] \arrow[d] & F_0(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Pf. TODO.

# EXERCISES (Projective Resolutions, etc.)

## PROVE

*Free Resolutions*

**Exercise 1.22:** Let $R = k[x]$. Use the structure theorem for finitely generated modules over a principal ideal domain to show that every finitely generated $R$-module has a finite free resolution.

**Exercise 1.23:** Let $R = k[x]/(x^n)$. Compute a free resolution of the $R$-module $R/(x^m)$, for any $m \leq n$. Show that the only $R$-modules with finite free resolutions are the free modules.

## VERIFY

Should we change rings to $\mathbb{C}[x_1, x_2]$, then the "projective dimension" is 2.

## PROVE the horseshoe lemma.

*A3.11.1 Exercises: Ext*

**Exercise A3.23:** If $x$ is a nonzerodivisor in a ring $R$, compute $\mathrm{Ext}^i_R(R/x, M)$. In particular, compute $\mathrm{Ext}^i_{\mathbf{Z}}(\mathbf{Z}/n, \mathbf{Z}/m)$ for any integers $n, m$.

**Exercise A3.24:** Show that a finitely generated Abelian group $A$ is free iff $\mathrm{Ext}^1_{\mathbf{Z}}(A, \mathbf{Z}) = 0$. It was conjectured by Whitehead that this would hold for all groups, but the truth turns out to depend on your set theory (Shelah [1974]).

# § Universal Coeff thm

We have the functors

$(\otimes)$ $\mathrm{Tor}(-,-) : Ab \times Ab \longrightarrow Ab$ — homological functor — Tor is right exact! "torsion"

$(\mathrm{Hom})$ $\mathrm{Ext}(-,-) : Ab^{op} \times Ab \longrightarrow Ab$ — cohomological functors — "extension"

**PROP** Let $G \in Ab$ and $0 \to A \to B \to C \to 0$.

Then
$$0 \longrightarrow \mathrm{Hom}(C,G) \longrightarrow \mathrm{Hom}(B,G) \longrightarrow \mathrm{Hom}(A,G) \longrightarrow \mathrm{Ext}(C,G) \longrightarrow \mathrm{Ext}(B,G) \longrightarrow \mathrm{Ext}(A,G) \longrightarrow 0$$
is exact,

and
$$0 \longrightarrow \mathrm{Tor}(G,A) \longrightarrow \mathrm{Tor}(G,B) \longrightarrow \mathrm{Tor}(G,C) \longrightarrow G\otimes A \longrightarrow G\otimes B \longrightarrow G\otimes C \longrightarrow 0$$
is exact too.

**Pf.** Horseshoe lemma.

**THM** (Universal Coeff thm.)

Let $\mathcal{C}_\bullet \in Ch$ for free Abelian gps, and $G \in Ab$, there's an exact seq$\underline{n}$

$H^n(\mathrm{Hom}(\mathcal{C}_\bullet, G))$ for upper indices...

$$0 \to \mathrm{Ext}(H_{n-1}(\mathcal{C}_\bullet), G) \to H^n(\mathcal{C}_\bullet; G) \xrightarrow{\beta} \mathrm{Hom}(H_n(\mathcal{C}_\bullet), G) \to 0$$

which is natural in $\mathcal{C}_\bullet$ and $G$, yet non-naturally split; $\beta([\![f]\!])([\![c]\!]) = f(c)$.

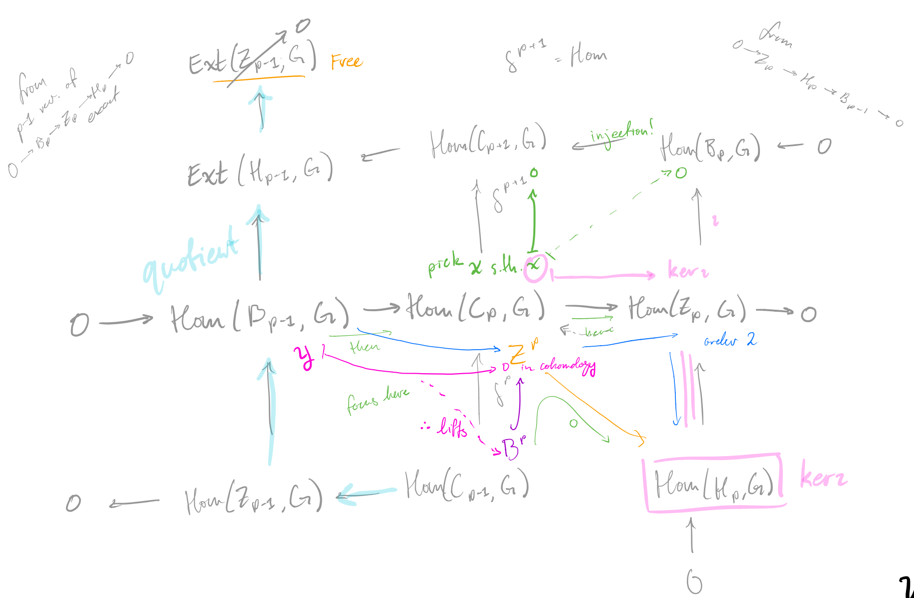
As well, there's a s.e.s (not naturally) that splits.

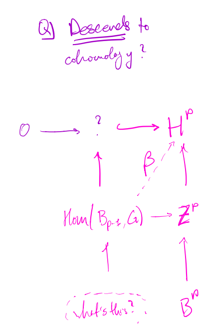
$$0 \to H_n(\mathcal{C}_\bullet)\otimes G \to H_n(\mathcal{C}_\bullet; G) \to \mathrm{Tor}(H_{n-1}(\mathcal{C}_\bullet), G) \to 0$$

**Proof.** Denote $H_p = H_p(\mathcal{C}_\bullet)$ also with $Z_p$, $B_p$.

Then $0 \to B_p \to Z_p \to H_p \to 0$ exact.

Also $0 \to Z_p \to C_p \to B_{p-1} \to 0$ is exact.



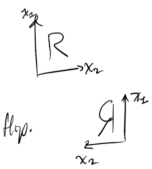




Video-proof. □

**Q]** Why do we care about coeffs? Cup products have ...

- Orientability defined in terms of cohomology...
- Definition depends on the coeffs.
- E.g. $GL(n,\mathbb{R})$ has either $\pm 1$ orientation.

  But then over $\mathbb{Z}/2$, we know nothing about this flip.



**EX]** For $(X,A) \in \mathrm{Top}_{\text{pairs}}$, there's a s.e.s. $0 \to \Delta_*(A) \to \Delta_*(X) \to \Delta_*(X,A) \to 0$ of free chain complexes, so at each level we have (by the UCT)

$$0 \to \mathrm{Ext}(H_{n-1}(\Delta_\cdot(X,A)), G) \to H^n(\Delta_\cdot(X,A); G) \xrightarrow{\beta} \mathrm{Hom}(H_n(\Delta_\cdot(X,A)), G) \to 0.$$

**EX]** We compute $H^*(\mathbb{R}P^4)$ and $H^*(\mathbb{R}P^5)$ knowing $H_*(\mathbb{R}P^4)$, $H_*(\mathbb{R}P^5)$.

| degree | 0 | 1 | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|---|---|
| homology groups | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $0$ | $\mathbb{Z}/2$ | $0$ | $0$ | $0$ | $\mathbb{R}P^4$ |
| | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $0$ | $\mathbb{Z}/2$ | $0$ | $\mathbb{Z}$ | $0$ | $\mathbb{R}P^5$ |
| cohomology groups | $\mathbb{Z}$ | $0$ | $\mathbb{Z}/2$ | $0$ | $\mathbb{Z}/2$ | $0$ | $0$ | $\mathbb{R}P^4$ |
| | $\mathbb{Z}$ | $0$ | $\mathbb{Z}/2$ | $0$ | $\mathbb{Z}/2$ | $\mathbb{Z}$ | $0$ | $\mathbb{R}P^5$ |

## § Tor version of U.C.T. (Bredon V.7)

**THM** For any f.g. abelian group $H_n(X,A)$, and $G \in Ab$, we have a natural ses in $H_n, G$, that unnaturally splits.

$$0 \to H_n(X,A) \otimes G \to H_n(X,A;G) \to \mathrm{Tor}^1_{\mathbb{Z}}(H_{n-1}, G) \to 0.$$

**EX** Non-naturality. $\varphi : \mathbb{R}P^2 \to \mathbb{R}P^2/\mathbb{R}P^1 \cong S^2$. (Recall the ses is natural.)

$$\begin{tikzcd}
 & & 0 \oplus \mathbb{Z}/2 \arrow[d, bend right, "?"'] \arrow[d, bend left, "\text{unnat'l.}"] & & \\
0 \arrow[r] & H_2(\mathbb{R}P^2) \otimes \mathbb{Z}/2 \arrow[r] \arrow[d] & H_2(\mathbb{R}P^2; \mathbb{Z}/2) \arrow[r] \arrow[d, "\cong"] & \mathrm{Tor}^1_{\mathbb{Z}}(H_1(\mathbb{R}P^2), \mathbb{Z}/2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_2(S^2) \otimes \mathbb{Z}/2 \arrow[r] & H_2(S^2; \mathbb{Z}/2) \arrow[r] & \mathrm{Tor}^1_{\mathbb{Z}}(H_1(S^2), \mathbb{Z}/2) \arrow[r] & 0 \\
 & & \mathbb{Z}/2 \oplus 0 & &
\end{tikzcd}$$

apply 5 lemma.

## EXERCISES

tensor by $\mathbb{Z}/2$

$$\begin{tikzcd}
\mathbb{R}P^2 & 0 & \mathbb{Z} \arrow[l] \arrow[d, "\cong"] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, "0"] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, "\cong"] & 0 \arrow[l] \\
S^2 & 0 & \mathbb{Z} \arrow[l] & 0 \arrow[l] & \mathbb{Z} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

COMPUTE $H_n(\mathbb{R}P^k; \mathbb{Z}/2)$

PROVE the U.C.T. Tor-version

## § Axiomatic cohomology

THM| Let $f: A_* \to B_*$ be a quasi-isomorphism (e.g. $f_*: H_*(A) \xrightarrow{\cong} H_*(B)$). Then it induces an isom on $H_*(-;G)$, $H^*(-;G)$ for all $G \in Ab$.

Pf.
$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}(H_{n-1}(A_*);G) & \to & H^n(A_*;G) & \to & \mathrm{Hom}_{\mathbb{Z}}(H_{n-1}(A_*),G) & \to 0 \\ & \downarrow \cong & & \therefore \downarrow \cong & & \downarrow \cong & \\ 0 \to & \mathrm{Ext}(H_{n-1}(B_*);G) & \to & H^n(B_*;G) & \to & \mathrm{Hom}_{\mathbb{Z}}(H_{n-1}(B_*),G) & \to 0 \end{array}$$
5 lemma!

Axiomatic development for coh.

les.
excision
homotopy
additivity
zeroth dimension

PROP| Homotopy axiom for cohomology. $\mathrm{Hom}(-;G)$, dual map, tensor it w/ stuff.

TODO| send $D$ in $D = z_3 \times X : \Delta_*(X) \to \Delta_*(\Delta_1 \times X)$
$\mathrm{Hom}(D,G)$ ✓ s.th $D\partial + \partial D = (\eta_1)_\Delta - (\eta_0)_\Delta$

Key. $X \overset{\eta_0}{\underset{\eta_1}{\rightrightarrows}} X \times \Delta_1$ to show

$H^*(\eta_0) = H^*(\eta_1)$

*ADDITIVITY If $(X, A)$ is the disjoint union of a set of pairs $(X_i, A_i)$, then the inclusions $(X_i, A_i) \longrightarrow (X, A)$ induce an isomorphism*

$$H^*(X, A; \pi) \longrightarrow \prod_i H^*(X_i, A_i; \pi).$$

A category $\mathcal{A}$ is called an **Ab**-*category* if every hom-set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ in $\mathcal{A}$ is given the structure of an abelian group in such a way that composition distributes over addition. In particular, given a diagram in $\mathcal{A}$ of the form

about hTop pairs
$$A \xrightarrow{f} B \overset{g'}{\underset{g}{\rightrightarrows}} C \xrightarrow{h} D$$
prove homotopy axiom

we have $h(g + g')f = hgf + hg'f$ in $\mathrm{Hom}(A, D)$. The category **Ch** is an **Ab**-category because we can add chain maps degreewise; if $\{f_n\}$ and $\{g_n\}$ are chain maps from $C_.$ to $D_.$, their sum is the family of maps $\{f_n + g_n\}$.

An *additive functor* $F: \mathcal{B} \to \mathcal{A}$ between **Ab**-categories $\mathcal{B}$ and $\mathcal{A}$ is a functor such that each $\mathrm{Hom}_{\mathcal{B}}(B', B) \to \mathrm{Hom}_{\mathcal{A}}(FB', FB)$ is a group homomorphism.

An *additive category* is an **Ab**-category $\mathcal{A}$ with a zero object (i.e., an object that is initial and terminal) and a product $A \times B$ for every pair $A, B$ of objects in $\mathcal{A}$. This structure is enough to make finite products the same as finite coproducts. The zero object in **Ch** is the complex "0" of zero modules and maps. Given a family $\{A_\alpha\}$ of complexes of $R$-modules, the product $\Pi A_\alpha$ and coproduct (direct sum) $\oplus A_\alpha$ exist in **Ch** and are defined degreewise: the differentials are the maps

$$\prod d_\alpha : \prod_\alpha A_{\alpha,n} \to \prod_\alpha A_{\alpha,n-1} \quad \text{and} \quad \oplus d_\alpha : \oplus_\alpha A_{\alpha,n} \to \oplus_\alpha A_{\alpha,n-1},$$

respectively. These suffice to make **Ch** into an additive category.

$\mathrm{Hom}(-;G)$, dual map, tensor it w/ stuff.

**Exercise 1.2.1** Show that direct sum and direct product commute with homology, that is, that $\oplus H_n(A_\alpha) \cong H_n(\oplus A_\alpha)$ and $\Pi H_n(A_\alpha) \cong H_n(\Pi A_\alpha)$ for all $n$.

## EXERCISES

READ

**Proposition 1.2.10.** *Let $M$ be an $R$-module, $(N_i)_{i \in I}$ be a family of $R$-modules, and $\pi_j : \prod_I N_i \to N_j$ for each $j$ be the projection map. Then the map*

$$\varphi : \mathrm{Hom}_R\Big(M, \prod_I N_i\Big) \to \prod_I \mathrm{Hom}(M, N_i)$$

*defined by $\varphi(f) = (\pi_i \circ f)_I$ is an isomorphism.*

*Proof.* $\varphi$ is clearly an $R$-homomorphism. Suppose $(f_i)_I \in \prod_I \mathrm{Hom}(M, N_i)$. Then $f_i$ is a map from $M$ to $N_i$ for each $i$. So we can define a map $f : M \to \prod N_i$ by $f(x) = (f_i(x))_I$. $f$ is clearly an $R$-homomorphism. Furthermore, $\pi_j \circ f(x) = \pi_j((f_i(x))_I) = f_j(x)$ for all $x \in M$ and so $\pi_j \circ f = f_j$ for each $j$. Hence $\varphi(f) = (\pi_i \circ f)_I = (f_i)_I$. That is, $\varphi$ is onto.

Now suppose $\varphi(f) = 0$. Then $(\pi_j \circ f)(x) = \pi_j(f(x)) = 0$ for each $j$ and each $x \in M$. But then $f(x) = 0$ for all $x \in M$. That is, $f = 0$ and so $\varphi$ is one-to-one. □

A similar proof gives the following.

**Proposition 1.2.11.** *Let $M$ be an $R$-module, $(N_i)_{i \in I}$ be a family of $R$-modules, and $e_j : N_j \to \bigoplus_I N_i$ be the $j$th embedding. Then the map*

$$\varphi : \mathrm{Hom}_R\Big(\bigoplus_I N_i, M\Big) \to \prod_I \mathrm{Hom}_R(N_i, M)$$

*defined by $\varphi(f) = (f \circ e_i)_I$ is an isomorphism.*

**Definition 1.2.12.** If $M, M', N, N'$ are $R$-modules and $f : M' \to M$, $g : N \to N'$ are $R$-homomorphisms, then define a map $\varphi : \mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N')$ by $\varphi(h) = ghf$. $\varphi$ is denoted by $\mathrm{Hom}(f, g)$. We have that $\mathrm{Hom}(f, g)(h_1 + h_2) = \mathrm{Hom}(f, g)(h_1) + \mathrm{Hom}(f, g)(h_2)$, that is, $\mathrm{Hom}(f, g)$ is *additive*. Furthermore, in the situation ${}_RM_S$, ${}_RM'_S$, ${}_RN$, ${}_RN'$, if $f : M' \to M$ is an $(R, S)$-homomorphism and $g : N \to N'$ is an $R$-homomorphism, then $\mathrm{Hom}(f, g)$ is an $S$-homomorphism between the two left $S$-modules.

