A Little Mathematics for Algorithm Analysis

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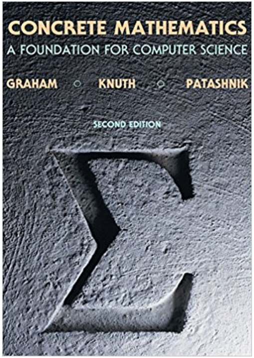
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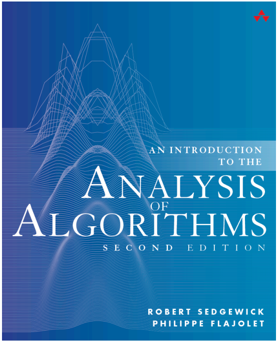
AN INTRODUCTION TO THE
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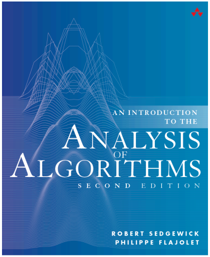
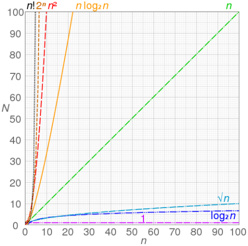
80% of the people are not good at math. I guess I belong to the other 25%

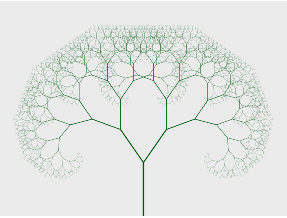


$A(n)$

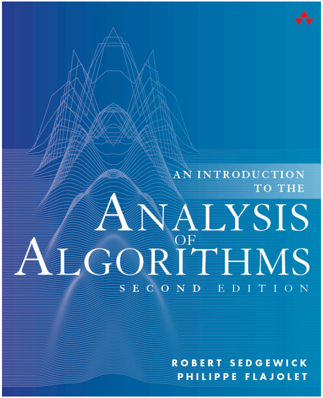
Mathematical Induction



Ω, Θ, O



$T(n) = aT(n/b) + f(n)$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Problem P Algorithm A

Inputs: $\mathcal{X}_n$ of size n

$$W(n) = \max_{x \in \mathcal{X}_n} T(x)$$

$$B(n) = \min_{x \in \mathcal{X}_n} T(x)$$

$$A(n) = \boxed{\sum_{x \in \mathcal{X}_n} T(x) \cdot P(x)} = \mathbb{E}[T] = \boxed{\sum_{t \in T(\mathcal{X}_n)} t \cdot P(T = t)}$$

Average-case Time Complexity (Problem 1.7)

$$r \in [1, n],\ r \in \mathbb{Z}^+$$

$$P\{r = i\} = \begin{cases} \frac{1}{n}, & 1 \le i \le \frac{n}{4} \\ \frac{2}{n}, & \frac{n}{4} < i \le \frac{n}{2} \\ \frac{1}{2n}, & \frac{n}{2} < i \le n \end{cases} \qquad T(r) = \begin{cases} 10, & r \le \frac{n}{4} \\ 20, & \frac{n}{4} < r \le \frac{n}{2} \\ 30, & \frac{n}{2} < r \le \frac{3n}{4} \\ n, & \frac{3n}{4} < r \le n \end{cases}$$

$$\begin{aligned} A &= \sum_{x \in \mathcal{X}} T(x) \cdot P(x) = \sum_{t \in T(\mathcal{X}_n)} t \cdot P(T = t) \\ &= T(1)P(1) + T(2)P(2) + \cdots + T(n)P(n) \\ &= 10 \times \frac{n}{4} \times \frac{1}{n} + 20 \times \frac{n}{4} \times \frac{2}{n} + 30 \times \frac{n}{4} \times \frac{1}{2n} + n \times \frac{n}{4} \times \frac{1}{2n} \\ &= \cdots \end{aligned}$$

Mathematical Induction

Horner's rule (Problem 1.6)

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

```
procedure HORNER(A[0...n], x)        ▷ A : {a_0 ... a_n}
    p ← A[n]
    for i ← n − 1 ⇓ 0 do
        p ← px + A[i]
    return p
```

Loop invariant (after the k-th loop):

$$\boxed{\mathcal{I}: \; p = \sum_{j=n}^{j=n-k} a_j x^{k-(n-j)}}$$

$$\boxed{\mathcal{I}: \; p = \sum_{j=n}^{j=n-k} a_j x^{k-(n-j)}}$$

When you are in an exam:

20% : Finding $\mathcal{I}$

80% : Proving $\mathcal{I}$ by PMI

Prove by mathematical induction on the number k of loops.

