# Dynamic Programming

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# Dynamic Programming

# Dynamic Programming

Q: What is DP?

- A: Smart scheduling of subproblems.

Q: What does DP look like?

1. Define subproblems (types)
2. Set the goal: what is the solution to the original problem
3. Define recurrence: (ask the right questions $\Rightarrow$ reduce to subproblems)
    - larger problem $\Leftarrow$ a number of "smaller" subproblems
4. Write pseudo-code (fill the array/table/matrix in order)
5. Analyze time complexity
6. Extract optimal solutions

# Common subproblems

1. 1-D subproblems
   - input: $x_1, x_2, \cdots, x_n$ (array, sequence, string)
   - subproblems: $x_1, x_2, \cdots, x_i$ (prefix/postfix)
   - $\# = O(n)$
   - examples: weighted interval scheduling, max-sum subarray, breaking into lines
2. 2-D subproblems

   2.1 input: $x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n$
   - subproblems: $x_1, x_2, \cdots, x_i; y_1, y_2, \cdots, y_j$
   - $\# = O(mn)$
   - examples: edit distance, LCS

   2.2 input: $x_1, x_2, \cdots, x_n$
   - subproblems: $x_i, \cdots, x_j$
   - $\# = (n^2)$
   - examples: multiplying a sequence of matrices, optimal binary search tree

# Common subproblems

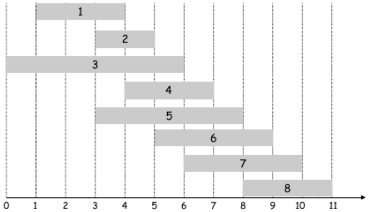
3. 3-D subproblems:
   - example: Floyd-Warshall algorithm, Bellman-Ford algorithm
4. DP on graphs (tree, DAG ...)
   - input: rooted tree
   - subproblems: rooted subtree
5. knapsack problem
   - example: changing coins
6. others ...

# Dynamic Programming

# 1-D DP

Weighted interval/class scheduling [Problem: 2.2.20]

- $\mathcal{C} = \{c_1, c_2, \cdots, c_n\}$
- $c_i$: $\langle g_i, s_i, f_i \rangle$
- choosing non-conflicting classes to maximize your grades



# 1-D DP

Solution.

- subproblem $G[i]$: the maximal grades obtained from $\{c_1, c_2, \cdots, c_i\}$
- goal: $G[n]$

# 1-D DP

Solution.

- question: choose $c_i$ or not in $G[i]$?
- recurrence:

$$G[i] = \max\{G[i-1], G[j] + g_i\}$$

  $c_j$: the last class which does not conflict with $c_i$

- initialization:

$$G[0] = 0$$

# 1-D DP

Solution.

- question: choose $c_i$ or not in $G[i]$?
- recurrence:

$$G[i] = \max\{G[i-1], G[j] + g_i\}$$

$c_j$: the last class which does not conflict with $c_i$

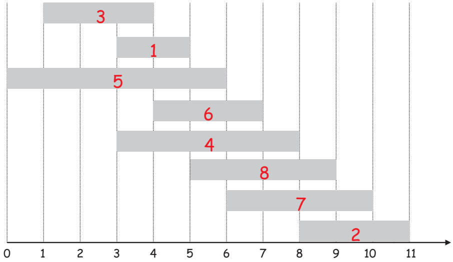
- initialization:

$$G[0] = 0$$

sort $\mathcal{C}$ by finishing time.

# 1-D DP

## Why is ordering necessary?

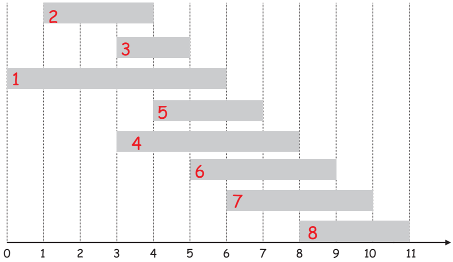



$$G[7] = \max\{G[6], G[\{1,3,5\}] + g_7\}$$

subproblems changed: all $O(2^n)$ subsets

# 1-D DP

## Sorting by starting time?




$$G[6] = \max\{G[5], G[\{2,3\}] + g_6\}$$

subproblems changed: all $O(2^n)$ subsets

# 1-D DP

Maximal sum subarray [Problem: 2.2.3, 2.2.13, Google Interview Problem]

