Assignment 1

Convex Optimization SS 2018

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Submission: Upload your solution as a PDF to the TeachCenter. You can also use your scanned hand-written manuscript.

Deadline: April 16^{th}, 2018 at 23:55h.

Convex Sets (13 points)

1. Let $C \subseteq \mathbb{R}^n$ be a convex set, $x_1, \dots, x_k \in C$ and let $\theta_1, \dots, \theta_k \in \mathbb{R}$ be factors that satisfy $\theta_i \geq 0$ and sum up to one $\sum_{i=1}^k \theta_i = 1$. Show that any combination of points $\theta_1 x_1 + \dots + \theta_k x_k$ is also element of the set C. Hint: Use induction on the definition of convexity.

2. Calculate the distance between two parallel hyperplanes $\{x \in \mathbb{R}^n \mid a^T x = b_1\}$ and $\{x \in \mathbb{R}^n \mid a^T x = b_2\}$. You can derive this geometrically. Draw a picture!

3. Prove or disprove the convexity of the following sets:
   a) The set $\{x \in \mathbb{R} \mid x \geq 1/2, x \leq -1/2\}$.
   b) The set of points closer to one set than another, i.e., $\{x \mid \mathrm{dist}(x, S) \leq \mathrm{dist}(x, T)\}$ with $S, T \subseteq \mathbb{R}^n$ (not necessarily convex) and $\mathrm{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}$.
   c) A *hyperrectangle*: $\{x \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i, i = 1, \dots, n\}$.
   d) A *wedge*, i.e., a set of form $\{x \in \mathbb{R}^n \mid a_1^T x \leq b_1, a_2^T x \leq b_2\}$. What happens if $b_1 = b_2 = 0$?

4. Let $x_0, x_1, \dots, x_k \in \mathbb{R}^n$. The set of points closer to a given point x_0 than the other points $x_1, \dots, x_k$
$$\{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x - x_i\|_2, \quad i = 1, \dots, k\} \tag{1}$$
is called a *Voronoi region* around x_0 with respect to the points $x_1, \dots, x_k$.
   a) Show that (1) is a convex set.
   b) Equation (1) can be expressed in terms of a polyhedron. Rewrite (1) in the form $\{x \mid Ax \preceq b\}$.
   c) Consider the case $k = 1$. Show that the set of all points closer to x_0 than to x_1 is a halfspace, $x_0 \neq x_1$. *Hint:* Recall the definition of a halfspace $\{x \mid a^T x \leq b\}$. Draw a picture!

Convex Functions (12 points)

1. Prove or disprove the convexity/concavity of the geometric mean
$$f(x) = \left(\prod_{i=1}^k x_i\right)^{\frac{1}{k}}$$
on $\mathbb{R}^k_{++}$.

2. Let $f : \mathbb{R}^n \to [-\infty, +\infty]$. Show that the perspective of f defined via

$$g(x, y) = yf(x/y),\ x \in \mathbb{R}^n,\ y \in \mathbb{R}_{++}$$

is convex if and only if $f(x)$ is a convex function.

3. Prove or disprove the convexity/concavity of the following functions:
   a) $f(x) = \log(x)$ for $x \in \mathbb{R}_{++}$
   b) $f(x) = |a|x| - x|^2$ for $a, x \in \mathbb{R}$
   c) $f(x_1, x_2, x_3) = -e^{(-x_1+x_2-2x_3)^2}$ for $x \in \mathbb{R}^3$
   d) $f(x) = \ln(e^{x_1} + e^{x_2} + \cdots + e^{x_n})$ for $x \in \mathbb{R}^n$