Math 6220 – Introduction - Topology 2

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Disclaimer. The following notes mainly follow *Topology and Geometry*, by Glen E. Bredon. They are based on a combination of that reference and

- *Topology and Geometry*, by Glen E. Bredon
- *An Introduction to Algebraic Topology*, by Joseph J. Rotman
- *A Concise Course in Algebraic Topology*, J. Peter May
- *Algebraic Topology*, A. Hatcher
- *An Introduction to Homological Algebra*, C. A. Weibel.

Chapter 1

Homology Theory (IV in Bredon)

Introduction

Algebraic topology is the study of algebraic invariants of topological spaces.

Example 1.0.1. 1. $\pi_0(X)$ is the set of path components of X

$$\pi_0 \colon \text{Top} \to \text{Sets}$$

Easy to compute for spaces like manifold.

2. $\pi_1(X, x)$ the fundamental group, is $\text{Map}_*(S^1, X)/\simeq$, where Map_* denotes continuous, based point preserving functions, $\simeq$ is the relation of based point preserving homotopy, and $*$-composition of path is the group operation:

$$\pi_1 \colon \text{Top}_* \to \text{Groups}$$

Moderately computable using Van Kampen.

3. For $n > 1$, $\pi_n(X, x)$, the nth homotopy group $\text{Map}_*(S^n, X)/\simeq$

$$\pi_n \colon \text{Top}_* \to \text{AbGroups}$$

Very hard.

4. Homology (and cohomology) $H_n(X)$ are also invariants. For $n \geqslant 0$,

$$H_n \colon \text{Top} \to \text{AbGroups}$$

Very computable compared to the π_ns.

There are many applications. In mathematics it has become a fundamental tool, but it now also is used in industry (Topological Data Analysis).

1.1 Homology Groups

The fundamental group and homotopy groups are defined in terms of maps $S^n \to X$. Similarly, homology is defined based on maps from $\Delta^n \to X$ where Δ^n is an n-dimensional tetrahedron.

Throughout, a *map* between topological spaces will always mean a continuous function unless otherwise specified. Similarly, a map between groups is a group homomorphism.

1.1.1 Simplices

Definition 1.1.1. Let $\{e_n\}_{n\geqslant 0}$ be the standard basis for $\mathbb{R}^\infty$. The *standard p-simplex* is

$$\Delta_p = \{x \in \mathbb{R}^\infty \mid x = \sum_{i=0}^{p} \lambda_i e_i,\ 0 \leqslant \lambda_i \leqslant 1,\ \sum_{i=0}^{p} \lambda_i = 1\}.$$

The λ_i are the *barycentric coordinates* of $x \in \Delta_p$.

Example 1.1.2. $\Delta_0 = \{e_0\}$, Δ_1 is the line segment between e_0 and e_1, Δ_2 is the triangle formed by the convex hull of e_0, e_1, e_2, etc.

Definition 1.1.3. For $v_0, \ldots, v_p \in \mathbb{R}^N$, we let $[v_0, \ldots, v_p]$ (called an *affine p-simplex*) be the map

$$\begin{aligned} \Delta_p &\to \mathbb{R}^N \\ \sum_{i=0}^{p} \lambda_i e_i &\mapsto \sum_{i=0}^{p} \lambda_i v_i. \end{aligned}$$

It is called *degenerate* if the v_i are not affine independent (i.e., $v_1 - v_0, \ldots, v_p - v_0$ are not linearly independent).

Example 1.1.4. $[e_0, e_0, e_1]$ is degenerate.

Definition 1.1.5. The ith face of Δ_p is the map

$$F_i^p \colon [e_0, \ldots, \hat{e}_i, \ldots, e_{p-1}] \colon \Delta_{p-1} \to \Delta_p$$

Example 1.1.6. $[e_1, e_2]$ is F_0^2, $[e_0, e_2]$ is F_1^2 and $[e_0, e_1]$ is F_2^2

Remark 1.1.7. The faces satisfy

$$F_j^{p+1} \circ F_i^p = \begin{cases} [e_0, \ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots, e_p] & i < j \\ [e_0, \ldots, \hat{e}_j, \ldots, \hat{e}_{i+1}, \ldots, e_p] & i \geqslant j. \end{cases}$$

In particular, $F_j^{p+1} \circ F_i^p = F_i^{p+1} \circ F_{j-1}^p$ if $i < j$. This is called a *simplicial identity.*

1.1.2 Singular Chains and Graded Abelian Groups

Definition 1.1.8. If X is a topological space, a *(singular) p-simplex* of X is a map

$$\sigma\colon \Delta_p \to X.$$

We write $\sigma(i) := \sigma(e_i)$ for $0 \leqslant i \leqslant p$. Let $\mathrm{Sing}_p(X)$ denote the set of all singular p-simplices. Let $\Delta_p(X)$ be the free abelian group on $\mathrm{Sing}_p(X)$:

$$\Delta_p(X) = \mathbb{Z}\{\mathrm{Sing}_p(X)\}$$

An element of $\Delta_p(X)$ is called a (singular) p-chain of X, and is of the form

$$\sum_\sigma n_\sigma \sigma$$

for $\sigma \in \mathrm{Sing}_p(X)$, $n_\sigma \in \mathbb{Z}$ with all but finitely many $n_\sigma = 0$.

Remark 1.1.9. Note that $\Delta_p(\varnothing) = 0$.

Example 1.1.10.

1. 0-simplices of X are maps $\sigma\colon \Delta_0 = \{e_0\} \to X$. So the map

   $$\mathrm{ev}\colon \mathrm{Sing}_0(X) \to X, \quad \sigma \mapsto \sigma(0)$$

   is a bijection, which induces an isomorphism

   $$\mathrm{ev}\colon \Delta_0(X) \xrightarrow{\cong} \mathbb{Z}\{X\}.$$

   Under this identification, a 0-chain looks like

   $$\sum_{x\in X} n_x x, \ \ n_x \in \mathbb{Z}, x \in X$$

   and $n_x = 0$ for all but finitely many $x \in X$. Here, "x" represents the simplex $\sigma_x\colon \Delta_0 \to X$ which sends e_0 to x.

2. 1-simplices of X are maps $\sigma\colon \Delta_1 \to X$. Since $\Delta_1 \cong [0,1]$, these are just paths in X.

Definition 1.1.11. A *graded abelian group* is a collection of abelian groups C_p indexed by the integers $p \in \mathbb{Z}$. A map $f\colon \{C_p\} \to \{D_p\}$ of graded abelian groups is a sequence of homomorphisms $f_p\colon C_p \to D_p$.

Example 1.1.12. Let $\Delta_p(X) = 0$ for $p < 0$. Then $\{\Delta_p(X)\}$ is a graded abelian group. Further, for $f\colon X \to Y$ be a map of topological spaces. Then

$$f_\Delta\colon \Delta_*(X) \to \Delta_*(Y)$$

given by

$$f_\Delta\left(\sum_\sigma n_\sigma \sigma\right) = \sum_\sigma n_\sigma f\sigma$$

is homomorphism of graded abelian groups.

Remark 1.1.13. Its clear that $f_\Delta \circ g_\Delta = (f \circ g)_\Delta$ and that $1_\Delta = 1$. Letting $\Delta_p(f) = f_\Delta$, we get a *functor*

$$\Delta_p(-)\colon \mathrm{Top} \to \mathrm{Ab}_*.$$

1.1.3 Boundary Map and Chain Complexes

Recall 1.1.14. If S is a set and A is an abelian group, then there is a bijection

$$\{\text{Functions } S \to A\} \Leftrightarrow \{\text{Homomorphisms } \mathbb{Z}\{S\} \to A\}$$

which sends $g\colon S \to A$ to

$$g\left(\sum_{s\in S} n_s s\right) := \sum_{s\in S} n_s g(s).$$

I'll call this *extending linearly.*

Definition 1.1.15. If $\sigma \in \mathrm{Sing}_p(X)$ is a p-simplex, then

$$\sigma^{(i)} = \sigma \circ F_i^p$$

is the *ith face* of σ. The *boundary* of a σ is the $p-1$-chain

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i \sigma^{(i)}.$$

Extending linearly gives a homomorphism

$$\partial_p\colon \Delta_p(X) \cong \mathbb{Z}\{\mathrm{Sing}_p(X)\} \to \Delta_{p-1}(X)$$

given by

$$\partial_p\left(\sum_\sigma n_\sigma \sigma\right) = \sum_\sigma n_\sigma \partial_p \sigma$$

Remark 1.1.16. Let $\sigma \in \Delta_1(X)$. Under the identification $\mathrm{ev} : \Delta_0(X) \cong \mathbb{Z}\{X\}$ of Example 1.1.10,

$$\partial_1(\sigma) = \sigma(1) - \sigma(0).$$

Proposition 1.1.17. $\partial_p \circ \partial_{p+1} = 0$

Proof. It's enough to check on the generators $\sigma\colon \Delta^{p+1} \to \Delta_p$. We have

$$\begin{aligned}
\partial_p \circ \partial_{p+1}(\sigma) &= \partial_p \sum_{j=0}^{p+1} (-1)^j \sigma \circ F_j \\
&= \sum_{j=0}^{p+1} (-1)^j \sum_{i=0}^{p} (-1)^i \sigma \circ F_j \circ F_i \\
&= \sum_{0\leqslant i<j\leqslant p+1} (-1)^{i+j} \sigma \circ F_j \circ F_i + \sum_{0\leqslant j\leqslant i\leqslant p} (-1)^{i+j} \sigma \circ F_j \circ F_i \\
&= \sum_{0\leqslant i<j\leqslant p+1} (-1)^{i+j} \sigma \circ F_i \circ F_{j-1} + \sum_{0\leqslant j\leqslant i\leqslant p} (-1)^{i+j} \sigma \circ F_j \circ F_i \\
&= \sum_{0\leqslant i\leqslant k\leqslant p} (-1)^{i+k+1} \sigma \circ F_i \circ F_k + \sum_{0\leqslant j\leqslant i\leqslant p} (-1)^{i+j} \sigma \circ F_j \circ F_i \qquad (k = j-1) \\
&= 0.
\end{aligned}$$

$\square$

We have the following structure:

$$\begin{tikzcd}
\dots \arrow[r] & \Delta_{p+1}(X) \arrow[r, "\partial_{p+1}"] \arrow[rr, bend right, "0"'] & \Delta_p(X) \arrow[r, "\partial_p"] & \Delta_{p-1}(X) \arrow[r] & \dots \arrow[r] & \Delta_2(X) \arrow[r, "\partial_2"] \arrow[rr, bend right, "0"'] & \Delta_1(X) \arrow[r, "\partial_1"] & \Delta_0(X) \arrow[r] & 0
\end{tikzcd}$$

Definition 1.1.18. A *chain complex* is a graded abelian group $\{C_p\}$ with a sequence of homomorphisms $\partial_p \colon C_p \to C_{p-1}$ such that $\partial_p \circ \partial_{p+1} = 0$.

$$\begin{tikzcd}
\dots \arrow[r] & C_{p+1} \arrow[r, "\partial_{p+1}"] \arrow[rr, bend right, "0"'] & C_p \arrow[r, "\partial_p"] & C_{p-1} \arrow[r] & \dots
\end{tikzcd}$$

The maps ∂_i are called the *differentials* or *boundary operators.* We use the notation $C_* = (\{C_p\}, \{\partial_p\})$, and often drop the indices and write $\partial = \partial_p$.

A map $f \colon C_* \to D_*$ of chain complexes is a map of graded abelian groups such that the diagram

$$\begin{tikzcd}
\dots \arrow[r] & C_{p+1} \arrow[r, "\partial_{p+1}"] \arrow[d, "f_{p+1}"] & C_i \arrow[r, "\partial_p"] \arrow[d, "f_p"] & C_{p-1} \arrow[r] \arrow[d, "f_{p-1}"] & \dots \\
\dots \arrow[r] & D_{p+1} \arrow[r, "\partial_{p+1}"] & D_p \arrow[r, "\partial_i"] & D_{p-1} \arrow[r] & \dots
\end{tikzcd}$$

commutes, abbreviated:

$$f \circ \partial = \partial \circ f.$$

These are called *chain maps.*

Example 1.1.19. $\Delta_*(X) = (\{\Delta_p(X)\}, \{\partial_p\})$ is the *singular chain complex of* X.

Proposition 1.1.20. *For $f \colon X \to Y$ a map of topological spaces, f_Δ is a chain map, that is*

$$f_\Delta \circ \partial = \partial \circ f_\Delta.$$

Proof. We show that $f_\Delta \circ \partial(\sigma) = \partial \circ f_\Delta(\sigma)$ on the generators $\sigma \in \mathrm{Sing}_p(X)$, which implies that they agree on all of $\Delta_p(X)$:

$$\begin{aligned}
f_\Delta(\partial\sigma) &= \sum_i (-1)^i f \circ \sigma^{(i)} \\
&= \sum_i (-1)^i f \circ \sigma \circ F_i^p \\
&= \sum_i (-1)^i (f \circ \sigma)^{(i)} \\
&= \partial(f_\Delta(\sigma)).
\end{aligned}$$

$\square$

Remark 1.1.21. In fact, this is a functor

$$\Delta_*(-) \colon \mathrm{Top} \to \mathrm{Ch}\,.$$

1.1.4 Homology and Functorial Properties

Definition 1.1.22. Let C_* be a chain complex. Let $c \in C_p$.

- If $c \in \ker(\partial_p)$, we call c a *p-cycle* and let
$$Z_p(C_*) = \ker(\partial_p) \subseteq C_p.$$
- If $c \in \operatorname{im}(\partial_{p-1})$, we call c a *p-boundary* and let
$$B_p(C_*) = \operatorname{im}(\partial_{p-1}) \subseteq Z_p(C_*).$$
- The *p homology group* of C_* is the group
$$H_p(C_*) = \frac{Z_p(C_*)}{B_p(C_*)} = \frac{\ker(\partial_p)}{\operatorname{im}(\partial_{p-1})}.$$
The book denotes by $[\![c]\!] \in H_p(C_*)$ the residue class of $c \in Z_p(C_*)$. It is called the *homology class* of c.
- If $c - c' \in B_p(C_*)$, c and c' are called *homologous*, denoted by $c \sim c'$. In this case, $[\![c]\!] = [\![c']\!]$.

We write $H_*(C_*)$ for the *graded abelian group* formed by the collection $\{H_p(C_*)\}$. We can also view this as a chain complex with zero differentials.

Definition 1.1.23. Let X be a topological space. The *p singular homology group* of X is
$$H_p(X) = H_p(\Delta_*(X)).$$
We also let
$$Z_p(X) = Z_p(\Delta_*(X)) \qquad\qquad B_p(X) = B_p(\Delta_*(X)).$$

Example 1.1.24. Let $X = *$ be the one point space. There is a unique map
$$\sigma_p \colon \Delta_p \to *$$
$\Delta_p(X) \cong \mathbb{Z}\{\sigma_p\}$ for every p. Further,
$$\partial_p(\sigma_p) = \sum_{i=0}^{p} (-1)^i \sigma_p^{(i)} = \sum_{i=0}^{p} (-1)^i \sigma_{p-1} = \begin{cases} 0 & p \text{ odd} \\ \sigma_{p-1} & p \text{ even.} \end{cases}$$
So the singular chain complex looks like
$$\dots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$
Therefore,
$$H_p(X) = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p > 0. \end{cases}$$

Exercise 1.1.25. Let $X = \bigcup_{i\in I} X_i$ where the X_i are the (disjoint) path components of X. Prove that

$$H_n(X) \cong \bigoplus_{i\in I} H_n(X_i).$$

Proposition 1.1.26. *A chain map $f\colon C_* \to D_*$ induces a map*

$$f_*\colon H_*(C_*) \to H_*(D_*)$$

given by $f_*(\llbracket c \rrbracket) = \llbracket f(c) \rrbracket$.

Proof. Let c be a cycle in C_p. Then

$$\partial(f(c)) = f(\partial(c)) = f(0) = 0.$$

So f induces a map

$$f\colon Z_p(C_*) \to Z_p(D_*).$$

Since $f(\partial(c)) = \partial(f(c))$, f also induces a map

$$f\colon B_p(C_*) \to B_p(D_*).$$

It follows that there is an induced map

$$f_*\colon H_p(C) = Z_p(C_*)/B_*(C_*) \to H_p(D) = Z_p(D_*)/B_p(D_*).$$

$\square$

Exercise 1.1.27. Check that $f_* \circ g_* = (f \circ g)_*$ and $1_* = 1$.

Remark 1.1.28. In fact, we have a homology is a *functor*:

$$H_*(-)\colon \mathrm{Ch} \to \mathrm{Ab}_*$$

where Ch is the category of chain complexes and Ab_* of graded abelian groups.

Remark 1.1.29. Since f_Δ is a chain map

$$f_\Delta\colon \Delta_*(X) \to \Delta_*(Y)$$

it induce a homomorphisms

$$f_*\colon H_*(X) \to H_*(Y).$$

Singular homology is a thus functor

$$H_*(-)\colon \mathrm{Top} \to \mathrm{Ab}_*$$

It is the composition of the functors

$$\mathrm{Top} \xrightarrow{\Delta_*(-)} \mathrm{Ch} \xrightarrow{H_*(-)} \mathrm{Ab}_*$$

Corollary 1.1.30. *Since functors preserve isomorphisms, $f\colon X \to Y$ is a homeomorphism, then $H_*(f)\colon H_*(X) \to H_*(Y)$ is an isomorphism.*

Later, we will discuss the following axiom, but use it in the meantime.

Theorem 1.1.31 (Homotopy Axiom)**.** *Let $f \simeq g\colon X \to Y$ be homotopic maps. Then $f_* = g_*\colon H_*(X) \to H_*(Y)$.*

Corollary 1.1.32. *If $X \simeq \mathrm{pt}$ is contractible, then $H_*(X) \cong H_*(\mathrm{pt})$.*

Proof. Since X is contractible, $\mathrm{id}\colon X \to X$ is homotopic to a map of the form

$$f\colon X \to \{x\} \hookrightarrow X.$$

So, $H_*(f) = \mathrm{id}_{H_*(X)}$. Since $H_*(f)$ factors through $H_*(\mathrm{pt})$, $\mathrm{id}_{H_p(X)} = 0$ for all $p > 0$, so $H_p(X) = 0$ for $p > 0$. Since a contractible space is path connected, $H_*(X) \cong H_*(\mathrm{pt})$. □

1.2 Zeroth Homology Group

Definition 1.2.1. The homomorphism $\epsilon\colon \Delta_0(X) \cong_{\mathrm{ev}} \mathbb{Z}\{X\} \to \mathbb{Z}$ defined by

$$\epsilon\left(\sum_{x \in X} n_x x\right) = \sum_{x \in X} n_x$$

is called the *augmentation.*

Remark 1.2.2. If $X \neq \varnothing$, the map ϵ is surjective.

For $\sigma \in \mathrm{Sing}_1(X)$,

$$\epsilon(\partial_1(\sigma)) = \epsilon(\sigma(1) - \sigma(0)) = 1 - 1 = 0$$

so the composite

$$\Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

is zero, (as it is zero on all the generators of $\Delta_1(X)$). So we get an induced homomorphism

$$\epsilon_*\colon H_0(X) = \Delta_0(X)/\operatorname{im}(\partial_1) \to \mathbb{Z}$$

which is also called the augmentation.

Notation. For $X \neq \varnothing$ a path connected space, fix $x_0 \in X$ and let $\lambda_x\colon \Delta_1 \to X$ a path from x_0 to x, with the convention that λ_{x_0} is the constant path. Note that this gives a group homomorphism

$$\lambda\colon \Delta_0(X) \cong \mathbb{Z}\{X\} \to \Delta_1(X)$$

determined by $\lambda(x) = \lambda_x$.

Theorem 1.2.3. *Let $X \neq \varnothing$ be a path connected space. Then ϵ_* is an isomorphism.*

Proof. Since ϵ_* is clearly surjective, it's enough to show that it is injective.

Suppose $\epsilon_*(\llbracket c \rrbracket) = 0$ for $c = \sum n_x x$, so that $\sum n_x = 0$. Then

$$c - \partial_1 \left(\sum n_x \lambda_x \right) = \sum n_x x - \sum n_x (x - x_0) = \left(\sum n_x \right) x_0 = 0.$$

So c is a boundary, i.e., $\llbracket c \rrbracket = 0$. □

Corollary 1.2.4. *Let $X \neq \varnothing$ be a path connected space. Then $\llbracket x \rrbracket = \llbracket y \rrbracket$ for any $x, y \in X$, and $H_0(X) \cong \mathbb{Z}$, generated by $\llbracket x \rrbracket$ for any point $x \in X$.*

Corollary 1.2.5. *Let $X = \bigcup_{i \in I} X_i$ for X_i path connected and non-empty. Let $x_i \in X_i$. Then*

$$H_0(X) \cong \bigoplus_{i \in I} \mathbb{Z}\{\llbracket x_i \rrbracket\}.$$

Exercise 1.2.6. Any map $f \colon X \to Y$ between path connected topological spaces induces an isomorphism $f_* \colon H_0(X) \xrightarrow{\cong} H_0(Y)$.

Definition 1.2.7. The *reduced* homology of X, denoted $\widetilde{H}_*(X)$ is the homology of the chain complex

$$\dots \longrightarrow \Delta_2(X) \xrightarrow{\partial_2} \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \longrightarrow \dots$$

In particular, for $X = \bigcup_{i \in I} X_i$ for X_i path connected and non-empty and $x_{i_0} \in X_{i_0}$ for any fixed $i_0 \in I$, then

$$\widetilde{H}_p(X) = \begin{cases} H_p(X) & p > 0 \\ \bigoplus_{i \in I \setminus \{i_0\}} \mathbb{Z}\{\llbracket x_i \rrbracket - \llbracket x_{i_0} \rrbracket\} & p = 0 \\ 0 & p < 0. \end{cases}$$

Note that if X is path connected, $\widetilde{H}_0(X) = 0$.

1.3 First Homology Group

In this section, $X \neq \varnothing$ is a path connected space and $x_0 \in X$. Let $\pi_1 = \pi_1(X, x_0)$ and

$$\widetilde{\pi}_1 = \widetilde{\pi}_1(X, x_0) = \pi_1(X, x_0)/[\pi_1, \pi_1]$$

be its abelianization. The main theorem we prove here is

Theorem 1.3.1 (Hurewicz)**.** *The map*

$$\phi = \phi_X \colon \pi_1(X, x_0) \to H_1(X)$$

which sends the homotopy class $[\gamma] \in \pi_1(X, x_0)$ of a loop $\gamma \colon \Delta_1 \to X$ to the homology class $\llbracket \gamma \rrbracket$ induces an homomorphism

$$\phi_* \colon \widetilde{\pi}_1(X, x_0) \to H_1(X).$$

Furthermore, ϕ_* is an isomorphism if X is path connected.

Corollary 1.3.2. *There are isomorphisms*

$$H_1(S^1) \cong \mathbb{Z}, \qquad H_1(S^n) \cong 0, \quad n \geqslant 2 \qquad H_1(\mathbb{R}P^n) \cong \mathbb{Z}/2, \quad n \geqslant 2.$$

Exercise 1.3.3. Use the Hurewicz theorem to solve the following problems:

(a) Compute $H_1(K)$ for the Klein bottle K.

(b) Compute H_1 of $X = \prod_{j\in J} X_j$ for topological spaces X_j in terms of $H_1(X_j)$.

(c) Let X_i with base points $x_i \in X_i$. Suppose that there are open sets $x_i \in U_i \subseteq X_i$ such that x_i is a deformation retract of U_i. Show that

$$H_1\left(\bigvee_{i\in I} X_i\right) \cong \bigoplus_{i\in I} H_1(X_i).$$

The fact that loops $\gamma\colon \Delta^1 \to X$ are 1-cycles implies that $[\![\gamma]\!] \in H_1(X)$. Since γ is a loop, it factors

$$\begin{tikzcd}
\Delta_1 \arrow[r, "\gamma"] \arrow[d, "\sigma_q"] & X \\
\Delta_1/(e_0 \sim e_1) \arrow[ur, dashed, "\overline{\gamma}"'] &
\end{tikzcd}$$

where σ_q is the quotient map. If $\gamma_1 \simeq \gamma_2$ through based point preserving homotopies, then $\overline{\gamma}_1 \simeq \overline{\gamma}_2$. By the homotopy axiom,

$$H_*(\overline{\gamma}_1) = H_*(\overline{\gamma}_2).$$

In particular,

$$[\![\gamma_1]\!] = H_*(\overline{\gamma}_1)([\![\sigma_q]\!]) = H_*(\overline{\gamma}_2)([\![\sigma_q]\!]) = [\![\gamma_2]\!].$$

So, ϕ is well defined.

Next, we show that ϕ respects composition.

Lemma 1.3.4. *If f and g are paths with $f(1) = g(0)$, then*

$$f + g - f * g$$

is a boundary.

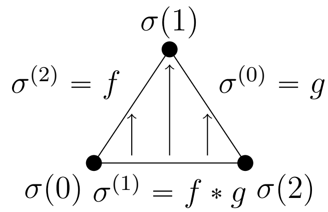
Proof. Define $\sigma\colon \Delta_2 \to X$ so that

- $f = \sigma^{(2)}$
- $g = \sigma^{(0)}$
- $f * g = \sigma^{(1)}$

- σ is constant on the line segments perpendicular to the image of $[e_0, e_2]$

Then

$$\partial_2(\sigma) = g - f * g + f$$



□

Corollary 1.3.5. *We have*

$$\phi([f][g]) = \phi([f * g]) = [\![f * g]\!] = [\![f + g]\!] = [\![f]\!] + [\![g]\!] = \phi([f]) + \phi([g]).$$

That is, ϕ is a group homomorphism to an abelian group, and thus factors through $\widetilde{\pi}_1$.

