Paths in Graphs

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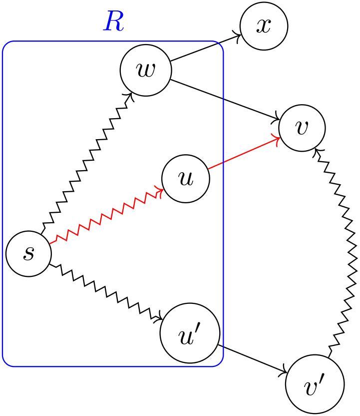
Dijkstra's Algorithm for SSSP

Finding shortest paths from s to other nodes t
in non-decreasing order of $\text{dist}(s, t)$.



$R \triangleq \{u \mid s \rightsquigarrow u \text{ is known}\}$

$v : \min_{u \in R} \text{dist}(u) + l(u, v)$

$l(v' \rightsquigarrow v) \geq 0$

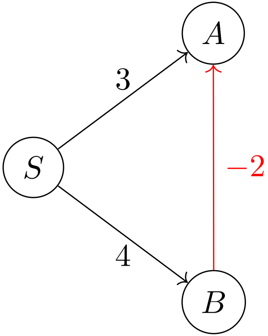
for all $v \in V$ **do**
    dist$[v] \leftarrow \infty$
dist$[s] \leftarrow 0$

$Q \leftarrow \textsc{MinPQ}(V)$
while $Q \neq \emptyset$ **do**
    $u \leftarrow \textsc{DeleteMin}(Q)$
    for all $(u, v) \in E \wedge v \notin Q$ **do**
        if dist$[v] >$ dist$[u] + l(u, v)$ **then**
            dist$[v] \leftarrow$ dist$[u] + l(u, v)$
            $\textsc{DecreaseKey}(Q, v)$

$$O(n + (n + m) \log n) \implies O((n + m) \log n) \implies O(m \log n)$$

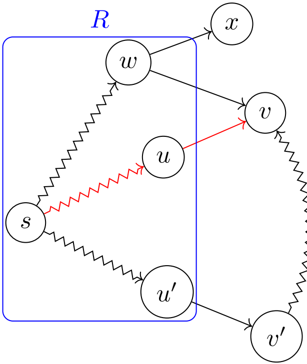
Negative Edges (Problem 11.1)

Dijkstra's algorithm may fail if $w(e) < 0$.



Negative Edges from s (Problem 11.9)

All negative edges are from s.



$$l(v' \rightsquigarrow v) \geq 0$$

Generalized Shortest Path (Problem 11.8)

Digraph $G = (V, E), \quad l_e > 0, \quad c_v > 0, \quad s \in V$

Shortest paths from s

$$\forall u \to v : l'(u, v) = l(u, v) + c_v$$

$$+ \, c_s$$

Shortest Paths Through v_0 (Problem 13.7)

Strongly connected digraph $G = (V, E), \quad w(e) > 0$

$$v_0 \in V$$

Find shortest paths $s \leadsto^{\textsf{SP}} t$ through v_0.

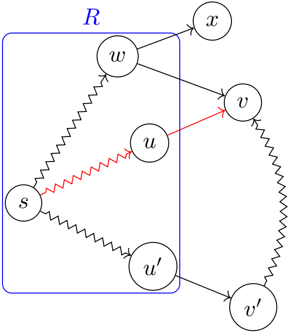
$$s \leadsto^{\textsf{SP}} v_0 \leadsto^{\textsf{SP}} t$$

$$\forall v : v_0 \leadsto^{\textsf{SP}} v$$

Dijkstra's Algorithm as a Framework

for all $v \in V$ **do**
  dist$[v] \leftarrow \infty$
dist$[s] \leftarrow 0$

$Q \leftarrow$ $\textsc{MinPQ}(V)$
while $Q \neq \emptyset$ **do**
  $u \leftarrow \textsc{DeleteMin}(Q)$
  for all $(u, v) \in E \wedge v \notin Q$ **do**
    if dist$[v] >$ dist$[u] + l(u, v)$ **then**
      dist$[v] \leftarrow$ dist$[u] + l(u, v)$
      $\textsc{DecreaseKey}(Q, v)$



