

Evasiveness of Graph Properties and Topological Fixed-Point Theorems

By Carl A. Miller

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Preface

Beginning with the paper *A Topological Approach to Evasiveness* by Kahn, Saks, and Sturtevant [18], there have been a number of interesting research papers that use topological methods to prove lower bounds on the complexity of graph properties. This is a fascinating topic that lies at the interface between mathematics and theoretical computer science. The goal of this text is to offer an integrated version of the underlying proofs in this body of research. While there are a number of very good expositions available on topological methods in decision-tree complexity, all those that I have seen refer to other sources for the proofs of some topological results (including the key fixed-point theorem of R. Oliver [32]). In this text I have attempted to give a completely self-contained account.

I have not assumed that the reader has any prior background in algebraic topology—all constructions from that subject are developed from scratch. The only prerequisite is a high level of comfort with discrete mathematics and linear algebra. Indeed, though I will sometimes refer to subsets of \mathbb{R}^n for intuition, all the results in this text finally rest on manipulations of finite sets.

While I was preparing this work for publication, I learned about the new book *A Course in Topological Combinatorics* by Mark de Longueville [27]. This book gives a similar treatment of topological methods for proofs of complexity of graph properties, including a proof of Oliver’s theorem. Whereas my text is more economical and is intended to offer as direct a route as possible to [18] and its related results, de Longueville’s book is broader in scope and encompasses topological methods for other combinatorial problems. I hope that the community will find both works beneficial.

The general flow of the text is to begin with foundational material and then to build up more complex results at a steady pace. The capstone results, which consist of three lower bounds on the complexity of graph properties, appear in the final part of the text. My undergraduate advisor Richard Hain once said that the final goal of mathematics is “to tell a good story.” That is what I have attempted to do here, and I hope the reader will enjoy the result.

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Abstract

Many graph properties (e.g., connectedness, containing a complete subgraph) are known to be difficult to check. In a decision-tree model, the cost of an algorithm is measured by the number of edges in the graph that it queries. R. Karp conjectured in the early 1970s that all monotone graph properties are evasive—that is, any algorithm which computes a monotone graph property must check all edges in the worst case. This conjecture is unproven, but a lot of progress has been made. Starting with the work of Kahn, Saks, and Sturtevant in 1984, topological methods have been applied to prove partial results on the Karp conjecture. This text is a tutorial on these topological methods. I give a fully self-contained account of the central proofs from the paper of Kahn, Saks, and Sturtevant, with no prior knowledge of topology assumed. I also briefly survey some of the more recent results on evasiveness.

1

Introduction

Let V be a finite set of size n , and let $\mathbf{G}(V)$ denote the set of undirected graphs on V . For our purposes, a **graph property** is simply a function

$$f: \mathbf{G}(V) \rightarrow \{0,1\} \tag{1.1}$$

which is such that whenever two graphs Z and Z' are isomorphic, $f(Z) = f(Z')$. A graph Z “has property f ” if $f(Z) = 1$.

We can measure the cost of an algorithm for computing f by counting the number of edge-queries that it makes. We assume that these edge-queries are adaptive (i.e., the choice of query may depend on the outcomes of previous queries). An algorithm for f can thus be represented by a binary decision-tree (see Figure 1.1). The **decision-tree complexity of f** , which we denote by $D(f)$, is the least possible depth for a decision-tree that computes f . In other words, $D(f)$ is the number of edge-queries that an optimal algorithm for f has to make in the worst case.

Some graph properties are difficult to compute. For example, let $h(Z) = 1$ if and only if Z contains a cycle. Suppose that an algorithm for h makes queries to an adversary whose goal is to maximize cost. The adversary can adaptively construct a graph Y to foil the algorithm: each

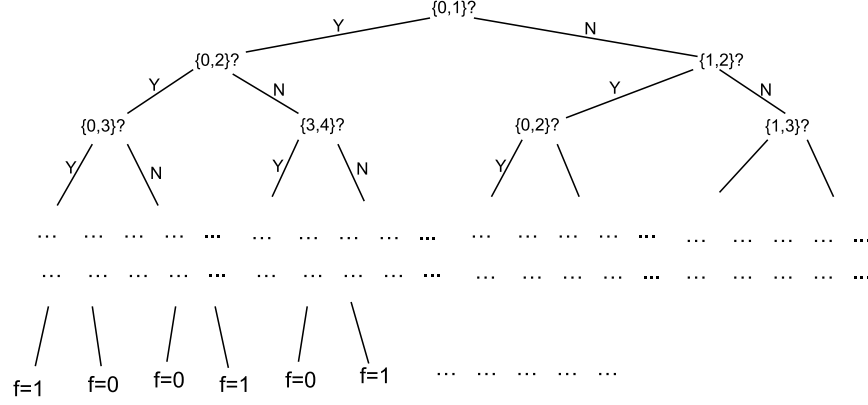


Fig. 1.1 A binary decision tree.

time a pair $(i, j) \in V \times V$ is queried, the adversary answers “yes,” unless the inclusion of that edge would necessarily make the graph Y have a cycle, in which case he answers “no.” After $\binom{n}{2} - 1$ edge-queries by the algorithm have been made, the known edges will form a tree on the elements of V . The algorithm at this point will have no choice but to query the last unknown edge to determine whether or not a cycle exists. We conclude from this argument that h is a graph property that has the maximal decision-tree complexity $\binom{n}{2}$. Such properties are called **evasive**.

A graph property is **monotone** if it is either always preserved by the addition of edges (monotone-increasing) or always preserved by the deletion of edges (monotone-decreasing). In 1973 the following conjecture was made [34].

Conjecture 1.1 (The Karp Conjecture). All nontrivial monotone graph properties are evasive.

To date, this conjecture is unproven and no counterexamples are known. However in 1984, a seminal paper was published by Kahn et al. [18] which proved the conjecture in some cases. This paper showed that evasiveness can be established through the use of topological fixed-point theorems. It has been followed by many more papers which exploited its method to prove better results.

This text is a tutorial on the topological method of [18]. My goal is to provide background on the problem and to take the reader through all of the necessary proofs. Let us begin with some history.

1.1 Background

Research on the decision-tree complexity of graph properties—including properties for both directed and undirected graphs—dates back at least to the early 1970s [4, 5, 15, 16, 21, 29, 34]. Proofs were given in early papers that certain specific graph properties are evasive (e.g., connectedness, containment of a complete subgraph of fixed size), and that other properties at least have decision-tree complexity $\Omega(n^2)$. Although it was known that there are graph properties whose decision-tree complexity is not $\Omega(n^2)$ (see Example 18 in [4]), Aanderdaa and Rosenberg conjectured that all **monotone** graph properties have decision-tree complexity $\Omega(n^2)$ [34]. This conjecture was proved by Rivest and Vuillemin [33] who showed that all monotone graph properties satisfy $D(f) \geq n^2/16$. Kleitman and Kwiatkowski [22] improved this bound to $D(f) \geq n^2/9$.

Underlying some of these proofs is the insight that if a graph property f has nonmaximal decision-tree complexity, then the collection of graphs that satisfy f have some special structure. For example, if f is not evasive, then in the set of graphs satisfying f there must be an equal number of graphs having an odd number of edges and an even number of edges. Rivest and Vuillemin [33] used the fact that if f has decision-tree complexity $\binom{n}{2} - k$, then the weight enumerator of f (i.e., the polynomial $\sum_j c_j t^j$, where c_j is the number of f -graphs containing j edges) must be divisible by $(1 + t)^k$.

A topological method for the evasiveness problem was introduced in [18]. Suppose that h is a monotone-increasing graph property on a vertex set $\{0, 1, \dots, n-1\}$. Let T be the collection of all graphs that do *not* satisfy h . The set T has the property that if G is in T , then all of its subgraphs are in T . This is a close analogy to the property which defines simplicial complexes in topology. Let $\{x_{ab} \mid 0 \leq i < j < n\}$ be a labeled collection of linearly independent vectors in some vector space \mathbb{R}^N . Each graph in T determines a simplex in \mathbb{R}^N : one takes the convex

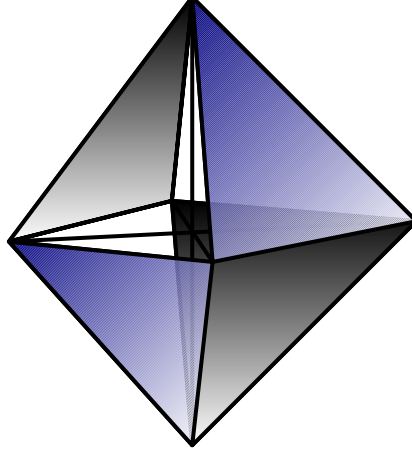


Fig. 1.2 The simplicial complex for “connectedness” on four vertices.

hull of the vectors x_{ab} corresponding to the edges $\{a,b\}$ that are in the graph. The union of these hulls forms a simplicial complex, Γ_h . The complex for “connectedness” on four vertices (represented in three dimensions) is shown in Figure 1.2.

A fundamental insight of [18] is that nonevasiveness can be translated to a topological condition. If h is not evasive, then Γ_h has a certain topological property called **collapsibility**. This property, which we will define formally later in this text, essentially means that Γ_h can be folded into itself and contracted to a single point. This property implies the even–odd weight–balance condition mentioned above, but it is stronger. In particular, it allows for the application of topological fixed-point theorems.

The following theorem is attributed to R. Oliver.

Theorem 1.2 (Oliver [32]). Let Γ be a collapsible simplicial complex. Let G be a finite group which satisfies the following condition:

- (*) There is a normal subgroup $G' \subseteq G$, whose size is a power of a prime, such that G/G' is cyclic.

Then, any action of G on Γ has a fixed point.

When $\Gamma = \Gamma_h$, the fixed points of G correspond to graphs, and this theorem essentially forces the existence of certain graphs that do not satisfy h . This theorem is the basis for the following result of [18]:

Theorem 1.3 (Kahn et al. [18]). Let f be a monotone graph property on graphs of size p^k , where p is prime. If f is not evasive, then it must be trivial.

The proof of this theorem essentially proceeds by demonstrating an appropriate group action G on the set of graphs of order p^k such that the only G -invariant graphs are the empty graph and the complete graph.

Thus evasiveness is known for all values of n that are prime powers. What about other values of n ? One could hope that if the decision-tree complexity is always $\binom{p}{2}$ when the vertex set is size p , then the quantity $\binom{p}{2}$ is a lower bound for the cases $p + 1$, $p + 2$, and so forth. Unfortunately there is no known way to show this. However, all is not lost. The following general theorem is also proved in [18].

Theorem 1.4 (Kahn et al. [18]). Let f be a nontrivial monotone graph property of order n . Then,

$$D(f) \geq \frac{n^2}{4} - o(n^2). \quad (1.2)$$

The paper [18] was then followed by several other papers on evasiveness by other authors who used the topological approach to prove new results on evasiveness [3, 8, 19, 23, 37, 38, 40]. Some of these papers found new group actions $G \curvearrowright \Delta_h$ to exploit in the nonprime cases.

The target results of this exposition are Theorems 1.3 and 1.4, and a theorem by Yao on evasiveness of bipartite graphs [40]. Now let us summarize what we need to do in order to get there.

1.2 Outline of Text

My goal in this exposition is to give a reader who does not know algebraic topology a complete tutorial on topological proofs of evasiveness. Therefore, a fair amount of space will be devoted to building

up concepts from algebraic topology. I have tended to be economical in my discussions and to develop concepts only on an as-needed basis. Readers who wish to learn more algebraic topology after this exposition may want to consult good references such as [14, 30].

We begin, in *Basic Concepts*, by formalizing the class of simplicial complexes and its relation to the class of graph properties. While we have presented a simplicial complex in this introduction as a subset of \mathbb{R}^n , it can also be defined simply as a collection of finite sets. (This is the notion of an **abstract simplicial complex**.) Although the definition in terms of subsets of \mathbb{R}^n is helpful for intuition, the definition in terms of finite sets is the one we will use in all proofs.

A critical construction in this monograph is the set of **homology groups** of a simplicial complex. These groups are algebraic objects which measure the shape of the complex, and also — crucially for our purposes — help us understand the behavior of the complex under automorphisms. *Chain Complexes* defines homology groups and provides some of the standard theory for them.

In *Fixed-Point Theorems* we prove some topological results. The first is the Lefschetz fixed-point theorem. One way to state this theorem is to say that any automorphism of a collapsible simplicial complex has a fixed point. However we instead prove a theorem which applies to the more general class of \mathbb{F}_p -**acyclic** complexes. A simplicial complex is \mathbb{F}_p -acyclic if its homology groups (over \mathbb{F}_p) are trivial. When a simplicial complex is \mathbb{F}_p -acyclic it behaves much like a collapsible complex (and in particular, any automorphism has a fixed point). Finally, we prove a version of Theorem 1.2. The proof of the theorem depends on finding a tower of subgroups

$$\{0\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = G, \quad (1.3)$$

where each quotient G_i/G_{i-1} is cyclic, and performing an inductive argument.

Results on Decision-Tree Complexity proves Theorem 1.3, a bipartite result of Yao [40], and Theorem 1.4. We conclude with an informal discussion of a few of the more recent results on decision-tree complexity of graph properties [3, 8, 19, 23, 37, 38].

My primary sources for this exposition were [10, 18, 30, 35, 40]. A particular debt is owed to Du and Ko [10], which was my first introduction to the subject.

1.3 Related Topics

I will briefly mention two alternative lines of research that are related to the one I cover here. One can change the measure of complexity that one is using to measure graph properties, and this leads to new problems requiring different methods. A natural variant is the **randomized decision-tree complexity**. Suppose that in our decision-tree model, our algorithm is permitted to make random choices at each step about which edges to check. We define the cost of the algorithm on a particular input graph to be the *expected* number of edge queries, and the cost of the algorithm as a whole to be the maximum of this quantity over all input graphs. The minimum of this quantity over all algorithms is the randomized decision-tree complexity, $R(f)$.

There is a line of research studying the randomized decision tree complexity of monotone graph properties [7, 11, 12, 13, 20, 31, 41]. While it is easy to see that $R(f)$ can be less than $\binom{n}{2}$, there are graph properties for which $R(f)$ is provably $\Omega(n^2)$ (such as the “emptiness property”—the property that the graph contains no edges). It is conjectured that $R(f)$ is always $\Omega(n^2)$ for monotone graph properties, just as in the deterministic model. The best proved lower bound [7, 13] is $\Omega(n^{4/3}(\log n)^{1/3})$.

Another variant of decision-tree complexity is **bounded-error quantum query complexity**. A quantum query algorithm for a graph property uses a quantum “oracle” in its computation. The oracle accepts a quantum state which is a superposition of edge-queries to a graph, and it returns a quantum state which encodes the answers to those queries. The algorithm is permitted to use this oracle along with arbitrary quantum operations to determine its result. The algorithm is permitted to make errors, but the likelihood of an error must be below a fixed bound on all inputs. (See [6].)

In the quantum case it is clear that a lower bound of $\Omega(n^2)$ does not hold: Grover’s algorithm [1] can search a space of size N in time

$\Theta(\sqrt{N})$ using an oracle model. With a modified version of Grover's algorithm, one can compute the emptiness property in time $\Theta(n)$. There are a number of other monotone properties for which the quantum query complexity is known to be $o(n^2)$ (see [9] for a good summary on this topic). It is conjectured that all monotone graph properties have quantum query complexity $\Omega(n)$. The best proved lower bound is $\Omega(n^{2/3})$, from an unpublished result attributed to Santha and Yao (see [36]).

1.4 Further Reading

Other expositions about topological proofs of evasiveness can be found in [10] (in the context of computational complexity theory) and [24] (in the context of algebraic topology), and also in Lovasz's lecture notes [26]. A reader who wishes to learn more about algebraic topology can consult [30], or, for a more advanced treatment, [14]. For the particular subject of the topology of complexes arising from graphs, there is an extensive treatment [17], which builds further on many of the concepts that I will discuss here. And finally, for readers who generally enjoy reading about applications of topology to problems in discrete mathematics, the excellent book [28] contains more material of the same flavor. It involves applications of a different topological result (the Borsuk–Ulam theorem) to some problems in elementary mathematics.

2

Basic Concepts

2.1 Graph Properties

This part of the text covers some preliminary material. We begin by formalizing some basic terminology for finite graphs.

For our purposes, a **finite graph** is an ordered pair of sets (V, E) , in which V (the **vertex set**) is a finite set, and E (the **edge set**) is a set of 2-element subsets of V . For example, the pair

$$(\{0, 1, 2, 3\}, \{\{0, 1\}, \{0, 2\}, \{1, 2\}, \{2, 3\}\}) \quad (2.1)$$

is a finite graph with four vertices, diagrammed in Figure 2.1.

An **isomorphism** between two finite graphs is a one-to-one correspondence between the vertices of the two graphs which matches up their edges. In precise terms, if $G = (V, E)$ and $G' = (V', E')$ are two

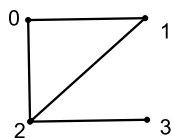


Fig. 2.1 A graph on four vertices.

graphs, then an isomorphism between G and G' is a bijective function $f: V \rightarrow V'$ which is such that the set

$$\{\{f(v), f(w)\} \mid \{v, w\} \in E\} \quad (2.2)$$

is equal to E' . For example, the graph in Figure 2.1 is isomorphic to the graph in Figure 2.2 under the map $f: \{0, 1, 2, 3\} \rightarrow \{0, 1, 2, 3\}$ defined by

$$f(0) = 1 \quad f(1) = 2 \quad (2.3)$$

$$f(2) = 3 \quad f(3) = 0. \quad (2.4)$$

We can now formalize the notion of a graph property. Briefly stated, a graph property is a function on graphs which is compatible with graph isomorphisms. Let V_0 be a finite set, and let $\mathbf{G}(V_0)$ denote the set of all graphs that have V_0 as their vertex set. Then a function

$$h: \mathbf{G}(V_0) \rightarrow \{0, 1\} \quad (2.5)$$

is a **graph property** (over V_0) if all pairs (G, G') of isomorphic graphs in $\mathbf{G}(V_0)$ satisfy $h(G) = h(G')$.

