Dynamic Programming

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June 11 - June 19, 2017



Dynamic Programming

- Overview
- (2) 1D DF
- 3 2D DF
- 4 3D DF
- 5 DP on Graphs
- 6 The Knapsack Problem
- Summary

What is DP?

 $\mathsf{DP} \approx \mathsf{"smarter brute force"}$

 $\mathsf{DP} \approx \mathsf{"smart} \mathsf{ scheduling} \mathsf{ of} \mathsf{ subproblems"}$

 $\mathsf{DP} \approx \text{``shortest/longest paths in some DAG''}$

What is not DP?

 ${\sf Programming} = {\sf Planning}$

Programming \neq Coding (Richard Bellman, 1940s)

Steps for applying DP

- 1. Define subproblems
 - # of subproblems
- 2. Set the goal
- 3. Define the recurrence
 - ▶ larger subproblem ← # smaller subproblems
 - init. conditions
- 4. Write pseudo-code: fill "table" in topo. order
- 5. Analyze the time complexity
- 6. Extract the optimal sulution

1D subproblems:

```
Input: x_1, x_2, \ldots, x_n (array, sequence, string)

Subproblems: x_1, x_2, \ldots, x_i (prefix/suffix)

#: \Theta(n)
```

Examples: Maximum-sum subarray, Longest increasing subsequence, Text justification (LATEX)

2D subproblems:

```
1. Input: x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n
Subproblems: x_1, x_2, \ldots, x_i; y_1, y_2, \ldots, y_j
#: \Theta(mn)
```

Examples: Edit distance, Longest common subsequence

```
2. Input: x_1, x_2, \ldots, x_n
Subproblems: x_i, \ldots, x_j
\#: \Theta(n^2)
```

Examples: Matrix chain multiplication, Optimal BST

3D subproblems:

► Floyd-Warshall algorithm

$$\mathsf{d}(i,j,k) = \min\{\mathsf{d}(i,j,k-1),\mathsf{d}(i,k,k-1) + \mathsf{d}(k,j,k-1)\}$$

DP on graphs:

1. On rooted tree

Subproblems: rooted subtrees

2. On DAG

Subproblems: nodes after/before in the topo. order

Knapsack problem:

▶ Subset sum problem, change-making problem

And Others . . .

Recurrences in DP

Make choices by asking yourself the right question:

- 1. Binary choice
 - whether . . .
- 2. Multi-way choices
 - ▶ where to ...
 - which one . . .

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$$f^{(S(n))} = 1$$

$$f^{(S(n))} = 1 \text{ (Problem 7.2)}$$

$$f(n) = \begin{cases} n-1 & \text{if } n \in \mathbb{Z}^+ \\ n/2 & \text{if } n\%2 = 0 \\ n/3 & \text{if } n\%3 = 0 \end{cases}$$

S(n): minimum number of steps taking n to 1.

S(i): minimum number of steps taking i to 1

$$S(i) = 1 + \min\{S(i-1), S(i/2) (\text{if } n\%2 = 0), S(i/3) (\text{if } n\%3 = 0)\}$$

$$S(1) = 0$$

$$f^{(S(n))} = 1$$

Collatz (3n+1) conjecture:

$$f(n) = \begin{cases} n/2 & \text{if } n\%2 = 0\\ 3n+1 & \text{if } n\%2 = 1 \end{cases}$$
$$f^*(n) = 1?$$

"Mathematics may not be ready for such problems."

— Paul Erdős

Longest increasing subsequence

Longest increasing subsequence (Problem 7.3)

- Given an integer array $A[1 \dots n]$
- ► To find (the length of) a longest increasing subsequence.

$$5, 2, 8, 6, 3, 6, 9, 7 \implies 2, 3, 6, 9$$

Longest increasing subsequence

Subproblem: L(i): the length of the LIS of A[1...i]

Goal: L(n)

Make choice: whether $A[i] \in LIS[1 \dots i]$?