Min-Max Path (Problem 11.12)

$$G = (V, E): \text{ network of highways}$$

$$l_e: \text{ road length} \quad L: \text{ tank capacity}$$

Given L, $\exists ? s \rightsquigarrow t$ in $O(n+m)$.

$$l_e > L \implies l_e = \infty$$

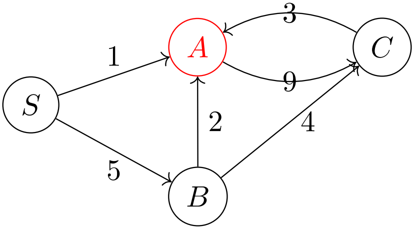
$$s \rightsquigarrow^{?} t$$

Min-Max Path (Problem 11.12)

$$G = (V, E) : \text{ network of highways}$$

$$l_e : \text{ road length} \quad L : \text{ tank capacity}$$

Given G, to compute $\min L$ in $O((n+m)\log n)$.



$$R \triangleq \{u \mid s \rightsquigarrow u \text{ is known}\}$$

for all $v \in V$ **do**
  $L[v] \leftarrow \infty$
$L[s] \leftarrow 0$

if $L[v] > \max(L[u], l(u, v))$ **then**
  $L[v] \leftarrow \max(L[u], l(u, v))$

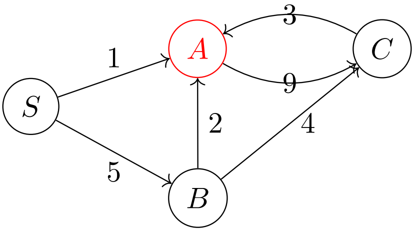
Max-Min Path (Problem 13.2 (1))

$$G = (V, E): \text{ network of oil pipelines}$$

$$c(u, v): \text{ capacity of } (u, v)$$

$$\mathsf{cap}(s, t): \max \min s \rightsquigarrow t$$

Given s, to compute $\mathsf{cap}(s, v)$.



$$R \triangleq \{u \mid s \rightsquigarrow u \text{ is known}\}$$

$$Q \leftarrow \mathsf{MaxPQ}(V)$$

for all $v \in V$ **do**
  $\mathsf{cap}[v] \leftarrow -\infty$
$\mathsf{cap}[s] \leftarrow 0$

if $\mathsf{cap}[v] < \min(\mathsf{cap}[u], c(u, v))$ **then**
  $\mathsf{cap}[v] \leftarrow \min(\mathsf{cap}[u], c(u, v))$

Max-Min Path (Problem 13.2 (2))

$$G = (V, E): \text{ network of oil pipelines}$$

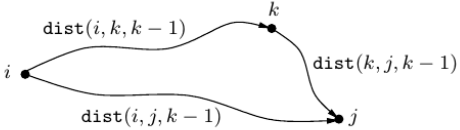
$$c(u, v): \text{ capacity of } (u, v)$$

$$\textsf{cap}(s, t) : \max \min s \rightsquigarrow t$$

Compute all-pair $\textsf{cap}(u, v)$.

Floyd-Warshall algorithm

$$\text{dist}[i,j,k] = \min\left(\text{dist}[i,j,k-1], \text{dist}[i,k,k-1] + \text{dist}[k,j,k-1]\right)$$



$$\text{dist}[i,j,0] = \begin{cases} 0 & i = j \\ w(i,j) & (i,j) \in E \\ \infty & \text{o.w.} \end{cases}$$

for $k \leftarrow 1$ **to** n **do**
  for $i \leftarrow 1$ **to** n **do**
    for $j \leftarrow 1$ **to** n **do**
      $\text{dist}(i, j, k) = \min\Big(\text{dist}(i, j, k-1), \text{dist}(i, k, k-1) + \text{dist}(k, j, k-1)\Big)$

