INCIDENCE NUMBERS IN CELLULAR HOMOLOGY

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This problem is set from Bredon [1, No. IV.11.12]. It demonstrates

- for a CW-complex K, the differential β of the chain complex $C_*^{\mathrm{CW}}(K)$ satisfies $\beta^2 = 0$, and thus
- for an $n+1$ cell σ and an $n-1$ cell ω, we've $\sum_\tau [\omega : \tau][\tau : \sigma] = 0$ with τ ranging over all n-cells.

Given. Let K be a CW-complex, with n-skeleton $K^{(n)}$ for $n \geq 0$. Because $K^{(n)}$ contains an open neighborhood around the closed subset $K^{(n-1)}$ that deform retracts onto $K^{(n-1)}$, we know:

- The relative homology $H_*(K^{(n)}, K^{(n-1)})$ is isomorphic to the reduced homology $\tilde{H}_*(K^{(n)}/K^{(n-1)})$.
- The quotient space $K^{(n)}/K^{(n-1)}$ is homeomorphic to the wedge $\vee(I^n/\partial I^{n-1}) \approx \vee S^n$, and thus

$$H_*(K^{(n)}, K^{(n-1)}) \xrightarrow[e_*]{\cong} \tilde{H}_*(K^{(n)}/K^{(n-1)}) \xrightarrow[\text{h.a.}]{\cong} \tilde{H}_*(\vee(I^n/\partial I^{n-1})) \xrightarrow{\cong} \bigoplus_{n\text{-cells of } K^{(n)}} \tilde{H}_*(S^n). \tag{1}$$

We may define a chain complex $C_*^{\mathrm{CW}}(K)$ associated to K as follows:

- Let the chain group $C_n^{\mathrm{CW}}(K)$ be $H_n(K^{(n)}, K^{(n-1)})$. This is the free abelian group (in the nth degree of the graded group) at the end of (1) whose basis is the set of n-cells attached to $K^{(n-1)}$.
- Let the differential $\beta_n \colon C_n^{\mathrm{CW}}(K) \to C_{n-1}^{\mathrm{CW}}(K)$ be the composite

$$\begin{tikzcd} C_n^{\mathrm{CW}}(K) = H_n(K^{(n)}, K^{(n-1)}) \arrow[r, "\delta_n"] \arrow[rr, bend right, "\beta_n"'] & H_{n-1}(K^{(n-1)}) \arrow[r, "j_{n-1}"] & H_{n-1}(K^{(n-1)}, K^{(n-2)}) = C_{n-1}^{\mathrm{CW}}(K). \end{tikzcd} \tag{2}$$

In (2), the boundary map δ_n arises from the long exact sequence for the pair $(K^{(n)}, K^{(n-1)})$, and the map of relative homology groups $j_{n-1} \colon H_{n-1}(K^{(n-1)}) \to H_{n-1}(K^{(n-1)}, K^{(n-2)})$ is induced by the inclusion of skeleta $j \colon (K^{(n-1)}, \varnothing) \hookrightarrow (K^{(n-1)}, K^{(n-2)})$.

From lecture [2, No. 1.11.3], we know δ_n respects the attaching maps; for an n-cell σ with attaching map $f_{\partial\sigma}$,

$$\delta_n[I^n_\sigma] = [f_{\partial\sigma}(\partial I^n_\sigma)].$$

And so, the differential β_n can be described with "incidence numbers" [3, No. 8.5]. For an n-cell σ and an $n-1$ cell τ, define

$$[\tau : \sigma] := \deg\left(\begin{tikzcd} S^{n-1} \arrow[r, "f_{\partial\sigma}"] \arrow[d, phantom, "\wr"] & K^{(n-1)} \arrow[r] \arrow[rrr, bend right, "p_\tau"'] & K^{(n-1)}/K^{(n-2)} \arrow[r, "\approx"] & \vee S^{n-1} \arrow[r, "\text{find } \tau"] & S^{n-1} \arrow[d, phantom, "\wr"] \\ \partial I^n_\sigma & & & & I^{n-1}_\tau/\partial I^{n-1}_\tau \end{tikzcd} \right). \tag{3}$$

To make three comments. First, we take for granted the rule $\sigma \mapsto \sum_\tau [\tau : \sigma]\tau$ on generators σ in $C_n^{\mathrm{CW}}(K)$ extends linearly and *is* the differential β_n in (2). See [1, p. 203]. So write $\beta_n(\sigma) := \sum_\tau [\tau : \sigma]\tau$. Second, all but finitely many terms in the sum $\sum_\tau [\tau : \sigma]\tau$ must be zero. This is because the compact set ∂I^n_σ is

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attached by $f_{\partial\sigma}$ to a *compact* subset of $K^{(n-1)}$. Third, the projection p_τ that "finds" τ in (3) is the unique map $p_\tau\colon K^{(n-1)} \to S^{n-1}$ satisfying:

i. $p_\tau \circ f_\tau = \gamma_{n-1} =$ smash product $\gamma \wedge \cdots \wedge \gamma$ of $n-1$ copies of the quotient map $\gamma\colon I^1 \to S^1$
ii. $p_\tau \circ f_{\tau'} =$ constant map to base point, for $\tau' \neq \tau$.

