Graph Decomposition

— DFS&BFS, Cycle, DAG, SCC, and Biconnectivity

hengxin0912@gmail.com

May 19, 2016

Graph Decomposition

Contents of Tutorials

1. Graph ~~Traversal~~ Decomposition
2. MST (Greedy Algorithm) & Path
3. DP: Dynamic Programming

Graph Decomposition

Graph decomposition vs. Graph traversal

Graph Decomposition

Classifying edges

Definition (Classifying edges)

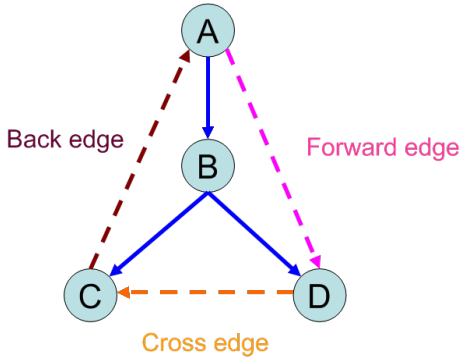
Given a dfs/bfs traversal:

- Tree edge
- Back edge
- Forward edge
- Cross edge

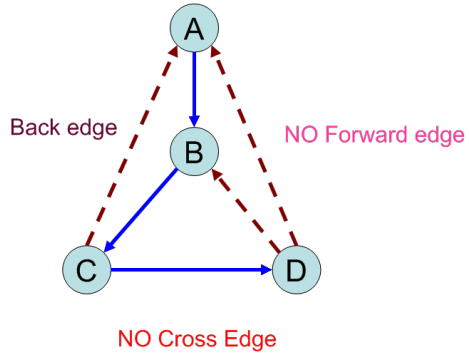
Remarks:

- Applicable to both DFS and BFS
- With respect to DFS/BFS trees

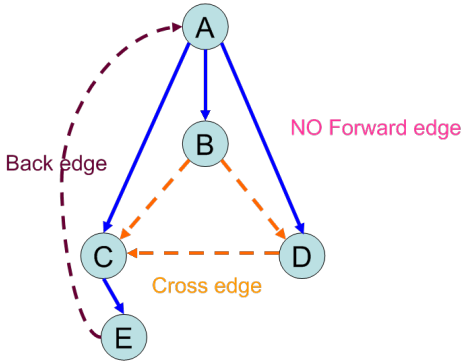
Classifying edges [Problem: 3.4.1]



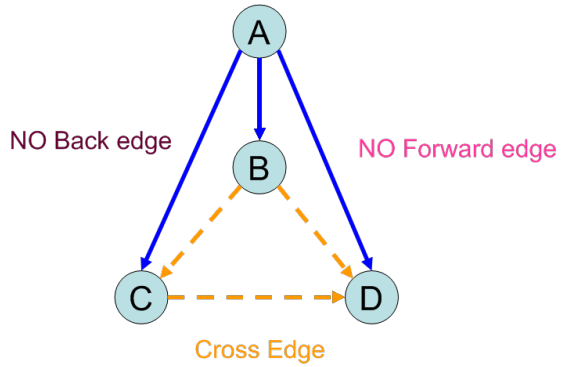
(a) DFS on directed graph.



(b) DFS on undirected graph.



(c) BFS on directed graph.



(d) BFS on undirected graph.

Classifying edges

DFS tree and BFS tree coincide [Problem: 3.4.30]

$G = (V, E), v \in V$. DFS tree T = BFS tree T'.

- G is an undirected graph $\Rightarrow G = T$.
- G is a digraph $\Rightarrow^{?} G = T$.

Solution.

- T: tree + back; T': tree + cross
- T: tree + back + forward + cross; T': tree + back + cross

Distance constraints for BFS

Distance constraints for BFS [Problem: 3.4.4]

BFS on digraph:

TE: $d[v] = d[u] + 1$

BE: $0 \leq d[v] \leq d[u]$

CE: $d[v] \leq d[u] + 1$

BFS on undirected graph:

TE: $d[v] = d[u] + 1$

CE: $d[v] = d[u] \vee d[v] = d[u] + 1$

Solution to "*CE* in BFS on *undirected* graph".

- $d[v] = d[u], d[v] = d[u] + 1$
- $d[v] < d[u], d[v] > d[u] + 1$

Remark.

- BFS tree defines a *shortest-path* from its root to every other node.
- Layers in BFS on *undirected* graph $\Rightarrow$ bipartite testing [Problem: 3.4.26]

Graph Decomposition

Cycle detection

Table: Cycle detection [Problem: 3.4.21]

	Digraph	Undirected graph
DFS	back edge $\iff$ cycle	back edge $\iff$ cycle
BFS	back edge $\Rightarrow$ cycle cycle $\not\Rightarrow$ back edge	cross edge $\iff$ cycle

Remark.

- cycle in undirected graphs
- cycle in digraphs
  - DAG
  - SCC

Orientation of undirected graph

Orientation of undirected graph [Problem: 3.4.13]

Undirected (connected) graph G, edge oriented s.t. $\forall v, \text{in}[v] \geq 1$.

Solution.

orientation $\Longleftrightarrow$ $\exists$ cycle; DFS

Bipartite graph

Bipartite graph [Problem: 3.4.26; 3.4.32]

To test bipartiteness of an undirected graph.

Solution.

BFS + Coloring:

- pick any s, c[s] $= 0$
- $u \leftarrow$ Dequeue(Q)
- $\forall (u, v)$:
  - tree edge
  - cross edge: check

Proof.

Check cross edge (u, v):

- $(\exists)$ $d[v] = d[u] \Rightarrow$ the same layer $\Rightarrow$ odd cycle ([Problem: 3.4.17]; EX) $\Rightarrow$ non-bipartite
- $(\forall)$ $d[v] = d[u] + 1 \Rightarrow$ different layers $\Rightarrow$ different colors

Graph Decomposition

DAG

$$\text{no back edge} \iff \text{DAG} \iff \exists \text{ topo. ordering}$$

Topo. sorting

DFS on digraph, $u \to v$:

- back edge: $f(u) < f(v)$
- others: $f(u) > f(v)$

$$u \to v \Rightarrow u \prec v$$

$$u \to v \Rightarrow f(u) > f(v)$$

Topo. sorting: sort vertices in *decreasing* order of their *finish* times.

Kahn's toposort algorithm

Kahn's toposort algorithm (1962) [Problem: 3.4.19]

- queue for source vertices ($\text{in}[v] = 0$)
- repeat: dequeue v, delete it, output it

Solution.

Lemma

Every DAG has at least one source and at least one sink vertex.

Remark

DFS on DAG:

- $\arg\max_v f(v) \Rightarrow$ source (used in SCC algorithm)
- $\arg\min_v f(v) \Rightarrow$ sink