Paths in Graphs

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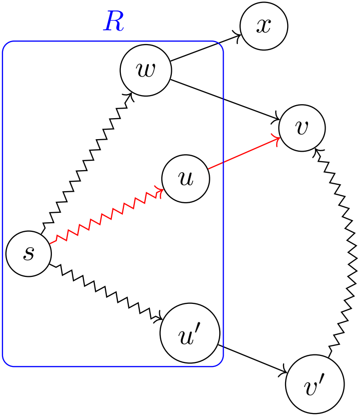
Dijkstra's Algorithm for SSSP

Finding shortest paths from s to other nodes t
in non-decreasing order of $\text{dist}(s,t)$.



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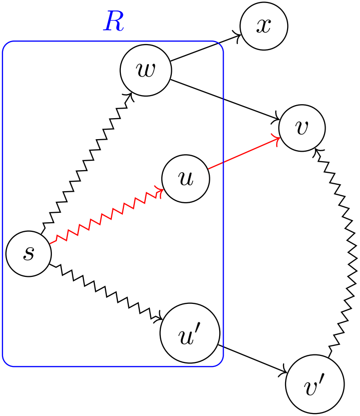

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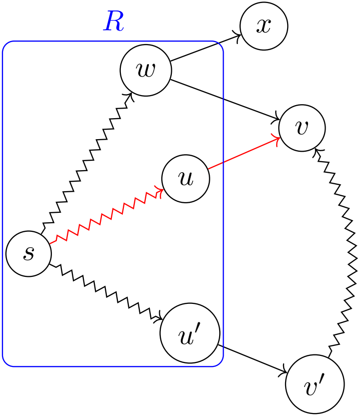

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$$v : \min_{u \in R} \text{dist}(u) + l(u, v)$$

$$l(v' \rightsquigarrow v) \geq 0$$

for all $v \in V$ **do**
  dist$[v] \leftarrow \infty$
dist$[s] \leftarrow 0$

$Q \leftarrow \textsc{MinPQ}(V)$
while $Q \neq \emptyset$ **do**
  $u \leftarrow \textsc{DeleteMin}(Q)$
  for all $(u, v) \in E \wedge v \notin Q$ **do**
    if dist$[v] >$ dist$[u] + l(u, v)$ **then**
      dist$[v] \leftarrow$ dist$[u] + l(u, v)$
      $\textsc{DecreaseKey}(Q, v)$

```
for all v ∈ V do
    dist[v] ← ∞
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while Q ≠ ∅ do
    u ← DeleteMin(Q)
    for all (u, v) ∈ E ∧ v ∉ Q do
        if dist[v] > dist[u] + l(u, v) then
            dist[v] ← dist[u] + l(u, v)
            DecreaseKey(Q, v)
```
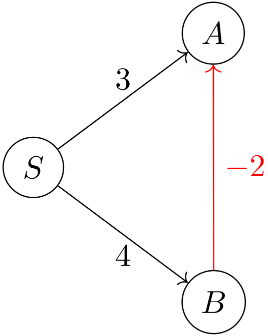

$$O(n + (n+m)\log n) \implies O((n+m)\log n) \implies O(m \log n)$$

Negative Edges (Problem 11.1)

Dijkstra's algorithm may fail if $w(e) < 0$.



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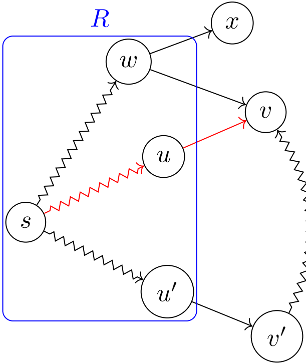

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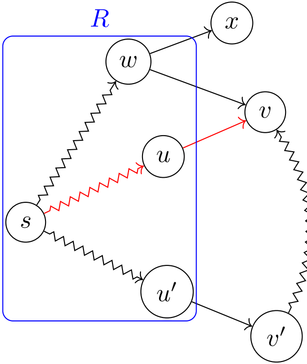

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Generalized Shortest Path (Problem 11.8)

$$\text{Digraph } G = (V, E), \quad l_e > 0, \quad c_v > 0, \quad s \in V$$

Shortest paths from s

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$$+ c_s$$

Shortest Paths Through v_0 (Problem 13.7)

$$\text{Strongly connected digraph } G = (V, E), \quad w(e) > 0$$

$$v_0 \in V$$

Find shortest paths $s \overset{\mathsf{SP}}{\rightsquigarrow} t$ through v_0.