Max-Min Path (Problem 13.2 (2))

$$G = (V, E): \text{ network of oil pipelines}$$

$$c(u, v): \text{ capacity of } (u, v)$$

$$\mathsf{cap}(s, t) : \max \min s \rightsquigarrow t$$

Compute all-pair $\mathsf{cap}(i, j)$.

$$\mathsf{cap}(i, j, k) = \max\Big(\mathsf{cap}(i, j, k-1), \min\big(\mathsf{cap}(i, k, k-1), \mathsf{cap}(k, j, k-1)\big)\Big)$$

Routing table (Problem 13.1)

$$\mathsf{Go}(i,j) = k \implies v_i \to v_k \rightsquigarrow v_j$$

Contruct routing table and extract shortest paths from it.

for all $k \leftarrow 1 \dots n$ **do**
  for all $i \leftarrow 1 \dots n$ **do**
    for all $j \leftarrow 1 \dots n$ **do**
      if $\mathsf{dist}[i,j,k-1] > \mathsf{dist}[i,k,k-1] + \mathsf{dist}[k,j,k-1]$ **then**
        $\mathsf{dist}[i,j,k] \leftarrow \mathsf{dist}[i,k,k-1] + \mathsf{dist}[k,j,k-1]$
        $\mathsf{Go}[i,j] \leftarrow \mathsf{Go}[i,k]$
      else
        $\mathsf{dist}[i,j,k] \leftarrow \mathsf{dist}[i,j,k-1]$

Routing table (Problem 13.1)

$$\text{Go}(i,j) = k \implies v_i \to v_k \rightsquigarrow v_j$$

Contruct routing table and extract shortest paths from it.

for all $i \leftarrow 1 \ldots n$ **do**
    for all $j \leftarrow 1 \ldots n$ **do**
        dist$[i,j] \leftarrow \infty$
        Go$[i,j] \leftarrow$ Nil

for all $(i,j) \in E$ **do**
    dist$[i,j] \leftarrow w(i,j)$
    Go$[i,j] \leftarrow j$

for all $i \leftarrow 1 \ldots n$ **do**
    dist$[i,i] \leftarrow 0$
    Go$[i,i] \leftarrow$ Nil

procedure PATH(i,j)
    if Go$[i,j] =$ Nil **then**
        Output "No Path."

    Output "i"
    while $i \neq j$ **do**
        $i \leftarrow$ Go$[i,j]$
    Output "i"

Shortest Cycle in Digraph (Problem 13.9)

Find shortest cycle in digraph $G = (V, E), \quad w(e) > 0.$

Initialize $\text{dist}[v][v] \leftarrow \infty$ in Floyd-Warshall algorithm

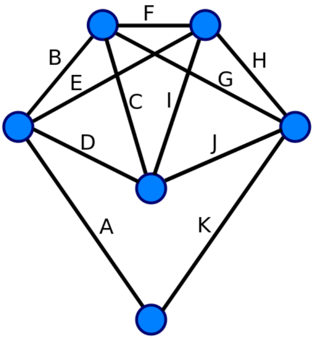
$$\forall v : \text{dist}[v][v] = \infty$$

$$\exists v : \text{dist}[v][v] < \infty \implies \min_i \text{dist}[v][v]$$

Eulerian Circuit (Problem 13.5)

To find an Eulerian circuit of a strongly connected digraph $G = (V, E)$ in $O(m)$ time.

$$\forall v \in V : \mathsf{in}[v] = \mathsf{out}[v]$$



$VE \leftarrow \emptyset$ ▷ Visited Edges

$C \leftarrow \emptyset$ ▷ Current Circuit

while $VE \neq E$ **do**

$u \leftarrow \textsc{Choose}(u : (u \to v) \notin VE)$

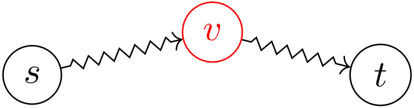
$C' \leftarrow \textsc{Circuit}(u, E \setminus VE)$

$VE \leftarrow VE \cup C'$

$C \leftarrow C \cup C'$

Data Structures?

Bitonic Shortest Path (Problem 11.7)




THANK YOU