# Dynamic Programming

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# Dynamic Programming


1. Overview
2. 1D DP
3. 2D DP
4. 3D DP
5. DP on Graphs
6. The Knapsack Problem
7. Summary

# What is DP?

DP $\approx$ " brute force"

DP $\approx$ "smart scheduling of subproblems"

DP $\approx$ "shortest/longest paths in some DAG"

# What is DP?

DP $\approx$ "smarter brute force"
DP $\approx$ "smart scheduling of subproblems"
DP $\approx$ "shortest/longest paths in some DAG"

# What is not DP?

Programming = Planning

# What is not DP?

Programming $=$ Planning

Programming $\neq$ Coding

(Richard Bellman, 1940s)

# Steps for applying DP

1. Define subproblems
   - # of subproblems
2. Set the goal
3. Define the recurrence
   - larger subproblem $\leftarrow$ # smaller subproblems
   - init. conditions
4. Write pseudo-code: fill "table" in topo. order
5. Analyze Time/Space complexity
6. Extract the optimal sulution

# Common subproblems in DP

1D subproblems:

Input: $x_1, x_2, \ldots, x_n$ (array, sequence, string)

Subproblems: $x_1, x_2, \ldots, x_i$ (prefix/suffix)

#: $\Theta(n)$

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#: $\Theta(n)$

Examples: Fib, Maximum-sum subarray, Longest increasing subsequence, Highway restaurants, Text justification

# Common subproblems in DP

2D subproblems:

1. Input: $x_1, x_2, \ldots, x_m; \quad y_1, y_2, \ldots, y_n$
   Subproblems: $x_1, x_2, \ldots, x_i; \quad y_1, y_2, \ldots, y_j$
   #: $\Theta(mn)$

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   Examples: Edit distance, Longest common subsequence

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   #: $\Theta(mn)$
   Examples: Edit distance, Longest common subsequence

2. Input: $x_1, x_2, \ldots, x_n$
   Subproblems: $x_i, \ldots, x_j$
   #: $\Theta(n^2)$

# Common subproblems in DP

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   #: $\Theta(mn)$
   Examples: Edit distance, Longest common subsequence

2. Input: $x_1, x_2, \ldots, x_n$
   Subproblems: $x_i, \ldots, x_j$
   #: $\Theta(n^2)$
   Examples: Matrix chain multiplication, Optimal BST

# Common subproblems in DP

3D subproblems:

- Floyd-Warshall algorithm

$$\mathsf{d}(i,j,k) = \min\{\mathsf{d}(i,j,k-1), \mathsf{d}(i,k,k-1) + \mathsf{d}(k,j,k-1)\}$$

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DP on graphs:

1. On rooted tree

   Subproblems: rooted subtrees

2. On DAG

   Subproblems: nodes after/before in the topo. order

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DP on graphs:

1. On rooted tree

   Subproblems: rooted subtrees
2. On DAG

   Subproblems: nodes after/before in the topo. order

Knapsack problem:

- Subset sum problem, change-making problem

# Common subproblems in DP

And Others . . .

# Recurrences in DP

Make choices by asking yourself the right question:

1. Binary choice
   - whether ...
2. Multi-way choices
   - where to ...
   - which one ...

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$$f(n) = \begin{cases} n - 1 & \text{if } n \in \mathbb{Z}^+ \\ n/2 & \text{if } n \% 2 = 0 \\ n/3 & \text{if } n \% 3 = 0 \end{cases}$$

$S(n)$ : minimum number of steps taking $n$ to 1.

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$S(n)$ : minimum number of steps taking $n$ to 1.

$S(i)$ : minimum number of steps taking $i$ to $1$

$$S(i) = 1 + \min\{N(i-1), N(i/2)(\text{if } n\%2 = 0), N(i/3)(\text{if } n\%3 = 0)\}$$

$$S(1) = 0$$

# $f^{(S(n))} = 1$

Collatz $(3n+1)$ conjecture:

$$f(n) = \begin{cases} n/2 & \text{if } n\%2 = 0 \\ 3n+1 & \text{if } n\%2 = 1 \end{cases}$$

$$f^*(n) = 1?$$

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$$f^*(n) = 1?$$

*"Mathematics may not be ready for such problems."*

*— Paul Erdős*

# Longest increasing subsequence

## Longest increasing subsequence (Problem 7.3)

- Given an integer array $A[1 \dots n]$
- To find (the length of) a longest increasing subseqence.

$$5, 2, 8, 6, 3, 6, 9, 7 \implies 2, 3, 6, 9$$

# Longest increasing subsequence

Subproblem: $L(i)$ : the length of the LIS of $A[1 \ldots i]$

Goal: $L(n)$

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Make choice: whether $A[i] \in LIS[1 \dots i]$?