For example, consider the graphs in Figure 2.3, which are members of $\mathbf{G}(\{0, 1, 2\})$. Then the function

$$h_1: \mathbf{G}(\{0, 1, 2\}) \rightarrow \{0, 1\} \quad (2.6)$$

defined by

$$h_1(G) = \begin{cases} 1 & \text{if } G = G_1 \\ 0 & \text{if } G \neq G_1 \end{cases} \quad (2.7)$$

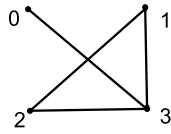


Fig. 2.2 A graph that is isomorphic to the graph in Figure 2.1.

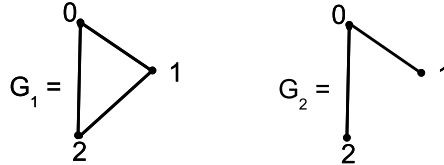


Fig. 2.3 Two graphs of size 3.

is a graph property. However, the function h_2 defined by

$$h_2(G) = \begin{cases} 1 & \text{if } G = G_2 \\ 0 & \text{if } G \neq G_2 \end{cases} \quad (2.8)$$

is *not* a graph property, since there exist graphs in $\mathbf{G}(\{0,1,2\})$ which are isomorphic to G_2 but not equal to G_2 .

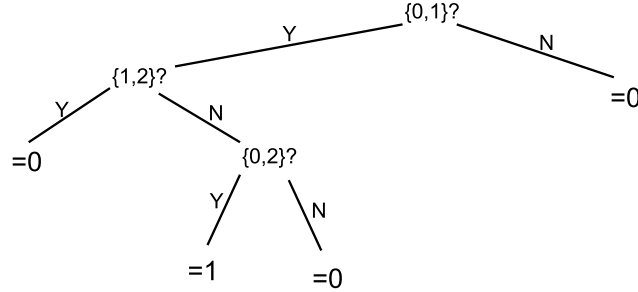
If $G, G' \in \mathbf{G}(V_0)$ are graphs such that the edge set of G' is a subset of the edge set of G , then we say that G' is a **subgraph** of G . Note that this relationship gives us a partial ordering on the set $\mathbf{G}(V_0)$. Let us say that a function $h: \mathbf{G}(V_0) \rightarrow \{0,1\}$ is **monotone increasing** if it respects this ordering. In other words, h is monotone increasing if it satisfies $h(G') \leq h(G)$ for all pairs (G', G) such that G' is a subgraph of G . Likewise, we say that the function h is **monotone decreasing** if it satisfies $h(G') \geq h(G)$ whenever G' is a subgraph of G .

If $h: \mathbf{G}(V_0) \rightarrow \{0,1\}$ is a function, then a **decision tree** for h is a step-by-step procedure for computing the value of h . An example is the decision tree in Figure 2.4, which computes the value of the function h_2 defined above. The diagram in Figure 2.4 describes an algorithm for computing h_2 . Each node in the tree specifies an “edge-query”, and each branch in the tree specifies how the algorithm responds to the results of the edge query. For example, suppose that we wish to apply the algorithm to compute the value of h_2 on the graph G_1 (from (2.1), above). The algorithm would first query the edge $\{0,1\}$, and it would find that this edge *is* contained in G_1 . It would then follow the “Y” branch from $\{0,1\}$, and query the edge $\{1,2\}$. It would then follow the “Y” branch from $\{1,2\}$, and determine that the value of h_2 is zero.

The **decision-tree complexity** of a function $h: \mathbf{G}(V_0) \rightarrow \{0,1\}$ is the smallest possible depth for a decision-tree which correctly computes h . We denote this quantity by $D(h)$. For example, the depth of the decision-tree in Figure 2.4 is 3. It can be shown that any decision-tree that computes h_2 must have depth at least 3. Therefore, $D(h_2) = 3$.

It is easy to prove that for any function $h: \mathbf{G}(V_0) \rightarrow \{0,1\}$, the inequality

$$D(h) \leq \binom{|V_0|}{2} \quad (2.9)$$

Fig. 2.4 A decision tree for the graph property h_2 .

is satisfied. If the function h satisfies

$$D(h) = \binom{|V_0|}{2}, \quad (2.10)$$

then we will say that the function h is **evasive**. Evasive functions are the functions that are the most difficult to compute via a decision-tree.¹

2.2 Simplicial Complexes

Now we give a brief introduction to the notion of a simplicial complex. We draw on [30] for definitions and terminology.

There are at least two natural ways of defining simplicial complexes—one is as a collection of finite sets, and another is as a collection of subsets of \mathbb{R}^n . The first definition is the easiest to work with (and it will be the one we use the most in this monograph). But the second definition is also important because it provides some indispensable geometric intuition. We will begin by building up the second definition.

Definition 2.1. Let N and n be positive integers, with $n \leq N$. Let $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^N$ be vectors satisfying the condition that

$$\{(\mathbf{v}_1 - \mathbf{v}_0), (\mathbf{v}_2 - \mathbf{v}_0), (\mathbf{v}_3 - \mathbf{v}_0), \dots, (\mathbf{v}_n - \mathbf{v}_0)\} \quad (2.11)$$

¹The concepts of “decision-tree complexity” and “evasiveness” can be defined for any Boolean function. See Chapter 5 of [10] for a more detailed treatment.

is linearly independent set. Then the n -simplex spanned by $\{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n\}$ is the set

$$\left\{ \sum_{i=0}^n c_i \mathbf{v}_i \mid 0 \leq c_i \leq 1 \text{ for all } i, \text{ and } \sum_{i=0}^n c_i = 1 \right\}. \quad (2.12)$$

When we refer to an “ n -simplex”, we simply mean a set which can be defined in the above form. Note that a 1-simplex is simply a line segment. A 2-simplex is a solid triangle, and a 3-simplex is a solid tetrahedron.

Definition 2.2. Let N and n be positive integers. Let $v_0, \dots, v_n \in \mathbb{R}^N$ be vectors which span an n -simplex V . Then the **faces** of V are the simplices in \mathbb{R}^N that are spanned by nonempty subsets of $\{v_0, v_1, \dots, v_n\}$.

So, for example, the 2-simplex in \mathbb{R}^3 shown in Figure 2.5 has seven faces (including itself): three of dimension zero, three of dimension 1, and one of dimension two. In general, an n -simplex has $\binom{n+1}{k+1}$ k -dimensional faces.

Definition 2.3. Let N be a positive integer. A **simplicial complex** in \mathbb{R}^N is a set S of simplices in \mathbb{R}^N which satisfies the following two conditions.

- (1) If V is a simplex that is contained in S , then all faces of V are also contained in S .

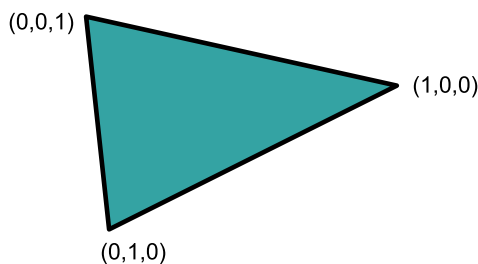


Fig. 2.5 A 2-simplex.

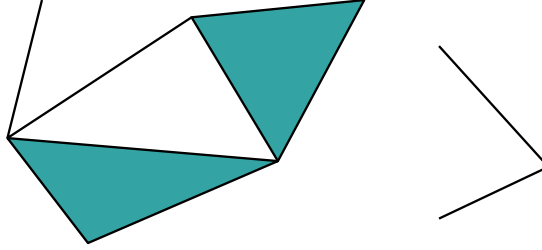


Fig. 2.6 A simplicial complex in \mathbb{R}^2 .

- (2) If V and W are simplices in S such that $V \cap W \neq \emptyset$, then $V \cap W$ is a face of both V and W .

An example of a simplicial complex in \mathbb{R}^2 is shown in Figure 2.6.

Now, as mentioned earlier, there is another definition of simplicial complexes which simply describes them as collections of finite sets. Following [30], we will use the term “abstract simplicial complex” to distinguish this definition from the previous one.

Definition 2.4. An **abstract simplicial complex** is a set Δ of finite nonempty sets which satisfies the following condition:

- If a set Q is an element of Δ , then all nonempty subsets of Q must also be elements of Δ .

Given a simplicial complex S in \mathbb{R}^N , one can obtain an abstract simplicial complex as follows. Let T be the set of all points in \mathbb{R}^N which occur as 0-simplices in S . Let Δ_S be the set of all subsets $T' \subseteq T$ which span simplices that are in S . Then, Δ_S is an abstract simplicial complex. (In a sense, Δ_S records the “gluing information” for the simplicial complex S .)

It is also easy to perform a reverse construction. Suppose that Δ is an abstract simplicial complex. Let

$$U = \bigcup_{Q \in \Delta} Q \quad (2.13)$$

be the union of all of the sets that are contained in Δ . Let $N = |U|$. Simply choose a set $V \subseteq \mathbb{R}^N$ consisting of N linearly independent vectors, and choose a one-to-one map $r: U \rightarrow V$. Every set in Δ determines a simplex in \mathbb{R}^N (via r), and the collection of all of these simplices is a simplicial complex.

We define some terminology for abstract simplicial complexes.

Definition 2.5. Let Δ be an abstract simplicial complex. Then,

- A **simplex in Δ** is simply an element of Δ . The **dimension** of a simplex $Q \in \Delta$, denoted $\dim(Q)$, is the quantity $(|Q| - 1)$. An **n -simplex** in Δ is an element of Δ of dimension n .
- If $Q, Q' \in \Delta$ and $Q' \subseteq Q$, then we say that Q' is a **face** of Q .
- The **vertex set of Δ** is the set

$$\bigcup_{Q \in \Delta} Q. \quad (2.14)$$

Elements of this set are called **vertices of Δ** .

Here is an initial example of how abstract simplicial complexes arise. Let F be a finite set. Let $\mathcal{P}(F)$ denote the power set of F . Let $t: \mathcal{P}(F) \rightarrow \{0, 1\}$ be a function which is “monotone increasing,” in the sense that any pair of sets (A, B) such that $A \subseteq B \subseteq F$ satisfies $t(A) \leq t(B)$. Then, the set

$$\{C \mid \emptyset \subset C \subseteq F \text{ and } t(C) = 0\} \quad (2.15)$$

is an abstract simplicial complex.

Thus, a monotone increasing function on a power set determines an abstract simplicial complex. This connection is the basis for what we will discuss next.

2.3 Monotone Graph Properties

Now we will establish a relationship between monotone graph properties and simplicial complexes. We also introduce a topological concept (“collapsibility”) which has an important role in this relationship.

Let V_0 be a finite set. Using notation from *Graph Properties*, let $\mathbf{G}(V_0)$ denote the set of all graphs that have vertex set V_0 . The elements of $\mathbf{G}(V_0)$ are thus pairs of the form (V_0, E) , where E can be any subset of the set

$$\{\{v, w\} \mid v, w \in V_0\}. \quad (2.16)$$

Let $h: \mathbf{G}(V_0) \rightarrow \{0, 1\}$ be a monotone increasing function. Then the **abstract simplicial complex associated with h** , denoted Δ_h , is the set of all nonempty subsets E of set (2.16) such that

$$h((V_0, E)) = 0. \quad (2.17)$$

Example 2.1. Consider the set $\mathbf{G}(\{0, 1, 2, 3\})$ of graphs on the vertex set $\{0, 1, 2, 3\}$. Define functions

$$h_1: \mathbf{G}(\{0, 1, 2, 3\}) \rightarrow \{0, 1\}, \quad (2.18)$$

$$h_2: \mathbf{G}(\{0, 1, 2, 3\}) \rightarrow \{0, 1\} \quad (2.19)$$

by

$$h_1(G) = \begin{cases} 1 & \text{if } G \text{ has at least three edges,} \\ 0 & \text{otherwise} \end{cases} \quad (2.20)$$

and

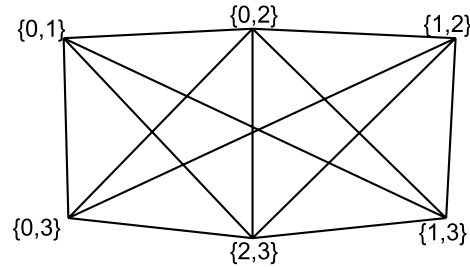
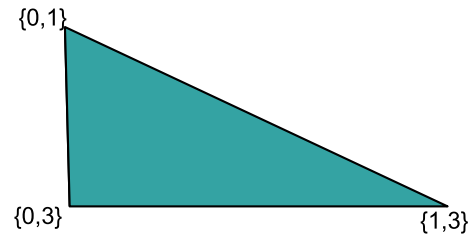
$$h_2(G) = \begin{cases} 1 & \text{if vertex "2" has at least one edge in } G, \\ 0 & \text{otherwise.} \end{cases} \quad (2.21)$$

Then the simplicial complexes for h_1 and h_2 are shown in Figures 2.7 and 2.8.²

Thus we have a way of associating with any monotone-increasing graph function

$$h: \mathbf{G}(V_0) \rightarrow \{0, 1\} \quad (2.22)$$

²Note: Ignore the apparent intersections in the interior of the diagram for h_1 . Imagine that the lines in the diagram only intersect at the labeled points $\{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 3\}, \{2, 3\}$, and $\{1, 3\}$. (To really draw this diagram accurately, we would need three dimensions.)

Fig. 2.7 The simplicial complex of h_1 .Fig. 2.8 The simplicial complex of h_2 .

an abstract simplicial complex Δ_h . The simplices of Δ_h correspond to graphs on V_0 . The vertices of Δ_h correspond to *edges* (not vertices!) of graphs on V_0 .

The association $[h \mapsto \Delta_h]$ is useful because it allows us to reinterpret statements about graph functions in terms of simplicial complexes. What we will do now is to prove a theorem (for later use) which exploits this association. The theorem relates a condition on graph functions (“evasiveness,” from *Graph Properties*) to a condition on simplicial complexes (“collapsibility”).

We begin with some definitions.

Definition 2.6. Let Δ be an abstract simplicial complex, and let $\alpha \in \Delta$ be a simplex. Then Δ is a **maximal** simplex if it is not contained in any other simplex in Δ .

Definition 2.7. Let Δ be an abstract simplicial complex, and let $\beta \in \Delta$ be a simplex. Then β is called a **free face** of Δ if it is nonmaximal

and it is contained in only one maximal simplex in Δ . If β is a free face and α is the unique maximal simplex that contains it, then we will say that β is a **free face of α** .

Definition 2.8. An **elementary collapse** of an abstract simplicial complex is the operation of choosing a single free face from the complex and deleting the face along with all the faces that contain it.

Here is an example of an elementary collapse: if

$$\Sigma_1 = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}, \quad (2.23)$$

then $\{0,1\}$ is a free face of $\{0,1,2\}$ in Δ . By deleting the simplices $\{0,1\}$ and $\{0,1,2\}$, we obtain the complex

$$\Sigma_2 = \{\{0\}, \{1\}, \{2\}, \{0,2\}, \{1,2\}\}. \quad (2.24)$$

The complex Σ_2 is an elementary collapse of the complex Σ_1 . See Figure 2.9.

The previous example is an instance of what we will call a **primitive** elementary collapse. An elementary collapse is primitive if the free face that is deleted has dimension one less than the maximal simplex in which it is contained. In such a case, the maximal simplex and free face itself are the only two simplices that are deleted. (Not all elementary collapses are primitive. An example of a nonprimitive elementary collapse would be deleting all of the simplices $\{0\}$, $\{0,1\}$, $\{0,2\}$, and $\{0,1,2\}$ from Σ_1 .)

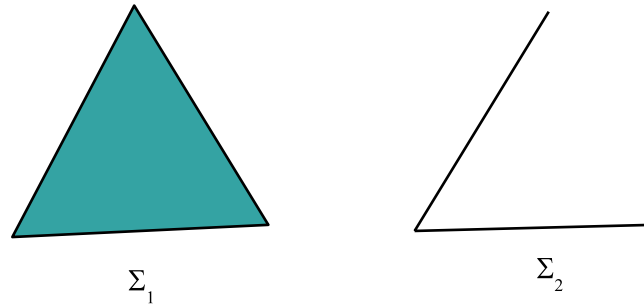


Fig. 2.9 An elementary collapse.

Definition 2.9. Let Δ be an abstract simplicial complex. Then Δ is **collapsible** if there exists a sequence of elementary collapses

$$\Delta, \Delta_1, \Delta_2, \Delta_3, \dots, \Delta_n \quad (2.25)$$

such that $|\Delta_n| = 1$.

In other words, Δ is collapsible if there exists a sequence of elementary collapses which reduce Δ to a single 0-simplex.

The abstract simplicial complexes Σ_1 and Σ_2 defined above are both collapsible. An example of an abstract simplicial complex that is not collapsible is the following:

$$\Sigma = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\}. \quad (2.26)$$

(This simplicial complex has no free faces, and therefore cannot be collapsed.)

The following theorem asserts that the simplicial complexes associated with certain monotone-increasing graph functions are collapsible. The theorem uses the concept of “evasiveness” from *Graph Properties*.

Theorem 2.1. Let V_0 be a finite set. Let

$$h: \mathbf{G}(V_0) \rightarrow \{0, 1\} \quad (2.27)$$

be a monotone-increasing function which is not evasive. If the complex Δ_h is not empty, then it is collapsible.

Proof. The theorem has an elegant visual proof. Essentially, what we do is to construct a decision-tree for h and then read off a collapsing-procedure for Δ_h from the decision-tree.³

Let $n = |V_0|$. Since we have assumed that the function h is not evasive, there must exist a decision tree of depth smaller than $n(n-1)/2$ which decides h . Let T be such a tree. (See Figure 2.10.) By modifying T if necessary, we can produce another decision-tree T' which decides h and which satisfies the following conditions. (See Figure 2.11.)

³Thanks to Yaoyun Shi, who suggested the nice visualization that appears in this proof.

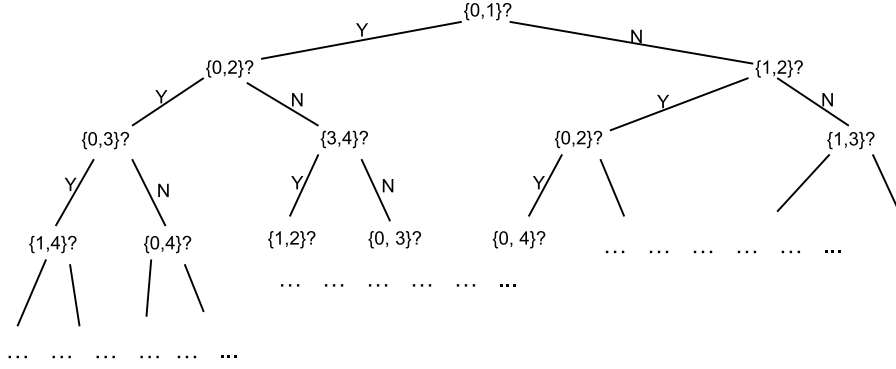


Fig. 2.10 A decision tree.

