$$\begin{cases} \underline{\underline{K}}^{-1}(\psi) q + \nabla\psi = -\nabla z \\ \theta(\psi)_t + div(q) = F \end{cases}$$

→ IMPLICIT EULER:

$$\begin{cases} \underline{\underline{K}}^{-1}(\psi^{n+1}) q^{n+1} + \nabla\psi^{n+1} = -\nabla z \\ \dfrac{\theta(\psi^{n+1})}{\Delta t} + div(q^{n+1}) = F^{n+1} + \dfrac{\theta(\psi^n)}{\Delta t} \end{cases}$$

→ PICARD:

V1: $$\begin{cases} \underline{\underline{K}}^{-1}(\psi^{n+1}_k) q^{n+1}_{k+1} + \nabla\psi^{n+1}_{k+1} = -\nabla z \\ div(q^{n+1}_{k+1}) = F^{n+1} + \dfrac{\theta(\psi^n) - \theta(\psi^{n+1}_k)}{\Delta t} \end{cases}$$

→ It doesn't converge...

V2: $$\begin{cases} \underline{\underline{K}}^{-1}(\psi^{n+1}_k) q^{n+1}_{k+1} + \nabla\psi^{n+1}_{k+1} = -\nabla z \\ \psi^{n+1}_{k+1} + div(q^{n+1}_{k+1}) = F^{n+1} + \dfrac{\theta(\psi^n) - \theta(\psi^{n+1}_k)}{\Delta t} + \psi^{n+1}_k \end{cases}$$

→ NEWTON:

$$\mathcal{L}(\psi, q) = \begin{bmatrix} \underline{\underline{K}}^{-1}(\psi) q + \nabla\psi \\ \dfrac{\theta(\psi)}{\Delta t} + div(q) \end{bmatrix}, \quad \underline{F} = \begin{bmatrix} -\nabla z \\ F^{n+1} + \dfrac{\theta(\psi^n)}{\Delta t} \end{bmatrix}$$

$$D\mathcal{L}_{(\psi^{n+1}_k, q^{n+1}_k)}(\delta\psi, \delta q) = \underline{F} - \mathcal{L}(\psi^{n+1}_k, q^{n+1}_k)$$

$$D\mathcal{L}_{(\dots)}(\dots) = \lim_{\varepsilon \to 0} \begin{bmatrix} \dfrac{\underline{\underline{K}}^{-1}(\psi + \varepsilon\delta\psi)(q + \varepsilon\delta q) - \underline{\underline{K}}^{-1}(\psi) q}{\varepsilon} + \nabla\delta\psi \\ \dfrac{\theta(\psi + \varepsilon\delta\psi) - \theta(\psi)}{\varepsilon \cdot \Delta t} + div(\delta q) \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{\partial \underline{\underline{K}}^{-1}(\psi)}{\partial\psi} \delta\psi\, q + \underline{\underline{K}}^{-1}(\psi) \delta q + \nabla\delta\psi \\ \dfrac{1}{\Delta t}\dfrac{\partial\theta}{\partial\psi}(\psi)\, \delta\psi + div(\delta q) \end{bmatrix}$$

$$D\mathcal{L}_{(\dots)}(\dots) = \underline{F} - \mathcal{L} \iff$$

$$\begin{cases} \dfrac{\partial \underline{\underline{K}}^{-1}(\psi^{n+1}_k)}{\partial\psi} (\psi^{n+1}_{k+1} - \psi^{n+1}_k) q^{n+1}_k + \underline{\underline{K}}^{-1}(\psi^{n+1}_k) q^{n+1}_{k+1} + \nabla\psi^{n+1}_{k+1} = -\nabla z \\ \dfrac{1}{\Delta t}\dfrac{\partial\theta}{\partial\psi}(\psi^{n+1}_k)(\psi^{n+1}_{k+1} - \psi^{n+1}_k) + div(q^{n+1}_{k+1}) = \dfrac{\theta(\psi^n) - \theta(\psi^{n+1}_k)}{\Delta t} + F^{n+1} \end{cases}$$

How should we deal the weak formulation?

$$\int_\Omega \frac{\partial\theta}{\partial\psi}(\psi^{n+1}_k)\, \psi^{n+1}_{k+1}\, \psi_i \, dx = D(\psi^{n+1}_k)\, \underline{\Psi}^{n+1}_{k+1}$$

where $D(\psi^{n+1}_k)_{ij} = \int_\Omega \frac{\partial\theta}{\partial\psi}(\psi^{n+1}_k)\, \psi_j\, \psi_i \, dx$ and

$$\psi^{n+1}_{k+1} = \sum_s [\underline{\Psi}^{n+1}_{k+1}]_s\, \psi_s$$

$$\int_\Omega \frac{\partial \underline{\underline{K}}^{-1}(\psi^{n+1}_k)}{\partial\psi}\, \psi^{n+1}_{k+1}\, q^{n+1}_k \cdot \phi_i \, dx = C(\psi^{n+1}_k, q^{n+1}_k)\, \underline{\Psi}^{n+1}_{k+1}$$

where:

$$C(\psi^{n+1}_k, q^{n+1}_k)_{ij} = \int_\Omega \frac{\partial \underline{\underline{K}}^{-1}(\psi^{n+1}_k)}{\partial\psi}\, \psi_j\, q^{n+1}_k \cdot \phi_i \, dx$$

