Minimum Spanning Tree (MST)

Hengfeng Wei

hfwei@nju.edu.cn

June 19, 2018

Cut Property

$$G = (V, E, w)$$

Cut Property (I)

$$
\begin{aligned}
X &: \text{ A part of some MST } T \text{ of } G \\
(S, V \setminus S) &: \text{ A \textit{cut} such that } X \text{ does \textit{not} cross } (S, V \setminus S) \text{ Âŋ} \\
e &: \text{ \textit{A} lightest edge across } (S, V \setminus S)
\end{aligned}
$$

Cut Property (I)

X : A part of some MST T of G

$(S, V \setminus S)$: A *cut* such that X does *not* cross $(S, V \setminus S)$ Âŋ

e : *A* lightest edge across $(S, V \setminus S)$

Then $X \cup \{e\}$ is a part of some MST T' of G.

Cut Property (I)

X : A part of some MST T of G

$(S, V \setminus S)$: A *cut* such that X does *not* cross $(S, V \setminus S)$ Âŋ

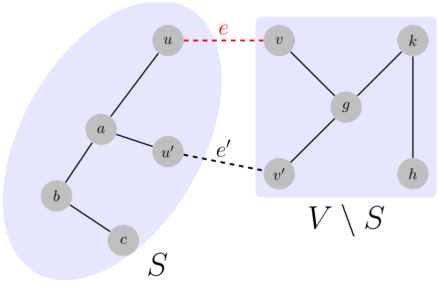
e : *A* lightest edge across $(S, V \setminus S)$

Then $X \cup \{e\}$ is a part of some MST T' of G.

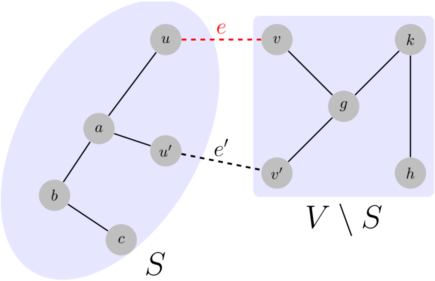
Correctness of Prim's and Kruskal's algorithms.

By Exchange Argument.

By Exchange Argument.

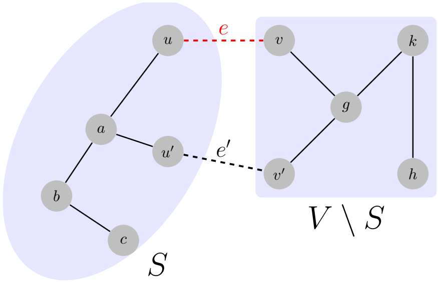


By Exchange Argument.



$$T' = \underbrace{T + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

By Exchange Argument.



$$T' = \underbrace{T + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

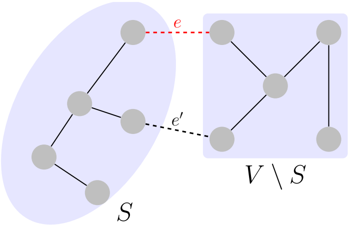
"a" $\rightarrow$ "the" $\Longrightarrow$ "some" $\rightarrow$ "all"

Cut Property (II)

A cut $(S, V \setminus S)$

Let $e = (u, v)$ be *a* lightest edge across $(S, V \setminus S)$

$\exists$ MST T of G : $e \in T$

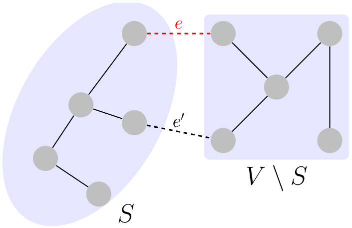


Cut Property (II)

A cut $(S, V \setminus S)$

Let $e = (u, v)$ be *a* lightest edge across $(S, V \setminus S)$

$\exists$ MST T of G : $e \in T$



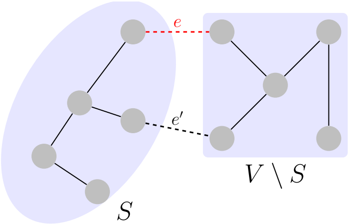
$$T' = \underbrace{T + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

Cut Property (II)