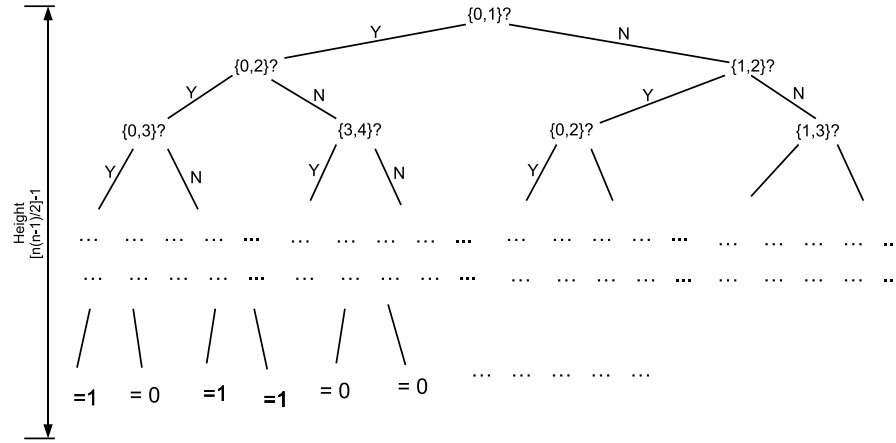


Fig. 2.11 A decision tree of uniform height.

- The paths in T' do not have repeated edges. (That is, no edge $\{i, j\}$ appears more than once on any path in T' .)
- Every path in T' has length exactly $[n(n-1)/2 - 1]$.

We can define a natural total ordering on the leaves of tree T' . The ordering is defined by asserting that for any parent-node in the tree, all leaves that can be reached through the “Y” branch of the node are smaller than all the leaves that can be reached through the “N” branch of the node. Since any two leaves share a common ancestor, this rule gives a total ordering.

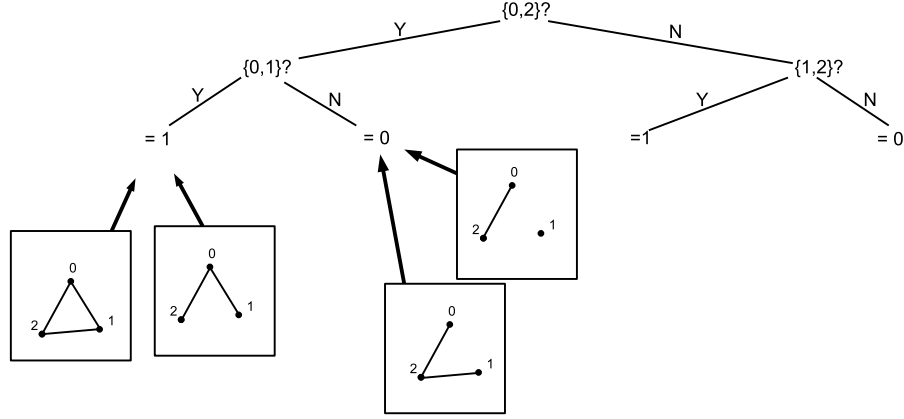


Fig. 2.12 A decision tree of height 2 for graphs of size 3.

For any leaf of tree T' , there are exactly two graphs which would cause the leaf to be reached during computation. Thus there is a one-to-two correspondence between leaves of T' and graphs on V_0 . An example is shown in Figure 2.12. Note that each leaf is labeled with either with a “1” or a “0”, depending on the value taken by the function h at the corresponding graphs. The simplicial complex Δ_h is composed out of the graphs that appear at the “0”-leaves of the tree.

The ordering of the leaves of T' provides a recipe for collapsing Δ_h . Simply find the smallest (i.e., leftmost) “0”-leaf that appears in tree T' . This leaf corresponds to a pair of simplices $\gamma_1, \gamma_2 \in \Delta_h$ with $\gamma_1 \subseteq \gamma_2$. From the ordering of the leaves, we can deduce that γ_1 and γ_2 are not contained in any simplices in Δ_h other than themselves. Thus γ_1 is a free face of Δ_h . We can therefore perform an elementary collapse: let

$$\Delta_1 = \Delta_h \setminus \{\gamma_1, \gamma_2\}. \quad (2.28)$$

Now find the second smallest 0-leaf that appears in T' . This leaf corresponds to another pair of simplices $\gamma'_1, \gamma'_2 \in \Delta_h$ which are not contained in any other simplices in Δ_h , except possibly γ_1 or γ_2 . Perform another elementary collapse:

$$\Delta_2 = \Delta_1 \setminus \{\gamma'_1, \gamma'_2\}. \quad (2.29)$$

Continuing in this manner, we can obtain a sequence of elementary collapses

$$\Delta_h, \Delta_1, \Delta_2, \Delta_3, \dots, \Delta_n \quad (2.30)$$

such that $|\Delta_n| = 1$. Therefore, Δ_h is collapsible. \square

2.4 Group Actions on Simplicial Complexes

Now we define the notion of a **simplicial isomorphism** between abstract simplicial complexes. This is a case of the more general notion of a simplicial map (see [30]).

Definition 2.10. Let Δ and Δ' be abstract simplicial complexes. A simplicial isomorphism from Δ to Δ' is a bijective map

$$f: \Delta \rightarrow \Delta' \quad (2.31)$$

which is such that for any $Q_1, Q_2 \in \Delta$,

$$Q_1 \subseteq Q_2 \iff f(Q_1) \subseteq f(Q_2). \quad (2.32)$$

In other words, a simplicial isomorphism between two abstract complexes Δ, Δ' is a one-to-one matching f between the simplices of Δ and Δ' which respects inclusion. We note the following assertions, which can be proven easily from this definition:

- If $f: \Delta \rightarrow \Delta'$ is a simplicial isomorphism, then f respects dimension (i.e., if $Q \in \Delta$ is an n -simplex, then $f(Q)$ must be an n -simplex).
- If $f: \Delta \rightarrow \Delta'$ is a simplicial isomorphism, then there is an associated map of vertex sets

$$\hat{f}: \bigcup_{Q \in \Delta} Q \rightarrow \bigcup_{Q' \in \Delta'} Q' \quad (2.33)$$

defined by $f(\{v\}) = \{\hat{f}(v)\}$. (Let us call this the **vertex map** of f .) The map \hat{f} uniquely determines f .

Let Δ be an abstract simplicial complex. A simplicial automorphism of Δ can be specified either as an inclusion preserving permutation of the elements of Δ , or simply as a permutation

$$b: \bigcup_{Q \in \Delta} Q \rightarrow \bigcup_{Q \in \Delta} Q \quad (2.34)$$

of the vertex set of Δ satisfying

$$Q \in \Delta \implies b(Q) \in \Delta. \quad (2.35)$$

When we speak of a **group action** $G \curvearrowright \Delta$, we mean an action of a group G on Δ by simplicial automorphisms.

In *Fixed-Point Theorems* we will be concerned with determining the “fixed points” of a group action on an abstract simplicial complex. As we will see, describing this set requires some care. One could simply take the set Δ^G of G -invariant simplices. But this set is not always subcomplex of Δ . Consider the two-dimensional complex Σ in Figure 2.13, which consists of the sets $\{0, 1, 2\}$ and $\{0, 2, 3\}$ and all of their proper nonempty subsets. If we let $f: \Sigma \rightarrow \Sigma$ be the simplicial automorphism which transposes $\{1\}$ and $\{3\}$ and leaves $\{0\}$ and $\{2\}$ fixed, then Σ^f is a subcomplex of Σ . However, if we let $h: \Sigma \rightarrow \Sigma$ be the simplicial automorphism which transposes $\{0\}$ and $\{2\}$ and leaves $\{1\}$ and $\{3\}$ fixed, then Σ^h is not a subcomplex of Σ , since it contains the set $\{0, 2\}$ but does not contain its subsets $\{0\}$ and $\{2\}$.

It is helpful to look at group actions on abstract simplicial complexes in terms of the geometric representation introduced in *Simplicial Complexes*. Let $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis vectors in \mathbb{R}^{n+1} .

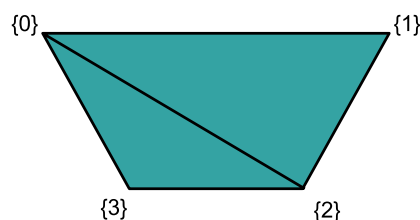


Fig. 2.13 The complex Σ .

These vectors span an n -simplex

$$\delta = \left\{ \sum_{i=0}^n c_i \mathbf{v}_i \mid 0 \leq c_i \leq 1, \sum_{i=0}^n c_i = 1 \right\}. \quad (2.36)$$

If $f: \{0, 1, \dots, n\} \rightarrow \{0, 1, \dots, n\}$ is a permutation with orbits $B_1, \dots, B_m \subseteq \{0, 1, \dots, n\}$, then f induces a bijective map on δ . The invariant set δ^f consists of those linear combinations $\sum c_i \mathbf{v}_i$ satisfying the condition that $c_i = c_j$ whenever i and j lie in the same orbit. The set δ^f is an $(m-1)$ -simplex which is spanned by the vectors

$$\left\{ \frac{\sum_{i \in B_k} \mathbf{v}_i}{|B_k|} \mid k = 1, 2, \dots, m \right\}. \quad (2.37)$$

This motivates the following definition.

Definition 2.11. Let Δ be a finite abstract simplicial complex with vertex set V , and let $G \curvearrowright \Delta$ be a group action. Let $A_1, \dots, A_m \subseteq V$ denote the orbits of the action of G on V . Then, let $\Delta^{[G]}$ denote the set of all subsets $T \subseteq \{A_1, \dots, A_m\}$ satisfying

$$\bigcup_{S \in T} S \in \Delta. \quad (2.38)$$

It is easy to see that the set $\Delta^{[G]}$ is always a simplicial complex. In the case of the complex Σ from Figure 2.13, if we let H be the group generated by the automorphism h which transposes $\{0\}$ and $\{2\}$, the complex $\Sigma^{[H]}$ is one-dimensional and consists of three zero simplices and two one-simplices. (See Figure 2.14.) The vertices of $\Sigma^{[H]}$ are the orbits $\{1\}$, $\{3\}$, and $\{0, 2\}$.

This complex $\Delta^{[G]}$ will be important in *Fixed-Point Theorems*.

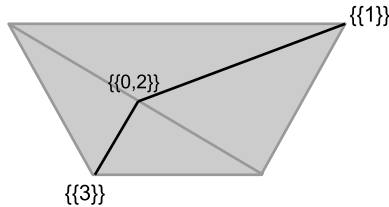


Fig. 2.14 The complex $\Sigma^{[H]}$.

3

Chain Complexes

In this part of the text we will introduce some algebraic objects which are crucial for measuring the behavior of simplicial complexes. The central objects of concern are **chain complexes** and **homology groups**. We will define these objects and develop some important tools for dealing with them.

3.1 Definition of Chain Complexes

A **complex of abelian groups** is a sequence of abelian groups

$$Z_0, Z_1, Z_2, \dots \quad (3.1)$$

together with group homomorphisms $d_i: Z_i \rightarrow Z_{i-1}$ for each $i > 0$, satisfying the condition

$$d_{i-1} \circ d_i = 0 \quad (3.2)$$

(or equivalently, $\text{im } d_i \subseteq \ker d_{i-1}$). The groups Z_i and the maps d_i are often expressed in a diagram like so:

$$\cdots \longrightarrow Z_3 \xrightarrow{d_3} Z_2 \xrightarrow{d_2} Z_1 \xrightarrow{d_1} Z_0 \quad (3.3)$$

We abbreviate the complex as Z_\bullet .

A chain complex is a particular complex of abelian groups that is obtained from a simplicial complex. The definition of chain complex that we will use requires first choosing a total ordering of the vertices of the abstract simplicial complex in question. If the vertices of the abstract simplicial complex happen to be elements of a totally ordered set (such as the set of integers), then our choice is already made for us. Otherwise, it is necessary before applying our definition to specify what ordering of vertices we are using. The particular choice of ordering is not terribly important, but it must be made consistently.

We introduce some new notation which takes this ordering issue into account.

Notation 3.1. Let V be a totally ordered set, and let Δ be an abstract simplicial complex whose vertices are all elements of V . For any sequence of distinct elements $v_0, v_1, \dots, v_n \in V$ such that

$$\{v_0, \dots, v_n\} \in \Delta \quad (3.4)$$

and

$$v_0 < v_1 < v_2 < \dots < v_n, \quad (3.5)$$

let

$$[v_0, v_1, \dots, v_n] \quad (3.6)$$

denote the n -simplex $\{v_0, \dots, v_n\}$ in Δ .

This notation allows us to cleanly handle the ordering on the vertices of an abstract simplicial complex. Note that if we say, “ $[v_0, v_1, \dots, v_n]$ is a simplex in Δ ”, we are implying both that $\{v_0, \dots, v_n\}$ is an element of Δ *and* that the sequence v_0, v_1, \dots, v_n is in ascending order.

Now we will define the sequence of groups which make up a chain complex.

Definition 3.1. Let V be a totally ordered set, and let Δ be an abstract simplicial complex whose vertices are elements of V . Let n be a nonnegative integer. Then, the **n th chain group of V over**

\mathbb{R} , denoted $K_n(\Delta, \mathbb{R})$, is the set of all formal \mathbb{R} -linear combinations of n -simplices in Δ .

Example 3.1. Let Σ be the simplicial complex

$$\Sigma = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}\}. \quad (3.7)$$

Then, Σ has three zero-simplices ($[0]$, $[1]$, and $[2]$) and three one-simplices ($[0, 1]$, $[1, 2]$, and $[0, 2]$). The chain group $K_0(\Sigma, \mathbb{R})$ is a three-dimensional real vector space, and its elements can be expressed in the form

$$r_1[0] + r_2[1] + r_3[2], \quad (3.8)$$

where r_1 , r_2 , and r_3 denote real numbers. The chain group $K_1(\Sigma, \mathbb{R})$ is a three-dimensional real vector space, and its elements can be expressed in the form

$$r_4[0, 1] + r_5[1, 2] + r_6[0, 2], \quad (3.9)$$

where r_4 , r_5 , and r_6 denote real numbers.

In general, if Δ is an abstract simplicial complex, then $K_n(\Delta, \mathbb{R})$ is a real vector space whose dimension is equal to the number n -simplices in Δ . (If Δ has no n -simplices, then $K_n(\Delta, \mathbb{R})$ is a zero vector space.)

Definition 3.2. Let V be a totally ordered set, and let Δ be an abstract simplicial complex whose vertices are elements of V . Let n be a positive integer. Then the **boundary map** on the n th chain group of Δ (over \mathbb{R}) is the unique \mathbb{R} -linear homomorphism

$$d_n: K_n(\Delta, \mathbb{R}) \rightarrow K_{n-1}(\Delta, \mathbb{R}) \quad (3.10)$$

defined by the equations

$$d_n([v_0, v_1, \dots, v_n]) = \sum_{i=0}^n (-1)^i [v_0, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n] \quad (3.11)$$

(where $[v_0, v_1, \dots, v_n]$ can be taken to be any n -simplex in Δ .)

Example 3.2. Let

$$\Sigma' = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, \{0, 1, 2\}\}. \quad (3.12)$$

Then the boundary map

$$d_2: K_2(\Sigma', \mathbb{R}) \rightarrow K_1(\Sigma', \mathbb{R}) \quad (3.13)$$

is defined by the equation

$$d_2([0, 1, 2]) = [1, 2] - [0, 2] + [0, 1]. \quad (3.14)$$

The boundary map

$$d_1: K_1(\Sigma', \mathbb{R}) \rightarrow K_0(\Sigma', \mathbb{R}) \quad (3.15)$$

is defined by the equations

$$d_1([0, 1]) = [0] - [1] \quad (3.16)$$

$$d_1([0, 2]) = [0] - [2] \quad (3.17)$$

$$d_1([1, 2]) = [1] - [2]. \quad (3.18)$$

Note that in equation (3.11), the simplicies that appear on the right side are precisely the $(n - 1)$ -simplex faces of the simplex $[v_0, v_1, \dots, v_n]$. Geometrically, if $U \subseteq \mathbb{R}^N$ is an n -simplex, then the codimension-1 faces of U make up the boundary (or exterior) of the set U . This gives us an idea of why d_n is called a “boundary” map.

Proposition 3.2. Let Δ be an abstract simplicial complex whose vertices are totally ordered. Let n be an integer such that $n \geq 2$. Then the map

$$d_{n-1} \circ d_n: K_n(\Delta, \mathbb{R}) \rightarrow K_{n-2}(\Delta, \mathbb{R}) \quad (3.19)$$

is the zero map.

Proof. Let $Q = [v_0, v_1, \dots, v_n]$ be an n -simplex in Δ . Then, applying Definition (3.2) twice, we find

$$\begin{aligned} d_{n-1}(d_n(Q)) &= \sum_{i=0}^n d_{n-1}((-1)^i[v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]) \\ &= \sum_{i=0}^n \left(\sum_{j=0}^{i-1} (-1)^{i+j}[v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_{i-1}, v_{i+1}, \dots, v_n] \right. \\ &\quad \left. + \sum_{j=i+1}^n (-1)^{i+j-1}[v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_{j+1}, \dots, v_n] \right). \end{aligned}$$

All terms in this double-summation cancel, and thus we find that

$$d_{n-1}(d_n(Q)) = 0. \quad (3.20)$$

Therefore by linearity, $d_{n-1} \circ d_n$ is the zero map. \square

If Δ is an abstract simplicial complex with ordered vertices, then the **chain complex of Δ over \mathbb{R}** is the set of \mathbb{R} -chain groups of Δ together with their boundary maps:

$$\dots \longrightarrow K_2(\Delta, \mathbb{R}) \xrightarrow{d_2} K_1(\Delta, \mathbb{R}) \xrightarrow{d_1} K_0(\Delta, \mathbb{R}) \xrightarrow{d_0} 0 \quad (3.21)$$

For any n , the **n th homology group** of Δ is defined by

$$H_n(\Delta, \mathbb{R}) = (\ker d_n) / (\text{im } d_{n+1}), \quad (3.22)$$

Consider the complex Σ from Example 3.1. The kernel of d_0 is the entire space $K_0(\Delta, \mathbb{R})$, while the image of d_1 is the set of all linear combinations $r_1[0] + r_2[1] + r_3[2]$ which are such that $r_1 + r_2 + r_3 = 0$. The quotient $H_0(\Delta, \mathbb{R}) = \ker d_0 / \text{im } d_1$ is a one-dimensional real vector space. The homology group $H_1(\Delta, \mathbb{R}) = \ker d_1 / \{0\}$ is also a one-dimensional real vector space, spanned by the element $[0, 1] - [0, 2] + [1, 2]$. All other homology groups of Σ are zero-dimensional.