Recurrence:

$$L(i) = \max\{L(i-1), 1 + \max_{j < i \land A[j] \le A[i]} L(j)\}$$

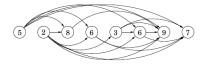
Init:

$$L(0) = 0$$

Time:

$$O(n^2) = \Theta(n) \cdot O(n)$$

Longest increasing subsequence



Longest path distance in the DAG!

Maximum-sum subarray (Google Interview)

- ightharpoonup Array $A[1\cdots n], a_i>=<0$
- lacktriangle To find (the sum of) a maximum-sum subarray of A

$$A[-2, 1, -3, 4, -1, 2, 1, -5, 4] \implies [4, -1, 2, 1]$$

Subproblem: MSS[i]: sum of an MS[i] of $A[1 \cdots i]$

Goal: mss = MSS[n]

Make choice: Is $a_i \in MS[i]$?

Recurrence:

$$\mathsf{MSS}[i] = \max\{\mathsf{MSS}[i-1], ???\}$$

Subproblem: MSS[i]: sum of an MS[i] ending with a_i

Goal: $mss = \max_{1 \le i \le n} MSS[i]$

Make choice: Where does the MS[i] start?

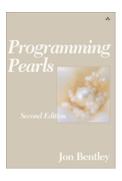
Recurrence:

$$MSS[i] = \max\{MSS[i-1] + a_i, a_i\}$$
 (Proof!)

Init:

$$\mathsf{MSS}[0] = 0$$

Time: $\Theta(n)$



Ulf Grenander $O(n^3) \Longrightarrow O(n^2)$ Michael Shamos $O(n\log n)$, onenight Jon Bentley Conjecture: $\Omega(n\log n)$ Michael Shamos Carnegie Mellon seminar Jay Kadane O(n), ≤ 1 minute

```
\begin{aligned} & \mathsf{MSS}[0] \leftarrow 0 \\ & \mathbf{for \ all} \ i \leftarrow 1 \dots n \ \mathbf{do} \\ & \mathsf{MSS}[i] \leftarrow \max\{\mathsf{MSS}[i-1] + a_i, 0\} \\ & \mathbf{return} \ \max_{i=1\dots n} \mathsf{MSS}[i] \end{aligned}
```

```
\begin{aligned} & \mathsf{mss} \leftarrow 0 \\ & \mathsf{MSS} \leftarrow 0 \\ & \mathbf{for\ all}\ i \leftarrow 1 \dots n\ \mathbf{do} \\ & & \mathsf{MSS} \leftarrow \max\{\mathsf{MSS} + a_i, 0\} \\ & & & \mathsf{mss} \leftarrow \max\{\mathsf{mss}, \mathsf{MSS}\} \end{aligned}
```

return mss

Maximum-product subarray

Maximum-product subarray (Problem 7.4)

- ▶ Array $A[1 \dots n]$
- ightharpoonup Find maximum-product subarray of A
- (1) $a_i \in \mathbb{N}$
- (2) $a_i \in \mathbb{Z}$
- (3) $a_i \in \mathbb{R}$

sum vs. product

Maximum-product subarray

 ${\sf Subproblem:}\ {\sf MaxP}[i], {\sf MinP}[i]$

		$\frac{1}{2}$	4	-2	5	$-\frac{1}{5}$	8
MaxP[i]	1	$\frac{1}{2}$	4	-2	5	8	64
MinP[i]	1	$\frac{1}{2}$	2	-8	-40	-1	-8

$$\begin{split} \mathsf{MaxP}[i] &= \max\{\mathsf{MaxP}[i-1] \cdot a_i, \mathsf{MinP}[i-1] \cdot a_i, a_i\} \\ \mathsf{MinP}[i] &= \min\{\mathsf{MaxP}[i-1] \cdot a_i, \mathsf{MinP}[i-1] \cdot a_i, a_i\} \end{split}$$

Reconstructing string

Reconstructing string (Problem 7.9)

- ▶ String $S[1 \cdots n]$
- ▶ Dict for lookup:

$$dict(w) = \begin{cases} \text{ true } & \text{if } w \text{ is a valid word} \\ \text{false } & \text{o.w.} \end{cases}$$

▶ Is $S[1 \cdots n]$ valid (reconstructed as a sequence of valid words)?