A cut $(S, V \setminus S)$

Let $e = (u, v)$ be *a* lightest edge across $(S, V \setminus S)$

$\exists$ MST T of G : $e \in T$



$$T' = \underbrace{T + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

"a" $\rightarrow$ "the" $\Longrightarrow$ "$\exists$" $\rightarrow$ "$\forall$"

Application of Cut Property [Problem: 10.15 (3)]

$$e = (u, v) \in G \text{ is a lightest edge } \implies e \in \exists \text{ MST of } G$$

Application of Cut Property [Problem: 10.15 (4)]

$$e = (u, v) \in G \text{ is the unique lightest edge } \implies e \in \forall \text{ MST}$$

Application of Cut Property [Problem: 10.15 (3)]

$$e = (u, v) \in G \text{ is a lightest edge } \implies e \in \exists \text{ MST of } G$$

$$\Big(S = \{u\}, V \setminus S\Big)$$

Application of Cut Property [Problem: 10.15 (4)]

$$e = (u, v) \in G \text{ is the unique lightest edge } \implies e \in \forall \text{ MST}$$

Wrong Divide&Conquer Algorithm for MST [Problem: 10.21]

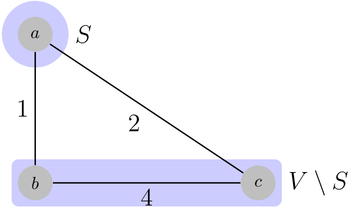
$$(V_1, V_2) : \Big| |V_1| - |V_2| \Big| \leq 1$$

$T_1 + T_2 + \{e\} : e$ is a lightest edge across (V_1, V_2)

Wrong Divide&Conquer Algorithm for MST [Problem: 10.21]

$$(V_1, V_2) : \Big| |V_1| - |V_2| \Big| \le 1$$

$$T_1 + T_2 + \{e\} : e \text{ is a lightest edge across } (V_1, V_2)$$

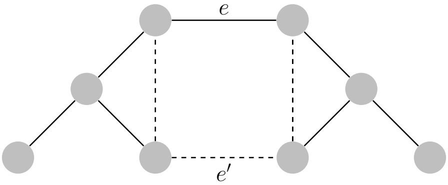


Cycle Property

Cycle Property [Problem: $10.19(b)$]

- Let C be any cycle in G
- Let $e = (u, v)$ be *a* maximum-weight edge in C

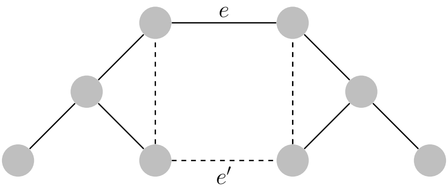
Then $\exists$ MST T of $G : e \notin T$.



Cycle Property [Problem: $10.19(b)$]

- Let C be any cycle in G
- Let $e = (u, v)$ be *a* maximum-weight edge in C

Then $\exists$ MST T of $G : e \notin T$.

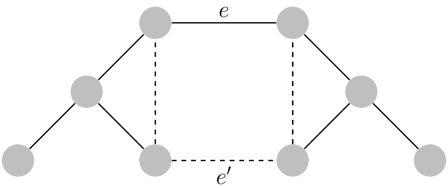


$$T' = \underbrace{T - \{e\}}_{\text{if } e \in T} + \{e'\}$$

Cycle Property [Problem: $10.19(b)$]

- Let C be any cycle in G
- Let $e = (u, v)$ be *a* maximum-weight edge in C

Then $\exists$ MST T of $G : e \notin T$.



$$T' = \underbrace{T - \{e\}}_{\text{if } e \in T} + \{e'\}$$

"a" $\to$ "the" $\implies$ "$\exists$" $\to$ "$\forall$"

Anti-Kruskal algorithm [Problem: $10.19\ (c)$]

Reverse-delete algorithm (wiki; clickable)

Anti-Kruskal algorithm [Problem: $10.19\ (c)$]

Reverse-delete algorithm (wiki; clickable)

$$O\Big(m \log n\ (\log \log n)^3\Big)$$

Anti-Kruskal algorithm [Problem: $10.19\ (c)$]

Reverse-delete algorithm (wiki; clickable)

$$O\Big(m \log n \ (\log \log n)^3\Big)$$

Proof.