- array $A[1 \cdots n], a_i >=< 0$
- to find (the sum of) an MS in $A$
  - special case: mss $= 0$ if all negative

$$A[-2, 1, -3, 4, -1, 2, 1, -5, 4] \Rightarrow [4, -1, 2, 1]$$

# 1-D DP

## Maximal sum subarray [Problem: 2.2.3, 2.2.13, Google Interview Problem]

- array $A[1 \cdots n], a_i >=< 0$
- to find (the sum of) an MS in $A$
  - special case: mss $= 0$ if all negative

$$A[-2, 1, -3, 4, -1, 2, 1, -5, 4] \Rightarrow [4, -1, 2, 1]$$

## Trial and error.

- try subproblem MSS$[i]$: the sum of the MS (MS[i]) in $A[1 \cdots i]$
- goal: mss $=$ MSS$[n]$
- question: Is $a_i \in$ MS$[i]$?
- recurrence:

$$\text{MSS}[i] = \max\{\text{MSS}[i-1], ???\}$$

# 1-D DP

Solution.

- subproblem $\mathsf{MSS}[i]$: the sum of the MS *ending with* $a_i$ or 0
- goal: mss $= \max_{1 \le i \le n} \mathsf{MSS}[i]$

# 1-D DP

**Solution.**

- subproblem MSS$[i]$: the sum of the MS *ending with* $a_i$ or 0
- goal: mss $= \max_{1 \le i \le n}$ MSS$[i]$
- question: where does the MS$[i]$ start?
- recurrence:

$$\text{MSS}[i] = \max\{\text{MSS}[i-1] + a_i, 0\} \text{ (prove it!)}$$

# 1-D DP

**Solution.**

- subproblem MSS$[i]$: the sum of the MS *ending with* $a_i$ or 0
- goal: mss $= \max_{1 \le i \le n}$ MSS$[i]$
- question: where does the MS$[i]$ start?
- recurrence:

$$\text{MSS}[i] = \max\{\text{MSS}[i-1] + a_i, 0\} \text{ (prove it!)}$$

- initialization: MSS$[0] = 0$

# 1-D DP

Code.

```
MSS[0] = 0
For i = 1 to n
  MSS[i] = max{MSS[i-1] + A[i], 0}
return max_{i = 1 to n} MSS[i]
```

# 1-D DP

**Reconstructing string** [Problem: 2.2.14]

- string $S[1 \cdots n]$
- dict for *lookup*:

$$\text{dict}(w) = \begin{cases} \text{true} & \text{if } w \text{ is a valid word} \\ \text{false} & \text{o.w.} \end{cases}$$

- Is $S[1 \cdots n]$ valid (reconstructed as a sequence of valid words)?

**Solution.**

- subproblem $V[i]$: is $S[1 \cdots i]$ valid?
- goal: $V[n]$

# 1-D DP

**Solution.**

- question: where does the last word start?
- recurrence:

$$V[i] = \bigvee_{j=1\ldots i} \left(V[j-1] \wedge \mathsf{dict}(S[j \cdots i])\right)$$

- initialization: $V[0] = \mathsf{true}$

# Dynamic Programming

# 2-D DP (part 1)

LCS: longest common subsequence [Problem: 2.2.7]

- $X = X_1 \cdots X_m; Y = Y_1 \cdots Y_n$
- find (the length of) an LCS of $X$ and $Y$

$$X = \langle A, B, C, B, D, A, B \rangle$$
$$Y = \langle B, D, C, A, B, A \rangle$$
$$Z = \langle B, C, B, A \rangle$$

Solution.

- subproblem: $L[i, j]$: the length of an LCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$
- goal: $L[m, n]$

# 2-D DP (part 1)

Solution.

- question: Is $X_i = Y_j$?
- recurrence:

$$L[i,j] = \begin{cases} L[i-1,j-1]+1 & \text{if } X_i = Y_j \\ \max\{L[i-1,j], L[i,j-1]\} & \text{if } X_i \neq Y_j \end{cases}$$

# 2-D DP (part 1)

**Solution.**

- question: Is $X_i = Y_j$?
- recurrence:

$$L[i,j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1,j], L[i,j-1]\} & \text{if } X_i \neq Y_j \end{cases}$$

- initialization:

$$L[i,0] = 0, 0 \leq i \leq m$$
$$L[0,j] = 0, 0 \leq j \leq n$$

# 2-D DP (part 1)

Solution.

