Decompositions of Graphs

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June 12, 2018

John Hopcroft

Robert Tarjan

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Robert Tarjan

"For fundamental achievements in the design and analysis of algorithms and data structures."

— *Turing Award, 1986*

SIAM J. COMPUT.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

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Key words. Algorithm, backtracking, biconnectivity, connectivity, depth-first, graph, search, spanning tree, strong-connectivity.

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"DFS is a powerful technique with many applications."

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Power of DFS:

Graph Traversal $\Longrightarrow$ Graph Decomposition

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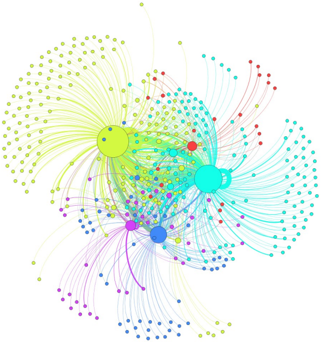
Structure! Structure! Structure!



Graph *structure* induced by DFS:

states of v

types of u v

Graph *structure* induced by DFS:

life time of v:

states of v

v : d$[v]$, f$[v]$

types of u — v

f$[v]$: DAG, SCC

d$[v]$: biconnectivity

Definition (Classifying edges)

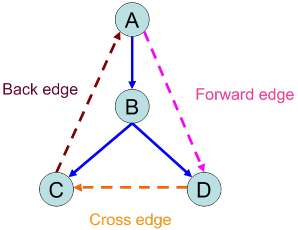
Given a DFS traversal $\implies$ DFS tree:

Tree edge: $\to$ child

Back edge: $\to$ ancestor

Forward edge: $\to$ *nonchild* descendant

Cross edge: $\to$ $(\neg$ancestor$) \wedge (\neg$descendant$)$



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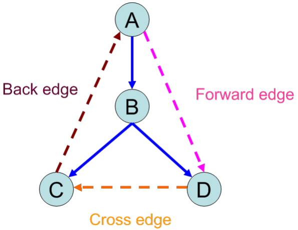
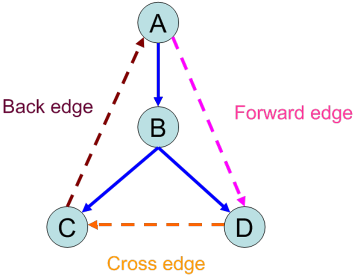
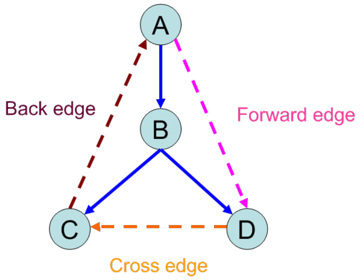


- also applicable to BFS
- w.r.t. DFS/BFS trees



DFS on directed graph



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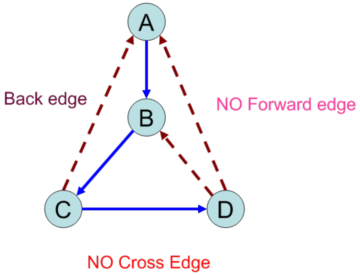
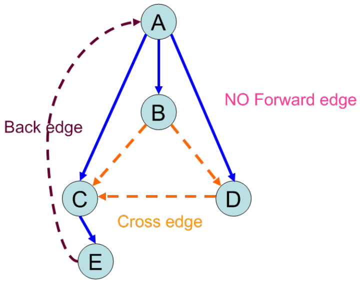
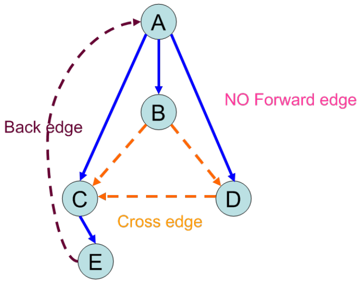


DFS on undirected graph



BFS on directed graph



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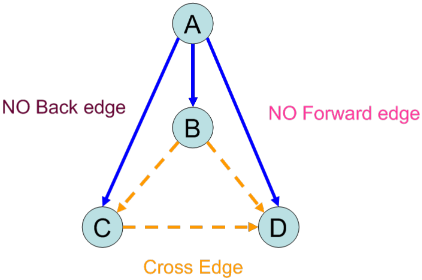

BFS on undirected graph

DFS tree and BFS tree coincide (Problem 5.7)

Undirected connected graph $G = (V, E), v \in V$

DFS tree T from $v \equiv$ BFS tree T' from v

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Proof.