If $M' \xrightarrow{f} M \xrightarrow{f'} M''$ and $N'' \xrightarrow{g'} N \xrightarrow{g} N'$ are homomorphisms, then it is easy to see that $\mathrm{Hom}(f, g) \circ \mathrm{Hom}(f', g') = \mathrm{Hom}(f' \circ f, g \circ g')$.

The maps $\mathrm{Hom}(f, \mathrm{id}_N)$, $\mathrm{Hom}(\mathrm{id}_M, g)$ are denoted by $\mathrm{Hom}(f, N)$, $\mathrm{Hom}(M, g)$ respectively. We note that if $f : M' \to M$ is an $R$-homomorphism, then we have a homomorphism of Abelian groups $\mathrm{Hom}(f, N) : \mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N)$. Similarly, for a map $g : N \to N'$, we get a map $\mathrm{Hom}(M, g) : \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N')$.

in Ab, its $\prod_i G_i \cong \bigoplus_i G_i$

PROVE ✓ in Rmod

- $\mathrm{Ext}(\bigoplus_i G_i, G') \cong \prod_i \mathrm{Ext}(G_i, G')$.
- $\mathrm{Ext}(G, \bigoplus_i G_i') \cong \prod_i \mathrm{Ext}(G_i, G)$
- $\mathrm{Tor}(\bigoplus_i G_i, G') \cong \bigoplus_i \mathrm{Tor}(G_i, G)$.

HINT
If you hom out of the direct sum, you get the product (because the direct sum is the coproduct)
If you tensor a direct sum, then you've the dir. sum of tensors...

PROVE

3. Let $M$ be an $R$-module and $(N_i)_{i \in I}$ be a family of $R$-modules. Consider the homomorphism of Abelian groups

$$\varphi : \bigoplus_{i \in I} \mathrm{Hom}_R(M, N_i) \to \mathrm{Hom}_R(M, \oplus N_i)$$

which maps $(f_i)_{i \in I}$ to $f$ where $f(x) = (f_i(x))_{i \in I}$

(a) Argue that $\varphi$ is an isomorphism if $I$ is finite or if $M$ is finitely generated.

(b) Find an example where $\varphi$ is not an isomorphism.

4. If $N$ is an $R$-module and $(M_i)_{i \in I}$ is a family of $R$-modules define

$$\varphi : \bigoplus_{i \in I} \mathrm{Hom}_R(M_i, N) \to \mathrm{Hom}_R\Big(\prod_{i \in I} M_i, N\Big)$$

where $\varphi(f_i)_{i \in I} = f$ with $f((x_i)_{i \in I}) = \sum_{i \in I} f_i(x_i)$

(a) Argue that $\varphi$ is an isomorphism if $I$ is finite.

(b) Find an example where $\varphi$ is not an isomorphism.

PROP! Homotopy axiom for cohomology.

TODO! send $D$ in $D = z_1 \times X : \Delta_*(X) \to \Delta_*(\Delta_1 \times X)$

$\mathrm{Hom}(D, G)$ s.th $D\partial + \partial D = (\eta_1)_\Delta - (\eta_0)_\Delta$

Key. $X \underset{\eta_1}{\overset{\eta_0}{\rightrightarrows}} X \times \Delta_1$ to show $H^*(\eta_0) = H^*(\eta_1)$

# MATH 6220 HOMEWORK 5

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We work in the category $\mathsf{Ch}$, *chain complexes* of graded abelian groups.

**1.** For a given chain complex $C$, let $\mathrm{cyl}(C)$ denote the *mapping cylinder* of the identity map on $\mathrm{id}\colon C \to C$. The chain complex $\mathrm{cyl}(C)$ encodes the following data: If $C$ is described by the level-wise groups and differentials

$$C = \left\{ C_{i+1} \xrightarrow{d_{i+1}} C_i : \text{for all } i \in \mathbb{Z} \right\},$$

then the mapping cylinder $\mathrm{cyl}(C)$ is described by level-wise groups and differentials

$$\mathrm{cyl}(C) = \left\{ \begin{tikzcd} C_i \arrow[r, "d_i"] & C_{i-1} \\ C_{i-1} \arrow[ur, "\mathrm{id}"] \arrow[r, "-d_{i-1}"] \arrow[dr, "-\mathrm{id}"] & C_{i-2} \\ C_i \arrow[r, "d_i"] & C_{i-1} \end{tikzcd} \ : \text{for all } i \in \mathbb{Z} \right\}.$$

Each level-wise group $(\mathrm{cyl}(C))_i$ is the direct sum of groups $C_i \oplus C_{i-1} \oplus C_i$, where $\mathrm{id}\colon C \to C$ is the identity, and where arrows indicate how a component group is mapped into a component group of lesser degree in the chain complex $\mathrm{cyl}(C)$.

Writing the level-wise groups as column vectors and the differential $[\partial]$ of $\mathrm{cyl}(C)$ as a matrix is elucidating, and will be helpful for computation:

$$[\partial] = \begin{bmatrix} d & \mathrm{id} & 0 \\ 0 & -d & 0 \\ 0 & -\mathrm{id} & d \end{bmatrix} : \begin{bmatrix} C_i \\ C_{i-1} \\ C_i \end{bmatrix} \longmapsto \begin{bmatrix} C_{i-1} \\ C_{i-2} \\ C_{i-1} \end{bmatrix}.$$

Now, say that $D$ is another chain complex that's the target of two chain maps $f, g\colon C \to D$.

**Claim** (Extending chain maps to a mapping cylinder)**.** Two chain maps $f, g\colon C \to D$ are chain homotopic if and only if they extend to a map

$$\begin{bmatrix} f & \mathcal{S} & g \end{bmatrix} : \mathrm{cyl}(C) \to D,$$

where $\mathcal{S}\colon C \to D$ is a sequence[1] of level-wise sections of homological degree $+1$.

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ C_i \arrow[d] \arrow[ur, "s_i"] & D_{i+1} \arrow[d] \\ C_{i-1} \arrow[d] \arrow[ur, "s_{i-1}"] & D_i \arrow[d] \\ C_{i-r} \arrow[d] \arrow[ur, "s_{i-2}"] & D_i \arrow[d] \\ \vdots \arrow[ur, "s_{i-3}"] & \vdots \end{tikzcd} .$$

[1]Note that $\mathcal{S}$ is *not* a chain map. For example, if $f = g$, then the maps in $\mathcal{S}$ satisfy $ds + sd = 0$, and *not* $ds = sd$.

*Proof.* By definition, the chain maps $f$ and $g$ are chain homotopic if and only if there exists a sequence $\{H_i\}_{i\in\mathbb{Z}}$ of group homomorphisms

$$\left\{C_i \xrightarrow{H_i} D_{i+1}\right\}_{i\in\mathbb{Z}}$$

such that, for each $i \in \mathbb{Z}$, and each chain $c \in C_i$, the homomorphisms $H_{i+1}$ and $H_i$ satisfy the homotopy condition

$$(f_i - g_i)(c) = (d_i H_{i-1} - H_i d_{i-1})(c). \tag{1.1}$$

On the other hand, $\begin{bmatrix} f & \mathcal{S} & g \end{bmatrix}$ is a chain map if and only if for each $i \in \mathbb{Z}$, the following diagram commutes:

$$\begin{tikzcd}
(\mathrm{cyl}(C))_i \arrow[r, "{[\partial]}"] \arrow[d, "{\begin{bmatrix} f & \mathcal{S} & g \end{bmatrix}}"'] & (\mathrm{cyl}(C))_{i-1} \arrow[d, "{\begin{bmatrix} f & \mathcal{S} & g \end{bmatrix}}"] \\
D_i \arrow[r, "d"] & D_{i-1}
\end{tikzcd} \tag{1.2}$$

We compute the composition $(\mathrm{cyl}(C))_i \to (\mathrm{cyl}(C))_{i-1} \to D_{i-1}$ along the upper right corner of (1.2),

$$\begin{bmatrix} f & \mathcal{S} & g \end{bmatrix} \begin{bmatrix} d & \mathrm{id} & 0 \\ 0 & -d & 0 \\ 0 & -\mathrm{id} & d \end{bmatrix} = \begin{bmatrix} fd & f - sd - g & gd \end{bmatrix}. \tag{1.3}$$

We also compute the composition $(\mathrm{cyl}(C))_i \to D_i \to D_{i-1}$ along the lower left corner of (1.2),

$$\begin{bmatrix} d \end{bmatrix} \begin{bmatrix} f & \mathcal{S} & g \end{bmatrix} = \begin{bmatrix} df & ds & dg \end{bmatrix}. \tag{1.4}$$

Now $df = fd$ and $dg = gd$ by the hypotheses that $f, g \colon C \to D$ are chain maps. Comparing entries in (1.4) and (1.3), the following are equivalent:

- $f, g \colon C \to D$ extends to a chain map $\begin{bmatrix} f & \mathcal{S} & g \end{bmatrix} \colon \mathrm{cyl}(C) \to D$.
- The diagram (1.2) commutes.
- For each $i$, the maps $(\mathrm{cyl}(C))_i$ to $D_{i-1}$ on the upper right and lower left of (1.2) are equal.
- For each $i$, the maps[2] $f_{i-1} - s_{i-2}d_{i-1} - g_{i-1} = d_i s_{i-1}$ from the center component of $(\mathrm{cyl}(C))_i$ to $D_{i-1}$ are equal.
- For each $i$, the levelwise maps satisfy $f - g = ds + sd$.
- $\mathcal{S}$ is a chain homotopy between $f, g \colon C \to D$.

We have proven that $f, g \colon C \to D$ are chain homotopic if and only if they extend to a chain map $\mathrm{cyl}(C) \to D$. □

**2.** Let $A$ and $B$ be chain complexes. Define the *tensor chain complex* $A \otimes B$ as the graded abelian group

$$(A \otimes B)_n := \bigoplus_{i+j=n} A_i \otimes B_j,$$

with differential, for each $i$-chain $a$ and each $j$-chain $b$ in $A_i$ and $B_j$,

$$\partial(a \otimes b) := d_i^A(a) \otimes b + (-1)^i a \otimes d_j^B(b).$$

**Claim** (Tensor product of complexes are complexes)**.** $A \otimes B$ with differential $\partial$ is a chain complex.

[2]Note the indices: the center component of $(\mathrm{cyl}(C))_i$ is $C_{i-1}$. Maps out of the center component are thus shifted $-1$ degree from what one might expect.

*Proof.* It suffices to show that $\partial^2 = 0$, which follows. Consider $a \otimes b$, where $a \in A_i$ and $b \in B_j$

$$\begin{aligned}
\partial^2(a \otimes b) &= \partial\big(d_i^A(a) \otimes b \quad + \quad (-1)^i a \otimes d_j^B(b)\big) \\
&= \partial\big(d_i^A(a) \otimes b\big) \quad + \quad (-1)^i \partial\big(a \otimes d_j^B(b)\big) \\
&= d_{i-1}^A d_i^A(a) \otimes b + (-1)^{i-1} d_i^A(a) \otimes d_j^B(b) \\
&\qquad\qquad + \quad (-1)^i \big(d_i^A(a) \otimes d_j^B(b) + (-1)^i a \otimes d_{j-1}^B d_j^B(b)\big) \\
&= (-1)^{i-1} d_i^A(a) \otimes d_j^B(b) + (-1)^i \big(d_i^A(a) \otimes d_j^B(b)\big) \\
&= 0.
\end{aligned}$$

We have shown that $\partial$ is order 2, because $\partial^2(a \otimes b) = 0$, and each chain in any levelwise group $(A \otimes B)_i$ is a linear combination of tensors of the form $a \otimes b$. $\square$

**Claim** (Mapping cylinders are realized as tensor products)**.** Let $I$ be the chain complex defined

- as the graded abelian group $I$ such that $I_0 = \mathbb{Z}\{\ell_0, \ell_1\}$, $I_1 = \mathbb{Z}\{\ell\}$, and $I_i = 0$ if $i \neq 0, 1$,
- with differential $d$ such that $d_1(\ell) = \ell_1 - \ell_0$ and $d_i = 0$ for all $i \neq 1$.

Then $\mathrm{cyl}(C) \cong I \otimes C$.

*Proof.* Recognize the free abelian group $\mathbb{Z}\{\ell_0, \ell_1\} \cong \mathbb{Z} \oplus \mathbb{Z}$. Because the tensor product commutes with direct sums, for an arbitrary abelian group $\mathcal{A}$, there's an natural isomorphism

$$\mathbb{Z}\{\ell_0, \ell_1\} \otimes \mathcal{A} \xrightarrow{\cong} (\mathbb{Z} \otimes \mathcal{A})^{\oplus 2}.$$

Moreover, this tensor product is over $\mathbb{Z}$, so $(\mathbb{Z} \otimes \mathcal{A})^{\oplus 2} \cong \mathcal{A} \oplus \mathcal{A}$. Accounting for $\mathbb{Z}\{\ell\}$ in a similar fashion, it follows that, for any degree $i \in \mathbb{Z}$, the abelian group $(I \otimes C)_i$ is naturally isomorphic to the direct sum

$$(\mathbb{Z}\ell_1 \otimes C_i) \oplus (\mathbb{Z}\ell \otimes C_{i-1}) \oplus (\mathbb{Z}\ell_0 \otimes C_i) \xrightarrow{\cong} C_i \oplus C_{i-1} \oplus C_i. \tag{2.1}$$

Therefore, as graded abelian groups, $I \otimes C \cong_{\mathsf{GrAb}} \mathrm{cyl}(C)$.

Considering the RHS and LHS of (2.1), we deduce that $d$ of $I$ induces the differential $\partial$ on $I \otimes C$ as follows:

$$\left\{ \begin{tikzcd}
\mathbb{Z}\ell_1 \otimes C_i \arrow[r, "\mathrm{id}\otimes d_i"] & \mathbb{Z}\ell_1 \otimes C_{i-1} \\
\mathbb{Z}\ell \otimes C_{i-1} \arrow[ur, "{[\ell \to \ell_0]\otimes\mathrm{id}}"] \arrow[r, "-\mathrm{id}\otimes d_{i-1}"] \arrow[dr, "{-[\ell \to \ell_1]\otimes\mathrm{id}}"] & \mathbb{Z}\ell \otimes C_{i-2} \\
\mathbb{Z}\ell_0 \otimes C_i \arrow[r, "\mathrm{id}\otimes d_i"] & \mathbb{Z}\ell_0 \otimes C_{i-1}
\end{tikzcd} : i \in \mathbb{Z} \right\} \rightsquigarrow \left\{ \begin{tikzcd}
C_i \arrow[r, "d_i"] & C_{i-1} \\
C_{i-1} \arrow[ur, "\mathrm{id}"] \arrow[r, "-d_{i-1}"] \arrow[dr, "-\mathrm{id}"] & C_{i-2} \\
C_i \arrow[r, "d_i"] & C_{i-1}
\end{tikzcd} : i \in \mathbb{Z} \right\}.$$

Hence, if $\varphi\colon I \otimes C \to \mathrm{cyl}(C)$ is the natural isomorphism of graded abelian groups in 2.1, then $\varphi \circ \partial = d \circ \varphi$. So $\varphi$ is an invertible chain map, thus $I \otimes C \cong_{\mathsf{Ch}} \mathrm{cyl}(C)$ as chain complexes. $\square$

*Note.* Say that $\Delta_1$ and $\Delta_2$ are abstract ordered simplices. The product $\Delta_1 \times \Delta_2$ contains 6 vertices, so is *not* a 3-simplex. However, there's an operator, call it $\times$, that takes $\Delta_1 \times \Delta_2$ and makes an ordered decomposition into 3 adjacent 3-simplices, each pairwise sharing 3 vertices. How does the rule $\times$ for the decomposition

$$\Delta_1 \times \Delta_2 \overset{\times}{\longmapsto} \Delta_3 \sqcup \Delta_3 \sqcup \Delta_3 / \sim$$

correspond to the rule on signs for the differential $[\partial]$ on the mapping cylinder? I really don't know. ◂

$$\begin{tikzcd}
 & & C_{i-1} \arrow[dd, "-d_{i-1}"] \arrow[dddll, "-\mathrm{id}"'] \arrow[dddrr, "\mathrm{id}"] & & \\
C_i \arrow[urr, dotted] \arrow[dd, "d_i"'] \arrow[ddrrrr, dotted] & & & & C_i \arrow[ull, dotted] \arrow[llll, dotted, bend left] \arrow[dd, "d_i"] \\
 & & C_{i-2} & & \\
C_{i-1} \arrow[urr, dotted] & & & & C_{i-1} \arrow[ull, dotted] \arrow[llll, dotted, bend left]
\end{tikzcd}$$

# § Ch. VI.1 Products in cohomology

**RECALL]** $(A_* \otimes B_*)_n = \bigoplus_{i+j=n} A_i \otimes B_j$ with $\partial_\otimes (\underset{\substack{\cap \\ A_p}}{a} \otimes b) = \partial a \otimes b + (-1)^{\deg(a)} a \otimes \partial b$. $\quad = \partial \otimes 1 + 1 \otimes \partial$

Consider $f : A_* \longrightarrow A'_{*+\deg f}$ in GrAb, $\deg f, \deg g \in \mathbb{Z}$

$g : B_* \longrightarrow B'_{*+\deg g}$

**THM]** $H_*(X \times Y; G) \overset{\text{def}}{=} H_*(\Delta_*(X \times Y) \otimes G)$

$\underset{\text{contract}}{\cong} H_*(\Delta_*(X) \otimes \Delta_*(Y) \otimes G)$

also $H^*(X \times Y; G) \cong H^*(\mathrm{Hom}(\Delta(X) \otimes \Delta(Y), G))$.

