Probability and Combinatorics Worksheet 6

Useful facts

- **Expected value**. For a discrete random variable with probability $p(x)$ this is

$$\langle g(X)\rangle = \sum_x p(x)g(x) \tag{1}$$

For a continuous random variable with density $f(x)$ this is

$$\langle g(X)\rangle = \int_{-\infty}^{\infty} f(x)g(x)dx \tag{2}$$

- **Mean and variance**. The mean is $\mu = \langle X\rangle$ and the variance is $\sigma^2 = \langle (X-\mu)^2\rangle = \langle X^2\rangle - \mu^2$.

- **Poisson distribution**. This has

$$p(r) = \frac{\lambda^r}{r!}e^{-\lambda} \tag{3}$$

where $\mu = \lambda$ and $\sigma^2 = \lambda$.

- **The limit of infinite compounding**

$$\left(1 - \frac{x}{n}\right)^n \to e^{-x} \tag{4}$$

as $n \to \infty$.

Questions

These are the questions you should make sure you work on in the workshop.

1. A typist makes on average two mistakes per page. What is the probability of a particular page having no errors on it?
2. Components are packed in boxes of 20. The probability of a component being defective is 0.1. What is the probability of a box containing 2 defective components?
3. A fisher catches on average one fish every 25 minutes. What is the probability that they catch two fish in an hour?
4. A random variable X gives the square of the face value of a six-sided dice. What are the mean and variance of X.

Extra questions

Do these in the workshop if you have time.

1. For a Poisson process let T be the interval for which the process has, on average, one event, so, for this interval $\lambda = 1$. What is the probability that there are no events for this interval?

2. Starting with the expression for the mean

$$\lambda = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} r e^{-\lambda} \tag{5}$$

calculate the variance of the Poisson distribution.

3. The **Fano factor** is sometimes used to describe distributions, it is

$$F = \frac{\sigma^2}{\mu} \tag{6}$$

What is the Fano factor for the Poisson distribution?

Another question

There is no need to do this question and it won't be considered in the workshop. A Poisson process represents true randomness in the sense that each event is unrelated to all the others. Consider a discretized two-dimensional Poisson process where the squares on a grid are black with a probability p and white with probability $1-p$; if you generate a grid like this does the distribution of black squares look random?