If $\psi \in \mathbb{P}_0$, then $\psi_j = \begin{cases} \alpha_j & \text{on element } \omega_j \\ 0 & \text{otherwise} \end{cases}$

Thus:

$$C(\psi^{n+1}_k, q^{n+1}_k)_{ij} = \int_{\omega_j} \alpha_j \frac{\partial \underline{\underline{K}}^{-1}(\psi^{n+1}_k)}{\partial\psi}\, q^{n+1}_k \cdot \phi_i \, dx =$$

$$= \alpha_j \frac{\partial \underline{\underline{K}}^{-1}(\psi^{n+1}_k)}{\partial\psi} \int_{\omega_j} q^{n+1}_k \cdot \phi_i \, dx =$$

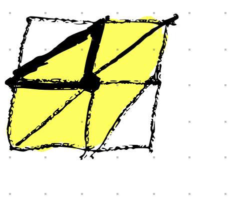
$$= \alpha_j \frac{\partial \underline{\underline{K}}^{-1}(\psi^{n+1}_k)}{\partial\psi} \int_\Omega \tilde{q}^{n+1}_{k,j} \cdot \phi_i \, dx =$$

$$= \alpha_j \frac{\partial \underline{\underline{K}}^{-1}(\psi^{n+1}_k)}{\partial\psi} [M_q\, \tilde{q}^{n+1}_{k,j}]_i =$$

$$= \frac{\partial \underline{\underline{K}}^{-1}(\psi^{n+1}_k)}{\partial\psi} [M_q\, \tilde{\tilde{q}}^{n+1}_{k,j}]_i$$

where $q^{n+1}_k \cdot I_{\omega_j}(x) \cdot \alpha_j = \sum_s [\tilde{\tilde{q}}^{n+1}_{k,j}]_s\, \phi_s$

Then, $[\tilde{\tilde{q}}^{n+1}_{k,j}]_s = \begin{cases} 0 & \text{if } supp(\phi_s) \cap \omega_j = \emptyset \\ \dots ? \end{cases}$



McDugall, Wotherospoon Newton $1+\sqrt{2}$

1) INITIALIZATION

$$x_0^* = x_0, \quad x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

2) ITERATIVE:

$$x_k^* = x_k - \frac{F(x_k)}{F'\left(\frac{x_{k-1}+x_{k-1}^*}{2}\right)}, \quad x_{k+1} = x_k - \frac{F(x_k)}{F'\left(\frac{x_k+x_k^*}{2}\right)}$$

MATRIX:

$$J\left(\frac{\underline{x}_{k-1}+\underline{x}_{k-1}^*}{2}\right)\delta \underline{x}_k^* = -F(\underline{x}_k), \quad \underline{x}_k^* = \underline{x}_k + \delta \underline{x}_k^*$$

$$J\left(\frac{\underline{x}_k+\underline{x}_k^*}{2}\right)\delta \underline{x}_k = -F(\underline{x}_k), \quad \underline{x}_{k+1} = \underline{x}_k + \delta \underline{x}_k$$

DIFFERENTIAL:

$$\begin{cases} \frac{\partial \underline{\underline{K}}^{-1}}{\partial \psi}\left(\frac{\psi_{k-1}^{n+1}+\psi_{k-1}^{*\,n+1}}{2}\right)\frac{q_{k-1}^{n+1}+q_{k-1}^{*\,n+1}}{2}\delta\psi + \underline{\underline{K}}^{-1}\left(\frac{\psi_{k-1}^{n+1}+\psi_{k-1}^{*\,n+1}}{2}\right)\delta q + \nabla\delta\psi = \\ \quad = -\nabla z - \underline{\underline{K}}^{-1}(\psi_k^{n+1})\, q_k^{n+1} - \nabla\psi_k^{n+1} \\ \frac{1}{\Delta t}\frac{\partial \theta}{\partial \psi}\left(\frac{\psi_{k-1}^{n+1}+\psi_{k-1}^{*\,n+1}}{2}\right)\delta\psi + \operatorname{div}(\delta q) = f^{n+1} + \frac{\theta(\psi^n)-\theta(\psi_k^{n+1})}{\Delta t} - \operatorname{div}(q_k^{n+1}) \end{cases}$$

$$\begin{cases} \frac{\partial \underline{\underline{K}}^{-1}}{\partial \psi}\left(\frac{\psi_k^{n+1}+\psi_k^{*\,n+1}}{2}\right)\frac{q_k^{n+1}+q_k^{*\,n+1}}{2}\delta\psi + \underline{\underline{K}}^{-1}\left(\frac{\psi_k^{n+1}+\psi_k^{*\,n+1}}{2}\right)\delta q + \nabla\delta\psi = \\ \quad = -\nabla z - \underline{\underline{K}}^{-1}(\psi_k^{n+1})\, q_k^{n+1} - \nabla\psi_k^{n+1} \\ \frac{1}{\Delta t}\frac{\partial \theta}{\partial \psi}\left(\frac{\psi_k^{n+1}+\psi_k^{*\,n+1}}{2}\right)\delta\psi + \operatorname{div}(\delta q) = f^{n+1} + \frac{\theta(\psi^n)-\theta(\psi_k^{n+1})}{\Delta t} - \operatorname{div}(q_k^{n+1}) \end{cases}$$