Paths in Graphs

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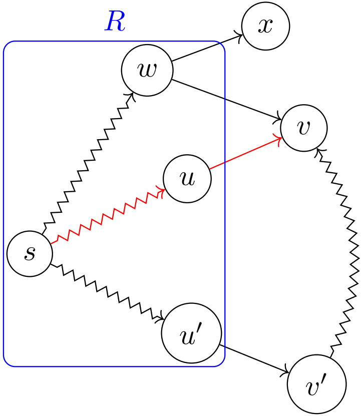
Dijkstra's Algorithm for SSSP

Finding shortest paths from s to other nodes t
in non-decreasing order of $\text{dist}(s, t)$.



$R \triangleq \{u \mid s \rightsquigarrow u \text{ is known}\}$

$v : \min_{u \in R} \text{dist}(u) + l(u, v)$

$l(v' \rightsquigarrow v) \geq 0$

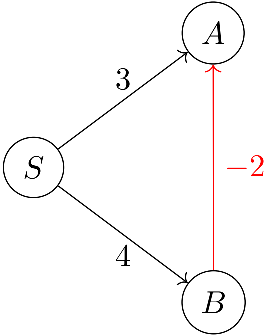
for all $v \in V$ **do**
    $\text{dist}[v] \leftarrow \infty$
$\text{dist}[s] \leftarrow 0$

$Q \leftarrow \textsc{MinPQ}(V)$
while $Q \neq \emptyset$ **do**
    $u \leftarrow \textsc{DeleteMin}(Q)$
    for all $(u, v) \in E \wedge v \notin Q$ **do**
        if $\text{dist}[v] > \text{dist}[u] + l(u, v)$ **then**
            $\text{dist}[v] \leftarrow \text{dist}[u] + l(u, v)$
            $\textsc{DecreaseKey}(Q, v)$

$$O(n + (n + m) \log n) \implies O((n + m) \log n) \implies O(m \log n)$$

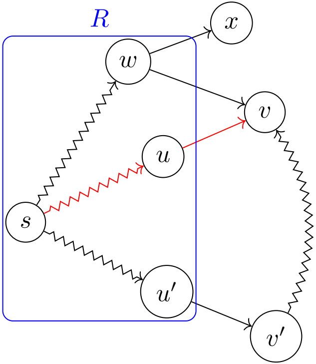
Negative Edges (Problem 11.1)

Dijkstra's algorithm may fail if $w(e) < 0$.



Negative Edges from s (Problem 11.9)

All negative edges are from s.



$$l(v' \rightsquigarrow v) \geq 0$$

Generalized Shortest Path (Problem 11.8)

Digraph $G = (V, E), \quad l_e > 0, \quad c_v > 0, \quad s \in V$

Shortest paths from s

$$\forall u \to v : l'(u, v) = l(u, v) + c_v$$

$$+ c_s$$

Shortest Paths Through v_0 (Problem 13.7)

Strongly connected digraph $G = (V, E), \quad w(e) > 0$

$$v_0 \in V$$

Find shortest paths $s \leadsto^{\text{SP}} t$ through v_0.

$$s \leadsto^{\text{SP}} v_0 \leadsto^{\text{SP}} t$$

$$\forall v : v_0 \leadsto^{\text{SP}} v$$

Dijkstra's Algorithm as a Framework

Min-Max Path (Problem 11.12)

$$G = (V, E): \text{ network of highways}$$

$$l_e: \text{ road length} \quad L: \text{ tank capacity}$$

Given L, $\exists ? s \rightsquigarrow t$ in $O(n + m)$.

$$l_e > L \implies l_e = \infty$$

$$s \rightsquigarrow^{?} t$$

Min-Max Path (Problem 11.12)

$$G = (V, E): \text{ network of highways}$$

$$l_e: \text{ road length} \quad L: \text{ tank capacity}$$

Given G, to compute $\min L$ in $O((n+m)\log n)$.

$$R \triangleq \{u \mid s \rightsquigarrow u \text{ is known}\}$$

for all $v \in V$ **do**
    $L[v] \leftarrow \infty$
$L[s] \leftarrow 0$

if $L[v] > \max(L[u], l(u,v))$ **then**
    $L[v] \leftarrow \max(L[u], l(u,v))$

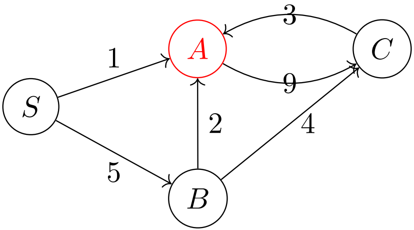
Max-Min Path (Problem 13.2 (1))

$$G = (V, E) : \text{ network of oil pipelines}$$

$$c(u, v) : \text{ capacity of } (u, v)$$

$$\text{cap}(s, t) : \max \min s \leadsto t$$

Given s, to compute $\text{cap}(s, v)$.



$$R \triangleq \{u \mid s \rightsquigarrow u \text{ is known}\}$$

$$Q \leftarrow \mathsf{MaxPQ}(V)$$

for all $v \in V$ **do**
    $\mathsf{cap}[v] \leftarrow -\infty$
$\mathsf{cap}[s] \leftarrow 0$

if $\mathsf{cap}[v] < \min(\mathsf{cap}[u], c(u,v))$ **then**
    $\mathsf{cap}[v] \leftarrow \min(\mathsf{cap}[u], c(u,v))$