Reconstructing string

Subproblem: V[i]: Is $S[1 \cdots i]$ valid?

Goal: V[n]

Make choice: Where does the last word start?

Recurrence:

$$V[i] = \bigvee_{j=1...i} (V[j-1] \wedge \operatorname{dict}(S[j\cdots i]))$$

Init:

$$V[0] = \mathsf{true}$$

Time:

$$O(n^2) = \Theta(n) \cdot O(n)$$

Hotel along a trip

Hotel along a trip (Problem 7.15)

- ▶ Hotel sequence (distance): $a_0 = 0, a_1, \dots, a_n$
- $ightharpoonup a_0 \leadsto a_n$
- Stop at only hotels
- ► Cost: $(200 x)^2$
- ► To minimize overall cost

Hotel along a trip

Subproblem: C[i]: minimum cost when the destination is a_i

Goal: C[n]

Make choice: What is the last but one hotel a_i to stop?

Recurrence:

$$C[i] = \min_{0 \le j < i} \{ C[j] + (200 - (a_i - a_j))^2 \}$$

Init:

$$C[0] = 0$$

Time:

$$O(n^2) = \Theta(n) \cdot O(n)$$

Highway restaurants

Highway restaurants (Problem 7.16)

- ▶ Locations: $L[1 \dots n]$
- ▶ Profits: $P[1 \dots n]$
- ▶ Any two hotels should be $\geq k$ miles apart
- ► To maximize the total profit

```
Subproblem: T[i]: max profit achievable using only L[1 \dots i]
```

Goal: T[n]

Make choice: Whether to open a restaurant at L_i ?

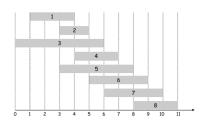
Recurrence:

$$T[i] = \max\{T[i-1], P_i + T[\mathsf{prev}(i)]\}$$

$$\mathsf{prev}(i) = \max\{j \mid j < i \land L_i - L_j \ge k\}$$

Weighted interval/class scheduling (Problem 7.14)

- ► Classes: $C = \{c_1, c_2, \cdots, c_n\}$ $c_i \triangleq \langle g_i, s_i, f_i \rangle$
- ► Choosing non-conflicting classes to maximize your grades



sort $\mathcal C$ by finishing time.

Greedy algorithms fail:

2

1

By finishing time.

6

1

1

1

By weights.

Subproblem: G[i]: the maximal grades obtained from $\{c_1, c_2, \cdots, c_i\}$

Goal: G[n]

Make choice: Choose c_i or not?

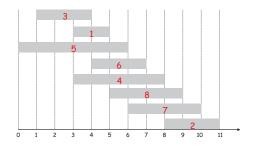
Recurrence:

$$G[i] = \max\{G[i-1], G[\mathsf{prev}(i)] + g_i\}$$
$$\mathsf{prev}(i) = \max\{j \mid j < i \land c_i \cap c_j = \emptyset\}$$

Init: G[0] = 0

Time: $O(n \log n) + T(p(i)) + O(n) \cdot O(1)$

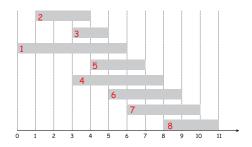
Why is ordering necessary?



$$G[7] = \max\{G[6], G[\{1, 3, 5\}] + g_7\}$$

subproblems changed: all $O(2^n)$ subsets

What about sorting by starting time?



$$G[6] = \max\{G[5], G[\{2,3\}] + g_6\}$$

subproblems changed: all $O(2^n)$ subsets

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Longest common subsequence

LCS: longest common subsequence (Problem 7.5)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

(1) Find (the length of) an LCS of X and Y

$$X = \langle A, B, C, B, D, A, B \rangle$$
$$Y = \langle B, D, C, A, B, A \rangle$$
$$Z = \langle B, C, B, A \rangle$$

Longest common subsequence

Subproblem: L[i,j]: the length of an LCS of $X[1\cdots i]$ and $Y[1\cdots j]$

Goal: L[m,n]

Make choice: Is $X_i = Y_i$?