Next, we prove that ϕ_* is an isomorphism when X is path connected. As before, let $\lambda \colon \Delta_0(X) \to \Delta_1(X)$ be the homomorphism

$$\sum n_x x \mapsto \lambda_{\sum n_x x} := \sum n_x \lambda_x$$

where $\lambda_x \colon \Delta_1 \to X$ is a path from x_0 to x and λ_{x_0} is the constant path. For $\sigma \in \mathrm{Sing}_1(X)$, let

$$\psi(\sigma) = [\lambda_{\sigma(0)} * \sigma * \lambda_{\sigma(1)}^{-1}] \in \widetilde{\pi}_1(X, x_0).$$

This extends to a map

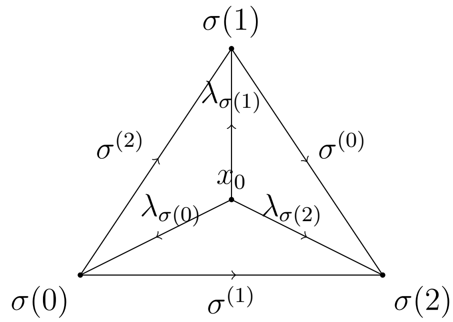
$$\psi \colon \Delta_1(X) \to \widetilde{\pi}_1(X, x_0)$$

Further, $B_1(X) \subseteq \ker(\psi)$ since, for $\sigma \in \mathrm{Sing}_2(X)$,

$$\begin{aligned}
\psi(\partial\sigma) &= \psi(\sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}) \\
&= [\lambda_{\sigma(1)} * \sigma^{(0)} * \lambda_{\sigma(2)}^{-1} * (\lambda_{\sigma(0)} * \sigma^{(1)} * \lambda_{\sigma(2)}^{-1})^{-1} * \lambda_{\sigma(0)} * \sigma^{(2)} * \lambda_{\sigma(1)}^{-1}] \\
&= [\lambda_{\sigma(1)} * \sigma^{(0)} * (\sigma^{(1)})^{-1} * \sigma^{(2)} * \lambda_{\sigma(1)}^{-1}] \\
&= 0.
\end{aligned}$$

So we get a homomorphism

$$\psi_* \colon H_1(X) \to \widetilde{\pi}_1(X, x_0)$$



Lemma 1.3.6. *ψ_* is the two sided inverse of ϕ_*.*

Proof. Let $[f] \in \widetilde{\pi}_1(X, x_0)$. Then

$$\psi_*\phi_*([f]) = \psi_*([\![f]\!]) = [\lambda_{x_0} * f * \lambda_{x_0}^{-1}] = [f]$$

since λ_{x_0} is constant.

On the other hand, if $\sigma \in \mathrm{Sing}_1(X)$, then

$$\begin{aligned} \phi_*(\psi(\sigma)) &= \phi_*([\lambda_{\sigma(0)} * \sigma * \lambda_{\sigma(1)}^{-1}]) \\ &= [\![\lambda_{\sigma(0)} * \sigma * \lambda_{\sigma(1)}^{-1}]\!] \\ &= [\![\sigma - (\lambda_{\sigma(1)} - \lambda_{\sigma(0)})]\!] \\ &= [\![\sigma - \lambda_{\partial\sigma}]\!]. \end{aligned}$$

Therefore, if $\partial c = 0$, then

$$\phi_*(\psi_*([\![c]\!])) = \phi_*(\psi(c)) = [\![c - \lambda_{\partial c}]\!] = [\![c]\!]. \qquad \square$$

Theorem 1.3.7. *The Hurewicz map is natural, i.e., if $f\colon X \to Y$ is a map of topological spaces such that $f(x_0) = y_0$, then the following diagram commutes:*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_\#"] \arrow[d, "\phi_X"] & \pi_1(Y, y_0) \arrow[d, "\phi_Y"] \\ H_1(X) \arrow[r, "f_*"] & H_1(Y). \end{tikzcd}$$

1.4 Homological Algebra

1.4.1 Long Exact Sequence on Homology

Definition 1.4.1. A diagram of abelian groups

$$A \xrightarrow{i} B \xrightarrow{j} C \tag{1.1}$$

is *exact (at B)* if

$$\ker(j) = \operatorname{im}(i).$$

A diagram

$$\dots \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow A_{n-2} \longrightarrow A_{n-3} \longrightarrow \dots$$

which is exact at every term is called an *exact sequence* (or *long exact sequence*).

If the diagram

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is exact, it is called a *short exact sequence*.

Example 1.4.2. The sequence

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$$

is exact.

Remark 1.4.3. The map i is injective if and only if

$$0 \longrightarrow A \xrightarrow{i} B$$

is exact (at A) and j is surjective if and only if

$$B \xrightarrow{j} C \longrightarrow 0$$

is exact (at C). Finally, $i \colon A \to B$ is an isomorphisms if and only if

$$0 \longrightarrow A \xrightarrow{i} B \longrightarrow 0$$

is an exact sequence.

Definition 1.4.4. A sequence

$$A_* \xrightarrow{i} B_* \xrightarrow{j} C_*$$

of chain complexes is exact (at B_*) if it is exact *levelwise*. That is,

$$A_p \xrightarrow{i_p} B_p \xrightarrow{j_p} C_p$$

is exact at B_p for each $p \in \mathbb{Z}$. The same holds for the definition of (long) exact sequences and short exact sequences.

Theorem 1.4.5. *Let*

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \longrightarrow 0$$

be a short exact sequence of chain complexes. There is a homomorphism

$$\delta_* \colon H_p(C_*) \to H_{p-1}(A_*)$$

called the connecting homomorphism, *given by*

$$\delta_*([\![c]\!]) = [\![i^{-1} \circ \partial^B \circ j^{-1}(c)]\!],$$

so that

$$\dots \xrightarrow{\delta_*} H_p(A_*) \xrightarrow{i_*} H_p(B_*) \xrightarrow{j_*} H_p(C_*) \xrightarrow{\delta_*} H_{p-1}(A_*) \xrightarrow{i_*} \dots$$

is a long exact sequence.

This theorem has a long and tedious proof. The most important part for applications is to remember how the map δ_* is defined. So I'll focus mostly on that.

Proof. The proof of this theorem is called a *diagram chase*, and the relevant diagram is

$$\begin{tikzcd}
0 \arrow[r] & A_{p+1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{p+1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{p+1} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_p \arrow[r, "i"] \arrow[d, "\partial"] & B_p \arrow[r, "j"] \arrow[d, "\partial"] & C_p \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{p-1} \arrow[r, "i"] & B_{p-1} \arrow[r, "j"] & C_{p-1} \arrow[r] & 0
\end{tikzcd}$$

First, fix $[\![c]\!] \in H_p(C)$. Then $\partial c = 0$. Since j is surjective, we choose b so that $j(b) = c$. But then

$$j\partial(b) = \partial j(b) = \partial c = 0,$$

so $\partial(b) = i(a)$ for $a \in A_{p-1}$. Since

$$i(\partial(a)) = \partial(i(a)) = \partial^2 b = 0$$

and i is injective, $\partial(a) = 0$. We let

$$\delta_*([\![c]\!]) := [\![a]\!].$$

Exercise 1.4.6. Check that it doesn't depend on the choice of lift b such that $j(b) = c$.

Solution. If $j(b') = c$, then $b' = b + i(a')$. The procedure

$$\partial(b') = \partial(b + i(a')) = \partial(b) + i(\partial(a')) = i(a + \partial(a'))$$

so that

$$\delta_*([\![c]\!]) := [\![a + \partial(a')]\!] = [\![a]\!].$$

It also doesn't depend on the choice of representative cycle c. If $c' = c + \partial(c'')$ so that $[\![c']\!] = [\![c]\!]$, then for b'' such that $j(b'') = c''$,

$$j(b + \partial(b'')) = c + \partial(c''),$$

so that the next step gives

$$\partial(b + \partial(b'')) = \partial(b) = i(a).$$

So, δ_* is well defined.

Exercise 1.4.7. Check that δ_* is a homomorphism.

Finally, we need to show exactness. We prove exactness at $H_p(C_*)$. That is, we prove that

$$H_p(B_*) \xrightarrow{j_*} H_p(C_*) \xrightarrow{\delta_*} H_{p-1}(A_*)$$

is exact.

Suppose that $[\![c]\!] \in \ker(\delta_*)$. Then $a = \partial(a')$, so that

$$\partial(b) = i(a) = i(\partial(a')) = \partial(i(a')).$$

So $b' = b - i(a')$ is a cycle, that is, $b' \in Z_p(B) = \ker(\partial)$. Therefore,

$$c = j(b) = j(i(a') + b') = j(b')$$

and $j_*([\![b']\!]) = [\![c]\!]$.

Suppose that $[\![c]\!] = j_*([\![b']\!])$ for some $[\![b']\!] \in H_p(B)$. Then we can choose

$$b = b' + \partial(b'')$$

for some $b'' \in B_{p+1}$ and $b' \in Z_p(B)$. So,

$$\partial(b) = \partial(b' + \partial(b'')) = 0$$

so $i(0) = \partial(b)$. Hence,

$$\delta_*([\![c]\!]) = 0.$$

Exercise 1.4.8. Prove exactness at $H_p(B_*)$ and $H_p(A_*)$.

$\square$

Exercise 1.4.9. Prove that the morphism δ_* is natural. That is, if

$$\begin{tikzcd}
0 \arrow[r] & A_* \arrow[r, "i"] \arrow[d, "f"] & B_* \arrow[r, "j"] \arrow[d, "g"] & C_* \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A'_* \arrow[r, "i"] & B'_* \arrow[r, "j"] & C'_* \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram with exact rows and $\delta'_* \colon H_*(C'_*) \to H_{*-1}(A'_*)$ the connecting homomorphism for the bottom row, then

$$\begin{tikzcd}
H_*(C_*) \arrow[r, "\delta_*"] \arrow[d, "h_*"] & H_{*-1}(A_*) \arrow[d, "f_*"] \\
H_*(C'_*) \arrow[r, "\delta'_*"] & H_{*-1}(A'_*)
\end{tikzcd}$$

commutes.

1.4.2 Homological Algebra Continued

First, some remarks from last class

Remark 1.4.10. (a) Let $f\colon A_* \to B_*$ be a chain map. Then f is an isomorphism (i.e. has a two sided inverse chain map g) if and only if $f_p\colon A_p \to B_p$ is an isomorphism for all $p \in \mathbb{Z}$. Note that an isomorphism of chain complexes induces an isomorphism on homology since homology is a functor.

(b) If $A_* = \bigoplus_i (A_i)_*$, then $H_*(A) \cong \bigoplus_i H_*((A_i)_*)$.

1.4.3 Relative Homology Groups

Definition 1.4.11. A pair of spaces (X, A) is a topological space X together with a subspace $A \subseteq X$. A map of pairs $f\colon (X, A) \to (Y, B)$ is a continuous function $f\colon X \to Y$ such that $f(A) \subseteq B$.

Definition 1.4.12. A subcomplex B_* of C_* is a sequence $B_p \subseteq C_p$ of subgroups such that $\partial_p(B_p) \subseteq B_{p-1}$. If B_* is a subcomplex, then the ∂_p induces a map $C_p/B_p \to C_{p-1}/B_{p-1}$. The quotient complex is C_*/B_* with the induced boundary map.

For example if A is a subspace of X, then $\Delta_*(A)$ is a subcomplex of $\Delta_*(X)$.

Definition 1.4.13. Let (X, A) be a pair of spaces. The *relative chains complex* is

$$\Delta_*(X, A) = \Delta_*(X)/\Delta_*(A).$$

The relative homology is

$$H_p(X, A) := H_p(\Delta_*(X, A)).$$

Theorem 1.4.14. *If (X, A) is a pair, there is a long exact sequence of homology groups*

$$\dots \xrightarrow{\delta_*} H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{j_*} H_p(X, A) \xrightarrow{\delta_*} H_{p-1}(A) \xrightarrow{i_*} \dots$$

Proof. This is the long exact sequence associated to

$$0 \to \Delta_*(A) \to \Delta_*(X) \to \Delta_*(X, A) \to 0.$$

□

1.4.4 Homology with coefficients and Cohomology

There is a functor

$$\mathrm{Hom}_{\mathrm{Ab}}(-,-)\colon \mathrm{Ab}^{\mathrm{op}} \times \mathrm{Ab} \to \mathrm{Ab},$$

which maps (A, B) to the abelian group $\mathrm{Hom}_{\mathrm{Ab}}(A, B)$ (group structure is given by adding values). If $\phi\colon A \to B$, $f\colon C \to A$ and $g\colon B \to D$, then

$$\mathrm{Hom}_{\mathrm{Ab}}(f, g)(\phi) = g \circ \phi \circ f.$$

There is also functor

$$(-) \otimes (-) \colon \text{Ab} \times \text{Ab} \to \text{Ab}$$

which takes (A, B) to

$$A \otimes B := \{a \otimes b : a \in A,\ b \in B\}/\sim$$

where the equivalence relation is

$$\begin{aligned} (a_1 + a_2) \otimes b &= a_1 \otimes b + a_2 \otimes b \\ a \otimes (b_1 + b_2) &= a \otimes b_1 + a \otimes b_2. \end{aligned}$$

For $f \colon A \to C$ and $g \colon C \to D$, we have a map

$$f \otimes g \colon A \otimes B \to C \otimes D$$

given by

$$f \otimes g(a \otimes b) = f(a) \otimes f(b).$$

This has the universal property that there are isomorphisms of abelian groups

$$\text{Hom}_{\text{Ab}}(A \otimes B, C) \cong \text{Bilin}_{\text{Ab}}(A \times B, C) \cong \text{Hom}_{\text{Ab}}(A, \text{Hom}_{\text{Ab}}(B, C)).$$

Definition 1.4.15. Let G be an abelian group and C_* be a chain complex:

(a) The *homology of C_* with coefficients in G*, denoted $H_*(C_*; G)$, is the homology of the chain complex $G \otimes C_*$, whose pth group is $G \otimes C_p$ and boundary $\partial_p^{C_*, G} = \text{id}_G \otimes \partial_p$.

(b) The *cohomology of C_* with coefficients in G*, denoted $H^*(C_*; G)$, is the homology of the cochain complex $\text{Hom}_{\text{Ab}}(C_*, G)$, whose pth group is $\text{Hom}_{\text{Ab}}(C_p, G)$ and boundary $\partial^p_{C_*, G} = \text{Hom}_{\text{Ab}}(\partial_p, \text{id}_G)$.

(c) The *homology of X with coefficients in G* is $H_*(X; G) := H_*(\Delta_*(X); G)$.

(d) The *cohomology of X with coefficients in G* is $H^*(X; G) = H^*(\Delta_*(X); G)$. We also write $H^*(X) = H^*(X; \mathbb{Z})$ and call it the cohomology of X.

Remark 1.4.16. Recall that

$$\Delta_*(X, A) = \Delta_*(X)/\Delta_*(A).$$

Note that

$$\text{Sing}_p(X) = \text{Sing}_p(A) \sqcup \{\sigma \in \text{Sing}_p(X) : \sigma(\Delta_p) \nsubseteq A\}.$$

Furthermore,

$$\mathbb{Z}\{\sigma \in \text{Sing}_p(X) : \sigma(\Delta_p) \nsubseteq A\} \to \Delta_p(X) \to \Delta_p(X, A)$$

is an isomorphism.

So, the exact sequence

$$0 \to \Delta_*(A) \xrightarrow{i} \Delta_*(X) \xrightarrow{j} \Delta_*(X,A) = \Delta_*(X)/\Delta_*(A) \to 0$$

is level-wise split.

It follows that

$$0 \to G \otimes \Delta_*(A) \to G \otimes \Delta_*(X) \to G \otimes \Delta_*(X,A) \to 0$$

and

$$0 \to \mathrm{Hom}_{\mathrm{Ab}}(\Delta_*(X,A),G) \to \mathrm{Hom}_{\mathrm{Ab}}(\Delta_*(X),G) \to \mathrm{Hom}_{\mathrm{Ab}}(\Delta_*(A),G) \to 0$$

are exact sequences (since they are also split), so there are long exact sequences

$$\ldots \xrightarrow{\delta_*} H_p(A;G) \xrightarrow{i_*} H_p(X;G) \xrightarrow{j_*} H_p(X,A;G) \xrightarrow{\delta_*} H_{p-1}(A;G) \xrightarrow{i_*} \ldots$$

and

$$\ldots \xrightarrow{\delta^*} H^p(X,A;G) \xrightarrow{i^*} H^p(X;G) \xrightarrow{j^*} H^p(A;G) \xrightarrow{\delta^*} H^{p+1}(X,A;G) \xrightarrow{i^*} \ldots$$

Similarly, if

$$0 \to G \to G' \to G'' \to 0$$

is an exact sequence of abelian groups, since $\Delta_p(X)$ is free, then

$$0 \to G \otimes \Delta_p(X) \to G' \otimes \Delta_p(X) \to G'' \otimes \Delta_p(X) \to 0$$

and

$$0 \to \mathrm{Hom}(\Delta_p(X),G) \to \mathrm{Hom}(\Delta_p(X),G') \to \mathrm{Hom}(\Delta_p(X),G'') \to 0$$

are exact (check it!). So, there are long exact sequences

$$\ldots \xrightarrow{\delta_*} H_p(X;G) \xrightarrow{i_*} H_p(X;G') \xrightarrow{j_*} H_p(X;G'') \xrightarrow{\delta_*} H_{p-1}(X;G) \xrightarrow{i_*} \ldots$$

and

$$\ldots \xrightarrow{\delta^*} H^p(X;G) \xrightarrow{i^*} H^p(X;G') \xrightarrow{j^*} H^p(X;G'') \xrightarrow{\delta^*} H^{p+1}(X;G) \xrightarrow{i^*} \ldots$$

1.5 Axioms for Homology

Now we consider homology as a functor of pairs. Note that $H_*(A,\varnothing) \cong H_*(A)$. So we can think of Top as being embedded in $\mathrm{Top}_{\mathrm{pairs}}$

Definition 1.5.1. $(X;A,B)$ is an excisive triad if X is the union of the interiors of A and B. An excisive triple is a triple (X,A,U) with $\overline{U} \subseteq \mathrm{int}(A)$.

Remark 1.5.2. One can go from an excisive triad to an excisive triple by letting $U = X - B$ and vice versa.

Definition 1.5.3. A *generalized* homology theory on the category of pairs of topological spaces is a covariant functor h_* to graded abelian groups together with a natural transformation

$$\delta_*\colon h_p(X, A) \to h_{p-1}(A)$$

that satisfies

1. (Homotopy Axiom) If $f \simeq g\colon (X, A) \to (Y, B)$, then

$$h_*(f) = h_*(g)\colon h_*(X, A) \to h_*(Y, B)$$

2. (Exactness axiom) For any pair (X, A), there is a long exact sequence

$$\dots \xrightarrow{\delta_*} h_p(A) \xrightarrow{i_*} h_p(X) \xrightarrow{j_*} h_p(X, A) \xrightarrow{\delta_*} h_{p-1}(A) \xrightarrow{i_*} \dots$$

   where $i\colon A \to X$ and $j\colon X \to (X, A)$ are the inclusions.

3. (Excision Axiom) If (X, A, U) is an excisive triple, then $(X - U, A - U) \to (X, A)$ induces an isomorphism

$$h_*(X - U, A - U) \xrightarrow{e_*} h_*(X, A)\ .$$

   Equivalently, given an excisive triad $(X; A, B)$, the inclusion $(B, A \cap B) \to (X, A)$ induces an isomorphism

$$h_*(B, A \cap B) \xrightarrow{e_*} h_*(X, A)\ .$$

4. (Additivity Axiom) If $(X, A) = \bigsqcup (X_i, A_i)$, then

$$h_*(X, A) \cong \bigoplus h_*(X_i, A_i)$$

   and the isomorphism is induced by the inclusions.

If h_* satisfies

6. (Dimension axiom) $h_*(\mathrm{pt}) = 0$ if $* > 0$.

then we say that h_* is *ordinary* and call the group $h_0(\mathrm{pt})$ the coefficients of h_*.

If $X \neq \varnothing$, then the *reduced homology* $\widetilde{h}_*(X)$ is defined by the exact sequence

$$0 \to \widetilde{h}_*(X) \to h_*(X) \to h_*(\mathrm{pt}) \to 0.$$

If $A \neq \varnothing$, we also let $\widetilde{h}_*(X, A) = h_*(X, A)$.

Exercise 1.5.4. Given a pair (X, A) with $A \neq \varnothing$, there is a long exact sequence

$$\dots \xrightarrow{\delta_*} \widetilde{h}_p(A) \xrightarrow{i_*} \widetilde{h}_p(X) \xrightarrow{j_*} h_p(X, A) \xrightarrow{\delta_*} \widetilde{h}_{p-1}(A) \xrightarrow{i_*} \dots$$

As an immediate consequence of the homotopy axiom, we have:

Proposition 1.5.5. *If $(X, A) \simeq (Y, B)$, then $h_*(X, A) \cong h_*(Y, B)$. If X is a contractible space, then $h_*(X) = h_*(\mathrm{pt})$, so that $\widetilde{h}_*(X) = 0$.*

In fact, one can show the following result (see Theorem IV.6.15 of Bredon):

Theorem 1.5.6. *Suppose that (X, A, B) is a triple so that $B \subseteq A \subseteq X$. Let*

$$\delta_* \colon h_p(X, A) \to h_{p-1}(A, B)$$

be the composite

$$h_p(X, A) \xrightarrow{\delta_*} h_{p-1}(A) \xrightarrow{i_*} h_{p-1}(A, B).$$

Then there is a long exact sequence

$$\dots \xrightarrow{\delta_*} h_p(A, B) \xrightarrow{i_*} h_p(X, B) \xrightarrow{j_*} h_p(X, A) \xrightarrow{\delta_*} h_{p-1}(A, B) \xrightarrow{i_*} \dots$$

Remark 1.5.7. As we see in the next section, it is excision which makes homology (and cohomology) computable. It means that homology depends only on local data, and allows one to compute the homology of a space from the homology of various subspaces.

1.6 Homology of Spheres and the Degree of a Map

We let $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere and $D^n \subseteq \mathbb{R}^n$ be the unit ball. Note that $S^{n-1} = \partial D^n \subseteq D^n$. Let

$$D^n_+ = \{(x_0, \dots, x_n) : x_n \geqslant 0\} \cong D^n \qquad D^n_- = \{(x_0, \dots, x_n) : x_n \leqslant 0\} \cong D^n$$

be the closed upper and closed lower hemisphere of S^n respectively. Let $S^{n-1} \subseteq S^n$ be embedded with the last coordinate $x_n = 0$, so that $S^{n-1} \subseteq D^n_{\pm 1}$

Proposition 1.6.1. *For $n \geqslant 0$, there are isomorphisms*

$$\widetilde{h}_*(S^n) \cong h_*(D^n, S^{n-1}) \cong h_*(S^n, D^n_+) \cong \begin{cases} G & i = n \\ 0 & i \neq n. \end{cases}$$

Proof. Consider the long exact sequence

$$\dots \xrightarrow{\delta_*} \widetilde{h}_p(S^{n-1}) \xrightarrow{i_*} \widetilde{h}_p(D^n_-) \xrightarrow{j_*} h_p(D^n_-, S^{n-1}) \xrightarrow{\delta_*} \widetilde{h}_{p-1}(S^{n-1}) \xrightarrow{i_*} \dots$$

Since $D^n \simeq \mathrm{pt}$, $\widetilde{h}_p(D^n_-) = 0$ for all p. So,

$$\boxed{h_p(D^n_-, S^{n-1}) \xrightarrow[\cong]{\delta_*} \widetilde{h}_{p-1}(S^{n-1})}$$

Similarly, using

$$\ldots \xrightarrow{\delta_*} \widetilde{h}_p(D^n_+) \xrightarrow{i_*} \widetilde{h}_p(S^n) \xrightarrow{j_*} h_p(S^n, D^n_+) \xrightarrow{\delta_*} \widetilde{h}_{p-1}(D^n_+) \xrightarrow{i_*} \ldots$$

we get that

$$\boxed{\widetilde{h}_p(S^n) \xrightarrow[\cong]{j_*} h_p(S^n, D^n_+)}$$

Furthermore, for U as small disk around the north pole of S^n, excision gives

$$h_p(S^n, D^n_+) \cong h_p(S^n - U, D^n_+ - U) \cong h_p(D^n_-, S^{n-1}).$$

That is,

$$\boxed{h_p(D^n_-, S^{n-1}) \xrightarrow[e_*]{\cong} h_p(S^n, D^n_+)}.$$

Now, if $n = 0$, S^0 is the disjoint union of two points so that the claim holds for $\widetilde{h}_*(S^0)$. The cases $h_*(D^0, \varnothing)$ and $h_*(S^0, D^0_+)$ are also simple to deduce from the axioms.

Now, the claim for $\widetilde{h}_*(S^0)$ gives the claim for $h_*(D^1_-, S^0)$, which then implies it for $h_*(S^1, D^1_+)$, which then gives is for $\widetilde{h}_*(S^1)$. Proceeding inductively proves the theorem. □

Corollary 1.6.2. *S^{n-1} is not a retract of D^n*

Corollary 1.6.3 (Brouwer's Fixed Point Theorem)**.** *Any map $f\colon D^n \to D^n$ has a fixed point.*

Proof. Let $r(x) \in S^{n-1}$ be the point where the ray from $f(x)$ to x passes through S^{n-1}. This is well defined since f has no fixed point and one can check it's continuous (see alternate approach in Chapter 2, Corollary 11.12 or Bredon). The map r is a retract of D^n onto S^{n-1}, which is a contradiction. □

Definition 1.6.4. Let h_* have coefficient $\mathbb{Z}$ and let $n \geqslant 0$. Let $f\colon S^n \to S^n$. The *degree* of f, denoted $\deg(f)$, is the unique integer d such that $f_*\colon \widetilde{h}_n(S^n) \to \widetilde{h}_n(S^n)$ is multiplication by d.

Note that $\deg(f \circ g) = \deg(f)\deg(g)$, and that if f is null-homotopic, then $\deg(f) = 0$. Furthermore, $\deg(\mathrm{id}) = 1$. Finally, if f is a homeomorphism, then $\deg(f) = \pm 1$, and $\deg(f) = \deg(f^{-1})$.