Recurrence:

$$L(i) = \max\{L(i-1), 1 + \max_{j<i \wedge A[j] \leq A[i]} L(j)\}$$

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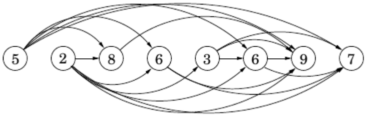
Init:

$$L(0) = 0$$

Time:

$$O(n^2) = \Theta(n) \cdot O(n)$$

# Longest increasing subsequence




Longest path distance in the DAG!

# Maximum-sum subarray

## Maximum-sum subarray (Google Interview)

- Array $A[1 \cdots n], a_i >=< 0$
- To find (the sum of) a maximum-sum subarray of $A$
  - mss $= 0$ if all negative

$$A[-2, 1, -3, 4, -1, 2, 1, -5, 4] \implies [4, -1, 2, 1]$$

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Subproblem: MSS$[i]$: the sum of the MS$[i]$ of $A[1 \cdots i]$

Goal: mss $=$ MSS$[n]$

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Goal: mss $=$ MSS$[n]$

Make choice: Is $a_i \in$ MS$[i]$?

Recurrence:

$$\text{MSS}[i] = \max\{\text{MSS}[i-1], ???\}$$

# Maximum-sum subarray

Subproblem: MSS$[i]$: the sum of the MS$[i]$ *ending with* $a_i$ or 0

Goal: mss $= \max_{1 \le i \le n}$ MSS$[i]$

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Make choice: where does the MS$[i]$ start?

Recurrence:

$$\text{MSS}[i] = \max\{\text{MSS}[i-1] + a_i, 0\} \text{ (prove it!)}$$

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$$\text{MSS}[0] = 0$$

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Recurrence:

$$\text{MSS}[i] = \max\{\text{MSS}[i-1] + a_i, 0\}$$ (prove it!)

Init:

$$\text{MSS}[0] = 0$$

Time: $\Theta(n)$

## Maximum-sum subarray

MSS$[0] \leftarrow 0$
**for all** $i \leftarrow 1 \dots n$ **do**
  MSS$[i] \leftarrow \max\{$MSS$[i-1] + a_i, 0\}$
**return** $\max_{i=1 \dots n}$ MSS$[i]$

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mss $\leftarrow 0$
MSS $\leftarrow 0$
**for all** $i \leftarrow 1 \ldots n$ **do**
    MSS $\leftarrow \max\{$MSS $+ a_i, 0\}$
    mss $\leftarrow \max\{$mss, MSS$\}$
**return** mss

# Maximum-product subarray

Maximum-product subarray (Problem 7.4)

- Array $A[1 \ldots n]$
- Find maximum-product subarray of $A$

(1) $a_i \in \mathbb{N}$

(2) $a_i \in \mathbb{Z}$

(3) $a_i \in \mathbb{R}$

# Maximum-product subarray

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(3) $a_i \in \mathbb{R}$

# Reconstructing string

## Reconstructing string (Problem 7.9)

- String $S[1 \cdots n]$
- Dict for *lookup*:

$$\text{dict}(w) = \begin{cases} \text{true} & \text{if } w \text{ is a valid word} \\ \text{false} & \text{o.w.} \end{cases}$$

- Is $S[1 \cdots n]$ valid (reconstructed as a sequence of valid words)?

## Reconstructing string

Subproblem: $V[i]$: is $S[1 \cdots i]$ valid?

Goal: $V[n]$

# Reconstructing string

Subproblem: $V[i]$: is $S[1 \cdots i]$ valid?

Goal: $V[n]$

Make choice: where does the last word start?

Recurrence:

$$V[i] = \bigvee_{j=1 \ldots i} (V[j-1] \wedge \mathsf{dict}(S[j \cdots i]))$$

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Subproblem: $V[i]$: is $S[1 \cdots i]$ valid?

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Init:

$$V[0] = \mathsf{true}$$

Time: $O(n^2)$

# Hotel along a trip

## Hotel along a trip (Problem 7.15)

- Hotel sequence (distance): $a_0 = 0, a_1, \cdots, a_n$
- $a_0 \rightsquigarrow a_n$
- Stop at only hotels
- Cost: $(200 - x)^2$
- To minimize overall cost

# Hotel along a trip

Subproblem: $C[i]$: minimum cost when the destination is $a_i$

Goal: $C[n]$

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$$C[i] = \min_{0 \le j < i} \{C[j] + (200 - (a_i - a_j))^2\}$$

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Init:

$$C[0] = 0$$

Time:

$$O(n^2) = \Theta(n) \cdot O(n)$$

# Highway restaurants

**Highway restaurants (Problem 7.16)**

- Locations: $L[1 \dots n]$
- Profits: $P[1 \dots n]$
- Any two hotels should be $\geq k$ miles apart
- To maximize the total profit