Recurrence: (Proof!)

$$L[i,j] = \begin{cases} L[i-1,j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1,j], L[i,j-1]\} & \text{if } X_i \neq Y_j \end{cases}$$

Init:

$$L[0, j] = 0, \ 0 \le j \le n$$

 $L[i, 0] = 0, \ 0 \le i \le m$

Time: $\Theta(mn)$

Longest common subsequence

Longest common subsequence (Problem 7.5)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

- (2) Allowing repetition of X
- (3) Allowing repetition $\leq k$ of X

$$L[i,j] = \begin{cases} L[i,j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1,j], L[i,j-1]\} & \text{if } X_i \neq Y_j \end{cases}$$
$$X \implies X^{(k)} \triangleq X_1^{(k)} \cdots X_m^{(k)}$$

Longest common substring

What about longest common substring?

Shortest common supersequence

Shortest common supersequence (Problem 7.6)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

lacktriangle Find (the length of) a shortest common subsequence of X and Y

Shortest common supersequence

Subproblem: L[i,j]: the length of an SCS of $X[1\cdots i]$ and $Y[1\cdots j]$

Goal: L[m, n]

Make choice: Is $X_i = Y_i$?

Recurrence:

$$L[i,j] = \begin{cases} L[i-1,j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1,j] + 1, L[i,j-1] + 1\} & \text{if } X_i \neq Y_j \end{cases}$$

Init:

$$L[0, j] = j, \ 0 \le j \le n$$

 $L[i, 0] = i, \ 0 \le i \le m$

Remark

$$\max(m, n) < L(m, n) < m + n$$

Shortest common supersequence

$$X[1 \dots m] \qquad Y[1 \dots n]$$

$$r = |\mathsf{LCS}(X,Y)|$$

$$t = |\mathsf{SCS}(X,Y)|$$

$$X \cap Y = \emptyset \implies r = 0 \quad t = m+n$$

$$X \prec Y \implies r = m \quad t = n$$

r+t=m+n

String shuffling (Problem 7.7)

- ▶ Strings X[1 ...m], Y[1 ...n], Z[1 ...k]
- ▶ Is $X \oplus Y = Z$?
- (1) Reduced to LCS:

$$(Z' \triangleq Z \setminus \mathsf{LCS}(X, Z)) = Y$$

- (2) O(mn)
- (3) Minimum # deleted characters to ensure $X \oplus Y = Z$ in O(mnk)

$$(Z' \triangleq Z \setminus \mathsf{LCS}(X, Z)) = Y$$

"YES"
$$\Longrightarrow X \oplus Y = Z$$

$$\text{``NO"} \implies X \oplus Y \neq Z$$

$$X = AC$$

$$Y = CB$$

$$Z = CBA$$

$$X = A$$

$$Y = AB$$

$$Z = ABA$$

Subproblem: S[i,j]: Is $X[1 \dots i] \oplus Y[1 \dots j] = Z[1 \dots r \triangleq i+j]$?

Goal: S[m,n]

Make choice: Is $Z[r] = X[i] \lor Z[r] = Y[j]$?

