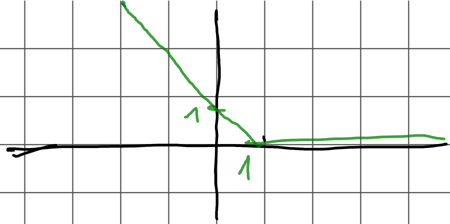
Infimal Convolution



1) $f(y) = \max(0, 1-y)$, $y \in \mathbb{R}$

$$f_\lambda(x) = \inf_y \max(0, 1-y) + \frac{1}{2\lambda}\|x-y\|^2$$

for $1-y > 0 \rightarrow y < 1$

$$= -1 + \frac{1}{\lambda}(x-y) = 0$$

$$-\lambda + y - x = 0$$

$$y = x + \lambda < 1$$

$$x < 1 - \lambda$$

Substitute Back:

$$1 - (x+\lambda) + \frac{1}{2\lambda}\|x-(x+\lambda)\|^2 =$$

$$1 - x - \lambda + \frac{1}{2\lambda}\|\lambda\|^2 = 1 - x - \lambda + \frac{\lambda}{2}$$

$$= 1 - x - \lambda \quad \text{for } x < 1 - \lambda$$

for $1-y < 0 \rightarrow y > 1$

$$= \inf_y 0 + \frac{1}{2\lambda}\|x-y\|^2$$

$$0 + \frac{1}{\lambda}(y-x) = 0$$

$$y = x > 1$$

Substitute Back:

$$= 0 + \frac{1}{2\lambda}\|x-x\|^2 = 0 \quad \text{for } x > 1$$

for $y = 1$: $\inf_y f(y) + \frac{1}{2\lambda}\|x-y\|^2$

$$[-1, 0] + \frac{1}{\lambda}(y-x) \in 0$$

$$y - x \in \lambda[0,1]$$

$$y \in x + \lambda[0,1] \Rightarrow \text{for} \begin{cases} x \leq 1 \\ \cancel{x} \geq 1-\lambda \end{cases}$$

Substitute Back:

$$= 0 + \frac{1}{2\lambda} \|x - 1\|^2 \Rightarrow \text{for } \begin{cases} x < 1 \\ x \geq 1 - \lambda \end{cases}$$

$$f_\lambda(x) = \begin{cases} 1 - x - \frac{1}{2}\lambda & \text{for } x < 1 - \lambda \\ 0 & \text{for } x > 1 \\ \frac{1}{2\lambda} \|x - 1\|^2 & \text{for } 1 - \lambda \leq x \leq 1 \end{cases}$$

2) $f(y) = |y| + \frac{1}{3}|y|^3 \quad , y \in \mathbb{R}$

$$f_\lambda(x) = \inf_y |y| + \frac{1}{3}|y|^3 + \frac{1}{2\lambda}\|x-y\|^2$$

$$= \partial|y| + |y|^2 \partial|y| + \frac{1}{\lambda}(x-y)(-1)$$

if $y > 0$:
$$1 + y^2 \cdot 1 - \frac{1}{\lambda}(x-y) = 0$$
$$y^2 + \frac{y}{\lambda} - \frac{x}{\lambda} + 1 = 0$$

$$y_{1/2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

$$y_{1/2} = -\frac{\frac{1}{\lambda}}{2} \pm \sqrt{\frac{\left(\frac{1}{\lambda}\right)^2}{4} + \frac{x}{\lambda} - 1}$$

$$= -\frac{1}{2\lambda} + \sqrt{\frac{1}{4\lambda^2} + \frac{x}{\lambda} - 1}$$ only positive case ! $y>0$

$$\frac{1}{2\lambda} < \sqrt{\frac{1}{4\lambda^2} + \frac{x}{\lambda} - 1}$$
$$\cancel{\frac{1}{4\lambda^2}} < \cancel{\frac{1}{4\lambda^2}} + \frac{x}{\lambda} - 1$$
$$1 < \frac{x}{\lambda}$$
$$\lambda < x$$

Substitute Back:

$$\left|-\frac{1}{2\lambda} + \sqrt{\frac{1}{4\lambda^2} + \frac{x}{\lambda} - 1}\right| + \frac{1}{3}|\ldots|^3 + \frac{1}{2\lambda}\|x - \ldots\|^2 \quad \text{for } x > \lambda$$

if $y < 0$:
$$-1 - y^2 - \frac{1}{\lambda}(x-y) = 0 \quad | \cdot (-1)$$
$$y^2 + \frac{1}{\lambda}(x-y) + 1 = 0$$
$$y^2 - \frac{y}{\lambda} + \frac{x}{\lambda} + 1 = 0$$
$$y_{1/2} = -\frac{-\frac{1}{\lambda}}{2} \pm \sqrt{\frac{\left(\frac{1}{\lambda}\right)^2}{4} - \left(\frac{x}{\lambda} + 1\right)} = 0$$

$$= \frac{1}{2\lambda} \pm \sqrt{\frac{1}{4\lambda^2} - \frac{x}{\lambda} - 1} = 0$$

