Matroids and the greedy algorithm

Rudi Pendavingh

March 7, 2013

Spanning forests and -trees

Let $G = (V, E)$ be an undirected graph, and let $F \subseteq E$

- F is a *forest* if (V, F) does not contain any cycles.
- F *spans* G if (V, F) and G have the same number of components.
- F is a *tree* if (V, F) is a forest with exactly one component.

The maximum spanning forest problem

Given: A graph $G = (V, E)$, a weight function $w : E \to \mathbb{R}$.
Find: A spanning forest F such that $w[F]$ is as large as possible.

Kruskal's algorithm

Given are an undirected graph $G = (V, E)$ and a weight function $w : E \to \mathbb{R}$.

Kruskal's algorithm

1. Sort the edges by weight, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.
2. $F \leftarrow \emptyset$, $i \leftarrow 1$
3. while $i < |E|$:
    1. if $F \cup \{e_i\}$ is a forest, put $F \leftarrow F \cup \{e_i\}$
    2. $i \leftarrow i + 1$

Theorem

Kruskal's algorithm finds a maximum-weight spanning forest.

Matroids

A matroid is determined by a finite set E, the *ground set*, and a partition of the set of subsets of E in *dependent* and *independent sets*.

Matroids

A matroid is determined by a finite set E, the *ground set*, and a partition of the set of subsets of E in *dependent* and *independent sets*.

Definition (Matroid)

A *matroid* is a pair $(E, \mathcal{I})$, where E is a finite set, and $\mathcal{I} \subseteq 2^E$, such that:

- I0 $\emptyset \in \mathcal{I}$
- I1 if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
- I2 if $I, J \in \mathcal{I}$ and $|I| < |J|$, then $\exists e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$

Matroids

A matroid is determined by a finite set E, the *ground set*, and a partition of the set of subsets of E in *dependent* and *independent sets*.

Definition (Matroid)

A *matroid* is a pair $(E, \mathcal{I})$, where E is a finite set, and $\mathcal{I} \subseteq 2^E$, such that:

- I0 $\emptyset \in \mathcal{I}$
- I1 if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
- I2 if $I, J \in \mathcal{I}$ and $|I| < |J|$, then $\exists e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$

Example (The Fano Matroid)

Let $E := \{a, b, c, d, e, f, g\}$ and let

$$\mathcal{I} := \{I \subseteq E \mid |I| \leq 3\} \setminus \{abc, cde, efa, adg, cfg, beg, bdf\}$$

Then $F_7 := (E, \mathcal{I})$ is the *Fano matroid*.

Matroids

A matroid is determined by a finite set E, the *ground set*, and a partition of the set of subsets of E in *dependent* and *independent sets*.

Definition (Matroid)

A *matroid* is a pair $(E, \mathcal{I})$, where E is a finite set, and $\mathcal{I} \subseteq 2^E$, such that:

- I0 $\emptyset \in \mathcal{I}$
- I1 if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
- I2 if $I, J \in \mathcal{I}$ and $|I| < |J|$, then $\exists e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$

Example (Graphic matroid)

Let $G = (V, E)$ be an undirected graph and let

$$\mathcal{I} := \{F \subseteq E \mid (V, F) \text{ is a forest}\}.$$

Then $M(G) := (E, \mathcal{I})$ is a *graphic matroid*.

Matroids

A matroid is determined by a finite set E, the *ground set*, and a partition of the set of subsets of E in *dependent* and *independent sets*.

Definition (Matroid)

A *matroid* is a pair $(E, \mathcal{I})$, where E is a finite set, and $\mathcal{I} \subseteq 2^E$, such that:

- I0 $\emptyset \in \mathcal{I}$
- I1 if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$
- I2 if $I, J \in \mathcal{I}$ and $|I| < |J|$, then $\exists e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$

Example (Linear matroid)

Let $\mathbb{F}$ be a field and let $E \subseteq \mathbb{F}^k$ be a finite set of vectors. Let

$$\mathcal{I} := \{F \subseteq E \mid F \text{ is linearly independent over } \mathbb{F}\}.$$

Then $M(E, \mathbb{F}) := (E, \mathcal{I})$ is a *linear matroid*.

