Algorithm to find diameter of a tree using BFS/DFS. Why does it work?

This link provides an algorithm for finding the diameter of an undirected tree **using BFS/DFS**. Summarizing:

> Run BFS on any node s in the graph, remembering the node u discovered last. Run BFS from u remembering the node v discovered last. d(u,v) is the diameter of the tree.

Why does it work ?

Page 2 of this provides a reasoning, but it is confusing. I am quoting the initial portion of the proof:

> Run BFS on any node s in the graph, remembering the node u discovered last. Run BFS from u remembering the node v discovered last. d(u,v) is the diameter of the tree.
>
> Correctness: Let a and b be any two nodes such that d(a,b) is the diameter of the tree. There is a unique path from a to b. Let t be the first node on that path discovered by BFS. If the paths p_1 from s to u and p_2 from a to b do not share edges, then the path from t to u includes s. So
>
> $d(t,u) \geq d(s,u)$
>
> $d(t,u) \geq d(s,a)$
>
> ....(more inequalities follow ..)

http://i61.tinypic.com/rji9uq.png

The inequalities do not make sense to me.

algorithms graphs search-algorithms trees graph-traversal

edited Apr 13 at 12:48
Community ♦
1

asked Mar 20 '14 at 7:09
curryage
191 1 2 8

I don't find the quote in the linked question. – Raphael ♦ Mar 20 '14 at 13:30

1 Try replacing "do not share edges" with "do not share vertices" in the solution. – Yuval Filmus Mar 20 '14 at 13:42

3 Answers

All parts of proving the claim hinge on 2 crucial properties of trees with undirected edges:

- 1-connectedness (ie. between any 2 nodes in a tree there is exactly one path)
- any node can serve as the root of the tree.

Choose an arbitrary tree node s. Assume $u, v \in V(G)$ are nodes with $d(u,v) = diam(G)$. Assume further that the algorithm finds a node x starting at s first, some node y starting at x next. wlog $d(s,u) \geq d(s,v)$. note that $d(s,x) \geq d(s,y)$ must hold, unless the algorithm's first stage wouldn't end up at x. We will see that $d(x,y) = d(u,v)$.

The most general configuration of all nodes involved can be seen in the following pseudo-graphics (possibly $s = z_{uv}$ or $s = z_{xy}$ or both):

```
(u)                                             (x)
  \                                             /
   \                                           /
    \                                         /
     ( z_uv )---------( s )----------( z_xy )
    /                                         \
   /                                           \
  /                                             \
(v)                                             (y)
```

we know that:

1. $d(z_{uv}, y) \leq d(z_{uv}, v)$. otherwise $d(u,v) < diam(G)$ contradicting the assumption.
2. $d(z_{uv}, x) \leq d(z_{uv}, u)$. otherwise $d(u,v) < diam(G)$ contradicting the assumption.
3. $d(s, z_{xy}) + d(z_{xy}, x) \geq d(s, z_{uv}) + d(z_{uv}, u)$, otherwise stage 1 of the algorithm wouldn't have stopped at x.
4. $d(z_{xy}, y) \geq d(v, z_{uv}) + d(z_{uv}, z_{xy})$, otherwise stage 2 of the algorithm wouldn't have stopped at y.

1) and 2) imply .

$$\begin{aligned} d(u,v) &= d(z_{uv},v) + d(z_{uv},u) \\ &\geq d(z_{uv},x) + d(z_{uv},y) = d(x,y) + 2\,d(z_{uv},z_{xy}) \\ &\geq d(x,y) \end{aligned}$$

3) and 4) imply

$$\begin{aligned} &d(z_{xy},y) + d(s,z_{xy}) + d(z_{xy},x) \\ &\quad \geq d(s,z_{uv}) + d(z_{uv},u) + d(v,z_{uv}) + d(z_{uv},z_{xy}) \end{aligned}$$

equivalent to .

$$\begin{aligned} d(x,y) &= d(z_{xy},y) + d(z_{xy},x) \\ &\geq 2 * \, d(s,z_{uv}) + d(v,z_{uv}) + d(u,z_{uv}) \\ &\geq d(u,v) \end{aligned}$$

therefore $d(u,v) = d(x,y)$.

analogue proofs hold for the alternative configurations

```
               (u)                      (x)
                 \                      /
                  \                    /
                   \                  /
   ( s )---------( z_uv )----------( z_xy )
                   /                  \
                  /                    \
                 /                      \
               (v)                      (y)
```

and

```
                       (x)        (u)
                       /            \
                      /              \
                     /                \
   ( s )---------( z_xy )----------( z_uv )
                     \                /
                      \              /
                       \            /
                       (y)        (v)
```

these are all possible configurations. in particular, $x \notin path(s,u), x \notin path(s,v)$ due to the result of stage 1 of the algorithm and $y \notin path(x,u), y \notin path(x,v)$ due to stage 2.