Subproblem:

Goal:

Make choice:

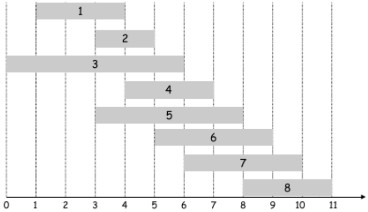
Recurrence:

Time:

# Weighted interval/class scheduling

## Weighted interval/class scheduling (Problem 7.14)

- Classes: $\mathcal{C} = \{c_1, c_2, \cdots, c_n\} \quad c_i \triangleq \langle g_i, s_i, f_i \rangle$
- Choosing non-conflicting classes to maximize your grades




sort $\mathcal{C}$ by finishing time.

# Weighted interval/class scheduling

Greedy algorithms by finishing time or weights fail.

# Weighted interval/class scheduling

Subproblem: $G[i]$: the maximal grades obtained from $\{c_1, c_2, \cdots, c_i\}$

Goal: $G[n]$

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Make choice: choose $c_i$ or not in $G[i]$?

Recurrence:

$$G[i] = \max\{G[i-1], G[p(i)] + g_i\}$$

$p(i)$: the largest index $j < i$ *s.t.* $c_i$ and $c_j$ are disjoint

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Init:

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Init:

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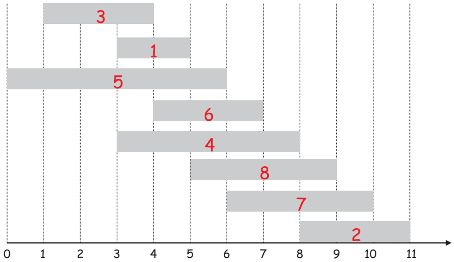
Time: $O(n \log n) + T(p(i)) + O(n) \cdot O(1)$

# Weighted interval/class scheduling

Why is ordering necessary?

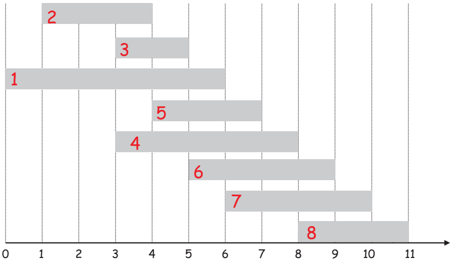



$$G[7] = \max\{G[6], G[\{1,3,5\}] + g_7\}$$

subproblems changed: all $O(2^n)$ subsets

# Weighted interval/class scheduling

What about sorting by starting time?




$$G[6] = \max\{G[5], G[\{2,3\}] + g_6\}$$

subproblems changed: all $O(2^n)$ subsets

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# Longest common subsequence

LCS: longest common subsequence (Problem 7.5)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

(1) Find (the length of) an LCS of $X$ and $Y$

$$X = \langle A, B, C, B, D, A, B \rangle$$
$$Y = \langle B, D, C, A, B, A \rangle$$
$$Z = \langle B, C, B, A \rangle$$

# Longest common subsequence

Subproblem: $L[i, j]$: the length of an LCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$

Goal: $L[m, n]$

## Longest common subsequence

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Goal: $L[m,n]$

Make choice: Is $X_i = Y_j$?

Recurrence: (Proof!)

$$L[i,j] = \begin{cases} L[i-1,j-1]+1 & \text{if } X_i = Y_j \\ \max\{L[i-1,j], L[i,j-1]\} & \text{if } X_i \neq Y_j \end{cases}$$

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Init:

$$L[0, j] = 0, \ 0 \leq j \leq n$$
$$L[i, 0] = 0, \ 0 \leq i \leq m$$

Time: $\Theta(mn)$

# Longest common subsequence

Longest common subsequence (Problem 7.5)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

(2) Allowing repetition of $X$

(3) Allowing repetition $\leq k$ of $X$

# Longest common subsequence

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$$X \implies X^{(k)} \triangleq X_1^{(k)} \cdots X_m^{(k)}$$

# Longest common substring

What about longest common *substring*?