Now, we almost done setting up results and rehashing definitions needed to make $C_*^{\mathrm{CW}}(K)$ a chain complex. It remains to argue that the differential β is of order 2, i.e., that $\beta^2 = 0$. So consider the following three long exact sequences in relative homology. (This is Ulrike Tillmann's argument [3, No. 8.6].)

(4)
$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(K^{(n+1)}, K^{(n)}) \arrow[r, "\delta_{n+1}"] & H_n(K^{(n)}) \arrow[r] \arrow[dl, equal] & H_n(K^{(n+1)}) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_n(K^{(n)}) \arrow[r, "j_n"'] & H_n(K^{(n)}, K^{(n-1)}) \arrow[r, "\delta_n"] & H_{n-1}(K^{(n-1)}) \arrow[r] \arrow[dl, equal] & \cdots \\
& \cdots \arrow[r] & H_{n-1}(K^{(n-1)}) \arrow[r, "j_{n-1}"'] & H_{n-1}(K^{(n-1)}, K^{(n-2)}) \arrow[r] & \cdots
\end{tikzcd}$$

Notice $\beta_n\beta_{n+1} = (j_{n-1}\delta_n)(j_n\delta_{n+1}) = j_{n-1}(\delta_n j_n)\delta_{n+1} = 0$, as $\delta_n j_n = 0$ by exactness of the middle row.

To prove. Let K be a CW-complex. For all $n+1$ cells σ and $n-1$ cells ω,

(5)
$$\sum_\tau [\omega : \tau][\tau : \sigma] = 0,$$

where τ ranges over all n-cells.

Proof. We require $\beta^2 = 0$, as in (4). We also require $\beta(\sigma) = \sum_\tau [\tau : \sigma]\tau$, as discussed after (3). Whence

$$\begin{aligned}
\beta^2(\sigma) &= \sum_\tau [\tau : \sigma]\beta(\tau) && (\beta \text{ is linear}) \\
&= \sum_\tau [\tau : \sigma] \sum_\omega [\omega : \tau]\omega && (\text{evaluate}) \\
&= \sum_{\forall \tau, \omega} [\omega : \tau][\tau : \sigma]\omega. && (\mathbf{Z} \text{ is a commutative ring})
\end{aligned}$$

$C_{n-1}^{\mathrm{CW}}(K)$ is the free abelian group whose basis is the set of $n-1$ cells in K. So if $\beta^2(\sigma) = 0$, then the coefficient $[\omega : \tau][\tau : \sigma]$ of each $n-1$ cell ω had better be zero. Thus, fixing ω, we conclude $\sum_\tau [\omega : \tau][\tau : \sigma]\omega = 0$. $\square$

Remarks.

i. Here's another way to remember (5) in the case that K is finite. Take the matrices $[\beta_{n+1}]$ and $[\beta_n]$ representing β_{n+1} and β_n with respect to the *finite* bases for $C_{n+1}^{\mathrm{CW}}(K)$ and $C_n^{\mathrm{CW}}(K)$. Because β is order two, $[\beta_n][\beta_{n+1}] = 0$ and (5) follows from matrix multiplication.

ii. If the coefficient $[\tau : \sigma]$ is the "incidence" of σ to τ, then the matrix $[\beta_n]$ suggests itself as the "incidence matrix" of the differential β_n.

iii. But the term "incidence matrix" is typically reserved for the following situation: Take the differential $\beta_1\colon C_1^{\mathrm{CW}}(K) \to C_0^{\mathrm{CW}}(K)$. How does the matrix $[\beta_1]$ describe the *directed graph* whose vertices are 0-cells in $K^{(0)}$ and whose directed edges are oriented 1-cells in $K^{(1)}$?

References

[1] G. E. Bredon, *Topology and Geometry.* New York: Springer-Verlag, 1993.

[2] A. Beaudry, "Math 6220 Class Notes." 2019.

[3] U. Tillmann, "Algebraic Topology Lecture Notes." 2013.