Minimum Spanning Tree (MST)

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Cut Property

$$G = (V, E, w)$$

Cut Property (I)

$$\begin{aligned} X &: \text{ A part of some MST } T \text{ of } G \\ (S, V \setminus S) &: \text{ A } \textit{cut} \text{ such that } X \text{ does } \textit{not} \text{ cross } (S, V \setminus S) \\ e &: \text{ } A \text{ lightest edge across } (S, V \setminus S) \end{aligned}$$

Cut Property (I)

X : A part of some MST T of G

$(S, V \setminus S)$: A *cut* such that X does *not* cross $(S, V \setminus S)$

e : *A* lightest edge across $(S, V \setminus S)$

Then $X \cup \{e\}$ is a part of some MST T' of G.

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Correctness of Prim's and Kruskal's algorithms.

By Exchange Argument.

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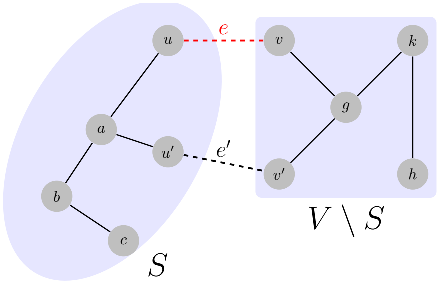

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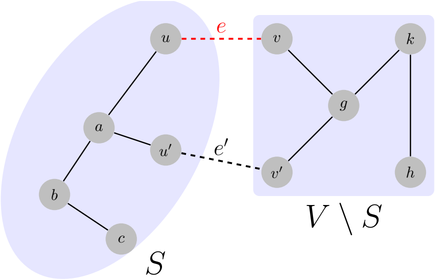

$$T' = \underbrace{T + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

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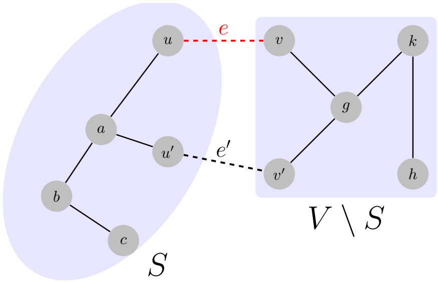

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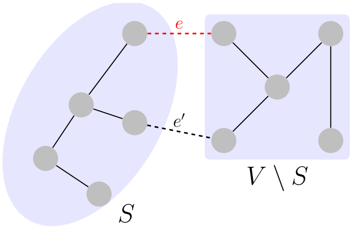
"a" → "the" $\Longrightarrow$ "some" → "all"

Cut Property (II)

A cut $(S, V \setminus S)$

Let $e = (u, v)$ be *a* lightest edge across $(S, V \setminus S)$

$\exists$ MST T of $G : e \in T$



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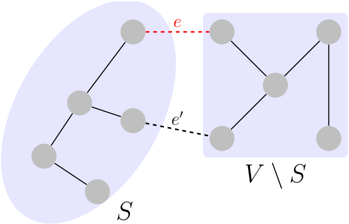

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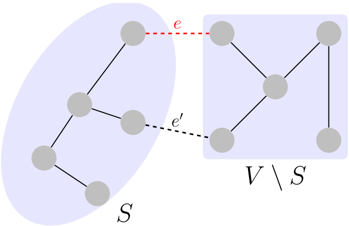

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"a" $\rightarrow$ "the" $\Longrightarrow$ "$\exists$" $\rightarrow$ "$\forall$"

Application of Cut Property [Problem: 10.15 (3)]

$$e = (u, v) \in G \text{ is a lightest edge } \implies e \in \exists \text{ MST of } G$$

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Cycle Property

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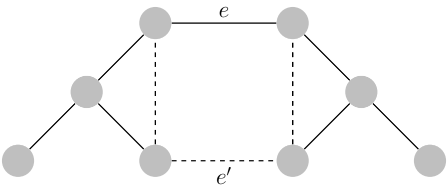
- Let C be any cycle in G
- Let $e = (u, v)$ be *a* maximum-weight edge in C

$$\text{Then } \exists \text{ MST } T \text{ of } G : e \notin T.$$



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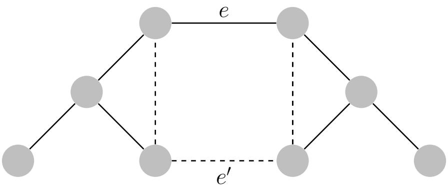

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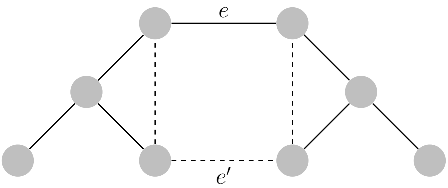

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Anti-Kruskal algorithm [Problem: 10.19 (c)]

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"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.