Eilenberg Zilber theorem

$\downarrow$

use the cross prod

Pf sketch. TODO

**CORO]** (Künneth theorem) $A_*, B_*$ free abelian chain complexes

There's a ses for all $n$, natural in $A, B$ and $G$, but not naturally splitting:

$$\boxed{0 \longrightarrow \bigoplus_{i+j=n} [H_i(A_*) \otimes H_j(B_*)] \longrightarrow H_n(A_* \otimes B_*) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}^1_{\mathbb{Z}}(H_i(A); H_j(G)) \longrightarrow 0}$$

## § Eilenberg Zilber thm

Recall $\times : \Delta_*(X) \otimes \Delta_*(Y) \longrightarrow \Delta_*(X \times Y)$ s.th. $\times$ is natural in both arguments.

acyclic model ⤳ last proof

**THM** There's a chain map $\Theta : \Delta_*(X \times Y) \longrightarrow \Delta_*(X) \otimes \Delta_*(Y)$ s.th. $\Theta \circ \times \simeq 1$, $\times \circ \Theta \simeq 1$. ($\Theta$ is functorial in $X, Y$ as spaces.)

**Lemma**: $X, Y$ contractible $\Rightarrow H_n(\Delta_*(X) \otimes \Delta_*(Y)) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{else} \end{cases}$

Pf. take $\partial_\otimes [D_\otimes] = \partial_\otimes [D_X \otimes 1 + \varepsilon \otimes D_Y]$ for a chain homotopy from $1 \otimes 1$ to $\varepsilon \otimes \varepsilon$, where $\varepsilon : F_* \to F_*$ is the augmentation map.

Pf. (Acyclic models)

**IDEA** To show $H_*(-)$ singular homology satisfies the homotopy axiom, we needed the "cross product".

(1) We assume $\sigma_x : pt \longrightarrow x \in X$. Then with $x \in Sing_0(X)$ and $\tau \in Sing_p(Y)$, $x \times \tau : \Delta_p \to X \times Y$ s.th. $(x \times \tau)(z) = (x, \tau(z)) \in X \times Y$. Moreover, for $\sigma \times y \in Sing_p(X)$ TODO

(2) Suppose $f : X \to X'$ and $g : Y \to Y'$ are morphisms in Top

$$\begin{tikzcd} \Delta_p(X) \times \Delta_q(Y) \arrow[r, "f_\Delta \times g_\Delta"] \arrow[d, "\times"] & \Delta_p(X') \times \Delta_q(Y') \arrow[d, "\times"] \\ \Delta_{p+q}(X \times Y) \arrow[r, "(f \times g)_\Delta"] & \Delta_{p+q}(X' \times Y') \end{tikzcd}$$

Pf sketch. Induct on $p+q$. Say $\Delta_i(X) \times \Delta_j(Y) \xrightarrow{\times} \Delta_{i+j}(X \times Y)$ for $i+j < p+q$ has been def'd and satisfies (1–3). Let $\iota_p : \Delta_p \to \Delta_p$ be the identity. Then $\iota_p \in \Delta_p(\Delta_p)$.

Construct $\iota_p \times \iota_q : \Delta_p(\Delta_p) \times \Delta_q(\Delta_q) \longrightarrow \Delta_{p+q}(\Delta_p \times \Delta_q)$. It's sufficient to define $\iota_p \times \iota_q$ so that by (2) with $\sigma : \Delta_p \to X$ s.th. $\sigma = \sigma_\Delta(\iota_p)$ and $\tau : \Delta_q \to Y$ s.th. $\tau = \tau_\Delta(\iota_q)$.

**TFDC**

$$\begin{tikzcd} \Delta_p(\Delta_p) \times \Delta_q(\Delta_q) \arrow[r, "\sigma_\Delta \times \tau_\Delta"] \arrow[d, "\times"] \arrow[dr, "\sigma \times \tau"] & \Delta_p(X) \times \Delta_q(Y) \arrow[d, "\times"] \\ \Delta_{p+q}(\Delta_p \times \Delta_q) \arrow[r, "(\sigma \times \tau)_\Delta"] & \Delta_{p+q}(X \times Y) \end{tikzcd}$$

$(\sigma \times \tau)_\Delta(\iota_p \times \iota_q)$

Yoneda lemma. TODO
$\Delta_p(-)$ is a representable functor, as a thm. $\Delta_p$ is a rep'ble obj. We get natural transformations.

Recall $\Delta_p$ contracts to a pt. Use this!

(Yoneda lemma in disguise)

Write the diagonal inclusion $d_k : \Delta_k \hookrightarrow \Delta_k \times \Delta_k$ where $d(x) = (x, x)$.

For $\sigma \in \Delta_k(X \times Y)$ s.th.

$$\begin{tikzcd} & & Y \\ \Delta_k \arrow[r, "\sigma"] \arrow[urr, bend left, "\pi_Y \sigma"] \arrow[drr, bend right, "\pi_X \sigma"'] & X \times Y \arrow[ur, "\pi_Y"] \arrow[dr, "\pi_X"'] & \\ & & X \end{tikzcd}$$

we see $\sigma(z) = (\pi_X \sigma(z), \pi_Y \sigma(z)) = (\pi_X \sigma, \pi_Y \sigma) \circ d_k$.

natural pt. of def$^n$

If $\Theta d_k$ is defined, then $\Theta\sigma \overset{\text{should}}{=} \Theta \circ (\pi_X\sigma, \pi_Y\sigma) \circ d_k = \boxed{((\pi_X\sigma)_\Delta \otimes (\pi_Y\sigma)_\Delta) \circ \Theta d_k}$

Assume $k=0$ holds and deg $< k$ is defined. Need $\Theta(d_k) \in \underbrace{\Delta_*(\Delta_k) \otimes \Delta_*(\Delta_k)}_{A}$

inductively defined

Lastly, should $\partial(\iota_p \times \iota_q) = 0$, then $\iota_p$ ... in $\Delta_p \times \Delta_q \simeq \{pt\}$.

$\Theta(\partial d_k)$ is defined. $\Theta(\partial d_k) = \Theta \partial^2 d_k = 0$. Thus $\Theta \partial d_k$ is a cycle in $A \Rightarrow$ a boundary.

$\boxed{\text{Note } \partial^2 \text{ should be } 0.}$ So $\partial^2(\iota_p \times \iota_q)$ ... $(\partial\iota_p \times \partial\iota_q) + (-1)^{p-1}(\partial\iota_p \times \partial\iota_q) + (-1)^p(\iota_p \times \partial^2 \iota_q)$

So, choose $\Theta d_k$ s.th. $\partial(\Theta d_k) = \Theta(\partial d_k)$

by (3) applied inductively. $\boxed{\text{We let } \partial(\iota_p \times \iota_q) = \xi}$. Since $\xi$ is a cycle ...

& use nat'l to define $k$ for any space.

the map s.th. $\partial(\iota_p \times \iota_q) = \xi$.

logical ordering here ???

**Part II** Test on $\iota_p \otimes \iota_q$ w/ $\iota_p : \Delta_p \longrightarrow \Delta_p$ and then defined $(\iota_p \times \iota_q)$ by the desired behavior for homotopy.

Any two maps $f, g : \Delta_*(X \times Y) \longrightarrow \Delta_*(X) \otimes \Delta_*(Y)$ which are canonical maps in deg 0, then they're chain homotopic.

Similarly, if $f, g : \Delta_*(X) \otimes \Delta_*(Y) \longrightarrow \Delta_*(X \times Y)$ s.th. $f, g$ are $\simeq$ in deg 0.

e.g. $\times \circ \Theta$, $id \in Aut(\Delta_*(X \times Y))$ and $\Theta \circ \times$, $id \in Aut(\Delta_*(X) \otimes \Delta_*(Y))$.

$\partial$ $\quad \partial_\otimes$

$f, g$ as above. WANT $\boxed{D \text{ s.th. } \partial_\otimes D + D\partial_\times = f - g}$ ① $\therefore$ ② should! $\boxed{\partial_\otimes D\tau + D\partial_\times \tau = f(\tau) - g(\tau)}$

Then for all $\tau \in \Delta_p(X \times Y)$, we should have ② from ①.

Say we have $D$ for all classes of deg $< k$.

...

TODO finish the def$^n$ $D$ on deg $k$. THX!

homol ... s.th. spaces ... then Eilenberg Zilber. show for relative chain complexes... pairs of spaces in Top

CORO] (Künneth theorem) $A_*$, $B_*$ free abelian chain complexes

There's a ses for all $n$, natural in $A, B$ and $G$, but not naturally splitting:

$$\boxed{0 \longrightarrow \bigoplus_{i+j=n} \left[H_i(A_*) \otimes H_j(B_*)\right] \longrightarrow H_n(A_* \otimes B_*) \longrightarrow \bigoplus_{i+j=n} \mathrm{Tor}^1_{\mathbb{Z}}\big(H_i(A); H_j(G)\big) \longrightarrow 0}$$

3.2| Given. The Klein bottle $\mathbb{K}^2$. $H_*(\mathbb{K}^2) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & * = 1 \\ 0 & \text{else} \end{cases}$

To find. Homology and cohomology of $\mathbb{K}^2$ in all dimensions, for coefficients $\mathbb{Z}$ and $\mathbb{F}_p$, for all primes.

3.3| **Given.** For $k, l$ indices ranging over $\{3, 6, 9\}$, let $X$ be the push out

$$\begin{tikzcd} D^2 \arrow[r, "f_k"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S^1 \arrow[d] \\ \partial D^2 \arrow[r, "f_{\partial k}"] & X \end{tikzcd}$$

s.th. $\deg f_{\partial k} = k$.

**To find.** Compute the homology of $X$ with coeffs $\mathbb{Z}_l$ for all $l$ ranging $\{3, 6, 9\}$.

**3.41** Given. Suppose $f, g : X \longrightarrow Y$ are maps s.th. $f_* = g_* : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$.

To find. Demonstrate there's a group $G$ s.th. $f_* \neq g_*$ as maps on homology for $H_*(X;G)$ to $H_*(Y;G)$.

Demo. (Idea to use $\mathbb{P}^2$ and $\mathbb{S}^2$ from Micky Steinberg, 2014, Wisconsin (Quals notes))

Let $\mathbb{P}^2$ be the pushout
$$\begin{tikzcd} S^1 \arrow[r, "\deg 2"] \arrow[d] & S^1 \arrow[d] \\ \mathbb{D}^2 \arrow[r] & \mathbb{P}^2 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$
, a CW complex with 1-cell in each dimension.

As 2 is even, $H_*(\mathbb{P}^2) = \{\mathbb{Z}$ if $* = 0$, $\mathbb{Z}/2$ if $* = 1$, $0$ else$\}$. Let $\mathbb{S}^2$ be $\mathbb{D}^2/\partial\mathbb{D}^2$, a CW complex with one 2-cell and one 0-cell. Recall $H_*(\mathbb{S}^2) = \{\mathbb{Z}$ if $* = 0$ or $2$, else $0\}$. Suppose $f : \mathbb{P}^2 \to S^2$ is a homeomorphism on the interior of the 2-cell in $\mathbb{P}^2$ to the interior of the 2-cell in $S^2$ with the boundary $\partial\mathbb{P}^2 = \mathbb{P}^1$ mapping to the 0-cell in $\mathbb{S}^2$.

Let $g : \mathbb{P}^2 \to S^2$ be the constant map onto the 0-cell of $S^2$. Since $g$ factors through a point, $H_*(g) : H_*(\mathbb{P}^2) \to H_*(S^2)$ is trivial for $* \neq 0$.

That $H_*(f) = H_*(g)$ can be seen as follows: both $\mathbb{P}^2$ and $S^2$ are cntd. So $H_0(f)$ and $H_0(g)$ are forced to agree on the generator $1 \in H_0(\mathbb{P}^2)$.

In each degree $n > 1$, either $H_n(\mathbb{P}^2)$ or $H_n(S^2)$ is $\{0\}$. Thus $H_n(f) = H_n(g) = 0$ for all $n > 1$. We've so far worked over $\mathbb{Z}$, let's change coeffs to $\mathbb{Z}/2$. Now $\mathrm{Tor}(H_n(S^2), \mathbb{Z}/2) = 0$ for all $n \geq 0$, so the UCT shows that $H_*(S^2) \otimes \mathbb{Z}/2 \cong H_*(S^2; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & * = 0, 2 \\ 0 & \text{else} \end{cases}$.

Now $\mathrm{Tor}(H_n(\mathbb{P}^2), \mathbb{Z}/2) = \{\mathbb{Z}/2$ if $n = 1$, else $0\}$. From the UCT, $0 \to H_n(\mathbb{P}^2) \otimes \mathbb{Z}/2 \to H_n(\mathbb{P}^2; \mathbb{Z}/2) \to \mathrm{Tor}(H_{n-1}(\mathbb{P}^2), \mathbb{Z}/2) \to 0$.

Therefore $H_*(\mathbb{P}^2; \mathbb{Z}/2) = \{\mathbb{Z}/2$ if $* = 0, 1, 2$; else $0\}$. Again, $H_*(g; \mathbb{Z}/2)$ is the trivial map for $* > 0$.

Let $\sigma$ be the 2-cell of $\mathbb{P}^2$ and $\tau$ the 2 cell of $\mathbb{S}^2$. In the cellular chain complexes we have $C_2(\sigma; \mathbb{Z}/2) = \langle \sigma \rangle$ and $C_2(\tau; \mathbb{Z}/2) = \langle \tau \rangle$. Because $f : \mathbb{P}^2 \to \mathbb{S}^2$ is cellular and homeomorphically maps $\mathring{\sigma}$ to $\mathring{\tau}$, $\deg f_{\sigma,\tau} = \pm 1$. Because $\pm 1 = 1$ in $\mathbb{Z}/2$, the induced map $\sigma \xmapsto{H_*(f;\mathbb{Z}/2)} \tau$ gives an isom $H_2(\mathbb{P}^2; \mathbb{Z}/2) \xrightarrow{H_*(f;\mathbb{Z}/2)} H_2(S^2; \mathbb{Z}/2)$. Therefore $H_*(f; \mathbb{Z}/2) \neq H_*(g; \mathbb{Z}/2)$. $\square$

# MATH 6220 HOMEWORK 6

COLTON GRAINGER
APRIL 16, 2019

**1.** (a) Chapter V, Section 7, 284: 5. *If $H_*(X)$ is finitely generated, then*

$$\chi(X) = \sum (-1)^i \dim H_i(X;\Lambda) \quad \textit{for any field } \Lambda.$$

(b) Chapter VI, Section 1, 321: 3. *For spaces $X$, $Y$ of bounded finite type,*

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

<u>Given</u>. Say $X, Y \in \mathrm{Top}$ with $H_*(X)$ and $H_*(Y)$ of bounded finite type. Then the total homology grp.

<u>WTS</u>

$$\chi(X\times Y) = \sum_{k=0}^{m+n} \mathrm{rank}(H_k(X\times Y))$$
$$= \left(\sum_{i=0}^{n} \mathrm{rank}\, H_i(X)\right)\left(\sum_{j=0}^{m} \mathrm{rank}\, H_j(X)\right) \quad ✪$$
$$= \chi(X)\chi(Y).$$

<u>Pf.</u> (✪) $\sum_{k\geq 0} \mathrm{rk}(H_k(X\times Y))$ can be computed from ...

① <u>Künneth</u>

$$H_k(X\times Y) \cong \frac{\bigoplus_{p+q=k} (H_p(X)\otimes H_q(Y))}{\bigoplus_{p+q=k-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(X), H_q(X))}.$$

finite gen'd $H_p$, $H_q$ imply rank Tor $= 0$.