# Shortest common supersequence

**Shortest common supersequence (Problem 7.6)**

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

- Find (the length of) a shortest common subsequence of $X$ and $Y$

# Shortest common supersequence

Subproblem: $L[i, j]$: the length of an SCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$

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Recurrence:

$$L[i, j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1, j] + 1, L[i, j-1] + 1\} & \text{if } X_i \neq Y_j \end{cases}$$

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Init:

$$L[0, j] = j, \; 0 \leq j \leq n$$
$$L[i, 0] = i, \; 0 \leq i \leq m$$

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**Remark**

$$\max(m, n) \leq L(m, n) \leq m + n$$

# Variants of LCS

Variants of LCS (Problem 7.7)

# Longest contiguous substring both forward and backward

**Longest contiguous substring both forward and backward (Problem 7.8)**

- String $T[1 \cdots n]$
- Find a longest contiguous substring (LCS) both forward and backward

dynamicprogrammingmanytimes

- try subproblem $L[i]$: the length of an LCS in $T[1 \cdots i]$
- try subproblem $L[i, j]$: the length of an LCS in $T[i \cdots j]$

# Longest contiguous substring both forward and backward

Subproblem: $L[i, j]$: the length of an LCS starting with $T_i$ and ending with $T_j$

Goal: $\max_{1 \leq i \leq j \leq n} L[i, j]$

# Longest contiguous substring both forward and backward

Subproblem: $L[i, j]$: the length of an LCS starting with $T_i$ and ending with $T_j$

Goal: $\max_{1 \leq i \leq j \leq n} L[i, j]$

Make choice: Is $T_i = T_j$?

Recurrence:

$$L[i, j] = \begin{cases} 0 & \text{if } T_i \neq T_j \\ L[i+1, j-1] + 1 & \text{if } T_i = T_j \end{cases}$$

# Longest contiguous substring both forward and backward

Subproblem: $L[i,j]$: the length of an LCS starting with $T_i$ and ending with $T_j$

Goal: $\max_{1 \le i \le j \le n} L[i,j]$

Make choice: Is $T_i = T_j$?

Recurrence:

$$L[i,j] = \begin{cases} 0 & \text{if } T_i \neq T_j \\ L[i+1, j-1] + 1 & \text{if } T_i = T_j \end{cases}$$

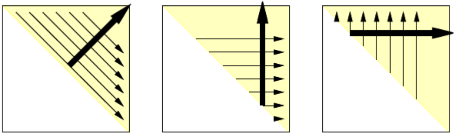
Init:

$$L[i,i] = 0, \; 0 \le i \le n$$

$$L[i,i+1] = \begin{cases} 1 & \text{if } T_i = T_{i+1} \\ 0 & \text{if } T_i \neq T_{i+1} \end{cases}$$

# Longest contiguous substring both forward and backward

Code: three ways of filling the table



```
for d = 2 to n-1
  for i = 1 to n-d
    j = i + d
    ...
return max_{1 <= i <= j <= n} L[i,j]
```

# Longest palindrome subsequence

Longest palindrome subsequence (Problem 7.10)

(1) Find (the length of) a longest palindrome subsequence of $S[1 \cdots n]$

Subproblem: $L[i, j]$: the length of an LSP of $S[i \cdots j]$

Goal: $L[1, n]$

# Longest palindrome subsequence

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Subproblem: $L[i,j]$: the length of an LSP of $S[i \cdots j]$

Goal: $L[1,n]$

Make choice: Is $S[i] = S[j]$?

Recurrence:

$$L[i,j] = \begin{cases} L[i+1, j-1] + 2 & \text{if } S[i] = S[j] \\ \max L[i+1,j], L[i,j-1] & \text{if } S[i] \neq S[j] \end{cases}$$

# Longest palindrome subsequence

**Longest palindrome subsequence (Problem 7.10)**

(1) Find (the length of) a longest palindrome subsequence of $S[1 \cdots n]$

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Recurrence:

$$L[i, j] = \begin{cases} L[i+1, j-1] + 2 & \text{if } S[i] = S[j] \\ \max L[i+1, j], L[i, j-1] & \text{if } S[i] \neq S[j] \end{cases}$$

Init:

$$L[i, i] = 1, \ \forall 1 \leq i \leq n$$

# Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \ldots n]$ into minimum number of palindromes ($\#$ cuts)

Subproblem: $C[i, j]$: minimum number of cuts for string $S[i \ldots j]$

Goal: $C[1, n] + 1$

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**Palindrome splitting (Problem 7.10)**

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes ($\#$ cuts)

Subproblem: $C[i,j]$: minimum number of cuts for string $S[i \dots j]$

Goal: $C[1,n] + 1$

Make choice: Where is the first cut?

Recurrence:

$$C[i,j] = \begin{cases} 0 & \text{if } S[i \dots j] \text{ is a palindrome} \\ \min\limits_{i+1 \le k \le j-1} C[i,k] + 1 + C[k+1,j] & \text{o.w.} \end{cases}$$

# Palindrome splitting

**Palindrome splitting (Problem 7.10)**

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Init: $C[i,i] = 0$

Time: $O(n^3)$

# Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes

Subproblem: $P[i]$: minimum number of palindromes for $S[1 \cdots i]$

Goal: $P[n]$

# Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes

Subproblem: $P[i]$: minimum number of palindromes for $S[1 \cdots i]$

Goal: $P[n]$

Make choice: Where does the last palindrome start from?

