How to solve this recurrence $T(n) = 2T(n/2) + n/\log n$

How can I solve the recurrence relation

$$T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$$

?

I am stuck up after few steps..

I arrive till

$$T(n) = 2^k T(1) + \sum_{i=0}^{\log(n-1)} \left(\frac{n}{\log n} - i\right)$$

How to simplify this log summation?

(recurrence-relations)

edited Feb 24 '14 at 22:21
TooTone
4,522 1 12 35

asked Feb 24 '14 at 22:16
user3112616
26 2

Just to check, the recurrence relation means, for example, that $T(8) = 2T(4) + 8/\log(8)$? If so, what happens when n isn't a power of 2, e.g. what is $T(7)$? – TooTone Feb 24 '14 at 22:27

is T evaluated at floor(n/2)? ceiling(n/2)? Or do you assume n is always a power of 2? – John Feb 24 '14 at 22:27

3 Answers

$$S(k) = 2^{-k} \cdot T(2^k) \implies S(k) = S(k-1) + \frac{1}{k \log 2}$$

$$S(k) = S(0) + \frac{H_k}{\log 2} \implies T(2^k) = \Theta(2^k \cdot \log k)$$

...Which **does not** imply that $T(n) = \Theta(n \cdot \log\log n)$, although this might be the conclusion suggested in your textbook.

answered Feb 24 '14 at 22:29
Did
223k 21 178 387

Can you elaborate why this does not imply that $T(n) = \Theta(n \log\log n)$? Are you referring to the case $n \neq 2^k$? (I suppose it then suffices to show that T is increasing in n to show that $T(n) = \Theta(n \log\log n)$.) – TMM Mar 9 '14 at 14:06

@TMM Yes this only gives access to $T(n)$ for n some power of 2 (and more generally to $T(i2^k)$ for some fixed odd i). – Did Mar 9 '14 at 14:13

Suppose you have $T(0) = 0$ and $T(1) = 1$ and your recurrence for $n \geq 2$ is

$$T(n) = 2T(\lfloor n/2 \rfloor) + \frac{n}{\lfloor \log_2 n \rfloor}.$$

This gives the following **exact** formula for $T(n)$ where $n \geq 2$:

$$T(n) = 2^{\lfloor \log_2 n \rfloor} + \sum_{j=0}^{\lfloor \log_2 n \rfloor - 1} \frac{1}{\lfloor \log_2 n \rfloor - j} \sum_{k=j}^{\lfloor \log_2 n \rfloor} d_k 2^k$$

where we suppose that the binary representation of n is

$$n = \sum_{k=0}^{\lfloor \log_2 n \rfloor} d_k 2^k.$$

The reader is invited to verify this formula which is not at all difficult.

Now for an upper bound consider a string of one digits which gives

$$
\begin{aligned}
T(n) &\le 2^{\lfloor \log_2 n \rfloor} + \sum_{j=0} \frac{1}{\lfloor \log_2 n \rfloor - j} \sum_{k=j} 2^k \\
&= 2^{\lfloor \log_2 n \rfloor} + \sum_{j=0}^{\lfloor \log_2 n \rfloor - 1} \frac{1}{\lfloor \log_2 n \rfloor - j} (2^{\lfloor \log_2 n \rfloor + 1} - 2^j) \\
&= 2^{\lfloor \log_2 n \rfloor} + 2^{\lfloor \log_2 n \rfloor + 1} \sum_{j=0}^{\lfloor \log_2 n \rfloor - 1} \frac{1}{\lfloor \log_2 n \rfloor - j} - \sum_{j=0}^{\lfloor \log_2 n \rfloor - 1} \frac{2^j}{\lfloor \log_2 n \rfloor - j} \\
&= 2^{\lfloor \log_2 n \rfloor} + 2^{\lfloor \log_2 n \rfloor + 1} H_{\lfloor \log_2 n \rfloor} - \sum_{j=1}^{\lfloor \log_2 n \rfloor} \frac{2^{\lfloor \log_2 n \rfloor - j}}{j} \\
&= 2^{\lfloor \log_2 n \rfloor} + 2^{\lfloor \log_2 n \rfloor + 1} H_{\lfloor \log_2 n \rfloor} - 2^{\lfloor \log_2 n \rfloor} \sum_{j=1}^{\lfloor \log_2 n \rfloor} \frac{2^{-j}}{j}.
\end{aligned}
$$