Recurrence:

$$S[i,j] = (Z[r] = X[i] \land S[i-1,j]) \lor$$

$$(Z[r] = Y[j] \land S[i,j-1])$$

Init:

$$S[0, j] = (Z = Y), \ \forall 0 \le j \le n$$

 $S[i, 0] = (Z = X), \ \forall 0 \le i \le m$

Time: O(mn)

Subproblem: D[i,j,r]: minimum # deleted characters to ensure

$$X[1 \dots i] \oplus Y[1 \dots j] = Z[1 \dots r]$$

Goal: D[m, n, k]

Make choice: Is $Z[r] = X[i] \vee Z[r] = Y[j]$?

Recurrence:

$$D[i,j,r] = \min \left\{ \begin{array}{ll} D[i-1,j,r-1] & \text{if } Z[r] = X[i] \\ D[i,j-1,r-1] & \text{if } Z[r] = Y[j] \\ 1 + D[i-1,j,r] \\ 1 + D[i,j-1,r] \\ 1 + D[i,j,r-1] \end{array} \right.$$

Init:

$$D[0, j, r] = j + r - 2|\mathsf{LCS}(Y[1 \dots j], Z[1 \dots r])|$$

 $D[i, 0, r] = i + r - 2|\mathsf{LCS}(X[1 \dots i], Z[1 \dots r])|$
 $D[i, j, 0] = i + j$

Longest contiguous substring both forward and backward

Longest contiguous substring both forward and backward (Problem 7.8)

- ▶ String $T[1 \cdots n]$
- ► Find a longest contiguous substring (LCS) both forward and backward

dynamicprogrammingmanytimes

- lacksquare Subproblem L[i]: the length of an LCS in $T[1\cdots i]$
- ▶ Subproblem L[i,j]: the length of an LCS in $T[i\cdots j]$

Longest contiguous substring both forward and backward

Subproblem: L[i,j]: the length of an LCS starting with T_i and ending with T_j

Goal: $\max_{1 \le i \le j \le n} L[i, j]$

Make choice: Is $T_i = T_j$?

Recurrence:

$$L[i,j] = \begin{cases} 0 & \text{if } T_i \neq T_j \\ L[i+1,j-1] + 1 & \text{if } T_i = T_j \end{cases}$$

Init:

$$\begin{split} L[i,i] &= 0, \ 0 \leq i \leq n \\ L[i,i+1] &= \left\{ \begin{array}{ll} 1 & \text{if } T_i = T_{i+1} \\ 0 & \text{if } T_i \neq T_{i+1} \end{array} \right. \end{split}$$

Longest contiguous substring both forward and backward

Code: three ways of filling the table







```
\begin{array}{l} \text{for all } d \leftarrow 2 \dots n-1 \text{ do} \\ \text{for all } i \leftarrow 1 \dots n-d \text{ do} \\ j \leftarrow i+d \\ \dots \\ \text{return } \max_{1 \leq i \leq j \leq n} L[i,j] \end{array}
```

Longest palindrome subsequence

Longest palindrome subsequence (Problem 7.10)

(1) Find (the length of) a longest palindrome subsequence of $S[1\cdots n]$

Subproblem: L[i,j]: the length of an LSP of $S[i\cdots j]$

Goal: L[1,n]

Make choice: Is S[i] = S[j]?

Recurrence:

$$L[i,j] = \begin{cases} L[i+1,j-1] + 2 & \text{if } S[i] = S[j] \\ \max\{L[i+1,j], L[i,j-1]\} & \text{if } S[i] \neq S[j] \end{cases}$$

Init:

$$L[i, i] = 1, \ \forall 1 \le i \le n$$

 $L[i, i+1] = 2, \ \text{if} \ S[i] = S[i+1], \ \forall 1 \le i \le n-1$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1\dots n]$ into minimum number of palindromes (# cuts)

Subproblem: C[i,j]: minimum number of cuts for string $S[i \dots j]$

Goal: C[1, n] + 1

Make choice: Where is the first cut?