Cycle Property

Anti-Kruskal algorithm [Problem: $10.19\ (c)$]

Reverse-delete algorithm (wiki; clickable)

$$O\Big(m \log n \, (\log\log n)^3\Big)$$

Proof.

Cycle Property

$$T \subseteq F \implies \exists\, T' : T' \subseteq F - \{e\}$$

□

Anti-Kruskal algorithm [Problem: $10.19\ (c)$]

Reverse-delete algorithm (wiki; clickable)

$$O\Big(m \log n \; (\log \log n)^3\Big)$$

Proof.

Cycle Property

$$T \subseteq F \implies \exists\, T' : T' \subseteq F - \{e\}$$

□

"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.

Application of Cycle Property [Problem: 10.15 (1)]

$$G = (V, E), \quad |E| > |V| - 1$$

e : the unique maximum-weighted edge of G

$$\overset{?}{\Longrightarrow}$$

$e \notin$ any MST

Application of Cycle Property [Problem: 10.15 (1)]

$$G = (V, E), \quad |E| > |V| - 1$$

e : the unique maximum-weighted edge of G

$$\overset{?}{\Longrightarrow}$$

$e \notin$ any MST

Bridge

Application of Cycle Property [Problem: $10.15\ (2)$]

$$C \subseteq G, \quad e \in C$$

$e:$ the unique maximum-weighted edge of G

$$\Longrightarrow$$

$e \notin$ any MST

Application of Cycle Property [Problem: 10.15 (2)]

$$C \subseteq G, \quad e \in C$$

$e:$ the unique maximum-weighted edge of G

$$\Longrightarrow$$

$e \notin$ any MST

Cycle Property

Application of Cycle Property [Problem: 10.15 (5)]

$$C \subseteq G, \quad e \in C$$

e : the unique lightest edge of C

$$\overset{?}{\Longrightarrow}$$

$e \in \forall$ MST

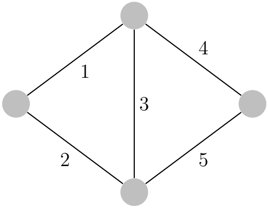
Application of Cycle Property [Problem: 10.15 (5)]

$$C \subseteq G, \quad e \in C$$

e : the unique lightest edge of C

$$\overset{?}{\Longrightarrow}$$

$e \in \forall$ MST



Uniqueness of MST

Uniqueness of MST [Problem: 10.18 (1)]

Distinct weights $\implies$ Unique MST.

Uniqueness of MST [Problem: 10.18 (1)]

Distinct weights $\implies$ Unique MST.

By Contradiction.

Uniqueness of MST [Problem: 10.18 (1)]

Distinct weights $\implies$ Unique MST.

By Contradiction.

$\exists$ MSTs $T_1 \neq T_2$

Uniqueness of MST [Problem: 10.18 (1)]

Distinct weights $\implies$ Unique MST.

By Contradiction.

$\exists$ MSTs $T_1 \neq T_2$

$$\Delta E = \{e \mid e \in T_1 \setminus T_2 \vee e \in T_2 \setminus T_1\}$$

Uniqueness of MST [Problem: 10.18 (1)]

Distinct weights $\implies$ Unique MST.

By Contradiction.

$$\exists \text{ MSTs } T_1 \neq T_2$$

$$\Delta E = \{e \mid e \in T_1 \setminus T_2 \lor e \in T_2 \setminus T_1\}$$

$$e = \min \Delta E$$

Uniqueness of MST [Problem: 10.18 (1)]

Distinct weights $\implies$ Unique MST.

By Contradiction.