Proposition 1.6.5. *Let $f_i\colon S^n \to S^n \subseteq \mathbb{R}^{n+1}$ be the map defined by*

$$f_i(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n).$$

Then $\deg(f_i) = -1$.

Proof. For $0 \leqslant i \leqslant n$, $f_i = s \circ f_0 \circ s^{-1}$ where

$$s(x_0, \ldots, x_n) = (x_i, x_1, \ldots, x_{i-1}, x_0, x_{i+1}, \ldots, x_n).$$

So, $\deg(f_i) = \deg(f_0)$ since $\deg(s) = \deg(s^{-1}) = \pm 1$. We show that $\deg(f_0) = -1$. We abbreviate $f = f_0$.

Note that

$$h_0(S^0) \cong h_0(\{(1)\}) \oplus h_0(\{(-1)\}).$$

Fix a generator of $h_0(\{1\})$ and call it a_+. Note that f_0 restricts to a map from $\{(1)\}$ to $\{(-1)\}$. Let $a_- = f_*(a_+)$. So,

$$h_0(S^0) \cong \mathbb{Z}\{a_+, a_-\}$$

and f_* interchanges a_- and a_+. There is an exact sequence

$$0 \to \widetilde{h}_0(S^0) \to h_0(S^0) \to h_0(\mathrm{pt}) \to 0.$$

Since $h_0(\{(\pm 1)\}) \to h_0(\mathrm{pt})$ is the identity, the kernel is

$$\widetilde{h}_0(S^0) \cong \mathbb{Z}\{a_+ - a_-\}.$$

So $f_* \colon \widetilde{h}_0(S^0) \to \widetilde{h}_0(S^0)$ maps $a_+ - a_-$ to $-(a_+ - a_-)$, and so $\deg(f) = -1$.

Assume that the claim holds for S^{n-1} and let $S^{n-1} \subseteq S^n$ be embedded as the points for which $x_n = 0$. The hemispheres D^n_+ and D^n_- are invariant under f. We have

$$\begin{tikzcd}
\widetilde{h}_n(S^n) \arrow[r, "\cong", "j_*"'] \arrow[d, "f_*"] & h_n(S^n, D^n_+) \arrow[d, "f_*"] & h_n(D^n_-, S^{n-1}) \arrow[l, "\cong"', "e_*"] \arrow[r, "\cong", "\delta_*"'] \arrow[d, "f_*"] & \widetilde{h}_{n-1}(S^{n-1}) \arrow[d, "(-1)"', "f_*"] \\
\widetilde{h}_n(S^n) \arrow[r, "\cong", "j_*"'] & h_n(S^n, D^n_+) & h_n(D^n_-, S^{n-1}) \arrow[l, "\cong"', "e_*"] \arrow[r, "\cong", "\delta_*"'] & \widetilde{h}_{n-1}(S^{n-1})
\end{tikzcd}$$

where the horizontal isomorphism are the ones we fixed in the boxes above. The claim follows. $\square$

Corollary 1.6.6. *Any reflection across a hyperplane which intersects the sphere at a great circle has degree -1. If $g \in O(n+1)$, then $\deg(g) = 1$ if it is int he component of the identity and -1 otherwise. In particular, any rotation has degree 1.*

Proof. Such a map of the sphere to itself differs from f_0 by conjugation by a rotation of $\mathbb{R}^{n+1}$.

The group $O(n+1)$ has two path components given by those matrices of determinant 1 and those matrices of determinant -1. The group $SO(n+1)$ is the component of the identity. If $g \in SO(n+1)$, then a path $\gamma \colon I \to SO(n+1)$ for g to the identity gives a homotopy

$$G \colon I \times S^n \to S^n$$

$G(t, x) = \gamma(t)(x)$ from g to the identity of S^n. So $\deg(g) = \deg(\mathrm{id}) = 1$ for $g \in SO(n+1)$. Similarly, $\deg(g) = -1$ for g in the path component of $-\mathrm{id}$. $\square$

Corollary 1.6.7. *The antipodal map $-1\colon S^n \to S^n$ has degree $(-1)^{n+1}$. In particular, if n is even, it is not homotopic to the identity map.*

Proof. It is the composite $f_0 \circ f_1 \circ \ldots \circ f_n$. □

Corollary 1.6.8. *If n is even, then for $f\colon S^n \to S^n$, there exists x such that $f(x) = \pm x$.*

Proof. See Corollary IV.6.13. The idea is to use f (assuming it doesn't send a point to itself or its antipode) to define homotopies that give $\mathrm{id} \simeq f \simeq -\mathrm{id}$. Namely,

$$F(x,t) = \frac{tf(x) + (1-t)x}{\|tf(x) + (1-t)x\|} \qquad G(x,t) = \frac{-tx + (1-t)f(x)}{\|-tx + (1-t)f(x)\|}$$

□

Corollary 1.6.9. *If n is even, the sphere S^n does not have a non-vanishing continuous tangent vector field.*

Proof. Let $\xi\colon S^n \to TS^n$ be a continuous nowhere vanishing vector field, write $x \mapsto \xi_x \in T_xS^n$, where we identify $T_xS^n \subseteq \mathbb{R}^{n+1}$ as the n-dimensional subspace orthogonal to x. Let

$$f(x) = \xi_x/\|\xi_x\|.$$

Then $f\colon S^n \to S^n$. Since $\langle f(x), x\rangle = 0$, $f(x) \neq \pm x$ for any $x \in S^n$, a contradiction. □

1.7 Computing the Degree Map

Proposition 1.7.1. *Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a non-singular linear map. We can define a map*

$$A^*\colon S^n \to S^n$$

by viewing S^n as the one point compactification of $\mathbb{R}^n$ and setting $A^*(\infty) = \infty$. Then,

$$\deg(A^*) = \mathrm{sgn}(\det(A)).$$

Proof. The matrix representing A is a product of elementary matrices. If the claim holds for elementary matrices, it will hold for A.

- Let E be an elementary matrix with one diagonal entry a real number $\alpha \neq 0$. Then, E^* is homotopic to the identity if $\alpha > 0$. If $\alpha < 0$, E^* is homotopic to a diagonal matrix with one diagonal -1 and the others all 1. This is one of the reflections f_i discussed before whose degree is -1.

- If E is the identity with one non-zero off diagonal entry, then E is homotopic to the identity, so $\deg(E^*) = 1$.

- If E is the identity with two columns swapped, then E is a reflection accross an $n-1$-dimensional hyperplane, so $\deg(E^*) = -1$.

$\square$

Theorem 1.7.2 (Taylor's Theorem)**.** *Let $U \subseteq \mathbb{R}^n$ and $f: U \to \mathbb{R}^m$ be a $\mathcal{C}^2$ function. Let $a \in U$. Let*

$$r(x) = f(a + x) - (f(a) + Df_a(x))$$

defined for x such that $a + x \in U$.

Then

$$\lim_{x \to 0} \frac{r(x)}{\|x\|} = 0$$

In particular, there exists $\epsilon > 0$ such that

$$\|r(x)\| < \frac{1}{2}\|x\|$$

for all x such that $\|x\| < \epsilon$.

Proposition 1.7.3. *Identify $S^n = \mathbb{R}^n \cup \{\infty\}$. Let $f: S^n \to S^n$ be a map which is smooth on $V = f^{-1}(U)$ for U an open neighborhood of 0 and such that*

- $f^{-1}(\{0\}) = \{0\}$,
- 0 *is a regular value for* $f|_V$ *and*
- $Df_0 : \mathbb{R}^n \to \mathbb{R}^n$ *is the identity.*

Then $\deg(f) = 1$.

Proof. The trick is to prove that $f \simeq \mathrm{id}$. First, we show that $f \simeq f_1$ for a map f_1 such that, for some small disk D around 0,

- $f_1|_D = \mathrm{id}$
- $f_1(D') \subseteq D'$ where $D' = S^n \backslash D$, and $f_1|_{\partial D'} = f_1|_{\partial D} = \mathrm{id}$.

For the disk D^n and a function $g: D^n \to D^n$ such that $g|_{\partial D^n} = \mathrm{id}$, the map $G: D^n \times I \to D^n$ given by

$$G(x, t) = tx + (1 - t)g(x)$$

is a homotopy from g to id relative to the boundary ∂D^n. So both $f_1|_D$ and $f_1|_{D'}$ are homotopic to the identity relative the boundary $\partial D = \partial D'$. So, $f_1 \simeq \mathrm{id}$ and we get

$$\deg(f) = \deg(f_1) = \deg(\mathrm{id}) = 1.$$

Now let's construct f_1. By Taylor's theorem, there exists $\epsilon > 0$ such that $f(x) = x + r(x)$ for $\|x\| \leqslant 2\epsilon$ for a continuous function r such that $\|r(x)\| < 1/2\|x\|$. Consider

$$F(x,t) = \begin{cases} f(x) & \|x\| \geqslant 2\epsilon \\ f(x) - t\left(2 - \frac{\|x\|}{\epsilon}\right) r(x) & \epsilon \leqslant \|x\| \leqslant 2\epsilon \\ f(x) - tr(x) & \|x\| \leqslant \epsilon. \end{cases}$$

Then, $f_1(x) = F(x,1)$ is given by

$$f_1(x) = \begin{cases} f(x) & \|x\| \geqslant 2\epsilon \\ x - \left(1 - \frac{\|x\|}{\epsilon}\right) r(x) & \epsilon \leqslant \|x\| \leqslant 2\epsilon \\ x & \|x\| \leqslant \epsilon. \end{cases}$$

In particular, if $\epsilon \leqslant \|x\| \leqslant 2\epsilon$, then letting $\lambda = \left(1 - \frac{\|x\|}{\epsilon}\right)$ and noting that $|\lambda| \leqslant 1$, we have

$$\begin{aligned} \|f_1(x)\| &= \|x - \lambda r(x)\| \\ &\geqslant |\|x\| - |\lambda|\|r(x)\|| \\ &\geqslant \|x\| - \|r(x)\| > \frac{1}{2}\epsilon. \end{aligned}$$

So, $f_1^{-1}(0) = \{0\}$ and f_1 is the identity on the closed disk of radius $\epsilon/2$ around 0. Let $C \subseteq S^n$ be the closed complement of the open ball of radius $\epsilon/2$. Since C is compact, $f_1(C)$ is compact, and in particular it is closed. Since $0 \notin C$, then $0 \in S^n \backslash f_1(C)$. So there is $0 < \delta \leqslant 1/2\epsilon$ such that the disk D of radius δ around the origin satisfies the conditions required above.

□

Remark 1.7.4. Let $f\colon S^n \to S^n$ be a smooth map. Suppose that p is a regular value. Suppose that $f^{-1}(p) = q$. Up to precomposing f by a rotation, we can assume that $p = q$. Again, by composing by rotations, we can assume that p is the origin if we model S^n by $\mathbb{R}^n \cup \{\infty\}$. We identify $T_0 S^n = \mathbb{R}^n \subseteq S^n$.

Let Df_0 be the derivative of f at 0. Since 0 is a regular value, $Df_0\colon \mathbb{R}^n \to \mathbb{R}^n$ is surjective, and so is an isomorphism. Then, $f \circ (Df_0^{-1})^*$ satisfies the condition of the previous theorem, so has degree 1. Therefore,

$$\deg(f) = \deg((Df_0)^*) = \operatorname{sgn}(\det(Df_0)).$$

In terms of the original function, this is the sign of the Jacobian determinant at q, computed coordinate systems at p and q which differ from a rotation. We write let $d(f,q)$ be this sign.

Corollary 1.7.5. *Let $f\colon S^n \to S^n$ be a continuous map which is smooth on $f^{-1}(U)$ for U a neighborhood of p. Suppose that p is a regular value and let*

$$f^{-1}(p) = \{q_1, \ldots, q_k\}.$$

Then,

$$\deg(f) = \sum_{i=1}^{k} d(f, q_i).$$

Proof. By the inverse function theorem, there is a disk D containing p such that $f^{-1}(D)$ is a disjoint union of disks D_i around q_i and each D_i maps diffeomorphically onto D under f. Consider the composite

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "g"] & S^n \arrow[d, "g'"] \\ \bigvee_i D_i/\partial D_i \arrow[r, "h"] & S^n \end{tikzcd}$$

where g sends every point outside of the interior of $f^{-1}(D)$ to a point, and g' maps every point outside of the interior of D to a point. The map h is the induced map.

Exercise 1.7.6. Use additivity and the fact that

$$\widetilde{h}\left(\bigvee_i D_i/\partial D_i\right) \cong \bigoplus_i \widetilde{h}(D_i/\partial D)$$

to prove that

$$\deg(f) = \deg(g' \circ f) = \deg(h \circ g) = \sum_{i=1}^{k} d(f, q_i).$$

$\square$

Corollary 1.7.7. *The degree of the k-fold cover $S^1 \to S^1$ is k.*

Proposition 1.7.8. *There is an isomorphism*

$$\widetilde{h}_p(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}/2 & p = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Let $\mathbb{R}P^2$ be constructed by attaching a disk D^2 to S^1 along the degree 2 map. Let D be the disk of radius $1/2$ in D^2 and let $U = \mathbb{R}P^2 - \{0\}$ where 0 is the center of D^2.

There's a long exact sequence

$$\begin{tikzcd} h_{p+1}(\mathbb{R}P^2, U) \arrow[r] & \widetilde{h}_p(U) \arrow[r] & \widetilde{h}_p(\mathbb{R}P^2) \arrow[r] & h_p(\mathbb{R}P^2, U) \\ h_{p+1}(\mathbb{R}P^2 - S^1, U - S^1) \arrow[r] \arrow[u, "\cong"] & \widetilde{h}_p(U - S^1) \arrow[r] \arrow[u] & \widetilde{h}_p(\mathbb{R}P^2 - S^1) \arrow[r] \arrow[u] & h_p(\mathbb{R}P^2 - S^1, U - S^1) \arrow[u, "\cong"'] \\ h_{p+1}(D, \partial D) \arrow[r] \arrow[u, "\cong"] & \widetilde{h}_p(\partial D) \arrow[r] \arrow[u] & \widetilde{h}_p(D) \arrow[r] \arrow[u] & h_p(D, \partial D) \arrow[u, "\cong"'] \end{tikzcd}$$

Note that
$$U \simeq U - S^1 \simeq \partial D \simeq S^1.$$
If $p \geqslant 3$, then this gives $\widetilde{h}_p(\mathbb{R}P^2) = 0$ and similarly if $p = 0$. So it remains to study the cases $p = 1, 2$.

$$\begin{tikzcd}
 & & h_2(D, \partial D) \arrow[r, "\cong"] \arrow[d, "\cong"] & \widetilde{h}_1(\partial D) \arrow[d] & & & \\
0 \arrow[r] & \widetilde{h}_2(\mathbb{R}P^2) \arrow[r] \arrow[d, "="] & h_2(\mathbb{R}P^2, U) \arrow[r] \arrow[d, "\cong"] & \widetilde{h}_1(U) \arrow[r] \arrow[d, "\cong"] & \widetilde{h}_1(\mathbb{R}P^2) \arrow[r] \arrow[d, "="] & h_1(\mathbb{R}P^2, U) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \widetilde{h}_2(\mathbb{R}P^2) \arrow[r] \arrow[d, "="] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "="] & \mathbb{Z} \arrow[r] \arrow[d, "="] & \widetilde{h}_1(\mathbb{R}P^2) \arrow[r] \arrow[d, "\cong"] & h_1(\mathbb{R}P^2, U) \arrow[r] \arrow[d, "="] & 0 \\
 & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 &
\end{tikzcd}$$

Since $S^1 \cong \partial D \to U \simeq S^1$ is the multiplication by 2 map, we get the result. $\square$

Remark 1.7.9. See the book for a proof of the fundamental theorem of algebra using the degree.

1.8 CW Complexes

Homework. Read Section 9, Chapter IV.

Definition 1.8.1 (Gluing along an image)**.** Let X and Y be a topological space. Let $f\colon A \to X$ and $g\colon A \to Y$ be continuous maps. Then we let $X \sqcup_A Y$ be the space whose underlying set is $X \cup Y/(f(a) \sim g(a))$. There are natural maps $i_X\colon X \to X \cup_A Y$ and $i_Y\colon Y \to X \cup_A Y$. We let $U \subseteq X \cup_A Y$ be open if and only if $i_X^{-1}(U)$ and $i_Y^{-1}(U)$ are open in X and Y respectively. Then given a commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \arrow[ddr, bend left] & \\
Y \arrow[r] \arrow[drr, bend right] & X \cup_A Y \arrow[dr, dotted] & \\
 & & Z
\end{tikzcd}$$

the dotted arrow exists in the category of topological spaces and is unique.

The space $X \cup_A Y$ is called the *pushout* of the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & X \\
Y &
\end{tikzcd}$$

We give the user's definition of CW-complexes. See Rotman or the Appendix of Hatcher for a more thorough treatment. (C=Closure finite, W=Weak topology.)

Definition 1.8.2. A CW complex X is the following data. First, let $K^{(0)}$ is a discrete set of points. Suppose that $K^{(n-1)}$ has been constructed for $n > 0$. Let

$$f_{\partial\sigma}\colon S_\sigma^{n-1} = S^{n-1} \to K^{n-1}$$

for $\sigma \in I_n$ be a collection of continuous functions. Then, $K^{(n)}$ is the pushout

$$\begin{tikzcd}
\bigsqcup_{\sigma\in I_n} S_\sigma^{n-1} \arrow[r, "\bigsqcup f_{\partial\sigma}"] \arrow[d] & K^{(n-1)} \arrow[d] \\
\bigsqcup_{\sigma\in I_n} D_\sigma^{n} \arrow[r, "\bigsqcup f_{\sigma}"] & K^{(n)}
\end{tikzcd}$$

where $D_\sigma^n = D^n$ and $S_\sigma^{n-1} \to D_\sigma^n$ are the inclusions of the boundary. Then,

$$K = \bigcup_{n\in\mathbb{N}} K^{(n)}$$

with the following topology (called the **weak topology**): $U \subseteq X$ is open (resp. closed) if and only if $U \cap K^{(n)}$ is open (resp. closed) for each $n \in \mathbb{N}$.

- We call $f_{\partial\sigma}$ an *attaching map.*
- We call f_σ a *characteristic map.*
- We often call σ an n-cell.
- We call $K_\sigma = f_\sigma(D_\sigma^n)$ a *closed n-cell* of K. It is a compact subset of K). Let $K_{\partial\sigma} = f_{\partial\sigma}(S^{n-1})$.
- We call $U_\sigma = f_\sigma(D_\sigma^n - S_\sigma^{n-1})$ an *open n-cell* of K. It is homeomorphic to $\text{int}(D^n)$ but may not be open in K. However, it is open in $K^{(n)}$. Note further that any $u \in K^{(0)}$ is an open cell.
- We call $K^{(n)}$ the nth skeleton of K.
- K has dimension n if $K^{(m)} = K^{(n)}$ for all $m \geqslant n$.
- K is finite if it is constructed from a finite number of cells.
- A *sub-complex* is a subspace of K which is a union of the subset of closed cells of K. This is a CW-complex whose attaching maps are a subset of those used for K.

Exercise 1.8.3. A subset $U \subseteq K$ is open (resp. closed), if and only if

- $f_\sigma^{-1}(U)$ is open (resp. closed) for every σ.
- $U \cap K_\sigma$ is open (resp. closed) for every σ.

A function $g\colon K \to X$ is continuous if and only if $g \circ f_\sigma$ is continuous for every σ.

Definition 1.8.4. A map $f\colon K \to L$ of CW complexes is *cellular* if $f(K^{(n)}) \subseteq L^{(n)}$.

1.8.1 Some important examples of CW complexes

The spheres

The sphere S^n has many different decompositions as a CW-complex. One example is to have $(S^n)^{(k)} = \{*\}$ for $0 \leqslant k < n$ and have one n-cell σ_n attached by the only map $f_{\partial\sigma_n}\colon S^{n-1} \to *$:

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & (S^n)^{(n-1)} \cong * \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd}$$

Another CW-structure has two cells in each dimensions $0 \leqslant k \leqslant n$. The k-skeleton of S^n can be taken to be

$$S^k = \{(x_0, \dots, x_k, 0, \dots, 0) \in S^n\}.$$

and the cells are

$$\begin{aligned} K_{\sigma_k} &= D^k_+ = \{(x_0, \dots, x_k, 0, \dots, 0) \in S^n : x_k \geqslant 0\} \\ K_{T\sigma_k} &= D^k_- = \{(x_0, \dots, x_k, 0, \dots, 0) \in S^n : x_k \leqslant 0\}. \end{aligned}$$

Let the two cells be denoted by σ_k and $T\sigma_k$ where T is the antipodal map. Then the attaching maps are chosen so that $f_{T\sigma_k} = T \circ f_{\sigma_k}$.

With this model,

$$S^0 \subseteq S^1 \subseteq S^2 \dots$$

and

$$S^\infty = \bigcup_{n \geqslant 0} S^n$$

with the weak topology.

Real Projective Space

Recall that $\mathbb{R}P^n$ is the space of real lines in $\mathbb{R}^{n+1}$. It can be described as a quotient space of $\mathbb{R}^{n+1} - \{0\}$ by the action of $\mathbb{R}^\times$. Equivalently, it's the quotient of S^n by the relation

$$(x_0, \dots, x_n) \sim T(x_0, \dots, x_n) = (-x_0, \dots, -x_n).$$

We denote the corresponding point of $\mathbb{R}P^n$ by $[x_0 : \dots : x_n]$. Note that $\mathbb{R}P^0$ is a point.

The space $\mathbb{R}P^n$ can be realized as the following CW-complex with one k-cell for each $0 \leqslant k \leqslant n$, obtained from $\mathbb{R}P^{n-1}$ by attaching one n-cell.

Note that $\mathbb{R}P^{n-1} \subseteq \mathbb{R}P^n$ as the subset of points of the form $[x_0 : \dots : x_{n-1} : 0]$. Each point $p \in \mathbb{R}P^n$ has a representative $(x_0, \dots, x_n) \in S^n$ with $x_n \geqslant 0$ and the representative is unique if the last coordinate is non zero.

For such a representative, letting $y = (x_0, \dots, x_{n-1})$, we have that $|y| \leqslant 1$ and

$$x_n = \sqrt{|x_n|^2} = \sqrt{1 - |y|^2}.$$

So, each point of $\mathbb{R}P^n$ can be represented as

$$(y, \sqrt{1-|y|^2}), \quad y \in \mathbb{R}^n, \quad |y| \leqslant 1.$$

Those points for which $|y| = 1$ are precisely the points of $\mathbb{R}P^{n-1}$. There is a homeomorphism

$$D^n - S^{n-1} \cong \{(y, \sqrt{1-|y|^2}) : y \in \mathbb{R}^n, |y| < 1\} \subseteq \mathbb{R}P^n.)$$

This is the open n-cell.

Let

$$f_{\partial\sigma_n} \colon S^{n-1} \to \mathbb{R}P^{n-1}$$

be the quotient map. The image of this map is those points represented by $(x_0, \ldots, x_n)$ such that $x_n = 0$. Then $\mathbb{R}P^n$ is obtained as the pushout:

$$\begin{tikzcd}
S^{n-1} \arrow[r, "f_{\partial\sigma_n}"] \arrow[d] & \mathbb{R}P^{(n-1)} \arrow[d] \\
D^n \arrow[r] & \mathbb{R}P^n
\end{tikzcd}$$

The k-skeleton of $\mathbb{R}P^n$ is then $\mathbb{R}P^k \subseteq \mathbb{R}P^n$ realized as those points of the form

$$[x_0 : \ldots : x_k : 0 : \ldots : 0].$$

In fact, for this structure and the structure of S^n with two k-cells for each k described above, the quotient map $S^n \to \mathbb{R}P^n$ is a cellular map and we have a commutative diagram

$$\begin{tikzcd}
S^0 \arrow[r] \arrow[d, "\varphi^1"] & S^1 \arrow[r] \arrow[d, "\varphi^2"] & S^2 \arrow[r] \arrow[d, "\varphi^3"] & \ldots \\
\mathbb{R}P^0 \arrow[r] & \mathbb{R}P^1 \arrow[r] & \mathbb{R}P^2 \arrow[r] & \ldots
\end{tikzcd}$$

Definition 1.8.5. The space $\mathbb{R}P^\infty$ is defined to be $\bigcup_{n\in\mathbb{N}} \mathbb{R}P^n$ with the weak topology.

Complex Projective Space

Similarly, $\mathbb{C}P^n$, which is the space of complex lines in $\mathbb{C}^{n+1}$. It is the quotient space of $\mathbb{C}^{n+1} - \{0\}$ by the action of $\mathbb{C}^\times$, or equivalently, the quotient of $S^{2n+1} \subseteq \mathbb{C}^n$ by the action of S^1,

$$(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n), \quad \lambda \in S^1 = \{x \in \mathbb{C} : |x| = 1\}.$$

Points of $\mathbb{C}P^n$ are denoted by $[z_0 : \ldots : z_n]$. Note that $\mathbb{C}P^0$ is a point. The points of the form $[z_0 : \ldots : z_{n-1} : 0]$ form a copy of $\mathbb{C}P^{n-1} \subseteq \mathbb{C}P^n$. Any other point of $\mathbb{C}P^n$ is of the form $[z_0 : \ldots : z_{n-1} : z_n]$ for $z_n \neq 0$ and $(z_0, \ldots, z_n) \in S^{2n+1}$.

We show that, as a CW-complex, $\mathbb{C}P^n$ has one cell in each even dimension $0 \leqslant k \leqslant n$ and is obtained from $\mathbb{C}P^{n-1}$ by attaching a $2n$-cell along the quotient map $f_{\partial\sigma_{2n}} \colon S^{2n-1} \to \mathbb{C}P^{n-1}$.

For each complex number z, there is a complex number $\lambda \in S^1$ such that $\lambda z \in \mathbb{R}_{\geqslant 0} \subseteq \mathbb{C}$ and this number is unique if $z \neq 0$.

So, each point of $\mathbb{C}P^n$ has a representative $(z_0, \dots, z_n) \in S^{2n+1}$ such that $z_n \in \mathbb{R}_{\geqslant 0} \subseteq \mathbb{C}$, and the representative is unique if $z_n \neq 0$.