Recurrence:

$$C[i,j] = \left\{ \begin{array}{l} 0 \ \ \text{if} \ S[i \ldots j] \ \text{is a palindrome} \\ \min_{i+1 \leq k \leq j-1} C[i,k-1] + 1 + C[k,j] \quad \text{ o.w.} \end{array} \right.$$

Init: C[i, i] = 0

Time: $O(n^3)$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes

Subproblem: P[i]: minimum number of palindromes for $S[1\cdots i]$

Goal: P[n]

Make choice: Where does the last palindrome start from?

Recurrence:

$$P[i] = \min_{\substack{1 \leq k \leq i \\ S[k...i] \text{ is a palindrome}}} P[k-1] + 1$$

Init: P[0] = 1Time: $O(n^2)$

String splitting

String splitting (Problem 7.11)

- ightharpoonup Split a string S into many pieces
- $ightharpoonup Cost |S| = n \implies n$
- ▶ Given locations of m cuts: $C_0, C_1, \cdots, C_m, C_{m+1}$
- lacktriangle Find the minimum cost of splitting S into m+1 pieces $S_0\cdots S_m$

String splitting

Subproblem: C[i,j]: the minimum cost of splitting substring $S_i \cdots S_{j-1}$ using cuts $C_{i+1} \cdots C_{j-1}$

Goal: C[0, m+1]

Make choice: What is the first cut in $C_{i+1} \cdots C_{i-1}$?

Recurrence:

$$C[i,j] = \min_{i < k < j} (C[i,k] + C[k,j] + l(S_i \cdots S_{j-1}))$$

Init: C[i, i+1] = 0

Subproblem:

Goal:

Make choice:

Recurrence:

Init:

Time:

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Floyd-Warshall algorithm

(1) DP for Floyd-Warshall algorithm for APSP on directed graphs

Subproblem: dist[i, j, k]: the length of the shortest path from i to j via only nodes in $v_1 \cdots v_k$

Goal: $dist[i, j, n], \forall i, j$

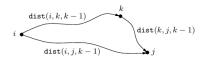
Floyd-Warshall algorithm

(1) DP for Floyd-Warshall algorithm for APSP on directed graphs

Make choice: Is v_k on the ShortestPath[i, j, k]?

Recurrence:

$$\mathsf{dist}[i,j,k] = \min\{\mathsf{dist}[i,j,k-1],\mathsf{dist}[i,k,k-1] + \mathsf{dist}[k,j,k-1]\}$$



Init:

$$\mathsf{dist}[i,j,0] = \left\{ \begin{array}{ll} 0 & i = j \\ w(i,j) & (i,j) \in E \\ \infty & \mathsf{o.w.} \end{array} \right.$$

Floyd-Warshall algorithm (Problem 6.25)

(2) Routing table for Floyd-Warshall algorithm

```
\begin{split} \text{for all } k \leftarrow 1 \dots n \text{ do} \\ \text{for all } i \leftarrow 1 \dots n \text{ do} \\ \text{for all } j \leftarrow 1 \dots n \text{ do} \\ \text{if } \operatorname{dist}[i,j] > \operatorname{dist}[i,k] + \operatorname{dist}[k,j] \text{ then} \\ \operatorname{dist}[i,j] \leftarrow \operatorname{dist}[i,k] + \operatorname{dist}[k,j] \\ \operatorname{Go}[i,j] \leftarrow \operatorname{Go}[i,k] \end{split}
```

Floyd-Warshall algorithm (Problem 6.25)

(2) Routing table

$$\begin{array}{l} \text{for all } i \leftarrow 1 \dots n \text{ do} \\ \text{ for all } j \leftarrow 1 \dots n \text{ do} \\ \text{ dist}[i,j] \leftarrow \infty \\ \text{ Go}[i,j] \leftarrow \text{Nil} \\ \\ \text{for all } (i,j) \in E \text{ do} \\ \text{ dist}[i,j] \leftarrow w(i,j) \\ \text{ Go}[i,j] \leftarrow j \\ \\ \text{for all } i \leftarrow 1 \dots n \text{ do} \\ \text{ dist}[i,i] \leftarrow 0 \\ \text{ Go}[i,i] \leftarrow \text{Nil} \\ \end{array}$$

```
\begin{array}{c} \textbf{procedure} \ \operatorname{Path}(i,j) \\ \textbf{if} \ \operatorname{Go}[i,j] = \operatorname{Nil} \ \textbf{then} \\ \operatorname{Output} \ \text{``No Path.''} \end{array}
```

```
Output "i" while i \neq j do i \leftarrow \operatorname{Go}[i,j] Output "i"
```