Application of Cycle Property [Problem: 10.15 (1)]

$$G = (V, E), \quad |E| > |V| - 1$$

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Application of Cycle Property [Problem: 10.15 (2)]

$$C \subseteq G, \quad e \in C$$

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Application of Cycle Property [Problem: 10.15 (2)]

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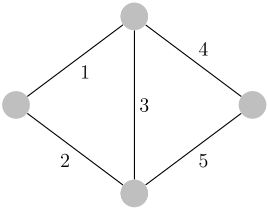

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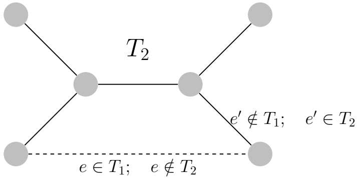

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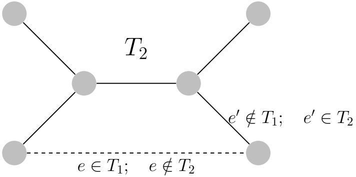

$$T_2 + \{e\} \implies C$$

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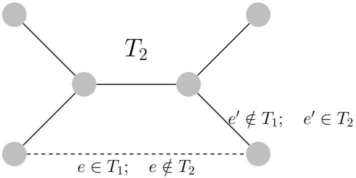

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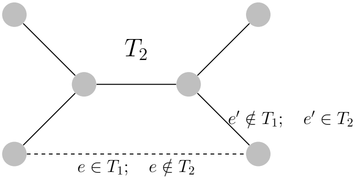

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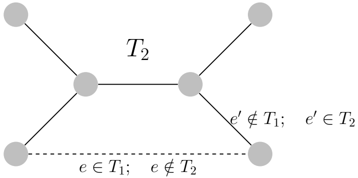

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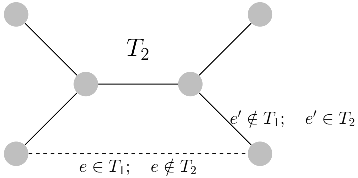

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$$T' = T_2 + \{e\} - \{e'\} \implies w(T') < w(T_2)$$

Condition for Uniqueness of MST [Problem: 10.18 (2)]

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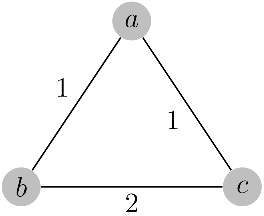

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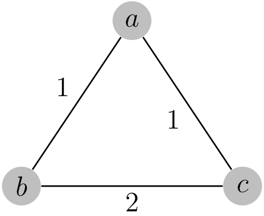

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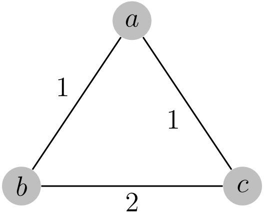

Theorem

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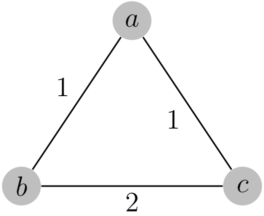

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Construct T by adding all such edges.

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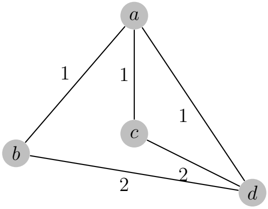

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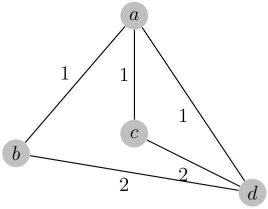

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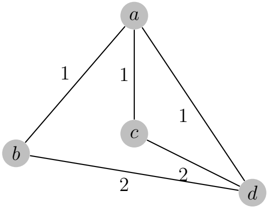

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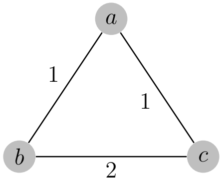

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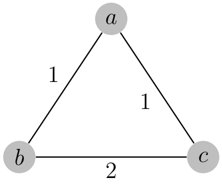

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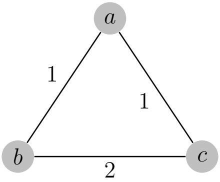

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By Kruskal Algorithm.

Variants of MST

Adding a Vertex v to MST T [Problem: 10.7]

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"On Finding and Updating Spanning Tress and Shortest Paths", 1975

"Algorithms for Updating Minimum Spanning Trees", 1978

MST with Specified Leaves: [Problem: 10.11]

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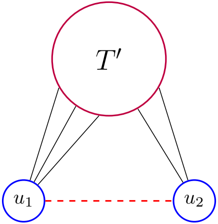

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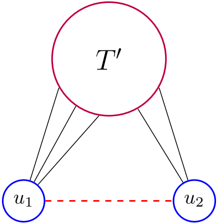

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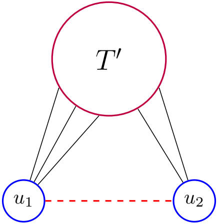
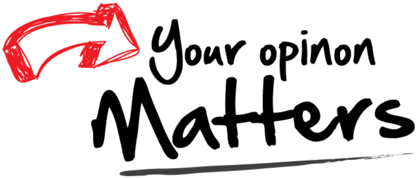


MST T' of $G' = G \setminus U$

Attach $\forall u \in U$ to T' (with lightest edge)


THANK YOU



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