Decompositions of Graphs
— DFS/BFS, DAG, SCC, Bicomp

Hengfeng Wei

hfwei@nju.edu.cn

June 12, 2019

John Hopcroft

Robert Tarjan

"For fundamental achievements in the design and analysis of algorithms and data structures."

— *Turing Award*, 1986

SIAM J. COMPUT.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

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"DFS is a powerful technique with many applications."

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Power of DFS:

Graph Traversal $\Longrightarrow$ Graph Decomposition

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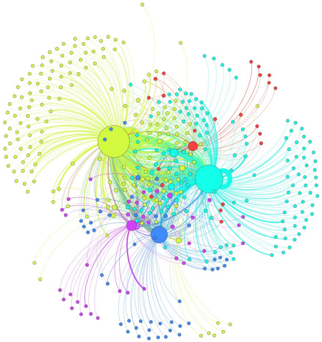
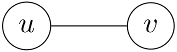
Structure! Structure! Structure!



Graph *structure* induced by DFS:

states of 

types of 

Graph *structure* induced by DFS:

states of v

types of u — v

life time of v:

v : d[v], f[v]

d[v]: Bicomp

f[v]: Toposort, SCC

Definition (Classifying edges)

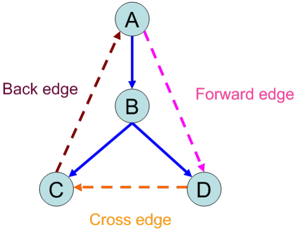
Given a DFS traversal $\implies$ DFS tree:

Tree edge: $\rightarrow$ child

Back edge: $\rightarrow$ ancestor

Forward edge: $\rightarrow$ *nonchild* descendant

Cross edge: $\rightarrow$ $(\neg$ancestor$) \wedge (\neg$descendant$)$



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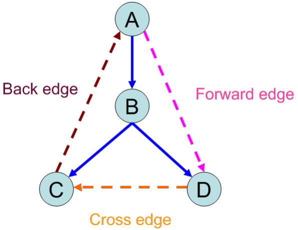
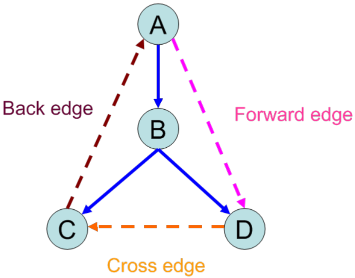
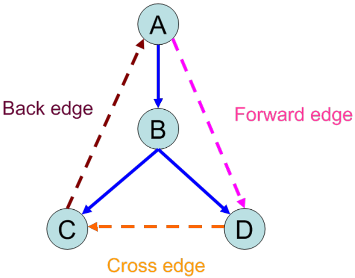


- Also applicable to BFS
- w.r.t. DFS/BFS trees



DFS on directed graph



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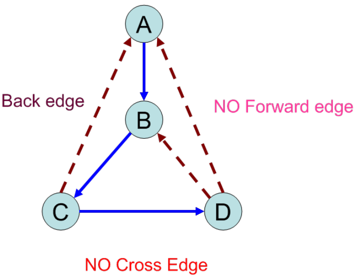
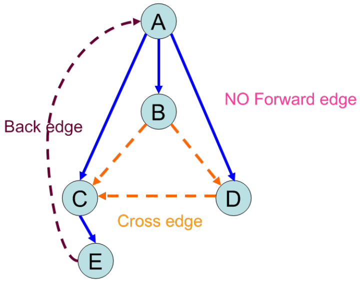
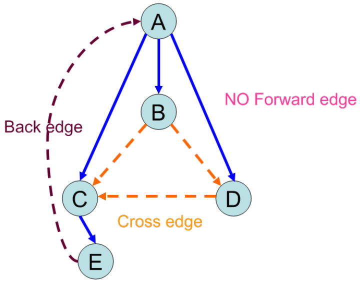