Floyd-Warshall algorithm (Problem 6.29)

(3) Find minimum-weighted cycle of directed graph (w(e) > 0)

$$\operatorname{dist}[i,i] \leftarrow 0 \implies \operatorname{dist}[i,i] \leftarrow \infty$$

$$\exists i : \mathsf{dist}[i,i] < 0$$

$$\forall i: \mathsf{dist}[i,i] \geq 0$$

Shortest paths on undirected graphs

Finding shortest paths in undirected graphs with possibly negative edge weights



The book "Algorithms" by Robert Sedgewick and Kevin Wayne hinted that (see the quote below) there are efficient algorithms for finding shortest paths in undirected graphs with possibly negative edge weights (not by treating an undirected edge as two directed one which means that a single negative edge implies a negative cycle). However, no references are given in the book. Are you aware of any such algorithms?



Q. How can we find shortest paths in undirected (edge-weighted) graphs?

A. For positive edge weights, Dijkstra's algorithm does the job. We just build an EdgeWeightedDigraph corresponding to the given EdgeWeightedGraph (by adding two directed edges corresponding to each undirected edge, one in each direction) and then run Dijkstra's algorithm. If edge weights can be negative (emphasis added), efficient algorithms are available, but they are more complicated than the Bellman-Ford algorithm.



https://cs.stackexchange.com/q/76578/4911

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- Summary

Minimum vertex cover on trees

Minimum vertex cover on trees [Problem: 2.2.18]

- ▶ Undirected tree T = (V, E); No designated root!
- ightharpoonup Compute (the size of) a minimum vertex cover of T



Minimum vertex cover on trees

Rooted T at any node r.

Subproblem: I(u): the size of an MVC of subtree T_u rooted at u

Goal: I(r)

Make choice: Is u in MVC[u]?

Recurrence:

$$I(u) = \min\{\# \text{ children of } u + \sum_{v: \text{ grandchildren of } u} I(v), \\ 1 + \sum_{v: \text{ children of } u} I(v)\}$$

Init: I(u) = 0, if u is a leave

Minimum vertex cover on trees

DFS on T from root r:

when u is "finished": if u is a leave then $I(u) \leftarrow 0$

else

$$I(u) \leftarrow \dots$$

Greedy algorithm:

Theorem

There is an MVC which contains no leaves.

DP on DAG

Longest path in DAG (Problem 7.17)

- ▶ Direction: \downarrow OR \rightarrow
- ► Score: >=< 0
- 1. digraph G
- 2. node weight \rightarrow edge weight
- 3. adding an extra sink s
- 4. $G \rightarrow G^T$

Compute a longest path from s in DAG

DP on DAG

Subproblem: dist[v]: longest distance from s to v

Goal: $\operatorname{dist}[v], \forall v \in V$

Make choice:

Recurrence:

$$\mathsf{dist}[v] = \max_{u \to v} \left(\mathsf{dist}[u] + w(u \to v) \right)$$

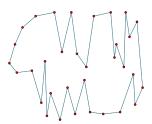
Init: dist[s] = 0

Compute dist[v] in topo. order

Bitonic tour

Bitonic tour (Problem 7.18)

- ▶ Points: $P[1...n], p_i = (x_i, y_i)$
- $x_1 < x_2 < \cdots < x_n$
- lacksquare Bitonic tour: $p_1 \leadsto^{x_i < x_{i+1}} p_n \leadsto^{x_i > x_{i+1}} p_1$
- ► Compute a shorest bitonic tour.