G_{DFS}: tree + back *vs.* G_{BFS}: tree + cross

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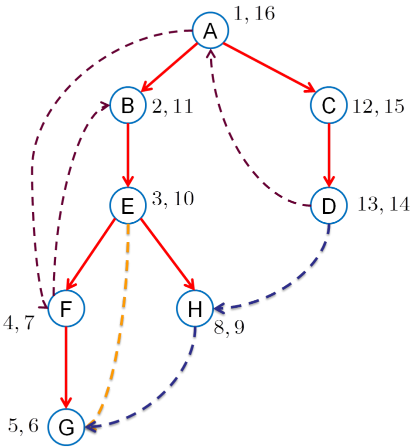
Q : What if G is a digraph?

Lift time of vertices in DFS



Theorem (Disjoint or Contained (Problem 4.2: (1) & (2)))

$$\forall u, v : [_u\ ]_u \cap [_v\ ]_v = \emptyset \bigvee \left([_u\ ]_u \subsetneqq [_v\ ]_v \vee [_v\ ]_v \subsetneqq [_u\ ]_u\right)$$

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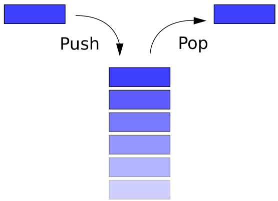
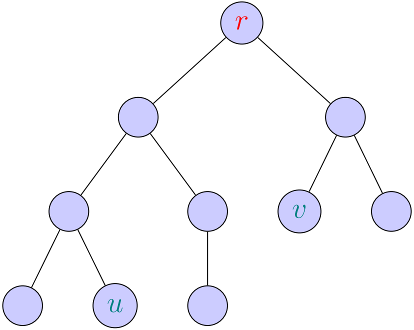


$\square$

Preprocessing for ancestor/descendant relation (Problem 5.23)



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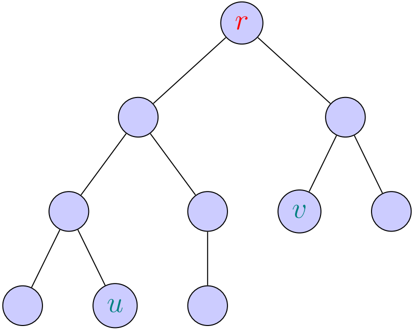

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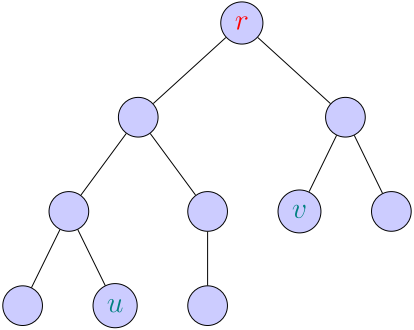

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Q : # of descendants of any v?

Edge types and life time of vertices in DFS (Problem 4.5)

$$\forall u \to v :$$

- tree/forward edge: $[_u\ [_v\ ]_v\ ]_u$
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$$\nexists \text{ cycle} \implies \boxed{u \to v \iff \mathsf{f}[v] < \mathsf{f}[u]}$$

Height and diameter of tree (Problem 5.4)

Binary tree $T = (V, E)$ with $|V| = n$ and the root r:

- height $H(T)$ in $O(n)$
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Your Job: Prove it!

Counting shortest paths (Problem 5.26)

Counting $\#$ of shortest paths in (un)directed graphs using BFS.

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Counting # of shortest paths in (un)directed graphs using BFS.

Maybe in the next class...

Cycle detection (Problem $5.8 - 1$)

	Digraph	Undirected graph
DFS		
BFS		

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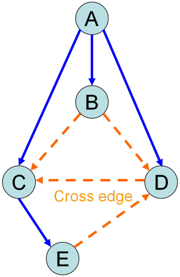

Evasiveness of acyclicity of undirected graphs (Problem $5.8 - 2$)

$$\text{Evasiveness} \triangleq \text{check} \binom{n}{2} \text{ edges (adjacency matrix)}$$

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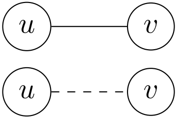

Algorithm $\mathbb{A}$:

$\textsc{CheckEdge}(u, v)$

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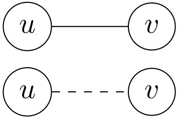

Algorithm $\mathbb{A}$:

$\textsc{CheckEdge}(u, v)$

Hint: Kruskal

$$\textsc{CheckEdge}(u, v) \leftarrow (u) \text{———} (v)$$

$$\iff$$

$$\nexists \text{ cycle } \in G + (u) \text{———} (v)$$

$\textsc{CheckEdge}(u, v) \leftarrow u$ — v

$$\Longleftrightarrow$$

$\nexists$ cycle $\in G +$ u — v

Q : Why adjacency matrix?

