Worksheet 4 - 2020/21 class test

This was set as a class test last year; this year the class test later so I am giving this as a worksheet. I think it is useful because it goes back over the important material we have already covered. When it was a test the first question was worth 20 marks, the second was worth 10.

1. This question is about calculating probabilities and conditional probabilities in the context of the dice game craps.

   a) A pair of fair six-sided dice is thrown. What is the probability their values will sum to seven? [4 marks]

   b) The game of craps involves repeated rolls of a pair of dice; it is a gambling game and participants can bet on different outcomes. For this question we will concentrate on the outcome called 'pass'. There are two phases to the pass game, the first is a single roll called a 'come out' roll. In the come out roll the player wins if the values sum to seven or 11, they lose if the values sum to two, three or 12; otherwise they progress on to the next phase of the game. What is the probability of these three possibilities? [4 marks]

   c) From now on we will call the summed value of the two dice 'the value of the roll'. If the value in the come out roll is not two, three, seven, 11 or 12, that is if the game progresses on to the next phase, the come out value is called the 'point'. The pair of dice will now be rolled repeatedly until the roll either gives the value seven, or its value is equal to the point. If it gives the point the player wins, if it gives seven the player loses. As an example, say in the come out round the value is six; six is now the point and the dice are rolled again, say in the next round the value is five the game continues, in the next round a two is rolled and, again, the dice are rolled again, at which point the value is seven, so the game ends with a loss for the player. If the value of the come out roll is four, what is the probability the player will win? It might be useful to think of this as $P(\text{value} = 4|\text{value is 4 or 7})$ [4 marks]

   d) If Q_i represents the chance of winning if the point is i: for example, the probability calculated above is Q_4 and P_i is the probability the point is i, so, $P_4 = 1/12$ what is the probability of winning, in terms of Q_i and P_i? [4 marks]

   e) What is the probability of winning? [4 marks]

Answer:
a) So there are 36 possible pairs of face values, each equally likes, of these 6+1, 5+2, 4+3, 3+4, 2+5 and 1+6 add to seven, so the probability of rolling a seven is 1/6.
b) win is 1/6+1/18 since there are two ways to make 11: 5+6 and 6+5, this is 4/18=2/9. lose is 1/36+1/18+1/36 which is 1/9 and by subtraction this leave 2/3 as the chance of going on to the next phase of the game.
c) So using Bayes rule

$$P(4|4\|7) = \frac{P(4\|7|4)P(4)}{P(4\|7)} \tag{1}$$

with $P(4) = 3/36$ and $P(7) = 6/36$ and $P(4\|7|4) = 1$ hence

$$P(4|4\|7) = 3/(3+6) = 1/3 \tag{2}$$

d) There are two ways of winning, either from the come out roll or a subsequent roll, which must be summed over the possible point values.

$$P(\text{first roll is 11 or seven}) + \sum_{4,5,6,8,9,10} P_i Q_i = \frac{2}{9} + \sum_{4,5,6,8,9,10} P_i Q_i \tag{3}$$

e) So

$$\frac{2}{9} + \frac{1}{3}\frac{1}{12} + \frac{2}{5}\frac{1}{9} + \frac{5}{11}\frac{5}{36} + \frac{5}{11}\frac{5}{36} + \frac{2}{5}\frac{1}{9} + \frac{1}{3}\frac{1}{12} \approx 0.49 \tag{4}$$

[each part has 4 marks with 4 for completely correct, 3 for correct with numerical error, 2 for right idea but not implemented, 1 for some idea].

2. This is a question about Bayes' theorem. Exactly a fifth of the people in a town have lycanthropy. There are two tests for lycanthropy, ASSAY1 and ASSAY2. When a person goes to a doctor to test for lycanthropy, with probability 2/3 the doctor conducts ASSAY1 on them and with probability 1/3 the doctor conducts ASSAY2 on them. When ASSAY1 is done on a person, the outcome is as follows: If the person has the disease, the result is positive with probability 3/4. If the person does not have the disease, the result is positive with probability 1/4. When ASSAY2 is done on a person, the outcome is as follows: If the person has the disease, the result is positive with probability 1. If the person does not have the disease, the result is positive with probability 1/2. A person is picked uniformly at random from the town and is sent to a doctor to test for lycanthropy. The result comes out positive. What is the probability that the person has the disease?[10 marks] [1]

Answer: So let A be the event of ASSAY1, L the event of lycanthropy and Y the event of a positive test. We want $P(L|Y)$. Now

$$P(L|Y) = \frac{P(Y|L)P(L)}{P(Y)} \tag{5}$$

so the easy bit is $P(L) = 1/5$, we are told that up front.

$$P(Y|L) = P(Y|L \cap A)P(A) + P(Y|L \cap \bar{A})P(\bar{A}) \tag{6}$$

and we know all those numbers so

$$P(Y|L) = \frac{3}{4}\frac{2}{3} + \frac{1}{3} = \frac{5}{6} \tag{7}$$

We will also need $P(Y|\bar{L})$ which is, by the same approach

$$P(Y|\bar{L}) = \frac{1}{4}\frac{2}{3} + \frac{1}{3}\frac{1}{2} = \frac{1}{3} \tag{8}$$

Hence

$$P(Y) = P(Y|L)P(L) + P(Y|\bar{L})P(\bar{L}) = \frac{5}{6}\frac{1}{5} + \frac{1}{3}\frac{4}{5} = \frac{13}{30} \tag{9}$$

so

$$P(L|Y) = \frac{1/6}{13/30} = \frac{5}{13} \tag{10}$$

[3 marks for recognizing $P(L|Y)$ as the goal and stating Bayes' theorem, $P(Y|L)$ 2 marks, $P(Y|\bar{L})$ 2 marks, 3 marks for $P(L|Y)$ or 1 mark if roughly right but with error]

[1] Taken from allendowney.blogspot.com/2017/02/a-nice-bayes-theorem-problem-medical.html