Probability and Combinatorics Worksheet 6 - outline solutions

Questions

These are the questions you should make sure you work on in the workshop.

1. A typist makes on average two mistakes per page. What is the probability of a particular page having no errors on it?

   Solutions: So $\lambda = 2$ and hence
$$p(0) = e^{-2} = 0.135 \tag{1}$$

2. Components are packed in boxes of 20. The probability of a component being defective is 0.1. What is the probability of a box containing 2 defective components?

   Solutions: This is a trick question in the sense that it is binomial distribution question not a Poisson distribution question; it is about discrete objects, not discrete events in continuous time. So $n = 20$ and $p = 0.1$ so
$$p(2) = \left(\begin{array}{c}20\\2\end{array}\right)0.1^2 0.9^{18} = 0.29 \tag{2}$$

3. A fisher catches on average one fish every 25 minutes. What is the probability that they catch two fish in an hour?

   Solution: If he catches a fish every 25 minutes then his rate for an hour is 60/25=2.4 so
$$p(0) = \frac{2.4^2}{2}e^{-2.4} \approx 0.26 \tag{3}$$

4. A random variable X gives the square of the face value of a six-sided dice. What are the mean and variance of X.

   Solution: So for the mean it is
$$\mu = \frac{1+4+9+16+25+36}{6} = 15.17 \tag{4}$$
   and the second moment is
$$\langle X^2 \rangle = \frac{1+16+81+256+625+1296}{6} = 379.17 \tag{5}$$
   so
$$\sigma^2 = \langle X^2 \rangle - \mu^2 = 149.1 \tag{6}$$

Extra questions

Do these in the workshop if you have time.

1. For a Poisson process let T be the interval for which the process has, on average, one event, so, for this interval $\lambda = 1$. What is the probability that there are no events for this interval?

   Solution

$$p(0) = \frac{1}{e} = 0.37 \tag{7}$$

2. Starting with the expression for the mean

$$\lambda = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} r e^{-\lambda} \tag{8}$$

   calculate the variance of the Poisson distribution.

   Solution: So differentiate both sides with respect to λ:

$$1 = \sum_{r=0}^{\infty} r^2 \frac{\lambda^{r-1}}{r!} e^{-lambda} - \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} r e^{-\lambda} \tag{9}$$

   We do the usual trick of multiplying and dividing by λ in the first term, and use the fact the second term is just the mean:

$$1 = \frac{1}{\lambda} \sum_{r=0}^{\infty} r^2 \frac{\lambda^r}{r!} e^{-\lambda} - \lambda \tag{10}$$

   Now, note the first term is the second moment, also multiply across by λ:

$$\lambda = \langle R^2 \rangle - \lambda^2 \tag{11}$$

   and since $\mu = \lambda$ the righthand side is σ^2 so

$$\sigma^2 = \lambda \tag{12}$$

3. The **Fano factor** is sometimes used to describe distributions, it is

$$F = \frac{\sigma^2}{\mu} \tag{13}$$

   What is the Fano factor for the Poisson distribution?

   Solution: well the mean and variance are both λ so this is one.