As we will see in *Picturing Homology Groups*, the homology groups are interesting because they supply structural information about the complex Δ . As an initial example, the reader is invited to prove the following fact as an exercise: for any finite abstract simplicial

complex Δ , the dimension of $H_0(\Delta, \mathbb{R})$ is equal to the number of connected components of Δ .

Although we defined chain groups using \mathbb{R} (the set of real numbers), it is possible to define them using other algebraic structures in place of \mathbb{R} . Here is a definition for chain groups over \mathbb{F}_p . Proposition 3.2 and the definition of homology groups carry over immediately to this case.

Definition 3.3. Let V be a totally ordered set, and let Δ be an abstract simplicial complex whose vertices are elements of V . Then $K_n(\Delta, \mathbb{F}_p)$ denotes the vector space of formal \mathbb{F}_p -linear combinations of n -simplices in V . For each $n \geq 1$, the map

$$d_n: K_n(\Delta, \mathbb{F}_p) \rightarrow K_{n-1}(\Delta, \mathbb{F}_p) \quad (3.23)$$

is the unique \mathbb{F}_p -linear map defined by

$$d_n([v_0, v_1, \dots, v_n]) = \sum_{i=0}^n (-1)^i [v_0, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n]. \quad (3.24)$$

For the rest of this exposition we will be focusing on homology groups with coefficients in \mathbb{F}_p , since these will eventually be the basis for our proofs of fixed-point theorems. Much of what we will do in this text with \mathbb{F}_p -homology could be done just as well with \mathbb{R} -homology, but there will be a key result (Proposition 4.4) which depends critically on the fact that we are using coefficients in \mathbb{F}_p .

3.2 Chain Complexes and Simplicial Isomorphisms

Suppose that

$$\dots \longrightarrow I_{n+1} \xrightarrow{d_{n+1}} I_n \xrightarrow{d_n} I_{n-1} \xrightarrow{d_{n-1}} \dots \quad (3.25)$$

and

$$\dots \longrightarrow J_{n+1} \xrightarrow{d_{n+1}} J_n \xrightarrow{d_n} J_{n-1} \xrightarrow{d_{n-1}} \dots \quad (3.26)$$

are two complexes of abelian groups. A **map of complexes** $F: I_\bullet \rightarrow J_\bullet$ is a family of homomorphisms

$$F_n: I_n \rightarrow J_n \quad (3.27)$$

such that

$$d_n \circ F_n = F_{n-1} \circ d_n. \quad (3.28)$$

Note that, as a consequence of this rule, the map F_n must send the kernel of d_n^I to the kernel of d_n^J . Moreover, the family F induces maps on homology groups

$$H_n(I_\bullet) \rightarrow H_n(J_\bullet). \quad (3.29)$$

for every n .

Let p be a prime. We are going to define the maps of chain complexes that are associated with simplicial isomorphisms. Some care must be taken in this definition. Let $f: \Delta \rightarrow \Delta'$ be a simplicial isomorphism. An obvious way to map $K_n(\Delta, \mathbb{F}_p)$ to $K_n(\Delta', \mathbb{F}_p)$ would be to naively apply f like so: $\sum c_i Q_i \mapsto \sum c_i f(Q_i)$. However, this definition does not necessarily give a map of complexes, because it is not necessarily compatible with the maps d_i . The reader will recall that the definition of d_i depends on the ordering of the vertices of the simplicial complex in question. The map f may not be compatible with the ordering of the vertices of Δ and Δ' . In our definition of the maps $K_n(\Delta, \mathbb{F}_p) \rightarrow K_n(\Delta', \mathbb{F}_p)$, we need to take this ordering issue into account.

Note that for any bijection $g: S_1 \rightarrow S_2$ between two totally ordered sets S_1 and S_2 , there is a unique permutation $\alpha: S_2 \rightarrow S_2$ which makes the composition $\alpha \circ g$ an order-preserving map. Let us say that the **sign** of the map g is the sign of its associated permutation α .¹

Definition 3.4. Suppose that Δ and Δ' are abstract simplicial complexes whose vertex sets are totally ordered. Suppose that $f: \Delta \rightarrow \Delta'$ is a simplicial isomorphism and that \hat{f} is its vertex map. Let p be a prime, and let n be a nonnegative integer. The **n th chain map associated with f** (over \mathbb{F}_p) is the unique \mathbb{F}_p -linear map

$$F_n: K_n(\Delta, \mathbb{F}_p) \rightarrow K_n(\Delta', \mathbb{F}_p) \quad (3.30)$$

¹ See [25], pp. 30–31 for a definition of the sign of a permutation. Briefly: if $\sigma: X \rightarrow X$ is a permutation of a finite set X , then we can write $\sigma = \tau_1 \circ \tau_2 \circ \dots \circ \tau_m$ for some m , where each of the maps $\tau_i: X \rightarrow X$ is a permutation which transposes two elements. The sign of σ is $(-1)^m$.

given by

$$Q \mapsto (\text{sign } (\hat{f}|_Q))f(Q). \quad (3.31)$$

for all $Q \in \Delta$. Here, $(\text{sign } (\hat{f}|_Q))$ denotes the sign of the bijection $(\hat{f})|_Q: Q \rightarrow f(Q)$.

Let Σ' be the complex from Example 3.2, and let $g: \Sigma' \rightarrow \Sigma'$ be the automorphism given by the permutation $[0 \mapsto 1, 1 \mapsto 2, 2 \mapsto 0]$. Then the chain maps G_n associated with g are as shown below.

$$\begin{array}{lll} G_0([0]) = [1] & G_1([0, 1]) = [1, 2] & \\ G_0([1]) = [2] & G_1([1, 2]) = -[0, 2] & G_2([0, 1, 2]) = [0, 1, 2] \\ G_0([2]) = [0] & G_1([0, 2]) = -[0, 1] & \end{array}$$

Proposition 3.3. The chain maps F_n of Definition 3.4 determine a map of complexes,

$$F: K_\bullet(\Delta, \mathbb{F}_p) \rightarrow K_\bullet(\Delta', \mathbb{F}_p). \quad (3.32)$$

Proof. It suffices to show that for any $n > 0$, and any n -simplex $Q \in \Delta$,

$$d_n(F_n(Q)) = F_n(d_n(Q)). \quad (3.33)$$

Let n be a positive integer, and let $Q \in \Delta$ be an n -simplex. Write the simplices Q and $f(Q)$ as

$$Q = [v_0, v_1, \dots, v_n], \quad f(Q) = [w_0, w_1, \dots, w_n]. \quad (3.34)$$

(Here, as usual, we assume that the sequences v_0, \dots, v_n and w_0, \dots, w_n are in ascending order.) The elements $d_n(F_n(Q))$ and $F_n(d_n(Q))$ are linear combinations of faces of the simplex $[w_0, \dots, w_n]$. We need simply to show that the coefficients in the expressions for $d_n(F_n(Q))$ and $F_n(d_n(Q))$ are the same.

Suppose that the face

$$[v_0, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n] \quad (3.35)$$

of Q maps to the face

$$[w_0, w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_n] \quad (3.36)$$

under f . Then, by applying the definitions of d_n and F_n we find that the coefficient of $[w_0, w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_n]$ in $d_n(F_n(Q))$ is

$$(-1)^j (\text{sign } \hat{f}|_Q), \quad (3.37)$$

whereas the coefficient of $[w_0, w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_n]$ in $F_n(d_n(Q))$ is

$$(\text{sign } \hat{f}|_{\{v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}}) (-1)^i. \quad (3.38)$$

It is a fact (easily proven from the definition of sign) that

$$(\text{sign } \hat{f}|_{\{v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}}) = (-1)^{j-i} (\text{sign } \hat{f}|_Q). \quad (3.39)$$

Therefore quantities (3.37) and (3.38) are equal. So the coefficients of $[w_0, w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_n]$ in $d_n(F_n(Q))$ and $F_n(d_n(Q))$ are the same. This reasoning can be repeated to show that all of the coefficients in $d_n(F_n(Q))$ and $F_n(d_n(Q))$ are the same. \square

We have proven that if $f: \Delta \rightarrow \Delta'$ is a simplicial isomorphism, then there is induced chain map (in fact, an isomorphism),

$$F: K_\bullet(\Delta, \mathbb{F}_p) \rightarrow K_\bullet(\Delta', \mathbb{F}_p). \quad (3.40)$$

This chain map induces vector space isomorphisms

$$H_n(\Delta, \mathbb{F}_p) \rightarrow H_n(\Delta', \mathbb{F}_p) \quad (3.41)$$

for every $n \geq 0$. (We may denote these maps using the same symbol, F .)

3.3 Picturing Homology Groups

Before continuing any further with our technical discussion of chain complexes, let us take a moment to explore some geometric interpretations for the concepts introduced so far. For convenience, we will assume in the following discussion that p is a prime greater than or equal to 5.

Consider the two-dimensional simplicial complex Γ shown in Figure 3.1. If $[v_0, v_1]$ is a 1-simplex (where we assume the existence of an ordering under which $v_0 < v_1$), then let us represent the chain element $[v_0, v_1] \in K_1(\Gamma, \mathbb{F}_p)$ by drawing an arrow from v_0 to v_1 , and let us represent the negation $-[v_0, v_1] \in K_1(\Gamma, \mathbb{F}_p)$ by drawing an arrow from v_1 to v_0 . We can likewise use double-headed arrows to represent

the elements $2[v_0, v_1]$ and $-2[v_0, v_1]$. Sums of such elements can be represented as collections of arrows. In this way we can draw some of the elements of $K_1(\Gamma, \mathbb{F}_p)$ as diagrams like the one in Figure 3.1.

An element $c \in K_1(\Gamma, \mathbb{F}_p)$ that is represented in this way will satisfy $dc = 0$ if and only if for every vertex v of Γ , the total multiplicity of incoming arrows at v is the same, mod p , as the total multiplicity of the outgoing arrows at v . The element a represented in Figure 3.1 is such a case.

Each element $c \in K_1(\Gamma, \mathbb{F}_p)$ satisfying $dc = 0$ represents an element of the quotient $H_1(\Gamma, \mathbb{F}_p) = \ker d_1 / \text{im } d_2$, and thus we can use this geometric interpretation to understand $H_1(\Gamma, \mathbb{F}_p)$. Note that, although there are many diagrams that we could draw which satisfy the balanced-multiplicity condition mentioned above, it will often occur that two diagrams represent the same element of $H_1(\Gamma, \mathbb{F}_p)$. Figure 3.2 gives an example. In fact, any two elements $u, v \in \ker d_1$ will lie in the same coset of $H_1(\Gamma, \mathbb{F}_p)$ if and only if the amount of flow around the missing center triangle of Γ is the same mod p for both u and v . This makes it easy to express the structure of $H_1(\Gamma, \mathbb{F}_p)$: if we let $\alpha \in H_1(\Gamma, \mathbb{F}_p)$ be

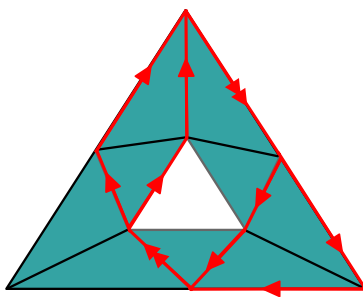


Fig. 3.1 A complex Γ and a chain element $a \in K_1(\Gamma, \mathbb{F}_p)$.

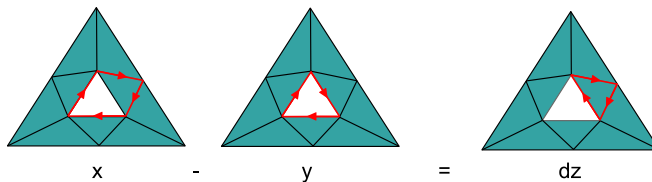


Fig. 3.2 Two elements $x, y \in K_1(\Gamma, \mathbb{F}_p)$ which are contained in the same coset of $H_1(\Gamma, \mathbb{F}_p)$.

the coset containing the element y from Figure 3.2, then $H_1(\Gamma, \mathbb{F}_p)$ is a one-dimensional \mathbb{F}_p -vector space that is spanned by α .

Meanwhile, it is easy to see that $\ker d_2 = \{0\}$ and hence $H_2(\Gamma, \mathbb{F}_p) = \{0\}$. We thus have the following:

$$H_0(\Gamma, \mathbb{F}_p) \cong \mathbb{F}_p \quad (3.42)$$

$$H_1(\Gamma, \mathbb{F}_p) \cong \mathbb{F}_p \quad (3.43)$$

$$H_i(\Gamma, \mathbb{F}_p) \cong \{0\} \quad \text{for all } i \geq 2. \quad (3.44)$$

This kind of reasoning can be used to describe the homology groups of any finite simplicial complex Π that is contained in \mathbb{R}^2 . The dimension of $H_1(\Pi, \mathbb{F}_p)$ for such a complex is always equal to the number holes enclosed by Π .

Such visualizations are also useful for understanding the behavior of homology groups under automorphisms. Figure 3.3 shows an example of a simplicial complex Γ' for which $H_1(\Gamma', \mathbb{F}_p) \cong \mathbb{F}_p^2$. Any automorphism of Γ' induces a linear automorphism of $H_1(\Gamma', \mathbb{F}_p)$. The figure describes a few such automorphism in terms of two chosen basis elements $\lambda, \beta \in H_1(\Gamma', \mathbb{F}_p)$.

To observe nontrivial automorphisms of higher homology groups, we need to consider simplicial complexes in three-dimensional space. Figure 3.4 shows a simplicial complex Λ in \mathbb{R}^3 which has the shape of a torus. Let $z \in K_2(\Lambda, \mathbb{F}_p)$ be a linear combination of all the 2-simplices

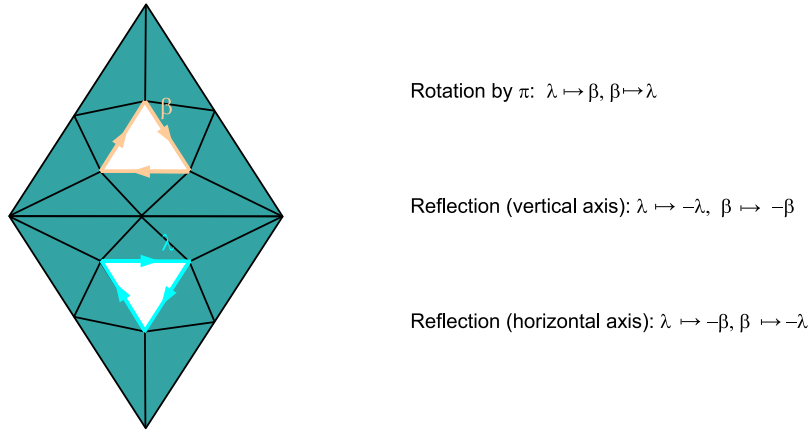


Fig. 3.3 The complex Γ' and the effect of three different automorphisms.

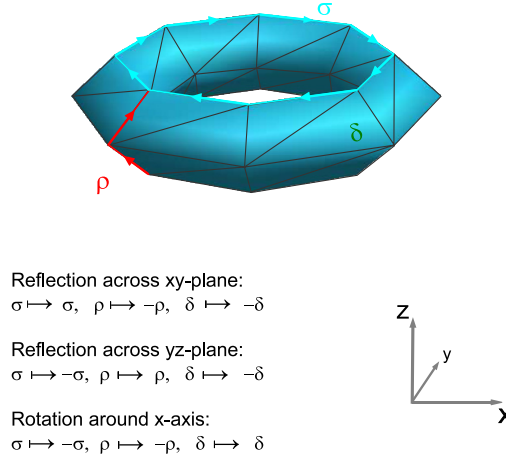


Fig. 3.4 The complex Λ and the effect of three different automorphisms.

in Λ in which the coefficient of the simplex $[v_0, v_1, v_2]$ in z is $(+1)$ if the vertices v_0, v_1 , and v_2 appear in clockwise order on the surface of the torus, and (-1) if they appear in counterclockwise order. When Definition 3.2 is applied to compute dz , all terms cancel and we find that $dz = 0$. The element z determines a coset $\delta \in H_2(\Lambda, \mathbb{F}_p)$, which spans the one-dimensional space $H_2(\Lambda, \mathbb{F}_p)$.

Figure 3.4 gives a basis $\{\sigma, \rho\}$ for the two-dimensional space $H_1(\Lambda, \mathbb{F}_p)$, and explains the effect of various automorphisms on $H_1(\Lambda, \mathbb{F}_p)$ and $H_2(\Lambda, \mathbb{F}_p)$.

3.4 Some Homological Algebra

We resume developing concepts from an algebraic standpoint. It is helpful now to take time to study homology groups in a more abstract setting, without reference to simplicial complexes. For any complex of abelian groups

$$\dots \longrightarrow K_{n+1} \xrightarrow{d_{n+1}} K_n \xrightarrow{d_n} K_{n-1} \xrightarrow{d_{n-1}} \dots, \quad (3.45)$$

the n th homology group of K_\bullet is defined by

$$H_n(K_\bullet, \mathbb{F}_p) = (\ker d_n) / (\text{im } d_{n+1}). \quad (3.46)$$

In this part of the text we will state a result (Proposition 3.4) which allows us to relate the homology groups of K_\bullet to the homology groups

of smaller complexes. This will be an essential building block in later proofs.