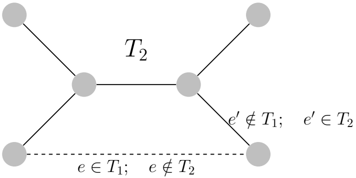
$$\exists \text{ MSTs } T_1 \neq T_2$$

$$\Delta E = \{e \mid e \in T_1 \setminus T_2 \lor e \in T_2 \setminus T_1\}$$

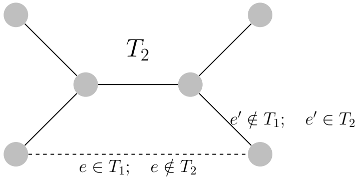
$$e = \min \Delta E$$

$$e \in T_1 \setminus T_2 \ \textit{(w.l.o.g)}$$

$$e \in T_1 \setminus T_2$$

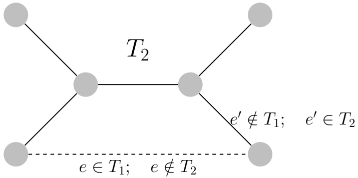


$$e \in T_1 \setminus T_2$$



$$T_2 + \{e\} \implies C$$

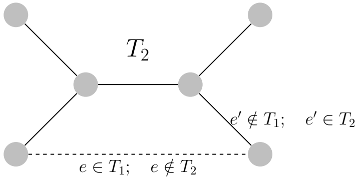
$$e \in T_1 \setminus T_2$$



$$T_2 + \{e\} \implies C$$

$\exists (e' \in C) \notin T_1$

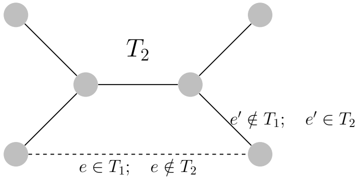
$$e \in T_1 \setminus T_2$$



$$T_2 + \{e\} \implies C$$

$$\exists (e' \in C) \notin T_1 \implies e' \in T_2 \setminus T_1 \implies e' \in \Delta E$$

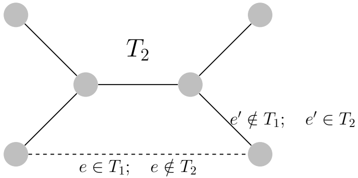
$$e \in T_1 \setminus T_2$$



$$T_2 + \{e\} \implies C$$

$$\exists (e' \in C) \notin T_1 \implies e' \in T_2 \setminus T_1 \implies e' \in \Delta E \implies w(e') > w(e)$$

$$e \in T_1 \setminus T_2$$



$$T_2 + \{e\} \implies C$$

$$\exists (e' \in C) \notin T_1 \implies e' \in T_2 \setminus T_1 \implies e' \in \Delta E \implies w(e') > w(e)$$

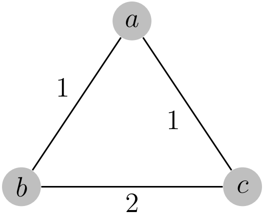
$$T' = T_2 + \{e\} - \{e'\} \implies w(T') < w(T_2)$$

Condition for Uniqueness of MST [Problem: 10.18 (2)]

$$\text{Unique MST} \;\not\Rightarrow\; \text{Equal weights.}$$

Condition for Uniqueness of MST [Problem: 10.18 (2)]

Unique MST $\not\Rightarrow$ Equal weights.

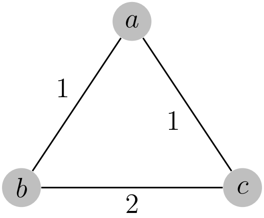


Unique MST [Problem: 10.21 (3)]

Unique MST $\not\Longrightarrow$ Minimum-weight edge across any cut is unique.

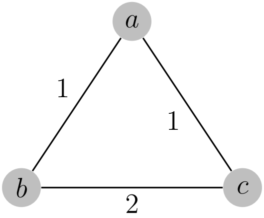
Unique MST [Problem: 10.21 (3)]

Unique MST $\not\Rightarrow$ Minimum-weight edge across any cut is unique.



Unique MST [Problem: 10.21 (3)]

Unique MST $\not\Rightarrow$ Minimum-weight edge across any cut is unique.



Theorem (After-class Exercise)

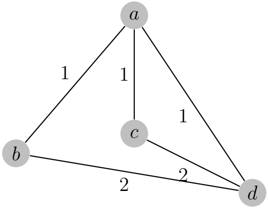
Minimum-weight edge across any cut is unique $\Longrightarrow$ *Unique MST.*

Unique MST [Problem: $10.21\ (3)$]

Unique MST $\not\Longrightarrow$ Maximum-weight edge in any cycle is unique.