$$y_1 = \frac{1}{2\lambda} + \sqrt{\frac{1}{4\lambda^2} - \frac{x}{\lambda} - 1}$$

$$\sqrt{\frac{1}{4\lambda^2} - \frac{x}{\lambda} - 1} < -\frac{1}{2\lambda}$$

$$\cancel{\frac{1}{4\lambda^2}} - \frac{x}{\lambda} - 1 < \cancel{\frac{1}{4\lambda^2}}$$

$$-\frac{x}{\lambda} < 1$$

$$-x < \lambda$$

$$y_2 = \frac{1}{2\lambda} - \sqrt{\frac{1}{4\lambda^2} - \frac{x}{\lambda} - 1}$$

$$\sqrt{\frac{1}{4\lambda^2} - \frac{x}{\lambda} - 1} < \frac{1}{2\lambda}$$

$$\cancel{\frac{1}{4\lambda^2}} - \frac{x}{\lambda} - 1 < \cancel{\frac{1}{4\lambda^2}}$$

$$-\frac{x}{\lambda} < 1$$

$$-x < \lambda$$

Substitute Back:

$$\left|\frac{1}{2\lambda} - \sqrt{\frac{1}{\lambda^2} - 4\left(1 + \frac{x}{\lambda}\right)}\right| + \frac{1}{3}\left|\dots\right|^3 + \frac{1}{\lambda}(\dots + x) \text{ for } x \le -\lambda$$

if $y = 0$: $[-1,1] + |y|^2[-1,1] + \frac{1}{\lambda}(y - x) \in 0$

$$[-1,1] + \frac{1}{\lambda}(-x) \in 0$$

$$x \in \lambda[-1,1] \Rightarrow |x| \le \lambda$$

Substitute Back:

$$|0| + \frac{1}{3}|0|^3 + \frac{1}{2\lambda}\|x-0\|^2 = \frac{1}{2\lambda}x^2 \quad \text{for } |x|\le\lambda$$

$$f_\lambda(x) \begin{cases} \left|-\frac{1}{2\lambda}+\sqrt{\frac{1}{4\lambda^2}+\frac{x}{\lambda}-1}\right| + \frac{1}{3}|\dots|^3 + \frac{1}{2\lambda}\|x-\dots\|^2 & \text{for } x>\lambda \\ 0 & \text{for } |x|\le\lambda \\ \left|\frac{1}{2\lambda}-\sqrt{\frac{1}{4\lambda^2}+\frac{-x}{\lambda}-1}\right| + \frac{1}{3}|\dots|^3 + \frac{1}{2\lambda}\|x-\dots\|^2 & \text{for } x\le-\lambda \end{cases}$$

3) $f(y) = \begin{cases} 0 & \text{if } y \in [a, b] \\ \infty & \text{else} \end{cases}$ $\quad a, b \in \mathbb{R}, y \in \mathbb{R}$

$\boxed{\partial f(y) \begin{cases} 0 & x \in [a, b] \\ [0, \infty] & x = b \\ [-\infty, 0] & x = a \end{cases}}$

$$f_\lambda(x) = \inf_y f(y) + g(x - y)$$

if $y = a$: $\quad \partial f(y) + \frac{1}{2\lambda} \|x - y\|^2 \in 0$

$$[-\infty, 0] + \frac{1}{\lambda}(x - y)(-1) \in 0$$

$$\lambda[-\infty, 0] + y - x \in 0$$

$$y \in x - \lambda[-\infty, 0]$$

$$a \in x - \lambda[-\infty, 0]$$

$$x \in a + \lambda[-\infty, 0]$$

Substitute Back:

$0 + \frac{1}{2\lambda}\|x - a\|^2$ $\quad$ for $-\infty < x < a$

if $a < y < b$: $\quad 0 + \frac{1}{\lambda}(y - x) = 0$ $\quad \begin{cases} y > a : x > a \\ y < b : x < b \end{cases}$

$$y = x$$

Substitute Back:

$0 + \frac{1}{2\lambda}\|x - x\|^2 = 0$ $\quad$ for $a < x < b$

if $y = b$: $\quad y \in x - \lambda[0, \infty]$

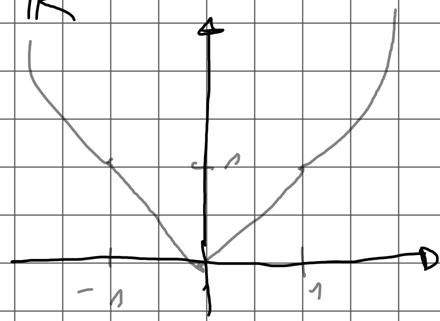
$$x \in b + \lambda[0, \infty]$$

Substitute Back:

$0 + \frac{1}{2\lambda}\|x - b\|^2$ $\quad$ for $b < x < \infty$

$$f_\lambda(x) = \begin{cases} \frac{1}{2\lambda}\|x - b\|^2 & \text{for } x \leq a \\ 0 & \text{for } a < x < b \\ \frac{1}{2\lambda}\|x - a\|^2 & \text{for } x \geq b \end{cases}$$