Let $w = (z_0, \dots, z_{n-1})$. Then, $(w, z_n) \in S^{2n+1}$, so $|w| \leqslant 1$ and

$$z_n = \sqrt{|z_n|^2} = \sqrt{1 - |w|^2}.$$

So, each point in $\mathbb{C}P^n$ has a representative of the form

$$(w, \sqrt{1 - |w|^2}), \quad w \in \mathbb{C}^n, \quad |w| \leqslant 1$$

The subspace of points for which $|w| = 1$ is precisely $\mathbb{C}P^{n-1}$. The subspace where the representatives are unique is

$$D^{2n} - S^{2n-1} \cong \{(w, \sqrt{1 - |w|}) : w \in \mathbb{C}^n, |w| < 1\} \subseteq \mathbb{C}P^n$$

That's the $2n$-cell.

Let $f_{\partial\sigma_{2n}} \colon S^{2n-1} \to \mathbb{C}P^{n-1}$ be the projection. Then $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by the following pushout:

$$\begin{tikzcd}
S^{2n-1} \arrow[r, "f_{\partial\sigma_{2n}}"] \arrow[d] & \mathbb{C}P^{n-1} \arrow[d] \\
D^{2n} \arrow[r] & \mathbb{C}P^n.
\end{tikzcd}$$

In fact, $\mathbb{C}P^k \subseteq \mathbb{C}P^n$, given by the points of the form $[z_0 : \dots : z_k : 0 \dots : 0]$, is the k-skeleton.

Definition 1.8.6. The space $\mathbb{C}P^\infty$ is defined to be $\bigcup_{n \in \mathbb{N}} \mathbb{C}P^n$ with the weak topology.

1.8.2 Closure finiteness

We won't prove the following statement (See Prop 8.1, Thm 8.2 of Bredon):

Proposition 1.8.7. *Let K be a CW-complex. Then*

- *Each cell of K is contained in a finite sub-complex of K. (This is* **closure finite***.)*
- *Any compact subset of K is contained in a finite subcomplex. In particular, for any σ, the image of $f_{\partial\sigma}$ is contained in a finite subcomplex.*

1.9 Conventions for CW Complexes

First, let's fix some things. First, we work with singular homology H_* with coefficients $\mathbb{Z}$, but some of this works more generally.

Definition 1.9.1. If X and Y are pointed spaces, we let their base point be $*$. Then wedge product is

$$X \vee Y = X \cup_* Y \cong X \times \{*\} \cup \{*\} \times Y.$$

The smash product is

$$X \wedge Y := X \times Y/(X \vee Y)$$

If $f\colon X \to Z$ and $g\colon Y \to W$ are maps of spaces which preserve the base points, then there is an induced map

$$f \wedge g\colon X \wedge Y \to Z \wedge W.$$

Let $I^1 = [0,1]$ with base point $\{0\}$ and $S^1 = I^1/\partial I^1 = I^1/\{0,1\}$ with base point ∂I^1. Let $\gamma_1\colon I^1 \to S^1$ be the quotient. Let

$$I^p = \underbrace{I^1 \times \ldots \times I^1}_{p}$$

and

$$S^p = \underbrace{S^1 \wedge \ldots \wedge S^1}_{p}$$

and

$$\gamma_p = \gamma_1 \wedge \ldots \wedge \gamma_1\colon I^p \to S^p.$$

Then,

$$\gamma_{p+q} = \gamma_p \wedge \gamma_q\colon I^p \times I^q \to S^p \wedge S^q = S^{p+q}.$$

So we have $I^p \times I^q = I^{p+q}$ and $\partial I^p \cong S^{p-1}$. In fact, we fix a homotopy equivalence

$$\begin{tikzcd}
\partial I^{p+1} \arrow[r, equal] \arrow[rrr, bend left, ":=\partial\gamma_{p+1}"] & \partial I^1 \times I^p \cup I^1 \times \partial I^p \arrow[r, dotted] \arrow[d, "\subseteq"] & S^0 \wedge S^p \arrow[r, "\cong"'] \arrow[d, "\subseteq"] & S^p \\
I^{p+1} \arrow[r, equal] & I^1 \times I^p \arrow[r, "1\wedge\gamma_p"] & I^1 \wedge S^p &
\end{tikzcd}$$

We also fix a homeomorphism

$$\begin{tikzcd}
I^p \arrow[r, "\gamma_p"] \arrow[d] & S^p \\
I_p/\partial I^p \arrow[ur, "\tilde{\gamma}_p"'] &
\end{tikzcd}$$

We choose generators of some relevant homology groups.

- Fix $[I^0] \in H_0(I^0) = H_0(I^0, \partial I^0) \cong \mathbb{Z}$ a generator and suppose that $[I^p] \in H_p(I^p, \partial I^p)$ has been determined.

- Let $[S^p] \in H_p(S^p, *)$ be given by $[S^p] = (\gamma_p)_*([I^p])$ where
$$(\gamma_p)_* \colon H_p(I^p, \partial I^p) \to H_p(S^p, *).$$
- Fix a generator $[\partial I^{p+1}] \in H_p(\partial I^{p+1}, *)$ such that
$$[S^p] = (\partial \gamma_{p+1})_*([\partial I^{p+1}]).$$
- Let $[I^{p+1}] \in H_{p+1}(I^{p+1}, \partial I^{p+1})$ be chosen so that
$$\delta_*([I^{p+1}]) = [\partial I^{p+1}].$$

Now that we have fixed all these generators, we can speak of the degree of a map between the relevant pairs. For example, a map $f \colon \partial I^p \to S^p$ has a degree defined by
$$f_*([I^p]) = \deg(f)[S^p].$$

Now, let K be a CW-complex. Fix characteristic maps
$$f_\sigma \colon I^n_\sigma \to K^{(n)}$$
and let $f_{\partial\sigma} = f_\sigma|_{\partial I^n_\sigma}$. Consider:
$$\begin{tikzcd} I^n_\sigma \arrow[r, "f_\sigma"] \arrow[rrr, bend right, "\gamma_n"'] & K^{(n)} \arrow[r] \arrow[rr, bend left, "p_\sigma"] & K^{(n)}/K^{(n-1)} = \bigvee_\tau I^n_\tau/\partial I^n_\tau \arrow[r, "\widetilde{\gamma}_n"] & S^n \end{tikzcd}$$
Here, p_σ is the projection onto the σ factor followed by $\widetilde{\gamma}_n$. Note that if $p_\sigma f_{\sigma'}$ is the constant map to the base point if $\sigma' \neq \sigma$.

Now, let τ be an n-cell and σ be an $n+1$-cell. Then define
$$[\tau : \sigma] := \deg(p_\tau f_{\partial\sigma})$$
for
$$\partial I^{n+1}_\sigma \xrightarrow{f_{\partial\sigma}} K^{(n)} \xrightarrow{p_\tau} S^n.$$

Finally, we let
$$K^{(0)}/K^{(-1)} = K^{(0)} \sqcup \{*\}.$$

1.10 Cellular Homology

Let K be a CW complex and A be a sub-complex. Let

$$K_A^{(n)} = K^{(n)} \cup A$$

for each $n \geqslant 0$.

We define a chain complex called the *cellular chain complex*, denoted $C_*(K, A)$. There will be more than one way to think about this chain complex.

Definition 1.10.1. The group $C_n(K, A)$ is the free abelian group on the n-cells of K which are not in A. Then boundary

$$\partial_{n+1} \colon C_{n+1}(K, A) \to C_n(K, A)$$

is given by

$$\partial\sigma = \sum_{\tau} [\tau : \sigma]\tau.$$

Our goal is to prove the following result:

Theorem 1.10.2. *There is an isomorphism*

$$C_n(K, A) \cong H_n(K_A^{(n)}, K_A^{(n-1)})$$

with boundary ∂_n identified with the composite

$$\begin{tikzcd} H_{n+1}(K_A^{(n+1)}, K_A^{(n)}) \arrow[r, "\delta_*"] \arrow[rr, bend right, "\beta_{n+1}"'] & H_n(K_A^{(n)}) \arrow[r, "j_*"] & H_n(K_A^{(n)}, K_A^{(n-1)}) \end{tikzcd}$$

Furthermore,

$$H_n(K, A) \cong H_n(C_*(K, A)).$$

We do this in a series of steps.

Lemma 1.10.3. *There is an isomorphim*

$$\bigoplus_{\sigma \notin A} H_*(I_\sigma^n, \partial I_\sigma^n) \cong H_*(\coprod_{\sigma \notin A} (I_\sigma^n, \partial I_\sigma^n)) \to H_*(K_A^{(n)}, K_A^{(n-1)})$$

which makes the diagram

$$\begin{tikzcd} H_*(K_A^{(n)}, K_A^{(n-1)}) \arrow[r, "\beta_{n+1}"] & H_{*-1}(K_A^{(n-1)}, K_A^{(n-2)}) \\ \bigoplus_{\sigma \notin A} H_*(I_\sigma^n, \partial I_\sigma^n) \arrow[u, "\cong"', "\oplus f_\sigma"] \arrow[r, "\delta_*"] & \bigoplus_{\sigma \notin A} H_{*-1}(\partial I_\sigma^n) \arrow[u, "\oplus f_{\partial\sigma}"] \end{tikzcd}$$

commute, where the top boundary comes from the triple $(K_A^{(n)}, K_A^{(n-1)}, K_A^{(n-2)})$.

Proof. The trick is to punch out the center of each cell. Let $p_\sigma \in \operatorname{int}(I^n_\sigma)$. Then we have

$$\begin{tikzcd}
H_*(\coprod_{\sigma\notin A}(I^n_\sigma, \partial I^n_\sigma)) \arrow[r] \arrow[d, "h.a."', "\cong"] & H_*(K_A^{(n)}, K_A^{(n-1)}) \arrow[d, "h.a."', "\cong"] \\
H_*(\coprod_{\sigma\notin A}(I^n_\sigma, I^n_\sigma - \{p_\sigma\})) \arrow[r] & H_*(K_A^{(n)}, K_A^{(n)} - \bigcup_{\sigma\notin A}\{f_\sigma(p_\sigma)\}) \\
H_*(\coprod_{\sigma\notin A}(\operatorname{int}(I^n_\sigma), \operatorname{int}(I^n_\sigma) - \{p_\sigma\})) \arrow[r, "\cong"] \arrow[u, "\cong"', "e_*"] & H_*(K_A^{(n)} - K_A^{(n-1)}, K_A^{(n)} - K_A^{(n-1)} - \bigcup_{\sigma\notin A}\{f_\sigma(p_\sigma)\}) \arrow[u, "\cong"', "e_*"]
\end{tikzcd}$$

For the commutativity of the diagram, we see that everything map in sight is natural. □

Lemma 1.10.4. *If $i > n$ or $i < 0$, then*

$$H_i(K_A^{(n)}, A) = 0.$$

Furthermore, if $k \geqslant 1$, then

$$H_n(K_A^{(n+1)}, A) \xrightarrow[i_*]{\cong} H_n(K_A^{(n+k)}, A)\ .$$

In particular, if $\dim(K) < \infty$*, then for any $k \geqslant 1$, then*

$$H_n(K, A) \cong H_n(K_A^{(n+k)}, A).$$

Proof. Consider the long exact sequence

$$H_{i+1}(K_A^{(n)}, K_A^{(n-1)}) \xrightarrow{\delta_*} H_i(K_A^{(n-1)}, A) \xrightarrow{i_*} H_i(K_A^{(n)}, A) \xrightarrow{j_*} H_i(K_A^{(n)}, K_A^{(n-1)}) \xrightarrow{\delta_*} H_{i-1}(K_A^{(n-1)}, A)$$

For the first claim, proceed by induction on n, using the fact that $H_i(K_A^{(n)}, K_A^{(n-1)}) = 0$ if $i \neq n$. For the second, start the induction with the triple $(K_A^{(n+2)}, K_A^{(n+1)}, A)$, looking at $i = n$ in the sequence. □

Remark 1.10.5. In fact, if $(X, A) = \bigcup_{n\in\mathbb{N}}(X_n, A_n)$ with the weak topology, then

$$H_*(X, A) = \operatorname{colim} H_*(X_n, A_n) = \bigoplus_n H_*(X_n, A_n)/\sim$$

where $\sim$ is generated by $[\![c_n]\!] - (i_n)_*[\![c_n]\!])$ for $i_n\colon (X_n, A_n) \to (X_{n+1}, A_{n+1})$ and $[\![c_n]\!] \in H_*(X_n, A_n)$. So the previous lemma holds for any CW-complex.

Lemma 1.10.6. *If* $\dim(K) < \infty$*, then for any $k \geqslant 1$,*

$$H_n(K, A) \cong H_n(K_A^{(n+1)}, A) \cong \ker(\beta_n)/\operatorname{im}(\beta_{n+1}).$$

Proof. Note that $H_n(K_A^{(n+1)}, A) \cong H_n(K_A^{(n+1)}, K_A^{(n-2)})$ by the long exact sequence

$$0 = H_n(K_A^{(n-2)}, A) \to H_n(K_A^{(n+1)}, A) \to H_n(K_A^{(n+1)}, K_A^{(n-2)}) \to H_{n-1}(K_A^{(n-2)}, A) = 0$$

So we need to show that

$$H_n(K_A^{(n+1)}, K_A^{(n-2)}) \cong \ker(\beta_n)/\operatorname{im}(\beta_{n+1}).$$

Note that β_{n+1} is the connecting homomorphism in the long exact sequence of the triple $(K_A^{(n+1)}, K_A^{(n)}, K_A^{(n-1)})$. The claim follows by inspecting the following commutative diagram where the vertical sequence is the long exact sequence for the triple $(K_A^{(n)}, K_A^{(n-1)}, K_A^{(n-2)})$ and the horizontal sequence the long exact sequence for $(K_A^{(n+1)}, K_A^{(n)}, K_A^{(n-2)})$.

$$\begin{tikzcd}
 & H_n(K_A^{(n-1)}, K_A^{(n-2)}) = 0 \arrow[d, "i_*"] & H_n(K_A^{(n+1)}, A) \arrow[d, "\cong"] & \\
H_{n+1}(K_A^{(n+1)}, K_A^{(n)}) \arrow[r, "\delta_*"] \arrow[d, "="] & H_n(K_A^{(n)}, K_A^{(n-2)}) \arrow[r, "i_*"] \arrow[d, "j_*"] & H_n(K_A^{(n+1)}, K_A^{(n-2)}) \arrow[r, "j_*"] & H_n(K_A^{(n+1)}, K_A^{(n)}) = 0 \\
H_{n+1}(K_A^{(n+1)}, K_A^{(n)}) \arrow[r, "\beta_{n+1}"] & H_n(K_A^{(n)}, K_A^{(n-1)}) \arrow[d, "\beta_n"] & & \\
 & H_{n-1}(K_A^{(n-1)}, K_A^{(n-2)}) & &
\end{tikzcd} \tag{1.2}$$

□

The last step is to prove that β_{n+1} is computed by the given formula. We have a commutative diagram

$$\begin{tikzcd}
 & H_*(S_\tau^{n-1}, *) & \\
H_*(K_A^{(n)}, K_A^{(n-1)}) \arrow[r, "\beta_{n+1}"] & H_{*-1}(K_A^{(n-1)}, K_A^{(n-2)}) \arrow[u, "(p_\tau)_*"] & \bigoplus_{\tau\notin A} H_{*-1}(I_\tau^{n-1}, \partial I_\tau^{n-1}) \arrow[l, "\cong", "\oplus(f_\tau)_*"'] \\
\bigoplus_{\sigma\notin A} H_*(I_\sigma^n, \partial I_\sigma^n) \arrow[u, "\cong"', "\oplus(f_\sigma)_*"] \arrow[r, "\delta_*"] & \bigoplus_{\sigma\notin A} H_{*-1}(\partial I_\sigma^n) \arrow[u, "\oplus(f_{\partial\sigma})_*"'] &
\end{tikzcd}$$

Note that

$$(f_\sigma)_*(\delta_*([I_\sigma^n])) = (f_{\partial\sigma})_*[\partial I_\sigma^n]$$

Furthermore, since the map

$$\bigoplus_{\tau\notin A} H_{*-1}(I_\tau^{n-1}, \partial I_\tau^{n-1}) \xrightarrow{\oplus(p_\tau f_\tau)_*} \bigoplus_{\tau\notin A} H_*(S_\tau^{n-1}, *)$$

is an isomorphism which maps $[I_\tau^{n-1}]$ to $[S_\tau^{n-1}]$, we conclude that

$$[I_\sigma^n] \mapsto \sum_\tau \deg(p_\tau f_{\partial\sigma})[I_\tau^{n-1}]$$

Next, we check that the construction is natural.

Definition 1.10.7. Let (K, A) and (L, B) be pairs of CW-complexes and a sub-complex. A cellular map of pairs $\phi\colon (K, A) \to (L, B)$ is a cellular map such that $\phi(A) \subseteq B$.

For such a map ϕ, the diagram (1.2) above is natural with respect to ϕ_*. In particular,

$$\phi_*\colon H_n(K, A) \to H_n(K, B)$$

is computed from the chain map

$$\begin{tikzcd}
H_n(K_A^{(n)}, K_A^{(n-1)}) \arrow[r, "\beta_n"] \arrow[d, "\phi_*"] & H_{n-1}(K_A^{(n-1)}, K_A^{(n-2)}) \arrow[d, "\phi_*"] \\
H_n(L_B^{(n)}, L_B^{(n-1)}) \arrow[r, "\beta_n"] & H_{n-1}(L_B^{(n-1)}, L_B^{(n-2)})
\end{tikzcd}$$

The book calls this chain map

$$\phi_\Delta\colon C_*(K, A) \to C_*(L, B).$$

Looking at the diagram

$$\begin{tikzcd}
I_\sigma^n \arrow[r, "f_\sigma"] \arrow[d, "\gamma_n"] & K^{(n)} \arrow[r, "\phi"] \arrow[d] & L^{(n)} \arrow[r, "p_\tau"] \arrow[d] & S_\tau^n \arrow[d, "="] \\
S_\sigma^n \arrow[r] \arrow[rrr, bend right, "\phi_{\tau,\sigma}"'] & K^{(n)}/K^{(n-1)} \arrow[r, "\phi"] & L^{(n)}/L^{(n-1)} \arrow[r, "p_\tau"] & S_\tau^n
\end{tikzcd}$$

we deduce that

$$\phi_\Delta(\sigma) = \sum_\tau \deg(\phi_{\tau,\sigma})\tau = \sum_\tau \deg(p_\tau \phi f_\sigma)\tau.$$

1.11 Products of CW complexes

Theorem 1.11.1. *Suppose that one of K or L is locally compact, or that K and L have countably many cells. Let K and L be CW-complexes. Then $K \times L$ has the structure of a CW-complex with cells given by the product of the cells of K and L. The characteristic maps are $f_\sigma \times f_\mu$ where f_σ and f_μ are characteristic maps for K and L respectively.*

Remark 1.11.2. The subtlety lies in whether or not the CW-topology induced on the product $K \times L$ corresponds with the product topology.

Example 1.11.3. Give I the CW structure with 0-cells $\{0,1\}$ and one 1-cell attached along the identity $S^0 \to S^0$. We denote the 1-cell by $[0,1]$ Then $I \times K$ is a CW complex with n-cells

$$\{0\} \times \sigma, \quad \{1\} \times \sigma \quad [0,1] \times \mu$$

where σ runs over the n-cell of K and τ over the $n-1$-cells of K.

Example 1.11.4. Draw the CW structure on $I \times I$.

We will come back to this topic in a bit, but first:

1.12 Cellular Approximation

Chapter 11 of the book has a collection of results on cellular maps. We will skip the proofs and look at their consequences.

Theorem 1.12.1. *Let (K, A) and (L, B) be pairs of CW-complexes. Suppose that $\phi \colon K \to L$ is a map with $\phi(A) \subseteq B$. Then ϕ is homotopic to a cellular map of pairs. Furthermore, any cellular map of pairs which are homotopic are homotopic via a homotopy $h \colon (I \times K, I \times A) \to (L, B)$ which is a cellular map.*

The take-away is that we may always assume that maps of CW-pairs are cellular, and we may always assume that homotopies between our cellular maps are cellular.

1.13 Homology of Real Projective Space

Another CW-structure has two cells in each dimensions $0 \leqslant k \leqslant n$. Let $T \colon S^n \to S^n$ be the antipodal map. Denote the cells by σ_k and $T\sigma_k$. The k-skeleton of S^n can be taken to be

$$S^k = \{(x_0, \ldots, x_k, 0, \ldots, 0) : x_0^2 + \ldots + x_k^2 = 1\}.$$

And attached the cells inductively so that

$$f_{T\sigma_k} = Tf_{\sigma_k}.$$

$$\begin{tikzcd}
\partial I^{k+1}_{\sigma_k} \cup \partial I^{k+1}_{T\sigma_k} \arrow[r, "f_{\partial\sigma_k} \cup Tf_{\partial\sigma_k}"] \arrow[d] & S^k \arrow[d] \\
I^{k+1}_{\sigma_k} \cup I^{k+1}_{T\sigma_k} \arrow[r, "f_{\sigma_k} \cup Tf_{\sigma_k}"'] & S^{k+1}
\end{tikzcd}$$

We have

$$p_{\sigma_k} f_{\sigma_k} = \gamma_k = p_{T\sigma_k} f_{T\sigma_k} = p_{T\sigma_k} T f_{\sigma_k}.$$

Since p_{σ_k} is uniquely determined by the fact that $p_{\sigma_k} f_{\sigma_k} = \gamma_k$, this means that

$$p_{\sigma_k} = p_{T\sigma_k} T, \quad p_{\sigma_k} T = p_{T\sigma_k}.$$

In particular,

$$p_{T\sigma_{k-1}} f_{\partial T\sigma_k} = p_{T\sigma_{k-1}} T f_{\partial\sigma_k} = p_{\sigma_{k-1}} f_{\partial\sigma_k}$$

and

$$p_{\sigma_{k-1}} f_{\partial T\sigma_k} = p_{\sigma_{k-1}} T f_{\partial\sigma_k} = p_{T\sigma_{k-1}} f_{\partial\sigma_k}.$$

We fix f_{σ_k} so that

$$\deg(p_{\sigma_{k-1}} f_{\partial\sigma_k}) = \deg(p_{T\sigma_{k-1}} f_{\partial T\sigma_k}) = 1$$

Then,

$$\begin{tikzcd} \partial I^k_{\sigma_k} \arrow[r, "f_{\partial\sigma_k}"] & S^{k-1} \arrow[r, "p_{\sigma_{k-1}}"] \arrow[d, "T"'] & S^{k-1} \\ & S^{k-1} \arrow[ur, "p_{T\sigma_{k-1}}"'] & \end{tikzcd}$$

implies that

$$\begin{aligned} \deg(p_{\sigma_{k-1}} f_{\partial\sigma_k}) &= \deg(p_{T\sigma_{k-1}} T f_{\partial\sigma_k}) \\ &= \deg(p_{T\sigma_{k-1}}) \deg(T) \deg(f_{\partial\sigma_k}) \\ &= \deg(T) \deg(p_{\sigma_{k-1}} f_{\partial\sigma_k}) = (-1)^k. \end{aligned}$$

So,

$$\begin{aligned} \partial(\sigma_k) &= \deg(p_{\sigma_{k-1}} f_{\partial\sigma_k})\sigma_{k-1} + \deg(p_{T\sigma_{k-1}} f_{\partial\sigma_k})T\sigma_{k-1} = \sigma_{k-1} + (-1)^k T\sigma_{k-1} \\ \partial(T\sigma_k) &= \deg(p_{\sigma_{k-1}} f_{\partial T\sigma_k})\sigma_{k-1} + \deg(p_{T\sigma_{k-1}} f_{\partial T\sigma_k})T\sigma_{k-1} = (-1)^k\sigma_{k-1} + T\sigma_{k-1} \end{aligned}$$

Now, the quotient map $\pi\colon S^n \to \mathbb{R}P^n$ gives $\mathbb{R}P^n$ a cell structure which has one cell τ_k in each dimension and characteristic maps

$$f_{\tau_k} = \pi f_{\sigma_k} = \pi T f_{\sigma_k} = \pi f_{T\sigma_k}.$$

With this, π_Δ is a cellular map and $\pi_\Delta\sigma_k = \pi_\Delta T\sigma_k = \tau_k$. So

$$\partial\tau_k = \partial\pi_\Delta\sigma_k = \pi_\Delta\partial\sigma_k = \pi_\Delta(\sigma_{k-1} + (-1)^k T\sigma_{k-1}) = (1 + (-1)^k)\tau_{k-1}.$$

We get

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2 & k \text{ odd}, k < n \\ \mathbb{Z} & n \text{ odd and } k = n \\ 0 & k \text{ is even, or } k > n. \end{cases}$$

1.14 Products of CW complexes revisited

We've established that $K \times L$ is a CW-complex with cells the product of the cells in K and L. So, $C_*(K \times L)$ is generated by $\sigma \times \tau$ where σ is a cell of K and μ a cell of L. So what's the differential?

