1 Problem Definitions and Statement

Definition 1. A *point cloud* A over $\mathbb{R}^n$ is a sequence of $|A| = k \in \mathbb{N}^+$ vectors in R^n, that is,

$$A = (a_1, \ldots, a_k) \quad \forall i \in [k],\, a_i \in \mathbb{R}^n$$

Definition 2. The *centroid* of a point cloud A is defined to be

$$\bar{a} = \frac{1}{|A|} \sum_{a \in A} a$$

Definition 3. A *rigid-body transformation* on a point cloud A is an affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ determined by an rotation matrix $R \in \mathbb{R}^{n \times n}$ and translation $t \in \mathbb{R}^n$ and given by

$$T : a \mapsto R(a - \bar{a}) + \bar{a} + t$$

that is, a rotation around the centroid of followed by a translation.

The Point-Cloud Registration Problem is, given two point clouds A, B and a pairing function $\phi(T, \cdot)$ based on a rigid-body transformation T such that $\forall a \in A,\, \phi(T, a) \in B$, to determine the rigid-body transformation T that optimizes

$$\min_T \sum_{a \in A} \|T(a) - \phi(T, a)\|^2 \tag{1}$$

Definition 4. The *point-to-point metric* for two point clouds A, B is given by

$$\phi(T, a) = \operatorname*{argmin}_{b \in B} \|T(a) - b\|^2 \tag{2}$$

2 Solution

In general, there is no closed-form solution that minimizes (**??**). When there is no translation, Horn (1987) derives a closed-form solution; therefore, if an initial odometry estimate is precise enough, this translation can be applied to all the points in the point cloud, producing a problem instance with zero translation.

Therefore, we must resort to heuristics

$$
\begin{aligned}
&\min_T \sum_{a \in A} \|T(a) - \phi(T, a)\|^2 \\
&= \min_T \sum_{a \in A} \|Ra' + \bar{a} + t - \phi(T, a)\|^2 && \text{(let } a' = a - \bar{a}\text{)} \\
&= \min_T \sum_{a \in A} \|Ra' + t - \phi(T, a)\|^2 && (\bar{a} \text{ is constant})
\end{aligned}
$$

$$
\begin{gathered}
-2b_x + 2t_x + 2({a'}_x)\cos(\theta) - 2({a'}_y)\sin(\theta) \\
-2b_y + 2t_y + 2({a'}_x)\sin(\theta) + 2({a'}_y)\cos(\theta) \\
(-2({a'}_x)\sin(\theta) - 2({a'}_y)\cos(\theta))(-b_x + t_x + ({a'}_x)\cos(\theta) - ({a'}_y)\sin(\theta)) + (2({a'}_x)\cos(\theta) - 2({a'}_y)\sin(\theta))(-b_y +
\end{gathered}
$$