Worksheet 3 - outline solutions

Questions

1. Two events A and B have probabilities $P(A) = 0.2$, $P(B) = 0.3$ and $P(A \cup B) = 0.4$. Find

   a) Find $P(A \cap B)$.

   b) Find $P(\bar{A} \cap \bar{B})$.

   c) Find $P(A|B)$.

   Solution: So
$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{1}$$
   so
$$P(A \cap B) = 0.2 + 0.3 - 0.4 = 0.1 \tag{2}$$
   If $P(A \cap B) = 0.1$ then $P(\bar{A} \cap B) = P(B) - P(A \cap B) = 0.2$. One the other hand $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) = 0.6$. Finally
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.3} = \frac{1}{3} \tag{3}$$

2. In a library where all books have blue or yellow spines, four fifths of books with yellow spines are about mathematics but only a fifth of books with blue spines are about mathematics. There are the same number of yellow and blue spined books, you come upom a book open on a table; the book is about mathematics. What is the chance it has a yellow spine?

   Solution: In the obvious notation we need $P(M)$:
$$P(M) = P(M|B)P(B) + P(M|Y)P(Y) = 0.1 + 0.4 = 0.5$$
   and using Bayes:
$$P(Y|M) = pP(M|Y)P(Y)/P(M) = 0.4/0.5 = 0.8$$

3. You want to go for a walk. However, when you wake up the day is cloudy and half of all raining days start off cloudy. On the other hand, two days in five start off cloudy and it's been rather dry recently with only rain only on one day in ten. What is the chance it will rain?

   Solution:
$$P(\text{rain}|\text{cloudy}) = \frac{P(\text{cloudy}|\text{rain})P(\text{rain})}{P(\text{cloudy})} = \frac{0.5 * 0.1}{0.4} = 0.125 \tag{4}$$

4. One night in a bar in Las Vegas you meet a dodgy character who tells you that there are two types of slot machine in the Topicana, one that pays out 10% of the time, the other 20%. One sort of machine is blue, the other red. Unfortunately the dodgy character is too drunk to remember which is which. The next day you randomly select red to try, you find a red machine and put in a coin. You lose. Assuming the dodgy character was telling the truth, what is the chance the red machine is the one that pays out more. If you had won instead of losing, what would the chance be?[1]

   Solution: So let J be the event of winning and R be the event that the red machine is the one that pays out more. We want

$$P(R|!J) = \frac{P(!J|R)P(R)}{P(!J)} \tag{5}$$

   where we are writing $!J$, not J, for $\bar{J}$. Now $P(!J|R) = 0.8$ according to the DC and $P(R) = 0.5$ because we chose between red and blue randomly. Finally

$$P(!J) = P(!J|R)P(R) + P(!J|B)P(B) = 0.8 \times 0.5 + 0.9 \times 0.5 = 0.85 \tag{6}$$

   and hence

$$P(R|\bar{J}) = \frac{0.8 \times 0.5}{0.85} \approx 0.47 \tag{7}$$

   so, consider the prior $P(R) = 0.5$ we haven't learned much from this lost coin. If we had won you'd have

$$P(R|J) = \frac{P(J|R)P(R)}{P(J)} = \frac{0.2 \times 0.5}{0.15} \approx 0.67 \tag{8}$$

   so you'd learn a lot more from a win, this makes sense since it is a rarer event.

Some more questions

1. $A = \{a, b, c\}$, $B = \{c, d, e\}$ and the sample space $X = A \cup B \cup \{f, g, h, i\}$. All outcomes, that is the letters, are equally likely. What is $P(A)$? What is $P(A|B)$? Are A and B independent?

   Solution: So $P(A) = 3/9 = 1/3$ and

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{3} \tag{9}$$

   so A and B are independent.

2. $A = \{a, b, c, j, k\}$, $B = \{c, d, e, l\}$, $C = \{a, b, c, d, e, f, g, h, i\}$ and $X = A \cup B \cup C \cup \{m, n\}$. Are A and B independent given C? Are they conditionally independent?

   Solution: well $P(A) = 5/14$ and $P(B) = 4/14$ but $P(A \cap B) = 1/14$, so no, they aren't independent. However, they are conditionally independent, in fact the calculations for conditional independence reduce to the previous calculation.

[1] I stole this problem from courses.smp.uq.edu.au/MATH3104/

3. A sample space consists of the words

$$\{\text{the,previous,example,had,a,mistake,in,it}\}.$$

   The event A consists of all words with four or fewer letters. The event B consists of all words ending in a vowel. What is $P(B)$, $P(B|A)$ and $P(A|B)$?

   Solution: So $A = \{the, had, a, in, it\}$ and $B = \{the, a, mistake, example\}$ so

$$P(B) = 1/2$$

   and

$$P(B|A) = 2/5$$

   and

$$P(A|B) = 1/2$$

4. In the xkcd cartoon above, what is the chance the Bayesian will win his or her bet if the chance the sun has exploded is one in a million? In reality the chance is, of course, much less than one in a million! Show the answer to six decimal places.

   Solution: Let N be the event that the sun has exploded and L be the event the machine says that the sun has exploded. Hence $P(L|N) = 35/36$ whereas $P(L|\bar{N}) = 1/36$ and $P(N) = 10^{-6}$. The Bayesian will win his or her bet is $P(\bar{N}|L)$:

$$P(\bar{N}|L) = \frac{P(L|\bar{N})P(\bar{N})}{P(L)} = \frac{1/36 \times (1 - 10^{-6})}{1/36 \times (1 - 10^{-6}) + 35/36 \times 10^{-6}} = 0.999965 \qquad (10)$$