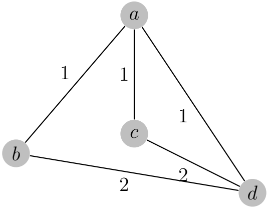
Unique MST [Problem: 10.21 (3)]

Unique MST $\not\Longrightarrow$ Maximum-weight edge in any cycle is unique.



Unique MST [Problem: 10.21 (3)]

Unique MST $\not\Longrightarrow$ Maximum-weight edge in any cycle is unique.



Theorem (After-class Exercise)

Maximum-weight edge in any cycle is unique $\Longrightarrow$ *Unique MST.*

Unique MST [Problem: 10.21 (4)]

To decide whether a graph has a unique MST.

Unique MST [Problem: 10.21 (4)]

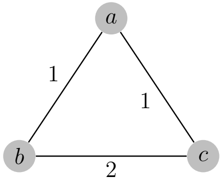
To decide whether a graph has a unique MST.

Ties in Prim's and Kruskal's algorithms

Unique MST [Problem: 10.21 (4)]

To decide whether a graph has a unique MST.

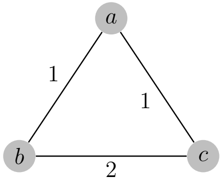
Ties in Prim's and Kruskal's algorithms



Unique MST [Problem: 10.21 (4)]

To decide whether a graph has a unique MST.

Ties in Prim's and Kruskal's algorithms



$$\underbrace{T}_{\text{Any MST}} + \underbrace{\{e\}, \forall e \notin T}_{\text{Cycle}}$$

Variants of MST

Adding a Vertex v to MST T [Problem: 10.7]

$$G' = (V', E') : V' = V + \{v\}, E' = E + E_v$$

To find an MST T' of G'.

Adding a Vertex v to MST T [Problem: 10.7]

$$G' = (V', E') : V' = V + \{v\}, E' = E + E_v$$

To find an MST T' of G'.

$$O\Big((m+n)\log n\Big) \quad \text{(recompute on } G'\text{)}$$

Adding a Vertex v to MST T [Problem: 10.7]

$$G' = (V', E') : V' = V + \{v\}, E' = E + E_v$$

To find an MST T' of G'.

$$O\Big((m+n)\log n\Big) \quad \text{(recompute on } G'\text{)}$$

Theorem

There exists an MST of G' that includes no edges in $G \setminus T$.

Adding a Vertex v to MST T [Problem: 10.7]

$$G' = (V', E') : V' = V + \{v\}, E' = E + E_v$$

To find an MST T' of G'.

$$O\Big((m+n)\log n\Big) \quad \text{(recompute on } G'\text{)}$$

Theorem

There exists an MST of G' that includes no edges in $G \setminus T$.

$$O(n \log n) \quad \text{(recompute on } G'' = (V + \{v\}, T + E_v)\text{)}$$

Adding a Vertex v to MST T [Problem: 10.7]

$$G' = (V', E') : V' = V + \{v\}, E' = E + E_v$$

To find an MST T' of G'.

$$O\Big((m+n)\log n\Big) \quad \text{(recompute on } G'\text{)}$$

Theorem

There exists an MST of G' that includes no edges in $G \setminus T$.

$$O(n \log n) \quad \text{(recompute on } G'' = (V + \{v\}, T + E_v)\text{)}$$

$$O(n)$$

Adding a Vertex v to MST T [Problem: 10.7]

$$G' = (V', E') : V' = V + \{v\}, E' = E + E_v$$

To find an MST T' of G'.

$$O\Big((m+n)\log n\Big) \quad \text{(recompute on } G'\text{)}$$

Theorem

There exists an MST of G' that includes no edges in $G \setminus T$.

$$O(n \log n) \quad \text{(recompute on } G'' = (V + \{v\}, T + E_v)\text{)}$$

$$O(n)$$

"On Finding and Updating Spanning Tress and Shortest Paths", 1975

"Algorithms for Updating Minimum Spanning Trees", 1978

Feedback Edge Set (FES): [Problem: 10.8]

$$\text{FES} \subseteq E : G' = (V, E \setminus \text{FES}) \text{ is acyclic}$$

To find a minimum FES.