By definition,

$$f_{\sigma \times \mu} = f_\sigma \times f_\mu.$$

To describe $p_{\sigma \times \mu}$, we note that

$$(K \times L)^{(p+q)} = \bigcup_{i=0}^{p+q} K^{(i)} \times L^{(p+q-i)}.$$

and that

$$(K \times L)^{(p+q-1)} = \bigcup_{i=0}^{p+q-1} K^{(i)} \times L^{(p+q-1-i)}.$$

Define $p_{\sigma \times \mu} \colon (K \times L)^{(p+q)} \to S^p \wedge S^q$ for $(x, y) \in K^{(i)} \times L^{(p+q-i)}$ by

$$p_{\sigma \times \mu}(x, y) = \begin{cases} (p_\sigma(x), p_\mu(y)) & i = p \\ * & \text{otherwise.} \end{cases}$$

Then, for $(x_1, \ldots, x_{p+q}) \in I^p \times I^q$ and σ' a p-cell and μ' a q-cell

$$\begin{aligned} p_{\sigma \times \mu} \circ f_{\sigma' \times \mu'}(x, y) &= (p_\sigma f_{\sigma'}(x_1, \ldots, x_p), p_\mu f_{\mu'}(x_{p+1}, \ldots, x_{p+q})) \\ &= \begin{cases} \gamma_{p+q}(x_1, \ldots, x_{p+q}) & \sigma = \sigma', \mu = \mu' \\ * & \text{otherwise} \end{cases} \end{aligned}$$

On the other hand, if σ' is an i-cell and μ' is a $p + q - i$-cell but $i \neq p$, then

$$\begin{aligned} p_{\sigma \times \mu} \circ f_{\sigma' \times \mu'}(x, y) &= (p_\sigma f_{\sigma'}(x_1, \ldots, x_i), p_\mu f_{\mu'}(x_{i+1}, \ldots, x_{p+q})) \\ &= * \end{aligned}$$

by our definition of $p_{\sigma \times \mu}$.

$$p_{\sigma \times \tau} = p_\sigma \wedge p_\mu.$$

Theorem 1.14.1. *For σ and μ as above*

$$\partial(\sigma \times \mu) = \partial\sigma \times \mu + (-1)^{\deg \sigma} \sigma \times \partial\mu.$$

Proof. Let $\deg(\sigma) = p$ and $\deg(\mu) = q$. We have to understand the composite

$$\partial(I^p \times I^q) = \partial I^p \times I^q \cup I^p \times \partial I^q \xrightarrow{f_{\partial\sigma} \times f_\mu \cup f_\sigma \times f_{\partial\mu}} (K \times L)^{p+q-1} \xrightarrow{p_\tau \wedge p_\phi} S_\tau^{\deg \tau} \wedge S_\phi^{\deg \phi} .$$

By the definition of $p_{\tau\times\phi}$, we see that the composite is trivial if $p_{\tau\times\phi} \circ f_{\partial(\sigma\times\mu)}$ is trivial when $\deg(\tau) < p-1$ or $\deg(\phi) < q-1$. Indeed, if $\deg(\tau) = i$ and $\deg(\phi) = j$, then for $(x, y) \in I^p \times \partial I^q \cup \partial I^p \times I^q$

$$\begin{aligned} p_{\tau\times\phi} \circ f_{\partial(\sigma\times\mu)}(x, y) &= p_{\tau\times\phi}(f_\sigma(x), f_\mu(y)) \\ &= \begin{cases} (p_\tau f_\sigma(x), p_\phi f_\mu(y)) & (f_\sigma(x), f_\mu(y)) \in K^{(i)} \times L^{(j)} \\ * & \text{otherwise.} \end{cases} \end{aligned}$$

Since $i + j = p + q - 1$, this can only be non-trivial if $(i, j) = (p, q-1)$ or $(i, j) = (p-1, q)$.

So, this leaves only the cases $\deg(\tau) = p-1$, $\deg(\phi) = q$ and $\deg(\tau) = p$, $\deg(\phi) = q-1$. Furthermore, if $\deg(\tau) = p$ and $\tau \neq \sigma$, then $p_\tau \circ f_\sigma = *$ so that

$$p_{\tau\times\phi} \circ f_{\partial(\sigma\times\mu)} = *$$

and similarly if $\deg(\phi) = q$. So that leaves:

$$\partial(\sigma \times \mu) = \sum_\tau [\tau \times \mu : \sigma \times \mu] \tau \times \mu + \sum_\phi [\sigma \times \phi : \sigma \times \mu] \sigma \times \phi.$$

We show:

- $\deg(p_{\tau\times\mu} \circ f_{\partial(\sigma\times\mu)}) = [\tau \times \mu : \sigma \times \mu] = [\tau : \sigma]$
- $\deg(p_{\phi\times\mu} \circ f_{\partial(\sigma\times\mu)}) = [\sigma \times \phi : \sigma \times \mu] = (-1)^{\deg \sigma}[\phi : \mu]$

First,

$$p_{\tau\times\mu} \circ f_{\partial(\sigma\times\mu)} = p_\tau \wedge p_\mu \circ (f_{\partial\sigma} \times f_\mu \cup f_\sigma \times f_{\partial\mu})$$

Since $p_\mu f_\mu = \gamma_q$, we have

$$\begin{aligned} p_{\tau\times\mu} \circ f_{\partial(\sigma\times\mu)}(x, y) &= p_{\tau\times\mu}(f_\sigma(x), f_\mu(y)) \\ &= (p_\tau f_\sigma(x), p_\mu f_\mu(y)) \\ &= (p_\tau f_\sigma(x), \gamma_q(y)) \\ &= (p_\tau f_\sigma \wedge 1) \circ (1_p \wedge \gamma_q)(x, y) \end{aligned}$$

so this map decomposes as

$$\partial(I^p \times I^q) \xrightarrow{1_p \wedge \gamma_q} \partial I^p \wedge S^q \xrightarrow{p_\tau f_{\partial\sigma} \wedge 1_q} S^{p-1} \wedge S^q$$

Similarly, $p_{\phi\times\mu} \circ f_{\partial(\sigma\times\mu)}$ gives

$$\partial(I^p \times I^q) \xrightarrow{\gamma_p \wedge 1_q} S^p \wedge \partial I^q \xrightarrow{1_p \wedge p_\phi f_{\partial\mu}} S^p \wedge S^{q-1}$$

So we show that, for $g\colon \partial I^p \to S^{p-1}$, respectively, $g\colon \partial I^q \to S^{q-1}$

$$\deg(g \wedge 1_q \circ 1_p \wedge \gamma_q) = \deg(g)$$
$$\deg(1_p \wedge g \circ \gamma_p \wedge 1_q) = (-1)^p \deg(g)$$

The suspension isomorphism inductively gives natural isomorphisms

$$H_{p-1}(X) \xrightarrow[\phi]{\cong} H_{p+q-1}(X \wedge S^q).$$

We get a commutative diagram

$$\begin{tikzcd}
H_{p-1}(\partial I^p) \arrow[r, "\cong", "\phi"'] \arrow[d, "g_*"] & H_{p+q-1}(\partial I^p \wedge S^q) \arrow[d, "(g\wedge 1_q)_*"] \\
H_{p-1}(S^{p-1}) \arrow[r, "\cong", "\phi"'] & H_{p+q-1}(S^{p-1} \wedge S^q)
\end{tikzcd}$$

Now, we orient $H_{p+q-1}(\partial I^p \wedge S^q)$ by

$$(1 \wedge \gamma_{p-1} \wedge 1_q)_*[\partial I^p \wedge S^q] := [S^{p+q-1}]$$

and we orient $H_{p+q-1}(S^p \wedge \partial I^q)$ by

$$(1_p \wedge 1 \wedge \gamma_{q-1})_*[S^p \wedge \partial I^q] := [S^{p+q-1}]$$

Now, let

$$\phi([S^{p-1}]) = \epsilon[S^{p+q-1}], \quad \phi([\partial I^p]) = \epsilon'[\partial I^p \wedge S^q].$$

The commutativity of the diagram

$$\begin{tikzcd}
H_{p-1}(\partial I^p) \arrow[r, "\cong", "\phi"'] \arrow[d, "(1\wedge\gamma_{p-1})_*"] & H_{p+q-1}(\partial I^p \wedge S^q) \arrow[d, "(1\wedge\gamma_{p-1}\wedge 1_q)_*"] \\
H_{p-1}(S^{p-1}) \arrow[r, "\cong", "\phi"'] & H_{p+q-1}(S^{p-1} \wedge S^q)
\end{tikzcd}$$

(and the fact that ∂I^p was oriented using $1 \wedge \gamma_{p-1}$) gives $\epsilon = \epsilon'$. In particular,

$$\deg(g) = \deg(g \wedge 1)$$

since

$$\deg(g)\epsilon[S^{p+q-1}] = \phi(g_*[\partial I^{p-1}]) = (g \wedge 1)_*\phi([\partial I^{p-1}]) = \epsilon \deg(g \wedge 1)[S^{p+q-1}].$$

Since

$$\begin{tikzcd}
\partial(I^p \times I^q) \arrow[r, "1_p \wedge \gamma_q"] \arrow[d, "1\wedge\gamma_{p+q-1}"'] & \partial I^p \wedge S^q \arrow[d, "1\wedge\gamma_{p-1}\wedge 1_q"] \\
S^{p+q-1} \arrow[r, "="] & S^{p-1} \wedge S^q
\end{tikzcd}$$

commutes, we also have that $\deg(1_p \wedge \gamma_q) = 1$. This gives

$$\deg(g \wedge 1 \circ 1_p \wedge \gamma_q) = \deg(g).$$

Exercise 1.14.2. The map $s_{p,q}\colon I^p \times I^q \to I^q \times I^p$ given by $s(x,y) = (y,x)$ induce maps of degree $(-1)^{pq}$.

Now, for the second claim, let $g\colon \partial I^q \to S^{q-1}$. Look at

$$\begin{tikzcd}
\partial(I^p \times I^q) \arrow[r, "\gamma_p\wedge 1"] \arrow[d, "s_{p,1}\times 1_{q-1}"'] & S^p \wedge \partial I^q \arrow[d, "1_p\wedge 1\wedge\gamma_{q-1}"] \\
\partial(I \times I^p \times I^{q-1}) \arrow[r, "1\wedge\gamma_{p+q-1}"] & S^p \wedge S^{q-1}
\end{tikzcd}$$

and note that $s_{p,1}$ has degree $(-1)^p$. Therefore, $\deg(\gamma_p \wedge 1) = (-1)^p$.

Next, we show that $\deg(1 \wedge g) = \deg(g \wedge 1) = \deg(g)$. Consider the diagram:

$$\begin{tikzcd}
H_{q-1}(\partial I^q) \arrow[r, "\cong", "\phi"'] \arrow[d, "g_*"] & H_{p+q-1}(\partial I^q \wedge S^p) \arrow[r] \arrow[d, "(g\wedge 1)_*"] & H_{p+q-1}(S^p \wedge \partial I^q) \arrow[d, "(1\wedge g)_*"] \\
H_{q-1}(S^{q-1}) \arrow[r, "\cong", "\phi"'] & H_{p+q-1}(S^{q-1} \wedge S^p) \arrow[r] & H_{p+q-1}(S^p \wedge S^{q-1})
\end{tikzcd}$$

From the way we've oriented things and the fact that the following digram commutes,

$$\begin{tikzcd}
H_{p+q-1}(\partial I^q \wedge S^p) \arrow[r] \arrow[d, "(1\wedge\gamma_{q-1}\wedge 1)_*"] & H_{p+q-1}(S^p \wedge \partial I^q) \arrow[d, "(1\wedge 1\wedge\gamma_{q-1})_*"] \\
H_{p+q-1}(S^{q-1} \wedge S^p) \arrow[r, "s_{q-1,p}"] & H_{p+q-1}(S^q \wedge S^{p-1}),
\end{tikzcd}$$

the degrees of the top and bottom arrows are equal. Therefore,

$$\deg(1 \wedge g) = \deg(g \wedge 1) = \deg(g).$$

So,

$$\deg(1 \wedge g \circ \gamma_p \wedge 1) = (-1)^p \deg(g).$$

□

1.15 The Homotopy Axiom and the Cross Product

Definition 1.15.1. Let $\phi, \psi \colon A \to B$ be chain maps. Then ϕ and ψ are *chain homotopic*, written as $\phi \simeq \psi$ if there exists a map of graded abelian groups D, $D_p \colon A_p \to B_{p+1}$ such that

$$\partial D + D\partial = \phi - \psi$$

Lemma 1.15.2. *If $\phi \simeq \psi \colon A \to B$, then $\phi_* = \psi_* \colon H_*(A) \to H_*(B)$.*

Proof. Let $[\![a]\!] \in H_p(A)$ and $a \in Z_p(A)$ be a cycle representative. Then,

$$\phi_*([\![a]\!]) = [\![\phi(a)]\!] = [\![\psi(a) + \partial D(a) + D\partial(a)]\!] = [\![\psi(a)]\!]$$

where here we used that $\partial(a) = 0$. □

Exercise 1.15.3. Chain homotopy is an equivalence relation on maps of chain complexes. Let Ch be the category of chain complex and chain maps, the homotopy category, denoted $\mathcal{K}(\mathrm{Ch})$ is the category with the same objects as Ch, but with homomorphisms the chain homotopy equivalence classes of maps. Check that this is indeed a category!

Definition 1.15.4. A chain map $\phi \colon A \to B$ is a *chain homotopy equivalence* if there is a chain map $\psi \colon B \to A$ such that $\phi \circ \psi \simeq 1_B$ and $\psi \circ \phi \simeq 1_A$.

In particular, if A and B are chain homotopy equivalent, they have isomorphic homology.

Theorem 1.15.5. *If X is contractible, the $H_i(X) = 0$ for $i \geqslant 1$ and $H_0(X) = \mathbb{Z}$.*

Proof. Let X contract onto $x_0 \in X$. Let $\epsilon \colon \Delta_*(X) \to \Delta_*(X)$ be zero for $* > 0$ and

$$\epsilon(\sum n_x x) = (\sum n_x) x_0$$

if $* = 0$. Note that ϵ is a chain map and that $\epsilon_* \colon H_*(X) \to H_*(X)$ is the identity in degree 0 and zero otherwise.

To prove the claim, we construct a chain homotopy

$$D \colon \Delta_{n-1}(X) \to \Delta_n(X)$$

with the property that

$$\partial D + D\partial = 1 - \epsilon.$$

This implies that $1_* = 0$ in positive degree, so the group $H_*(X)$ must be itself zero.

First, let

$$F \colon X \times I \to X$$

be a contraction to x_0, so that $F(x, 0) = x$ and $F(x, 1) = x_0$ for all $x \in X$. Let $\sigma \colon \Delta_{n-1} \to X$ and

$$D\sigma \colon \Delta_n \to X$$

be given by

$$D\sigma\left(\sum_{i=0}^{n}\lambda_i e_i\right) = \begin{cases} F\left(\sigma\left(\sum_{i=1}^{n}\frac{\lambda_i}{1-\lambda_0}e_{i-1}\right), \lambda_0\right) & \lambda_0 \neq 1 \\ x_0 & \lambda_0 = 1. \end{cases}$$

This is like doing σ on the 0-face of Δ_n and using the fact the latter is the cone on Δ_{n-1} with vertex e_0, together with the contraction to extend the $n-1$ simplex to an n-simplex. Extend linearly to $\Delta_{n-1}(X)$.

Now, it's an exercise to check that

$$(D\sigma)^{(j)} = \begin{cases} D\sigma^{(j-1)} & 0 < j \leqslant n \\ \sigma & j = 0. \end{cases}$$

So, if $n > 1$, then

$$\partial D\sigma = \sum_{i=0}^{n}(-1)^i (D\sigma)^{(i)} = \sigma - \sum_{j=0}^{n-1}(-1)^j D(\sigma^{(j)}) = \sigma - D\partial\sigma.$$

So,

$$\partial D\sigma + D\partial\sigma = \sigma = \sigma - \epsilon(\sigma).$$

If $n = 1$, then

$$\partial D\sigma = (D\sigma)^{(0)} - (D\sigma)^{(1)} = \sigma - x_0 = \sigma - \epsilon(\sigma).$$

Since $D\partial\sigma = D(0) = 0$, we see that in all cases,

$$D\partial + \partial D = 1 - \epsilon.$$

□

Theorem 1.15.6. *There is a bilinear map*

$$\times\colon \Delta_p(X) \times \Delta_q(Y) \to \Delta_{p+q}(X \times Y)$$

called the cross product such that

(1) If $x \in X$, then $\times\colon \Delta_0(X) \times \Delta_q(Y) \to \Delta_q(X \times Y)$ is the given by

$$x \times \tau\colon \Delta_p \to X \times Y$$

defined by $(x \times \tau)(p) = (x, \tau(p))$. Similarly for $y \in Y$ and $\sigma \in \Delta_p(X)$, $(\sigma \times y)(p) = (\sigma(p), y)$.

(2) The cross product is natural: That is, the diagram

$$\begin{tikzcd} \Delta_p(X) \times \Delta_q(Y) \arrow[r, "f_\Delta \times g_\Delta"] \arrow[d, "\times"] & \Delta_p(X') \times \Delta_q(Y') \arrow[d, "\times"] \\ \Delta_{p+q}(X \times Y) \arrow[r, "(f\times g)_\Delta"] & \Delta_{p+q}(X' \times Y') \end{tikzcd}$$

is commutative for $f\colon X \to X'$ and $g\colon Y \to Y'$.

(3) $\partial(a \times b) = \partial a \times b + (-1)^p a \times \partial b$ *for any chains* $a \in \Delta_p(X)$ *and* $b \in \Delta_q(Y)$.

Before we construct the cross product, let's look at why we care.

Exercise 1.15.7. If (X, A) and (Y, B) are pairs. The cross product induces a bilinear map

$$\times\colon H_p(X, A) \times H_q(Y, B) \to H_{p+q}(X \times Y, X \times B \cup A \times Y)$$

given by $[\![a]\!] \times [\![b]\!] := [\![a \times b]\!]$. We call $(X, A) \times (Y, B) := (X \times Y, X \times B \cup A \times Y)$.

Lemma 1.15.8. *For* $i = 0, 1$, *let* $\eta_i\colon X \to \Delta_1 \times X$ *be given by* $\eta_i(x) = (e_i, x)$. *Then,*

$$(\eta_0)_* = (\eta_1)_*\colon H_*(X) \to H_*(\Delta_1 \times X).$$

Proof. We construct a chain homotopy between $(\eta_0)_\Delta$ and $(\eta_1)_\Delta$. Note that

$$(\eta_i)_\Delta(\sigma)(x) = \eta_i \circ \sigma(x) = (e_i, \sigma(x)) = (e_i \times \sigma)(x).$$

So,

$$(\eta_i)_\Delta = e_i \times -\colon \Delta_p(X) \to \Delta_p(\Delta_1 \times X).$$

Let $\iota_1\colon \Delta_1 \to \Delta_1$ be the identity. Define

$$D\colon \Delta_n(X) \to \Delta_{n+1}(\Delta_1 \times X)$$

by $D(\sigma) = \iota_1 \times \sigma$. Then

$$\begin{aligned}\partial(\iota_1 \times \sigma) &= \partial(\iota_1) \times \sigma - \iota_1 \times \partial\sigma\\ &= e_1 \times \sigma - e_0 \times \sigma - \iota_1 \times \partial\sigma\\ &= (\eta_1)_\Delta(\sigma) - (\eta_0)_\Delta(\sigma) - \iota_1 \times \partial\sigma.\end{aligned}$$

That is,

$$\partial D + D\partial = (\eta_1)_\Delta - (\eta_0)_\Delta,$$

which proves the claim. □

Theorem 1.15.9 (Homotopy Axiom)**.** *If* $f, g : X \to Y$ *are homotopic, then* $H_*(f) = H_*(g)$.

Proof. Let $h\colon \Delta_1 \times X \to Y$ be a homotopy, so that $h\eta_0 = f$ and $h\eta_1 = g$. Then

$$H_*(f) = H_*(h\eta_0) = H_*(h)H_*(\eta_0) = H_*(h)H_*(\eta_1) = H_*(h\eta_1) = H_*(g).$$

□

So it remains to construct the cross product.

Proof of Theorem 2.3.2. We proceed by induction on $p+q$. Suppose that the cross product has been constructed in dimensions less than $p+q$. Let

$$\iota_p \colon \Delta_p \to \Delta_p$$

be the identity. This is an element of $\Delta_p(\Delta_p)$. Note that if we construct

$$\iota_p \times \iota_q \colon \Delta_p(\Delta_p) \times \Delta_q(\Delta_q) \to \Delta_{p+q}(\Delta_p \times \Delta_q),$$

then naturally and the fact that $\sigma = \sigma_\Delta(\iota_p)$ and $\tau = \tau_\Delta(\iota_q)$ forces the definition of $\sigma \times \tau$ via

$$\sigma \times \tau = (\sigma \times \tau)_\Delta(\iota_p \times \iota_q) = \sigma(\iota_p) \times \tau(\iota_q)$$

This is seen by chasing the diagram

$$\begin{tikzcd}
\Delta_p(\Delta_p) \times \Delta_q(\Delta_q) \arrow[r, "\sigma_\Delta \times \tau_\Delta"] \arrow[d, "\times"] & \Delta_p(X) \times \Delta_q(Y) \arrow[d, "\times"] \\
\Delta_{p+q}(\Delta_p \times \Delta_q) \arrow[r, "(\sigma\times\tau)_\Delta"] & \Delta_{p+q}(X \times Y)
\end{tikzcd}$$

Applying the formula for the boundary and using naturality in one dimension lower than $p+q$, you can check that $\sigma \times \tau$ so defined satisfies the boundary formula. So, it's enough to define $\iota_p \times \iota_q$. Again, suppose that the bilinear map $\times$ has been defined when $p'+q' < p+q$ and satisfies the boundary formula. Then it has been defined for

$$\partial\iota_p \times \iota_q, \qquad \iota_p \times \partial\iota_q$$

so

$$\begin{aligned}
\partial(\partial\iota_p \times \iota_q + (-1)^p \iota_p \times \partial\iota_q) &= \partial\partial\iota_p \times \iota_q + (-1)^{p-1}\partial\iota_p \times \partial\iota_q + (-1)^p(\partial\iota_p \times \partial\iota_q + (-1)^p \iota_p \times \partial\partial\iota_q)) \\
&= 0.
\end{aligned}$$

Now, since $\Delta_p \times \Delta_q$ is contractible, $(\partial\iota_p \times \iota_q + (-1)^p\iota_p \times \partial\iota_q)$ must be a boundary. Define $\iota_p \times \iota_q$ to be any element of $\Delta_{p+q}(\Delta_p \times \Delta_q)$ such that

$$\partial(\iota_p \times \iota_q) = (\partial\iota_p \times \iota_q + (-1)^p\iota_p \times \partial\iota_q).$$

The rest is easy to check. □

1.16 Sketch of Excision Proof

This is meant to be an aide to read Chapter IV, Section 17 of Bredon. I'm skipping the details.

Let X is a space and $\mathcal{U} = \{U_\alpha\}$ be an open cover. Let $\Delta_*^{\mathcal{U}}(X)$ be the subcomplex of $\Delta_*(X)$ generated by chains whose images are completely contained in U_α for some α. The key is to show that

$$\Delta_*^{\mathcal{U}}(X) \to \Delta_*(X)$$

induces an isomorphism on homology. We let $H_*^{\mathcal{U}}(X) = H_*(\Delta_*^{\mathcal{U}}(X))$. Note, using the 5-lemma, we also get an isomorphism

$$H_*^{\mathcal{U}}(X, A) = H_*(\Delta_*^{\mathcal{U}}(X)/\Delta_*^{\mathcal{U}\cap A}(A)) \to H_*(X, A)$$

where $\mathcal{U} \cap A = \{U_\alpha \cap A\}$.

If we show this fact, then we can prove excision as follows:

Theorem 1.16.1 (Excision)**.** *If $U, V \subseteq X$ are such that $X = \mathrm{int}(U) \cup \mathrm{int}(V)$, then*

$$H_*(V, U \cap V) \xrightarrow{e_*} H_*(X, U)$$

is an isomorphism.

To get the usual statement of excision for $B \subseteq A \subseteq X$, let $U = A$ and $V = X - B$, then we get an iso

$$H_*(X - B, A - B) = H_*(V, U \cap V) \to H_*(X, U) = H_*(X, A).$$

Proof. Let $\mathcal{U} = \{U, V\}$. We have that

$$\Delta_*^{\mathcal{U}}(X) = \Delta_*(U) + \Delta_*(V).$$

Further,

$$\Delta_*(U \cap V) = \Delta_*(U) \cap \Delta_*(V).$$

So, there is a diagram with short exact columns:

$$\begin{tikzcd}
\Delta_*(V \cap U) = \Delta_*(U) \cap \Delta_*(V) \arrow[r] \arrow[d] & \Delta_*^{\mathcal{U}\cap U}(U) = \Delta_*(U) \arrow[d] \\
\Delta_*(V) \arrow[r] \arrow[d] & \Delta_*^{\mathcal{U}}(X) = \Delta_*(U) + \Delta_*(V) \arrow[d] \\
\Delta_*(V)/\Delta_*(U \cap V) \arrow[r, "\cong"] & \Delta_*^{\mathcal{U}}(X)/\Delta_*(U).
\end{tikzcd}$$

We get

$$\begin{tikzcd}
H_*(V, V \cap U) \arrow[rr, "\cong"] \arrow[dr, "e_*"'] & & H_*^{\mathcal{U}}(X, U) \arrow[dl, "\cong"] \\
& H_*(X, U). &
\end{tikzcd}$$

$\square$

The key idea in the proof that $\Delta_*^{\mathcal{U}}(X) \to \Delta_*(X)$ is a homology isormorphism is called *barycentric subdivision.* The idea is to construction a natural transformation

$$\Upsilon\colon \Delta_*(-) \to \Delta_*(-)$$

which "shrinks" the simplices, but with the property that Υ is homotopic to the identity via a natural transformation

$$T\colon \Delta_*(-) \to \Delta_{*+1}(-)$$

so that

$$\partial T + T\partial = \Upsilon - 1.$$

To define Υ, we need a bit of setup:

Remark 1.16.2. Let $v \in \Delta_q$ and $\sigma = [v_0, \dots, v_p]$ so that $\sigma(\sum \lambda_i e_i) = \sum \lambda_i v_i$. Then

$$v\sigma = [v, v_0, \dots, v_p]\colon \Delta_{p+1} \to \Delta_q$$

is called the cone on σ with vertex v. We denote the group of affine p-simplices by $L_p(\Delta_q)$. The definition of $v(-)$ is extended to chains c linearly. Further, one verifies that for $c \in L_p(\Delta_q)$,

$$\partial(vc) = \begin{cases} c - v(\partial c) & p > 0 \\ c - \epsilon(c)[v] & p = 0. \end{cases}$$

In other words, the boundary of the cone on c is c minus the cone on its boundary.