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$$\forall v : v_0 \leadsto^{\textsf{SP}} v$$

Dijkstra's Algorithm as a Framework

Min-Max Path (Problem 11.12)

$$G = (V, E): \text{ network of highways}$$

$$l_e: \text{ road length} \quad L: \text{ tank capacity}$$

Given L, $\exists ? s \rightsquigarrow t$ in $O(n+m)$.

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$$l_e > L \implies l_e = \infty$$

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$$s \rightsquigarrow^? t$$

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$$R \triangleq \{u \mid s \rightsquigarrow u \text{ is known}\}$$

for all $v \in V$ **do**
  $L[v] \leftarrow \infty$
$L[s] \leftarrow 0$

if $L[v] > \max(L[u], l(u,v))$ **then**
  $L[v] \leftarrow \max(L[u], l(u,v))$

Max-Min Path (Problem 13.2 (1))

$$G = (V, E): \text{ network of oil pipelines}$$

$$c(u, v): \text{ capacity of } (u, v)$$

$$\mathsf{cap}(s, t) : \max \min s \rightsquigarrow t$$

Given s, to compute $\mathsf{cap}(s, v)$.

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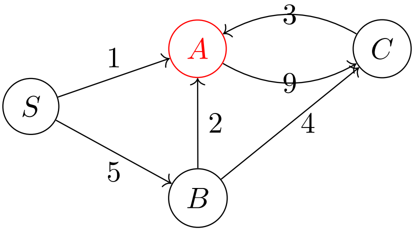

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for all $v \in V$ **do**
    $\text{cap}[v] \leftarrow -\infty$
$\text{cap}[s] \leftarrow 0$

if $\text{cap}[v] < \min(\text{cap}[u], c(u, v))$ **then**
    $\text{cap}[v] \leftarrow \min(\text{cap}[u], c(u, v))$

Max-Min Path (Problem 13.2 (2))

$$G = (V, E): \text{ network of oil pipelines}$$

$$c(u, v): \text{ capacity of } (u, v)$$

$$\mathsf{cap}(s, t): \max \min s \leadsto t$$

Compute all-pair $\mathsf{cap}(u, v)$.

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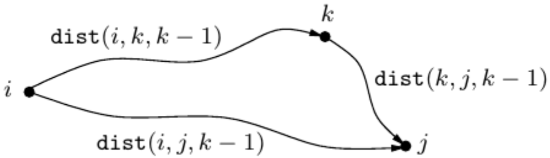

$$\mathsf{dist}(i,j,k) = \min\Big(\mathsf{dist}(i,j,k-1), \mathsf{dist}(i,k,k-1) + \mathsf{dist}(k,j,k-1)\Big)$$

$$\#k's = 1$$

for all (i, j) **do**

  if $(i, j) \in E$ **then**

    $\text{dist}(i, j, 0) \leftarrow l(i, j)$

  else

    $\text{dist}(i, j, 0) \leftarrow \infty$

for $k \leftarrow 1$ **to** n **do**

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Routing table (Problem 13.1)

$$\mathsf{Go}(i, j) = k \implies v_i \to v_k \rightsquigarrow v_j$$

Contruct routing table and extract shortest paths from it.

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$$\mathsf{Go}(i,j) \gets \mathsf{Nil}$$

$$\forall (i,j) \in E : \mathsf{Go}(i,j) \gets j$$

if ... **then**
    $\mathsf{Go}(i,j) \gets \mathsf{Go}(i,k)$

if $Go(i,j) = \mathsf{Nil}$ **then**
    $\cdots$

while $i \neq j$ **do**
    $i \gets \mathsf{Go}(i,j)$

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$\quad i \leftarrow \mathsf{Go}(i, j)$

$$\mathsf{Prev}(i, j) \leftarrow \mathsf{Prev}(k, j)$$

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while $i \neq j$ **do**
$\quad i \leftarrow \text{Go}(i,j)$

$$\text{Prev}(i,j) \leftarrow \text{Prev}(k,j)$$

$$\text{Intermediate}(i,j) \leftarrow k$$

Shortest Cycle in Digraph (Problem 13.9)

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THANK YOU