DFS on undirected graph



BFS on directed graph



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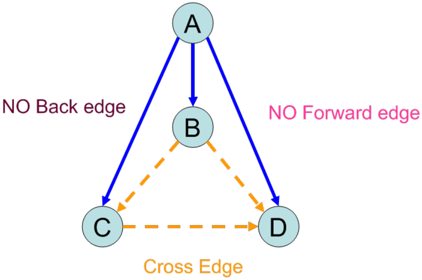
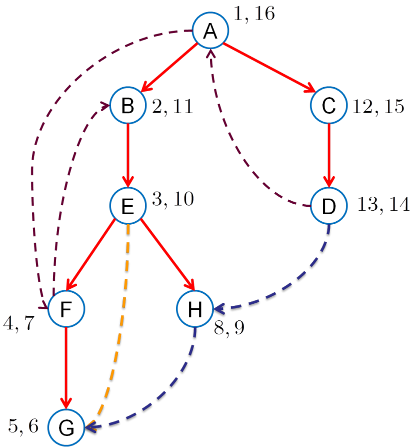


BFS on undirected graph (Problem 5.1)

Life time of vertices in DFS



Theorem (Disjoint or Contained (Problem 4.2 : (1)&(2)))

$$\forall u, v : [_u\ ]_u \cap [_v\ ]_v = \emptyset \bigvee \Big([_u\ ]_u \subset [_v\ ]_v \vee [_v\ ]_v \subset [_u\ ]_u \Big)$$

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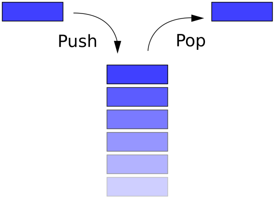
Proof.



□

Edge types and life time of vertices in DFS (Problem 4.5)

$$\forall u \to v :$$

- tree/forward edge: $[_u \; [_v \;]_v \;]_u$
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$$\text{f}[u] < \text{f}[v] \iff \text{back edge}$$

$$\nexists \text{ cycle} \implies \boxed{u \to v \iff \text{f}[v] < \text{f}[u]}$$

Counting shortest paths (Problem 5.10)

Counting # of shortest paths in (un)directed graphs using BFS.

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Maybe in the next class...

Cycle detection (Problem 5.8 – 1)

	Digraph	Undirected graph
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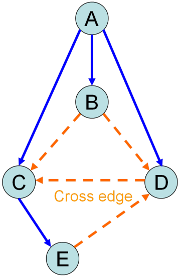

Evasiveness of acyclicity of undirected graphs (Problem 5.8 – 2)

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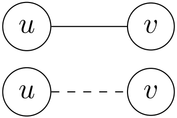

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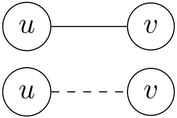

Hint: Kruskal

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$$\Longleftrightarrow$$

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$$\iff$$

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Q : Why adjacency matrix?

After-class Exercise: Evasiveness of connectivity of undirected graphs

$$\text{Evasiveness} \triangleq \text{check} \binom{n}{2} \text{ edges (adjacency matrix)}$$

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Hint: Anti-Kruskal

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$\nexists$ back edge $\iff$ DAG

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Sort vertices in *decreasing* order of their *finish* times.

Kahn's TOPOSORT algorithm (1962; Problem 4.16)

- Queue Q for source vertices ($\text{in}[v] = 0$)
- Repeat: DEQUEUE($\exists u \in Q$), output u

  delete u and $u \to v$ from Q,

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Every DAG has at least one source (and at least one sink vertex).

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Lemma (Correctness of Kahn's TOPOSORT)

Every DAG has at least one source (and at least one sink vertex).

Q : What if G is *not* a DAG?

Hamiltonian path in DAG (Problem 4.14)

HP: path visiting each vertex once

Q : $\exists$ HP in a DAG in $O(n+m)$

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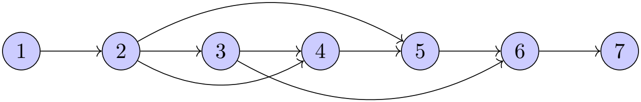

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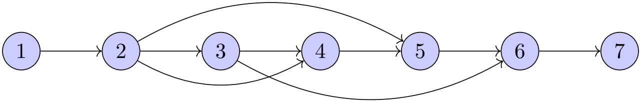

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Tarjan's TOPOSORT + Check edges (v_i, v_{i+1})