Definition 1.16.3. Let $\sigma = [v_0, \dots, v_p]$ be an affine p-simplex. Then

$$\Upsilon\sigma = \begin{cases} \sigma & p = 0 \\ \underline{\sigma}\Upsilon(\partial\sigma) & p > 0. \end{cases}$$

Here, $\underline{\sigma}$ is the barycenter of σ, namely,

$$\underline{\sigma} = \frac{1}{p+1}\sum_{i=0}^{p} v_i,$$

and $\underline{\sigma}\Upsilon(\partial\sigma)$ denotes the cone construction.

Now, let $T\colon L_p(\Delta_q) \to L_{p+1}(\Delta_q)$ be given by $T = 0$ if $p = 0$, and inductively by

$$T\sigma = \underline{\sigma}(\Upsilon\sigma - \sigma - T(\partial\sigma)).$$

Proposition 1.16.4. $\Upsilon\colon L_*(\Delta_q) \to L_*(\Delta_q)$ *is a chain map and T is a chain homotopy*

$$\partial T + T\partial = \Upsilon - 1.$$

Proof.

$$\begin{aligned}
\partial T\sigma &= \partial(\underline{\sigma}(\Upsilon\sigma - \sigma - T(\partial\sigma))) \\
&= (\Upsilon\sigma - \sigma - T(\partial\sigma)) - \underline{\sigma}\partial(\Upsilon\sigma - \sigma - T(\partial\sigma)) \\
&= (\Upsilon\sigma - \sigma - T(\partial\sigma)) - \underline{\sigma}(\Upsilon\partial\sigma - \partial\sigma - \partial T(\partial\sigma)) \\
&= (\Upsilon\sigma - \sigma - T(\partial\sigma)) - \underline{\sigma}(\Upsilon\partial\sigma - \partial\sigma + (T(\partial\partial\sigma) - \Upsilon(\partial\sigma) + \partial\sigma)) \\
&= \Upsilon\sigma - \sigma - T(\partial\sigma).
\end{aligned}$$

□

Noting that for any $\sigma \in \mathrm{Sing}_p(X)$, $\sigma = \sigma_\Delta(\iota_p)$, we extend the definition of Υ and T to X via

$$\Upsilon(\sigma) = \Upsilon(\sigma_\Delta(\iota_p)) := \sigma_\Delta\Upsilon(\iota_p)$$
$$T(\sigma) = T(\sigma_\Delta(\iota_p)) := \sigma_\Delta T(\iota_p).$$

The definition is natural by design and it's straightforward to check that T is a chain homotopy from Υ to the identity.

Proposition 1.16.5. *For any X and any $k \geqslant 1$, $\Upsilon^k\colon \Delta_*(X) \to \Delta_*(X)$ is chain homotopic to the identity via*

$$T^k = (\Upsilon^{k-1} + \Upsilon^{k-2} + \ldots + \Upsilon + 1),$$

so that

$$\partial T^k + T^k\partial = \Upsilon^k - 1.$$

We can now state what we mean by "shrink":

Lemma 1.16.6. *If $\sigma = [v_0, \ldots, v_p]$, then the diameter of any simplex in the chain $\Upsilon\sigma$ is at most*

$$\frac{p}{p+1}\mathrm{diam}(\sigma).$$

In particular, $\Upsilon^k(\iota_p) \in \Delta_p(\Delta_p)$ has diameter at most

$$\left(\frac{p}{p+1}\right)^k \mathrm{diam}(\Delta_p).$$

Proposition 1.16.7. *If X is a space and $\mathcal{U} = \{U_\alpha\}$ is an open cover, then given any simplex $\sigma\colon \Delta_p \to X$, there exists $k \geqslant 1$ such that $\Upsilon^k(\sigma) \in \Delta_*^{\mathcal{U}}(X)$.*

Proof. This is an application of the Lebesgue Lemma (Lemma 9.11 of Chapter I of Bredon) and uses the compactness of Δ_p. □

Now, to prove that

$$\Delta_*^{\mathcal{U}}(X) \to \Delta_*(X)$$

induces an isomorphism on homology, the idea for surjectivity is that, if $c \in \Delta_*(X)$, then $\Upsilon^k(c) \in \Delta^{\mathcal{U}}(X)$ for some k. Similarly, if $\partial e = c$, for $c \in \Delta^{\mathcal{U}}(X)$ and $e \in \Delta_*(X)$, then

$$c = \partial(\Upsilon^k(e) - T^k(c)) \in \Delta^{\mathcal{U}}(X).$$

1.17 Mayer-Vietoris

From excision, we have the following consequence. See the book for a proof.

Theorem 1.17.1 (Mayer-Vietoris)**.** *Let $(X; X_1, X_2)$ be an excisive triad. Let $i_k\colon X_1 \cap X_2 \to X_k$ and $j_k\colon X_k \to X$ for $k = 1, 2$. There is a long exact sequence*

$$\dots \longrightarrow H_n(X_1 \cap X_2) \xrightarrow{i_1+i_2} H_n(X_1) \oplus H_n(X_2) \xrightarrow{j_1-j_2} H_n(X) \xrightarrow{D} H_{n-1}(X_1 \cap X_2) \longrightarrow \dots$$

where $D = \partial e^{-1} p_2$ for $p_2\colon H_n(X) \to H_n(X, X_2)$, $e\colon H_n(X_1, X_1 \cap X_2) \to H_n(X, X_2)$ the excision isomorphism and $\partial\colon H_n(X_1, X_1 \cap X_2) \to H_{n-1}(X_1 \cap X_2)$ the connecting homomorphism.

Exercise **1.17.2.** There is also a version of Mayer-Vietoris for reduced homology. If $(X; X_1, X_2)$ be an excisive triad with $X_1 \cap X_2 \neq \varnothing$, then the sequence

$$\dots \longrightarrow \widetilde{H}_n(X_1 \cap X_2) \xrightarrow{i_1+i_2} \widetilde{H}_n(X_1) \oplus \widetilde{H}_n(X_2) \xrightarrow{j_1-j_2} \widetilde{H}_n(X) \xrightarrow{D} \widetilde{H}_{n-1}(X_1 \cap X_2) \longrightarrow \dots$$

is exact.

Chapter 2

Cohomology

2.1 Cohomology revisted

Let G be an abelian group and let

$$\Delta^p(X;G) = \mathrm{Hom}(\Delta_p(X), G), \quad \delta = \mathrm{Hom}(\partial_p, G)\colon \Delta^{p-1}(X;G) \to \Delta^p(X;G).$$

We call $\Delta^p(X;G)$ the cochain complex of X with coefficients in G and let

$$H^p(X;G) = H^p(\Delta^*(X;G)) = \frac{\ker(\delta\colon \Delta^p(X;G) \to \Delta^{p+1}(X;G))}{\mathrm{im}(\delta\colon \Delta^{p-1}(X;G) \to \Delta^p(X;G))}.$$

This is called the *cohomology of X with coefficients in G*. Similarly,

$$H^*(X,A;G) \cong H^*(\Delta^*(X,A;G))$$

where

$$\Delta^*(X,A;G) = \mathrm{Hom}(\Delta_*(X,A), G).$$

We also defined the homology of X with coefficients in G, denoted $H_*(X;G)$, as the homology of the the chain complex

$$\Delta_p(X;G) = \Delta_p(X) \otimes G, \quad \partial = \partial \otimes G$$

and the relative homology as the homology of the complex $\Delta_*(X,A;G) = \Delta_*(X,A) \otimes G$. We've already discussed that there are long exact sequences for a pair on homology with coefficients and on cohomology. There are also long exact sequences for coefficients on homology and cohomology associated to an exact sequence

$$0 \to G \to G' \to G'' \to 0$$

if abelian groups.

2.2 Users guide to Ext and Tor groups in the category of abelian groups

Everything I will do here can be done in more generality, but that would take us astray.

Recall or check:

Exercise 2.2.1. Suppose that G is an abelian group and that

$$0 \to A \to B \to C \to 0$$

is an exact sequence of abelian group. Then there are exact sequences

$$0 \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, B) \to \mathrm{Hom}(G, C)$$
$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$$

and similarly,

$$G \otimes A \to G \otimes B \to G \otimes C \to 0.$$

We call G

- *injective* if $\mathrm{Hom}(-, G)$ is exact,
- *projective* if $\mathrm{Hom}(G, -)$ is exact, and
- *flat* if $G \otimes -$ is exact.

Check that

- G is injective if and only if G is *divisible*, that is, for every $d \in G$ and integer $n \neq 0$, there exists $e \in G$ such that $ne = d$.
- G is projective if and only if G is free,
- G is flat if and only if G is torsion-free.

Note that free abelian groups are torsion-free, so in particular, they are flat.

The Tor and Ext functors are the correction terms in the failure of exactness of Hom and $\otimes$. For each $0 \leqslant n \leqslant 1$ exist bi-functors

$$\mathrm{Tor}(-,-) \colon \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$$

and

$$\mathrm{Ext}(-,-) \colon \mathrm{Ab}^{\mathrm{op}} \times \mathrm{Ab} \to \mathrm{Ab}$$

that saitsify the following properties

(a) $\mathrm{Tor}(A, B) = \mathrm{Tor}(B, A)$.

(b) Let G be an abelian group and

$$0 \to A \to B \to C \to 0$$

be an exact sequence of abelian, then there is are long exact sequences

$$0 \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, B) \to \mathrm{Hom}(G, C) \xrightarrow{\delta} \mathrm{Ext}(G, A) \to \mathrm{Ext}(G, B) \to \mathrm{Ext}(G, C) \to 0,$$

$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G) \xrightarrow{\delta} \mathrm{Ext}(C, G) \to \mathrm{Ext}(B, G) \to \mathrm{Ext}(A, G) \to 0$$

and

$$0 \to \mathrm{Tor}(G, A) \to \mathrm{Tor}(G, B) \to \mathrm{Tor}(G, C) \xrightarrow{\delta} G \otimes A \to G \otimes B \to G \otimes C \to 0$$

Further, the connecting homomorphisms δ are natural.

Note in particular, that

- If G is projective or G' is injective $\mathrm{Ext}(G, G') = 0$.
- G is flat or G' is flat, then $\mathrm{Tor}(G, G') = 0$.

Here's how one way to compute Ext and Tor. Let G be an abelian group. Note that G has a presentation

$$0 \longrightarrow F_1(G) \xrightarrow{d} F_0(G) \longrightarrow G \longrightarrow 0$$

where $F_1(G)$ and $F_0(G)$ are free abelian groups. From this, we get a cochain complexe

$$0 \longrightarrow \mathrm{Hom}(F_0(G), G') \xrightarrow{\mathrm{Hom}(d, G')} \mathrm{Hom}(F_1(G), G') \longrightarrow 0$$

and a chain complex

$$0 \longrightarrow F_1(G) \otimes G' \xrightarrow{d \otimes G'} F_0(G) \otimes G' \longrightarrow 0$$

Then, Ext and Tor are computed as the first homology/cohomology groups of these complexes

Definition 2.2.2. We have

$$\mathrm{Ext}(G, G') = \frac{\mathrm{Hom}(F_1(G), G')}{\mathrm{im}(\mathrm{Hom}(d, G'))} = H^1(\mathrm{Hom}(F_*(G), G'))$$
$$\mathrm{Tor}(G, G') = \ker(d \otimes G') \subseteq F_1(G) \otimes G' = H_1(F_*(G) \otimes G').$$

Note further that

$$\mathrm{Hom}(G, G') \cong \ker(\mathrm{Hom}(d, G')) \subseteq \mathrm{Hom}(F_0(G), G')$$
$$G \otimes G' \cong \frac{F_0(G) \otimes G'}{\mathrm{im}(d \otimes G')} \ker(d \otimes G').$$

Recall from last time that, for a presentation $F_*(G) \to G$, giving rise to a chain complex

$$0 \longrightarrow F_1(G) \xrightarrow{d} F_0(G) \longrightarrow 0$$

where $F_1(G)$ and $F_0(G)$ are free abelian groups, we defined

$$\mathrm{Ext}(G, G') = H^1(\mathrm{Hom}(F_*(G), G')), \quad \mathrm{Tor}(G, G') = H_1(F_*(G) \otimes G').$$

Lemma 2.2.3. *There are isomorphisms*

$$\mathrm{Hom}(G, G') \cong H^0(\mathrm{Hom}(F_*(G), G')), \quad G \otimes G' \cong H_0(F_*(G) \otimes G').$$

Proof. Note that $H^0(\mathrm{Hom}(F_*(G), G')) = \ker(\mathrm{Hom}(d_1, G')) = \ker(- \circ d_1)$. This is expressed in the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & F_1(G) \arrow[r, "d_1"] \arrow[dr, "f \circ d_1 = 0"'] & F_0(G) \arrow[r, "d_0", "\cong"'] \arrow[d, "f"] & G \arrow[r] \arrow[dl, dotted, "f'"] & 0 \\
& & G' & &
\end{tikzcd}$$

A map $f \in \mathrm{Hom}(F_0(G), G')$ is in the kernel of $\mathrm{Hom}(d_1, G')$ if and only if $f \circ d_1 = 0$. Since

$$\mathrm{coker}(d_1) = F_0(G)/F_1(G) \xrightarrow[\cong]{d_0} G\,,$$

the map

$$\mathrm{Hom}(G, G') \xrightarrow{- \circ d_0} \mathrm{Hom}(F_0(G), G')$$

induces an isomorphism

$$\mathrm{Hom}(G, G') \cong \ker(d_1, G') \subseteq \mathrm{Hom}(F_0(G), G').$$

The proof for $G \otimes G'$ is left as an exercise. $\square$

We need to check that the definition of Ext and Tor doesn't depend on the choice of presentation.

Remark 2.2.4. If F is a free abelian group, then the dotted arrow always exists in the following commutative diagram:

$$\begin{tikzcd}
& F \arrow[dl, dotted] \arrow[d] & \\
M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Proposition 2.2.5. *Let $F_*(G) \to G$ and $F_*(G') \to G'$ be presentations and $f = f_{-1} \colon G \to G'$ be a homomorphism. Then, there exists a chain map $f_* \colon F_*(G) \to F_*(G')$ which extends f_{-1}. That is, there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & F_1(G) \arrow[r, "d_1^G"] \arrow[d, "f_1"] & F_0(G) \arrow[r, "d_0^G"] \arrow[d, "f_0"] & G \arrow[r] \arrow[d, "f_{-1}"] & 0 \arrow[d] \\
0 \arrow[r] & F_1(G') \arrow[r, "d_1^{G'}"] & F_0(G') \arrow[r, "d_0^{G'}"] & G' \arrow[r] & 0
\end{tikzcd}$$

Furthermore, any two choices of lift are chain homotopic.

Proof. To get f_0, we use the fact that $F_0(G') \to G'$ is surjective and $F_0(G)$ is free. Then the dotted arrow exists and we let it be f_0:

$$\begin{tikzcd}
0 \arrow[r] & F_1(G) \arrow[r, "d_1^F"] \arrow[d, dotted, "f_1"] & F_0(G) \arrow[r, "d_0^F"] \arrow[d, dotted, "f_0"] \arrow[dr] \arrow[dr, phantom, "\circ", near start, shift left=2] & G \arrow[r] \arrow[d, "f_{-1}"] & 0 \arrow[d] \\
0 \arrow[r] & F_1(G') = \ker(d_0^{G'}) \arrow[r, "d_1^{G'}"'] & F_0(G') \arrow[r, "d_0^{G'}"'] & G' \arrow[r] & 0
\end{tikzcd}$$

The map f_1 then is the canonical map induced on kernels.

If g_* is another extension, we need a map $D \colon F_0(G) \to F_1(G')$ such that

$$Dd_1^F = f_1 - g_1, \quad d_1^{G'} D = f_0 - g_0$$

To get the chain homotopy, proceed similarly as before, using the picture

$$\begin{tikzcd}
0 \arrow[r] & F_1(G) \arrow[r, "d_1^F"] \arrow[d, "f_1 - g_1"'] & F_0(G) \arrow[r, "d_0^F"] \arrow[d, "f_0 - g_0"] \arrow[dl, dotted, "D"'] & G \arrow[r] \arrow[d, "0"] & 0 \arrow[d] \\
0 \arrow[r] & F_1(G') \arrow[r, "d_1^{G'}"'] & F_0(G') \arrow[r, "d_0^{G'}"'] & G' \arrow[r] & 0
\end{tikzcd}$$

Since $d_0^{G'}(f_0 - g_0) = 0$, there is a unique lift D to $F_1(G') = \ker(d_0^{G'})$, and it's easy to check that the above identities for D.

$\square$

Corollary 2.2.6. *The* Ext *and* Tor *groups are independent of the choice of resolution. Furthermore, they are functorial in G and G'.*

Lemma 2.2.7. *If G is free, then* $\mathrm{Tor}(G, G') = 0$ *and* $\mathrm{Ext}(G, G') = 0$.

Proof. This follows from the fact that

$$\begin{tikzcd}
0 \arrow[r] & F_1(G) = 0 \arrow[r, "d"] & F_0(G) = G \arrow[r, "\mathrm{id}"] & G \arrow[r] & 0
\end{tikzcd}$$

is a presentation for G. $\square$

Exercise 2.2.8. There are isomorphisms

$$\operatorname{Ext}(\bigoplus_\alpha G_\alpha, G) \cong \prod_\alpha \operatorname{Ext}(G_\alpha, G)$$

and

$$\operatorname{Ext}(G, \prod_\alpha G_\alpha) \cong \prod_\alpha \operatorname{Ext}(G, G_\alpha)$$

and

$$\operatorname{Tor}(\bigoplus_\alpha G_\alpha, G) \cong \bigoplus_\alpha \operatorname{Tor}(G_\alpha, G).$$

Example 2.2.9. Let $G = \mathbb{Z}/p^n$ presentation for G is given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p^n} \mathbb{Z} \longrightarrow G \longrightarrow 0$$

Let $G' = \mathbb{Z}/q^m$ for another prime q. Let $k = \min(m,n)$. Tensoring the presentation with $\mathbb{Z}/q^m$ we see that

$$\operatorname{Tor}(\mathbb{Z}/p^n, \mathbb{Z}/q^m) \cong \begin{cases} \mathbb{Z}/p^k & p = q \\ 0 & \text{otherwise.} \end{cases}.$$

Applying $\operatorname{Hom}(-, \mathbb{Z}/q^m)$, we see that

$$\operatorname{Ext}(\mathbb{Z}/p^n, \mathbb{Z}/q^m) = \begin{cases} \mathbb{Z}/p^k & p = q \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.2.10 (Horseshoe Lemma)**.** *Suppose that*

$$0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$$

is an exact sequence of abelian groups and that we are given presentations $F_*(A) \to A$ and $F_*(C) \to C$. Then there exists a presentation $F_*(B)$ of B and lifts $f_*^F \colon F_*(B) \to F_*(C)$ and $g_*^F \colon F_*(A) \to F_*(B)$ such that the following diagram is commutative with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_1(A) \arrow[r, "g_1^F"] \arrow[d, "d_1"] & F_1(B) \arrow[r, "f_1^F"] \arrow[d, "d_1"] & F_1(C) \arrow[r] \arrow[d, "d_1"] & 0 \\
0 \arrow[r] & F_0(A) \arrow[r, "g_0^F"] \arrow[d, "d_0"] & F_0(B) \arrow[r, "f_0^F"] \arrow[d, "d_0"] & F_0(C) \arrow[r] \arrow[d, "d_0"] & 0 \\
0 \arrow[r] & A \arrow[r, "g"] \arrow[d] & B \arrow[r, "f"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Proof. Exercise. □

Proposition 2.2.11. *Suppose that G is an abelian group and*

$$0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$$

is an exact sequence of abelian groups. Then, there is a long exact sequence

$$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G) \xrightarrow{\delta} \mathrm{Ext}(C,G) \to \mathrm{Ext}(B,G) \to \mathrm{Ext}(A,G) \to 0$$

Proof. Apply $\mathrm{Hom}(-,G)$ to the exact sequence

$$0 \to F_*(A) \to F_*(B) \to F_*(C) \to 0$$

from the diagram of Proposition 2.2.10 Since the rows are all split exact, we get an exact sequence of chain complexes

$$0 \to \mathrm{Hom}(F_*(C),G) \to \mathrm{Hom}(F_*(B),G) \to \mathrm{Hom}(F_*(A),G) \to 0.$$

The long exact sequence we are after is the associated long exact sequence in homology. □

2.2.1 Universal Coefficient Theorem

Theorem 2.2.12. *Let C_* be a chain complex of free abelian groups. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C_*),G) \longrightarrow H^n(\mathrm{Hom}(C_*,G)) \xrightarrow{\beta} \mathrm{Hom}(H_n(C_*),G) \longrightarrow 0.$$

The sequence is natural in C_* and natural in G. Furthermore, it splits, but not naturally in C_*. Here,

$$\beta([\![f]\!])([\![c]\!]) = f(c).$$

Proof. Let $H_p = H_p(C_*)$, $Z_p = Z_p(C_*)$ and $B_p = B_p(C_*)$. Recall that there is a (split) short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Z_p \arrow[r] & C_p \arrow[l, bend right, "s"'] \arrow[r, "\partial"] & B_{p-1} \arrow[r] & 0 \end{tikzcd}$$

and a short exact sequence

$$0 \longrightarrow B_p \longrightarrow Z_p \longrightarrow H_p \longrightarrow 0$$

Let G be an abelian group. Consider the diagram

$$
\begin{tikzcd}
 & \operatorname{Ext}(Z_{p-1}, G) = 0 & & & \\
 & \operatorname{Ext}(H_{p-1}, G) \arrow[u] & \operatorname{Hom}(C_{p+1}, G) & \operatorname{Hom}(B_p, G) \arrow[l] & 0 \arrow[l] \\
0 \arrow[r] & \operatorname{Hom}(B_{p-1}, G) \arrow[r] \arrow[u] & \operatorname{Hom}(C_p, G) \arrow[r] \arrow[u, "\delta^{p+1}"] \arrow[ur, dashed] & \operatorname{Hom}(Z_p, G) \arrow[r] \arrow[u] \arrow[l, bend left] & 0 \\
0 & \operatorname{Hom}(Z_{p-1}, G) \arrow[l] \arrow[u] & \operatorname{Hom}(C_{p-1}, G) \arrow[l] \arrow[u, "\delta_p"'] & \operatorname{Hom}(H_p, G) \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}
$$

The indicated splitting is $\operatorname{Hom}(s, G)$. Further, $\operatorname{Ext}(Z_{p-1}, G) = 0$ since $Z_{p-1} \subseteq C_{p-1}$ is free. The middle row is short exact, and the right and left columns are exact.

A map $f \in \operatorname{Hom}(C_p, G)$ satisfies $\delta^{p+1} f = 0$ if and only if its image in $\operatorname{Hom}(B_p, G)$ under the dashed composite is zero. Therefore, $\ker(\delta^{p+1})$ surjects onto the kernel of $\operatorname{Hom}(Z_p, G) \to \operatorname{Hom}(B_p, G)$, which is isomorphic to $\operatorname{Hom}(H_p, G)$. We get a canonical map $\ker(\delta^{p+1}) \to \operatorname{Hom}(H_p, G)$. The commutativity of the left bottom square gives that this composite

$$\operatorname{im}(\delta^p) \to \ker(\delta^{p+1}) \to \operatorname{Hom}(H_p, G) \to 0.$$

is zero, and we let

$$H^p(C_*; G) \xrightarrow{\beta} \operatorname{Hom}(H_p, G) \to 0$$

be the induced map. If $\beta(f) = 0$, then f lifts to a map in $\operatorname{Hom}(B_{p-1}, G)$. However, since $\operatorname{Hom}(C_{p-1}, G) \to \operatorname{Hom}(Z_{p-1}, G)$ is surjective, the image of

$$\operatorname{Hom}(B_{p-1}, G) \to H^p(C_*; G)$$

is isomorphic to $\operatorname{Hom}(B_{p-1}, G)/\operatorname{Hom}(Z_{p-1}, G) \cong \operatorname{Ext}(H_{p-1}, G)$.

The splitting induces the desired splitting of the universal coefficient short exact sequence. However, this choice (as opposed to everything else) is not natural in C_*. □

Corollary 2.2.13. *Let (X, A) be a pair of topological space. Then there is an exact sequence*

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(X, A), G) \longrightarrow H^n(X, A; G) \xrightarrow{\beta} \operatorname{Hom}(H_n(X, A), G) \longrightarrow 0.$$

The sequence is natural in (X, A) and natural in G. Furthermore, it splits, but not naturally in (X, A).

Proof. This follows since $\Delta_*(X, A)$ is a chain complex of free abelian groups. □

Example 2.2.14. There is an isomorphism

$$H^p(\mathbb{R}P^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & p = 0 \text{ or, if } p = n \text{ and } n \text{ is odd} \\ \mathbb{Z}/2 & p \text{ is even, } 0 < p \leqslant n \\ 0 & \text{otherwise} \end{cases}$$

This follows from the fact that $\mathrm{Ext}(\mathbb{Z}/2,\mathbb{Z}) \cong \mathbb{Z}/2$.

Remark 2.2.15. So, in the universal coefficient theorem for spaces whose cohomology are finitely generated abelian groups, we have that the torsion shifts up and the free summands stay put.

Corollary 2.2.16. *If $H_{n-1}(X)$ is free, then*

$$H^n(X,A;G) \cong \mathrm{Hom}(H_n(X,A),G).$$

In particular, $H^1(X,A;G) \cong \mathrm{Hom}(H_1(X,A),G)$ for all X,G. Furthermore, if $H_n(X,A)$ and $H_{n-1}(X,A)$ are finitely generated abelian groups, then

$$H^n(X,A;\mathbb{Z}) \cong \mathrm{free}(H_n(X,A)) \oplus \mathrm{torsion}(H_{n-1}(X,A)).$$

Theorem 2.2.17 (Universal Coefficient Theorem)**.** *For a chain complex C_* of free abelian groups, there is a natural exact sequence*

$$0 \longrightarrow H_n(C_*) \otimes G \xrightarrow{\alpha} H_n(C_* \otimes G) \longrightarrow \mathrm{Tor}(H_{n-1}(C_*),G) \longrightarrow 0$$

which splits (naturally in G but not in C_*). Here, $\alpha([\![c]\!] \otimes g) = [\![c \otimes g]\!]$.