4) $f(y) = \max(|y|, y^2) \quad , \ y \in \mathbb{R}$



$\inf_y f(y) + \frac{1}{2\lambda}\|x-y\|^2$

if $y > 1$: $\quad \inf_y y^2 + \frac{1}{2\lambda}\|x-y\|^2$

$$2y + \frac{1}{\lambda}(x-y)(-1) = 0$$

$$2y\lambda + y - x = 0$$

$$y(2\lambda+1) - x = 0$$

$$y = \frac{x}{2\lambda+1} > 1$$

$$x > 2\lambda+1$$

Substitute Back:

$$\left(\frac{x}{2\lambda+1}\right)^2 + \frac{1}{2\lambda}\left\|x - \frac{x}{2\lambda+1}\right\|^2 \quad \text{for } x > 2\lambda+1$$

if $y < -1$: $\quad \inf_y y^2 + \frac{1}{2\lambda}\|x-y\|^2$

$$2y + \frac{1}{\lambda}(x-y) = 0$$

$$2y\lambda + y - x = 0$$

$$y(2\lambda+y) - x = 0$$

$$y = \frac{x}{2\lambda+y} < -1$$

$$x < -2\lambda - y$$

Substitute Back:

$$\left(\frac{x}{2\lambda+1}\right)^2 + \frac{1}{2\lambda}\|x-y\|^2 \quad \text{for } x < -2\lambda - y$$

if $\underline{y=1}$: $\partial|x| + \frac{1}{2\lambda}\|x-y\|^2 = 0$

$$[1,2] + \frac{1}{\lambda}(x-y) = 0$$

$$\lambda[1,2] - (x-y) = 0$$

$$y = x - \underline{\lambda[1,2] = 1}$$

$$x = 1 + \lambda[1,2]$$

$$\Rightarrow \begin{cases} x > 1+\lambda \\ x < 1+2\lambda \end{cases}$$

Substitute Back:

$$1 + \frac{1}{2\lambda}\|x-1\|^2 \quad \text{for } 1+\lambda < x < 1+2\lambda$$

if $\underline{y=-1}$: $\inf_y 1 + \frac{1}{2\lambda}\|x-y\|^2$

$$[-2,-1] + \frac{1}{\lambda}(x-y)(-1) = 0$$

$$y = \underline{x+\lambda[1,2] = -1} \qquad \begin{cases} x > -1-2\lambda \\ x < -1-\lambda \end{cases}$$

Substitute Back:

$$1 + \frac{1}{2\lambda}\|x+1\|^2 \quad \text{for } -1-2\lambda < x < -1-\lambda$$

if $\underline{y=0}$: $\inf 0 + \frac{1}{2\lambda}\|x-y\|^2$

$$[-1,1] + \frac{1}{\lambda}(x-y) = 0$$

$$y = \underline{x+\lambda[-1,1] = 0} \qquad \begin{cases} x < \lambda \\ x > -\lambda \end{cases}$$

Substitute Back:

$$0 + \frac{1}{2\lambda}\|x-0\|^2 = \frac{1}{2\lambda}\|x\|^2 \quad \text{for } \begin{cases} x < \lambda \\ x > -\lambda \end{cases}$$

if $y<1$ && $y>0$:

$$\inf_y |y| + \frac{1}{2\lambda}\|x-y\|^2$$

$$1+\frac{1}{\lambda}(y-x)=0 \Rightarrow y=x-\lambda \begin{cases} <1 & x<1+\lambda \\ >0 & x>\lambda \end{cases}$$

Substitute Back:

$$|x-\lambda| + \frac{1}{2\lambda}\|x-x+\lambda\|^2 = |x-\lambda| + \frac{1}{2\lambda}\lambda^2 =$$

$$= |x-\lambda| + \frac{\lambda}{2} \quad \text{for} \begin{cases} x<1+\lambda \\ x>\lambda \end{cases}$$

if $y>-1$ && $y<0$: $\inf_y |y| + \frac{1}{2\lambda}\|x-y\|^2$

$$-1+\frac{1}{\lambda}(y-x)=0$$

$$-\lambda + y - x = 0$$

$$y = x+\lambda \begin{cases} y>-1 \\ y<0 \end{cases}$$

$$\Rightarrow \begin{cases} x>-1-\lambda \\ x<-\lambda \end{cases}$$

Substitute Back:

$$|x+\lambda| + \frac{1}{2\lambda}|x-x-\lambda\|^2 = |x+\lambda| + \frac{\lambda}{2} \quad \text{for} \begin{cases} x>-1-\lambda \\ x<-\lambda \end{cases}$$

if $y<-1$ && $y<0$: $-1+\frac{1}{\lambda}(y-x)=0$

$$y = x+\lambda \begin{cases} >-1 \\ >0 \end{cases} \quad \text{for} \begin{matrix} x>-1-\lambda \\ x<-\lambda \end{matrix}$$

