Probability and Combinatorics Worksheet 8 - solutions

Questions

These are the questions you should make sure you work on in the workshop.

1. The length of human pregnancies from conception to birth approximates a normal distribution with a mean of 266 days and a standard deviation of 16 days. What proportion of all pregnancies will last between 240 and 270 days (roughly between 8 and 9 months)?

   Solution: So
   $$z_1 = \frac{240 - 266}{16} = -1.625/\sqrt{2} \approx -1.15 \tag{1}$$
   and
   $$z_2 = \frac{270 - 266}{16} = 0.25/\sqrt{2} =\approx 0.18 \tag{2}$$
   giving
   $$p(\text{between 250 and 270}) = \frac{1}{2}[\text{erf}\,(0.18) - \text{erf}\,(-1.15)] \approx 0.55 \tag{3}$$

2. Starting from the expression for the mean
   $$\mu = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \tag{4}$$
   show the Gaußian distribution has variance σ^2.

   Solution: Differenciate with respect to μ to get
   $$1 = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x \frac{x-\mu}{\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \tag{5}$$
   and multiply across by the σ^2 to get
   $$\sigma^2 = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x(x-\mu) e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \tag{6}$$
   and then a bit of algebra gives
   $$\sigma^2 = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx - \mu \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \tag{7}$$

3. The size of a standard croquet ball is 3 5/8 inches[1]. The height of a croquet hoop is 3 3/4 inches. If a not very good croquet-ball making machine makes croquet balls whose mean matches the standard and with standard deviation 1/8 inch, what is the chance it will make a ball too large to fit through the hoop? You can write the solution in terms of the error function.

[1] Everything in croquet is measured in old timey units

Solution: So

$$z = \frac{x-\mu}{\sqrt{2}\sigma} \tag{8}$$

so for $x_1 = 3.75$ in, we have

$$z_1 = \frac{1/8}{\sqrt{2}/8} = \frac{1}{\sqrt{2}} \tag{9}$$

Any height bigger than this will not fit, so $z_2 = \infty$ and $\text{erf}\infty = 1$ so

$$\text{Prob}(x > 3.75) = \frac{1}{2}[1 - \text{erf}(1/\sqrt{2})] \approx 0.16 \tag{10}$$

where the 0.16 is given for interest, it wasn't expected as part of the answer.

4. If X and Y are two independent random variables what is $\text{var}(X+Y)$?

Solution: So, using the obvious notation

$$\text{var}(X+Y) = \langle (X+Y)^2 \rangle - (\mu_X + \mu_Y)^2 \tag{11}$$

or

$$\text{var}(X+Y) = \langle X^2 \rangle + 2\langle XY \rangle + \langle Y^2 \rangle - \mu_X^2 - 2\mu_x\mu_Y + \mu_Y^2 \tag{12}$$

and by independence $\langle XY \rangle = \langle X \rangle \langle Y \rangle = \mu_X\mu_Y$ giving

$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) \tag{13}$$

Extra questions

1. This will look like a long question but it is almost all background and the question is not too bad when you actually read through it. In particle physics when a collider is being used to find a new particle like the Higgs boson or the top squark scientists don't detect the sought after particle directly since it usually decays almost straight away, instead they detect the more common particles that particle will decay into, for example, a Higgs boson can decay in to two photons and these can be detected. Roughly speaking scientists count these events. However, the whole situation is very messy and there will always be some events even if the particle doesn't exist at the energy being examined. The amount of these background events will fluctuate from experiment to experiment, typically like a Gaußian. The scientific team is allowed to claim they have discovered the particle if the number of events they measure is more than five standard deviations above what would be expected if the particle didn't exist. What is the probability of this 'discovery' happening by chance?

