P and NP

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P and NP

Computability theory first

Theorem

Halting problem is undecidable.

Proof.

□

Complexity theory to follow

Is a given Sodoku configuration solvable?

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Is $3 \times 3 \times 3$ Rubik's Cube solvable in 20 moves?

Decision problems

Definition (Decision problems.)

Decision problems are problems whose solution is "Yes/No".

Remarks.

- decision problem vs. optimization problem
- decision problem is "hard" $\Rightarrow$ its optimization problem is "hard"

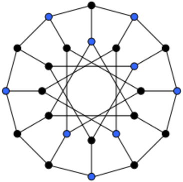
Decision problems vs. optimization problem

INDEPENDENT SET

Optimization problem.

Instance: Undirected graph $G = (V, E)$.

Question: Find the maximal independent set in G.



Decision problem.

Instance: Undirected graph $G = (V, E)$ and an integer k.

Question: Does G has an independent set of size (at least) k?

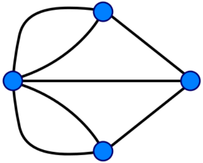
The class P

Definition (The class P)

P is the class of decision problems that are solvable in Polynomial time.

Examples for the class P

Euler path.



Maze problem.

Primality testing.

1	**2**	**3**	4	**5**	6	**7**	8	9	10
11	12	**13**	14	15	16	**17**	18	**19**	20
21	22	**23**	24	25	26	27	28	**29**	30
31	32	33	34	35	36	**37**	38	39	40
41	42	**43**	44	45	46	**47**	48	49	50
51	52	**53**	54	55	56	57	58	**59**	60
61	62	63	64	65	66	**67**	68	69	70
71	72	**73**	74	75	76	77	78	**79**	80
81	82	**83**	84	85	86	87	88	**89**	90
91	92	93	94	95	96	**97**	98	99	100

The class NP

Definition (The class NP)

NP is the class of decision problems that are solvable in Polynomial time by Non-deterministic algorithm.

$$\text{NP} \neq \text{Non-Polynomial}$$

$$\text{NP} \neq \text{No Problem}$$

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Definition (Non-deterministic polynomial algorithm.)

Given an instance $\mathcal{I}$ of a decision problem:

Guessing: generate a certificate c for $\mathcal{I}$

Verifying: $V(\mathcal{I}, c)$

$$O(\text{Guessing}) + O(\text{Verifying}) = O(n^c)$$

Proof of being in NP

Theorem

INDEPENDENT SET $\in$ *NP.*

Proof.

Given $G = (V, E)$ and k:

Guessing: Nondeterministically select a subset c of k vertices of G.

Verifying: Test whether G contains no edges for all vertices pairs in c.

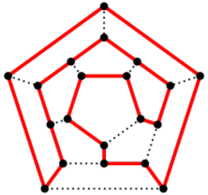
Output: If the test passes, ouput "yes"; otherwise, output "no".

The complexity is $O(n^2)$. □

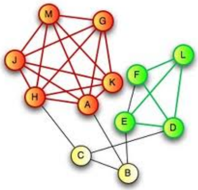
The class NP

Examples for the class NP

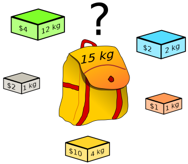
Hamiltonian path.



Clique problem.



Knapsack problem.

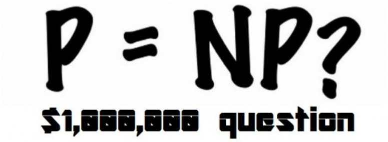


P vs. NP

P vs. NP

P: polynomially solvable

NP: polynomially verifiable



P vs. NP

Theorem

$$P \subseteq NP.$$

Proof.

To design a non-deterministic polynomial algorithm given a deterministic polynomial algorithm (PA).

- Guessing: generate a certificate c for instance $\mathcal{I}$.
- Verifying: ignore c; output PA$(\mathcal{I})$.

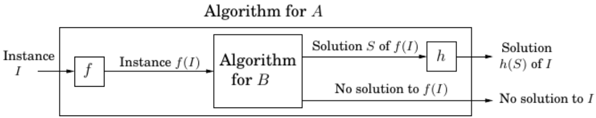
$\square$

Reduction

Why reduction?

To relate complexities of individual problems.

Definition (Reduction.)



Polynomial reduction

Definition (Polynomial reduction.)

$T(x)$ in polynomial time.

Remarks.

Theorem

Transitivity: $A \leq_P B, B \leq_P C \Rightarrow A \leq_p C.$

Theorem

$A \leq_P B, B \in P \Rightarrow A \in P.$

NP-hard

Definition (NP-hard)

B is NP-hard if

$$\forall A \in \text{NP} : A \leq_P B.$$

Remark.

NP-hard problems are at least as hard as any problem in NP.

NP-complete

Definition (NP-complete)

B is NP-complete if:

1. $B \in \mathrm{NP}$
2. B is NP-hard.

Remark.

NP-complete problems are the hardest problems in NP.

NP-complete

Theorem

If B is NP-complete and $B \in P$, then $P = NP$.

Theorem

If B is NP-complete and $B \leq_P C \in NP$, then C is NP-complete.

Proof of being in NP-complete

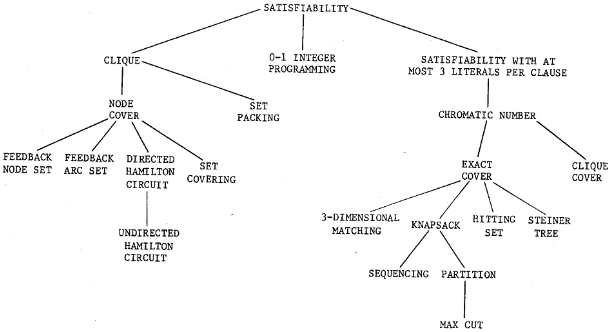
To prove C is NP-Complete.

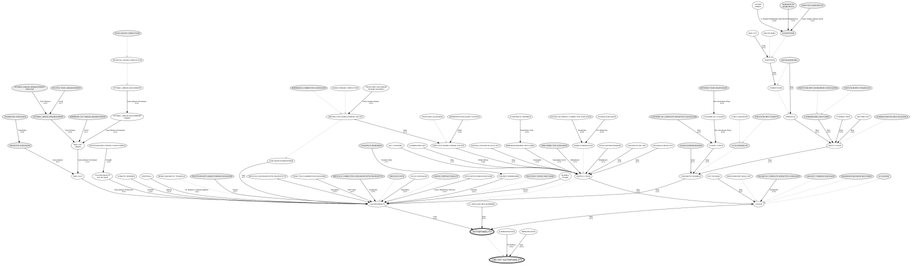
1. $C \in$ NP:
   - non-deterministic polynomial algorithm
2. C is NP-hard:
   - choose a known NP-complete problem B
   - prove $B \leq_P C$

NP-complete problem

The first known NP-complete problem.

SAT (circuit satisfiablity) is NP-complete.





http://adriann.github.io/npc/npc.html

https://github.com/hengxin/algorithm-ta-tutorial.git