$$f_\lambda(x) \begin{cases} \left(\frac{x}{2\lambda+1}\right)^2 + \frac{1}{2\lambda}\|x-\frac{x}{2\lambda+1}\|^2 \rightarrow \text{for } x>2\lambda+1 \\ \text{———— '' ————} \rightarrow \text{for } x<-2\lambda-1 \\ 1+\frac{1}{2\lambda}\|x-1\|^2 \rightarrow \text{for } 1+\lambda<x<1+2\lambda \\ 1+\frac{1}{2\lambda}\|x+1\|^2 \rightarrow \text{for } -2\lambda-1<x<-\lambda-1 \\ |x-\lambda|+\frac{\lambda}{2} \rightarrow \text{for } \lambda<x\ 1+\lambda \\ |x+\lambda|+\frac{\lambda}{2} \rightarrow \text{for } -1-\lambda<x\ -\lambda \\ \frac{1}{2\lambda}x^2 \rightarrow \text{for } -\lambda<x<\lambda \end{cases}$$

Proximal Operators

1.a) $f(x) = \|x\|_\infty$, $x \in \mathbb{R}^n$

$f^*(y) = \sup_x (x^T y - f(x))$

$\omega = \text{prox}_f(\omega) + \text{prox}_{f^*}(\omega)$,

where $f^* = I_B$ is the convex conjugate and B is the Unit Ball for the dual norm.

$$f^*(x) = \mathbb{1}_{\{\|x\|_1 \leq 1\}}(x)$$

where

$$\mathbb{1}_S(x) = \begin{cases} 0 & \text{if } x \in S \\ \infty & \text{if } x \notin S \end{cases}$$

~~Rule: $f(x) = \|x\|_p$, $p \geq 1$~~
~~$f^*(y) = \mathbb{1}_{\{\|y\|_q \leq 1\}}$, $\frac{1}{p} + \frac{1}{q} = 1$~~

We know that by the Moreau Identity

$$\text{prox}_{\lambda f}(v) = v - \text{prox}_{f^*/\lambda}(v/\lambda)$$
$$= v - \Pi_B(v/\lambda)$$

(Ref.: Proximal Algorithms, Steven Boyd p70)

Therefore, we have to solve

$$\text{prox}_{\lambda f^*} = \arg\min_z \left(\mathbb{1}_{\{\|z\|_1 \leq 1\}} + \|z - x\|_2^2 \right)$$

The Unit Ball B of l_∞ is a box.

$$(\Pi_B(v))_i = \begin{cases} 1 & v_i > 1 \\ v_i & |v_i| \leq 1 \\ -1 & v_i < -1 \end{cases}$$

l_∞ norm is the dual norm of the l_1 norm and we can evaluate the proximal operator of $f = \|\cdot\|_1$

$$(\text{prox}_{\lambda f}(v))_i = \begin{cases} v_i - \lambda & v_i \geq \lambda \\ 0 & |v_i| \leq \lambda \\ v_i + \lambda & v_i \leq \lambda \end{cases}$$

"elementwise softthreshold operator"

Now, we replace back into Moreau's Identity

$$v = \text{prox}_{\lambda f}(v) + \Pi_B$$

$$\underline{\text{prox}_{\lambda f}(v) = v - \Pi_B}$$

b) $f(x) = \|x\|_0$, $x \in \mathbb{R}^n$ "Zero Norm"

$= \lambda \|x\|_0$, $\lambda = 1$ $\Rightarrow$ hard thresholding

$$f(x) = \sum_{i=1}^{n} I(x_i), \text{ where } I(t) = \begin{cases} \lambda & t \neq 0 \\ 0 & t = 0 \end{cases}$$

$$\text{def. } I = J - \lambda, \text{ where } J(t) = \begin{cases} 0 & t \neq 0 \\ -\lambda & t = 0 \end{cases}$$

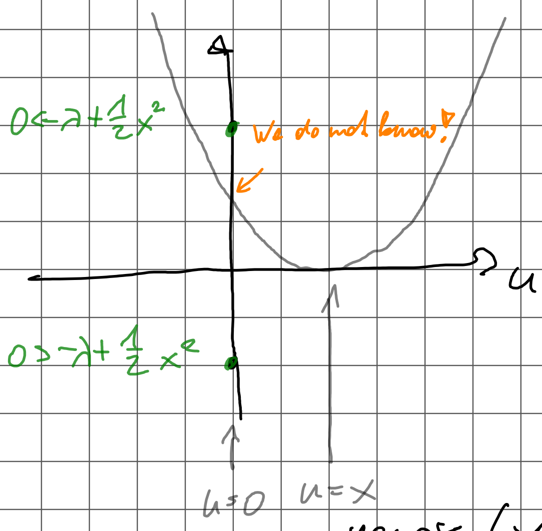
$$\text{prox}_J(x) = \underset{u}{\text{argmin}}\ J(u) + \frac{1}{2}(u - x) =$$

$$= \begin{cases} -\lambda + \frac{x^2}{2} & u = 0 \\ \frac{1}{2}(u-x)^2 & u \neq 0 \end{cases}$$