The greedy algorithm

if $M = (E, \mathcal{I})$ is a matroid, then $F \subseteq E$ is a *basis* if F is an inclusionwise maximal independent set.

The maximum-weight basis problem

Given: A matroid $M = (E, \mathcal{I})$, a weight function $w : E \to \mathbb{R}$.
Find: A basis F such that $w[F]$ is as large as possible.

The greedy algorithm

1. Sort the edges by weight, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.
2. $F \leftarrow \emptyset$, $i \leftarrow 1$
3. while $i < |E|$:
    1. if $F \cup \{e_i\}$ is independent, put $F \leftarrow F \cup \{e_i\}$
    2. $i \leftarrow i + 1$

The greedy algorithm characterizes matroids

Theorem

Let $M = (E, \mathcal{I})$ be such that

- I0 $\emptyset \in \mathcal{I}$, *and*
- I1 *if* $J \in \mathcal{I}$ *and* $I \subseteq J$, *then* $I \in \mathcal{I}$.

Then M is a matroid if and only if the greedy algorithm finds a basis B of maximum weight $w[B]$, for each weight function $w : E \to \mathbb{R}_+$.

The greedy algorithm characterizes matroids

Theorem

Let $M = (E, \mathcal{I})$ be such that

- I0 $\emptyset \in \mathcal{I}$, *and*
- I1 *if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$.*

Then M is a matroid if and only if the greedy algorithm finds a basis B of maximum weight $w[B]$, for each weight function $w : E \to \mathbb{R}_+$.

Proof outline: We first prove sufficiency, '$\Leftarrow$'.

The greedy algorithm characterizes matroids

Theorem

Let $M = (E, \mathcal{I})$ be such that

- I0 $\emptyset \in \mathcal{I}$, *and*
- I1 *if* $J \in \mathcal{I}$ *and* $I \subseteq J$, *then* $I \in \mathcal{I}$.

Then M is a matroid if and only if the greedy algorithm finds a basis B of maximum weight $w[B]$, for each weight function $w : E \to \mathbb{R}_+$.

Proof outline: We first prove sufficiency, '$\Leftarrow$'.

- Suppose $M = (E, \mathcal{I})$ is not a matroid. Then there exist $I, J \in \mathcal{I}$ such that $|I| < |J|$, but $\nexists e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$

The greedy algorithm characterizes matroids

Theorem

Let $M = (E, \mathcal{I})$ be such that

- I0 $\emptyset \in \mathcal{I}$, *and*
- I1 *if* $J \in \mathcal{I}$ *and* $I \subseteq J$, *then* $I \in \mathcal{I}$.

Then M is a matroid if and only if the greedy algorithm finds a basis B of maximum weight $w[B]$, for each weight function $w : E \to \mathbb{R}_+$.

Proof outline: We first prove sufficiency, '$\Leftarrow$'.

- Suppose $M = (E, \mathcal{I})$ is not a matroid. Then there exist $I, J \in \mathcal{I}$ such that $|I| < |J|$, but $\nexists e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$
- Let $k := |I|$. Define $w : E \to \mathbb{R}_+$ by $w(e) := k + 2$ if $e \in I$, $w(e) := k + 1$ if $e \in J \setminus I$, $w(e) := 0$ if $e \notin J$.

The greedy algorithm characterizes matroids

Theorem

Let $M = (E, \mathcal{I})$ be such that

I0 $\emptyset \in \mathcal{I}$, *and*

I1 *if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$.*

Then M is a matroid if and only if the greedy algorithm finds a basis B of maximum weight $w[B]$, for each weight function $w : E \to \mathbb{R}_+$.

Proof outline: We first prove sufficiency, '$\Leftarrow$'.

