MATH 6220 HOMEWORK 6

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1. (a) Chapter V, Section 7, 284: 5. *If $H_*(X)$ is finitely generated, then*

$$\chi(X) = \sum (-1)^i \dim H_i(X; \Lambda) \quad \textit{for any field } \Lambda.$$

(b) Chapter VI, Section 1, 321: 3. *For spaces X, Y of bounded finite type,*

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

2. Chapter VI, Section 1, 321: 2. *Let X_p be the space resulting from attaching an n-cell to S^{n-1} by a map of degree p. Use the Künneth Theorem to compute the homology of $X_p \times X_q$ for any p, q.*

Note. By a *graded commutative ring*, we will mean a graded abelian group R^* together with a homomorphism of graded abelian groups

$$\mu \colon R^* \otimes R^* \to R^*$$

such that,

- There exists $1 \in R^0$ which is a two sided unit for μ.
- $\mu(a \otimes b) = (-1)^{\deg(a)\deg(b)} \mu(b \otimes a)$.

The cup product gives a graded commutative ring structure on the cohomology of a space. (See Chapter VI, Section 4 and Example 4.12.) ◀

3. (a) *Write down the ring structure of $H^*(S^n)$ and of $H^*(S^n \times S^m)$.*

(b) *We will see later that $H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x]/\langle x^{n+1} \rangle$ for $x \in H^1(\mathbb{R}P^n; \mathbb{Z}/2)$. Use this to prove that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.*

(c) Chapter VI, 334: 3. *Show that any map $S^4 \to S^2 \times S^2$ must induce the zero homomorphism on $H_4(-)$.*

4. Chapter VI, 334: 5. *Any two chain maps $\Phi, \Psi \colon \Delta_*(X) \to \Delta_*(X \otimes X)$ that agree with the diagonal approximation*

$$\Delta(x) = x \otimes x \quad \textit{in the 0th degree}$$

are chain homotopic: $\Phi \simeq \Psi$.

Definition (Alexander–Whitney diagonal approximation)**.** Let $\sigma \colon \Delta_n \to X$ be a singular n-simplex in X. The *Alexander–Whitney diagonal approximation* explicitly computes the image of σ under the chain map $\Delta \colon \Delta_*(X) \to \Delta_*(X) \otimes \Delta_*(X)$ from the *front and back faces* of σ.

$$\Delta\sigma = \sum_{p+q=n} \|\sigma\|^p_{\text{front}} \otimes \|\sigma\|^q_{\text{back}}.$$

Proposition (Computing the cup product)*.* Say f and g are in the cochains with degrees p and q respectively, such that $p+q=n$. Then

$$\begin{aligned}(f \smile g)(\sigma) &= (f \otimes g)(\Delta\sigma)\\ &= (f \otimes g)\left(\sum_{i+j=n} \|\sigma\|^i_{\text{front}} \otimes \|\sigma\|^j_{\text{back}}\right)\\ &= (f \otimes g)(\|\sigma\|^p_{\text{front}} \otimes \|\sigma\|^q_{\text{back}})\\ &= (-1)^{\deg g \deg f} f(\|\sigma\|^p_{\text{front}}) g(\|\sigma\|^q_{\text{back}}) \quad \text{(an element of } \Lambda).\end{aligned}$$