Thus

$$\mathrm{rk}(H_k(X\times Y)) = \sum_{p+q=k} \mathrm{rank}\,(H_p(X)\otimes H_q(Y)) \quad ② \text{ Linear Algebra}$$

$$= \sum_{p+q=k} \mathrm{rk}\, H_p \times \mathrm{rk}\, H_q \quad ③ \text{ since } \otimes \text{ and } \oplus \text{ commute... with } H_p, H_q \text{ fin'd gen'd.}$$

$$= \sum_{k\geq 0}\left(\sum_{p+q=k} \mathrm{rk}\, H_p\, \mathrm{rk}\, H_q\right) \quad ④ \text{ Cauchy Product}$$



$$= \sum_{k\geq 0} \mathrm{rk}\, H_p\, \mathrm{rk}\, H_q. \quad \square$$

$\square$

- I just read another proof online and hated it (indices, yuck).
- But I also found (and admire) this comment. Has all the ammo I needed.
  - *Yes, that's true, by the (i) universal coefficient theorem and (ii) the fact that for any finitely generated abelian group $A$ and any prime $p$, we have*

    $$\dim_{\mathbb{Q}} A \otimes_{\mathbb{Z}} \mathbb{Q} = \operatorname{rank}_{\mathbb{Z}} A = \dim_{\mathbb{F}_p} A \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} - \dim_{\mathbb{F}_p} \operatorname{Tor}^1_{\mathbb{Z}}(A, \mathbb{Z}/p\mathbb{Z}).$$
  - `https://math.stackexchange.com/posts/996525/`

**Given.** Suppose $H_*(X)$ is finitely generated, and consider the Euler characteristic $\chi(X)$.

**To prove.** For any field $F$, $\chi(X) = \sum_i (-1)^i \dim H_i(X;F)$.

**Proof.** We define $\mathrm{Tor}(H_i(X),F)$ in the exact sequence $0 \to \mathrm{Tor}(H_i(X),F) \to R \otimes F \to P \otimes F \to H_i(X) \otimes F \to 0$, where $R \hookrightarrow P \twoheadrightarrow H_i(X)$ is a projective presentation of the ($\mathbb{Z}$ module) abelian gp $H_i(X)$. By the FTFGAB, $H_i(X) \cong \mathbb{Z}^\beta \oplus \left(\frac{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}{n_1\mathbb{Z} \oplus \cdots \oplus n_r\mathbb{Z}}\right)$ with invariant factors $n_1 | \cdots | n_r$. We may assume $R = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r \text{ copies for } r \text{ elm. divisors}}$, $P = \mathbb{Z}^\beta \oplus \underbrace{(\mathbb{Z} \oplus \cdots \oplus \mathbb{Z})}_{\text{terms to receive } n_1\mathbb{Z} \oplus \cdots \oplus n_r\mathbb{Z}}$, $\beta = \mathrm{rank}(H_i(X))$.

This presentation yields the s.e.s.:

denoted "diag" or "diag$(n_1, \dots, n_r)$", and $n_1 | \cdots | n_r$ given by FTFGAB.

$$0 \longrightarrow \mathbb{Z}^{\oplus r} \xrightarrow{\begin{bmatrix} 0 \\ \begin{bmatrix} n_1 & & \\ & \ddots & \\ & & n_r \end{bmatrix} \end{bmatrix}} \mathbb{Z}^b \oplus \mathbb{Z}^{\oplus r} \longrightarrow H_i(X) \longrightarrow 0.$$

We apply $- \otimes_{\mathbb{Z}} F$ to determine $\mathrm{Tor}_1^{\mathbb{Z}}(H_i(X), F) := \ker(R \otimes F \to P \otimes F)$. The induced s.e.s. is (by Lemma 0),

lemma 0

$$0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_i(X), F) \longrightarrow \underbrace{F^r \xrightarrow{\begin{bmatrix} 0 \\ \mathrm{diag} \end{bmatrix}} F^\beta \oplus F^r}_{\text{what's the kernel of } [\mathrm{diag}]\,?} \longrightarrow H_i(X) \otimes F \to 0.$$

**Lemma 0** | Map $\mathbb{Z} \to F$, so that $\mathbb{Z}^\ell \otimes_{\mathbb{Z}} F \cong (\mathbb{Z} \otimes_{\mathbb{Z}} F)^{\oplus \ell} \cong F^\ell$.

**CASE I** Suppose char $F = 0$. Then diag$(n_1, \dots, n_r)$ is a general linear transf. $F^r \to F^r$. Thus $F^r \xrightarrow{\begin{bmatrix} 0 \\ \mathrm{diag} \end{bmatrix}} F^\beta \oplus F^r$ is a (right) injection. So $0 = \ker\begin{bmatrix} 0 \\ \mathrm{diag} \end{bmatrix} = \mathrm{Tor}(H_i, F)$.

Applying the UCT in this case, for each $i \geq 0$, $\mathrm{Tor}(H_i, F) = 0$ implies $H_i(X;F) \cong H_i(X) \otimes_{\mathbb{Z}} F$.

Applying lemma 1, $\mathrm{rank}(H_i(X)) = \dim_F(H_i(X) \otimes_{\mathbb{Z}} F) = \dim(H_i(X;F))$, and so $\chi(X) = \sum_{i \geq 0} (-1)^i \dim(H_i(X;F))$.

**Lemma 1** | Let $F$ be a field of char $0$. If $A$ is an abelian group, then $\mathrm{rank}(A) = \dim_F(A \otimes_{\mathbb{Z}} F)$.

**CASE I** Suppose char $F = p$. Because the inv. factors of $H_i$ satisfy $n_1 | \cdots | n_r$ and b/c $p$ is prime, if $p | n_{k+1}$, then $p | n_{k+1} | \cdots | n_r$. Hence diag$(n_1, \dots, n_r) = $ diag$(n_1, \dots, n_k, 0, \dots, 0)$ (mod $p$). I claim $\mathrm{Tor}(H_i, F) \cong F^{r-k}$. To see this:

$$\mathrm{Tor}(H_i, F) := \ker\left(F^r \xrightarrow{\begin{bmatrix} 0 \\ \mathrm{diag} \end{bmatrix}} F^\beta \oplus F^r\right) = \underbrace{0 \oplus \cdots \oplus 0}_{\beta \text{ terms}} \oplus \overbrace{0^k \oplus \underbrace{F_{k+1} \oplus \cdots \oplus F_r}_{r-k \text{ terms}}}^{r \text{ terms}} \cong F^{r-k}.$$

Now let $i$ index homology, let $n_{1_i} | \cdots | n_{r_i}$ be the inv. factors of each $H_i(X)$, and let $k_i$ be the least index s.th. $p | n_{k_i+1} | \cdots | n_{r_i}$. For $\forall i \geq 0$, consider a proj. res. $R_i \hookrightarrow P_i \twoheadrightarrow H_i$ of the form above studied. ($H_{-1} = \{0\}$ so $R_{-1} = \{0\}$ and $r_{-1} = k_{-1} = 0$.)

Applying the UCT, for each $i \geq 0$, we have the s.e.s.

$$\begin{array}{ccccccccc} 0 & \to & H_i(X) \otimes F & \to & H_i(X;F) & \to & \mathrm{Tor}(H_{i-1}, F) & \to & 0. \\ & & \| & & \| & & \downarrow \cong & & \\ 0 & \to & H_i \otimes F & \to & H_i(X;F) & \to & F^{r_i - k_i} & \to & 0. \end{array}$$

With rank-nullity, $\dim(H_i \otimes F) = \dim(H_i(X;F)) - (r_{i-1} - k_{i-1})$. Also, from lem. 2, $\dim_{\mathbb{F}_p}(H_i \otimes \mathbb{F}_p) = \beta_i + r_i - k_i$. Whence

$$\dim(H_i(X;\mathbb{F}_p)) = \beta_i + (r_i + r_{i-1}) - (k_i + k_{i-1}).$$

**Lemma 2.** $\dim_{\mathbb{F}_p}(H_i \otimes \mathbb{F}_p) = \operatorname{rank}_{\mathbb{Z}}(H_i) + \dim_{\mathbb{F}_p}(\operatorname{Tor}^1_{\mathbb{Z}}(H_i, \mathbb{F}_p))$ (for any finitely gen'd $H_i \in Ab$)

**Pf.** $0 \to \operatorname{Tor}_{\mathbb{Z}}(H_i, \mathbb{F}_p) \to R_i \otimes_{\mathbb{Z}} \mathbb{F}_p \to P_i \otimes_{\mathbb{Z}} \mathbb{F}_p \to H_i \otimes_{\mathbb{Z}} \mathbb{F}_p \to 0$ is a ses. in $\mathrm{Vec}_{\mathbb{F}_p}$.

Thus $\overset{(\star)}{} H_i \otimes_{\mathbb{Z}} \mathbb{F}_p \cong (P_i \otimes_{\mathbb{Z}} \mathbb{F}_p) \Big/ \left(\frac{R_i \otimes_{\mathbb{Z}} \mathbb{F}_p}{\operatorname{Tor}_{\mathbb{Z}}(H_i, \mathbb{F}_p)}\right)$ are isom in $\mathrm{Vec}_{\mathbb{F}_p}$. Since $R_i$ and $P_i$ are free $\mathbb{Z}$-modules, their dimension doesn't change when tensored by $\mathbb{F}_p$. (This is lemma 0.) With $r_i, k_i, \beta_i$ as in the main proof, the isom $(\star)$ implies $\dim_{\mathbb{F}_p}(H \otimes \mathbb{F}_p) = \beta_i + r_i - r_i + r_i - k_i. = \operatorname{rank}_{\mathbb{Z}} H_i + \dim_{\mathbb{F}_p}(\operatorname{Tor}(H_i, \mathbb{F}_p))$. $\square$

To compute $\sum_{i \geq 0} \dim(H_i(X; \mathbb{F}_p))$, note $\sum_{i \geq 0} (-1)^i (r_i + r_{i-1}) = r_{-1} = 0$ and $\sum_{i \geq 0} (-1)^i (k_i + k_{i-1}) = k_{-1} = 0$.

Therefore, $\sum (-1)^i \dim_{\mathbb{F}}(H_i(X; \mathbb{F})) = \sum_{i \geq 0} [\beta_i + (r_i + r_{i-1}) - (k_i + k_{i-1})] = \sum (-1)^i \beta_i = \sum (-1)^i \operatorname{rank}(H_i) = \chi(X)$. QED.

*Note* (See Chapter VI, Section 4 and Example 4.12.). By a *graded commutative ring*, we will mean a graded abelian group $R^*$ together with a homomorphism of graded abelian groups $\mu\colon R^* \otimes R^* \to R^*$ such that,

- There exists $1 \in R^0$ which is a two sided unit for $\mu$, and
- $\mu(a \otimes b) = (-1)^{\deg(a)\deg(b)}\mu(b \otimes a)$.

The cup product gives a graded commutative ring structure on the cohomology of a space. ◁

**2.** Chapter VI, Section 1, 321: 2. *Let $X_p$ be the space resulting from attaching an $n$-cell to $S^{n-1}$ by a map of degree $p$. Use the Künneth Theorem to compute the homology of $X_p \times X_q$ for any $p, q$.*

Recall that $\varepsilon \times 1$ is the pull back of the augmentation,

hence with $\varepsilon \in H^0(S^n)$

$$0 \longrightarrow \bigoplus_{i+j=k} H_i(S^n) \otimes H_j(S^m) \longrightarrow H_k(S^n \times S^m) \longrightarrow 0 \quad \text{as}$$

so but!

$$0 \to \mathrm{Ext}(\cancel{H_k(S^n \times S^m)}^{\,0}, \mathbb{Z}) \to H^k(S^n \times S^m) \overset{\therefore \cong}{\longrightarrow} \mathrm{Hom}(H_k(S^n \times S^m), \mathbb{Z}) \to 0$$

We can dualize with Ext?

**EX 2|** Choose $\alpha, \beta \in H^p(S^p), H^q(S^q)$ sth $\alpha(a) = \beta(b) = 1$ for $a, b$ generators in $H_p(S^p), H_q(S^q)$.

$X \xrightarrow{f} Y$ induces "natural"

$$f^*(a \smile b) = f^*(a) \smile f^*(b).$$

Warning! The Künneth theorem occasionally fails

**Given.** $X_p$ is the pushout

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\varphi_{\partial\sigma}} & S^{n-1} \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{\varphi_\sigma} & X_p \end{array}$$

$\deg \varphi_\partial = p \iff \tilde{H}_*(\varphi_\partial) : \tilde{H}_*(S^{n-1}) \xrightarrow{\times p} \tilde{H}_*(S^{n-1})$.

$$\Delta_*^{cw}(X_p) = \cdots 0 \to \underbrace{\mathbb{Z}\{\sigma\}}_{\substack{n\text{th degree} \\ \sigma = D^n}} \xrightarrow{[p]} \underbrace{\mathbb{Z}\{\tau\}}_{\substack{n-1\text{ th} \\ \tau = D^{n-1}}} \xrightarrow{[0]} \cdots \xrightarrow{[0]} \underbrace{\mathbb{Z}\{e_0\}}_{0^{th}\text{ degree}} \to 0 \to \cdots$$

$$H_*^{cw}(X_p) = \{0, \text{else};\ \mathbb{Z}/p\mathbb{Z}, * = n-1;\ \cdots,\ \mathbb{Z}, * = 0\}$$

So $H_*(X_p) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & * = n-1 \\ \mathbb{Z} & * = 0 \\ 0 & \text{else} \end{cases}$

We want $H_*(X_p \times X_q)$ for any $p, q$.

$$0 \longrightarrow \bigoplus_{i+j=k} H_i(X_p) \otimes H_j(X_q) \longrightarrow H_k(X_p \times X_q) \longrightarrow \bigoplus_{i+j=k-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_i(X_p), H_j(X_q)) \to 0$$

$n \geq 3$

$k = 0$ $\quad H_0(X_p \times X_q) \cong \mathbb{Z}$

$k = 1$ $\quad 0$

$k = n-1$

$k = 2$ $\quad \mathbb{Z}/p \oplus \mathbb{Z}/q$

$k = n$ $\quad 0$

$\vdots$

$k = 2n-2$ $\quad \mathbb{Z}/p \otimes \mathbb{Z}/q \cong \mathbb{Z}/(p,q)\mathbb{Z}$

$k = 2n-1$ $\quad \mathbb{Z}/p \oplus \mathbb{Z}/q$

$k \geq 2n$ $\quad 0$

$n = 2$

$k = 0$ $\quad \mathbb{Z}$

$k = 1$ $\quad \mathbb{Z}/p \oplus \mathbb{Z}/q$

$k = 2$ $\quad \mathbb{Z}/(p,q)$

$k = 3$ $\quad \mathbb{Z}/(p,q)$

$k \geq 3$ $\quad 0$

$n = 1$

nope!

(Sanity check: $X_p \times X_q$ as a CW complex)

$n \geq 3$

$$\left.\begin{array}{ll} a_0 \times b_0 & 0 \\ \vdots & \\ a_{n-1} \times b_0,\ a_0 \times b_{n-1} & n-1 \\ a_n \times b_0,\ a_0 \times b_n & n \end{array}\right\}$$
$$\left.\begin{array}{ll} \vdots & \\ a_{n-1} \times b_{n-1} & 2n-2 \\ a_{n-1} \times b_n,\ a_n \times b_{n-1} & 2n-1 \\ a_n \times b_n & 2n \\ \vdots & \end{array}\right\}$$

$$\begin{array}{ccccccccccc} & 2n & & 2n-1 & & 2n-2 & & n & & n-1 & & 0 \\ 0 \to & \mathbb{Z} & \xrightarrow{\begin{bmatrix} p \\ (-1)^n q \end{bmatrix}} & \mathbb{Z}^{\oplus 2} & \xrightarrow{\begin{bmatrix} (-1)^{n-1} q & p \end{bmatrix}} & \mathbb{Z} & \to 0 \cdots 0 \to & \mathbb{Z}^{\oplus 2} & \xrightarrow{\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}} & \mathbb{Z}^{\oplus 2} & \to 0 \cdots 0 \to & \mathbb{Z} \xrightarrow{} 0 \end{array}$$

$$\begin{array}{cccccc} 2n & 2n-1 & 2n-2 & n & n-1 & 0 \\ 0 & \mathbb{Z}/\gcd(p,q) & (p,q) & 0 & \mathbb{Z}/p \oplus \mathbb{Z}/q & \mathbb{Z} \end{array}$$

$6\mathbb{Z}/12\mathbb{Z}$

$(12/2)\mathbb{Z}/12\mathbb{Z}$

$\mathbb{Z}/\gcd(p,q)$

$\rightsquigarrow \ker \begin{bmatrix} (-1)^{n-1} q & p \end{bmatrix}$
$= \mathrm{span} \begin{bmatrix} (-1)^n p/\gcd(p,q) \\ q/\gcd(p,q) \end{bmatrix}$
$= \mathrm{span} \begin{bmatrix} p/\gcd(p,q) \\ (-1)^n q/\gcd(p,q) \end{bmatrix}$

$= \begin{bmatrix} (-1)^n q/\mathrm{lcm}(q,p) \\ p/\mathrm{lcm}(q,p) \end{bmatrix} = \gcd(p,q) \begin{bmatrix} (-1)^n/p \\ 1/ \end{bmatrix}$

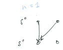
$pq/\gcd(p,q) = \mathrm{lcm}(p,q)$

$\gcd(p,q) = pq/\mathrm{lcm}(p,q)$

$\frac{\gcd(p,q)}{p} = \frac{q}{\mathrm{lcm}(p,q)}$

$\mathrm{lcm}(p,q)/q$
$= p/\gcd(p,q)$

$n = 1$



$\deg 0$

$f : S^0 \to S^0$ $\qquad \deg 1$

$g : S^0 \to S^0$

$$0 \to P_1 \to P_0 \to A \to 0$$

$$0 \longrightarrow \mathrm{Tor}_{\mathbb{Z}}^1(A, G) \to P_1 \otimes_{\mathbb{Z}} G \to P_0 \otimes_{\mathbb{Z}} G \to A \otimes_{\mathbb{Z}} G \to 0$$

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$$

$$\mathbb{Z} \otimes \mathbb{Z}/q \to \mathbb{Z} \otimes \mathbb{Z}/q \to \mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Z}/q \to 0$$

**3.** (a) *Write down the ring structure of* $H^*(S^n)$ *and of* $H^*(S^n \times S^m)$.

(b) *We will see later that* $H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x]/\langle x^{n+1} \rangle$ *for* $x \in H^1(\mathbb{R}P^n; \mathbb{Z}/2)$. *Use this to prove that* $\mathbb{R}P^3$ *is not homotopy equivalent to* $\mathbb{R}P^2 \vee S^3$.