Proof. Exercise. It's a proof completely analogous to the cohomological version. $\square$

Corollary 2.2.18. *Let (X,A) be a pair of topological space. Then there is an exact sequence*

$$0 \longrightarrow H_n(X,A) \otimes G \longrightarrow H_n(X,A;G) \longrightarrow \mathrm{Tor}(H_{n-1}(X,A),G) \longrightarrow 0.$$

The sequence is natural in (X,A) and natural in G. Furthermore, it splits, but not naturally in (X,A).

Example 2.2.19. $H_n(\mathbb{R}P^k;\mathbb{Z}/2) \cong \mathbb{Z}/2$ for $0 \leqslant n \leqslant k$. Indeed,

$$0 \longrightarrow H_n(\mathbb{R}P^k) \otimes \mathbb{Z}/2 \longrightarrow H_n(\mathbb{R}P^k;\mathbb{Z}/2) \longrightarrow \mathrm{Tor}(H_{n-1}(\mathbb{R}P^k),\mathbb{Z}/2) \longrightarrow 0$$

gives a $\mathbb{Z}/2$ from $H_n(\mathbb{R}P^k) \otimes \mathbb{Z}/2$ when n is odd and less than or equal to k, and a $\mathbb{Z}/2$ coming from $\mathrm{Tor}(H_{n-1}(\mathbb{R}P^k),\mathbb{Z}/2)$ when n is even and less than k.

Example 2.2.20 (Non-naturality of the splitting)**.** Consider the map

$$\phi\colon \mathbb{R}P^2 \to \mathbb{R}P^2/\mathbb{R}P^1 \cong S^2.$$

We have a map of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d, "="] & \mathbb{Z}/2 \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & H_2(\mathbb{R}P^2)\otimes\mathbb{Z}/2 \arrow[r] \arrow[d] & H_2(\mathbb{R}P^2,\mathbb{Z}/2) \arrow[r] \arrow[d] & \mathrm{Tor}(H_1(\mathbb{R}P^2),\mathbb{Z}/2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_2(S^2)\otimes\mathbb{Z}/2 \arrow[r] & H_2(S^2,\mathbb{Z}/2) \arrow[r] & \mathrm{Tor}(H_1(S^2),\mathbb{Z}/2) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[u, "="'] & \mathbb{Z}/2 \arrow[r] \arrow[u, "="'] & 0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

We have

$$\begin{tikzcd}
H_2(\mathbb{R}P^2,\mathbb{Z}/2) \arrow[r, "\cong"] \arrow[d, "\cong"] & H_2(S^2,\mathbb{Z}/2) \arrow[d, "\cong"] \\
(H_2(\mathbb{R}P^2)\otimes\mathbb{Z}/2)\oplus\mathrm{Tor}(H_1(\mathbb{R}P^2),\mathbb{Z}/2) \arrow[r, "0\oplus 0"] & (H_2(S^2)\otimes\mathbb{Z}/2)\oplus\mathrm{Tor}(H_1(S^2),\mathbb{Z}/2)
\end{tikzcd}$$

so the splitting is not natural.

Theorem 2.2.21. *Suppose that $\phi\colon A_* \to B_*$ is a quasi-isomorphism of free chain complexes, then*

$$\mathrm{Hom}(\phi, G)\colon\ \mathrm{Hom}(B_*, G) \to \mathrm{Hom}(A_*, G)$$

and

$$\phi\otimes G\colon A_*\otimes G \to B_*\otimes G$$

are quasi-isomorphisms.

Proof. The map ϕ induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(A_*),G) \arrow[r] \arrow[d, "\cong"] & H^n(\mathrm{Hom}(A_*,G)) \arrow[r, "\beta"] \arrow[d] & \mathrm{Hom}(H_n(A_*),G) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(B_*),G) \arrow[r] & H^n(\mathrm{Hom}(B_*,G)) \arrow[r, "\beta"] & \mathrm{Hom}(H_n(B_*),G) \arrow[r] & 0.
\end{tikzcd}$$

So the claim follows from the 5-lemma. The claim for the tensor product is the same, except using the homological version of the universal coefficient theorem. □

Corollary 2.2.22. *If $\phi\colon (X, A) \to (Y, B)$ induces an isomorphism on homology, then it does on cohomology and homology with arbitrary coefficients. In particular, using the inclusion $i\colon (X - U, A - U) \to (X, A)$, we see that excision holds on cohomology and homology with arbitrary coefficients.*

We also get Mayer-Vietoris with coefficients and on cohomology. See the Chapter V, Section 8. Since

$$\operatorname{Hom}(\bigoplus_\alpha A_\alpha, G) \cong \prod_\alpha \operatorname{Hom}(A_\alpha, G)$$

for any abelian groups A_α and G, we have:

Theorem 2.2.23. *Suppose that $X = \bigsqcup X_\alpha$, then*

$$H^p(X,G) \cong \prod_\alpha H^p(X_\alpha, G)$$
$$H_p(X,G) \cong \bigoplus_\alpha H_p(X_\alpha, G).$$

Proof sketch. From

$$\Delta_*(X) \cong \bigoplus_\alpha \Delta_*(X_\alpha),$$

we get

$$\operatorname{Hom}(\Delta_*(X), G) \cong \prod_\alpha \operatorname{Hom}(\Delta_*(X_\alpha), G)$$
$$\Delta_*(X) \otimes G \cong \bigoplus_\alpha (\Delta_*(X_\alpha) \otimes G)$$

So what one needs to show is that homology (and cohomology) commutes with direct products and direct sums. □

Theorem 2.2.24 (Homotopy axiom)**.** *Let $f_0 \simeq f_1 \colon X \to Y$ be homotopic. Then, they induce equal maps on cohomology and homology with arbitrary coefficients.*

Proof. We do cohomology. Homology with coefficients is easier. As for homology, it suffices to prove that

$$\eta_0, \eta_1 \colon X \to \Delta_1 \times X$$

induce the same map. We proved the homotopy axiom by constructing a natural (in X) chain homotopy

$$D = \iota_1 \times X \colon \Delta_*(X) \to \Delta_*(\Delta_1 \times X)$$

so that

$$D\partial + \partial D = (\eta_1)_\Delta - (\eta_0)_\Delta$$

where f_Δ is the chain map induced by f. Applying $\operatorname{Hom}(-, G)$ to D gives a chain homotopy

$$\operatorname{Hom}(D,G)\delta + \delta \operatorname{Hom}(D,G) = (\eta_1)^\Delta - (\eta_0)^\Delta$$

where f^Δ is the cochain map induced by f. □

2.3 Products in Cohomology

Recall that

$$A_* \otimes B_* = \bigoplus_{i+j=n} A_i \otimes B_j$$

with

$$\partial(a \otimes b) = \partial a \otimes b + (-1)^{\deg a} a \otimes \partial b$$

To get a sense for the sign, note that in general, if $f\colon A_* \to A'_{*+\deg(f)}$ and $g\colon B_* \to B'_{*+\deg(g)}$ be chain maps, where $\deg(f), \deg(g)$ are integers. We let

$$f \otimes g(a \otimes b) = (-1)^{\deg(a)\deg(g)} f(a) \otimes g(b)$$

The idea is that we had to commute g and a to apply g to b, so we pick up this sign.

Our next goal is to give a sense of why the following result is true:

Theorem 2.3.1. *There are isomorphisms*

$$H_*(X \times Y; G) \cong H_*(\Delta_*(X) \otimes \Delta_*(Y) \otimes G)$$

and

$$H^*(X \times Y; G) \cong H^*(\mathrm{Hom}(\Delta_*(X) \otimes \Delta_*(Y); G)).$$

Recall:

Theorem 2.3.2. *There is a map*

$$\times\colon \Delta_p(X) \otimes \Delta_q(Y) \to \Delta_{p+q}(X \times Y)$$

called the cross product such that

(1) If $x \in X$ and $y \in Y$, then $\times\colon \Delta_0(X) \otimes \Delta_q(Y) \to \Delta_q(X \times Y)$ is the map

$$x \otimes y \mapsto (x, y).$$

(2) The cross product is natural: That is, the diagram

$$\begin{tikzcd} \Delta_p(X) \otimes \Delta_q(Y) \arrow[r, "f_\Delta \times g_\Delta"] \arrow[d, "\times"] & \Delta_p(X') \otimes \Delta_q(Y') \arrow[d, "\times"] \\ \Delta_{p+q}(X \times Y) \arrow[r, "(f\times g)_\Delta"] & \Delta_{p+q}(X' \times Y') \end{tikzcd}$$

is commutative for $f\colon X \to X'$ and $g\colon Y \to Y'$.

(3) $\partial(a \times b) = \partial a \times b + (-1)^p a \times \partial b$ for any chains $a \in \Delta_p(X)$ and $b \in \Delta_q(Y)$.

The boundary formula for $\times$

$$\partial_\otimes(a \times b) = (\partial \otimes 1 + 1 \otimes \partial)(a \otimes b) = \partial a \times b + (-1)^{\deg a} a \times \partial b$$

implies that this is a chain map.

The justification for the previous theorem is that $\times$ is a chain homotopy equivalence, so induces an isomorphism on homology. The content of this statement is the following theorem:

Theorem 2.3.3 (Eilenberg–Zilber Theorem)**.** *There exists a natural chain map*

$$\theta\colon \Delta_*(X \times Y) \to \Delta_*(X) \otimes \Delta_*(Y)$$

which, in degree 0*, is given by* $\theta((x, y)) = x \otimes y$*. Any two natural chain maps*

$$f, g\colon \Delta_*(X \times Y) \to \Delta_*(X) \otimes \Delta_*(Y)$$

which are the canonical map in degree zero are chain homotopic. A similar claim holds for chain maps

$$f, g\colon \Delta_*(X) \otimes \Delta_*(Y) \to \Delta_*(X \times Y)$$
$$f, g\colon \Delta_*(X) \otimes \Delta_*(Y) \to \Delta_*(X) \otimes \Delta_*(Y)$$
$$f, g\colon \Delta_*(X \times Y) \to \Delta_*(X \times Y)$$

In particular, $\theta \circ \times \simeq 1$ *and* $\times \circ \theta \simeq 1$*.*

The idea is the same trick we used to define $\times$ in the first place, called *acyclic models.* First, one proves the following lemma.

Lemma 2.3.4. *If* X *and* Y *are contractible, then* $H_n(\Delta_*(X) \otimes \Delta_*(Y)) = 0$ *for* $n > 0$ *and* $\mathbb{Z}$ *in degree* 0*.*

Proof. To show that the homology of a contractible space is concentrated in degree zero, we produced a chain homotopy

$$D\colon \Delta_p(X) \to \Delta_{p+1}(X)$$

such that $\partial D + D\partial = 1 - \epsilon$ where ϵ was the augmentation (zero in positive degrees and $\epsilon(\sum n_x x) = (\sum n_x)x_0$ in degree zero). The map

$$E = D_X \otimes 1 + \epsilon \otimes D_Y$$

is a chain homotopy from $1 \otimes 1$ to $\epsilon \otimes \epsilon$. □

Proof sketch for E-Z theorem. The proofs all have the same flavor. Let's do the construction of θ. For more details, see Theorem 1.2, 1.3 and 1.4 of Bredon, Chapter VI.

Assume that the claim holds in degree $< k$. Let

$$d_k\colon \Delta_k \to \Delta_k \times \Delta_k, \quad d_k(z) = (z, z).$$

be the diagonal map. Note that for any $\sigma \in \Delta_k(X \times Y)$,

$$\sigma(z) = (\pi_X\sigma(z), \pi_Y\sigma(z)) = (\pi_X\sigma, \pi_Y\sigma) \circ d_k(z)$$

So, if we define $\theta(d_k)$, then naturality will force

$$\begin{aligned}\theta(\sigma) &= \theta((\pi_X\sigma \times \pi_Y\sigma)_\Delta(d_k)) \\ &= (\pi_X\sigma)_\Delta \otimes (\pi_Y\sigma)_\Delta(\theta(d_k)).\end{aligned}$$

Consider $\theta(\partial d_k)$. By induction, this has been defined. Then,

$$\partial\theta(\partial d_k) = \theta(\partial\partial d_k) = 0.$$

Since, the complex $\Delta_*(\Delta_k) \otimes (\Delta_k)$ is acyclic, $\theta(\partial d_k)$ must be a boundary. Define $\theta(d_k)$ by choosing any element which satisfies

$$\partial\theta(d_k) = \theta(\partial d_k).$$

Similarly, let $\phi, \psi \colon \Delta_*(X \times Y) \to \Delta_*(X) \otimes \Delta_*(Y)$. Suppose D has been defined in degrees less than k so that

$$\partial D + D\partial = \phi - \psi$$

Then,

$$\partial_\otimes(\phi - \psi - D\partial)(d_k) = (\phi - \psi)(\partial d_k) - (\phi - \psi - D\partial)\partial(d_k) = 0.$$

So, there is a class Dd_k such that

$$\partial_\otimes Dd_k = \partial_\otimes(\phi - \psi - D\partial)(d_k).$$

$\square$

Now, we have sufficient motivation to study tensor products of chain complexes.

Theorem 2.3.5 (Künneth Theorem)**.** *Let A_* and B_* be free chain complexes. There is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(A_*) \otimes H_j(B_*) \xrightarrow{\times} H_n(A_* \otimes B_*) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(A_*), H_j(B_*)) \longrightarrow 0$$

The sequence is non-naturally split. Note that

$$\times([\![a]\!] \otimes [\![b]\!]) = [\![a \otimes b]\!].$$

We won't prove this theorem, but the proof is of a similar flavor as that of the Universal Coefficient Theorem. In fact, the version of the Universal Coefficient Theorem involving Tor is a special case of the Künneth Theorem, taking B_* to be the chain complex defined by $B_0 = G$ and $B_n = 0$ otherwise.

Corollary 2.3.6. *Let (X, A), (Y, B) be pairs of spaces. Then there is a natural exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X,A) \otimes H_j(Y,B) \xrightarrow{\times} H_n(X \times Y, X \times B \cup A \times Y) \to$$

$$\bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(X,A), H_j(Y,B)) \longrightarrow 0$$

The sequence is non-naturally split.

2.4 Pairings

The first thing the book does is introduce a new sign convention, so let's adopt also.

Let

$$\mathrm{Hom}(A_*, B_*)_{-p} = \mathrm{Hom}(A_*, B_*)^p = \{f : A_i \to B_{i-p}\}$$

These are just maps of graded abelian groups. Call $p = \deg(f)$. We define

$$(\delta f)(a) = \partial(f(a)) - (-1)^{\deg(f)} f(\partial(a))$$

Note that with this convention, when $p = 0$, $\delta f = 0$ if and only if f is a chain map.

So, in the singular cochains of X with coefficient in G, we think of G as a chain complex concentrated in degree zero with all zero differentials. This gives the sign:

$$\Delta^p(X; G) = \mathrm{Hom}(\Delta_*(X), G)^p, \quad (\delta f) = (-1)^{p+1} f\partial.$$

2.4.1 The cross product

Definition 2.4.1. Let Λ be a commutative, unital ring. The cohomological cross-product is defined as

$$\begin{tikzcd} \Delta^p(X;\Lambda) \otimes \Delta^q(Y;\Lambda) \arrow[rr, "\times"] \arrow[dr] & & \Delta^{p+q}(X \times Y;\Lambda) \\ & \Delta^p(X \times Y; \Lambda \times \Lambda) \arrow[ur] & \end{tikzcd}$$

by

$$f \times g = (f \otimes g) \circ \theta$$

where $f \otimes g : \Delta_p(X) \otimes \Delta_q(Y) \to \Lambda$ is the map

$$(f \otimes g)(x \otimes y) = f(x)g(y).$$

In general, we extend $f \in \Delta^p(X; G)$ to $f : \Delta_*(X) \to G$ by letting $f(x) = 0$ if $\deg(x) \neq \deg(f)$. This allows us to extend the cross product to a homomorphism

$$\times : \Delta^*(X;\Lambda) \otimes \Delta^*(Y;\Lambda) \to \Delta^*(X \times Y;\Lambda).$$

The following lemma is easy to believe so I will skip the proof. See p.322 of Bredon.

Lemma 2.4.2.

$$\delta(f \times g) = \delta f \times g + (-1)^{\deg f} f \times \delta g.$$

Theorem 2.4.3. *There is a natural homomorphism*

$$\times \colon H^p(X;\Lambda) \otimes H^q(Y;\Lambda) \to H^{p+q}(X \times Y;\Lambda).$$

This can also be defined on relative cohomology, so that there is a natural product

$$\times \colon H^p(X,A;\Lambda) \otimes H^q(Y,B;\Lambda) \to H^{p+q}((X,A) \times (Y,B);\Lambda).$$

2.4.2 The Kronecker pairing

The evaluation

$$\Delta^*(X;G) \otimes \Delta_*(X) \to G$$

defined by

$$f \otimes c \mapsto f(c).$$

We extend $f \colon \Delta_p(X) \to G$ to a map of graded abelian groups by letting it be zero on chains of degree different than p. This extends the pairing to a map

$$\Delta^*(X;G) \otimes \Delta_*(X) \to G.$$

Proposition 2.4.4. *There is a well-defined pairing*

$$H^*(X;G) \otimes H_*(X) \to G$$

is defined by

$$[\![f]\!] \otimes [\![c]\!] \mapsto f(c) =: \langle [\![f]\!], [\![c]\!] \rangle.$$

called the Kronecker pairing.

Proof. Check that $f(c)$ doesn't depend on the choice of representatives f and c. □

It interacts with the cross product as follows:

Lemma 2.4.5. *The evaluation*

$$Z^*(X \times Y;G) \otimes Z_*(X \times Y) \to G$$

satisfies

$$(f \times g)(a \times b) = (-1)^{\deg(f)\deg(a)} f(a)g(b).$$

Proof. See p.323 of Bredon. □

Definition 2.4.6. Let $1 \in H^0(X)$ be the class of the augmentation $\epsilon \colon \Delta_0(X) \to \mathbb{Z}$.

The canonical maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ give maps

$$H^*(X) \xrightarrow{\pi_X^*} H^*(X \times Y) \xleftarrow{\pi_Y^*} H^*(Y)$$

Lemma 2.4.7.

$$\alpha \times 1 = \pi_X^*(\alpha) \in H^*(X \times Y)$$

and similarly, for $1 \times \beta$ and relative classes.

Proof. Recalling the definition of the cross-product, the map

$$\Delta_p(X) \otimes \Delta_0(Y) \xrightarrow{\times} \Delta_p(X \times Y) \xrightarrow{(\pi_X)_\Delta} \Delta_p(X)$$

sends $\tau \otimes y$ to τ, and so $\tau \otimes \sigma$ to $\tau\epsilon(\sigma)$. So on cohomology, we get

$$\times^* \pi_X^*(\alpha) = \alpha \otimes \epsilon$$

Applying θ^* to both side, noting that θ^* is an inverse to $\times^*$, we get

$$\pi_X^*(\alpha) = \theta^* \times^* \pi_X^*(\alpha) = \theta^*(\alpha \otimes \epsilon) = \alpha \otimes \epsilon \circ \theta = \alpha \times 1.$$

$\square$

Finally, we turn to the question of commutativity.

There is a canonical homeomorphism

$$T : X \times Y \to Y \times X$$

which maps (x, y) to (y, x).

Proposition 2.4.8. *For $\alpha \in H^*(X; \Lambda)$ and $\beta \in H^*(Y; \Lambda)$, we have*

$$\alpha \times \beta = (-1)^{\deg(\alpha)\deg(\beta)} T^*(\beta \times \alpha)$$

Proof. There is a natural chain map

$$\tau : A_* \otimes B_* \to B_* \otimes A.$$

defined by $\tau(a \otimes b) = (-1)^{\deg(a)\deg(b)} b \otimes a$. Consider the diagram

$$\begin{tikzcd}
\Delta_*(X \times Y) \arrow[r, "\theta_{X,Y}"] \arrow[d, "T_\Delta"] & \Delta_*(X) \otimes \Delta_*(Y) \\
\Delta_*(Y \times X) \arrow[r, "\theta_{Y,X}"] & \Delta_*(Y) \otimes \Delta_*(X) \arrow[u, "\tau"']
\end{tikzcd}$$

Since both $\theta_{X,Y}$ and $\tau\theta_{Y,X}T_\Delta$ are the canonical map in degree zero, they are chain homotopic.

Let $D\colon \Delta_*(X \times Y) \to \Delta_*(X) \otimes \Delta_*(Y)$ be a chain homotopy

$$\partial_\otimes D + D\partial = \tau\theta_{Y,X}T_\Delta - \theta_{X,Y}.$$

Let $\alpha = [\![f]\!]$ and $\beta = [\![g]\!]$ so that f, g are cocycles. We have

$$\begin{aligned} T*(\beta \times \alpha) &= [\![g \times f \circ T_\Delta]\!] \\ &= [\![g \otimes f \circ \theta_{Y,X} \circ T_\Delta]\!] \\ &= (-1)^{\deg(\alpha)\deg(\beta)}[\![f \otimes g \circ \tau \circ \theta_{Y,X} \circ T_\Delta]\!] && \text{(We use commutativity of } \Lambda\text{)} \\ &= (-1)^{\deg(\alpha)\deg(\beta)}[\![f \otimes g \circ (\theta_{X,Y} + \partial_\otimes D + D\partial)]\!] \end{aligned}$$

Let's examine

$$(f \otimes g) \circ (\partial_\otimes D + D\partial)$$

First, since both f and g are cocycles,

$$(f \otimes g) \circ \partial_\otimes = 0.$$

Further,

$$(f \otimes g) \circ D\partial = \pm\delta((f \otimes g) \circ D)$$

so this is a coboundary. Therefore,

$$\begin{aligned} T^*(\beta \times \alpha) &= (-1)^{\deg(\alpha)\deg(\beta)}[\![f \otimes g \circ (\theta_{X,Y})]\!] \\ &= (-1)^{\deg(\alpha)\deg(\beta)}[\![f \times g]\!] \\ &= (-1)^{\deg(\alpha)\deg(\beta)}\alpha \times \beta. \end{aligned}$$

□

2.4.3 Remarks over fields

Note that

$$\mathrm{Hom}(\Delta_*(X); \Lambda) \cong \mathrm{Hom}_\Lambda(\Delta_*(X) \otimes \Lambda; \Lambda).$$

If Λ is a field, the functor

$$\mathrm{Hom}_\Lambda(-, \Lambda)$$

is exact, so it commutes with cohomology. Therefore,

$$\begin{aligned} H^*(X; \Lambda) &= H^*(\mathrm{Hom}(\Delta_*(X); \Lambda)) \\ &= H^*(\mathrm{Hom}_\Lambda(\Delta_*(X) \otimes \Lambda; \Lambda)) \\ &\cong \mathrm{Hom}_\Lambda(H_*(\Delta_*(X) \otimes \Lambda); \Lambda) \\ &\cong \mathrm{Hom}_\Lambda(H_*(X; \Lambda); \Lambda) \end{aligned}$$

So, cohomology is the dual of homology over field coefficients. Furthermore, over a field,

$$H_*(X;\Lambda) \otimes_\Lambda H_*(Y;\Lambda) \xrightarrow[\cong]{\times} H_*(X \times Y;\Lambda)$$

The idea is that, if we are working in the category of Λ-vector spaces, there are no non-trivial Ext and Tor functors since the functors

$$(-) \otimes_\Lambda (-) \colon \mathrm{Vect}_\Lambda \times \mathrm{Vect}_\Lambda \to \mathrm{Vect}_\Lambda$$

and

$$\mathrm{Hom}_\Lambda(-,-) \colon \mathrm{Vect}_\Lambda^{\mathrm{op}} \times \mathrm{Vect}_\Lambda \to \mathrm{Vect}_\Lambda$$

are exact in both variables.

Furthermore, if $H_*(X;\Lambda)$ is of finite type (f.d. in each degree), then

$$H^*(X;\Lambda) \otimes_\Lambda H^*(Y;\Lambda) \xrightarrow[\cong]{\times} H^*(X \times Y;\Lambda)\ .$$

The finiteness condition is to ensure that tensor product and dualization commute.

2.4.4 The Cup Product

Let $d \colon X \to X \times X$ be the diagonal. The cup product

$$\cup \colon \Delta^*(X;\Lambda) \otimes \Delta^*(X;\Lambda) \to \Delta^*(X;\Lambda)$$

is defined as

$$f \cup g = d^\Delta(f \times g) = f \otimes g \circ \theta \circ d_\Delta.$$

It satisfies the formula

$$\delta(f \cup g) = \delta f \cup g + (-1)^{\deg f} f \cup \delta g.$$

Therefore, we get a well-defined product

$$\cup \colon H^*(X;\Lambda) \otimes H^*(X;\Lambda) \to H^*(X;\Lambda)$$

For cohomology classes α, β, we have

$$\alpha \cup \beta = (-1)^{\deg \alpha \deg \beta} \beta \cup \alpha.$$

Remark 2.4.9. Let 1 be the class represented by the augmentation ϵ. Then $1 \cup \alpha = \alpha = \alpha \cup 1$ for every α. This follows from the fact that $\alpha \times 1 = \pi_1^* \alpha$ where π_1^* is projection onto the first factor, and similarly for the second factor.

Remark 2.4.10. If A and B are open subsets of X, there is also a relative cup product

$$\cup \colon H^*(X, A;\Lambda) \otimes H^*(X, B;\Lambda) \to H^*(X, A \cup B;\Lambda)$$

Definition 2.4.11. The Alexander-Whitney diagonal map $\Delta_*(X) \to \Delta_*(X) \otimes \Delta_*(X)$ is defined as

$$\Delta\sigma = \sum_{p+q=n} \sigma_{[0,\ldots,p]} \otimes \sigma_{[n-q,\ldots,n]}$$

for $\sigma\colon \Delta_n \to X$.

Remark 2.4.12. Any two chain maps $\Delta_*(X) \to \Delta_*(X) \otimes \Delta_*(X)$ which map x to $x \otimes x$ are chain homotopic. This can be shown using the method of acyclic models.