Let us say that a sequence of maps of abelian groups

$$\dots \longrightarrow A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \dots \quad (3.47)$$

is **exact** if it satisfies the condition $\ker f_n = \operatorname{im} f_{n+1}$ for every n . Thus, a sequence of the form

$$0 \longrightarrow P \xrightarrow{f} Q \xrightarrow{g} R \longrightarrow 0 \quad (3.48)$$

is exact if and only if f is injective, g is surjective, and $\operatorname{im} f = \ker g$. (Note that this makes R isomorphic to the quotient $Q/f(P)$.) Suppose that a sequence of maps of complexes

$$0 \longrightarrow X_\bullet \xrightarrow{F} Y_\bullet \xrightarrow{G} Z_\bullet \longrightarrow 0 \quad (3.49)$$

is such that

$$0 \longrightarrow X_n \xrightarrow{F_n} Y_n \xrightarrow{G_n} Z_n \longrightarrow 0 \quad (3.50)$$

is an exact sequence for every n . Then we will say that (3.49) is an exact sequence of complexes.

I claim that if

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X_{n+1} & \xrightarrow{F} & Y_{n+1} & \xrightarrow{G} & Z_{n+1} \longrightarrow 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d \\ 0 & \longrightarrow & X_n & \xrightarrow{F} & Y_n & \xrightarrow{G} & Z_n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array} \quad (3.51)$$

is an exact sequence of complexes, then

$$H_n(X_\bullet) \rightarrow H_n(Y_\bullet) \rightarrow H_n(Z_\bullet) \quad (3.52)$$

is an exact sequence. This can be seen through a “diagram-chasing” argument. It is obvious that

$$\operatorname{im} [H_n(X_\bullet) \rightarrow H_n(Y_\bullet)] \subseteq \ker [H_n(Y_\bullet) \rightarrow H_n(Z_\bullet)], \quad (3.53)$$

and so we only need to prove the reverse inclusion. Suppose that $(y + \text{im } d_n^Y)$ is a coset in $H_n(Y_\bullet)$ that is killed by the map to $H_n(Z_\bullet)$. Then $G(y) \in \text{im } d_{n+1}^Z$, so we can find $z' \in Z_{n+1}$ such that $dz' = G(y)$. Choosing an arbitrary element $y' \in G^{-1}\{z'\}$, we have $y - dy' \in \ker G$, and therefore by exactness, $F(x) = y - dy'$ for some x . Since $dy = 0$ and $d(dy') = 0$, we have $F(dx) = dF(x) = 0$ and therefore $dx = 0$. Thus $(x + \text{im } d_n^X)$ is a coset in $H_n(X_\bullet)$ which maps to $y + \text{im } d_n^Y$, and the claim is proved.

While it might be tempting to assume that the maps $H_n(X_\bullet) \rightarrow H_n(Y_\bullet)$ are injective and the maps $H_n(Y_\bullet) \rightarrow H_n(Z_\bullet)$ are surjective, this is not generally true. The homology groups of X_\bullet , Y_\bullet , and Z_\bullet have a more complex relationship which is expressed by the following proposition.

Proposition 3.4. Let X_\bullet , Y_\bullet , and Z_\bullet be complexes of abelian groups, and let $F: X_\bullet \rightarrow Y_\bullet$ and $G: Y_\bullet \rightarrow Z_\bullet$ be maps of complexes such that for any n , the sequence

$$0 \longrightarrow X_n \xrightarrow{F_n} Y_n \xrightarrow{G_n} Z_n \longrightarrow 0 \quad (3.54)$$

is an exact sequence. Then, there exist homomorphisms

$$\gamma_n: H_n(Z_\bullet) \rightarrow H_{n-1}(X_\bullet) \quad (3.55)$$

for every n which are such that the sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_2(Y_\bullet) & \longrightarrow & H_2(Z_\bullet) & & \\ & & & & \searrow \gamma_2 & & \\ & & H_1(X_\bullet) & \longrightarrow & H_1(Y_\bullet) & \longrightarrow & H_1(Z_\bullet) \\ & & & & \searrow \gamma_1 & & \\ & & H_0(X_\bullet) & \longrightarrow & H_0(Y_\bullet) & \longrightarrow & H_0(Z_\bullet) \longrightarrow 0 \end{array} \quad (3.56)$$

is exact.

Since the proof of this proposition is fairly technical, we have placed it in Appendix. (See Proposition A.1.) The maps γ_n can be briefly described like so: let $\overline{G}_n: Z_n \rightarrow Y_n$ be a function (not necessarily a homomorphism) which is such that $G_n \circ \overline{G}_n$ is the identity map, and let $\overline{F}_n: F(X_n) \rightarrow X_n$ be the inverse of F . Then, for any coset

$$z + \text{im } d_{n+1}^Z \in H_n(Z_\bullet), \quad (3.57)$$

the image under $\gamma_n: H_n(Z_\bullet) \rightarrow H_{n-1}(X_\bullet)$ is given by

$$\overline{F}_{n-1}(d(\overline{G}_n(z))) + \text{im } d_n^X \in H_{n-1}(X_\bullet). \quad (3.58)$$

As we will see, the above proposition is very useful because it allows us to draw conclusions about the homology groups of a complex Y_\bullet based on the homology groups of its subcomplexes and quotient complexes.

We close with a few additional constructions. Note that for any map of complexes $F: I_\bullet \rightarrow J_\bullet$, there exist the complexes

$$\dots \longrightarrow \text{im } F_{n+1} \xrightarrow{d_{n+1}} \text{im } F_n \xrightarrow{d_n} \text{im } F_{n-1} \xrightarrow{d_{n-1}} \dots \quad (3.59)$$

and

$$\dots \longrightarrow \ker F_{n+1} \xrightarrow{d_{n+1}} \ker F_n \xrightarrow{d_n} \ker F_{n-1} \xrightarrow{d_{n-1}} \dots \quad (3.60)$$

We write these complexes as $(\text{im } F)$ and $(\ker F)$, respectively. Note that these complexes fit into an exact sequence

$$0 \rightarrow \ker F \rightarrow I_\bullet \rightarrow \text{im } F \rightarrow 0. \quad (3.61)$$

The **direct sum** of I_\bullet and J_\bullet , written $I_\bullet \oplus J_\bullet$, is the complex

$$\dots \longrightarrow I_{n+1} \oplus J_{n+1} \longrightarrow I_n \oplus J_n \longrightarrow I_{n-1} \oplus J_{n-1} \longrightarrow \dots, \quad (3.62)$$

where the maps in this complex are simply the maps induced by $d_k: I_k \rightarrow I_{k-1}$ and $d_k: J_k \rightarrow J_{k-1}$. Note that the homology groups of this complex are simply $H_n(I_\bullet) \oplus H_n(J_\bullet)$.

3.5 Collapsibility Implies Acyclicity

Now we will offer our first application of Proposition 3.4. In *Graph Properties*, we defined the notion of *collapsibility* for simplicial

complexes. In this part of the text we will see how the condition of collapsibility for a simplicial complex Δ implies that the homology groups of Δ are trivial.

We begin with a useful definition.

Definition 3.5. Let Δ be an abstract simplicial complex whose vertex-set is totally ordered. Let p be a prime, and let n be a nonnegative integer. Define the map

$$s: K_0(\Delta, \mathbb{F}_p) \rightarrow \mathbb{F}_p \quad (3.63)$$

by asserting that $s(\gamma)$ is the sum of the coefficients of γ . That is, if

$$\gamma = c_1 Q_1 + c_2 Q_2 + \dots + c_r Q_r, \quad (3.64)$$

with $c_i \in \mathbb{F}_p$ and $Q_i \in \Delta$, then

$$s(\gamma) = c_1 + c_2 + \dots + c_r \in \mathbb{F}_p. \quad (3.65)$$

The **reduced n th homology group of Δ over \mathbb{F}_p** , denoted $\tilde{H}_n(\Delta, \mathbb{F}_p)$, is the n th homology group of the complex

$$\dots \longrightarrow K_2(\Delta, \mathbb{F}_p) \xrightarrow{d_2} K_1(\Delta, \mathbb{F}_p) \xrightarrow{d_1} K_0(\Delta, \mathbb{F}_p) \xrightarrow{s} \mathbb{F}_p \longrightarrow 0$$

The reduced homology groups $\{\tilde{H}_n(\Delta, \mathbb{F}_p)\}$ of an abstract simplicial complex Δ are the same as the ordinary homology groups $\{H_n(\Delta, \mathbb{F}_p)\}$, except that the dimension of $\tilde{H}_0(\Delta, \mathbb{F}_p)$ is one less than the dimension of $H_0(\Delta, \mathbb{F}_p)$. Note that the reduced homology groups of the trivial complex $\{\{0\}\}$ are all zero.

Definition 3.6. Let Δ be an abstract simplicial complex whose vertex-set is totally ordered. Then, Δ is \mathbb{F}_p -**acyclic** if

$$\dim_{\mathbb{F}_p} \tilde{H}_n(\Delta, \mathbb{F}_p) = 0 \quad (3.66)$$

for all nonnegative integers n .

Stated differently, a complex is \mathbb{F}_p -acyclic if its \mathbb{F}_p -homology is the same as that of single point. An example of an \mathbb{F}_p -acyclic simplicial complex is this one, from Example 3.2.

$$\Sigma' = \{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, \{0, 1, 2\}\}.$$

One can check by direct calculation that all of the reduced homology groups of this simplicial complex are trivial. On the other hand, the simplicial complex Σ of Example 3.1 is *not* \mathbb{F}_p -acyclic, since $\tilde{H}_1(\Sigma, \mathbb{F}_p) \cong \mathbb{F}_p$.

Another way of expressing Definition 3.6 is this: an abstract simplicial complex Δ is \mathbb{F}_p -acyclic if

$$\dots \longrightarrow K_2(\Delta, \mathbb{F}_p) \xrightarrow{d_2} K_1(\Delta, \mathbb{F}_p) \xrightarrow{d_1} K_0(\Delta, \mathbb{F}_p) \xrightarrow{s} \mathbb{F}_p \longrightarrow 0$$

is an exact sequence.

When a complex forms an exact sequence, let us refer to it as an **exact complex**. The following algebraic lemma is useful for proving exactness of complexes.

Lemma 3.5. Let

$$0 \rightarrow X_\bullet \rightarrow Y_\bullet \rightarrow Z_\bullet \rightarrow 0 \quad (3.67)$$

be an exact sequence of complexes of abelian groups. Then,

- (1) If X_\bullet and Y_\bullet are exact complexes, then Z_\bullet is an exact complex.
 - (2) If Y_\bullet and Z_\bullet are exact complexes, then X_\bullet is an exact complex.
 - (3) If X_\bullet and Z_\bullet are exact complexes, then Y_\bullet is an exact complex.
-

Proof. We prove (1). Suppose that X_\bullet and Y_\bullet are exact complexes. By Proposition 3.4, there is an exact sequence

$$\begin{aligned} \dots \rightarrow H_{n+1}(Z_\bullet) \rightarrow H_n(X_\bullet) \rightarrow H_n(Y_\bullet) \rightarrow H_n(Z_\bullet) \\ \rightarrow H_{n-1}(X_\bullet) \rightarrow H_{n-1}(Y_\bullet) \rightarrow \dots \end{aligned}$$

The reader will observe that since the groups $\{H_n(X_\bullet)\}$ and $\{H_n(Y_\bullet)\}$ are all zero, the groups $\{H_n(Z_\bullet)\}$ must all be zero as well. Therefore Z_\bullet is an exact complex.

Assertions (2) and (3) follow similarly. \square

Now we are ready to prove our main theorem.

Theorem 3.6. Let p be a prime. Let Δ be an abstract simplicial complex which has a total ordering on its vertex set. If Δ is collapsible, then Δ is \mathbb{F}_p -acyclic.

Proof. Recall (from *Monotone Graph Properties*) the definition of **primitive elementary collapse**. For any elementary collapse (Σ, Σ') , there is a sequence of primitive elementary collapses which reduces Σ to Σ' :

$$\Sigma, \Sigma_1, \Sigma_2, \dots, \Sigma_t, \Sigma' \quad (3.68)$$

(This is an elementary fact which the reader is invited to prove as an exercise.)

Suppose that the complex Δ is collapsible. There exists a sequence of elementary collapses which collapse Δ to a single 0-simplex. Therefore, there exists a sequence of *primitive* elementary collapses which collapse Δ to a single 0-simplex. Let

$$\Delta, \Delta_1, \Delta_2, \dots, \Delta_r \quad (3.69)$$

be such a sequence, with $|\Delta_r| = 1$.

Let Z_\bullet be the complex formed by the quotient groups

$$Z_n = K_n(\Delta, \mathbb{F}_p) / K_n(\Delta_1, \mathbb{F}_p) \quad (3.70)$$

The structure of the complex Z_\bullet is quite simple: it is isomorphic to the following complex:

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{F}_p \xrightarrow{Id} \mathbb{F}_p \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots \quad (3.71)$$

There is an exact sequence of complexes

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots & (3.72) \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & K_1(\Delta_1, \mathbb{F}_p) & \longrightarrow & K_1(\Delta, \mathbb{F}_p) & \longrightarrow & Z_1 & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & K_0(\Delta_1, \mathbb{F}_p) & \longrightarrow & K_0(\Delta, \mathbb{F}_p) & \longrightarrow & Z_0 & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \mathbb{F}_p & \longrightarrow & \mathbb{F}_p & \longrightarrow & 0 & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow 0
 \end{array}$$

The complex Z_\bullet is clearly exact. So by Lemma 3.5, the complex

$$\dots \longrightarrow K_2(\Delta, \mathbb{F}_p) \xrightarrow{d_2} K_1(\Delta, \mathbb{F}_p) \xrightarrow{d_1} K_0(\Delta, \mathbb{F}_p) \xrightarrow{s} \mathbb{F}_p \longrightarrow 0$$

is exact iff

$$\dots \longrightarrow K_2(\Delta_1, \mathbb{F}_p) \xrightarrow{d_2} K_1(\Delta_1, \mathbb{F}_p) \xrightarrow{d_1} K_0(\Delta_1, \mathbb{F}_p) \xrightarrow{s} \mathbb{F}_p \longrightarrow 0$$

is exact. Therefore Δ is \mathbb{F}_p -acyclic iff Δ_1 is \mathbb{F}_p -acyclic.

Similar reasoning shows that for any i , Δ_i is \mathbb{F}_p -acyclic iff Δ_{i+1} is \mathbb{F}_p -acyclic. The theorem follows by induction. \square

4

Fixed-Point Theorems

We are now ready to put the theory from *Chain Complexes* to use to study group actions $G \curvearrowright \Delta$ on simplicial complexes.

4.1 The Lefschetz Fixed-Point Theorem

Theorem 4.1. Let Δ be a finite abstract simplicial complex with ordered vertices. Suppose that Δ is \mathbb{F}_p -acyclic for some prime number p . Let $f: \Delta \rightarrow \Delta$ be a simplicial automorphism. Then, there exists a simplex $Q \in \Delta$ such that $f(Q) = Q$.

Proof. Let us introduce some notation: if Y is a finite-dimensional vector space over \mathbb{F}_p , and $h: Y \rightarrow Y$ is a linear endomorphism, then let $\text{Tr}_h(Y)$ denote the trace of h on Y . Note that the trace function is additive over exact sequences. That is, if

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \quad (4.1)$$

is an exact sequence, and h acts on X , Y , and Z in a compatible manner, then

$$\text{Tr}_h(Y) = \text{Tr}_h(X) + \text{Tr}_h(Z). \quad (4.2)$$

Let F denote the chain map associated with f . Consider the values of

$$\mathrm{Tr}_F(H_n(\Delta, \mathbb{F}_p)) \quad (4.3)$$

for $n = 0, 1, 2, \dots$. Since Δ is \mathbb{F}_p -acyclic, these are easy to compute. If $n > 0$, then $H_n(\Delta, \mathbb{F}_p)$ is a zero vector space. The vector space $H_0(\Delta, \mathbb{F}_p)$ is a one-dimensional \mathbb{F}_p -vector space on which F acts trivially. Therefore,

$$\mathrm{Tr}_F(H_0(\Delta, \mathbb{F}_p)) = 1, \quad (4.4)$$

$$\mathrm{Tr}_F(H_n(\Delta, \mathbb{F}_p)) = 0 \quad \text{for } n > 0. \quad (4.5)$$

Now we can carry out the proof using the additivity of the trace function. Suppose, for the sake of contradiction, that there is *no* simplex in Δ which is stabilized by F . Then, for any n , the chain map F acts on $K_n(\Delta, \mathbb{F}_p)$ by permuting the basis elements in a fixed-point free manner, possibly changing signs. A matrix representation of this action would be a matrix with entries from the set $\{-1, 0, 1\}$, having only zeroes on the main diagonal. Thus we see that

$$\mathrm{Tr}_F(K_n(\Delta, \mathbb{F}_p)) = 0. \quad (4.6)$$

Observe the following chain of equalities.

$$\begin{aligned} 0 &= \sum_{n \geq 0} (-1)^n \mathrm{Tr}_F(K_n(\Delta, \mathbb{F}_p)) \\ &= \mathrm{Tr}_F(K_0(\Delta, \mathbb{F}_p)) + \sum_{n \geq 1} (-1)^n [\mathrm{Tr}_F(\mathrm{im} \, d_n) + \mathrm{Tr}_F(\ker d_n)] \\ &= \mathrm{Tr}_F(K_0(\Delta, \mathbb{F}_p)) - \mathrm{Tr}_F(\mathrm{im} \, d_1) \\ &\quad + \sum_{n \geq 1} (-1)^n [\mathrm{Tr}_F(\ker d_n) - \mathrm{Tr}_F(\mathrm{im} \, d_{n+1})] \\ &= \mathrm{Tr}_F(H_0(\Delta, \mathbb{F}_p)) + \sum_{n \geq 1} (-1)^n \mathrm{Tr}_F(H_n(\Delta, \mathbb{F}_p)) \\ &= 1. \end{aligned}$$

We obtain a contradiction. Therefore, there must exist a simplex Q in Δ such that $f(Q) = Q$. \square

Corollary 4.2. Let Σ be a finite abstract simplicial complex which is collapsible. Let $g: \Sigma \rightarrow \Sigma$ be a simplicial automorphism. Then there must exist a simplex $T \in \Sigma$ such that $g(T) = T$.