After-class Exercise: Evasiveness of connectivity of undirected graphs

$$\text{Evasiveness} \triangleq \text{check} \binom{n}{2} \text{edges (adjacency matrix)}$$

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Hint: Anti-Kruskal

Orientation of undirected graph (Problem 4.13)

- undirected (connected) graph G
- edges oriented *s.t.*

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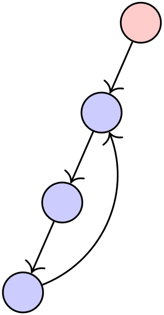

Shortest cycle of undirected graph (Problem 4.12)

A WRONG DFS-based algorithm:

$$\forall v : \text{level}[v]$$

$$\text{Back edge } u \to v : \text{level}[u] - \text{level}[v] + 1$$

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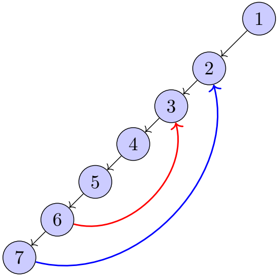

Shortest cycle of digraph (Problem 4.12)

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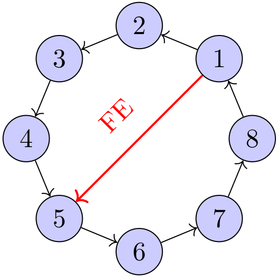

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TOPOSORT by Tarjan (probably), 1976

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Sort vertices in *decreasing* order of their *finish* times.

Kahn's TOPOSORT algorithm (1962; Problem 4.16)

- Queue Q for source vertices (in$[v] = 0$)
- Repeat: $\textsc{Dequeue}(u \in Q)$, delete u and $u \to v$ from Q, output u, $\textsc{Enqueue}(v)$ if in$[v] = 0$

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Lemma (Correctness of Kahn's TOPOSORT)

Every DAG has at least one source (and at least one sink vertex).

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Q : What if G is *not* a DAG?

Taking courses in few semesters (Problem 5.14)

- n courses
- m of $c_1 \to c_2$: prerequisite
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Critical path *OR* Longest path using DFS in $O(n+m)$

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Critical path *OR* Longest path using DFS in $O(n+m)$

For general digraph, LONGEST-PATH is NP-hard.

Line up (Problem 4.22)

1. i hates j: $i \succ j$
2. i hates j: $\#i < \#j$

TOPOSORT

Critical path

Hamiltonian path in DAG (Problem 4.14)

HP: path visiting each vertex once

Q : $\exists$ HP in a DAG in $O(n+m)$

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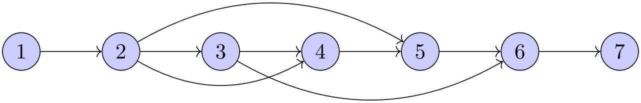

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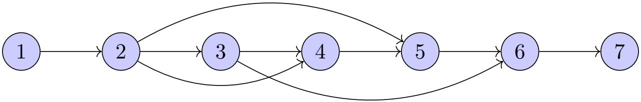

$\boxed{\text{DAG: } \exists \text{ HP} \iff \exists! \text{ topo. ordering}}$

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Tarjan's TOPOSORT + Check edges (v_i, v_{i+1})

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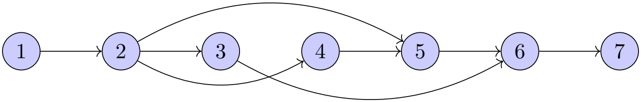
Tarjan's $\textsc{Toposort}$ + Check edges (v_i, v_{i+1})



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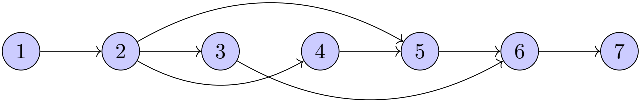
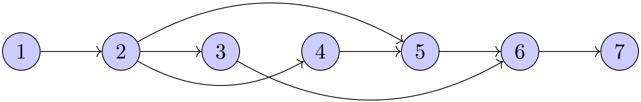
Tarjan's Toposort + Check edges (v_i, v_{i+1})



Kahn's Toposort (Problem 4.16)

$$\boxed{\text{DAG: } \exists \text{ HP} \iff \exists! \text{ topo. ordering}}$$

Tarjan's $\textsc{Toposort}$ + Check edges (v_i, v_{i+1})



Kahn's $\textsc{Toposort}$ (Problem 4.16)

$$|Q| \leq 1$$


THANK YOU