Recurrence:

$$P[i] = \min_{\substack{1 \le k \le i \\ S[k \dots i] \text{ is a palindrome}}} P[k-1] + 1$$

# Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \ldots n]$ into minimum number of palindromes

Subproblem: $P[i]$: minimum number of palindromes for $S[1 \cdots i]$

Goal: $P[n]$

Make choice: Where does the last palindrome start from?

Recurrence:

$$P[i] = \min_{\substack{1 \le k \le i \\ S[k...i] \text{ is a palindrome}}} P[k-1] + 1$$

Init: $P[0] = 1$

Time: $O(n^2)$

# String splitting

String splitting (Problem 7.11)

- Split a string $S$ into many pieces
- Cost $|S| = n \implies n$
- Given locations of $m$ cuts: $C_0, C_1, \cdots, C_m, C_{m+1}$
- Find the minimum cost of splitting $S$ into $m+1$ pieces $S_0 \cdots S_m$

## String splitting

Subproblem: $C[i, j]$: the minimum cost of splitting substring $S_i \cdots S_{j-1}$ using cuts $C_{i+1} \cdots C_{j-1}$

Goal: $C[0, m+1]$

# String splitting

Subproblem: $C[i,j]$: the minimum cost of splitting substring $S_i \cdots S_{j-1}$ using cuts $C_{i+1} \cdots C_{j-1}$

Goal: $C[0, m+1]$

Make choice: What is the first cut in $C_{i+1} \cdots C_{j-1}$?

Recurrence:

$$C[i,j] = \min_{i<k<j} \left( C[i,k] + C[k,j] + l(S_i \cdots S_{j-1}) \right)$$

## String splitting

Subproblem: $C[i,j]$: the minimum cost of splitting substring $S_i \cdots S_{j-1}$ using cuts $C_{i+1} \cdots C_{j-1}$

Goal: $C[0, m+1]$

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Recurrence:

$$C[i,j] = \min_{i<k<j} \left( C[i,k] + C[k,j] + l(S_i \cdots S_{j-1}) \right)$$

Init: $C[i, i+1] = 0$

Subproblem:

Subprblem:

Goal:

Make choice:

Subprblem:
Goal:
Make choice:
Recurrence:
Init:
Time:

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# 3-D DP

**Floyd-Warshall algorithm**

(1) DP for Floyd-Warshall algorithm for APSP on directed graphs

**Subproblem:** dist$[i, j, k]$: the length of the shortest path from $i$ to $j$ via only nodes in $v_1 \cdots v_k$

**Goal:** dist$[i, j, n], \forall i, j$

# 3-D DP

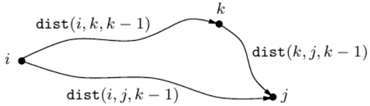
Floyd-Warshall algorithm

(1) DP for Floyd-Warshall algorithm for APSP on directed graphs

Make choice: Is $v_k$ on the ShortestPath$[i, j, k]$?

Recurrence:

$$\text{dist}[i,j,k] = \min\{\text{dist}[i,j,k-1], \text{dist}[i,k,k-1] + \text{dist}[k,j,k-1]\}$$



# 3-D DP

## Floyd-Warshall algorithm

(1) DP for Floyd-Warshall algorithm for APSP on directed graphs

Make choice: Is $v_k$ on the ShortestPath$[i, j, k]$?

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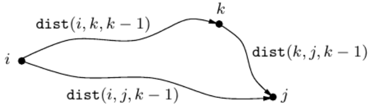


Init:

$$\text{dist}[i, j, 0] = \begin{cases} 0 & i = j \\ w(i, j) & (i, j) \in E \\ \infty & \text{o.w.} \end{cases}$$

# 3-D DP

**Floyd-Warshall algorithm (Problem 6.25)**

(2) Routing table for Floyd-Warshall algorithm

**for all** $k \leftarrow 1 \dots n$ **do**
  **for all** $i \leftarrow 1 \dots n$ **do**
    **for all** $j \leftarrow 1 \dots n$ **do**
      **if** dist$[i,j] >$ dist$[i,k] +$ dist$[k,j]$ **then**
        dist$[i,j] \leftarrow$ dist$[i,k] +$ dist$[k,j]$

# 3-D DP

Floyd-Warshall algorithm (Problem 6.25)

(2) Routing table for Floyd-Warshall algorithm

**for all** $k \leftarrow 1 \dots n$ **do**
  **for all** $i \leftarrow 1 \dots n$ **do**
    **for all** $j \leftarrow 1 \dots n$ **do**
      **if** $\text{dist}[i, j] > \text{dist}[i, k] + \text{dist}[k, j]$ **then**
        $\text{dist}[i, j] \leftarrow \text{dist}[i, k] + \text{dist}[k, j]$
        $\text{Go}[i, j] \leftarrow \text{Go}[i, k]$