Base Step: $k = 0$.

Inductive Hypothesis: $\mathcal{I}$ is valid after the k-th ($k \geq 0$) loop.

Inductive Step: $\mathcal{I}$ maintains for the $(k+1)$-th loop:

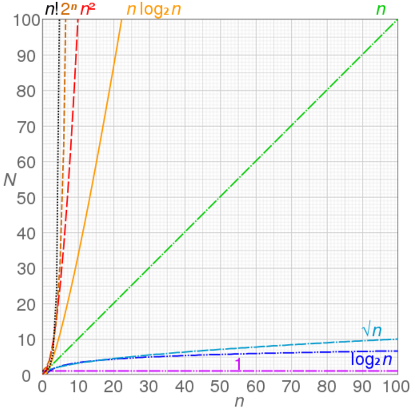
$$\boxed{\left(\sum_{j=n}^{j=n-k} a_j x^{k-(n-j)}\right) \cdot x + A[n-k-1] = \sum_{j=n}^{j=n-(k+1)} a_j x^{(k+1)-(n-j)}}$$

Termination

(a) $i \leftarrow n-1 \Downarrow 0$

(b) $k = n \implies p = \sum_{i=0}^{i=n} a_i x^i$

Asymptotics



$$Q : \theta(f)\ ?$$

$$O(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \Big\}$$

$$\Big\{ \quad \Big\}$$

$$\exists n_0 > 0, \forall n \geq n_0$$

$$\exists c > 0$$

$$O(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \ge n_0 : 0 \le f(n) \le cg(n) \Big\}$$

$$\Omega(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \ge n_0 : 0 \le cg(n) \le f(n) \Big\}$$

$$\Theta(g(n)) = \Big\{ f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \ge n_0 : \\ 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \Big\}$$

$$o(g(n)) = \Big\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \ge n_0 : 0 \le f(n) < cg(n) \Big\}$$

$$\omega(g(n)) = \Big\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \ge n_0 : 0 \le cg(n) < f(n) \Big\}$$

$$f(n) \sim g(n) \iff \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$$

Asymptotics (Problem 2.6 (4))

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \land f(n) = \Omega(g(n))$$

Asymptotics (Problem 2.6 (6))

$$\Theta(g(n)) \cap o(g(n)) = \emptyset$$

$$Q: f(n) = O(g(n)) \lor f(n) = \Omega(g(n)) \ ?$$

$$f(n) = n, \quad g(n) = n^{1+\sin n}$$

Asymptotics (Problem 2.7 (2))

$$(\log n)^2 \quad vs. \quad \sqrt{n}$$

$$(\log n)^{c_1} = O(n^{c_2}) \quad c_1, c_2 > 0$$

Summation (Problem 2.20)

```
procedure CONUNDRUM(n)
    r ← 0
    for i ← 1 to n do
        for j ← i + 1 to n do
            for k ← i + j − 1 to n do
                r ← r + 1
    return r
```

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \frac{n^2 - n}{2} = \Theta(n^2)$$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \frac{n^2 - n}{2} = \Theta(n^2)$$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \frac{1}{48}\Big(3\big(-1 + (-1)^n\big) + 2n(n+2)(2n-1)\Big) = \Theta(n^3)$$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} (n-i-j+2) \; [j \le n-i+1, i \le \frac{n}{2}]$$

$$= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=i+1}^{n-i+1} (n-i-j+2)$$

$$\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=i+j-1}^{n} 1$$

$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n} (n-(i+j-1)+1)\,[i+j-1\le n]$$

$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n} (n-i-j+2)\,[j\le n-i+1] \qquad n-i+1 \geq i+1 \Rightarrow i > \frac{n}{2}$$

$$= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor}\sum_{j=i+1}^{n-i+1} (n-i-j+2)$$

$$= \frac{1}{2}\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (n-2i+1)(n-2i+2)$$