Feedback Edge Set (FES): [Problem: 10.8]

$$\text{FES} \subseteq E : G' = (V, E \setminus \text{FES}) \text{ is acyclic}$$

To find a minimum FES.

$$G \text{ is connected} \implies G' \text{ is connected}$$

$$G' \text{ is connected} + \text{acyclic} \implies G' \text{ is an ST}$$

Feedback Edge Set (FES): [Problem: 10.8]

$$\text{FES} \subseteq E : G' = (V, E \setminus \text{FES}) \text{ is acyclic}$$

To find a minimum FES.

$$G \text{ is connected} \implies G' \text{ is connected}$$

$$G' \text{ is connected} + \text{acyclic} \implies G' \text{ is an ST}$$

$$\text{FES} \iff G \setminus \text{Max-ST}$$

MST with Specified Leaves: [Problem: 10.11]

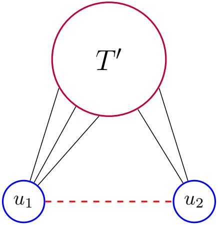
$$G = (V, E), \quad U \subset V$$

To find an MST with U as leaves.

MST with Specified Leaves: [Problem: 10.11]

$$G = (V, E), \quad U \subset V$$

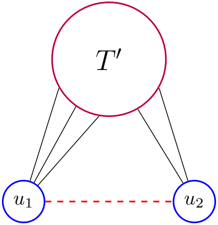
To find an MST with U as leaves.



MST with Specified Leaves: [Problem: 10.11]

$$G = (V, E), \quad U \subset V$$

To find an MST with U as leaves.

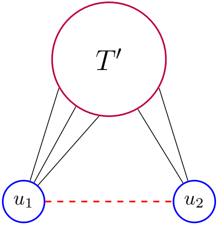


MST T' of $G' = G \setminus U$

MST with Specified Leaves: [Problem: 10.11]

$$G = (V, E), \quad U \subset V$$

To find an MST with U as leaves.



MST T' of $G' = G \setminus U$

Attach $\forall u \in U$ to T' (with lightest edge)

MST with Specified Edges: [Problem: 10.13]

$$G = (V, E), \quad S \subset E \text{ (no cycle in } S)$$

To find an MST with S as edges.

MST with Specified Edges: [Problem: 10.13]

$$G = (V, E), \quad S \subset E \text{ (no cycle in } S\text{)}$$

To find an MST with S as edges.

$G \to G'$: contract each component of S to a vertex

MST with Specified Edges: [Problem: 10.13]

$$G = (V, E), \quad S \subset E \text{ (no cycle in } S)$$

To find an MST with S as edges.

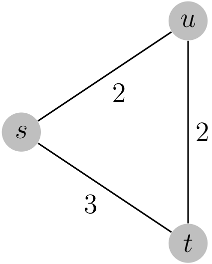
$G \to G'$: contract each component of S to a vertex

Compute MST on G'

MST *v.s.* Shortest Path

MST vs. Shortest Paths [Problem: 10.15 (6)]

✗ The shortest path between s and t is necessarily part of some MST.



Sharing Edges [Problem: 10.9]

$$G = (V, E, w), \quad w(e) > 0, \quad s \in V$$

All sssp trees from s must share some edge with all (some) MSTs of G.

Sharing Edges [Problem: 10.9]

$$G = (V, E, w), \quad w(e) > 0, \quad s \in V$$

All sssp trees from s must share some edge with all (some) MSTs of G.

$E' \subseteq E :$ lightest edges leaving s

Sharing Edges [Problem: 10.9]

$$G = (V, E, w), \quad w(e) > 0, \quad s \in V$$

All sssp trees from s must share some edge with all (some) MSTs of G.

$$E' \subseteq E : \text{ lightest edges leaving } s$$

$$E' \subseteq \forall \text{ sssp tree from } s$$

Sharing Edges [Problem: 10.9]

$$G = (V, E, w), \quad w(e) > 0, \quad s \in V$$

All sssp trees from s must share some edge with all (some) MSTs of G.

$$E' \subseteq E : \text{ lightest edges leaving } s$$

$$E' \subseteq \forall \text{ sssp tree from } s$$

$$\forall \text{ MST } T \text{ of } G : T \cap E' \neq \emptyset$$


THANK YOU