# 3-D DP

**Floyd-Warshall algorithm (Problem 6.25)**

(2) Routing table

**for all** $i \leftarrow 1 \dots n$ **do**
  **for all** $j \leftarrow 1 \dots n$ **do**
    dist$[i, j] \leftarrow \infty$
    Go$[i, j] \leftarrow$ Nil
**for all** $(i, j) \in E$ **do**
  dist$[i, j] \leftarrow w(i, j)$
  Go$[i, j] \leftarrow j$
**for all** $i \leftarrow 1 \dots n$ **do**
  dist$[i, i] \leftarrow 0$
  Go$[i, j] \leftarrow$ Nil

# 3-D DP

Floyd-Warshall algorithm (Problem 6.25)

(2) Routing table

**for all** $i \leftarrow 1 \dots n$ **do**
  **for all** $j \leftarrow 1 \dots n$ **do**
    dist$[i, j] \leftarrow \infty$
    Go$[i, j] \leftarrow$ Nil
**for all** $(i, j) \in E$ **do**
  dist$[i, j] \leftarrow w(i, j)$
  Go$[i, j] \leftarrow j$
**for all** $i \leftarrow 1 \dots n$ **do**
  dist$[i, i] \leftarrow 0$
  Go$[i, j] \leftarrow$ Nil

**procedure** $\textsc{Path}(i, j)$
  **if** Go$[i, j]$ = Nil **then**
    Output "No Path."

  Output "$i$"
  **while** $i \neq j$ **do**
    $i \leftarrow$ Go$[i, j]$
  Output "$i$"

# 3-D DP

**Floyd-Warshall algorithm (Problem 6.29)**

(3) Find minimum-weighted cycle of directed graph $(w(e) > 0)$

$$\text{dist}[i, i] \leftarrow 0 \implies \text{dist}[i, i] \leftarrow \infty$$

# 3-D DP

Floyd-Warshall algorithm (Problem 6.29)

(3) Find minimum-weighted cycle of directed graph $(w(e) > 0)$

$$\text{dist}[i,i] \leftarrow 0 \implies \text{dist}[i,i] \leftarrow \infty$$

$\exists i : \text{dist}[i,i] < 0$

$\forall i : \text{dist}[i,i] \geq 0$

# Shortest paths on undirected graphs

## Finding shortest paths in undirected graphs with possibly negative edge weights

2

The book "Algorithms" by Robert Sedgewick and Kevin Wayne hinted that (*see the quote below*) there are efficient algorithms for finding shortest paths in undirected graphs with possibly negative edge weights (***not*** by treating an undirected edge as two directed one which means that a single negative edge implies a negative cycle). However, no references are given in the book. Are you aware of any such algorithms?

2

> Q. How can we find shortest paths in undirected (edge-weighted) graphs?
>
> A. For positive edge weights, Dijkstra's algorithm does the job. We just build an EdgeWeightedDigraph corresponding to the given EdgeWeightedGraph (by adding two directed edges corresponding to each undirected edge, one in each direction) and then run Dijkstra's algorithm. ***If edge weights can be negative*** (***emphasis added***), efficient algorithms are available, but they are more complicated than the Bellman-Ford algorithm.

algorithms graph-theory shortest-path weighted-graphs reference-question

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edited Jun 9 at 14:15

asked Jun 9 at 13:58
hengxin
6,056 11 38

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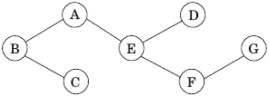
# Minimum vertex cover on trees

Minimum vertex cover on trees [Problem: 2.2.18]

- Undirected tree $T = (V, E)$; No designated root!
- Compute (the size of) a minimum vertex cover of $T$



# Minimum vertex cover on trees

Rooted $T$ at any node $r$.

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Subproblem: $I(u)$: the size of an MVC of subtree $T_u$ rooted at $u$

Goal: $I(r)$

# Minimum vertex cover on trees

Rooted $T$ at any node $r$.

Subproblem: $I(u)$: the size of an MVC of subtree $T_u$ rooted at $u$

Goal: $I(r)$

Make choice: Is $u$ in MVC$[u]$?

Recurrence:

$$I(u) = \min\{\#\text{ children of } u + \sum_{v:\text{ grandchildren of } u} I(v), \quad 1 + \sum_{v:\text{ children of } u} I(v)\}$$

# Minimum vertex cover on trees

Rooted $T$ at any node $r$.

Subproblem: $I(u)$: the size of an MVC of subtree $T_u$ rooted at $u$

Goal: $I(r)$

Make choice: Is $u$ in MVC$[u]$?