当n为偶时 $= \frac{1}{2}\left[\sum_{i=1}^{\frac{n}{2}} (n^2+3n+2) + 4\sum_{i=1}^{\frac{n}{2}} i^2 - \sum_{i=1}^{\frac{n}{2}} (4n+6)i\right)$

$$= \frac{1}{2}\times\left(\frac{1}{2}(n^3+3n^2+2n) + \frac{n(\frac{n}{2}+1)(n+1)}{3} - \frac{(2n+3)(\frac{n}{2}+1)n}{2}\right)$$

$= \underline{化简} = \dfrac{2n^3+3n^2-2n}{24} = \dfrac{1}{48}\left(0+2n(2+n)(2n-1)\right)$

当n为奇时，$\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$，代入，可化得 $= \frac{1}{48}\left(-6+2n(2+n)(2n-1)\right)$

(这个我懒的化了，谁有兴趣化一下，多个常数项).

*通解 $\frac{1}{48}\left(3(-1+(-1)^n)+2n(2+n)(2n-1)\right)$

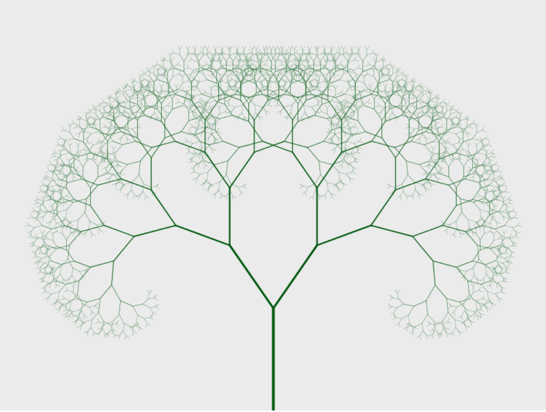
* $\lfloor \frac{n}{2} \rfloor = \dfrac{n+\frac{(-1)^n+(-1)}{2}}{2}$，代入理应可直接得结果，太繁

From Zheng (171860658)

Reference:

"Big Omicron and Big Omega and Big Theta" by Donald E. Knuth, 1976.

Recurrences



$$T(n) = aT(n/b) + f(n) \quad (a > 0, b > 1)$$

Assume that $T(n)$ is constant for sufficiently small n.

$$\left.\begin{array}{r} f(n) \\ af(\frac{n}{b}) \\ a^2 f(\frac{n}{b^2}) \\ \vdots \\ a^{\log_b n} T(1) = \Theta(n^{\log_b a}) \end{array}\right\} \sum \overset{f(n) \underline{vs.}\ n^E}{=} \begin{cases} n^{\log_b a}, & f(n) = O(n^{E-\epsilon}) \\ n^{\log_b a} \log n, & f(n) = \Theta(n^E) \\ f(n), & f(n) = \Omega(n^{E+\epsilon}) \end{cases}$$

Solving Recurrences (Problem 2.15)

(1) $\Theta(n^{\log_3 2})$

(2) $\Theta(\log^2 n)$

(3) $\Theta(n)$

(4) $\Theta(n \log n)$

(5) $\Theta(n \log^2 n)$

(6) $\Theta(n^2)$

(7) $\Theta(n^{\frac{3}{2}} \log n)$

(8) $\Theta(n)$

(9) $\Theta(n^{c+1})$

(10) $\Theta(c^{n+1})$

(11) $\cdots$

$$T(n) = T(n/2) + \log n$$

$$T(n) = 2T(n/2) + n \log n$$

Reference:

$$f(n) = \Theta(n^{\log_b a} \log^k n) \implies \Theta(n^{\log_b a} \log^{k+1} n)$$

Gaps in Master Theorem (Problem 2.18)

$$T(n) = 2T(n/2) + \frac{n}{\log n} = \Theta(n \log \log n)$$

Solving Recurrences (Problem 2.15)