Solution: So we are interested in the probability of a results bigger than $\mu + 5\sigma$. Now

$$z_1 = \frac{\mu + 5\sigma - \mu}{\sqrt{2}\sigma} = \frac{5}{\sqrt{2}} \tag{14}$$

and

$$\text{Prob}(x > \mu + 5\sigma) = \frac{1}{2}[1 - \text{erf}(5/\sqrt{(2)})] \tag{15}$$

which is about one chance in 3.5 million.

2. Australorp hens weigh on average 4kg with a standard deviation 0.25kg; in one farm australorps who weigh less than 3.5kg are fed *patent chicken spicer*, a mixture of chalk, corn and pepper. What fraction of these hens are fed patent chicken spicer?

   Solution: Here is a picture of one of my hens, an Australorp, sitting on the kitchen window ledge.

   

   So we are asking for $P(x < 3.5)$ and since

   $$z = \frac{3.5 - 4}{0.25\sqrt{2}} = \sqrt{2}$$

   the probability, and hence the proportion, is

   $$P(x < 3.5) = \frac{1}{2}[1 - \text{erf}\,(\sqrt{2})] \approx 0.02275$$

3. The Beta distribution for a random variable X has non-zero probability for $x \in (0, 1)$ and probability mass function

   $$p(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1 - x)^{\beta-1} \tag{16}$$

where α and β are shape parameters and $B(\alpha, \beta)$ is the Beta function, a special function[2] be defined as

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} \tag{17}$$

and serves to normalize the distribution so $\int_0^1 p(x)dx = 1$. With some fiddling with the integrals and some integrating by parts you can show

$$B(\alpha, \beta) = B(\alpha+1, \beta) + B(\alpha, \beta+1) \tag{18}$$

and from this it follows, again you don't need to show this, that

$$B(\alpha+1, \beta) = \frac{\alpha}{\alpha+\beta} B(\alpha, \beta) \tag{19}$$

Find the mean of the Beta distribution.

Solution: So

$$\mu = \frac{1}{B(\alpha,\beta)} \int_0^1 x x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{1}{B(\alpha,\beta)} \int_0^1 x^{(\alpha+1)-1}(1-x)^{\beta-1} dx = \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} \tag{20}$$

so

$$\mu = \frac{\alpha}{\alpha+\beta} \tag{21}$$

4. This is going to be complicated but it introduces the concept of a conjugate prior, so bear with me! You go to a casino, the Baysiana, and the slot machines have a probability x of paying out. You don't know this probability because it varies from casino to casino, in fact as an international gambler you have been to many casinos and you know that the distribution of x for casinos satisfies a Beta distribution with some α and β you have calculated from your many casino visits. In other words your prior for the slot machines at the Baysiana, before you play one, is

$$p(x) \sim \text{Beta}(\alpha, \beta) \tag{22}$$

or

$$p(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} \tag{23}$$

and α and β describe your understanding of what p is for the Baysiana. Now you put a coin in and pull the lever, you win! What is your posterior distribution for x?

Solution: So we want $p(x|w)$ where w is a win and hence, by definition $p(w|x) = x$. We use the usual rule

$$p(x|w) = \frac{p(w|x)p(x)}{p(w)} \tag{24}$$

[2] Roughly speaking, in mathematics a special function is an integral or solution to a differential equation that could not, in the nineteenth century, be related to a function that had already been names, so instead it is given a name of its own

where $p(x)$ is the original prior so

$$p(x|w) = \frac{x x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)p(w)} \tag{25}$$

now

$$p(w) = \int_0^1 p(w|x)p(x)dx \tag{26}$$

or

$$p(w) = \frac{1}{B(\alpha,\beta)}\int_0^1 x^{\alpha}(1-x)^{\beta-1}dx = \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} \tag{27}$$

Hence

$$p(x|w) = \frac{x^{(\alpha+1)-1}(1-x)^{\beta-1}}{B(\alpha+1,\beta)} \tag{28}$$

or

$$p(x|w) \sim \mathrm{Beta}(\alpha+1,\beta) \tag{29}$$

This means the consequence of winning is to change your prior by adding one to α!