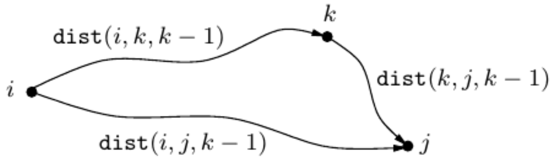
Max-Min Path (Problem 13.2 (2))

$$G = (V, E): \ \text{network of oil pipelines}$$

$$c(u, v): \ \text{capacity of } (u, v)$$

$$\mathsf{cap}(s, t) : \max \min s \rightsquigarrow t$$

Compute all-pair $\mathsf{cap}(u, v)$.



$$\mathsf{dist}(i,j,k) = \min\Big(\mathsf{dist}(i,j,k-1), \mathsf{dist}(i,k,k-1) + \mathsf{dist}(k,j,k-1)\Big)$$

$$\#k's = 1$$

for all (i, j) **do**
    if $(i, j) \in E$ **then**
        dist$(i, j, 0) \leftarrow l(i, j)$
    else
        dist$(i, j, 0) \leftarrow \infty$

for $k \leftarrow 1$ **to** n **do**
    for $i \leftarrow 1$ **to** n **do**
        for $j \leftarrow 1$ **to** n **do**
            dist$(i, j, k) = \min\Big(\text{dist}(i, j, k-1), \text{dist}(i, k, k-1) + \text{dist}(k, j, k-1)\Big)$

Max-Min Path (Problem 13.2 (2))

$$G = (V, E) : \text{ network of oil pipelines}$$

$$c(u, v) : \text{ capacity of } (u, v)$$

$$\mathsf{cap}(s, t) : \max \min s \rightsquigarrow t$$

Compute all-pair $\mathsf{cap}(u, v)$.

$$\mathsf{cap}(u, v, k) = \max\Big(\mathsf{cap}(u, v, k-1), \min\big(\mathsf{cap}(u, k, k-1), \mathsf{cap}(k, v, k-1)\big)\Big)$$

Routing table (Problem 13.1)

$$\mathsf{Go}(i,j) = k \implies v_i \to v_k \rightsquigarrow v_j$$

Contruct routing table and extract shortest paths from it.

$$\mathsf{Go}(i,j) \leftarrow \mathsf{Nil}$$

$$\forall (i,j) \in E : \mathsf{Go}(i,j) \leftarrow j$$

if $\ldots$ **then**
  $\mathsf{Go}(i,j) \leftarrow \mathsf{Go}(i,k)$

if $Go(i,j) = \mathsf{Nil}$ **then**
  $\cdots$

while $i \neq j$ **do**
  $i \leftarrow \mathsf{Go}(i,j)$

$$\mathsf{Prev}(i,j) \leftarrow \mathsf{Prev}(k,j)$$

$$\mathsf{Intermediate}(i,j) \leftarrow k$$

Shortest Cycle in Digraph (Problem 13.9)

Find shortest cycle in digraph $G = (V, E), \quad w(e) > 0.$

Initialize $\text{dist}[v][v] \leftarrow \infty$ in Floyd-Warshall algorithm

$$\exists v : \text{dist}[v][v] < \infty \implies \min_i \text{dist}[v][v]$$

$$\forall v : \text{dist}[v][v] = \infty$$

Hamiltonian Path in Tournament Graph (Problem 11.10)

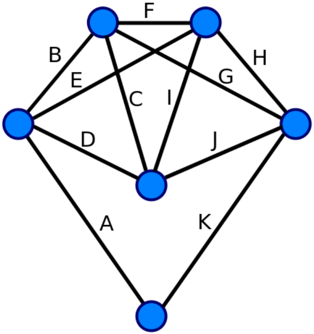
$$\forall u, v : (u \to v \lor v \to u) \\ \land \neg(u \to v \land v \to u)$$

By mathematical induction on n.

Eulerian Circuit (Problem 13.5)

To find an Eulerian circuit of a strongly connected digraph $G = (V, E)$ in $O(m)$ time.

$$\forall v \in V : \text{in}[v] = \text{out}[v]$$



$VE \leftarrow \emptyset$
$C \leftarrow \emptyset$

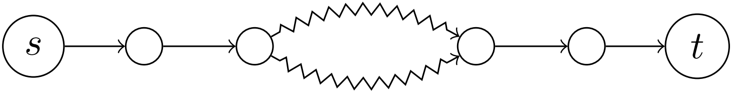
while $VE \neq E$ **do**
  $u \leftarrow \textsc{Choose}(u : (u \to v) \notin VE)$
  $C' \leftarrow \textsc{Circuit}(u, E \setminus VE)$
  $VE \leftarrow VE \cup C'$
  $C \leftarrow C \cup C'$

Data Structures?

Most Critical Edge (Problem 11.3)

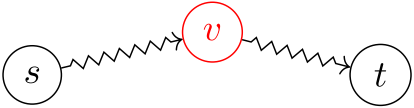
$$s, t \in V$$

$$e : E \setminus \{e\} \implies \text{dist}(s, t) \text{ increases most}$$



$$O(n\ (m \log n)) = O(mn \log n)$$

Bitonic Shortest Path (Problem 11.7)




THANK YOU