- Suppose $M = (E, \mathcal{I})$ is not a matroid. Then there exist $I, J \in \mathcal{I}$ such that $|I| < |J|$, but $\nexists e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{I}$
- Let $k := |I|$. Define $w : E \to \mathbb{R}_+$ by $w(e) := k + 2$ if $e \in I$, $w(e) := k + 1$ if $e \in J \setminus I$, $w(e) := 0$ if $e \notin J$.
- The greedy algorithm outputs $B \supseteq I$ with $w[B] = w[I] = k(k + 2) < (k + 1)(k + 1) \leq w[J]$. So B is not optimal.

The greedy algorithm characterizes matroids

Theorem

Let $M = (E, \mathcal{I})$ *be such that*

- I0 $\emptyset \in \mathcal{I}$, *and*
- I1 *if* $J \in \mathcal{I}$ *and* $I \subseteq J$, *then* $I \in \mathcal{I}$.

Then M *is a matroid if and only if the greedy algorithm finds a basis* B *of maximum weight* $w[B]$, *for each weight function* $w : E \to \mathbb{R}_+$.

Proof outline: We next prove necessity, '$\Rightarrow$'.

The greedy algorithm characterizes matroids

Theorem

Let $M = (E, \mathcal{I})$ be such that

- I0 $\emptyset \in \mathcal{I}$, *and*
- I1 *if* $J \in \mathcal{I}$ *and* $I \subseteq J$, *then* $I \in \mathcal{I}$.

Then M is a matroid if and only if the greedy algorithm finds a basis B of maximum weight $w[B]$, for each weight function $w : E \to \mathbb{R}_+$.

Proof outline: We next prove necessity, '$\Rightarrow$'.

- Suppose $M = (E, \mathcal{I})$ is a matroid. Let $w : E \to \mathbb{R}_+$ be a weight function.

The greedy algorithm characterizes matroids

Theorem

Let $M = (E, \mathcal{I})$ be such that

- I0 $\emptyset \in \mathcal{I}$, *and*
- I1 *if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$.*

Then M is a matroid if and only if the greedy algorithm finds a basis B of maximum weight $w[B]$, for each weight function $w : E \to \mathbb{R}_+$.

Proof outline: We next prove necessity, '$\Rightarrow$'.

- Suppose $M = (E, \mathcal{I})$ is a matroid. Let $w : E \to \mathbb{R}_+$ be a weight function.
- Call an independent set $I \in \mathcal{I}$ *greedy* if there is a maximum-weight basis B so that $I \subseteq B$.

The greedy algorithm characterizes matroids

Theorem

Let $M = (E, \mathcal{I})$ be such that

- I0 $\emptyset \in \mathcal{I}$, *and*
- I1 *if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$.*

Then M is a matroid if and only if the greedy algorithm finds a basis B of maximum weight $w[B]$, for each weight function $w : E \to \mathbb{R}_+$.

Proof outline: We next prove necessity, '$\Rightarrow$'.

- Suppose $M = (E, \mathcal{I})$ is a matroid. Let $w : E \to \mathbb{R}_+$ be a weight function.
- Call an independent set $I \in \mathcal{I}$ *greedy* if there is a maximum-weight basis B so that $I \subseteq B$.
- To prove: if I is greedy, and e attains the maximum in $\max\{w(e) \mid I \cup \{e\} \in \mathcal{I}, e \in E \setminus I\}$, then $I \cup \{e\}$ is greedy.

Transversal matroids

Let E be a finite set, and let $\mathcal{A}$ be a finite set of subsets of E.
A *transversal* of $\mathcal{A}$ is a set $F \subseteq E$ so that there exists an injection $\phi : F \to \mathcal{A}$ with $e \in \phi(e)$ for all $e \in F$.

Example (Transversal matroids)

Let E be a finite set, and let $\mathcal{A}$ be a finite set of subsets of E. Put

$$\mathcal{I} := \{F \subseteq E \mid F \text{ is a transversal of } \mathcal{A}\}.$$

Then $M(E, \mathcal{A}) := (E, \mathcal{I})$ is a *transversal matroid*.

