

## 1 Problem Definitions and Statement

**Definition 1.** A *point cloud*  $A$  over  $\mathbb{R}^n$  is a sequence of  $|A| = k \in \mathbb{N}^+$  vectors in  $\mathbb{R}^n$ , that is,

$$A = (a_1, \dots, a_k) \quad \forall i \in [k], a_i \in \mathbb{R}^n$$

**Definition 2.** The *centroid* of a point cloud  $A$  is defined to be

$$\bar{a} = \frac{1}{|A|} \sum_{a \in A} a$$

**Definition 3.** A *rigid-body transformation* on a point cloud  $A$  is an affine map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  determined by an rotation matrix  $R \in \mathbb{R}^{n \times n}$  and translation  $t \in \mathbb{R}^n$  and given by

$$T : a \mapsto R(a - \bar{a}) + \bar{a} + t$$

that is, a rotation around the centroid of followed by a translation.

The POINT-CLOUD REGISTRATION PROBLEM is, given two point clouds  $A, B$  and a pairing function  $\phi(T, \cdot)$  based on a rigid-body transformation  $T$  such that  $\forall a \in A, \phi(T, a) \in B$ , to determine the rigid-body transformation  $T$  that optimizes

$$\min_T \sum_{a \in A} \|T(a) - \phi(T, a)\|^2 \quad (1)$$

**Definition 4.** The *point-to-point metric* for two point clouds  $A, B$  is given by

$$\phi(T, a) = \operatorname{argmin}_{b \in B} \|T(a) - b\|^2 \quad (2)$$

## 2 Solution

In general, there is no closed-form solution that minimizes (1). When there is no translation, Horn (1987) derives a closed-form solution; therefore, if an initial odometry estimate is precise enough, this translation can be applied to all the points in the point cloud, producing a problem instance with zero translation.

Therefore, we must resort to heuristics

$$\begin{aligned}
& \min_T \sum_{a \in A} \|T(a) - \phi(T, a)\|^2 \\
&= \min_T \sum_{a \in A} \|Ra' + \bar{a} + t - \phi(T, a)\|^2 && \text{(let } a' = a - \bar{a}) \\
&= \min_T \sum_{a \in A} \|Ra' + t - \phi(T, a)\|^2 && (\bar{a} \text{ is constant})
\end{aligned}$$

$$\begin{aligned}
& -2b_x + 2t_x + 2(a'_x) \cos(\theta) - 2(a'_y) \sin(\theta) \\
& -2b_y + 2t_y + 2(a'_x) \sin(\theta) + 2(a'_y) \cos(\theta) \\
& (-2(a'_x) \sin(\theta) - 2(a'_y) \cos(\theta))(-b_x + t_x + (a'_x) \cos(\theta) - (a'_y) \sin(\theta)) + (2(a'_x) \cos(\theta) - 2(a'_y) \sin(\theta))(-b_y +
\end{aligned}$$