Recurrence:

$$I(u) = \min\{\#\text{ children of } u + \sum_{v\text{: grandchildren of } u} I(v), \quad 1 + \sum_{v\text{: children of } u} I(v)\}$$

Init: $I(u) = 0$, if $u$ is a leave

# Minimum vertex cover on trees

DFS on $T$ from root $r$:

when $u$ is "finished":
**if** $u$ is a leave **then**
$I(u) \leftarrow 0$
**else**
$I(u) \leftarrow \ldots$

# Minimum vertex cover on trees

DFS on $T$ from root $r$:

when $u$ is "finished":
**if** $u$ is a leave **then**
$I(u) \leftarrow 0$
**else**
$I(u) \leftarrow \ldots$

Greedy algorithm:

**Theorem**

*There is an MVC which contains no leaves.*

# DP on DAG

Longest path in DAG (Problem 7.17)

- Direction: $\downarrow$ OR $\rightarrow$
- Score: $>=<0$

# DP on DAG

**Longest path in DAG (Problem 7.17)**

- Direction: $\downarrow$ OR $\rightarrow$
- Score: $>=<0$

1. digraph $G$
2. node weight $\rightarrow$ edge weight
3. adding an extra sink $s$
4. $G \rightarrow G^T$

# DP on DAG

### Longest path in DAG (Problem 7.17)

- Direction: $\downarrow$ OR $\rightarrow$
- Score: $>=< 0$

1. digraph $G$
2. node weight $\rightarrow$ edge weight
3. adding an extra sink $s$
4. $G \rightarrow G^T$

Compute a longest path from $s$ in DAG

# DP on DAG

Subproblem: dist$[v]$: longest distance from $s$ to $v$

Goal: dist$[v], \forall v \in V$

# DP on DAG

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Make choice:

Recurrence:

$$\text{dist}[v] = \max_{u \to v} \left( \text{dist}[u] + w(u \to v) \right)$$

# DP on DAG

Subproblem: dist$[v]$: longest distance from $s$ to $v$

Goal: dist$[v], \forall v \in V$

Make choice:

Recurrence:

$$\text{dist}[v] = \max_{u \to v} \left( \text{dist}[u] + w(u \to v) \right)$$

Init: dist$[s] = 0$

# DP on DAG

Subproblem: dist$[v]$: longest distance from $s$ to $v$

Goal: dist$[v], \forall v \in V$

Make choice:

Recurrence:

$$\text{dist}[v] = \max_{u \to v} \left( \text{dist}[u] + w(u \to v) \right)$$

Init: dist$[s] = 0$

Compute dist$[v]$ in topo. order

# Bitonic tour

Bitonic tour (Problem 7.18)

# Bitonic tour

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# The change-making problem

**The change-making problem (Problem 7.12)**

- Coins values: $x_1 \dots x_n$
- Amount: $v$
- Is it possible to make change for $v$?

# The change-making problem

The change-making problem (Problem 7.12(2), Problem 7.1 (Subset sum))

(2) Without repetition $(0/1)$

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The change-making problem (Problem 7.12(2), Problem 7.1 (Subset sum))

(2) Without repetition $(0/1)$

Subproblem: $C[i, w]$: Possible to make change for $w$ using only $x_1 \dots x_n$?

Goal: $C[n, v]$

# The change-making problem

The change-making problem (Problem 7.12(2), Problem 7.1 (Subset sum))

(2) Without repetition $(0/1)$

Subproblem: $C[i, w]$: Possible to make change for $w$ using only $x_1 \ldots x_n$?

Goal: $C[n, v]$

Make choice: Using $x_i$ or not?

Recurrence:

$$C[i, w] = C[i-1, w] \vee (C[i-1, w - x_i] \wedge w \geq x_i)$$

# The change-making problem

The change-making problem (Problem 7.12(2), Problem 7.1 (Subset sum))

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Goal: $C[n, v]$

Make choice: Using $x_i$ or not?

Recurrence:

$$C[i, w] = C[i-1, w] \vee (C[i-1, w - x_i] \wedge w \geq x_i)$$

Init:

$$C[i, 0] = \text{true}$$
$$C[0, w] = \text{false, if } w > 0$$
$$C[0, 0] = \text{true}$$

Time: $O(nv)$

# The change-making problem

The change-making problem (Problem 7.12(1))

(1) Unbounded repetition ($\infty$)

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Subproblem: $C[i, w]$: Possible to make change for $w$ using only $x_1 \ldots x_n$?

Goal: $C[n, v]$

Make choice: Using $x_i$ or not?

Recurrence:

$$C[i, w] = C[i-1, w] \lor (C[i, w - x_i] \land w \geq x_i)$$

# The change-making problem

The change-making problem (Problem 7.12(1))

(1) Unbounded repetition ($\infty$)

Subproblem: $C[i, w]$: Possible to make change for $w$ using only $x_1 \ldots x_n$?