(c) Chapter VI, 334: 3. *Show that any map* $S^4 \to S^2 \times S^2$ *must induce the zero homomorphism on* $H_4(-)$.

a) Say $a \in H_p(S^p)$ with dual generator $\alpha \in H^p(S^p)$ s.th. $\alpha(a) = 1$.

let $\varepsilon \in H^0(S^p)$ be

**4.** Chapter VI, 334: 5. *Any two chain maps $\alpha, \beta \colon \Delta_*(X) \to \Delta_*(X \otimes X)$ that agree with the diagonal approximation*

$$\Delta(x) = x \otimes x \quad \textit{in the } 0\textit{th degree}$$

*are chain homotopic: $\alpha \simeq \beta$.*

*Proof.* Let $X$ be a topological space. We will manipulate the functor:

$$\begin{tikzcd} \mathsf{Top} \arrow[r, "\mathrm{Maps}(\Delta^n_{\mathsf{Top}}, -)"] \arrow[rrr, bend right, "\Delta_*(-)"'] & \mathsf{sSet} \arrow[r, "\mathbb{Z}\langle - \rangle"] & \mathsf{sAb} \arrow[r, "C(-)"] & \mathsf{Ch}_*^+ \end{tikzcd} \tag{4.1}$$

that eventually[1] sends $X$ to its singular chain complex $\Delta_*(X)$.

So, let $\alpha, \beta \colon \Delta_*(X) \to \Delta_*(X) \otimes \Delta_*(X)$ be chain homomorphisms such that $\alpha = \beta$ on $\Delta_0(X)$:

$$\alpha(x) = \beta(x) = x \otimes x \quad \text{for } x \colon \Delta_0 \to X.$$

To see that $\alpha \simeq \beta$ are chain homotopic, we need to construct a sequence of homomorphisms

$$\left\{ h_n \colon \Delta_n(X) \to (\Delta_*(X) \otimes \Delta_*(X))_{n+1} \right\}_{n \in \mathbb{Z}_{\geq 0}}$$

such that, for each $n \in \mathbb{Z}_{\geq 0}$ and each singular simplex $\sigma \in \Delta_n(X)$, the homomorphisms $h_n$ and $h_{n-1}$ satisfy the homotopy condition

$$(\alpha_n - \beta_n)(\sigma) = (\delta_{n+1} h_n - h_{n-1} \partial_n)(\sigma), \tag{4.2}$$

which is an equality of chains in $(\Delta_*(X) \otimes \Delta_*(X))_n$.

For the BASE CASE, let $X$ be *any topological space* and $\alpha$, $\beta$ *any diagonal approximations.* Define

$$h_0 \colon \Delta_0(X) \to \Delta_0(X) \otimes \Delta_0(X)$$

by $h_0(x) = 0$. Then (trivially)

$$0 = \alpha_0 - \beta_0 = \delta_1 h_0 - h_{-1} \partial_0$$

as $\alpha$ and $\beta$ are diagonal approximations that agree on 0-simplices.

For the INDUCTIVE STEP, let $X$, $\alpha$, and $\beta$ be as above, and say the first $p < n$ homomorphisms

$$\left\{ h_p \colon \Delta_p(X) \to (\Delta_*(X) \otimes \Delta_*(X))_{p+1} \right\}_{p<n}$$

and satisfy the homotopy condition (4.2).

We will complete the inductive step finding a chain homotopy $\eta$ on an acyclic model, $\Delta_n$. Consider the identity map $\iota_n \colon \Delta_n \to \Delta_n$ from the topological $n$-simplex $\Delta_n$ to itself. Note $\iota_n$ is both a singular $n$-simplex

$$\iota_n \in \mathrm{Sing}_n := \mathrm{Maps}(\Delta^n_{\mathsf{Top}}, \Delta_n),$$

and a continuous map between topological spaces

$$\Delta_n \xrightarrow{\iota_n} \Delta_n.$$

In particular,

$$\delta \iota_n \in \Delta_{n-1}(\Delta_n)$$

is a $n-1$-chain. So say that $A$ and $B$ are diagonal approximations for the singular chain complex $\Delta_*(\Delta)_n$. By the inductive hypothesis, the homomorphisms $\eta_p$ for $p < n$ are defined. Therefore

$$\begin{aligned} &(A_{n-1} - B_{n-1})(\partial \iota_n) = (\delta_n \eta_{n-1} - \eta_{n-2} \partial_{n-1})(\partial \iota_r) \\ \iff\ &(A_{n-1} - B_{n-1} - \eta_{n-2} \partial_{n-1})(\partial \iota_n) = (\delta_n \eta_{n-1})(\partial \iota_r) \\ \iff\ &(A_{n-1} - B_{n-1})(\partial \iota_n) = (\delta_n \eta_{n-1})(\partial \iota_r) \end{aligned}$$

is an equality of chains in $(\Delta_*(\Delta)_r \otimes \Delta_*(\Delta)_n)_{n-1}$.

[1]Here $\mathrm{Maps}(\Delta^n_{\mathsf{Top}}, -)$ takes the space $X$ to the simplicial set $\mathrm{Sing}X$, which $\mathbb{Z}\langle - \rangle$ takes to the simplicial (free) abelian group $\mathbb{Z}\langle \mathrm{Sing}X \rangle$, which then $C(-)$ takes to the alternating face map complex $\Delta_*(X)$.

# OFFICE HOURS 2019-04-09

COLTON GRAINGER

## 1. Prerequisites

- General Topology
- Homological Algebra
- Homotopy Theory
- Stable Homotopy Theory
- Cobordism and Complex Oriented Cohomology

*Date*: 2019-04-09.
*url:* `https://mathoverflow.net/questions/27282/is-bredons-topology-a-sufficient-prelude-to-bredons-sheaf-theory`.

I intend to try working through Bredon's seminal sheaf theory text prior to graduating (I am currently a second year undergraduate), but it is at a level which is far beyond my own (friends of mine who study algebraic topology have gone until their second and third years as graduate students before touching it). However, I am interested in algebraic geometry (though the material treated in Bredon's text is certainly of relative interest to me) and find Bredon's "Topology and Geometry" to be a superb treatment of the algebro-topological tools which may have some utility in my future studies (Bredon takes a more geometric approach). . . .

asked Jun 6 '10 at 22:52

> To be frank, my reaction to this is "what's the rush?" I am perhaps blinded by my own personal limitations, but I believe that the undergraduate years are best spent doing as much mathematics as possible using only your "bare hands" and learning as many concrete examples as possible, whether in topology or algebraic geometry. All the fancy abstract machinery is much more meaningful, if you have first tried to do things without it. Also, I have seen too many precocious students try to answer relatively simple questions using too much machinery. – Deane Yang Jun 6 '10 at 23:43

> Deane, I was going to write exactly the same thing. Learning something as dry as sheaf theory before encountering a real need for it (in alg. geom., several complex variables, etc.) is a very unwise idea (like learning homological algebra in the absence of applications). Retention will be negligible. Sheaf theory is a powerful body of techniques for solving certain kinds of problems, but this stuff is best understood only in the service of an application (e.g., cup products, deRham comparison isom, etc.). Anyway, Godement's sheaf theory book (in French) is better than Bredon's. :) – BCnrd Jun 7 '10 at 1:15

> Alright, well I am not unsure as to my capacity to at least learn some geometry/topology; is the Bredon text mentioned above a good place to start learning these topics in a 'more advanced' light? I have the basic results of point-set topology and analytic geometry in my ken, so to speak. – lambdafunctor Jun 7 '10 at 1:56

> So learn more geometry and topology! There are plenty of books that aim to teach you complex differential or algebraic geometry, where just enough sheaf theory is introduced as needed. If you have not already learned everything in the context of Riemann surfaces, that's a really nice easy place to start. Also, if you have not read Bott-Tu (which really is a graduate text), I suggest studying that before Bredon.

# MATH 6220 NOTES: TOPOLOGY 2

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These notes were taken in University of Colorado's MATH 6220 (Topology 2) class in Spring 2019, taught by Dr. Agnés Beaudry. I live-TEXed them with `vim`, so there may be typos and failures of understanding. Any mistakes are my own. Please send questions, comments, complaints, and corrections to `colton.grainger@colorado.edu`. Thanks to `adebray` for the LATEX template, which I have forked from `https://github.com/adebray/latex_style_files`.

## Contents



Alas, I only have the last 3 weeks of lectures.

Lecture 1.

## 2019-04-12

*"We'll have a reading course in the Fall to introduce stable/chromatic homotopy theory."*

Today, $\Lambda$ is a commutative unital ring. We'll aim to define the cross product and cup product on cohomology.

**Definition 1.1** (Internal hom of chain complexes, Schreiber)**.** Let $A$ and $B$ be chain complexes. Define a chain complex $[A, B]$ to have components

$$[A, B]_n := \prod_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{Mod}}(A_i, B_{i+n})$$

(the collection of *homological degree-$n$* maps between the underlying graded modules) and whose differential is defined on homogeneously graded elements $f \in [A, B]_n$ by

$$\delta f = \partial_B f - (-1)^n f \partial_A.$$

This complex is variously denoted

$$[A, B]_n = \mathrm{Hom}(A_\bullet, B_\bullet)_n = \mathrm{hom}(A, B) = \{f_i \colon A_i \to B_{i+n}\}_{i \in \mathbb{Z}}.$$

(Dually, maps of *cohomological degree-$n$* belong to the chain complex $[A, B]_{-n} = [A, B]^n$. Note the differential $\delta f$ is well defined regardless of whether we're working with homological or cohomological degree.)

> *"One algebraic way to motivate this is to observe that the signs in the differential for the Hom are precisely what is needed for 0-cycles in the* $\mathrm{hom}(A, B)$ *complex to be the set of morphisms of complexes* $A \to B$ *(and also, that the 0th homology group* $H_0(\mathrm{hom}(A, B))$ *is the set of homotopy classes of morphisms* $A \to B$*). This is quite great. Once you decide you want this, all the other signs you mention follow because you need various things to hold. For example, you want the adjunction between* hom *and* $\otimes$ *to hold for the internal versions, so this forces you to add signs to the* $\otimes$*, and so on."*
> –*Mariano Suárez-Álvarez,* `https://math.stackexchange.com/questions/40468/`

**Exercise 1.2** (Internal hom is a functor, Schreiber)**.**

(1) Verify $[-,-] \colon \mathsf{Ch} \times \mathsf{Ch} \to \mathsf{Ch}$ is the internal hom functor on $\mathsf{Ch}$.
(2) Verify a map $f \in [A, B]$ of homological degree 0 is a cycle if and only if $f$ is a chain map.
(3) Verify a map $h \in [A, B]$ of homological degree 1 such that $\delta h = f - g$ is a chain homotopy from $g$ to $f$.
(4) Verify $H_0([A, B])$ is the group of homotopy-equivalence classes of chain maps in $[A, B]_0$.

> *"If one thinks of chain complexes as algebraic analogues of topological spaces and internal hom as an algebraic analogue of the internal hom in a nice category of topological spaces, then 0-cycles are analogues of points and 1-cycles are analogues of homotopies between points, so a 0-cycle in the internal hom is a continuous map and a 0-cycle up to boundaries is a homotopy class of continuous maps." –Qiaochu Yuan,* math.stackexchange.com/questions/40468/

**Corollary 1.3.** *Say $G$ is a chain complex concentrated at degree $0$. Then the cohomological degree of*

$$f\colon A_p \to G_0$$

*is $p$. Since $\partial_G = 0$ (because it maps out of the concentration in degree $0$), we see that the chain complex $[A,G]$ has differential $\delta f = (-1)^{\deg f+1} f\partial_A$.*

*Note* (Concentrated complexes and flat resolutions)*.* For any PID $R$ and finitely generated $M \in \mathsf{RMod}$, the finite flat resolution $F_\bullet(M)$ of $M$ is quasi isomorphic to the complex $M$ concentrated in degree 0:

$$\begin{array}{ccccccc} \cdots \to & 0 & \to & F_1 & \to & F_0 \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots \to & 0 & \to & 0 & \to & M \end{array} \quad \text{is quasi-isomorphic if and only if} \quad 0 \to F_0 \to F_1 \to M \to 0 \quad \text{is exact.}$$

◀

To handle products (especially products of CW-complexes) in singular homology, we proved there exists a natural bilinear map $\times$ (the cross product) from the product of two singular chain complexes to the chain complex of the product of two spaces. With $\times$, we computed the boundary map of the chain complex $\Delta_*(X \times Y)$ using $\Delta_*(X)\times\Delta_*(Y)$ and the *degree of incidence* between $n$ and $n-1$ cells. Then, taking the interval $I = Y$ as the second space gave proof that (CW homology, and thus) singular homology satisfied the homotopy axiom.

Now we seek to define the cross product $\times$ on cochain complexes. (TODO Motivate.) Consider the cochains $f\colon \Delta_p(X) \to \Lambda$ and $g\colon \Delta_q(X) \to \Lambda$. There's a map induced by the ring structure

$$f \otimes g\colon \Delta_p(X) \otimes \Delta_q(Y) \to \Lambda \otimes \Lambda.$$

The tensor product is over $\Lambda$, so multiplication $\Lambda \otimes \Lambda \xrightarrow{m} \Lambda$ gives an isomorphism. Composing the previous two maps, for any $p$- and $q$-simplices $\sigma$ and $\tau$,

$$\sigma \otimes \tau \longmapsto f(\sigma) \otimes g(\tau) \longmapsto f(\sigma)g(\tau).$$

Recall also, from the Eilenberg-Zilber theorem, there's a natural chain equivalence $\theta\colon \Delta_*(X\times Y) \to \Delta_*(X)\otimes\Delta_*(Y)$. (The content of this theorem was in constructing a chain homotopies $\theta \circ \times \simeq \mathrm{id}$).

**Definition 1.4** (Cross product on cochain complexes)**.** Define the cross product

$$\times\colon \Delta^p(X;\Lambda) \otimes \Delta^q(Y;\Lambda) \to \Delta^{p+q}(X \times Y;\Lambda),$$

by the rule

$$f \times g = f \otimes g \circ \theta.$$

*Remark* 1.5. Yu asked how

$$\theta\colon \Delta_p(X \times Y;\Lambda) \to (\Delta_*(X) \otimes \Delta_*(Y))_p$$

could be well defined. TODO Write $f\colon \Delta_p(X) \to \Lambda$, then extend to all of $\Delta_*(X)$ (by varying $p$?). ◀

**Lemma 1.6.** *$\delta(f \times g) = \delta f \times g + (-1)^{\deg f} f \times \delta g$.*

*Proof.* Acyclic models. □

**Proposition 1.7.** *There's a natural (linear map out of the) product of homology groups*

$$\times\colon H^p(X;\Lambda) \otimes H^q(Y;\Lambda) \to H^{p+q}(X \times Y;\Lambda)$$

*such that*

$$[f] \otimes [g] \longmapsto [f] \times [g],$$

*which is induced by*

$$\Delta_*(X \times Y) \xrightarrow{\theta} \Delta_*(X) \otimes \Delta_*(Y) \xrightarrow{f\otimes g} \Lambda\,.$$

*Proof.* Let the unit $1 \in H^0(X;\mathbb{Z})$ be the class of the augmentation $\varepsilon\colon \Delta_0(X) \to \mathbb{Z}$. We appeal to the universal property of the product space $X \times Y$. The component projections induce chain maps on the cochain complexes.