$u \neq 0$: $\frac{1}{2}(u-x)^2 = 0$

$u = x$ ... there is a minimum

$$u = 0: \begin{cases} 0 > -\lambda + \frac{1}{2}x^2 \to \lambda > \frac{1}{2}x^2 \\ 0 < -\lambda + \frac{1}{2}x^2 \to \lambda < \frac{1}{2}x^2 \end{cases}$$



$\lambda > \frac{1}{2}x^2 \quad | \cdot 2 \quad | \sqrt{\ }$

$\sqrt{2\lambda} > |x|$

$$\text{prox}_J(x) = \begin{cases} \{0\} & |x| < \sqrt{2\lambda} \\ \{x\} & |x| > \sqrt{2\lambda} \\ \{0, x\} & |x| = \sqrt{2\lambda} \end{cases}$$

c) $f(x) = \|x\|_1 + \frac{a}{2}\|x\|_2^2 \quad , \; a > 0 \; , \; x \in \mathbb{R}^n$

$$\text{prox}_f(x) = \underset{u}{\text{argmin}} \left(\|x\| + \frac{a}{2}\|x\|_2^2 + \frac{1}{2\lambda}\|u-x\|_2^2 \right)$$

$$\rightarrow \partial|u_i| + a \cdot \partial\|u_i\| + u_i - x_i$$

$$v_i \in \partial|u_i| = \begin{cases} [-1,1] & \text{if } u_i = 0 \\ \text{sign}(u) & \text{if } u_i \neq 0 \end{cases}$$

$$w_i \in \partial\|u_i\|_2 = \begin{cases} \{z \mid \|z\|_2 \leq 1\} & \text{if } u_i = 0 \\ \frac{u_i}{\|u_i\|_2} & \text{if } u_i \neq 0 \end{cases}$$ (Dual Norm)

if $u_i = 0$: $\lambda_1 v_i + \lambda_2 w_i - x_i = 0$

$x_i = \lambda_1 v_i + \lambda_2 w_i$

$\lambda_2 w_i = x - \lambda_1 v_i$

- $v_i \in [-1,1]$ and $\|w_i\|_2 \leq 1$

  $\hookrightarrow \|x_i - \lambda_1 v_i\|_2 \leq \lambda_2$

- $w_i = \text{sign}(x_i)$: $\|x_i - \lambda_1 \text{sign}(x_i)\|_2 \leq \lambda_2$

  $\Leftrightarrow \| S_{\lambda_1}(x_i) \| \leq 1$

  ↳ Soft Threshold function!

if $u_i \neq 0$: $\lambda_1 v_i + \lambda_2 \frac{u_i}{\|u_i\|} + u_i - x_i = 0$

$u_i\left(1 + \frac{\lambda_2}{\|u_i\|}\right) = \underline{x_i - \lambda_1 v_i}$

For elements $u_i = 0$: $x_i \leq \lambda_1$, $\omega_i \in [-1,1]$

Assume : $\forall i \in \{j \mid u_i \neq 0\}$: $\mathrm{sign}(u_i) = \mathrm{sign}(x_i)$

$$u_i\left(1+\frac{\lambda_2}{\|u_i\|}\right) = S_{\lambda_1}(x_i)$$

$$\|u_i\|_2\left(1+\frac{\lambda_2}{\|u_i\|_2}\right) = \|S_{\lambda_1}(x_i)\|_2$$

$$\|u_i\|_2 + \lambda_2 = \|S_{\lambda_1}(x_i)\|_2$$

$$\|u_i\|_2 = \|S_{\lambda_1}(x_i)\|_2 - \lambda_2$$

Set it back to the line

$$u_i = \frac{S_{\lambda_1}(x_i)}{1+\frac{\lambda_2}{\|x\|}} = \frac{S_{\lambda_1}(x_i)}{1+\frac{\lambda_2}{(\|S_{\lambda_1}(x)\|_2 - \lambda_2)}}$$

$$= \frac{S_{\lambda_1}(x_i)}{1+\frac{\lambda_2 - \|S_{\lambda_1}(x_i)\| + \|S_{\lambda_1}(x)\|}{\|S_{\lambda_1}(x)\| - \lambda_2}}$$

$$= \frac{S_{\lambda_1}(x_i)}{1-1+\frac{\|S_{\lambda_1}(x)\|}{\|S_{\lambda_1}(x)\| - \lambda_2}}$$

$$= \frac{S_{\lambda_1}(x_i)\left(\|S_{\lambda_1}(x_i)\| - \lambda_2\right)}{\|S_{\lambda_1}(x_i)\|} = S_{\lambda_1}(x_i) - \frac{\lambda_2}{\|S_{\lambda_1}(x)\|} S_{\lambda_1}(x_i)$$