Gammoids

Let $D = (V, A)$ be a directed graph and let $S, T \subseteq V$. Then a subset $F \subseteq T$ is *linked* to S in D if there is a set of vertex-disjoint directed paths with starting points in S and with endpoints F.

Example (Gammoids)

Let $D = (V, A)$ be a directed graph, and let $S, T \subseteq V$. Let

$$\mathcal{I} := \{F \subseteq T \mid F \text{ is linked to } S \text{ in } D\}.$$

Then $M(D, S, T) := (V, \mathcal{I})$ is a *gammoid*.

Algebraic matroids

Definition

Let $\mathbb{K}$ be an extension field of $\mathbb{F}$. A set $\{x_1, \ldots, x_n\} \subseteq \mathbb{K}$ is *algebraically dependent over* $\mathbb{F}$ if there exists a polynomial p with coefficients in $\mathbb{F}$ such that $p(x_1, \ldots, x_n) = 0$.

Algebraic matroids

Definition

Let $\mathbb{K}$ be an extension field of $\mathbb{F}$. A set $\{x_1, \ldots, x_n\} \subseteq \mathbb{K}$ is *algebraically dependent over* $\mathbb{F}$ if there exists a polynomial p with coefficients in $\mathbb{F}$ such that $p(x_1, \ldots, x_n) = 0$.

Example (Algebraic matroids)

Let $\mathbb{K}$ be an extension field of $\mathbb{F}$, and let $E \subseteq \mathbb{K}$ be finite. Let

$$\mathcal{I} := \{F \subseteq E \mid F \text{ algebraically independent over } \mathbb{F}\}$$

Then $M(E, \mathbb{F}) := (E, \mathcal{I})$ is an *algebraic matroid*.

‘Half-plane property‘ matroids

Definition

Let $H := \{z \in \mathbb{C} \mid \Re(z) > 0\}$. A complex polynomial p in n variables has the *half-plane property* if $p(x_1, \ldots, x_n) \neq 0$ for all $x_1, \ldots, x_n \in H$.

‘Half-plane property‘ matroids

Definition

Let $H := \{z \in \mathbb{C} \mid \Re(z) > 0\}$. A complex polynomial p in n variables has the *half-plane property* if $p(x_1, \ldots, x_n) \neq 0$ for all $x_1, \ldots, x_n \in H$.

Let $\{x_e \mid e \in E\}$ be variables. For $F \subseteq E$, we write $x^F := \prod_{e \in F} x_e$.

‘Half-plane property‘ matroids

Definition

Let $H := \{z \in \mathbb{C} \mid \Re(z) > 0\}$. A complex polynomial p in n variables has the *half-plane property* if $p(x_1, \ldots, x_n) \neq 0$ for all $x_1, \ldots, x_n \in H$.

Let $\{x_e \mid e \in E\}$ be variables. For $F \subseteq E$, we write $x^F := \prod_{e \in F} x_e$.

Theorem

Let $p = \sum_{F \subseteq E} p_F x^F$ be a homogeneous complex polynomial. If p has the half-plane property, then

$$\{F \subseteq E \mid p_F \neq 0\}$$

is the set of bases of a matroid on E.

‘Half-plane property‘ matroids

Definition

Let $H := \{z \in \mathbb{C} \mid \Re(z) > 0\}$. A complex polynomial p in n variables has the *half-plane property* if $p(x_1, \ldots, x_n) \neq 0$ for all $x_1, \ldots, x_n \in H$.

Let $\{x_e \mid e \in E\}$ be variables. For $F \subseteq E$, we write $x^F := \prod_{e \in F} x_e$.

Theorem

Let $p = \sum_{F \subseteq E} p_F x^F$ be a homogeneous complex polynomial. If p has the half-plane property, then

$$\{F \subseteq E \mid p_F \neq 0\}$$

is the set of bases of a matroid on E.

... so these are the *half-plane-property (HPP) matroids*.