Bitonic tour

$$P_{i,j} (i \leq j)$$
: bitonic path $p_i \leadsto p_1 \leadsto p_j$ includes all p_1, p_2, \ldots, p_j

Subproblem: d[i,j]: the length of a shortest bitonic path $P_{i,j}$

Goal:
$$d[n,n] = d[n-1,n] + l(p_{n-1}p_n)$$

Make choice: Is p_{j-1} on the increasing path or the decreasing path?

Recurrence:

$$\begin{aligned} d[i,j] &= d[i,j-1] + l(p_{j-1}p_j) & \forall i < j-1 \\ d[i,j] &= \min_{1 \le k < j-1} \{ d[k,j-1] + l(p_kp_j) \} & \forall i = j-1 \end{aligned}$$

Init:
$$d[1,2] = l(p_1p_2)$$

Time:

$$O(n^2) = O(n \log n) + O(n^2) \cdot O(1) + O(n) \cdot O(n)$$

Dynamic Programming

- Overview
- (2) 1D DP
- 3 2D DF
- 4 3D DP
- 5 DP on Graphs
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The change-making problem (Problem 7.12)

- ightharpoonup Coins values: $x_1 \dots x_n$
- ► Amount: v
- \blacktriangleright Is it possible to make change for v?

The change-making problem (Problem 7.12(2), Problem 7.1 (Subset sum)) (2) Without repetition (0/1)

Subproblem: C[i, w]: Make change for w using only values of $x_1 \dots x_i$?

Goal: C[n,v]

Make choice: Using value x_i or not?

Recurrence:

$$C[i,w] = C[i-1,w] \lor (C[i-1,w-x_i] \land w \ge x_i)$$

Init:

$$\begin{split} C[i,0] &= \mathsf{true} \\ C[0,w] &= \mathsf{false}, \mathsf{if} \ w > 0 \\ C[0,0] &= \mathsf{true} \end{split}$$

Time: O(nv)

The change-making problem (Problem 7.12(1))

(1) Unbounded repetition (∞)

Subproblem: C[i, w]: Make change for w using only values of $x_1 \dots x_i$?

Goal: C[n,v]

Make choice: Using value x_i or not?

Recurrence:

$$C[i, w] = C[i - 1, w] \lor (C[i, w - x_i] \land w \ge x_i)$$

Init:

$$\begin{split} C[i,0] &= \mathsf{true} \\ C[0,w] &= \mathsf{false}, \mathsf{if} \ w > 0 \\ C[0,0] &= \mathsf{true} \end{split}$$

Time: O(nv)

The change-making problem (Problem 7.12(1))

(1) Unbounded repetition (∞)

Subproblem: C[w]: Possible to make change for w?

Goal: C[v]

Make choice: Suppose x_i is used.

Recurrence:

$$C[w] = \bigvee_{i: \ x_i \le w} C[w - x_i]$$

Time: O(nv)

 \mathbb{Q} : C[i,w] vs. C[w]

The change-making problem (Problem 7.12(3))

(3) Unbounded repetition with $\leq k$ coins

Subproblem: C[i, w, l]: Possible to make change for w with $\leq l$ coins of values of $x_1 \dots x_i$?

Goal: C[n, v, k]

Make choice: Using value x_i or not?

Recurrence:

$$C[i, w, l] = C[i-1, w, l] \lor (C[i, w - x_i, l-1] \land w \ge x_i)$$

Init:

$$\begin{split} C[0,0,l] &= \mathsf{true}, \quad C[0,w,l] = \mathsf{false}, \mathsf{if} \ w > 0 \\ C[i,0,l] &= \mathsf{true}, \quad C[i,w,0] = \mathsf{false}, \mathsf{if} \ w > 0 \end{split}$$

Dynamic Programming

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More DPs . . .

Algorithms that use dynamic programming [edit | edit source]



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