$\lambda_2 < \|S_{\lambda_1}(b)\|_2$

$$= S_{\lambda_1}(x_i)\left(1-\frac{\lambda_2}{\|S_{\lambda_1}(x)\|_2}\right) \longrightarrow \text{positive}$$

$$\mathrm{prox}_{\lambda_1\|\cdot\|_1 + \lambda_2\|\cdot\|_2}(x) = \begin{cases} 0 & \text{if } \|S_{\lambda_1}(b)\|_2 \leq \lambda_2 \\ S_{\lambda_1}(x)\left(1-\frac{\lambda_2}{\|S_{\lambda_1}(x)\|_2}\right) & \text{if } \|S_{\lambda_1}(x)\|_2 > \lambda_2 \end{cases}$$

$$= \mathrm{prox}_{\lambda_2\|\cdot\|_2}\left(\mathrm{prox}_{\lambda_1\|\cdot\|_1}(x)\right)$$

d)

$$f(x) = \begin{cases} -a \ln(x) + \frac{x^2}{2} & \text{for } x > 0 \\ \infty & \text{for } x \leq 0 \end{cases} \quad , a > 0$$

$$\text{prox}_{f\lambda}(x) = \underset{u}{\text{argmin}} \left\{ -a \ln(u) + \frac{u^2}{2} + \frac{1}{2\lambda} \|u - x\|^2 \right\}$$

$$= -\frac{a}{u} + u + \frac{1}{\lambda} u^2 - ux = 0 \quad \Big| \cdot u$$

$$= -a + u^2 + \frac{1}{\lambda} u^2 - ux = 0$$

$$= -a\lambda + \lambda u^2 + u^2 - \lambda ux = 0$$

$$= u^2(1+\lambda) - ux - a = 0$$

$$u_{1,2} = \frac{x \pm \sqrt{x^2 + 4a(1+\lambda)}}{2(1+\lambda)}$$

if $u > 0$: $\frac{x \pm \sqrt{x^2 + 4a(1+\lambda)}}{2(1+\lambda)} > 0$

$$\rightarrow x > \sqrt{x^2 + 4a(1+\lambda)}$$

if $u = 0$ ~~$u \leq 0$~~: $-\infty + \frac{1}{\lambda}(u - x) = 0$

$$u - x = +\lambda\infty$$

$$u = x + \lambda\infty \overset{=}{\neq} 0$$

$$\rightarrow x = -\lambda\infty$$

$$\text{prox}_{\lambda f}(x) = \begin{cases} x + \lambda\infty & \text{for } x = -\lambda\infty \\ \frac{x + \sqrt{x^2 + 4a(1+\lambda)}}{2(1+\lambda)} & \text{for } x > 0 \end{cases}$$

e) $f(x) = \max(|x| - a, 0) \qquad a > 0, \; x \in \mathbb{R}$

if $u < -a$: $\qquad -1 + \frac{1}{\lambda}(u - x) = 0$

$$u = x + \lambda < -a$$

$$x < -a - \lambda$$

if $u = -a$: $\qquad [1, 0] + \frac{1}{\lambda}(u - x) \in 0$

$$u \in x + \lambda[0, 1] = -a$$

$$\begin{cases} x < -a \\ x > -a - \lambda \end{cases}$$

if $-a < u < a$: $\qquad 0 + \frac{1}{\lambda}(u - x) = 0$

$$u = x \begin{cases} u > -a: & x > -a \\ u < a: & x < a \end{cases}$$

if $u = a$: $\qquad [0, 1] + \frac{1}{\lambda}(u - x) \in 0$

$$u \in x - \lambda[0, 1] = a$$

$$\begin{cases} x > a \\ x < a + \lambda \end{cases}$$

if $u > a$: $\qquad 1 + \frac{1}{\lambda}(u - x) = 0$

$$u = x - \lambda > a$$

$$x > a - \lambda$$

$$\mathrm{prox}_{\lambda f}(x): \begin{cases} x + \lambda & \text{if } x < -a - \lambda \\ -a & \text{if } -a - \lambda < x < -a \\ x & \text{if } |x| < a \\ a & \text{if } a < x < a + \lambda \\ x - \lambda & \text{if } x > a - \lambda \end{cases}$$

f)

$$f(x) = \begin{cases} \ln(a) - \ln(a - |x|) & \text{for } |x| < a \\ \infty & \text{else} \end{cases}$$

$$\operatorname{prox}_{\lambda f}(x) = \operatorname{argmin}_u f(u) + \frac{1}{2\lambda}\|u-x\|_2^2$$

subdiffs

$$\partial f(u) + \frac{1}{\lambda}(u-x) \in 0$$

$$\lambda \partial f(u) + u - x \in 0$$

$$u \in x - \lambda \partial f(u)$$

if $u = a$: $-\infty + \frac{1}{\lambda}(u-x) = 0$

$$-\lambda\infty + u - x = 0$$

$$u = x + \lambda\infty \Rightarrow \infty$$

$$x + \lambda\infty = -a$$

$$x = -a - \lambda\infty$$

$$-\infty$$

if $|x| < a$:

if $u > 0$: $\frac{1}{a-|u|} + \frac{1}{\lambda}(u-x) \in 0$

$$\frac{\lambda}{a-u} = x - u$$

$$\lambda = (x-u)(a-u)$$

$$0 = xa - ux - ua + u^2 - \lambda$$

$$0 = u^2 - u(x+a) + xa - \lambda$$

$$u_{1,2} = \frac{a+x}{2} \pm \sqrt{\left(\frac{a+x}{2}\right)^2 - ax + \lambda} > 0 \quad |^2$$