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_X"] \arrow[dr, "\pi_Y"'] & X \\ & Y \end{tikzcd} \quad \overset{\Delta^*(-)}{\rightsquigarrow} \quad \begin{tikzcd} \Delta^*(X \times Y) & \Delta^*(X) \arrow[l, "\pi_X^*"'] \\ & \Delta^*(Y) \arrow[ul, "\pi_Y^*"] \end{tikzcd}$$

I claim that the maps $\pi_X^*$ and $\pi_Y^*$ descend to cohomology, i.e., that there're well-defined homomorphisms $H^*(X) \to H^*(X \times Y)$ sending $\alpha$ under $\pi_X^*$ to $\alpha \times 1$ and symmetrically $H^*(Y) \to H_*(X \times Y)$ sending $\beta \mapsto 1 \times \beta$.

To see this, consider the "slice map"

$$\begin{tikzcd} \Delta_p(X) \otimes \Delta_0(Y) \arrow[r, "\times"] & \Delta_p(X \times Y) \arrow[r, "\pi_{X\Delta}"] & \Delta_p(X) \\ \tau \otimes y \arrow[r, mapsto] & (\tau \times y)(x) = (\tau(z), y) \arrow[r, mapsto] & \tau \end{tikzcd}$$

Extend the slice map linearly to a map on chains, then obtain an induced map on cochains. $\square$

**Example 1.8** (The twist map)**.** How can we compare $\alpha \times \beta \in H^*(X \times Y;\Lambda)$ to $\beta \times \alpha \in H^*(Y \times X;\Lambda)$?

Consider the twist map $T\colon X \times Y \to Y \times X$ taking $(\alpha,\beta) \mapsto (\beta,\alpha)$, which induces a map on cohomology.

$$H^*(X \times Y;\Lambda) \xleftarrow[T^*]{} H^*(Y \times X;\Lambda)\ .$$

$\blacktriangleleft$

If $X = Y$, we apply proposition 1.7 and pull back the diagonal map $d\colon X \to X \times X$ (where $x \mapsto (x,x)$) to obtain the cup product.

**Definition 1.9** (Cup product)**.** The *cup product* is the bilinear map

$$\begin{tikzcd} H^*(X;\Lambda) \otimes H^*(X;\Lambda) \arrow[r, "\times"] & H^*(X \times X;\Lambda) \arrow[r, "d^*"] & H^*(X;\Lambda)\ , \\ \alpha \otimes \beta \arrow[rr, mapsto] & & \alpha \smile \beta \end{tikzcd}$$

given by the pullback of the diagonal map

$$d^*\colon H^*(X \times X) \to H^*(X)$$

precomposed with the cross product $\times$. (To compute the cup product on cochains $f^p$ and $g^q$ in $\Delta^p(X)$ and $\Delta^q(X)$, see definition 2.7.)

**Exercise 1.10.** From example 1.8, prove that the cup product is a graded-commutative operation.

Lecture 2.

## 2019-04-15

*"Does the torus $S^1 \times S^1$ have a cohomology ring*[1] *that's not a polynomial ring? Does any space?"*

Recall that last week, we proved the Eilenberg-Zilber chain map $\theta$ was a natural homotopy inverse of the cross product $\times$. And last lecture, knowing $\theta$ and $\times$ are (chain-)homotopy equivalences, we exploited the chain equivalence $\theta\colon \Delta_*(X \times Y) \to \Delta_*(X) \otimes \Delta_*(Y)$ to define the cross product and the cup product on cohomology groups.

*Note.* It's not at all apparent, however, what map $\times$ induces on cohomology:

$$\times\colon H^*(X;\Lambda) \otimes H^*(Y;\Lambda) \overset{?}{\to} H^*(X \times Y;\Lambda). \tag{2.1}$$

In general, there's no Künneth theorem for the cohomology cross product. But, if the ring of coefficients $\Lambda$ is a field and either $H_*(X;\Lambda)$ or $H_*(Y;\Lambda)$ is of finite type, then the map in (2.1) is an isomorphism. (E.g., if $\Lambda = \mathbb{R}$ and $X$ is a compact $n$-manifold.) $\blacktriangleleft$

This lecture we'll give details for making computations with the cup product. Let $X$ and $Y$ be topological spaces and $\Lambda$ a commutative unital ring.

[1] Hint. The cohomology ring for $S^1$ is to $\mathbb{RP}^2$ as – is to – (?) with spheres adjoined (!?). Consider the homogeneous elements of the ring. Just describe the ring. For example, when algebraic topologists write $\mathbb{Z}[x]/\langle x^4 \rangle$, it is understood to be *a homogeneous ring*. Addition is *strictly levelwise*. Products are defined between two homogeneous elements of non-homogeneous degrees.

**Exercise 2.2.** Say $f\colon A_* \to \Lambda$ and $g\colon B_* \to \Lambda$ are cochains in the complexes $A^*(\Lambda)$ and $B^*(\Lambda)$. Check that the ring structure of $\Lambda \otimes \Lambda = \Lambda$ forces the definition

$$f \otimes g\colon A_* \otimes B_* \to \Lambda$$
$$(f \otimes g)(a \otimes b) = (-1)^{\deg a \deg g} f(a)g(b),$$

for chains $a \in A_*$ and $b \in B_*$.

**Corollary 2.3.** *For the singular chain complexes over a topological space $X$, let $f, g$ be cochains in $\Delta^*(X)$ and $\alpha, \beta$ chains in $\Delta_*(X)$. Then, $(f \otimes g)(\alpha \times \beta) = (-1)^{\deg \alpha \deg g} f(\alpha)g(\beta)$.*

Now, we certainly have an *evaluation* ev from the group of $p$-cochains $f\colon \Delta_p(X) \to \Lambda$ tensored with the group $\Delta_p(X)$ of $p$-chains,

$$\Delta^p(X;\Lambda) \otimes \Delta_p(X) \xrightarrow{\text{ev}} \Lambda,$$

defined by

$$\text{ev}\colon f \otimes c \longmapsto f(c).$$

In fact, ev induces a map on cohomology, which is denoted by the angle brackets

$$\langle [f], [c] \rangle \in \Lambda.$$

**Lemma 2.4** (Krönecker pairing)**.** *The evaluation $\Delta^p(X;\Lambda) \otimes \Delta_p(X) \xrightarrow{\text{ev}} \Lambda$ induces a $\Lambda$-linear map*

$$H^p(X;\Lambda) \otimes H_p(X) \to \Lambda \tag{2.5}$$

*such that $\langle |f|, |c| \rangle \mapsto f(c)$.*

*Proof.* (TODO: revise) The *Kronecker pairing* is the argument that $f(c)$ does not depend on representatives $f$ or $c$ (from the cochain, resp, chain complexes). Consider that in the proof of the universal coefficient theorem, we found a map $\beta$ from $H^*(X;\Lambda)$ to $\text{Hom}(H_p(X), \Lambda)$ such that $|f| \mapsto \{|c| \mapsto f(c)\}$ gave a group homomorphism. Use this. ☐

*Remark* 2.6. The *cap product* operation over a topological space $X$ is the above (2.5) pairing, which is "given by combining the *Kronecker pairing* of the cohomology class with the image of the homology class under diagonal and using the Eilenberg-Zilber theorem." (See https://ncatlab.org/nlab/show/cap+product.) ⋞

**Definition 2.7** (The cup product on cochains)**.** Let $X \in \mathsf{Top}$. The diagonal map $d\colon X \to X \times X$, induces $d_\Delta\colon \Delta_*(X) \to \Delta_*(X)$. Define a natural *diagonal approximation* $\Delta = d_\Delta \circ \theta$, where $\theta$ is the chain equivalence from the Eilenberg-Zilber theorem.

$$\begin{tikzcd} 0 \arrow[r] & \Delta_*(X) \arrow[r, "d_\Delta"] \arrow[rr, bend right, "\Delta"'] & \Delta_*(X \times X) \arrow[r, "\theta"] & \Delta_*(X) \otimes \Delta_*(X) \end{tikzcd}$$

The *cup product of homogeneous cochains* $f$ and $g$ is

$$f \smile g = (f \otimes g)\theta d_\Delta.$$

*Note.* The equation

$$\delta(f \smile g) = \delta f \smile g + (-1)^{\deg} f \smile \delta g$$

follows from the boundary formula for the cross product $\times$. ⋞

**Proposition 2.8.**

(1) *The cup product is natural for $X$ in* $\mathsf{Top}$ *and $\Lambda$ in* $\mathsf{Ring}$. *Given a continuous map $\varphi\colon X \to Y$ in* $\mathsf{Top}$*, the induced map on cohomology satisfies*

$$\varphi^*(\alpha \smile \beta) = \varphi^*(\alpha) \smile \varphi^*(\beta)$$

*for all homogeneous cochains $\alpha, \beta$ in $\Delta^*(X)$.*

(2) $\alpha \smile 1 = \alpha = 1 \smile \alpha$*, where 1 is the class of the augmentation $\varepsilon$ (TODO. Be specific.)*

(3) *The cup product $\smile$ is associative.*

(4) *The cup product is* skew-commutative*:*

$$\alpha \smile \beta = (-1)^{\deg \alpha \deg \beta}(\beta \smile \alpha).$$

**Definition 2.9** (Alexander–Whitney diagonal approximation)**.** Let $\sigma\colon \Delta_n \to X$ be a singular $n$-simplex in $X$. The *Alexander–Whitney diagonal approximation* explicitly computes the image of $\sigma$ under the chain map $\Delta\colon \Delta_*(X) \to \Delta_*(X) \otimes \Delta_*(X)$ from the *front and back faces* of $\sigma$.

$$\Delta\sigma = \sum_{p+q=n} \|\sigma\|^p_{\text{front}} \otimes \|\sigma\|^q_{\text{back}}.$$

**Exercise 2.10.** Any two chain maps $\Phi, \Psi \colon \Delta_*(X) \to \Delta_*(X \otimes X)$ that agree with the diagonal approximation

$$\Delta(x) = x \otimes x \quad \text{in the 0th degree}$$

are chain homotopic: $\Phi \simeq \Psi$.

**Proposition 2.11** (Computing the cup product)**.** *Say $f$ and $g$ are in the cochains with degrees $p$ and $q$ respectively, such that $p+q=n$. Then*

$$\begin{aligned}(f \smile g)(\sigma) &= (f \otimes g)(\Delta\sigma)\\ &= (f \otimes g)\left(\sum_{i+j=n} \|\sigma\|^i_{front} \otimes \|\sigma\|^j_{back}\right)\\ &= (f \otimes g)\left(\|\sigma\|^p_{front} \otimes \|\sigma\|^q_{back}\right)\\ &= (-1)^{\deg g \deg f} f(\|\sigma\|^p_{front}) g(\|\sigma\|^q_{back}) \quad (\textit{an element of } \Lambda).\end{aligned}$$

**Exercise 2.12** (A derivation from the cup product)**.** Let $A, B \subset X$ in $\mathsf{Top}$ be open in $X$. Verify the following:

(1) $\Delta_*(A) + \Delta_*(B) \twoheadrightarrow \Delta_*(A \smile B)$.
(2) $H^*(X, A; \Lambda) \otimes H^*(X, B; \Lambda) \to H^*(X, A; \Lambda)$. .
(3) From the snake lemma and (2), there's a long exact sequence

$$\cdots \longleftarrow H^{*+1}(X, A; \Lambda) \xleftarrow[\delta^*]{} H^*(A; \Lambda) \xleftarrow[i^*]{} H^*(X; \Lambda) \longleftarrow \cdots$$

that's natural in $X, A, B$ and $\Lambda$.
(4) The connecting homomorphism $\delta$ is a derivation (TODO of what?) defined by

$$\delta^*(\alpha \smile i^*(\beta)) = \alpha \smile \delta^*(\beta).$$

Lecture 3.

## 2019-04-19

*"À bas Euclide! Mort aux triangles![2] [Down with Euclid! Death to triangles!]"*

Today, we'll define the cap product and sketch Poincaré duality for (nice) topological manifolds. Let $X \in \mathsf{Top}$. To reference the degrees of cochains in $\Delta^p(X)$ and chains in $\Delta_q(X)$ let $p, n$ be nonnegative integers with $q = n - p$. We'll regard a $p$-cochain $f$ to "be defined but equal to zero on $i$-simplices when $i \neq p$". (So for $c \in \Delta_i(X)$, $f(c) = 0$ if $i \neq p$.)

**Definition 3.1** (The cap product on cochains and chains)**.** Suppose $f \in \Delta^p(X; \Lambda)$ and $c \in \Delta_n(X)$. The *cap product* on the chain–cochain level is the linear map

$$\begin{aligned}\cap \colon \Delta^p(X) \otimes \Delta_n(X) &\to \Delta_{n-p=q}(X)\\ f \otimes c &\overset{\cap}{\longmapsto} f \cap c := (1 \otimes f)\Delta c\end{aligned}$$

where $\Delta \colon \Delta_*(X) \to \Delta_*(X) \otimes \Delta_*(X)$ is some diagonal approximation.

As a consequence, if $\Delta$ is the Alexander–Whitney diagonal approximation and $\sigma$ is an $n$-simplex in $\Delta_n(X)$, then

$$\begin{aligned}f \cap \sigma &= (1 \otimes f)\Delta\sigma\\ &= (1 \otimes f) \sum_{p+q=n} (\sigma\rfloor_q \otimes {}_p\lfloor\sigma)\\ &= (-1)^{pq} f({}_p\lfloor\sigma) \cdot \sigma\rfloor_{n-p}.\end{aligned}$$

**Proposition 3.2** (Properties of the cap product with respect to other operations)**.**

(1) *(Augmentation) For the augmentation $\varepsilon \colon \Delta_0(X) \to \Lambda$ and any 0-chain $c \in \Delta_0(X)$,*

$$\varepsilon \cap c = c.$$

[2] Jean Dieudonné, keynote address at the Royaumont Seminar (1959)

(2) *(Krönecker pairing) For any cochain $f \in \Delta^p(X)$ and chain $c \in \Delta_p(X)$ (of the same degree),*

$$\begin{array}{ccccc} \Delta^p(X) \otimes \Delta_p(X) & \xrightarrow{\cap} & \Delta_0(X) & \xrightarrow{\varepsilon} & \Lambda \\ f \otimes c & \longmapsto & f \cap c & \longmapsto & \varepsilon(f \cap c) = f(c) \end{array}$$

*That is, the cap product coincides with the Krönecker pairing (if interpreted correctly).*

(3) *(Cup product) For any two cochains $f \in \Delta^p(X)$, $g \in \Delta^k(X)$, and any chain $c \in \Delta_{n+k}(X)$,*

$$(f \cup g) \cap c = f \cap (g \cap c) \quad (\text{which is } 0 \text{ if } p + k \geq n).$$

(4) *(Induced chain maps) For any map of spaces $X \xrightarrow{\varphi} Y$, any cochain $f \in \Delta^p(X)$, and any chain $c \in \Delta_n(X)$, the chain maps $\varphi_\Delta$ and $\varphi^\Delta$ satisfy*

$$\varphi_\Delta\big(\varphi^\Delta(f) \cap c\big) = f \cap \varphi_\Delta(c).$$

(5) *(Boundary maps) For any cochain $f$ and chain $c$,*

$$\partial(f \cap c) = \delta(f) \cap c + (-1)^{\deg f} f \cap \partial c.$$

The cap product on chains and cochains descends to (co)homology by the boundary formula above (2.7).

$$\frown : H^p(X) \otimes H_n(X) \to H_{n-p}(X).$$

**Proposition 3.3** (Relation of the cap product on homology classes to other operations)**.**

(1) *(Triviality) For $1 \in H^0(X)$ the class of the augmentation, and any $\gamma \in H_n(X)$,*

$$1 \frown \gamma = \gamma.$$

(2) *(Krönecker pairing) For $\varepsilon_*$ induced from the augmentation, $\alpha \in H^p$, and $\gamma \in H_p$,*

$$\varepsilon_*(\alpha \frown \gamma) = \langle \alpha, \gamma \rangle .$$

(3) *(Cup product) $(\alpha \smile \beta) \frown \gamma = \alpha \frown (\beta \frown \gamma)$.*
(4) *(Naturality) $\varphi_*(\varphi^*\alpha \frown \gamma) = \alpha \frown \varphi_*\gamma$.*
(5) *(Annihilation) For $\alpha \smile \beta$ in $H^p$, $\gamma \in H_p$, TODO.*
(6) *(Cross product) We have $(\alpha \times \beta) \cap (a \times b) = (-1)^{\deg \alpha \deg \beta}(\alpha \cap a) \times (\beta \cap b)$.*

The cup and cap product have an *adjoint-ish* relationship with each other (but, on facebook, `it's complicated`).

*Note.* Manifolds will have a canonical class to cap with. The purpose of the development now is to reach Poincaré duality for manifolds. Suppose I have a symmetric monoidal category. Then I have a notion of "a ring object" in the category. The cap product then gives a *pairing.* ◀

*Remark* 3.4 (Fundamental class of a manifold). Say $M \in \mathsf{Man}$ is a (closed, compact, orientable) $n$-manifold. Then with $[M] \in H_n(M)$ the fundamental class, the pairing

$$\begin{aligned} H^p(M) \otimes H_n(M) &\to H_{n-p}(M) \\ \alpha &\longmapsto \alpha \cap [M] \end{aligned}$$

will induce an isomorphism. ◀

**Definition 3.5** (Topological manifolds)**.** An $n$-manifold is a (second-countable) Hausdorff topological space $M$ such that every $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^n$.