Proof. This follows immediately from the above theorem and Theorem 3.6. \square

Let us consider what Theorem 4.1 means geometrically. Suppose that Θ is an ordinary simplicial complex in \mathbb{R}^N (see *Simplicial Complexes*). Then a simplicial automorphism of Θ is simply a continuous permutation of the points of Θ which maps every n -simplex of Θ to another n -simplex of Θ in an affine-linear manner.

Suppose that $V \subset \mathbb{R}^N$ is a single n -simplex spanned by $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^N$. Note that any affine-linear map of V onto itself must fix the point

$$\sum_{i=0}^n \left(\frac{1}{n+1} \right) \mathbf{v}_i \in V. \quad (4.7)$$

Thus, any simplicial map which stabilizes V must have a fixed point in V . Therefore, when we establish that a simplicial automorphism maps a particular simplex to itself, we have in fact proved that it has a fixed point. This justifies our calling Theorem 4.1 a “fixed-point theorem.”

Let $f: \Delta \rightarrow \Delta$ be a simplicial automorphism which satisfies the assumptions of Theorem 4.1. We can use the reasoning from the proof of Theorem 4.1 to draw further conclusions about the set Δ^f . Note that the quantity

$$\text{Tr}_F(K_n(\Delta, \mathbb{F}_p)) \quad (4.8)$$

is equal to the number of n -simplices $Q \in \Delta$ that satisfy $f(Q) = Q$. By the reasoning from the proof of Theorem 4.1, we have

$$\sum_{n \geq 0} (-1)^n \text{Tr}_F(K_n(\Delta, \mathbb{F}_p)) = 1. \quad (4.9)$$

This implies a different version of Theorem 4.1. For any subset S of a simplicial complex Δ , let

$$\chi(S) = \sum_{n \geq 0} (-1)^n |\{Q \in S \mid \dim(Q) = n\}|. \quad (4.10)$$

The quantity $\chi(S)$ is called the **Euler characteristic** of S .

Theorem 4.3. Let Δ be a finite abstract simplicial complex with ordered vertices, and suppose that Δ is \mathbb{F}_p -acyclic for some prime number p . Let $f: \Delta \rightarrow \Delta$ be a simplicial automorphism. Then,

$$\chi(\Delta^f) = 1. \quad (4.11)$$

4.2 A Nonabelian Fixed-Point Theorem

In this part of the text we will prove a nonabelian fixed-point theorem which is attributed to R. Oliver [32].

Let Δ be a collapsible abstract simplicial complex. Let G be a finite group which acts on Δ via simplicial automorphisms. By Corollary 4.2, we know that for any element $g \in G$, there must be a simplex $Q \in \Delta$ such that $g(Q) = Q$. We will prove that, under certain conditions, a stronger statement can be made: there must exist a single simplex Q which is stabilized by all the elements of G .

Our method of proof for this result is essentially an inductive one. We require that the automorphism group G has a certain filtration by subgroups,

$$\{0\} = G_0 \subset G_1 \subset G_2 \subset \dots \subset G_r = G, \quad (4.12)$$

and we inductively deduce conditions on the G_i -fixed subsets of Δ , for $i = 0, 1, 2, \dots, r$. The key to this argument is the first result that we will prove, Proposition 4.4, which tells us that the property of “ \mathbb{F}_p -acyclicity” can be carried forward along this filtration. The proof of Proposition 4.4 is the most difficult part of the argument; once that proposition is proved, the other elements of the argument fall into place easily.

For now, we will be focusing our attention on simplicial automorphisms $f: \Delta \rightarrow \Delta$ for which Δ^f is a subcomplex of Δ . That is, we will be focusing on those maps f satisfying the condition

$$Q \in \Delta^f \text{ and } Q' \subseteq Q \implies Q' \in \Delta^f. \quad (4.13)$$

for any $Q, Q' \in \Delta$. Geometrically, what this condition implies is that if f stabilizes a simplex Q , then it also fixes all of the vertices of Q .

The **order** of a simplicial automorphism $f: \Delta \rightarrow \Delta$ is the least $n \geq 1$ such that f^n is the identity. (If no such n exists, then the order of f is ∞ .)

Proposition 4.4. Let Δ be a finite abstract simplicial complex with ordered vertices. Let p be a prime, and suppose that Δ is \mathbb{F}_p -acyclic. Suppose that $f: \Delta \rightarrow \Delta$ is an order- p automorphism of Δ such that Δ^f is a subcomplex of Δ . Then, the complex Δ^f must be \mathbb{F}_p -acyclic.

Proof. Suppose that Δ is \mathbb{F}_p -acyclic. We know, by Theorem 4.1, that the subcomplex Δ^f must be nonempty. To prove the proposition, we must show that the homology groups $H_n(\Delta^f, \mathbb{F}_p)$ are trivial for $n > 0$, and that $H_0(\Delta^f, \mathbb{F}_p)$ is one-dimensional.

The proof that follows is based on the paper “Fixed-point theorems for periodic transformations” by Smith [35]. The approach of the proof is to define some special subcomplexes of $K_\bullet(\Delta, \mathbb{F}_p)$ and then exploit relationships between these subcomplexes.

Let

$$F: K_\bullet(\Delta, \mathbb{F}_p) \rightarrow K_\bullet(\Delta, \mathbb{F}_p) \quad (4.14)$$

denote the chain map associated with f . Note that since F is a map of complexes, any linear combination of the maps F, F^2, F^3, \dots is also a map of complexes. Define

$$\delta: K_\bullet(\Delta, \mathbb{F}_p) \rightarrow K_\bullet(\Delta, \mathbb{F}_p) \quad (4.15)$$

by

$$\delta = \mathbb{I} - F. \quad (4.16)$$

(Here \mathbb{I} denotes the identity map.) Define

$$\sigma: K_\bullet(\Delta, \mathbb{F}_p) \rightarrow K_\bullet(\Delta, \mathbb{F}_p) \quad (4.17)$$

by

$$\sigma = \mathbb{I} + F + F^2 + \dots + F^{p-1}. \quad (4.18)$$

The maps δ and σ determine four subcomplexes of $K_\bullet(\Delta, \mathbb{F}_p)$:

$$(\operatorname{im} \delta), (\ker \delta), (\operatorname{im} \sigma), \quad \text{and} \quad (\ker \sigma). \quad (4.19)$$

We can describe these four complexes explicitly. Let $\Delta' = \Delta \setminus \Delta^f$. Let $S \subseteq \Delta'$ be a set which contains exactly one element from every f -orbit in Δ' . Then the following assertions hold (as the reader may verify):

- The set

$$\left\{ \sum_{i=0}^{p-1} F^i(Q) \mid Q \in S \right\}$$

is a basis¹ for $(\operatorname{im} \sigma)$.

- The set

$$\{F^i(Q) - F^{i+1}(Q) \mid Q \in S, 0 \leq i \leq p-2\}$$

is a basis for $(\operatorname{im} \delta)$.

- The set

$$\{F^i(Q) - F^{i+1}(Q) \mid Q \in S, 0 \leq i \leq p-2\} \cup \{Q \mid Q \in \Delta^f\}$$

is a basis for $(\ker \sigma)$.

- The set

$$\left\{ \sum_{i=0}^{p-1} F^i(Q) \mid Q \in S \right\} \cup \{Q \mid Q \in \Delta^f\}$$

is a basis for $(\ker \delta)$.

From these bases, we can see that there are the following isomorphisms of complexes:

$$(\ker \sigma) \cong (\operatorname{im} \delta) \oplus K_\bullet(\Delta^f, \mathbb{F}_p) \quad (4.20)$$

and

$$(\ker \delta) \cong (\operatorname{im} \sigma) \oplus K_\bullet(\Delta^f, \mathbb{F}_p). \quad (4.21)$$

¹When we say that a set T is a basis for a complex X_\bullet , we mean that T is a union of bases for the vector spaces $\{X_i\}$.

These imply isomorphisms of homology groups:

$$H_n(\ker \sigma) \cong H_n(\operatorname{im} \delta) \oplus H_n(\Delta^f, \mathbb{F}_p), \quad (4.22)$$

$$H_n(\ker \delta) \cong H_n(\operatorname{im} \sigma) \oplus H_n(\Delta^f, \mathbb{F}_p). \quad (4.23)$$

Now, consider the exact sequences

$$0 \rightarrow (\ker \sigma) \rightarrow K_\bullet(\Delta, \mathbb{F}_p) \rightarrow (\operatorname{im} \sigma) \rightarrow 0, \quad (4.24)$$

$$0 \rightarrow (\ker \delta) \rightarrow K_\bullet(\Delta, \mathbb{F}_p) \rightarrow (\operatorname{im} \delta) \rightarrow 0 \quad (4.25)$$

By Proposition 3.4, these imply the existence of two long exact sequences:

$$\begin{aligned} \dots \rightarrow H_{n+1}(\operatorname{im} \sigma) \rightarrow H_n(\ker \sigma) \rightarrow H_n(\Delta, \mathbb{F}_p) \rightarrow H_n(\operatorname{im} \sigma) \\ \rightarrow H_{n-1}(\ker \sigma) \rightarrow \dots \end{aligned}$$

$$\begin{aligned} \dots \rightarrow H_{n+1}(\operatorname{im} \delta) \rightarrow H_n(\ker \delta) \rightarrow H_n(\Delta, \mathbb{F}_p) \rightarrow H_n(\operatorname{im} \delta) \\ \rightarrow H_{n-1}(\ker \delta) \rightarrow \dots \end{aligned}$$

Let us step through the terms in these sequences, starting from the left. Let c be the dimension of the complex Δ (that is, the dimension of the largest simplex in Δ). The exact sequences take the form

$$\dots \rightarrow 0 \rightarrow H_c(\ker \sigma) \rightarrow H_c(\Delta, \mathbb{F}_p) \rightarrow H_c(\operatorname{im} \sigma) \rightarrow H_{c-1}(\ker \sigma) \rightarrow \dots$$

$$\dots \rightarrow 0 \rightarrow H_c(\ker \delta) \rightarrow H_c(\Delta, \mathbb{F}_p) \rightarrow H_c(\operatorname{im} \delta) \rightarrow H_{c-1}(\ker \delta) \rightarrow \dots$$

Since Δ is acyclic, we know that $H_c(\Delta, \mathbb{F}_p) = \{0\}$, which clearly implies that both $H_c(\ker \sigma)$ and $H_c(\ker \delta)$ are zero. So the exact sequences take the form

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow H_c(\operatorname{im} \sigma) \rightarrow H_{c-1}(\ker \sigma) \rightarrow \dots$$

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow H_c(\operatorname{im} \delta) \rightarrow H_{c-1}(\ker \delta) \rightarrow \dots$$

But isomorphisms (4.22) and (4.23) imply that $H_c(\operatorname{im} \sigma)$ and $H_c(\operatorname{im} \delta)$ are also zero. So the exact sequences are like so:

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow H_{c-1}(\ker \sigma) \rightarrow \dots$$

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow H_{c-1}(\ker \delta) \rightarrow \dots$$

We can apply the same reasoning to show that all terms in the sequences with index $(c-1)$ are likewise zero. Continuing in this manner, we eventually find that *all* the homology groups in the sequences that have a positive index are zero. We are left with the exact sequences in the following form:

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow H_0(\ker \sigma) \rightarrow H_0(\Delta, \mathbb{F}_p) \rightarrow H_0(\operatorname{im} \sigma) \longrightarrow 0 \quad (4.26)$$

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow H_0(\ker \delta) \rightarrow H_0(\Delta, \mathbb{F}_p) \rightarrow H_0(\operatorname{im} \delta) \longrightarrow 0 \quad (4.27)$$

We have shown that all of the homology groups $H_n(\ker \sigma)$, $n > 0$ are trivial. This implies by isomorphism (4.22) that $H_n(\Delta^f, \mathbb{F}_p)$ is trivial for all $n > 0$. Also, we know from isomorphism (4.22) and sequence (4.26) that

$$\dim H_0(\Delta^f, \mathbb{F}_p) \leq \dim H_0(\ker \sigma) \leq \dim H_0(\Delta, \mathbb{F}_p) = 1. \quad (4.28)$$

The dimension of $H_0(\Delta^f, \mathbb{F}_p)$ cannot be zero (since Δ^f is nonempty). So $H_0(\Delta^f, \mathbb{F}_p)$ must be one-dimensional. Therefore, Δ^f is \mathbb{F}_p -acyclic. \square

Corollary 4.5. Suppose that H is a group of order p^m , with $m \geq 1$, which acts on Δ in such a way that Δ^h is a subcomplex of Δ for any $h \in H$. Then, Δ^H is \mathbb{F}_p -acyclic.

Proof. Since $|H| = p^m$, there exists a filtration of H by normal subgroups,

$$\{0\} = H_0 \subset H_1 \subset \dots \subset H_m = H \quad (4.29)$$

such that $H_i/H_{i-1} \cong \mathbb{Z}/p\mathbb{Z}$ for any $i \in \{1, 2, \dots, m\}$. (See Chapter I, Corollary 6.6 in [25].) For any $i \in \{1, 2, \dots, m\}$, we can choose an element $a_i \in H_i$ which generates H_i/H_{i-1} . Then,

$$\Delta^{H_i} = (\Delta^{H_{i-1}})^{a_i}. \quad (4.30)$$

By Proposition 4.4, if $\Delta^{H_{i-1}}$ is \mathbb{F}_p -acyclic, so is Δ^{H_i} . The corollary follows by induction. \square

Now we are ready to prove the main theorem.

Theorem 4.6. Let G be a finite group satisfying the following condition:

- There is a normal subgroup $G' \subseteq G$ such that $|G'|$ is a prime power and G/G' is cyclic.

Let Δ be a collapsible abstract simplicial complex on which G acts, satisfying the condition that Δ^g is a simplicial complex for any $g \in G$. Then, $\chi(\Delta^G) = 1$.

Proof. We are given that $|G'| = p^m$ for some prime p and $m \geq 0$. Choose a total ordering on the vertices of Δ . By Theorem 3.6, Δ is \mathbb{F}_p -acyclic. By Corollary 4.5, $\Delta^{G'}$ is \mathbb{F}_p -acyclic.

Choose an element $b \in G$ which generates G/G' . By Theorem 4.1, the complex

$$(\Delta^{G'})^b = \Delta^G \tag{4.31}$$

has Euler characteristic equal to 1. □

Note that Theorem 4.6 implies in particular that the invariant subcomplex Δ^G is nonempty.

4.3 Barycentric Subdivision

In *A Nonabelian Fixed-Point Theorem* we proved Theorem 4.6, which asserts that if a group action $G \curvearrowright \Delta$ satisfies certain requirements, then Δ^G must be nonempty. The theorem as stated is unfortunately not general enough for our purposes. Indeed the condition that all of the subsets $\{\Delta^g \mid g \in G\}$ are subcomplexes will not be satisfied by the simplicial complexes arising from graph properties, except in trivial cases. Therefore we need a theorem which can be applied to group actions that do not satisfy this condition.

Barycentric subdivision is a process of dividing up the simplices in a simplicial complex into smaller simplices. Barycentric subdivision

replaces an abstract simplicial complex Δ with a larger complex Δ' that has similar properties. The advantage of this construction is that for any simplicial automorphism $g: \Delta \rightarrow \Delta$, there is an induced automorphism $g: \Delta' \rightarrow \Delta'$ which satisfies the condition that $(\Delta')^g$ is an abstract simplicial complex. Working within this larger complex will allow us to prove a generalization of Theorem 4.6.

Definition 4.1. Let Δ be an abstract simplicial complex. Then the **barycentric subdivision** of Δ , denoted $\text{bar}(\Delta)$, is the simplicial complex

$$\text{bar}(\Delta) = \{ \{Q_1, Q_2, \dots, Q_r\} \mid r \geq 1, Q_i \in \Delta, Q_1 \subset Q_2 \subset \dots \subset Q_r \}.$$

Here is another way to phrase the above definition. Let Δ be an abstract simplicial complex. Then the subset relation \subset gives a partial ordering on the elements of Δ . The complex $\text{bar}(\Delta)$ is the set of all \subset -chains in Δ .

As an example, let

$$\Sigma = \{ \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\} \}. \quad (4.32)$$

The complex Σ and its barycentric subdivision $\text{bar}(\Sigma)$ are shown in Figure 4.1.

Geometrically, the operation $[\Delta \mapsto \text{bar}(\Delta)]$ has the effect of splitting every simplex of dimension n in Δ into $(n+1)!$ simplices of

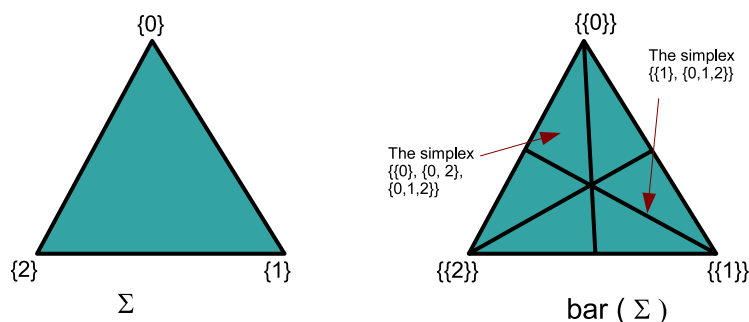


Fig. 4.1 The complexes Σ and $\text{bar}(\Sigma)$.

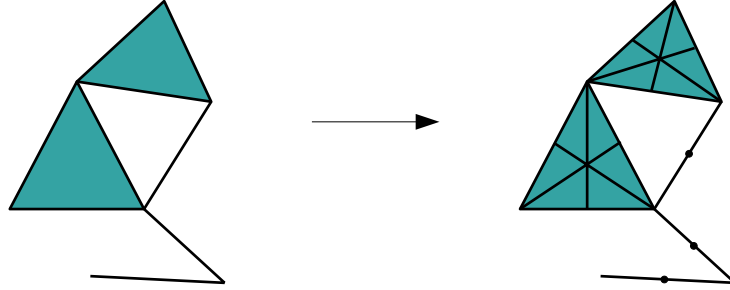


Fig. 4.2 An example of barycentric subdivision.

dimension n . Note that vertices in $\text{bar}(\Delta)$ are in one-to-one correspondence with the simplices of Δ . Figure 4.2 shows another example of barycentric subdivision.

