MATH 6210 NOTES: TOPOLOGY 2

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These notes were taken in University of Colorado's MATH 6210 (Topology 2) class in Spring 2019, taught by Agnés Beaudry. I live-TEXed them with `vim`, so there may be typos and failures of understanding. Any mistakes are my own. Please send questions, comments, complaints, and corrections to colton.grainger@colorado.edu. Thanks to `adebray` for the LATEX template, which I have forked from https://github.com/adebray/latex_style_files.

Contents

Lecture 1.

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"Does the torus $S^1 \times S^1$ have a cohomology ring[1] that's not a polynomial ring?"

Last time, we defined (extended actually) the cross product

$$\times\colon H^*(X;\Lambda) \otimes H^*(Y;\Lambda) \to H^*(X \times Y;\Lambda),$$

which arises from the map on cochains (say $f\colon A_* \to \Lambda$ and $g\colon B_* \to \Lambda$) given by

$$\begin{aligned} f \otimes g\colon A_* \otimes B_* &\to \Lambda \\ (f \otimes g)(a \otimes b) &= (-1)^{\deg a \deg g} f(a)g(b). \end{aligned}$$

Definition 1.1. Let $X \in \mathsf{Top}$ and $\Lambda \in \mathsf{Ring}$. Shouldn't there be an *evaluation* ev from chain complex of p-cochains $f\colon \Delta_p(X) \to \Lambda$ tensored with the chain complex $\Delta_p(X)$ of p-chains?

$$\Delta^p(X;\Lambda) \otimes \Delta_p(X) \xrightarrow{\text{ev}} \Lambda$$

In fact, there is such an evaluation:

$$\text{ev}\colon f \otimes c \longmapsto f(c).$$

We'll see that ev induces a map on cohomology, which is denoted by the brackets (in Halmos style)

$$\langle |f|, |c| \rangle \in \Lambda.$$

Lemma 1.2 (Kronecker pairing)**.** *The evaluation $\Delta^p(X;\Lambda) \otimes \Delta_p(X) \xrightarrow{\text{ev}} \Lambda$ induces a homomorphism out of the graded ring*

$$H^p(X;\Lambda) \otimes H_p(X) \to \Lambda \tag{1.3}$$

such that $\langle |f|, |c| \rangle \mapsto f(c)$.

Proof. The *Kronecker pairing* is the argument that $f(c)$ does not depend on representatives f or c (from the cochain, resp, chain complexes). Consider that in the proof of the universal coefficient theorem, we found a map β from $H^*(X;\Lambda)$ to $\text{Hom}(H_p(X), \Lambda)$ such that $|f| \mapsto \{|c| \mapsto f(c)\}$ gave a group homomorphism. Use this. □

[1] Hint. The cohomology ring for S^1 is to $\mathbb{RP}^2$ as – is to – (?) with spheres adjoined (!?). Moral. Consider the homogeneous elements of the ring. Just describe the ring. For example, when algebraic topologists write $\mathbb{Z}[x]/\langle x^4 \rangle$, it is understood to be *a homogeneous ring*, with addition *strictly levelwise* but with products (via the cup product) allowed between non-homogeneous degrees.

Lemma 1.4 (A sign convention)**.** *For the singular chain complexes over a topological space X, let f, g be cochains in $\Delta^*(X)$ and α, β chains in $\Delta_*(X)$. Then, $(f \otimes g)(\alpha \times \beta) = (-1)^{\deg \alpha \deg g} f(\alpha) g(\beta)$.*

Definition 1.5 (Cap product)**.** The *cap product* operation over a topological space X is the above (1.3) pairing, which is "given by combining the *Kronecker pairing* of the cohomology class with the image of the homology class under diagonal and using the Eilenberg-Zilber theorem." (See `https://ncatlab.org/nlab/show/cap+product`.)