- question: Is $X_i = Y_j$?
- recurrence:

$$L[i,j] = \begin{cases} L[i-1,j-1]+1 & \text{if } X_i = Y_j \\ \max\{L[i-1,j], L[i,j-1]\} & \text{if } X_i \neq Y_j \end{cases}$$

- initialization:

$$L[i,0] = 0, 0 \le i \le m$$
$$L[0,j] = 0, 0 \le j \le n$$

It *may be* correct. But I feel quite uncomfortable without a proof.

# 2-D DP (part 1)

## Counterexample?

$$L[i,j] = L[i-1,j-1] + 1 \text{ if } X_i = Y_j$$

$$X = a, b, c, c, c \qquad X = a, b, c, c, c$$
$$Y = a, b, c \qquad Y = a, b, c$$
$$Z = a, b, c \qquad Z = a, b, c$$

# 2-D DP (part 1)

## Correctness proof (I).

**Theorem**

$L[i, j] = L[i-1, j-1] + 1$ *if* $X_i = Y_j$.

**Theorem**

$Z[1 \cdots k]$ *with* $Z_k \equiv X_i \wedge Z_k \equiv Y_j$ *is an LCS of* $X[1 \cdots i]$ *and* $Y[1 \cdots j]$.

**Proof.**

1. $Z_k = X_i = Y_j$ (by contradiction)
2. $Z_k = X_i = Y_j \Rightarrow$ either $Z_k \equiv X_i$ or $Z_k \equiv Y_j$ (by contradiction)
   2.1 $Z_k \equiv X_i \wedge Z_k \equiv Y_j$
   2.2 $Z_k \not\equiv X_i \wedge Z_k \equiv Y_j$
   2.3 $Z_k \equiv X_i \wedge Z_k \not\equiv Y_j$

# 2-D DP (part 1)

Correctness proof (II).

**Theorem**

$L[i,j] = \max\{L[i-1,j], L[i,j-1]\}$ *if* $X_i \neq Y_j$

**Theorem**

*If* $X_i \neq Y_j$*, then either* $X_i \notin LCS[i,j]$ *or* $Y_j \notin LCS[i,j]$*.*

**Proof.**

By contradiction. □

# 2-D DP (part 1)

## Edit distance revisited

$$\mathsf{ED}[i,j] = \min \begin{cases} \mathsf{ED}[i-1,j]+1 \\ \mathsf{ED}[i,j-1]+1 \\ \mathsf{ED}[i-1,j-1]+\mathsf{I}\{X_i = Y_j\} \end{cases}$$

# 2-D DP (part 1)

## Edit distance revisited

$$\text{ED}[i,j] = \min \begin{cases} \text{ED}[i-1,j]+1 \\ \text{ED}[i,j-1]+1 \\ \text{ED}[i-1,j-1]+\text{I}\{X_i = Y_j\} \end{cases}$$

$$\text{ED}[i,j] = \begin{cases} \text{ED}[i-1,j-1] & \text{if } X_i = Y_j \\ \min \begin{cases} \text{ED}[i-1,j]+1 \\ \text{ED}[i,j-1]+1 \\ \text{ED}[i-1,j-1]+1 \end{cases} & \text{if } X_i \neq Y_j \end{cases}$$

# 2-D DP (part 1)

## Edit distance revisited

$$\text{ED}[i,j] = \min \begin{cases} \text{ED}[i-1,j]+1 \\ \text{ED}[i,j-1]+1 \\ \text{ED}[i-1,j-1]+\mathsf{I}\{X_i = Y_j\} \end{cases}$$

$$\text{ED}[i,j] = \begin{cases} \text{ED}[i-1,j-1] & \text{if } X_i = Y_j \\ \min \begin{cases} \text{ED}[i-1,j]+1 \\ \text{ED}[i,j-1]+1 \\ \text{ED}[i-1,j-1]+1 \end{cases} & \text{if } X_i \neq Y_j \end{cases}$$

## Theorem

*If $X_i = Y_j$, then $ED[i-1,j-1] \leq ED[i-1,j]+1$.*

# 2-D DP (part 2)

**Longest contiguous substring both forward and backward** [Problem: 2.2.9]

- string $T[1 \cdots n]$
- to find LCS both forward and backward

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**Trial and error.**

- try subproblem $L[i]$: the length of an LCS in $T[1 \cdots i]$
- try subproblem $L[i,j]$: the length of an LCS in $T[i \cdots j]$

# 2-D DP (part 2)

Solution.

