Problem Sheet 1 - outline solutions

1. In the poker hand two pair there are two pairs of cards with each card in the pair matched by value; the fifth card is a different value. What is the probability of two pairs when five cards are drawn randomly.

   Solution: There are 13 choose two choices for the two values for the two pairs and for each pair there are four choose two possible cards. For the remaining card there are 11 possible values and four possible suits. Thus, the number of possible pairs is

   $$\left(\begin{array}{c}13\\2\end{array}\right)\left(\begin{array}{c}4\\2\end{array}\right)^2 \times 44 = \frac{13\times 12}{1\times 2}\times 36\times 44 = 123552 \tag{1}$$

   and hence the probability is 123552/2598960=0.0475.

2. In a full house there is one pair and one triple, what is the probability of getting a full house?

   Solution: For full house, there are 13 possible values for the pair and 12 for the triple; including the choice of suits we have

   $$13\times 12\times \left(\begin{array}{c}4\\2\end{array}\right)\left(\begin{array}{c}4\\3\end{array}\right) = 3744 \tag{2}$$

   and the probability is 0.0014.

3. How many anagrams are there of the word 'COVID'?

   Solution: The number of ways to reorder five letters is $5! = 5\times 4\times 3\times 2 = 120$, so, if you don't count the word itself, there are 119 anagrams.

4. How many distinct anagrams are there of the word 'CUMMINGS'?

   Solution: This one is harder because of the repeated letter, if you just look at the reorderings that gives 8! but that double counts each word since you can swap the 'M's, so the answer, leaving out the word itself, is $8!/2 - 1 = 20159$. Another way to think about it that allows you to generalize to more complicated examples is to think that there are eight slots, two are allocated to 'M's and one to each of the other letters, so n the number of anagrams is

   $$n = \left(\begin{array}{c}8\\2,1,1,1,1,1,1\end{array}\right) - 1 = 20159 \tag{3}$$

Extra questions

1. When it started in 1987 the Irish lottery has 36 numbers; participants paid 50 Irish pence to buy a combination of six different numbers; they would win if these numbers matched the six drawn. In the last week in May in 1992 a syndicate tried to buy all combinations of numbers. If they had succeeded how many numbers would they have bought?

   Solution: Well this is just 36 choose six:
$$\begin{pmatrix} 36 \\ 6 \end{pmatrix} = 1947792 \tag{4}$$
   so they would've spend $973,896$ Irish pounds.

2. From a group of three undergraduates and five graduate students, four students are randomly selected to act as TAs. What is the chance there will be exactly two undergraduate TAs?

   Solution: So this is another counting exercise, the total ways of selecting four out of eight is
$$\begin{pmatrix} 8 \\ 4 \end{pmatrix} = frac8 \times 7 \times 6 \times 51 \times 2 \times 3 \times 4 = 70 \tag{5}$$
   Now the number of ways of choosing two undergraduates out of three is three and the number of ways of picking two graduates out of five is 10. Hence the answer is $3/7$.

3. Prove
$$\begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n \\ n-r \end{pmatrix} \tag{6}$$

   Solution: This follows from the definition
$$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!} \tag{7}$$
   which stays the same if you swap r and $n-r$.

4. How many distinct anagrams has the word 'OROONOKO'?

   Solution: So there are eight letters, five of which are 'O', so following the answer to 'CUMMINGS' the anwer is
$$n = \begin{pmatrix} 8 \\ 5,1,1,1 \end{pmatrix} - 1 = 8 \times 7 \times 6 - 1 = 335 \tag{8}$$