(1) $\Theta(n^{\log_3 2})$

(2) $\Theta(\log^2 n)$

(3) $\Theta(n)$

(4) $\Theta(n \log n)$

(5) $\Theta(n \log^2 n)$

(6) $\Theta(n^2)$

(7) $\Theta(n^{\frac{3}{2}} \log n)$

(8) $\Theta(n)$

(9) $\Theta(n^{c+1})$

(10) $\Theta(c^{n+1})$

(11) $\cdots$

$$T(n) = T(n-1) + c^n \quad c > 1$$

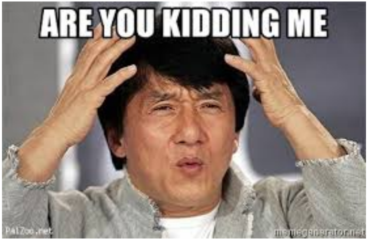
$$T(n) = T(n-1) + n^c \quad c \geq 1$$

$$\int$$

$$(\frac{n}{2}) \cdot (\frac{n}{2})^c \leq T(n) \leq n \cdot n^c$$

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8)$$



Where is $f(n)$?

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8)$$

$$\boxed{T(n) = \Theta(n^{0.879146})}$$

$$\boxed{T(n) = \Theta(n^{\alpha})}$$

$$2^{-\alpha} + 4^{-\alpha} + 8^{-\alpha} = 1$$

```
Solve[2^{-x} + 4^{-x} + 8^{-x} == 1, x] // N
```

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8) + n$$

By recursion-tree.

$$\boxed{T(n) = \Theta(n)}$$

Exercise: Prove it by mathematical induction.

Reference:

"On the Solution of Linear Recurrence Equations" by Akra & Bazzi, 1996.

$$T(n) = \sum_{i=1}^{k} a_i T(n/b_i) + f(n)$$

Solving Recurrences (Problem 2.17)

$$
\begin{aligned}
\mathrm{T}(n) &= \sqrt{n}\ \mathrm{T}(\sqrt{n}) + n \\
&= n^{\frac{1}{2}}\ \mathrm{T}\left(n^{\frac{1}{2}}\right) + n \\
&= n^{\frac{1}{2}} \left(n^{\frac{1}{2^2}}\ \mathrm{T}\left(n^{\frac{1}{2^2}}\right) + n^{\frac{1}{2}}\right) + n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}}\ \mathrm{T}\left(n^{\frac{1}{2^2}}\right) + 2n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}} \left(n^{\frac{1}{2^3}}\ \mathrm{T}\left(n^{\frac{1}{2^3}}\right) + n^{\frac{1}{2^2}}\right) + 2n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}}\ \mathrm{T}\left(n^{\frac{1}{2^3}}\right) + 3n \\
&= \cdots \\
&= n^{\sum_{i=1}^{k} \frac{1}{2^i}}\ \mathrm{T}\left(n^{\frac{1}{2^k}}\right) + kn
\end{aligned}
$$

$$\mathrm{T}(n) = n^{\sum_{i=1}^{k} \frac{1}{2^i}} \ \mathrm{T}\left(n^{\frac{1}{2^k}}\right) + kn$$

$$n^{\frac{1}{2^k}} = 1$$

$$n^{\frac{1}{2^k}} = 2 \implies k = \log\log n$$

$$\begin{aligned}
\mathrm{T}(n) &= n^{\sum_{i=1}^{k} \frac{1}{2^i}}\ \mathrm{T}\left(n^{\frac{1}{2^k}}\right) + kn \\
&= n^{\sum_{i=1}^{\log\log n} \frac{1}{2^i}}\ \mathrm{T}(2) + n \log\log n
\end{aligned}$$

$$\sum_{i=1}^{\log\log n} \frac{1}{2^i} < 1 \implies T(n) = \Theta(n \log\log n)$$

Exercise: Prove it by mathematical induction.

$$\boxed{T(n) = \sqrt{n}T(\sqrt{n}) + n}$$

$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

$$n \leftrightarrow 2^m$$

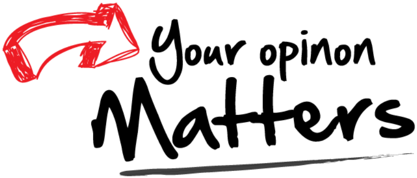
$$\frac{T(2^m)}{2^m} = \frac{T(2^{m/2})}{2^{m/2}} + 1$$

$$S(m) \leftrightarrow \frac{T(2^m)}{2^m}$$

$$S(m) = S(m/2) + 1 = \Theta(\log m)$$

$$\boxed{T(n) = n \log \log n}$$


Thank
You!



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