- $L[i,j]$: the length of an LCS starting with $T_i$ and ending with $T_j$
- goal: $\max_{1 \le i \le j \le n} L[i,j]$ (simply $O(n^3)$)

# 2-D DP (part 2)

Solution.

- $L[i, j]$: the length of an LCS starting with $T_i$ and ending with $T_j$
- goal: $\max_{1 \le i \le j \le n} L[i, j]$ (simply $O(n^3)$)
- question: Is $T_i = T_j$?
- recurrence:

$$L[i, j] = \begin{cases} 0 & \text{if } T_i \neq T_j \\ L[i+1, j-1] + 1 & \text{if } T_i = T_j \end{cases}$$

# 2-D DP (part 2)

Solution.

- $L[i, j]$: the length of an LCS starting with $T_i$ and ending with $T_j$
- goal: $\max_{1 \le i \le j \le n} L[i, j]$ (simply $O(n^3)$)
- question: Is $T_i = T_j$?
- recurrence:

$$L[i, j] = \begin{cases} 0 & \text{if } T_i \neq T_j \\ L[i+1, j-1] + 1 & \text{if } T_i = T_j \end{cases}$$

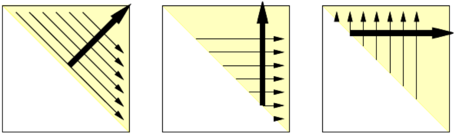
- initialization:

$$L[i, i] = 0, 0 \le i \le n$$

$$L[i, i+1] = \begin{cases} 1 & \text{if } T_i = T_{i+1} \\ 0 & \text{if } T_i \neq T_{i+1} \end{cases}$$

# 2-D DP (part 2)

Code: three ways of filling the table.



```
for d = 2 to n-1
  for i = 1 to n-d
    j = i + d
    ...
return max_{1 <= i <= j <= n} L[i,j]
```

# 2-D DP (part 2)

Code: three ways of filling the table.

```
for i = n-2 to 1
  for j = i+2 to n
    ...
return ...
```

```
for j = 3 to n
  for i = j-2 to 1
    ...
return ...
```

# 2-D DP (part 2)

## String split problem [Problem: 2.2.16]

- split a string $S$ into many pieces
- cost $|S| = n \Rightarrow n$
- given locations of $m$ cuts: $C_0, C_1, \cdots, C_m, C_{m+1}$
- to find the MinCost of splitting $S$ into $m+1$ pieces $S_0 \cdots S_m$

## Solution.

- subproblem: $\text{MinCost}[i, j]$: the minimum cost of splitting substring $S_i \cdots S_{j-1}$ using cuts $C_{i+1} \cdots C_{j-1}$
- goal: $\text{MinCost}[0, m+1]$

# 2-D DP (part 2)

**Solution.**

- question: what is the first cut in $C_{i+1} \cdots C_{j-1}$?
- recurrence:

$$\text{MinCost}[i, j] = \min_{i<k<j} \left( \text{MinCost}[i, k] + \text{MinCost}[k, j] + l(S_i \cdots S_{j-1}) \right)$$

- initialization:

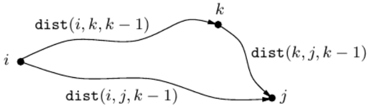
$$\text{MinCost}[i, i+1] = 0$$

# Dynamic Programming

# 3-D DP

## Floyd-Warshall algorithm

- subproblem dist$[i, j, k]$: the length of the shortest path from $i$ to $j$ via only nodes $v_1 \cdots v_k$
- goal: dist$[i, j, n], \forall i, j$
- question: Is $v_k$ in ShortestPath$[i, j, k]$?
- recurrence:



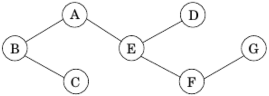

$$\text{dist}[i, j, k] = \min\{\text{dist}[i, j, k-1], \text{dist}[i, k, k-1] + \text{dist}[k, j, k-1]\}$$

# Dynamic Programming

# DP on graphs

## Minimum vertex cover [Problem: 2.2.18]

- tree $T$
- compute (the size of) a minimum vertex cover of $T$




## Solution.

- rooted $T$ at $r$
- subproblem $I(u)$: the size of a MVC of $T_u$ subtree
- goal: $I(r)$

# DP on graphs

**Solution.**

- question: Is $u$ in MVC$[u]$?
- recurrence:

$$I(u) = \max\{|\text{children of } u| + \sum_{v:\text{grandchildren of } u} I(v), 1 + \sum_{v:\text{children of } u} I(v)\}$$