*Remark* 3.6 ("Feeling" version of dual triangulations). Say that I triangulate $M$ with simplices $\sigma_i$ such that the alternating face maps are coherent with the triangulation. If it is possible to define coherently, the *fundamental class* of the manifold $M$ is the boundary $\partial(\sum_i \sigma_i) = [M]$. We'll define this rigorously in a bit.

For example, $\mathbb{RP}^2$ can be triangulated, but fails to admit oriented 2-cells. ◀

**Example 3.7** (Dual cells as indicator functions)**.** Let $M \in \mathsf{Man}$ be sufficiently nice (compact, orientable) of dimension $n$ (for example, 2). Then the 0-cells (say, $u$, $w$, and $v$) are *paired* to the *indicator functions* on the 0-cells.

$$D(u)^*, D(w)^*, D(v)^* : C_2^D(M) \to \Lambda. \tag{3.8}$$

The dual cell structure arises from assigning each $k$-cell to a $n-k$-cell by the rule

$$\langle u_0, u_1, \ldots, u_k \rangle \rightsquigarrow D(u_0) \cap D(u_1) \cap \ldots \cap D(u_k).$$

◀

**Exercise 3.9.** Find a planar graph $(V, E, F)$, and algorithmically compute the dual graph.

Lecture 4.

## 2019-04-22

In the last section of the course, for orientations and Poincaré duality, we'll make the lateral move to Peter May's Concise.

Fix an $n \in \mathbb{N}$ today for the dimension of our topological manifolds. E.g., we work in $\mathsf{Man}^n$.

*Remark* 4.1 (Local excision). Let $x \in U \subset M$ be a point of the manifold $M \in \mathsf{Man}^n$, with $U$ a chart domain. From an application of excision, the coordinate balls in $M$ have trivial homology groups:

$$H_i(M, M-x; G) \cong H_i(U, U-x; G) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{else.} \end{cases}$$

That is, for any $x \in U$, there's an isomorphism $H_n(M, M \setminus U) \xrightarrow{\cong} H_n(M, M-x)$. ⊲

Fix coefficients over a commutative unital ring $\Lambda$.

**Definition 4.2** (Fundamental class)**.** A *fundamental class* of $M$ at $X \subset M$ is a class $z \in H_n(M, M \setminus X; \Lambda)$ such that the image of $z$ in $H_n(M, M-x; \Lambda)$ is a generator for all $x \in X$.

$$j_{x,X} \colon H_n(M, M \setminus X) \to H_n(M, M-x) \cong \Lambda$$

$$z \xmapsto{j_{x,X}} j_{x,:}(z) \text{ which needs to be a unit in } \Lambda.$$

*Note.* Observe that the fundamental class at $X$ doesn't necessarily exist, but if $U$ is a "fundamental" open set, then this class *always* exists. ⊲

**Definition 4.3** (Orientation (May))**.** An $\Lambda$-orientation is an open cover $\{U_i\}$ of $M$ with fundamental classes $z_i$ of $M$ at $U_i$

$$z_i \in H_i(M, M \setminus U_i; \Lambda)$$

such that $z_i$ and $z_j$ map to the same class in the intersected "local homology"

$$H_i\left(M, M \setminus \underbrace{(U_i \cap U_j)}_{\text{nonempty}}; \Lambda\right)$$

(Note that the small neighborhood $U_i \cap U_j$ is more susceptible to *receive* induced homology maps than are either of the larger open neighborhoods $U_i$ or $U_j$.)

Here's a differential geometric interpretation. Pick a frame in some $n$-manifold over some point $a \in M$. Consider the set $S_a$ of all frames over points $b$ such that there's a continuous path from $a$ to $b$ in $M$. Either, for some homotopy class of paths in $M$ that return to $a$, there exist frames in $S_a$ whose determinant differs by $-1$, or no such homotopy class of paths exists. In the later case, $M$ is orientable. *It's an $n$-finger rule!*

**Example 4.4** (Orientation double cover)**.** Suppose now that the coefficient ring is $\mathbb{Z}$. Define the *orientation double cover of $M$* to be the set of pairs

$$\tilde{M}_{\mathbb{Z}} = \{(x, a) : x \in M, a \in H_n(M, M-x)\}$$

En masse, say we have an open cover $\mathcal{U}$ of $M$. For each open set $U \subset M$, we need to define a *topology basis* for the orientation bundle $\tilde{m}$. But for $U$ in the open cover $\mathcal{U}$, there's a fundamental class $a \in H_n(M, M \setminus U)$. So take a open set $U_a \subset \tilde{M}$ (thought of as the open set $U \subset M$ "evaluated" at the class $a$) defined by

$$U_a = \{(x, b) \in \tilde{M} : x \in U, b = j_{x,U}(a)\}$$

as a basis element for the topology of the orientation bundle $\tilde{M}$. ⊲

**Proposition 4.5** (The orientation bundle is a double cover)**.** *As defined in 4.4, the projection from the bundle $\tilde{M} \twoheadrightarrow M$ such that $(x, a) \mapsto x$ is a double cover. (It's a trivial, path disconnected, double cover if $M$ is orientable; it's* always *a double cover.)*

*Proof.* TODO. Here's the sketch: consider the lift of an open set $U \subset M$. Then $p^{-1}(U)$ should be isomorphic to the group of units $(\mathbb{Z}/(2))$ of the local homology. We want to establish the double cover on $U$.

$$\begin{tikzcd} p^{-1}(U) \arrow[dr] & U \times \mathbb{Z}^\times \arrow[l] \arrow[d] \\ & U \end{tikzcd}$$

□

**Definition 4.6** (Orientation (Bredon))**.** Let $X \subset M$ be any subset of the manifold $M$. Let $\tilde{M}$ be the orientation bundle. Then *an orientation of $M$ along $X$* is a continuous section $X \xrightarrow{s} \tilde{M} \xrightarrow{p} M$ with $p \circ s = \mathrm{id}$.

**Exercise 4.7** ($SU(2)$ is a cover of $SO(3)$)**.** Show that either $SU(2)$ is or is not an orientation bundle of $S0(3)$.

**Proposition 4.8** (Sufficient conditions for orientability)**.** *Let $M$ be a connected $n$-manifold. The following are equivalent.*

(1) *$M$ is orientable.*
(2) *$M$ is orientable along any compact subset.*
(3) *$\tilde{M}$ is a trivial double cover, and $\tilde{M} \cong M \sqcup M$.*

Note the two definitions are almost equivalent, but the former definition was a bit too *raw*, as in, May gave us an actual bundle, not an equivalence class of bundles.

*Proof.* 1 implies 2. Consider the definition of continuous sections, then march along any compact subset (get a Lebesgue cover). 2 implies 3. Assume also that $\tilde{M}$ is connected. Let $x \in M$. Let $\sigma\colon [0,1] \to \tilde{M}$ be a path, starting at $(x,a)$ and ending at $(x,-a)$. Then project $\sigma([0,1])$ down to $M$. Consider that $p\sigma([0,1]) \subset M$ is compact, so by (2) there's a section $s\colon p\sigma([0,1]) \to \tilde{M}$ along the image of the path.

$$\begin{tikzcd}
 & \tilde{M} \\
{[0,1]} \arrow[ur, "\sigma"] \arrow[r, "p\sigma"'] & M \arrow[u, "s"']
\end{tikzcd}$$

3 implies 1. If there's a trivial double cover of $M$, then it's easy (since the projection is a local homeomorphism) to obtain a continuous section from $M$ back into the orientation bundle $\tilde{M}$. $\square$

**Proposition 4.9** ($\mathbb{Z}/2$ orientation)**.** *Let the ring of coefficients $\Lambda = \mathbb{Z}/(2)$. (TODO, is this true for* any *2-torsion ring?) Running the same argument as in the proof of 4.8, the projection from $\tilde{M}_{\mathbb{Z}}$ to $M$ induces a projection from the $\mathbb{Z}/(2)$ orientation bundle. TFDC:*

$$\begin{tikzcd}
\tilde{M}_{\mathbb{Z}} \arrow[rr] \arrow[dr] & & \tilde{M}_{\mathbb{Z}/2} \arrow[dl] \\
 & M &
\end{tikzcd}$$

*It follows that $\tilde{M}_{\mathbb{Z}/2} \cong M \sqcup M$, so that $M$ is orientable* $\mod 2$.

## Wednesday, May 1

(1) Bredon Chapter VI p.348 Problem 2 and 5

(2) May (Concise algebraic topology) Chapter 20.

https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf

Read section 1 of Chapter 20 and do Problems 5 on p. 164 of the link.

(3) Bredon Chapter VI p.355, Problem 6 and 7.

(4) May, Chapter 20, Problem 6

2. If $M$ is a connected manifold such that $\pi_1(M)$ has no subgroups of index 2 then show that $M$ is orientable.

Given. $M \in \mathrm{Man}^n$, cntd. $\pi_1(M) = G$.

$$\begin{array}{cc} K & 1 \\ | & | \\ \mathbb{Q}(\alpha) & H \\ | & | \\ \mathbb{Q} & G \end{array} \qquad \begin{array}{l} \pi_1(M) = G \\ \pi_0(M) = \{1\} \end{array} \qquad \begin{array}{l} H_1(M) = G^{ab} \\ H_0(M) = \mathbb{Z} \end{array}$$

If $[G:H] = 2$, then $H \trianglelefteq G$.

$\forall H \trianglelefteq G$, $G/H \neq 2$

So it suffices to show $\forall g \in G$, $gHg^{-1} = H$

Say $H_1 \in \mathrm{Ab}$, and $H_0 \cong \mathbb{Z}$.

Say $K \overset{i}{\hookrightarrow} M$, w/ $K$ cmpt. $i_*: \pi_1(K) \hookrightarrow \pi_1(M)$

Say $M$ ain't orientable. Then $\tilde{M}$ is a connected double cover of $M$.

Pf. Upon verifying that $p: \tilde{M} \longrightarrow M$ is a covering map, we have that $\pi_1(\tilde{M}) \hookrightarrow \pi_1(M)$ with

$$\begin{array}{ccc} \{1\} & & \downarrow \\ | & & \\ \pi_1(\hat{M}) & & \tilde{M} \\ | \, 2 & & \downarrow 2 \\ \pi_1(M) & & M \end{array}$$

So $\pi_1(M)/\pi_1(\tilde{M}) \cong \mathbb{Z}/2$

THM| The fund. grp. of a second ctbl. manifold is ctbl. ?

so $X, Y$ are path cntd, locally path cntd, Hsdf

THM| If $p: Y \to X$ is a covering map in Top and $Y$ is cntd, then $\pi_1(Y, y) \xhookrightarrow{p_*} \pi_1(X, x)$ for $p(y) = x$.

Pf. Directly. Say that $\pi_1(M)$ has no subgps of index 2. Let $\mathcal{M}$ be a cntd comp. of $\tilde{M}$. Then $p: \mathcal{M} \to M$ is a cover, and so for $\hat{M}$ the universal covering space of $M$, $\exists! q: \hat{M} \to \mathcal{M}$, a cover too.

$$\begin{array}{ccc} \hat{M} & & \pi_1(\hat{M}) = 1 \\ \downarrow & & | \\ \mathcal{M} & & \pi_1(\mathcal{M}) \\ \downarrow & & | \\ M & & \pi_1(M) \end{array} \qquad \pi_1(M)/\pi_1(\mathcal{M})$$

What's the orbit of a pt $x \in \mathcal{M}$ under the action of $\pi_1(M)$?

Just the fiber! And we know $|p^{-1}(x)| \leq 2$, so by orbit stabilizer, $2 \geq$

Pf. Let $N$ be a component of $\tilde{M}$, and let $p: N \longrightarrow M$ be a covering map (it's a local homeo and surjective, since any component is the union of $\{U_\alpha\}$ for a cover $\{U\}$ of $M$).

~~Let id: $M \to M$ be the identity covering map.~~

THEOREM. *Let $p: E \longrightarrow B$ and $p': E' \longrightarrow B$ be coverings and choose $b \in B$, $e \in E$, and $e' \in E'$ such that $p(e) = b = p'(e')$. There exists a map $g: E \longrightarrow E'$ of coverings with $g(e) = e'$ if and only if*

$$p_*(\pi_1(E, e)) \subset p'_*(\pi_1(E', e')),$$

*and there is then only one such $g$. In particular, two maps of covers $g, g': E \longrightarrow E'$ coincide if $g(e) = g'(e)$ for any one $e \in E$. Moreover, $g$ is a homeomorphism if and only if the displayed inclusion of subgroups of $\pi_1(B, b)$ is an equality. Therefore $E$ and $E'$ are homeomorphic if and only if $p_*(\pi_1(E, e))$ and $p'_*(\pi_1(E', e'))$ are conjugate whenever $p(e) = p'(e')$.*

Choose $b \in M$ and $e \in N$ s.th. $p(e) = b$. Then $p_\#(\pi_1(N, e)) \subset \pi_1(M, b)$ by the fund. thm. of covering space theory.

PROPOSITION 10 *Let $p: \tilde{X} \to X$ be a covering map, and put $\pi = \pi_1(X, x_0)$. Then the fibre $F = p^{-1}(x_0)$ is a $\pi$-set, and the stabilizer of any point $\tilde{x} \in F$ is $H_{\tilde{x}} = p_*(\pi_1(\tilde{X}, \tilde{x}))$.*

Recall the fiber $F = p^{-1}(b)$ is a $\pi(M, b)$ set, acting by unique lifting.

Since $N$ is (path) cntd, $\pi$ acts transitively on $p^{-1}(b)$, the fiber $p^{-1}(b)$ is the orbit of $e$ under $\pi$. The size of the orbit $|\pi(e)|$ is the index of the stabilizer $\mathrm{Stab}_\pi(e) = p_\#(\pi_1(N, e))$ in $\pi$.

So $|p^{-1}(b)| = |\pi(e)| = \pi_1(M, b) / \!\! / p_\#(\pi_1(N, e)) \neq 2$.

Yet $|p^{-1}(b)| \leq 2$. So $p^{-1}(b) = e$, and the cover $p: N \to M$ is a homeomorphism. So $p^{-1}: M \to N \hookrightarrow \tilde{M}$ an orientation ☺. $\square$

# Homework 7 Problem 1

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Math 6220

1. Do Bredon Chapter VI p.348 Problem 2 and 5

**Problem 2** If $M$ is a connected manifold such that $\pi_1(M)$ has no subgroups of index 2 then show that $M$ is orientable.

*Proof.* Suppose that $M$ is not orientable. Then $\tilde{M}$ is connected and $p_*(\pi_1(\tilde{M}))$ is a subgroup of $\pi_1(M)$ of index 2, which is contradiction. $\square$

**Problem 5** For a connected nonorientable manifold $M^n$ show that there exists a unique orientable double covering space of $M^n$.

*Proof.* We know that the typical construction of the orientable double cover only depends on the local orientation of $M$. So the main proof consists of showing that this double cover is unique. So suppose we have two double covers, $\tilde{M}$ and $\bar{M}$. Both of these covers are connected because M is nonorientable. We will fix an orientation of $\tilde{M}$ given by a local orientation $\mu_{\bar{x}} \in H_n(\bar{M}, \bar{M} - \bar{x})$. We now define a map $\phi : \bar{M} \mapsto \tilde{M}$ in the following way: Given $\bar{x} \in \bar{M}$, denote by $\bar{p}_*\mu_{\bar{x}} \in H_n(M, M - x)$ the local orientation at $x = \bar{p}(\bar{x})$ induced by $\mu_{\bar{x}}$. Then define $\phi : \bar{M} \mapsto \tilde{M}$, $\bar{x} \mapsto (\bar{p}(\bar{x}), \bar{p}_*\mu_{\bar{x}})$. $\phi$ is a continuous map, and $\tilde{p} \circ \phi = \bar{p}$ and hence $\bar{p}_*\pi_1(\bar{M}) \subset \tilde{p}_*\pi_*(\tilde{M})$. Both are index 2 subgroups so $\bar{p}_*\pi_1(\bar{M}) = \tilde{p}_*\pi_*(\tilde{M})$. This implies that $\tilde{M} \cong \bar{M}$. $\square$

# May 1 Problem 3

Justin Willson

May 2019

## Bredon Page 355

### 6

**Problem.** If $M$ is a compact connected $(2n + 1)$-manifold, possibly nonorientable, show that the Euler characteristic of $M^{2n+1}$ is 0.