$$\left(\frac{a+x}{2}\right)^2 + \left(\frac{a+x}{2}\right)^2 - ax + \lambda > 0$$

$$x > \frac{\lambda}{a}$$

if $u < 0$: $-\frac{1}{a+u} + \frac{1}{\lambda}(u-x) \stackrel{!}{=} 0$

$$(a+u)(u-x) = \lambda$$

$$au - ax + u^2 - ux = \lambda$$

$$u^2 + u(a-x) - ax - \lambda = 0$$

$$u_{1,2} = -\frac{a-x}{2} \pm \sqrt{\left(\frac{a-x}{2}\right)^2 + ax + \lambda} < 0 \quad |^2$$

$$\cancel{\left(\frac{a-x}{2}\right)^2} - \cancel{\left(\frac{a-x}{2}\right)^2} + ax + \lambda < 0$$

$$x < -\frac{\lambda}{a}$$

if $u = 0$: $\frac{1}{a}[-1,1] - \frac{x}{\lambda} = 0$

$$x = \frac{\lambda}{a}[-1,1]$$

if $u = a$: $\quad \infty + \frac{1}{\lambda}(u-x) = 0$

$$\lambda\infty + u - x = 0$$

$$u = x - \lambda\infty \quad \longrightarrow x = a + \lambda\infty$$

$$\operatorname{prox}_{\lambda f}(x) = \begin{cases} -\infty & \text{for } x = -a - \lambda\infty \\ \frac{a+x}{2} + \sqrt{\left(\frac{a+x}{2}\right)^2 + ax + \lambda} & \text{for } x > \frac{\lambda}{a} \\ 0 & \text{for } -\frac{\lambda}{a} \le x \le \frac{\lambda}{a} \\ -\frac{a-x}{2} - \sqrt{\frac{(a-x)^2}{4} + ax + \lambda} & \text{for } x < -\frac{\lambda}{a} \\ \infty & \text{for } x = a + \lambda\infty \end{cases}$$

Proximal Operator

2) $\boxed{x = \mathrm{prox}_{\lambda f^*}(x) + \lambda\, \mathrm{prox}_{f/\lambda}\left(\frac{x}{\lambda}\right)}$

(Skriptum p 35 / Theorem 6)

$$\mathrm{prox}_{\lambda f^*}(x) = x - \lambda\, \mathrm{prox}_{f/\lambda}\left(\frac{x}{\lambda}\right)$$

$$y = \mathrm{prox}_{\lambda f^*}(x) = (I + \lambda \partial f^*)^{-1}(x)$$

$\Leftrightarrow \quad y(I + \lambda \partial f^*) \ni x$

$\Leftrightarrow \quad y + (y)\lambda \partial f^* \ni x$

$\Leftrightarrow \quad \lambda \partial f^*(y) \ni x - y$

$\Leftrightarrow \quad \partial f^*(y) \ni \frac{x-y}{\lambda}$

$\Leftrightarrow \quad y \in \partial f\left(\frac{x-y}{\lambda}\right)$

$\Leftrightarrow \quad y + x - y \in \partial f\left(\frac{x-y}{\lambda}\right) + x - y$

$\Leftrightarrow \quad \frac{x}{\lambda} \in \frac{\partial f\left(\frac{x-y}{\lambda}\right)}{\lambda} + \left(\frac{x}{\lambda} - \frac{y}{\lambda}\right)$

$\Leftrightarrow \quad \frac{x}{\lambda} \in \left(\frac{\partial f}{\lambda} + I\right)\left(\frac{x-y}{\lambda}\right)$

$\Leftrightarrow \quad \left(\frac{\partial f}{\lambda} + I\right)^{-1}\left(\frac{x}{\lambda}\right) \ni \frac{x-y}{\lambda}$

$\Leftrightarrow \quad \lambda\left(\frac{\partial f}{\lambda} + I\right)^{-1}\left(\frac{x}{\lambda}\right) \ni x - y$

$\Leftrightarrow \quad y = x - \lambda\left(\frac{\partial f}{\lambda} + I\right)^{-1}\left(\frac{x}{\lambda}\right)$

$\Leftrightarrow \quad \boxed{y = x - \lambda\, \mathrm{prox}_{f/\lambda}\left(\frac{x}{\lambda}\right)}$

$\square$

Proximal Operator:

2) Moreau Identity (Proof) : $x = \mathrm{prox}_{\lambda f^*}(x) + \lambda\, \mathrm{prox}_{f/\lambda}(x/\lambda)$

To better understand the Moreau Identity, we want to illustrate an example → Convex Cone

$f(x) = I_K(x)$ ... indicator fct of a convex cone K

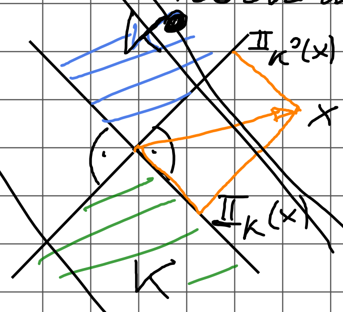
$f(x) = 0$ ... on dom$(f) = K$

Then $f^*(x) = \sup_{y\in K} x^T y$.