‘Half-plane property‘ matroids

Definition

Let $H := \{z \in \mathbb{C} \mid \Re(z) > 0\}$. A complex polynomial p in n variables has the *half-plane property* if $p(x_1, \ldots, x_n) \neq 0$ for all $x_1, \ldots, x_n \in H$.

Let $\{x_e \mid e \in E\}$ be variables. For $F \subseteq E$, we write $x^F := \prod_{e \in F} x_e$.

Theorem

Let $p = \sum_{F \subseteq E} p_F x^F$ be a homogeneous complex polynomial. If p has the half-plane property, then

$$\{F \subseteq E \mid p_F \neq 0\}$$

is the set of bases of a matroid on E.

... so these are the *half-plane-property (HPP) matroids*.

The matroid polytope

If $A \subseteq E$, then its *incidence vector* $x^A \in \mathbb{R}^E$ is determined by

$$x^A_e = \begin{cases} 1 & \text{if } e \in A \\ 0 & \text{if } e \notin A \end{cases}$$

Definition (Matroid polytope)

Let $M = (E, \mathcal{I})$ be a matroid. The *matroid polytope* is

$$P(M) := \text{conv.hull}\{x^I \mid I \in \mathcal{I}\}.$$

The *rank* of $F \subseteq E$ in $M = (E, \mathcal{I})$ is $r_M(F) := \max\{|I| \mid I \in \mathcal{I}, I \subseteq F\}$.

Theorem

$P(M) = \{x \in \mathbb{R}^E \mid x[F] \leq r_M(F)$ *for all* $F \subseteq E,\ x \geq 0\}$

Theorem

$P(M) = \{x \in \mathbb{R}^E \mid x[F] \leq r_M(F) \textit{ for all } F \subseteq E,\ x \geq 0\}$

Proof outline: It suffices to prove that for any $w : E \to \mathbb{R}$, the problem

$$\max\{w^T x \mid x \in P(M)\}$$

has an optimal solution $x^* = x^I$, where I is an independent set of M.

- Let $f_1, f_2, \ldots, f_m$ be the elements of E as chosen by the greedy algorithm.
- Let $F_i := \{e \in E \mid r_M\{f_1, \ldots, f_i, e\} = r_M\{f_1, \ldots, f_i\}\}$.
- Let $p = \max\{i \mid w(f_i) > 0\}$, and put $I := \{f_1, \ldots, f_p\}$
- If $x \in P(M)$, then

$$w^T x \leq \sum_{i=1}^{p} u_i x[F_i] \leq \sum_{i=1}^{p} u_i r_M(F_i) \leq \sum_{i=1}^{p} w(f_i) = w^T x^I$$

  for an appropriate choice of $u_i \geq 0$. So x^I is an optimal solution.

Some proof details

We choose $u_i := w(f_i) - w(f_{i+1})$ for $i = 1, \ldots, p-1$, $u_p := w(f_p)$.

- note: $r(F_i) = i$ for each i
- $\sum_{i=1}^{p} u_i r_M(F_i) = \sum_{i=1}^{p} w(f_i)$
- if $x \in P(M)$, then $x[F_i] \leq r(F_i)$ by definition of $P(M)$, hence

$$\sum_{i=1}^{p} u_i x[F_i] \leq \sum_{i=1}^{p} u_i r_M(F_i)$$

- to prove $w^T x \leq \sum_{i=1}^{p} u_i x[F_i]$, we need to argue for each e that

$$w(e) \leq \sum_{i=k}^{p} u_i = w(f_k)$$

where $k := \min\{i \mid e \in F_i\}$. But if $e \in F_k \setminus F_{k-1}$, then $w(f_k) \geq w(e)$.

Homework

- Determine if the Fano matroid is graphic/ linear/ algebraic/ a gammoid/ transversal/ HPP.
- Read sections 10.1, 10.2, 10.3, and 10.7 (until Thm. 10.14) of the handout.
- Make exercises 10.1, 10.5, 10.18 of the handout.