- initialization:

$$I(u) = 0, \text{if } u \text{ is a leave}$$

# DP on graphs

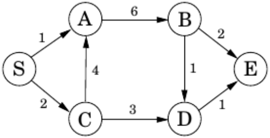
Code.

```
DFS on T from root r:
  when u is ``finished'':
    I(u) = 0, if u is a leave
    I(u) = ..., otherwise
```

# DP on graphs

## Shortest paths in dags

- dag $G = (V, E, w)$
- $s \in V$
- SSSP from $s$




## Solution.

- subproblem $\mathrm{dist}[v]$: shortest distance from $s$ to $v$
- goal: all $\mathrm{dist}[v]$

# DP on graphs

**Solution.**

- question: What is the relation between dist$[v]$ and dist$[u]$ of its predecessors $u$?
- recurrence:

$$\text{dist}[v] = \min_{u \to v} (\text{dist}[u] + w(u \to v))$$

# DP on graphs

Code.

```
dist[s] = 0
dist[v] = infty for others

for v != s in linearized order
  dist[v] = min_{u -> v} dist[u] + w(u \to v)
```

Remarks.

1. longest path
2. negative edges

# Dynamic Programming

# The knapsack problem

The change-making problem [Problem: 2.2.17 (b), 2.2.4 (subset sum)]

- coins values: $x_1 \ldots x_n$
- amount: $v$
- possible to make change for $v$?
- without repetition

# The knapsack problem

## The change-making problem [Problem: 2.2.17 (b), 2.2.4 (subset sum)]

- coins values: $x_1 \dots x_n$
- amount: $v$
- possible to make change for $v$?
- without repetition

## Trial and error.

- subproblem $C[i]$: is it possible to make change for $v$ using only $x_1 \cdots x_n$
- goal: $C[n]$
- question: using $x_i$ or not?
- recurrence:

$$C[i] = C[i-1] \vee ???$$

# The knapsack problem

Solution.

- subproblem $C[i, w]$: is it possible to make change for $w$ using only $x_1 \dots x_n$
- goal: $C[n, v]$
- question: using $x_i$ or not?
- recurrence:

$$C[i, w] = C[i-1] \vee (C[i-1, v-w] \wedge v \geq w)$$

# The knapsack problem

Solution.

- subproblem $C[i, w]$: is it possible to make change for $w$ using only $x_1 \dots x_n$
- goal: $C[n, v]$
- question: using $x_i$ or not?
- recurrence:

$$C[i, w] = C[i-1] \vee (C[i-1, v-w] \wedge v \geq w)$$

- initialization:

$$C[i, 0] = \text{true}$$
$$C[0, w] = \text{false, if } w > 0$$
$$C[0, 0] = \text{true}$$

# The knapsack problem

The change-making problem [Problem: 2.2.17 (a)]

- coins values: $x_1 \dots x_n$
- amount: $v$
- possible to make change for $v$?
- unbounded repetition

Solution.

- subproblem $C[i, w]$: is it possible to make change for $w$ using only $x_1 \dots x_n$
- goal: $C[n, v]$
- question: using $x_i$ or not?
- recurrence:

$$C[i, w] = C[i-1] \vee (C[i, w - x_i] \wedge w \geq x_i)$$

# The knapsack problem

The change-making problem [Problem: 2.2.17 (c)]

- coins values: $x_1 \dots x_n$
- amount: $v$
- possible to make change for $v$?
- $\leq k$-coins

Solution.

- subproblem $C[i, w, l]$: is it possible to make change for $w$ with $\leq l$ coins of $x_1 \dots x_i$
- goal: $C[n, v, k]$

# The knapsack problem

Solution.

- question: using $x_i$ or not?
- recurrence:

$$C[i, w, l] = C[i-1, w, l] \vee (C[i, w - x_i, l - 1] \wedge w \geq x_i)$$

# The knapsack problem

Solution.

- question: using $x_i$ or not?
- recurrence:

$$C[i, w, l] = C[i-1, w, l] \vee (C[i, w - x_i, l-1] \wedge w \geq x_i)$$

- initialization:

$$\begin{aligned} C[0,0,l] &= \text{true} \\ C[0,w,l] &= \text{false, if } w > 0 \\ C[i,0,l] &= \text{true} \\ C[i,w,0] &= \text{false, if } w > 0 \end{aligned}$$


https://github.com/hengxin/algorithm-ta-tutorial.git