Corollary 2.4.13. *For $f \in \Delta^p(X;\Lambda)$ and $g \in \Delta^q(X;\Lambda)$ and $\sigma \in \Delta_n(X)$,*

$$f \cup g(\sigma) = (-1)^{pq} f(\sigma_{[0,\ldots,p]}) g(\sigma_{[n-q,\ldots,n]})$$

Example 2.4.14. One can compute directly using this formula the product structure on $H^*(\mathbb{R}P^2;\mathbb{Z}/2)$. See Example 4.7, p.330 of Bredon.

The cup product satisfies the following properties:

(a) If $\phi\colon X \to Y$ is a continuous map, then $\phi^*(\alpha \cup \beta) = \phi^*(\alpha) \cup \phi^*(\beta)$. That is, ϕ^* is a ring homomorphism.

(b) The cup product is associative.

(c) The cup product is graded commutative, $\alpha \cup \beta = (-1)^{\deg\alpha \deg\beta} \beta \cup \alpha$

(d) Let $A \subseteq X$ and consider the exact sequence

$$H^*(X;\Lambda) \xrightarrow{i^*} H^*(A;\Lambda) \xrightarrow{\delta^*} H^{*+1}(X,A;\Lambda) .$$

Then

$$\delta^*(\alpha \cup i^*(\beta)) = \delta^*(\alpha) \cup \beta.$$

Proposition 2.4.15. *Let $X = U \cup V$ for U, V acyclic open sets. Then $\alpha \cup \beta = 0$ for positive degree classes α and β.*

Proof. This follows since $\alpha \cup \beta$ is the image of $\overline{\alpha} \cup \overline{\beta}$ in the following diagram, where $\overline{\alpha}$ and $\overline{\beta}$ are the lifts of α and β in the exact sequences displayed:

$$\begin{array}{ccccc} & & \overline{\alpha} \in H^p(X,U) \longrightarrow \alpha \in H^p(X) \longrightarrow H^p(U) = 0 \\ \overline{\alpha} \cup \overline{\beta} \in H^{p+q}(X, U \cup V) & \longrightarrow & \alpha \cup \beta \in H^{p+q}(X) \\ & & \overline{\beta} \in H^q(X,V) \longrightarrow \beta \in H^q(X) \longrightarrow H^q(V) = 0 \end{array}$$

$\square$

Corollary 2.4.16. *All products of elements in positive degree of $H^*(\Sigma X)$ are zero.*

It follows that, as rings,

$$H^*(\Sigma^k(\mathbb{R}P^2 \vee S^3)) \cong H^*(\Sigma^k \mathbb{R}P^3).$$

In fact, it turns out

$$\Sigma^2(\mathbb{R}P^2 \vee S^3) \simeq \Sigma^2 \mathbb{R}P^3.$$

We say that these spaces are *stably equivalent*.

2.4.5 The cap product

I will leave the proof out of this section. For the most part, they are straightforward verifications. See Bredon for the details.

Definition 2.4.17. The *cap product* is the pairing

$$\cap \colon \Delta^p(X; G) \otimes \Delta_n(X) \to \Delta_{n-p}(X; G)$$

specified by

$$f \cap c = (1 \otimes f)\Delta c$$

where Δ is the Alexander-Whitney map.

Remark 2.4.18. For $\sigma \in \Delta_n(X)$ and $f \in \Delta^p(X; G)$, we get the formula

$$f \cap \sigma = (-1)^n f(\sigma_{[n-p,\dots,n]})\sigma_{[0,\dots,n-p]}.$$

Proposition 2.4.19. *The cap product satisfies the following properties:*

(a) $\epsilon \cap c = c$

(b) $\epsilon(f \cap c) = f(c)$ *if* $\deg f = \deg c$

(c) $(f \cup g) \cap c = f \cap (g \cap c)$

(d) If $\phi\colon X \to Y$ *is a continuous function,* $f \in \Delta^p(Y)$ *and* $c \in \Delta_n(X)$*, then*

$$\phi_\Delta(\phi^\Delta(f) \cap c) = f \cap \phi_\Delta(c).$$

(e) If $f \in \Delta^p(X)$*, then* $\partial(f \cap c) = \delta f \cap c + (-1)^p f \cap \partial c$.

Because of the boundary formula, we get a well defined product:

$$\cap \colon H^p(X; G) \otimes H_n(X) \to H_{n-p}(X; G)$$

which then satisfies

(a) $1 \cap \gamma = \gamma$

(b) $\epsilon_*(\alpha \cap \gamma) = \langle \alpha, \gamma \rangle$ if $\deg \alpha = \deg \gamma$.

(c) $(\alpha \cup \beta) \cap \gamma = \alpha \cap (\beta \cap \gamma)$

(d) If $\phi\colon X \to Y$ is a continuous function,

$$\phi_*(\phi^*(\alpha) \cap \gamma) = \alpha \cap \phi_*(\gamma).$$

Further, for $\alpha \in H^p(X)$, $\beta \in H^q(Y)$, $a \in H_n(X)$, $b \in H_m(Y)$,

$$(\alpha \times \beta) \cap (a \times b) = (-1)^{\deg a \deg \beta} (\alpha \cap a) \times (\beta \cap b).$$

Corollary 2.4.20. *Let $\alpha \cup \beta \in H^n(X)$ and $\gamma \in H_n(X)$, $\mu \in H^n(Y)$ and $\phi\colon X \to Y$. Then*

(a) $\langle \alpha \cup \beta, \gamma \rangle = \langle \alpha, \beta \cap \gamma \rangle$

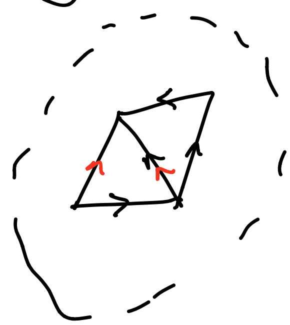
(b) $\langle f^*(\mu), \gamma \rangle = \langle \alpha, f_*(\gamma) \rangle$

Remark 2.4.21. There is also a relative cap product:

$$\cap\colon H^p(X, A; G) \otimes H_n(X, A \cup B; G) \to H_{n-p}(X, B; G)$$

2.5 Classical Outlook on Duality

Duality



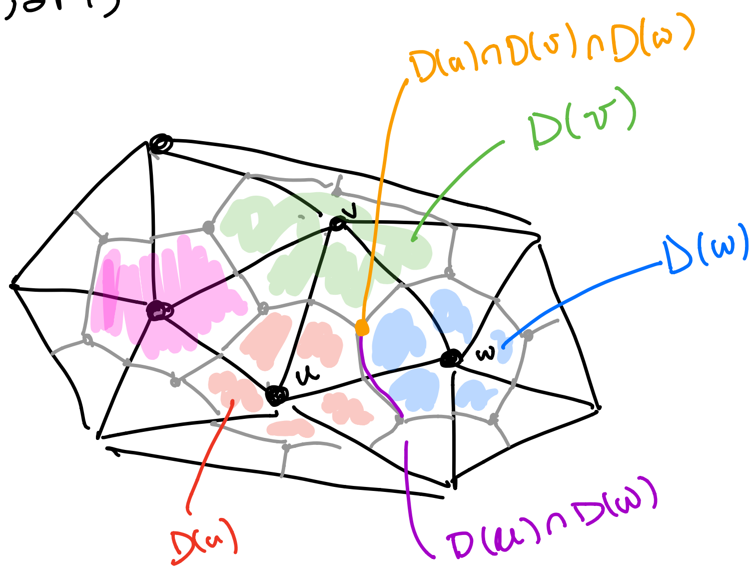
$=$ sum of the simplices in the triangulation

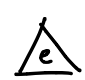
$\partial c = 0$ provided we can orient all the simplices.

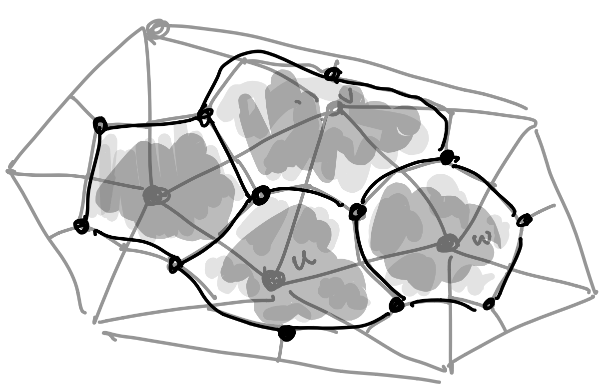
$$\rightsquigarrow \quad H_n(M) \cong \mathbb{Z}\{[c]\}$$

$\uparrow$ fundamental class. $[M]$

$[c] \in H_n(M;\partial M)$







$D(u)$: $C_2^D(M) \to \Lambda$

indicator function of the cell $D(u)$

$D(\langle u,v\rangle)$: $C_1^D(M) \to \Lambda$

indicator function of the cell $D(u) \cap D(v)$

$D(\langle u,v,w\rangle)$: $C_0^D(M) \to \Lambda$

ind. fn. of the cell $D(u) \cap D(v) \cap D(w)$

$$\boxed{H_p(M) \cong H^{n-p}(M)}$$

Eg $D([M]) = \underline{\underline{\varepsilon}}$

$D([M]) = [M]^*$ (evaluate as 1 on exactly one of the elements is sum of all cells.)

2.6 Orientations

At this point, I will follow a mix of Bredon, May and Hatcher.

Definition 2.6.1. A topological n-manifold is a second countable Hausdorff space M such that every point $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^n$.

In the rest of this section, M will be a connected n-manifold. We will write U for coordinate charts so that $U \subseteq \mathbb{R}^n \subseteq M$ as the open unit ball. Note that $U \cong \mathbb{R}^n$ in this case.

If $x \in M$, then

$$H_n(M, M - \{x\}; G) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; G) \cong G$$

for any coefficient group G. We let

$$j_{x,X} \colon H_n(M, M - X; G) \to H_n(M, M - \{x\}; G)$$

be the map induced by the inclusion of the pairs.

Suppose that is a coordinate chart as above, then

$$H_n(M, M - U; G) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - U; G) \cong G$$

So every point has a neighborhood U so that $H_n(M, M - U; G) \cong G$.

Definition 2.6.2. A Λ-fundamental class of M at $X \subseteq M$ is a class $\alpha \in H_n(M, M - X)$ with the property that $j_{x,X}(\alpha) \in H_n(M, M - x)$ is a generator for every $x \in X$. If $X = M$, we simply call α a fundamental class.

There are various definitions out there, but they all capture the same concept. I'll go with Hatcher/Bredon's definition since it is most precise. First, we introduce the orientation cover.

Definition 2.6.3. Let

$$M_\Lambda = \{(x, a) : x \in M, a \in H_n(M, M - x; \Lambda)\}.$$

The space M_Λ is topologized using the basis of open sets indexed by open sets $U \subseteq M$ and classes $\alpha \in H_n(M, M - U)$ where

$$U_\alpha = \{(x, a) : x \in U, a = j_{x,U}(\alpha)\}.$$

Proposition 2.6.4. *The map $p \colon M_\Lambda \to M$ which sends (x, a) to x is a covering space.*

Proof idea. Let $x \in M$ and U be one of our coordinate charts at x so that $H_n(M, M - U; \Lambda) \cong \Lambda$. Then, the following diagram gives the required maps to check that p is a covering space:

$$\begin{tikzcd}
M_\Lambda \supseteq p^{-1}(U) \arrow[dr, "p"'] & & U \times H_n(M, M - U; \Lambda) \arrow[ll, "(y,\alpha)\mapsto(y,j_{y,U}(\alpha))"'] \arrow[rr, "\cong", "(y,\alpha)\mapsto(y,j_{x,U}(\alpha))"'] & & U \times H_n(M, M - x; \Lambda) \arrow[dll] \\
& M \supseteq U & & &
\end{tikzcd}$$

$\square$

Definition 2.6.5. The orientation double cover of M is the subcover $\widetilde{M} \subseteq M_{\mathbb{Z}}$ consisting of pairs $(x, a) \in M_{\mathbb{Z}}$ such that a is a generator of $H_n(M, M - x)$.

Definition 2.6.6. A Λ-orientation of M along $X \subseteq M$ is a continuous section $s\colon X \to M_\Lambda$ such that $s(x) \in H_n(M, M - x; \Lambda)$ is a generator for every $x \in X$. If there exists a $\mathbb{Z}$-orientation of M along M, we simply call it an orientation and say that M is orientable.

Note that an orientation is the same as a continuous section $s\colon M \to \widetilde{M}$.

Remark 2.6.7. If $\widetilde{M}$ is disconnected, then $\widetilde{M} \to M$ restrict to a bijective local homeomorphism, a to a homeomorphism, on each connected component. Therefore, $\widetilde{M} = M \sqcup M$ and $\widetilde{M} \to M$ is the trivial double cover. In this case, there are two sections $s_{\pm 1}\colon M \to \widetilde{M}$.

Theorem 2.6.8. *The following are equivalent:*

(1) M is orientable

(2) M is orientable along any compact subset

(3) $\widetilde{M} \to M$ is a trivial double cover.

Proof. Suppose that M is orientable, then an orientation along any compact subset is obtained by restricting an orientation of M. We've already discussed why (3) implies (1). So, suppose that M is orientable along every compact subset. Let $x \in M$ and (x, a) and $(x, -a)$ be the two fibers of x in $\widetilde{M}$. We may assume that $\widetilde{M}$ is path connected, since otherwise (3) holds. Let $\lambda\colon [0, 1] \to \widetilde{M}$ be a path from (x, a) to $(x, -a)$. Then, $X = p\lambda([0, 1])$ is a compact subset of M, so there is a section $s\colon X \to \widetilde{M}$. Note that both $sp\lambda$ and λ are lifts of the loop $p\lambda$. By the unique path lifting property, they must be equal, a contradiction since $sp\lambda$ is a loop and λ is not. $\square$

Remark 2.6.9. For any ring Λ, there is a commutative diagram

$$\begin{tikzcd} M_{\mathbb{Z}} \arrow[rr] \arrow[dr] & & M_\Lambda \arrow[dl] \\ & M & \end{tikzcd}$$

Note that there is a canonical isomorphism

$$H_n(M, M - x; \Lambda) \cong H_n(M, M - x) \otimes \Lambda$$

induced by the map

$$\Delta_n(M, M - x) = \Delta_n(M, M - x) \otimes \Lambda$$

which sends c to $c \otimes 1$. So, the canonical isomorphism maps $(x, a) \to (x, a \otimes 1)$.

Lemma 2.6.10. *For each $x \in M$, let $a_x \in H_n(M, M - x)$ be any choice of generator. Let*

$$M_\lambda = \{(x, a_x \otimes \lambda) : x \in M, \lambda \in \Lambda\}$$

Consider the set $\overline{\Lambda} = \Lambda/\lambda \sim -\lambda$. As spaces

$$M_\Lambda = \coprod_{[\lambda] \in \overline{\Lambda}} M_\lambda \cup M_{-\lambda}.$$

Proof. The M_λ are disjoint, so we just need to show they are open. Up to a choice of sign

$$(x, a_x \otimes \lambda) \in M_\lambda$$

Choose one of our coordinate charts U and a lift α of a_x in $H_n(M, M - U)$. Consider the basic open set

$$(x, a_x \otimes \lambda) \in U_{\alpha \otimes \lambda}.$$

For $y \in U$, $j_{y,U}(\alpha) = \pm a_y$. Therefore, $j_{y,U}(\alpha \otimes \lambda) = \pm a_y \otimes \lambda \in M_\lambda \cup M_{-\lambda}$. So, $M_\lambda \cup M_{-\lambda}$ is open. □

Proposition 2.6.11. *Let M be a connected n-manifold. If M is orientable, then it is Λ-orientable for any Λ. If M is not orientable, it is Λ-orientable if and only if Λ contains a unit of order 2.*

Proof. For the first claim, one can compose the section $s \colon M \to M_{\mathbb{Z}}$ with the map $M_{\mathbb{Z}} \to M_\Lambda$. Suppose M is not orientable, but λ is unit of order two, then $x \mapsto a_x \otimes \lambda$ defines an orientation as, in this case, $M_\lambda = M_{-\lambda} \cong M$. Conversely, suppose that M is Λ-orientable and $s \colon M \to M_\Lambda$ is a section, then $s(M) \subseteq M_\lambda \cup M_{-\lambda}$ for some unit $\lambda \in \Lambda$. The map

$$\phi \colon \widetilde{M} \to M_\lambda \cup M_{-\lambda}$$

which maps $(x, a_x) \to (x, a_x \otimes \lambda)$ is a map of covering space. If it is injective, then it is a homeomorphism, which would give a section $\phi^{-1} \circ s \colon M \to \widetilde{M}$. Therefore, there must be a point x such that (x, a_x) and $(x, -a_x)$ have the same image. That is, $a_x \otimes \lambda = -a_x \otimes \lambda$. This implies that $2\lambda = 0$, so that λ is a unit of order two. □

Now we state a series of hard theorems and we will discuss parts of their proof.

Theorem 2.6.12 (Vanishing Theorem)**.** *Let M be a connected n-manifold for $n \geqslant 1$. For any G, $H_i(M; G) = 0$ if $i > n$. If M is not compact, then $H_n(M; G) = 0$.*

For now, we assume this and use it to finish the discussion on orientability.

Theorem 2.6.13 (Compact orientation theorem)**.** *Let K be a compact subset of M.*

- *$H_i(M, M - K; G) = 0$ for $i > n$*

- *A continuous section* $s\colon M \to M_\Lambda$*, with* $s(x) = (x, z_x)$*, uniquely determines a class* $z_K \in H_n(M, M - K; \Lambda)$ *which maps to* $z_x \in H_n(M, M - x; \Lambda)$ *under* $j_{x,K}$ *for every* $x \in K$*.*

Theorem 2.6.14. *Let* M *be a compact connected* n*-manifold. Then*

(a) If M *is* Λ*-orientable, then* $H_n(M; \Lambda) \to H_n(M, M - x; \Lambda) \cong \Lambda$ *is an isomorphism for any* $x \in M$*.*

(b) If M *is not* Λ*-orientable, then* $H_n(M; \Lambda) \to H_n(M, M - x; \Lambda) \cong \Lambda$ *is an isomorphism onto the* 2*-torsion subgroup of* Λ*.*

Proof of Theorem 2.6.14. Take $M = K$ and suppose that M is orientable. Consider

$$0 = H_n(M - x; \Lambda) \to H_n(M; \Lambda) \to H_n(M, M - x; \Lambda) \cong \Lambda$$

where the vanishing comes from the fact that $M - x$ is not compact. Since M is Λ-orientable, there is a Λ-fundamental class z_M which maps to a generator z_x. Since this is a map of Λ-modules, the map is also surjective, so is an isomorphism,

Suppose that M is not orientable. Again, $H_n(M; \Lambda) \to H_n(M, M - x; \Lambda) \cong \Lambda$ is injective. A class $\alpha \in H_n(M; \Lambda)$ determines a section

$$s_\alpha\colon M \to M_\Lambda$$

by $x \mapsto (x, \alpha_x) = (x, a_x \otimes \lambda)$, where α_x is the image of α in $H_n(M, M - x; \Lambda)$. So, the image of s_α lands in M_λ. As before, there is a map $\widetilde{M} \to M_\lambda$, and it cannot be injective since otherwise M would be orientable. Therefore, $\lambda = -\lambda$, so the image is inside the 2 torsion. If λ is a two torsion element, then there is a section $M \to M_\lambda \cong M$. This determines a class $z_M \in H_n(M; \Lambda)$ which maps to $z_x = a_x \otimes \lambda$. That is, z_M maps to λ under $H_n(M; \Lambda) \to H_n(M, M - x; \Lambda)$.

□

Before we discuss the proof of the Compact orientation theorem Theorem 2.6.13, we prove the following lemma:

Lemma 2.6.15. *A section* $s\colon M \to M_\Lambda$*,* $s(x) = (x, z_x)$*, uniquely determines a unique class of* $z_U \in H_n(M, M - U; \Lambda)$ *such that* $j_{x,U}(z_U) = z_x$ *for* U *one of our coordinate charts.*

Proof. The section determines a class $z_x \in H_n(M, M - x; \Lambda)$. For any $x \in U$, $H_n(M, M - U; \Lambda) \to H_n(M, M - x; \Lambda)$ is an isomorphism, so specifying a class z_x determines a unique class z_U. □

Proof sketch of Theorem 2.6.13. Any compact subset of M is contained in a finite union $K_1 \cup \ldots \cup K_n$, where each K_i is contained in a coordinate chart. So the proof has two steps

(1) The claim holds for $K \subseteq U$ a coordinate chart.

(2) If the claim holds for K, L and $K \cap L$, then it holds for $K \cup L$.

For (1), letting $i > n$, excision gives

$$H_i(M, M-K; G) \cong H_i(U, U-K; G) \cong \widetilde{H}_{i-1}(U-K; G) = 0$$

where the last equality is the vanishing theorem since $U-K$ is an open subset of $\mathbb{R}^n$, so in particular, isn't compact. Let z the class of $H_n(M, M-U; \Lambda)$ determined by the orientation, then we define z_K to be the image of z in $H_n(M, M-K; \Lambda)$.

For (2), use Mayer-Vietoris:

$$H_{i+1}(M, M-K\cap L) \longrightarrow H_i(M, M-K\cup L) \longrightarrow H_i(M, M-K) \oplus H_i(M, M-L)$$

$$\to H_i(M, M-K\cup L) \longrightarrow \ldots$$

The vanishing statement is clear when $i > n$.

Now, suppose that the section uniquely determines classes z_K and z_L and $z_{K\cap L}$. Then note that z_K and z_L both restrict to $z_{K\cap L}$ since they agree with the section on every point x of $K \cap L$. The sequence

$$0 = H_{n+1}(M, M-K\cap L) \longrightarrow H_n(M, M-K\cup L) \longrightarrow H_n(M, M-K) \oplus H_n(M, M-L)$$

$$\to H_n(M, M-K\cup L) \longrightarrow \ldots$$

implies that there is a unique class $z_{K\cup L}$ which maps to the class (z_K, z_L), since that element is in the kernel of the next map. It follows that $z_{K\cup L}$ is determined by the section.

For the induction step, assume the claim holds for any union of i compact sets contained in coordinate charts. Then it holds for $K_1 \cup \ldots \cup K_i$ and for

$$K_{i+1} \cap (K_1 \cup \ldots \cup K_i) = (K_{i+1} \cap K_1) \cup \ldots \cup (K_{i+1} \cap K_i)$$

so it does for $K_1 \cup \ldots \cup K_{i+1}$. □

We now discuss the proof of the vanishing theorem. Recall

Theorem 2.6.16 (Vanishing Theorem)**.** *Let M be a connected n-manifold for $n \geqslant 1$. For any G, $H_i(M; G) = 0$ if $i > n$. If M is not compact, then $H_n(M; G) = 0$.*

Remark 2.6.17. Consider a cycle $c = \sum \sigma_j$ representing a class in $[\![c]\!] \in H_i(M)$. Let $K = \bigcup_j \operatorname{im}(\sigma_j)$. Then, $\Delta_i(K)$ is a subcomplex of $\Delta_i(M)$ so c is a cycle for K and $[\![c]\!]$ is the image of this cycle under the natural map:

$$H_i(K) \to H_i(M).$$

In other words, homology is *compactly supported*. So, to prove that $H_i(M) = 0$, it's enough to prove that for any compact set K, there is a subspace U with

$$K \subseteq U \subseteq M$$

such that $H_i(U) = 0$. Then $H_i(K) \to H_i(M)$ factors through $H_i(U) = 0$.

Since any compact set K is contained in a finite union of coordinate charts $U_1 \cup \ldots \cup U_n$, the proof of the vanishing theorem then reduces to proving:

Lemma 2.6.18. *If $U_1 \cup \ldots \cup U_n$ is a union of coordinate charts as above, then $H_i(U_1 \cup \ldots \cup U_m) = 0$ for $i > n$. If M is not compact, then we also have that $H_n(U_1 \cup \ldots \cup U_m) = 0$.*

Proof Sketch.
Base Case. If U is an open subset of $\mathbb{R}^n$, then $H_i(U) = 0$ for $i \geqslant n$. To show this, take $K \subseteq U$ as above. Give $\mathbb{R}^n$ a CW structure given by a grid with tiny squares. Then let $K \subseteq L \subseteq U$ be a subcomplex of this CW structure on $\mathbb{R}^n$. However $H_i(L) = 0$ for $i \geqslant n$, since for $i \geqslant n$, we have

$$H_{i+1}(\mathbb{R}^n) = 0 \longrightarrow H_{i+1}(\mathbb{R}^n, L) \longrightarrow H_i(L) \longrightarrow H_i(\mathbb{R}^n) = 0$$

but $H_{i+1}(\mathbb{R}^n, L) = 0$ since $i + 1 > n$ and $(\mathbb{R}^n, L)$ is a CW pair of dimension n.

Induction hypothesis. If U is a coordinate chart and $V \subseteq M$ is such that $H_i(V) = 0$ if $i > n$ and $H_n(V) = 0$ if M is compact, then $H_i(U \cup V)$ satisfies these properties too.

For $i > n$ this is an easy application of Mayer-Vietoris sequence. Proving that $H_n(U \cup V) = 0$ when M is not compact is the hard part. I won't do this part as it is very well presented in May, but the key is that, if M is not compact, then

$$H_i(M) \to H_i(M, M - x)$$

is the zero map for every $x \in M$. To see this, not that choosing a path L between x and y, we have

$$\begin{tikzcd}
H_n(M, M-y) & H_n(M, M-L) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H_n(M, M-x) \\
& H_n(M) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

So, mapping to zero in $H_n(M, M - x)$ implies mapping to zero in $H_n(M, M - y)$ for all $y \in M$. But now, choose a class in $H_n(M)$ and a compact set K as above. We can pick a point $x \in M - K$. Then, the diagram

$$\begin{tikzcd}
H_n(K) \arrow[r] \arrow[d] & H_n(M) \arrow[d] \\
H_n(M - x, M - x) \arrow[r] & H_n(M, M - x)
\end{tikzcd}$$

proves the claim that $[\![c]\!]$ maps to zero in $H_n(M, M - x)$. $\square$