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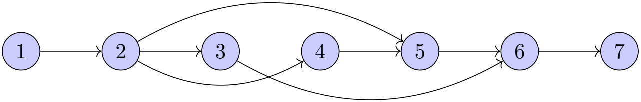

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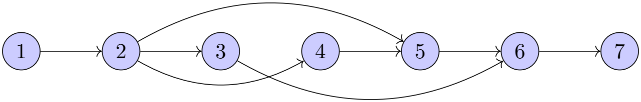

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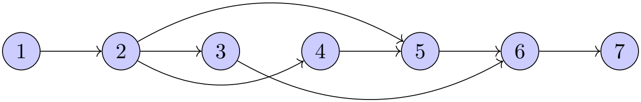

Kahn's TOPOSORT (Problem 4.16)

$$|Q| \leq 1$$

Theorem (Digraph as DAG (Problem 4.6))

Every digraph is a dag of its SCCs.

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Two tiered structure of digraphs:

digraph $\equiv$ a dag of SCCs

SCC: equivalence class over reachability

$\boxed{\text{digraph} \equiv \text{a dag of SCCs}}$

Kosaraju's SCC algorithm, 1978

"SCCs can be topo-sorted

in decreasing order of their highest finish time."

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(II) DFS on G^T; DFS/BFS on G

One-to-all reachability in a digraph (Problem 5.12)

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$$\Longleftarrow : \exists !\ \text{source}$$

$$\Longrightarrow : \text{ By contradiction.}$$

$$\exists u : v \not\leadsto u \land \text{in}[u] > 0 \implies \exists \text{ cycle}$$

Impacts of vertices in a digraph (Problem 4.18)

$$\text{impact}(v) = |\{w \neq v : v \leadsto w\}|$$

- $\arg\min_v \text{impact}(v)$
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$Q : \forall v$, computing $\text{impact}(v)$

2SAT (Problem 4.23)

$$I : (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (\overline{x_3} \vee x_4) \wedge (\overline{x_1} \vee x_4)$$

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Theorem (2SAT)

$$\exists\ SCC\ \exists x : v_x \in SCC \wedge v_{\overline{x}} \in SCC \iff I \textit{ is not satisfiable.}$$

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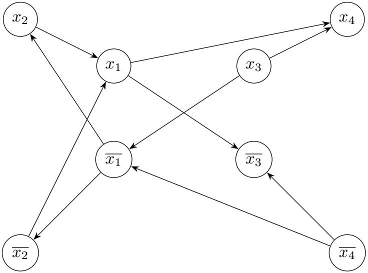
Implication graph G_I.

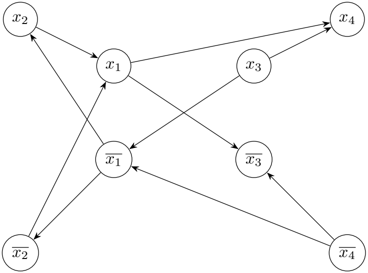
Theorem (2SAT)

$$\exists\ SCC\ \exists x : v_x \in SCC \wedge v_{\overline{x}} \in SCC \iff I \textit{ is not satisfiable.}$$

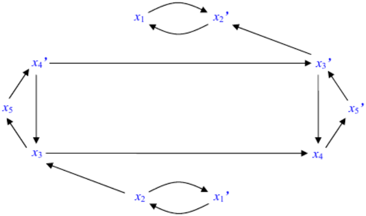
Reference:

- *"A Linear-time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas"* by Bengt Aspvall, Michael Plass, and Robert Tarjan, 1979.


x_2
x_4
x_1
x_3
$\overline{x_1}$
$\overline{x_3}$
$\overline{x_2}$
$\overline{x_4}$

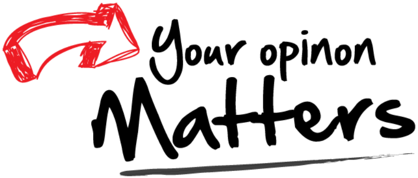


x_2
x_4
x_1
x_3
$\overline{x_1}$
$\overline{x_3}$
$\overline{x_2}$
$\overline{x_4}$



x_1
x_2'
x_4'
x_3'
x_5
x_5'
x_3
x_4
x_2
x_1'


THANK YOU



Office 302

Mailbox: H016

hfwei@nju.edu.cn