If we consider the polar cone $K^o = \{x : x^T y \leq 0, \forall y \in K\}$. The conjugate function $f^*(x)$ results as

$$I_{K}^{o} = f^*(x) = \begin{cases} 0 & x \in K_o \\ \infty & \text{otherwise.} \end{cases}$$

Proximal Operator of the indicator function is an Euclidean Projection. The Moreau's identity can be defined as



$$x = \Pi_K(x) + \Pi_{K_o}(x),$$

where $\Pi_K(x)$ is the projection on K.

Proof: Minimizing the Moreau's Envelope

$$f(x) = \min_y \left\{ f(y) + \frac{1}{2}\|x-y\|^2 \right\}$$
$$= \min_y \left\{ f(y) + \frac{1}{2}\|u\|^2 \right\} \quad \text{s.t. } u = x-y$$

$$\mathcal{L}(y,u,\lambda) = f(y) + \frac{1}{2}\|u\|^2 + \lambda^T(x-y-u)$$
$$= [f(y) - \lambda^T y] + \left[\frac{1}{2}\|u\|^2 - \lambda^T u\right] + \lambda^T x$$

$$g(y) = \inf_{y,z} \mathcal{L}(y,z,\lambda)$$
$$= \inf_y \{f(y) - \lambda^T y\} - \frac{1}{2}\|\lambda\|^2 + \lambda^T x$$

$$= -f^*(\lambda) - \frac{1}{2}\|\lambda\|^2 + \lambda^T x$$

strong duality

$$f(x) = \sup_\lambda g(\lambda) = \sup_\lambda \{-f^*(\lambda) - \frac{1}{2}\|\lambda\|^2 + \lambda^T x\}$$

$$= \left(f^* + \frac{1}{2}\|\cdot\|^2\right)^*(x)$$

The conjugate of a closed proper function is smooth. The moreau envelope is smooth.

Moreaus Identity:

$$f(x); \quad \frac{1}{2}\|x\|^2 - \left(f + \frac{1}{2}\|\cdot\|^2\right)^*_{(x)} = \left(f^* + \frac{1}{2}\|\cdot\|^2\right)^*(x)$$

Rearrange ||

$$\frac{1}{2}\|x\|^2 = \left(f + \frac{1}{2}\|\cdot\|^2\right)(x) + \left(f^* + \frac{1}{2}\|\cdot\|^2\right)^*(x)$$

$$x = \text{prox}_f(x) + \text{prox}_{f^*}(x) \qquad \square$$

(F) ISTA Algorithm for Lasso:

2) Compute the proximal map for (4) analytically

$g(u) = \lambda \|u\|_1$

$$\text{prox}_{g/L}(x) = \underset{u}{\text{argmin}} \left(\lambda \|u\|_1 + \frac{L}{2} \|u - x\|_2^2 \right)$$

$$\frac{\partial}{\partial u} \begin{cases} \lambda + L(u-x) & |u| > 0 \\ \lambda[-1,1] + L(u-x) & |u| = 0 \\ -\lambda + L(u-x) & |u| < 0 \end{cases} \overset{!}{=} 0$$

$|u|>0$: $\frac{\lambda}{L} + u - x = 0$

$$u = x - \frac{\lambda}{L} > 0, \quad x > \frac{\lambda}{L}$$

$|u| = 0$: $\lambda[-1,1] + L \cdot (u - x) = 0 \quad | : L$

$$\frac{\lambda}{L}[-1,1] + u - x = 0$$

$$u = x - \frac{\lambda}{L}[-1,1] = 0$$

$$x = \frac{\lambda}{L}[-1,1]$$

$|u| < 0$: $-\lambda + L(u-x) = 0 \quad | : L$

$$-\lambda/L + u - x = 0$$

$$u = x + \frac{\lambda}{L} < 0$$

$$x < -\lambda/L$$

$$\text{prox}_{g/L}(x) = \begin{cases} x - \frac{\lambda}{L} & \text{for } x > \frac{\lambda}{L} \\ 0 & \text{for } -\frac{\lambda}{L} \leq x \leq \frac{\lambda}{L} \\ x + \frac{\lambda}{L} & \text{for } x < -\lambda/L \end{cases}$$

$\longrightarrow$ Soft threshold Th

$$x^{k+1} = Th\left(x^k - \frac{1}{L} D^T(DC - B)\right)$$

Lipschitz Continuous Gradient:

Assume, that f is convex and has Lipschitz continuous gradients, ie there exists some $L > 0$ s.t.

$$\| D^T(DC_1 - B) - D^T(DC_2 - B) \| \leq L \| C_1 - C_2 \|$$

$$\| D^T D C_1 - \cancel{D^T B} - D^T D C_2 + \cancel{D^T B} \| \leq L \| C_1 - C_2 \|$$

Cauchy Schwarz

$|\langle x, y \rangle| \leq \|x\| \|y\|$

$$\| D^T D (C_1 - C_2) \| \leq L \| C_1 - C_2 \|$$

$$\| D^T D \| \leq L$$