edited Mar 20 '14 at 23:37

answered Mar 20 '14 at 23:28
collapsar
780 4 8

(1) Regarding the first graphic, shouldn't the path from s to x always contain vertices u and v in some order since they are present on tree generated by BFS? (2) Could you clarify how the inequalities are obtained ? (3) Since the BFS starting from s and that starting from x contain u,v somewhere on the path, I believe the graphic should be as shown in the link imgur.com/jQ94erY . How does the reasoning you provided apply here ? – curryage Mar 21 '14 at 5:39

@curryage note that the tree is given and not being constructed by the bfs! specific answers: ad 1) no. imagine a refinement of the tree in graphics (1) by adding arbitrarily many nodes on the edge (s, z_{xy}) and exactly 1 node on the edge (z_{xy}, x). the first stage bfs will then end at x. ad 2) which inequality/ies are unclear? we are always assuming that (u, v) be a path the length of the graph's diameter $diag(G)$. this is well defined as G is 1-connected. ad 3) no: 3.1 there is more than 1 path between any 2 nodes apart from (s, y), so the graph is not a tree. ... – collapsar Mar 21 '14 at 15:27

@curryage ... 3.2 $d(x,y) > d(u,v)$; this is impossible as $d(u,v) = diam(G)$ by assumption and a graph's diameter is the maximal minimum distance between any two nodes. in the case of a tree there is exactly 1 path between any 2 nodes, so the definition reduces to 'maximum distance between any two nodes'. – collapsar Mar 21 '14 at 15:31

By the definition of BFS, the distance (from the starting node) of each node explored is either equal to the distance of the previous node explored or greater by 1. Thus, the last node explored by BFS will be among those farthest from the starting node.

Thus, the algorithm of using BFS twice amounts to "Pick an arbitrary node x. Find the node a farthest from x (last node found by BFS starting from x). Find the node b farthest from a (last node found by BFS starting from a).", which thus finds two nodes of maximum distance from eachother.

answered Jan 3 at 16:15
Extrarius
641 2 9

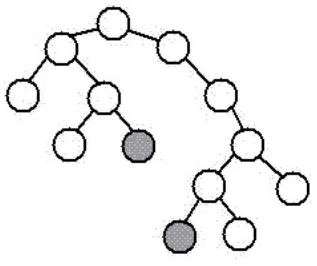
1 Thanks for the answer with the intuition. However, the "thus" in the your last sentence is not obvious. Why does that follow? Why does the node farthest from x have to be one of the two nodes at maximum distance from each other? It seems like that needs some proof. – D.W. ♦ Jan 3 at 16:41

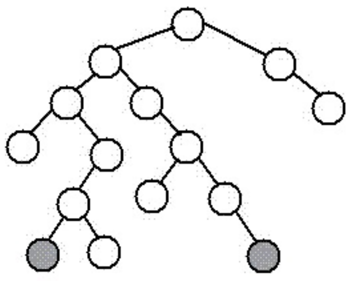
I'm not sure how to construct such a proof. I feel like the converse is intuitively true: if two nodes are at maximum distance from each other, then, for any given node, one of the two is at the greatest possible distance from it. – Extrarius Jan 3 at 20:41

The "intuitively true" claim isn't true in general for general graphs. See the graph in cs.stackexchange.com/a/213/755, and imagine starting the BFS from node v (i.e., let $x = v$); then it will pick $a = u$ and find the node b at greatest distance from a, but that doesn't find the two nodes of maximum distance from each other. So the claimed statement, if true, must rely on some special property of trees that doesn't hold for general graphs. – D.W. ♦ Jan 3 at 23:53

Yes, but this question specifies undirected trees, which is the context I'm intuiting in. Barring cycles and directed edges makes many graph problems significantly simpler to reason about. – Extrarius Jan 4 at 0:41

First run a DFS from a random node then the diameter of a tree is the path between the deepest leaves of a node in its DFS subtree:


diameter, 9 nodes, through root


diameter, 9 nodes, NOT through root

edited Dec 16 '16 at 9:30
Yuval Filmus
138k 6 125 261

answered Dec 16 '16 at 8:48
seddik11
1

2 Why does this work? – Yuval Filmus Dec 16 '16 at 9:30