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Time: $O(nv)$

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(1) Unbounded repetition ($\infty$)

Subproblem: $C[w]$: Possible to make change for $w$?

Goal: $C[v]$

Make choice: Suppose $x_i$ is used.

Recurrence:

$$C[w] = \bigvee_{i:\ x_i \le w} C[w - x_i]$$

# The change-making problem

The change-making problem (Problem 7.12(1))

(1) Unbounded repetition ($\infty$)

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Goal: $C[v]$

Make choice: Suppose $x_i$ is used.

Recurrence:

$$C[w] = \bigvee_{i:\, x_i \le w} C[w - x_i]$$

Time: $O(nv)$

Q: $C[i, w]$ *vs.* $C[w]$

# The change-making problem

The change-making problem (Problem 7.12(3))

(3) Unbounded repetition with $\leq k$ coins

# The change-making problem

**The change-making problem (Problem 7.12(3))**

(3) Unbounded repetition with $\leq k$ coins

Subproblem: $C[i, w, l]$: Possible to make change for $w$ with $\leq l$ coins of $x_1 \dots x_i$?

Goal: $C[n, v, k]$

# The change-making problem

The change-making problem (Problem 7.12(3))

(3) Unbounded repetition with $\leq k$ coins

Subproblem: $C[i, w, l]$: Possible to make change for $w$ with $\leq l$ coins of $x_1 \dots x_i$?

Goal: $C[n, v, k]$

Make choice: Using $x_i$ or not?

Recurrence:

$$C[i, w, l] = C[i-1, w, l] \vee (C[i, w - x_i, l - 1] \wedge w \geq x_i)$$

# The change-making problem

## The change-making problem (Problem 7.12(3))

(3) Unbounded repetition with $\leq k$ coins

Subproblem: $C[i, w, l]$: Possible to make change for $w$ with $\leq l$ coins of $x_1 \ldots x_i$?

Goal: $C[n, v, k]$

Make choice: Using $x_i$ or not?

Recurrence:

$$C[i, w, l] = C[i-1, w, l] \lor (C[i, w - x_i, l - 1] \land w \geq x_i)$$

Init:

$$C[0, 0, l] = \text{true}, \quad C[0, w, l] = \text{false, if } w > 0$$
$$C[i, 0, l] = \text{true}, \quad C[i, w, 0] = \text{false, if } w > 0$$

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# More DPs . . .

## Algorithms that use dynamic programming [ edit | edit source ]

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- Recurrent solutions to lattice models for protein-DNA binding
- Backward induction as a solution method for finite-horizon discrete-time dynamic optimization problems
- Method of undetermined coefficients can be used to solve the Bellman equation in infinite-horizon, discrete-time, discounted, time-invariant dynamic optimization problems
- Many string algorithms including longest common subsequence, longest increasing subsequence, longest common substring, Levenshtein distance (edit distance)
- Many algorithmic problems on graphs can be solved efficiently for graphs of bounded treewidth or bounded clique-width by using dynamic programming on a tree decomposition of the graph.
- The Cocke–Younger–Kasami (CYK) algorithm which determines whether and how a given string can be generated by a given context-free grammar
- Knuth's word wrapping algorithm that minimizes raggedness when word wrapping text
- The use of transposition tables and refutation tables in computer chess
- The Viterbi algorithm (used for hidden Markov models)
- The Earley algorithm (a type of chart parser)
- The Needleman–Wunsch algorithm and other algorithms used in bioinformatics, including sequence alignment, structural alignment, RNA structure prediction
- Floyd's all-pairs shortest path algorithm
- Optimizing the order for chain matrix multiplication
- Pseudo-polynomial time algorithms for the subset sum, knapsack and partition problems
- The dynamic time warping algorithm for computing the global distance between two time series
- The Selinger (a.k.a. System R) algorithm for relational database query optimization
- De Boor algorithm for evaluating B-spline curves
- Duckworth–Lewis method for resolving the problem when games of cricket are interrupted
- The value iteration method for solving Markov decision processes
- Some graphic image edge following selection methods such as the "magnet" selection tool in Photoshop
- Some methods for solving interval scheduling problems
- Some methods for solving the travelling salesman problem, either exactly (in exponential time) or approximately (e.g. via the bitonic tour)
- Recursive least squares method
- Beat tracking in music information retrieval
- Adaptive-critic training strategy for artificial neural networks
- Stereo algorithms for solving the correspondence problem used in stereo vision
- Seam carving (content-aware image resizing)
- The Bellman–Ford algorithm for finding the shortest distance in a graph
- Some approximate solution methods for the linear search problem
- Kadane's algorithm for the maximum subarray problem


THANK YOU