Observe that the remaining sum term converges to a number, so that we get the following asymptotics for the upper bound:

$$2^{\lfloor \log_2 n \rfloor} (1 + 2H_{\lfloor \log_2 n \rfloor} - \log 2).$$

This upper bound is actually attained and hence cannot be improved upon, just like the lower bound, which we now compute and which occurs for a one digit followed by a string of zero digits, giving

$$T(n) \ge 2^{\lfloor \log_2 n \rfloor} + \sum_{j=0}^{\lfloor \log_2 n \rfloor - 1} \frac{1}{\lfloor \log_2 n \rfloor - j} 2^{\lfloor \log_2 n \rfloor} = 2^{\lfloor \log_2 n \rfloor} (1 + H_{\lfloor \log_2 n \rfloor}).$$

Joining the dominant terms of the upper and the lower bound we get a complexity of

$$\Theta\left(2^{\lfloor \log_2 n \rfloor} \times H_{\lfloor \log_2 n \rfloor}\right) = \Theta\left(2^{\log_2 n} \log \log n\right) = \Theta\left(n \log \log n\right).$$

This MSE link points to a chain of similar computations.

answered Feb 25 '14 at 1:40
Marko Riedel
29.7k 2 25 90

Your summation is wrong, and you should have replaced k with $log(n)$ in your last expression.

Here are detailed steps of this recurrence relation:

$$T(n) = 2T(\frac{n}{2}) + \frac{n}{\log n}$$

$$T(n) = 2(2T(\frac{n}{4}) + \frac{\frac{n}{2}}{\log \frac{n}{2}}) + \frac{n}{\log n} = 2^2 T(\frac{n}{2^2}) + \frac{n}{\log(n) - 1} + \frac{n}{\log n}$$

$$T(n) = 2(2(2T(\frac{n}{8}) + \frac{\frac{n}{4}}{\log \frac{n}{4}}) + \frac{\frac{n}{2}}{\log \frac{n}{2}}) + \frac{n}{\log n} = 2^3 T(\frac{n}{2^3}) + \frac{n}{\log(n) - 2} + \frac{n}{\log(n) - 1} + \frac{n}{\log n}$$

$$\cdots$$

$$T(n) = 2^k T(\frac{n}{2^k}) + \sum_{i=0}^{k-1} \frac{n}{\log(n) - i}$$

$$\text{When } \frac{n}{2^k} = 1 \Rightarrow n = 2^k \Rightarrow k = \log_2(n)$$

$$T(n) = 2^{\log_2(n)} T(1) + \sum_{i=0}^{\log_2(n)-1} \frac{n}{\log(n) - i}$$

$$T(n) = \Theta(n) + \sum_{i=0}^{\log_2(n)-1} \frac{n}{\log_2(n) - i}$$

$$T(n) \approx \Theta(n) + \sum_{j=1}^{\log_2(n)} \frac{n}{j}$$

$$T(n) \approx \Theta(n) + nH_{\log_2(n)}$$

$$T(n) \approx \Theta(n) + n \ln(\log_2(n))$$

$$\text{And finally } T(n) \in \Theta(n \ln(\log_2(n)))$$

edited Mar 22 '14 at 14:13 answered Mar 9 '14 at 13:19
Mohamed Ennahdi El Idrissi
210 1 13

could you please elaborate on your last 2 passages? (how do you substitute H for log(log(n))... Thx – jreing Mar 22 '14 at 12:31

1
@user1685224, if you look at page 7 of these slides, it asserts that Hn = ln n. I should rectify then, from log(log(n)) to ln(log(n)), which is practically the same thing. This link is useful too. – Mohamed Ennahdi El Idrissi Mar 22 '14 at 13:42