As the reader can observe, the simplicial complex $\text{bar}(\Delta)$ has some similarities with the original simplicial complex Δ . It can be shown that the homology groups of $\text{bar}(\Delta)$ are isomorphic to those of Δ , although we will not need to prove that here. The following propositions are proved in the Appendix (as Proposition A.4 and Proposition A.7).

Proposition 4.7. Let Δ be an abstract simplicial complex. If Δ is collapsible, the $\text{bar}(\Delta)$ is also collapsible.

Proposition 4.8. Let Δ be a finite abstract simplicial complex. Then, $\chi(\text{bar}(\Delta)) = \chi(\Delta)$.

Now, let us consider how this construction behaves under group actions. Let $f: \Delta \rightarrow \Delta$ be a simplicial automorphism of Δ . Then there is an induced simplicial automorphism,

$$f: \text{bar}(\Delta) \rightarrow \text{bar}(\Delta). \quad (4.33)$$

The invariant subset $\text{bar}(\Delta)^f$ can be expressed like so:

$$\text{bar}(\Delta)^f = \{ \{Q_1, Q_2, \dots, Q_r\} \mid r \geq 1, Q_i \in \Delta^f, Q_1 \subset Q_2 \subset \dots \subset Q_r \}.$$

It is easy to see that this set is always a simplicial complex. Thus the following lemma holds true:

Lemma 4.9. Let Δ be an abstract simplicial complex, and let $G \curvearrowright \Delta$ be a group action on Δ . Then, for any $g \in G$, the set

$$(\text{bar}(\Delta))^g \quad (4.34)$$

is a subcomplex of $\text{bar}(\Delta)$.

Lemma 4.9 can be observed in the example complex Σ which we discussed above (4.32). As we can see in Figure 4.1, any permutation of the set $\{0, 1, 2\}$ fixes a subcomplex of the complex $\text{bar}(\Sigma)$.

With the aid of barycentric subdivision, we can now prove the following fixed-point theorem.

Theorem 4.10. Let Δ be a collapsible abstract simplicial complex. Let $G \curvearrowright \Delta$ be a group action on Δ . Suppose that G has a normal subgroup G' which is such that $|G'|$ is a prime power and G/G' is cyclic. Then, the set Δ^G is nonempty.

Proof. By Proposition 4.7, $\text{bar}(\Delta)$ is collapsible. By Theorem 4.6, $\chi(\text{bar}(\Delta)^G) = 1$. Therefore $\text{bar}(\Delta)^G$ is nonempty, and thus Δ^G is likewise nonempty. \square

Now let $\Delta^{[G]}$ denote the complex constructed in *Group Actions on Simplicial Complexes*. The set $\Delta^{[G]}$ is very similar to Δ^G ; indeed, there is a one-to-one inclusion preserving map

$$i: \Delta^{[G]} \rightarrow \Delta^G \quad (4.35)$$

which is given simply by mapping any $S \in \Delta^{[G]}$ to the union of the elements of S . (The main difference between Δ^G and $\Delta^{[G]}$ is that $\Delta^{[G]}$ is a simplicial complex, whereas Δ^G generally is not.)

The map (4.35) induces a simplicial isomorphism

$$\text{bar}(\Delta^{[G]}) \rightarrow \text{bar}(\Delta)^G \quad (4.36)$$

Figure 4.3 illustrates the relationship between $\Delta^{[G]}$ and $\text{bar}(\Delta)^G$. Isomorphism (4.36) enables our final generalization of Theorem 4.6.



Fig. 4.3 A continuation of the example from Figures 2.13 and 2.14. The barycentric subdivision of $\Sigma^{[H]}$ is isomorphic to $\text{bar}(\Sigma)^H$.

Theorem 4.11. Let Δ be a collapsible abstract simplicial complex. Let $G \curvearrowright \Delta$ be a group action on Δ . Suppose that G has a normal subgroup G' which is such that $|G'|$ is a prime power and G/G' is cyclic. Then,

$$\chi(\Delta^{[G]}) = 1. \quad (4.37)$$

Proof. By Proposition 4.7, $\text{bar}(\Delta)$ is collapsible. Therefore the Euler characteristic of $\text{bar}(\Delta)^G \cong \text{bar}(\Delta^{[G]})$ is 1. By Proposition 4.8, the Euler characteristic of $\Delta^{[G]}$ is likewise equal to 1. \square

5

Results on Decision-Tree Complexity

In this part of the text, we will give the proofs of three lower bounds on the decision-tree complexity of graph properties, due to Kahn, Saks, Sturtevant, and Yao. Then we will sketch (without proof) some more recent results.

Let

$$h: \mathbf{G}(V) \rightarrow \{0,1\} \tag{5.1}$$

be a nontrivial monotone-increasing graph property. The function h satisfies two conditions: it is increasing (meaning that if Z is a subgraph of Z' then $h(Z) \leq h(Z')$) and it is also isomorphism-invariant ($Y \cong Y' \implies h(Y) = h(Y')$). Proofs of evasiveness exploit the interaction between these two conditions.

As we saw in *Basic Concepts*, the monotone-increasing condition implies that h determines a simplicial complex, Δ_h , whose simplices correspond to graphs Z that satisfy $h(Z) = 0$. The isomorphism-invariant property implies that this complex Δ_h is highly symmetric. If σ is any permutation of V , and

$$E \subseteq \{\{v,w\} \mid v,w \in V\} \tag{5.2}$$



Fig. 5.1 Let V be a set of size 9, and let h be a nontrivial increasing graph property. If h is not evasive, then at least one of the graphs above must fail to satisfy h .

is an edge set such that $h((V, E)) = 0$, then the edge set

$$\sigma(E) = \{\{\sigma(v), \sigma(w)\} \mid \{v, w\} \in E\} \quad (5.3)$$

also satisfies $h((V, \sigma(E))) = 0$. Thus there is an induced automorphism $\sigma: \Delta_h \rightarrow \Delta_h$.

If h were nonevasive, then Δ_h would be collapsible, and we could apply fixed-point theorems to Δ_h . Corollary 4.2 would imply that Δ_h must have a simplex which is stabilized by σ . Therefore, we have the following interesting result: if h is a nonevasive graph property, then for any permutation $\sigma: V \rightarrow V$ there must be a nontrivial σ -invariant graph which does not satisfy h . Figure 5.1 shows what we can deduce when $|V| = 9$ and σ is chosen to be a cyclic permutation.

When we go further and consider the actions of finite groups on Δ_h , we get stronger results. Note that the entire symmetric group $\text{Sym}(V)$ acts on Δ_h . Unfortunately this group is too big for the application of any fixed-point theorems that we have proved, and so we must restrict the action to some appropriate subgroup of $\text{Sym}(V)$. Making this choice of subgroup is a key step for many of the results that we will discuss.

5.1 Graphs of Order p^k

Theorem 5.1. (Kahn et al. [18]) Let V be a finite set of order p^k , where p is prime and $k \geq 1$. Let

$$h: \mathbf{G}(V) \rightarrow \{0, 1\} \quad (5.4)$$

be a nontrivial monotone-increasing graph property. Then, h must be evasive.

Proof. Without loss of generality, we may assume that V is the set of elements of the finite field \mathbb{F}_{p^k} . For any $a, b \in \mathbb{F}_{p^k}$ with $a \neq 0$, there is a permutation of V given by

$$x \mapsto ax + b. \quad (5.5)$$

Let $G \subseteq \text{Sym}(V)$ be the group of all such permutations. Let $G' \subseteq G$ be the subgroup consisting of permutations of the form $x \mapsto x + b$.

We make the following observations:

- (1) **The subgroup G' is an abelian group of order p^k .** It is isomorphic to the additive group of \mathbb{F}_{p^k} .
- (2) **The subgroup G' is normal.** This is apparent from the fact that for any $x, a, b \in \mathbb{F}_{p^k}$, with $a \neq 0$,

$$a^{-1}(ax + b) = x + a^{-1}b. \quad (5.6)$$

- (3) **The quotient group G/G' is cyclic.** The quotient group G/G' is isomorphic to the multiplicative group of \mathbb{F}_{p^k} , which is known to be cyclic (see Theorem IV.1.9 from [25]).
- (4) **The action of G is transitive on pairs of distinct elements $(x, x') \in V \times V$.** This is a consequence of the fact that for any pairs (x, x') and (y, y') with $x \neq x'$ and $y \neq y'$, the system of equations

$$ax + b = y \quad (5.7)$$

$$ax' + b = y' \quad (5.8)$$

has a solution, with $a \neq 0$.

Consider the group action

$$G \curvearrowright \Delta_h \quad (5.9)$$

Suppose that the graph property h is nonevasive. By Theorem 2.1, the simplicial complex Δ_h is collapsible.¹ By Theorem 4.10, the set $(\Delta_h)^G$ is

¹Technically, this is not true if Δ_h is empty, and so we need to address that case separately. If Δ_h is empty, then h must be the function that maps the empty graph to zero and all other graphs to 1. This graph property is easily seen to be evasive.

nonempty. Therefore there is a nonempty G -invariant graph which does not satisfy h . But by property (4) above, the only nonempty G -invariant graph is the complete graph. This makes h a trivial graph property, and thus we obtain a contradiction.

We conclude that h must be an evasive graph property. \square

5.2 Bipartite Graphs

Let V be a finite set which is the disjoint union of two subsets, Y and Z . Then a bipartite graph on (Y, Z) is a graph whose edges are all elements of the set

$$\{\{y, z\} \mid y \in Y, z \in Z\}. \quad (5.10)$$

A bipartite isomorphism between such graphs is a graph isomorphism which respects the partition (Y, Z) .

Let $\mathbf{B}(Y, Z)$ denote the set of all bipartite graphs on (Y, Z) . A **bipartite graph property** is a function

$$f: \mathbf{B}(Y, Z) \rightarrow \{0, 1\} \quad (5.11)$$

which respects bipartite isomorphisms. If this function is monotone increasing, it determines a simplicial complex Δ_f whose vertices are elements of the set (5.10).

Naturally, we say that the bipartite graph property (5.11) is evasive if its decision-tree complexity $D(f)$ is equal to $|Y| \cdot |Z|$. The following proposition can be proved by the same method that we used to prove Theorem 2.1.

Proposition 5.2. Let Y and Z be disjoint finite sets, and let

$$f: \mathbf{B}(Y, Z) \rightarrow \{0, 1\} \quad (5.12)$$

be a monotone-increasing bipartite graph property which is not evasive. If the complex Δ_f is not empty, then it is collapsible.

Note that the complex Δ_f always has a group action,

$$(\text{Sym}(Y) \times \text{Sym}(Z)) \curvearrowright \Delta_f. \quad (5.13)$$

Theorem 5.3 (Yao [40]). Let Y and Z be disjoint finite sets, and let

$$f: \mathbf{B}(Y, Z) \rightarrow \{0, 1\} \quad (5.14)$$

be a nontrivial bipartite graph property which is monotone increasing. Then, f is evasive.

Proof. Let $\sigma: Y \rightarrow Y$ be a cyclic permutation of the elements of Y , and let $G \subseteq \text{Sym}(Y)$ be the subgroup generated by σ . The edge set of any G -invariant bipartite graph has the form

$$H_S := \{\{y, z\} \mid y \in Y, z \in S\} \quad (5.15)$$

where S is a subset of Z (see Figure 5.2). Since f is isomorphism-invariant and monotone-increasing, the behavior of f on such graphs can be easily described: there is some integer $k \in \{1, 2, \dots, |Z|\}$ such that

$$(V, H_S) \text{ has property } f \iff |S| > k. \quad (5.16)$$

Let $\Delta = \Delta^{[G]}$. The vertices of $\Delta^{[G]}$ are the sets of the form

$$H_z := \{\{y, z\} \mid y \in Y\}, \quad (5.17)$$

with $z \in Z$, and the simplices are precisely the subsets of $\{H_z \mid z \in Z\}$ whose union forms a graph that does not have property f . Thus we

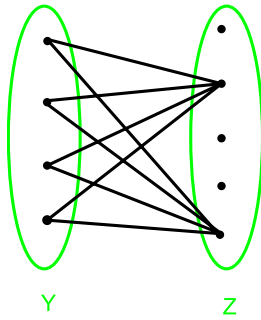


Fig. 5.2 An example of a set H_S .

can calculate the Euler characteristic directly:

$$\chi(\Delta^{[G]}) = \sum_{j=0}^{k-1} (-1)^j \binom{|Z|}{j+1} \quad (5.18)$$

$$= 1 + (-1)^{k-1} \binom{|Z|-1}{k}. \quad (5.19)$$

Suppose that f is nonevasive. Then Δ is collapsible and by Theorem 4.11,

$$\chi(\Delta^{[G]}) = 1. \quad (5.20)$$

But this is possible only if $k = |Z|$ and f is trivial. \square

5.3 A General Lower Bound

Now we prove a lower bound on decision-tree complexity which applies to graphs of arbitrary size. Our method of proof is based on [18].

Proposition 5.4. Let V be a finite set and let

$$h: \mathbf{G}(V) \rightarrow \{0, 1\} \quad (5.21)$$

be a nontrivial monotone-increasing graph property. Let p be the largest prime that is less than or equal to $|V|$. Then,

$$D(h) \geq \frac{p^2}{4}. \quad (5.22)$$

Proof. Assume that $|V| = n$. For any $r, s \geq 0$, let us write K_r for the complete graph on $\{1, 2, \dots, r\}$, and let us write $K_{r,s}$ for the complete bipartite graph on the sets $\{1, 2, \dots, r\}$ and $\{r+1, \dots, r+s\}$. For any two graphs $H = (V, E)$ and $H' = (V', E')$, let us abuse notation slightly and write $H \cup H'$ for the graph $(V \cup V', E \cup E')$.

For any $k \geq 1$, let C_k denote the least decision-tree complexity that occurs for nontrivial monotone-increasing graph properties on graphs of size k . We prove a lower bound for $D(h)$ in three cases.

Case 1: $\mathbf{h}(\mathbf{K}_{1,n-1}) = \mathbf{0}$. In this case, the function h induces a non-trivial graph property h' on the vertex set $\{2, 3, \dots, n\}$, given by

$$h'(P) = h(P \cup K_{1,n-1}). \quad (5.23)$$

This function has decision-tree complexity at least C_{n-1} , and therefore $D(h) \geq C_{n-1}$.

Case 2: $\mathbf{h}(\mathbf{K}_{n-1}) = \mathbf{1}$. In this case the function h induces a non-trivial graph property h' on the vertex set $\{2, 3, \dots, n\}$ given by

$$h'(P) = h(P \cup K_1), \quad (5.24)$$

which is likewise nontrivial. This function has decision-tree complexity at least C_{n-1} , and so $D(h) \geq C_{n-1}$.

Case 3: $\mathbf{h}(\mathbf{K}_{1,n-1}) = \mathbf{1}$ and $\mathbf{h}(\mathbf{K}_{n-1}) = \mathbf{0}$. Let $m = \lfloor n/2 \rfloor$. The property h induces a bipartite graph property on the sets $\{1, 2, \dots, m\}$ and $\{m+1, m+2, \dots, n\}$ defined by

$$h'(P) = h(P \cup K_m). \quad (5.25)$$

Since $h(K_m) \leq h(K_{n-1}) = 0$ and $h(K_m \cup K_{m,n-m}) \geq h(K_{1,n-1}) = 1$, the property h' is nontrivial. Therefore it has decision-tree complexity at least $m(n-m)$. The decision-tree complexity of h is likewise bounded by $m(n-m) \geq (n-1)^2/4$.

In all cases, we have

$$D(h) \geq \min \left\{ C_{n-1}, \frac{(n-1)^2}{4} \right\}. \quad (5.26)$$

The same reasoning shows that

$$C_k \geq \min \left\{ C_{k-1}, \frac{(k-1)^2}{4} \right\} \quad (5.27)$$

for every $k \in \{p+1, p+2, \dots, n-1\}$. Therefore by induction,

$$D(h) \geq \min \left\{ C_p, \frac{p^2}{4} \right\}. \quad (5.28)$$

The quantity C_p is $\binom{p}{2}$, and the desired result follows. \square

Theorem 5.5 (Kahn et al. [18]). Let C_n denote the least decision-tree complexity that occurs among all nontrivial monotone-increasing graph properties of order n . Then,

$$C_n \geq \frac{n^2}{4} - o(n^2). \quad (5.29)$$

Proof. By the prime number theorem, there is a function $z(n) = o(n)$ such that for any n , the interval $[n - z(n), n]$ contains a prime.² By Proposition 5.4,

$$C_n \geq \frac{(n - z(n))^2}{4} \quad (5.30)$$

$$\geq \frac{n^2}{4} - o(n^2). \quad (5.31)$$

as desired. \square

5.4 A Survey of Related Results

Much work on the decision-tree complexity of graph properties has followed the papers of Kahn, Saks, Sturtevant, and Yao. We briefly sketch some of the newer results in this area.

V. King proved a lower bound for properties of **directed** graphs.

Theorem 5.6 (King [19]). Let C'_n denote the least decision-tree complexity that occurs among all nontrivial monotone **directed** graph properties of order n . Then,

$$C'_n \geq \frac{n^2}{2} - o(n^2). \quad (5.32)$$

Triesch [37, 38] proved multiple results about the evasiveness of particular subclasses of monotone graph properties.

²The prime number theorem [42] asserts that if $\pi(n)$ denotes the number of primes less than or equal to n , then $\lim_{n \rightarrow \infty} \pi(n) (n / \ln n)^{-1} = 1$. If there were an infinite number of linearly sized gaps between the primes, this limit could not exist.

Korneffel and Triesch improved on the asymptotic bound of Theorem 5.5 by using a different group action on the set of vertices. Let V be a set of size n , and let p be a prime that is close to $(\frac{2}{5})n$. Break the set V up into disjoint subsets V_1 , V_2 , and V_3 , with $|V_1| = |V_2| = p$ and $|V_3| = n - 2p$. Let \mathbf{P} be the class of tripartite graphs on (V_1, V_2, V_3) which, when taken together with the complete graphs on the sets V_i , do not satisfy property h . The abelian group

$$G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/(n - 2p)\mathbb{Z} \quad (5.33)$$

acts on the class \mathbf{P} by cyclically permuting the elements of V_1 , V_2 , and V_3 . From this action and some other arguments, the authors are able to prove the following.

