Probability and Combinatorics Worksheet 7 - outline solutions

Questions

These are the questions you should make sure you work on in the workshop.

1. A distribution x has the form

$$p(x) = \begin{cases} x & 0 \leq x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

What is the probability $x < 1$; what is the probability $x < 1.5$? What is the probability $0.5 < x < 1.5$?

Solution: The first bit is easy since clearly half the probability mass is below one and the rest is above, so $P(x < 1) = 0.5$, the next is harder

$$P(x < 1.5) = \int_0^{1.5} p(x)dx \tag{2}$$

Now $p(x)$ is a complicated function but we can split this up

$$P(x < 1.5) = \int_0^1 p(x)dx + \int_1^{1.5} p(x)dx = 0.5 + \int_1^{1.5} (2-x)dx \tag{3}$$

and now we can do the integral

$$P(x < 1.5) = 0.5 + \left(2x - \frac{1}{2}x^2\right)_1^{1.5} = 0.5 + 0.375 = 0.875 \tag{4}$$

Similarly

$$P(0.5 < x < 1.5) = \int_{0.5}^{1.5} p(x)dx = \int_{0.5}^1 p(x)dx + \int_1^{1.5} p(x)dx \tag{5}$$

and substituting in the mass function in each subinterval

$$P(0.5 < x < 1.5) = \int_{0.5}^1 xdx + \int_1^{1.5} (2-x)dx = \left.\frac{x^2}{2}\right]_{0.5}^1 + 0.375 = 0.75 \tag{6}$$

2. The distribution of tree heights in a pine tree forest is

$$p(h) = \begin{cases} 0.3 & 0 \leq h < 2 \\ 0.2 & 2 \leq h < 4 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

What is the mean height of trees in the forest?

Solution: So

$$\mu = \langle H \rangle = \int_{-\infty}^{\infty} hf(h) = \int_0^2 0.3hdh + \int_2^4 0.2hdh \tag{8}$$

and

$$\int_0^2 0.3h dh = 0.3\frac{h^2}{2}\bigg|_0^2 = 0.6 \tag{9}$$

and

$$\int_2^4 0.2h dh = 0.2\frac{h^2}{2}\bigg|_2^4 = 0.2(8-2) = 1.2 \tag{10}$$

so $\mu = 1.8$.

3. Work out the mean and variance for the distribution

$$p(x) = \begin{cases} 1/2a & x \in [-a, a] \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Solution: So in this case

$$\langle X \rangle = \int_{-\infty}^{\infty} xp(x)dx = \frac{1}{2a}\int_{-a}^{a} xdx = 0 \tag{12}$$

and

$$\langle X^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x)dx = \frac{1}{2a}\int_{-a}^{a} x^2 dx = \frac{2x^3}{3a}\bigg]_{-a}^{a} = \frac{a^2}{3} \tag{13}$$

and here the variance is the same as the second moment because the mean is zero.

4. Another useful distribution is the exponential distribution:

$$p(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability $\text{Prob}(x_1 < x < x_2)$ where x_1 and x_2 are both positive.

Solution:

$$\text{Prob}(x_1 < x < x_2) = \lambda \int_{x_1}^{x_2} e^{-\lambda y} dy = -\, e^{-\lambda y}\bigg]_{x_1}^{x_2} = e^{-\lambda x_1} - e^{-\lambda x_2} \tag{14}$$

Extra questions

Do these in the workshop if you have time.

1. By integrating, what is the mean of the exponential distribution?

Solution: So

$$< X >= \lambda \int_0^{\infty} x e^{-\lambda x} dx \tag{15}$$

so this is an integration by parts with $u = x$ and $dv = \exp(-\lambda x)dx$ so

$$du = 1 \tag{16}$$

and

$$v = -\frac{1}{\lambda}e^{-\lambda x} \tag{17}$$

giving

$$< X >= \int_0^\infty e^{-\lambda x}dx = \frac{1}{\lambda} \tag{18}$$

2. Work out the mean of the exponential distribution by differenciating

$$1 = Z = \int_0^\infty p(x)dx$$

Solution: Well

$$1 = \lambda \int_0^\infty e^{-\lambda x}dx \tag{19}$$

and differentiate both sides with respect to λ to get

$$0 = \int_0^\infty e^{-\lambda x}dx - \lambda \int_0^\infty xe^{-\lambda x}dx \tag{20}$$

so

$$\langle X \rangle = \frac{1}{\lambda} \tag{21}$$

3. What is the variance of the exponential distribution?

Solution: So same again, using the formula for the mean

$$1 = \lambda^2 \int_0^\infty xe^{-\lambda x}dx \tag{22}$$

and differentiate both sides with respect to λ to get

$$0 = 2\lambda \int_0^\infty xe^{-\lambda x}dx - \lambda^2 \int_0^\infty x^2e^{-\lambda x}dx \tag{23}$$

so

$$\langle X^2 \rangle = \frac{2}{\lambda} \tag{24}$$

and hence

$$\mathrm{var}(X) = \frac{1}{\lambda} \tag{25}$$