Definition 1.6 (Cup product)**.** Let $X \in \mathsf{Top}$ and consider the diagonal map $d \colon X \to X \times X$. There's an induced chain map $d_\Delta \colon \Delta_*(X) \to \Delta_*(X)$. Then d_Δ precomposed with θ (from Eilenberg-Zilber) is the natural *diagonal approximation* Δ. Schematically,

$$\begin{tikzcd} 0 \arrow[r] & \Delta_*(X) \arrow[r, "d_\Delta"] \arrow[rr, bend right, "\Delta"'] & \Delta_*(X \times X) \arrow[r, "\theta"] & \Delta_*(X) \otimes \Delta_*(X) \end{tikzcd} .$$

The *cup product* of two homogeneous cochains f and g is thus defined to be

$$f \smile g = (f \otimes g)\theta d_\Delta.$$

Note. The equation

$$\delta(f \smile g) = \delta f \smile g + (-1)^{\deg} f \smile \delta g$$

follows from the boundary formula for the cross product $\times$. ◁

Proposition 1.7.

(1) *The cup product is natural for X in Top and Λ in Ring. a continuous map $\varphi \colon X \to Y$ in Top, the induced map on cohomology satisfies*

$$\varphi^*(\alpha \smile \beta) = \varphi^*(\alpha) \smile \varphi^*(\beta)$$

for all homogeneous cochains α, β in $\Delta^*(X)$.

(2) *$\alpha \smile 1 = \alpha = 1 \smile \alpha$, where 1 is the class of the augmentation ε (TODO. Be specific.)*

(3) *The cup product $\smile$ is associative.*

(4) *The cup product is* skew-commutative*:*

$$\alpha \smile \beta = (-1)^{\deg \alpha \deg \beta}(\beta \smile \alpha).$$

Definition 1.8 (Alexander–Whitney diagonal approximation)**.** Let $\sigma \colon \Delta_n \to X$ be a singular n-simplex in X. The *Alexander–Whitney diagonal approximation* explicitly computes the image of σ under the chain map $\Delta \colon \Delta_*(X) \to \Delta_*(X) \otimes \Delta_*(X)$ from the *front and back faces* of σ.

$$\Delta\sigma = \sum_{p+q=n} \|\sigma\|^p_{\text{front}} \otimes \|\sigma\|^q_{\text{back}}.$$

Exercise 1.9. Any two chain maps $\Phi, \Psi \colon \Delta_*(X) \to \Delta_*(X \otimes X)$ that agree with the diagonal approximation

$$\Delta(x) = x \otimes x \quad \text{in the 0th degree}$$

are chain homotopic: $\Phi \simeq \Psi$.

Proposition 1.10 (Computing the cup product)**.** *Say f and g are in the cochains with degrees p and q respectively, such that $p + q = n$. Then*

$$\begin{aligned} (f \smile g)(\sigma) &= (f \otimes g)(\Delta\sigma) \\ &= (f \otimes g)\left(\sum_{i+j=n} \|\sigma\|^i_{front} \otimes \|\sigma\|^j_{back} \right) \\ &= (f \otimes g)\left(\|\sigma\|^p_{front} \otimes \|\sigma\|^q_{back} \right) \\ &= (-1)^{\deg g \deg f} f(\|\sigma\|^p_{front}) g(\|\sigma\|^q_{back}) \quad (\textit{an element of } \Lambda). \end{aligned}$$

Exercise 1.11 (A derivation from the cup product)**.** Let $A, B \subset X$ in Top be open in X. Verify the following:

(1) $\Delta_*(A) + \Delta_*(B) \twoheadrightarrow \Delta_*(A \smile B)$.

(2) $H^*(X, A; \Lambda) \otimes H^*(X, B; \Lambda) \to H^*(X, A; \Lambda)$. .

(3) From the snake lemma and (2), there's a long exact sequence

$$\cdots \longleftarrow H^{*+1}(X, A; \Lambda) \xleftarrow[\delta^*]{} H^*(A; \Lambda) \xleftarrow[i^*]{} H^*(X; \Lambda) \longleftarrow \cdots$$

that's natural in X, A, B and Λ.

(4) The connecting homomorphism δ is a derivation (TODO of what?) defined by

$$\delta^*(\alpha \smile i^*(\beta)) = \alpha \smile \delta^*(\beta).$$