Lower bound for maxima on 2D plane

> Given n points $(x_1, y_1), \dots, (x_n, y_n)$ on a 2-dimensional plane.
> A point (x_1, y_1) *dominates* (x_2, y_2) if $x_1 > x_2 \land y_1 > y_2$.
> A point is called a *maxima* if no other points dominate it.

I can come up with an $O(n \log n)$ algorithm to find all the maxima. However, I failed to solve:

> **Problem:** What is the lower bound for the "finding all 2D maxima" problem (say, on the comparison model)? And how to prove it?

complexity-theory reference-request computational-geometry lower-bounds

edited May 6 '15 at 13:36

asked May 5 '15 at 14:05
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See also the skyline problem: cs.stackexchange.com/q/41382/755 – D.W. ♦ May 6 '15 at 0:50

1 Answer

When asking for a lower bound, you should let us know what computation model you're interested in. A good first model to prove lower bounds on is the comparison model. In this case, usually you use the fact that given n values $x_1, \dots, x_n$, determining whether all of them are unique takes $\Omega(n \log n)$ comparisons. This problem is known as *element distinctness* or *element uniqueness*.

In your case, the natural candidate reduction is to map x_i to $(x_i, -x_i)$. Note that no pair dominates each other, but if $x_i = x_j$ then $(x_i, -x_i)$ "weakly dominates" $(x_j, -x_j)$. Therefore any algorithm outputting the number of "weak maxima" can be used to decide element uniqueness, and so requires $\Omega(n \log n)$ comparisons.

You are interested in maxima rather than weak maxima, so you need to tweak this idea a bit. Suppose that all x_i were integers. Then you map x_i to $p_i = (x_i + i/(2n), -x_i + i/(2n))$. Now p_i dominates p_j iff $x_i = x_j$ and $i > j$. So the number of maxima again allows you to decide element uniqueness.

While you can't really assume that the x_i are integers in the comparison model, you can tweak the reduction even further to implement the same effect: you map x_i to $p_i = ((x_i, i), (-x_i, i))$, and you compare pairs of the form (x, i) according to the following rule: $(x, i) > (y, j)$ if $x > y$ or $x = y$ and $i > j$. Every algorithm for computing the number of maximas in the comparison model using m comparisons can be converted to an algorithm for deciding element uniqueness using $O(m)$ comparisons, hence requires $\Omega(n \log n)$ comparisons.

answered May 5 '15 at 16:27
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Answer Your Question