**Solution.** Recall that

$$\chi(M) = \sum_i (-1)^i \dim(H_i(M; \mathbb{Z}_2)).$$

From Poincare duality, we have that

$$H^i(M; \mathbb{Z}_2) = H_{2n+1-i}(M; \mathbb{Z}_2)$$

Putting these two facts together, we see that if we show that

$$H_i(M; \mathbb{Z}_2) \cong H^i(M; \mathbb{Z}_2),$$

we will have shown that $\chi(M) = 0$. This is because $H_i(M; \mathbb{Z}_2)$ and $H_{2n+1-i}(M; \mathbb{Z}_2)$ would have the same rank, but would appear with opposite signs in the sum. To that end, we apply the generalized universal coefficent theorem to prove $H_i(M; \mathbb{Z}_2)$ is isomorphic to $H^i(M; \mathbb{Z}_2)$. We see

$$H^i(M; \mathbb{Z}) \cong \operatorname{Ext}^1_{\mathbb{Z}_2}(H_{i-1}(M; \mathbb{Z}_2), \mathbb{Z}_2) \oplus \hom(H_i(M; \mathbb{Z}), \mathbb{Z}_2).$$

The Ext term vanishes because $H_{i-1}(M; \mathbb{Z}_2)$ is a vector space and therefore free. Further, we see that $H_i(M; \mathbb{Z})$ is isomorphic to $\hom(H_i(M; \mathbb{Z}), \mathbb{Z}_2)$ as $\hom(H_i(M; \mathbb{Z}), \mathbb{Z}_2)$ is its dual vector space, and these spaces are finite dimensional vector spaces. Thus we have show

$$H_i(M; \mathbb{Z}_2) \cong H^i(M; \mathbb{Z}_2)$$

and thus the Euler characteristic of $M$ is zero.

## 7

**Problem.** If $M$ is a compact, connected, and nonorientable 3-manifold, show that $H_1(M)$ is infinite.

**Solution.** If we apply problem 6, and the formula for Euler characteristic, we see

$$0 = \text{rank}H_0(M) - \text{rank}H_1(M) + \text{rank}H_2(M) - \text{rank}H_3(M).$$

Then note that because $M$ is connected, $H_0(M) = \mathbb{Z}$. Also, because $M$ is not orientable, $H_3(M) = 0$ because $\mathbb{Z}$ has no 2-torsion. Thus we have

$$0 = 1 - \text{rank}H_1(M) + \text{rank}H_2(M).$$

Thus, $\text{rank}H_1(M)$ must be greater than 1 as otherwise this sum would be positive. Hence, $H_1(M)$ is infinite.

5 For a connected nonorientable manifold $M^n$ show that there exists a unique orientable double covering space of $M^n$.

PROPOSITION. *Let $M$ be a connected $n$-manifold. Then there is a 2-fold cover $p : \tilde{M} \longrightarrow M$ such that $\tilde{M}$ is connected if and only if $M$ is not orientable.*

PROOF. Define $\tilde{M}$ to be the set of pairs $(x, \alpha)$, where $x \in M$ and where $\alpha \in H_n(M, M - x) \cong \mathbb{Z}$ is a generator. Define $p(x, \alpha) = x$. If $U \subset M$ is open and $\beta \in H_n(M, M - U)$ is a fundamental class of $M$ at $U$, define

$$\langle U, \beta \rangle = \{(x, \alpha) | x \in U \text{ and } \beta \text{ maps to } \alpha\}.$$

The sets $\langle U, \beta \rangle$ form a base for a topology on $\tilde{M}$. In fact, if $(x, \alpha) \in \langle U, \beta \rangle \cap \langle V, \gamma \rangle$, we can choose a coordinate neighborhood $W \subset U \cap V$ such that $x \in W$. There is a unique class $\alpha' \in H_n(M, M - W)$ that maps to $\alpha$, and both $\beta$ and $\gamma$ map to $\alpha'$. Therefore

$$\langle W, \alpha' \rangle \subset \langle U, \beta \rangle \cap \langle V, \gamma \rangle.$$

Clearly $p$ maps $\langle U, \beta \rangle$ homeomorphically onto $U$ and

$$p^{-1}(U) = \langle U, \beta \rangle \cup \langle U, -\beta \rangle.$$

Therefore $\tilde{M}$ is an $n$-manifold and $p$ is a 2-fold cover. Moreover, $\tilde{M}$ is oriented. Indeed, if $U$ is a coordinate chart and $(x, \alpha) \in \langle U, \beta \rangle$, then the following maps all induce isomorphisms on passage to homology:

$$\begin{tikzcd}
(\tilde{M}, \tilde{M} - \langle U, \beta \rangle) \arrow[d] & (M, M - U) \arrow[d] \\
(\tilde{M}, \tilde{M} - (x, \alpha)) & (M, M - x) \\
(\langle U, \beta \rangle, \langle U, \beta \rangle - (x, \alpha)) \arrow[u] \arrow[r, "p", "\cong"'] & (U, U - x). \arrow[u]
\end{tikzcd}$$

Via the diagram, $\beta \in H_n(M, M - U)$ specifies an element $\tilde{\beta} \in H_n(\tilde{M}, \tilde{M} - \langle U, \beta \rangle)$, and $\tilde{\beta}$ is independent of the choice of $(x, \alpha)$. These classes are easily seen to specify an orientation of $\tilde{M}$. Essentially by definition, an orientation of $M$ is a cross section $s : M \longrightarrow \tilde{M}$: if $s(U) = \langle U, \beta \rangle$, then these $\beta$ specify an orientation. Given one section $s$, changing the signs of the $\beta$ gives a second section $-s$ such that $\tilde{M} = \mathrm{im}(s) \amalg \mathrm{im}(-s)$, showing that $\tilde{M}$ is not connected if $M$ is oriented. □

Pf. Recall if $G$ is a gp with $N \leq G$ sth $[G : N] = 2$, then for all $g \in G$, and any $n \in N$, $gn \notin N$ or $gn \in N$. If $gn \notin N$, then $gn = h \notin N$, and $h$ is unique. Since $h \in G$, we have $h^{-1}gn = 1 \in N$. So $n^{-1} = h^{-1}g$. $hn^{-1} = g$, $gn = hn^{-1}n = h \notin N$.

$gn \in N$ if $g \in N$. If $g \notin N$, then $gn = h$ is unique. $h^{-1} = n^{-1}g^{-1} \Rightarrow gh^{-1} = gn^{-1}g^{-1}$

$G/\!/N \overset{\text{sets}}{=} \mathbb{Z}/2$. So let $gN = \bar{1}$ and $hN = \bar{0}$ $g, h \in G$. $gNhN = gN$ as $h \in N$, $gNgN =$

$gN = Ng^{-1}$ ? $gn \overset{?}{=} \tilde{n}g^{-1}$ iff $gn(g\tilde{n}^{-1}) \overset{?}{=} 1$

**Lemma.**

Say $H \leq G$ with index 2. Then Cayley's thm gives $\rho : G \xrightarrow[\text{mult.}]{\text{left}} \mathrm{Aut}(G/\!/H) \cong S_2$. Now the normal core of $H$

$$\ker \rho = \{g \in G : gxH = xH \text{ for all } x \in G\}$$
$$= \{g \in G : x^{-1}gx \in H \text{ for all } x \in G\}$$
$$= \bigcap_{\forall x \in G} xHx^{-1} \leq H.$$

Since $G/\ker\rho \hookrightarrow C_2$, $|G/\ker\rho| \leq |C_2| = 2$

Since $2 = |G : H| \leq |G : \ker\rho| \leq 2$

we see $|G : H| = |G : \ker\rho|$.

Hence $H = \ker\rho \iff \bigcap_{x \in G} xHx^{-1} = H$

$\iff H \trianglelefteq G$. □

Pf. Let $\tilde{M}$ and $\tilde{M}'$ be orientation bundles sth $p : \tilde{M} \to M$ and $p' : \tilde{M}' \to M$ w/ $M$ non-orientable. Then choose $b \in M$ and $e \in \tilde{M}$, $e' \in \tilde{M}'$ sth $p(e) = b = p'(e')$. Because $\tilde{M}$ and $\tilde{M}'$ are 2-fold cntd covers, the fund gp $\pi_1(M, b)$ contains $p_*(\pi_1(\tilde{M}, e))$ and $p_*(\pi_1(\tilde{M}', e'))$, both of index 2.

If $[\gamma] \in \pi_1(\tilde{M}, e)$, then $[p \circ \gamma]$ lifts uniquely to $\pi_1(\tilde{M}', e')$ via $p'$. So $p_*(\pi_1(\tilde{M}, e)) \subset p'_*(\pi_1(\tilde{M}', e'))$.

a normal core of... a normal subgp...

$\therefore p_*(\pi_1(\tilde{M}, e)) = p'_*(\pi_1(\tilde{M}', e'))$. By the fund. thm of covering spaces;

So $\tilde{M} \longrightarrow \tilde{M}'$ is a cover, and $\tilde{M}' \longrightarrow \tilde{M}$ is too. Hence $\exists! \, \tilde{M} \xrightarrow{\cong} \tilde{M}'$. By inspection of the defns of

$$\tilde{M} = \{\langle x, \alpha \rangle : x \in U \text{ is a chart}, \langle \alpha \rangle = H_n(M, M - U)\}$$

we have $\tilde{M} = \text{"}\pm\tilde{M}'\text{"}$, given be a unique homeo.

Lastly, let $q : E \to \tilde{M}$ be the orientation bundle on $\tilde{M}$. Define $s : \tilde{M} \to E$ by choosing any chart $U \subset M$. Then any generator $\beta \in H_n(M, M \setminus U)$, which specifies

note $\tilde{M}$ that ... here

is cntd. //

a unique chart $\langle U, \beta \rangle \in \tilde{M}$. (Conversely choose the chart then push) Since (May p. 161)

[redacted], if $U$ is a coordinate chart and $(x, \alpha) \in \langle U, \beta \rangle$, then the following maps all induce isomorphisms on passage to homology:

$$\begin{tikzcd}
(\tilde{M}, \tilde{M} - \langle U, \beta \rangle) \arrow[d] & (M, M - U) \arrow[d] \\
(\tilde{M}, \tilde{M} - (x, \alpha)) & (M, M - x) \\
(\langle U, \beta \rangle, \langle U, \beta \rangle - (x, \alpha)) \arrow[u] \arrow[r, "p", "\cong"'] & (U, U - x). \arrow[u]
\end{tikzcd}$$

Via the diagram, $\beta \in H_n(M, M - U)$ specifies an element $\tilde{\beta} \in H_n(\tilde{M}, \tilde{M} - \langle U, \beta \rangle)$,

let $s : (x, \alpha) \mapsto ((x, \alpha), \tilde{\beta})$ iff $(x, \alpha) \in \langle U, \beta \rangle$.

$s^{-1}(\langle \langle U, \beta \rangle, \tilde{\beta} \rangle) = \langle U, \beta \rangle$ for all open sets in $E$.

Also $q \circ s : \tilde{M} \to \tilde{M}$ is the identity on $\tilde{M}$

$q(s(x, \alpha)) = q((x, \alpha), \tilde{\beta}) = (x, \alpha)$

So $s$ is a cts. section from $\tilde{M} \to E$, hence an orientation of the orientation bundle. $\square$

5. Let $M$ be oriented with fundamental class $z$. Let $f : S^n \longrightarrow M$ be a map such that $f_*(i_n) = qz$, where $i_n \in H_n(S^n; \mathbb{Z})$ is the fundamental class and $q \neq 0$.
   (a) Show that $f_* : H_*(S^n; \mathbb{Z}_p) \longrightarrow H_*(M; \mathbb{Z}_p)$ is an isomorphism if $p$ is a prime that does not divide $q$.
   (b) Show that multiplication by $q$ annihilates $H_i(M; \mathbb{Z})$ if $1 \leq i \leq n - 1$.

a) Given. $M$ cntd, cmpt, orientable. $f: S^n \longrightarrow M$
maps $f_* : i_n \longmapsto qz$ where $i_n \in H_n(S^n;\mathbb{Z})$ is the fund. class, and $z \in H_n(M;\mathbb{Z})$ is too.

To prove. For $q \in \mathbb{N}$ and $p$ prime sth $(p,q)=1$,

$f_*(S^n;\mathbb{Z}_p) \longrightarrow H_*(M;\mathbb{Z}_p)$ is an isom.

$H_k(X;\mathbb{Z}_p) \cong H_k(X;\mathbb{Z}) \otimes \mathbb{Z}_p$ with $\mathbb{Z}_p$ a field ($- \otimes \mathbb{Z}_p$ is exact)

Pf. Since $H_k(S^n;\mathbb{Z}) = \mathbb{Z}$ for $k=0,n$, else $0$, and $\mathbb{Z}\otimes\mathbb{Z}_p = \mathbb{Z}_p$, we have $H_k(S^n;\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & k=0,n \\ 0 & \text{else} \end{cases}$

Because $M$ is connected, $\mathbb{Z}$-orientable, $H_k(M, M-x) \cong H_k(M) = \begin{cases} \mathbb{Z} & k=0,n \\ 0 & \text{else} \end{cases} \Rightarrow H_k(M;\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & k=0,n \\ 0 & \text{else} \end{cases}$

Becuz $M$ is cmpt, and $\mathbb{Z}$-orientable, poincaré duality applies, so we only need to show $H_k(M;\mathbb{Z}_p) = H^{n-k}(M;\mathbb{Z}_p) = 0$ for all $0<k<n$.

By way of contra — say not. Then let $\alpha \neq 0$ be an elem of $H^k(M;\mathbb{Z}_p)$.

$\mathbb{Z}_p$ is a field, so $H^{n-k}(M;\mathbb{Z}_p) \otimes H^k(M;\mathbb{Z}_p) \longrightarrow \mathbb{Z}_p$ is a perfect pairing.

So there exists $\beta \in H^{n-k}(M;\mathbb{Z}_p)$ sth $\langle \alpha \smile \beta \rangle = H^n(M;\mathbb{Z}_p)$, so $\alpha \smile \beta$ is $\pm[z]$ $\in H^n(M;\mathbb{Z}_p)$ dual to the fund. cocycle the top class.

Let $f^*$ be the pull back induced on cohomology. Then

since these are in $H^{n-k}(S^n;\mathbb{Z}_p), H^k(S^n;\mathbb{Z}_p) \cong 0$

$f^*([z]) = f^*(\alpha \smile \beta) = \underbrace{f^*(\alpha) \smile f^*(\beta)}_{\text{naturality}} = \underline{0 \smile 0 = 0}$.

But $f^*([z]) \in H^n(S^n;\mathbb{Z}_p)$ acts on $i_n \in H_n(S^n;\mathbb{Z}_p)$ by $f^*([z])(i_n) = [z]f_*(i_n) = [z]qz = q$.

Since $(q,p)=1$, $\langle q \rangle = \mathbb{Z}_p$. Thence $\underline{\langle f^*[z] \rangle = H^n(S^n;\mathbb{Z}_p)}$. $\boxed{\divideontimes}$ Absurd!

Given. $H_*(M;\mathbb{Z}_p) \xleftarrow{\cong} H_*(S^n;\mathbb{Z}_p)$

$qz \longleftarrow i_n$

WTS. $q\mathbb{Z} \subset \mathrm{Ann}(H_k(M;\mathbb{Z}))$ for $0<k<n$.

Contrapositive.

if $z \longmapsto qz$ in $H_k(M;\mathbb{Z})$ non-trivial

then $z \longmapsto qz$ in $H_k(M;\mathbb{Z}_p) \cong H_k(M;\mathbb{Z}) \otimes \mathbb{Z}_p$ is non-trivial as $(p,q)=1$ □

# 2019-05-01 OFFICE HOUR

- singular homology; face and boundary maps; categories Top, Ab, GrAb, Ch; functors Map, Sing_n, \Delta_*, H_*

- group theory review; Hurewicz homomorphism; examples with 0-, 1-, and 2-dim manifolds; functors H_0, H_1, \pi_1, (--)^ab

- relative homology; s.e.s.'s; l.e.s. of a pair; snake lemma; categories Top_pairs, (hTop_pairs), (triangulated Ch)

- Eilenberg--Steenrod axioms; definition of a co/homology theory with coefficients

- derivation of properties in an ordinary homology theory; definition of degree; homotopy classes of maps of spheres; example with antipodal map

- CW-complexes; cellular homology computes singular homology; examples with RP^n, CP^n, surfaces; (equivariant CW-complexes); (cellular maps); (cellular approximation)

- product CW-complexes; a cross product exists; acyclic models; verification of the homotopy axiom

- verification of excision; Euler characteristic; examples with 2-dim manifolds; (computations with simplicial homology); (proof of Meyer--Vietoris)

- transition to cohomology; free resolutions; (structure theorems for Ab, modules over PIDs); Ext and Tor; chain homotopies; (homotopy category of chain complexes)

- universal coefficients; Künneth theorem; examples with Euler characteristic, surfaces, (finite fields), non-natural splitting

- Eilenberg--Zilber theorem; Alexander--Whitney diagonal approximation; (Krönecker pairings); cup products; cap products

- topological manifolds; orientations; (maps of fiber bundles); Poincaré duality