Theorem 5.7(Korneffel and Triesch [23]). Let C_n denote the least decision-tree complexity that occurs among all nontrivial monotone-increasing graph properties of order n . Then,

$$C_n \geq \frac{8n^2}{25} - o(n^2). \quad (5.34)$$

The work of Chakrabati et al. [8] considers the **subgraph containment property**. For any finite graph X , let $h_{X,n}$ denote the graph property for graphs of size n which assigns a value of 1 to a graph if and only if it contains a subgraph isomorphic to X . This property is studied using another group action. For appropriate values of n , the vertex set V can be partitioned into sets V_1, \dots, V_m , where $|V_i| = q^{\alpha_i}$ for some prime power q which is greater than or equal to the number of vertices in X . Choose isomorphisms $V_i \cong \mathbb{F}_{q^{\alpha_i}}$. Let G be the group of permutations of V that is generated by the group $\mathbb{F}_{q^{\alpha_1}}^+ \times \dots \times \mathbb{F}_{q^{\alpha_m}}^+$ (acting on the sets V_1, \dots, V_m in a component-wise manner) and the group \mathbb{F}_q^* (acting simultaneously on all the sets V_i). If h_X were non-evasive, then there would exist nontrivial G -invariant graphs which do not satisfy h_X . Such graphs would have a uniform structure and would correspond simply to graphs on the set $\{1, 2, \dots, m\}$.

With this reduction the authors are able to prove that $h_{X,n}$ is evasive for all n within a set of positive density. In general, the following

asymptotic bound holds:

$$D(h_{X,n}) \geq \frac{n^2}{2} - O(n). \quad (5.35)$$

This approach was further developed by Babai et al. [3], who proved that $h_{X,n}$ is evasive for almost all n , and that

$$D(h_{X,n}) \geq \binom{n}{2} - O(1). \quad (5.36)$$

As one can observe from recent papers on evasiveness, advances in the strength of results are paralleled by substantial increases in the difficulty of the proofs! The increase in difficulty has become fairly steep at this point. Perhaps a new basic insight, like the one in [18], will be necessary to proceed further toward the Karp conjecture.

A

Appendix

A.1 Long Exact Sequences of Homology Groups

The goal of this part of the appendix is to give a complete proof of the following proposition. Our method is based on [2].

Proposition A.1. Suppose that there is an exact sequence of complexes:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X_m & \xrightarrow{F} & Y_m & \xrightarrow{G} & Z_m \longrightarrow 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d \\
 0 & \longrightarrow & X_{m-1} & \xrightarrow{F} & Y_{m-1} & \xrightarrow{G} & Z_{m-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X_0 & \xrightarrow{F} & Y_0 & \xrightarrow{G} & Z_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array} \tag{A.1}$$

Then, there exist homomorphisms $\gamma_n: H_n(Z_\bullet) \rightarrow H_{n-1}(X_\bullet)$ for $n \in \{1, 2, \dots, m\}$ such that the sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_m(X_\bullet) & \longrightarrow & H_m(Y_\bullet) & \longrightarrow & H_m(Z_\bullet) \\
 & & & & \searrow \gamma_m & & \\
 & & H_{m-1}(X_\bullet) & \longrightarrow & H_{m-1}(Y_\bullet) & \longrightarrow & H_{m-1}(Z_\bullet) \\
 & & & & \searrow \gamma_{m-1} & & \\
 & & & & & & \vdots \\
 & & & & & & \searrow \gamma_1 \\
 & & H_0(X_\bullet) & \longrightarrow & H_0(Y_\bullet) & \longrightarrow & H_0(Z_\bullet) \longrightarrow 0
 \end{array} \tag{A.2}$$

is exact.

We begin by addressing the case in which $m = 1$. Suppose that we have an exact sequence of complexes

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X_1 & \xrightarrow{F_1} & Y_1 & \xrightarrow{G_1} & Z_1 \longrightarrow 0 \\
 & & \downarrow d^X & & \downarrow d^Y & & \downarrow d^Z \\
 0 & \longrightarrow & X_0 & \xrightarrow{F_0} & Y_0 & \xrightarrow{G_0} & Z_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array} \tag{A.3}$$

Then, $H_1(X_\bullet)$ is equal to $\ker d^X$, and $H_0(X_\bullet)$ is equal to the group $X_0/\text{im } d^X$, which we denote by $\text{coker } d^X$. (The latter group is called the “cokernel” of d^X .) Similar statements hold for Y_\bullet and Z_\bullet .

Define a function

$$\gamma: \ker d^Z \rightarrow \operatorname{coker} d^X \quad (\text{A.4})$$

as follows. Suppose that z_1 is an element of $\ker d^Z$. Choose an element $y_1 \in Y_1$ which maps to z_1 . The element $dy_1 \in Y_0$ maps to zero under G_0 , and thus we can find a (unique) element $x_0 \in X_0$ which maps to dy_1 . Let $\gamma(x) \in \operatorname{coker} d^X$ be the coset containing x_0 .

Note that the value of $\gamma(x)$ does not depend on the choice of y_1 , since if we choose a different element \bar{y}_1 and obtain a different element $\bar{x}_0 \in X_0$, then we will have $\bar{y}_1 - y_1 = F_1(x_1)$ for some x_1 , and thus $\bar{x}_0 - x_0 = dx_1$, and \bar{x}_0 and x_0 will lie in the same coset of $\operatorname{coker} d^X$. Note also that $\gamma(x)$ is a homomorphism: if $z_1 = z'_1 + z''_1$, then for any chosen pre-images y_1, y'_1 , and y''_1 , the quantities y_1 and $(y'_1 + y''_1)$ will differ by an element of $\operatorname{im} F_1$, and this difference will likewise vanish when we map to $\operatorname{coker} d^X$.

Consider the sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker d^X & \longrightarrow & \ker d^Y & \longrightarrow & \ker d^Z \\ & & & & & \searrow \gamma & \\ & & & & & \operatorname{coker} d^X & \longrightarrow \operatorname{coker} d^Y \longrightarrow \operatorname{coker} d^Z \longrightarrow 0 \end{array} \quad (\text{A.5})$$

It is easy to see that this sequence is a complex. (The verification of this is left to the reader.) We wish to prove that the sequence is in fact exact. We do this in six steps.

- (1) **Exactness at $\ker d^X$.** Immediate.
- (2) **Exactness at $\ker d^Y$.** Suppose that $y_1 \in \ker d^Y$ is an element that is killed by the map to $\ker d^Z$. Then, there exists an element $x_1 \in X_1$ which maps to y_1 . We must have $dx_1 = 0$ (since otherwise y_1 could not be in the kernel of d^Y) and so y_1 lies in the image of $\ker d^X$.
- (3) **Exactness at $\ker d^Z$.** Suppose that $z_1 \in \ker d^Z$ is such that $\gamma(z_1) = 0$. Then, if we let y_1 and x_0 be the elements chosen in the definition of γ , we must have $x_0 = dx_1$ for

some $x_1 \in X_1$. The element $y'_1 := y_1 - F(x_1)$ maps to z_1 and satisfies $dy'_1 = 0$. Therefore, z_1 is in the image of $[\ker d^Y \rightarrow \ker d^Z]$.

- (4) **Exactness at $\operatorname{coker} d^X$.** Suppose that a coset of the form $x_0 + \operatorname{im} d^X \in \operatorname{coker} d^X$ maps to zero in $\operatorname{coker} d^Y$. Then, there exists $y_1 \in Y_1$ such that $dy_1 = F(x_0)$. Therefore the element $z_1 := G(y_1)$ maps to $(x_0 + \operatorname{im} d^X)$ under γ .
- (5) **Exactness at $\operatorname{coker} d^Y$.** Suppose that a coset of the form $y_0 + \operatorname{im} d^Y \in \operatorname{coker} d^Y$ maps to zero in $\operatorname{coker} d^Z$. Then, there exists $z_1 \in Z_1$ such that $dz_1 = G(y_0)$. If we let $y_1 \in Y_1$ be an element which maps to z_1 , then we have $y_0 - dy_1 = F(x_0)$ for some x_0 . Therefore the coset $(x_0 + \operatorname{im} d^X) \in \operatorname{coker} d^X$ maps to $(y_0 + \operatorname{im} d^Y)$.
- (6) **Exactness at $\operatorname{coker} d^Z$.** Immediate.

We conclude that sequence (A.5) is indeed exact.

To prove Proposition A.1 in general, we will need the following lemma, which is a modified version of what we just proved.

Lemma A.2. Suppose that the following is a diagram of maps of abelian groups.

$$\begin{array}{ccccccc}
 A_1 & \xrightarrow{f_1} & B_1 & \xrightarrow{g_1} & C_1 & \longrightarrow & 0 \\
 \downarrow d^A & & \downarrow d^B & & \downarrow d^C & & \\
 0 \longrightarrow & A_0 & \xrightarrow{f_0} & B_0 & \xrightarrow{g_0} & C_0 &
 \end{array} \tag{A.6}$$

Suppose that the maps are compatible ($d \circ f = f \circ d$ and $d \circ g = g \circ d$) and that the top and bottom rows are both exact. Then, there exists a homomorphism $\lambda: \ker f \rightarrow \operatorname{coker} d$ such that the sequence

$$\ker d \longrightarrow \ker e \longrightarrow \ker f \xrightarrow{\lambda} \operatorname{coker} d \longrightarrow \operatorname{coker} e \longrightarrow \operatorname{coker} f$$

is exact.

Proof. This follows by repeating the previous proof with steps (1) and (6) omitted. \square

Now we are ready to prove Proposition A.1 for arbitrary m . Take any $n \in \{1, 2, \dots, m\}$. As we know, in the diagram

$$\begin{array}{ccccccc} \text{coker } d_{n+1}^X & \longrightarrow & \text{coker } d_{n+1}^Y & \longrightarrow & \text{coker } d_{n+1}^Z & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker d_{n-1}^X & \longrightarrow & \ker d_{n-1}^Y & \longrightarrow & \ker d_{n-1}^Z \end{array} \quad (\text{A.7})$$

induced by diagram (A.1), both rows are exact. Therefore Lemma A.2 implies that

$$\begin{array}{ccccc} H_n(X_\bullet) & \longrightarrow & H_n(Y_\bullet) & \longrightarrow & H_n(Z_\bullet) \\ & & \searrow \gamma_n & & \\ H_{n-1}(X_\bullet) & \longrightarrow & H_{n-1}(Y_\bullet) & \longrightarrow & H_{n-1}(Z_\bullet) \end{array} \quad (\text{A.8})$$

is an exact sequence for some homomorphism γ_n . This completes the proof.

A.2 Properties of Barycentric Subdivision

This part of the appendix is a supplement to *Barycentric Subdivision*. Our purpose here is to prove two facts:

- (1) For any finite abstract simplicial complex Δ , the Euler characteristic of $\text{bar}(\Delta)$ is the same as that of Δ .
- (2) If Δ is collapsible, then $\text{bar}(\Delta)$ is also collapsible.

A more extensive discussion of the relationship between collapsibility and barycentric subdivision can be found in [39].

We begin by considering the Euler characteristic property. Recall that the Euler characteristic of a subset S of a simplicial complex Δ is given by

$$\chi(S) = \sum_{n \geq 0} (-1)^n |\{Q \in S, \dim(Q) = n\}|. \quad (\text{A.9})$$

Let us define three basic abstract simplicial complexes. For any $n \geq 1$, let Π_n denote the abstract simplicial complex consisting of the set of all nonempty subsets of $\{0, 1, \dots, n\}$. Let $\Theta_n = \Pi_n \setminus \{[0, 1, 2, \dots, n]\}$ and

$\Lambda_n = \Theta_n \setminus \{[0, 1, 2, \dots, n-1]\}$. It is easy to see that Π_n and Λ_n are both collapsible.

If Δ is an abstract simplicial complex and t is an element not contained in the vertex set of Δ , then let us say that **cone** of Δ over t , denoted by $t \star \Delta$, is the simplicial complex that arises from adding to Δ all sets of the form $\{t\} \cup Q$ with $Q \in \Delta$ or $Q = \emptyset$. Note that $n \star \Pi_{n-1} = \Pi_n$.

Lemma A.3. Let Δ be a finite abstract simplicial complex, and let t be an element that is not contained in the vertex set of Δ . Then, $\chi(t \star \Delta) = 1$.

Proof. For any $Q \in \Delta$, the set

$$\{Q, Q \cup \Delta\} \tag{A.10}$$

has Euler characteristic zero. The set $(t \star \Delta) \setminus \{\{t\}\}$ can be partitioned into such two-member sets. Therefore $\chi(t \star \Delta) = \chi(\{\{t\}\}) = 1$. \square

Proposition A.4. Let Σ be finite abstract simplicial complex of dimension n . Then, $\chi(\text{bar}(\Sigma)) = \chi(\Sigma)$.

Proof. We induct on n . The base case ($n = 0$) is trivial. Suppose that $n \geq 1$ and that the statement is known to hold for all complexes of dimension less than n .

The Euler characteristic of Θ_n is

$$\chi(\Theta_n) = \sum_{k=0}^{n-1} (-1)^k \binom{n+1}{k+1} = 1 - (-1)^n, \tag{A.11}$$

therefore by inductive assumption, $\chi(\text{bar}(\Theta_n))$ is also equal to $1 - (-1)^n$. The Euler characteristic of $\text{bar}(\Pi_n) = [0, 1, \dots, n] \star \text{bar}(\Theta_n)$ is 1 by Lemma A.3. Therefore,

$$\chi(\text{bar}(\Pi_n) \setminus \text{bar}(\Theta_n)) = 1 - (1 - (-1)^n) = (-1)^n. \tag{A.12}$$

An easy consequence of equation (A.12) is that if Δ is an n -dimensional abstract simplicial complex and $\Delta' \subset \Delta$ is a subcomplex that arises from deleting a single n -simplex from Δ , then

$$\chi(\text{bar}(\Delta) \setminus \text{bar}(\Delta')) = (-1)^n. \quad (\text{A.13})$$

Let Σ be a finite n -dimensional abstract simplicial complex. Let d be the number of n -simplices in Σ , and let $\Sigma^{(n-1)} \subseteq \Sigma$ be the subcomplex that arises from deleting all n -simplices. Then,

$$\chi(\text{bar}(\Sigma)) = \chi(\text{bar}(\Sigma^{(n-1)})) + d \cdot (-1)^n. \quad (\text{A.14})$$

Since $\chi(\text{bar}(\Sigma^{(n-1)})) = \chi(\Sigma^{(n-1)})$ by inductive assumption, we have $\chi(\text{bar}(\Sigma)) = \chi(\Sigma)$ as desired. \square

Now we turn to collapsibility. If Σ is an abstract simplicial complex and $\Sigma' \subseteq \Sigma$ is a subcomplex, then let us say that Σ **can be collapsed onto** Σ' if there exists a sequence of elementary collapses (or equivalently, a sequence of primitive elementary collapses),

$$\Sigma = \Sigma_0, \Sigma_1, \dots, \Sigma_m = \Sigma'. \quad (\text{A.15})$$

Lemma A.5. If Σ is a collapsible abstract simplicial complex and t is not in the vertex set of Σ , then the cone $t \star \Sigma$ can be collapsed onto Σ .

Proof. There must exist a collapsing procedure which collapses Σ to a single 0-simplex $\{s\} \in \Sigma$. The same procedure collapses $t \star \Sigma$ onto the subcomplex

$$\Sigma \cup \{t, s\} \cup \{s\}, \quad (\text{A.16})$$

which can be collapsed onto Σ . \square

Let $\text{bar}(\Pi_n)$ denote the barycentric subdivision of Π_n (see Definition 4.1). The set $\text{bar}(\Pi_n)$ is the set of all \subset -chains of nonempty subsets of $\{0, 1, \dots, n\}$. The cone

$$[0, 1, 2, \dots, n] \star \text{bar}(\Lambda_n) \quad (\text{A.17})$$

is a subcomplex of $\text{bar}(\Pi_n)$.

Lemma A.6. The abstract simplicial complex $\text{bar}(\Pi_n)$ can be collapsed onto $[0, 1, \dots, n] \star \text{bar}(\Lambda_n)$.

Proof. Let

$$\Gamma = \{Q \in \text{bar}(\Pi_n) \mid [0, 1, \dots, n-1] \in S, [0, 1, \dots, n] \notin S\}. \quad (\text{A.18})$$

Write the elements of Γ as a sequence

$$Q_1, Q_2, \dots, Q_m \in \Gamma \quad (\text{A.19})$$

so that $\dim(Q_i) \geq \dim(Q_{i+1})$. Let

$$\text{bar}(\Pi_n) = \Sigma_0, \Sigma_1, \Sigma_2, \dots, \Sigma_m \quad (\text{A.20})$$

be the sequence of subcomplexes of $\text{bar}(\Pi_n)$ that arises from deleting the pairs $(Q_i, Q_i \cup \{[0, 1, \dots, n]\})$ from $\text{bar}(\Pi_n)$ one at a time. This is a sequence of primitive elementary collapses, and its final term is $[0, 1, \dots, n] \star \text{bar}(\Lambda_n)$. \square

Proposition A.7. Let Δ be an abstract simplicial complex of dimension $n \geq 0$. Suppose that Δ' is a subcomplex of Δ , and suppose that Δ can be collapsed onto Δ' . Then, $\text{bar}(\Delta)$ can be collapsed onto $\text{bar}(\Delta')$.

Proof. Again we induct on n . The base case ($n = 0$) is immediate. Suppose that $n \geq 1$ and that the proposition is known to hold for all simplicial complexes of dimension smaller than n .

Every collapsing sequence can be expanded into a sequence of primitive elementary collapses. Therefore, to prove the proposition, it suffices to show that for any $k \geq n$, the complex $\text{bar}(\Pi_k)$ can be collapsed onto $\text{bar}(\Lambda_k)$. By inductive assumption, we know this to be true for $k < n$, and so we need only prove it for $k = n$.

By Lemma A.6, the complex $\text{bar}(\Pi_n)$ can be collapsed onto $[0, 1, \dots, n] \star \text{bar}(\Lambda_n)$. The complex Λ_n is collapsible, and therefore $\text{bar}(\Lambda_n)$ is collapsible by inductive assumption (since $\dim \Lambda_n = n - 1$). By Lemma A.5, the cone $[0, 1, \dots, n] \star \text{bar}(\Lambda_n)$ can be collapsed to $\text{bar}(\Lambda_n)$. This completes the proof. \square

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