Paths in Graphs

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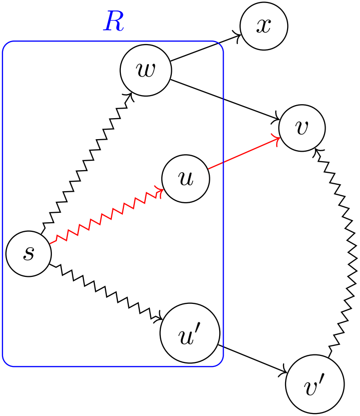
Dijkstra's Algorithm for SSSP

Finding shortest paths from s to other nodes t
in non-decreasing order of $\text{dist}(s, t)$.



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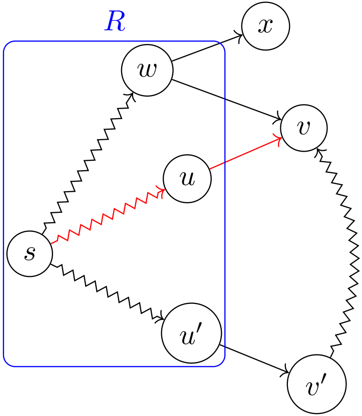

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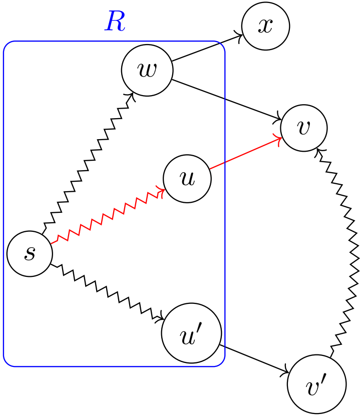

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$l(v' \rightsquigarrow v) \geq 0$

for all $v \in V$ **do**
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$Q \leftarrow \textsc{MinPQ}(V)$
while $Q \neq \emptyset$ **do**
    $u \leftarrow \textsc{DeleteMin}(Q)$
    for all $(u, v) \in E \wedge v \notin Q$ **do**
        if $\text{dist}[v] > \text{dist}[u] + l(u, v)$ **then**
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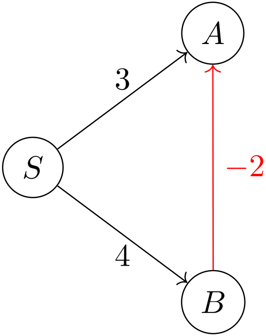
$$O(n + (n + m) \log n) \implies O((n + m) \log n) \implies O(m \log n)$$

Negative Edges (Problem 11.1)

Dijkstra's algorithm may fail if $w(e) < 0$.



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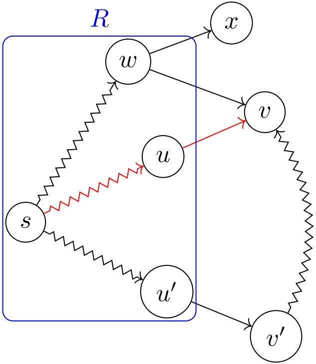

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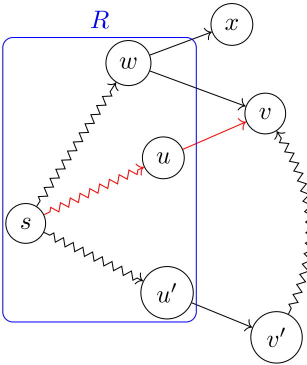

$$l(v' \rightsquigarrow v) \geq 0$$

Generalized Shortest Path (Problem 11.8)

$$\text{Digraph } G = (V, E), \quad l_e > 0, \quad c_v > 0, \quad s \in V$$

Shortest paths from s

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$$+ c_s$$

Shortest Paths Through v_0 (Problem 13.7)

$$\text{Strongly connected digraph } G = (V, E), \quad w(e) > 0$$

$$v_0 \in V$$

Find shortest paths $s \overset{\text{SP}}{\rightsquigarrow} t$ through v_0.

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$$s \rightsquigarrow^{\textsf{SP}} v_0 \rightsquigarrow^{\textsf{SP}} t$$

$$\forall v : v_0 \rightsquigarrow^{\textsf{SP}} v$$

Dijkstra's Algorithm as a Framework

for all $v \in V$ **do**
  $\text{dist}[v] \leftarrow \infty$
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$Q \leftarrow \textsc{MinPQ}(V)$
while $Q \neq \emptyset$ **do**
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  for all $(u, v) \in E \land v \notin Q$ **do**
    if $\text{dist}[v] > \text{dist}[u] + l(u, v)$ **then**
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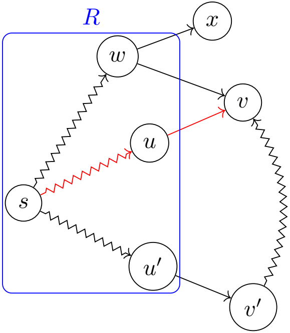
```
for all v ∈ V do
    dist[v] ← ∞
dist[s] ← 0

Q ← MinPQ(V)
while Q ≠ ∅ do
    u ← DeleteMin(Q)
    for all (u, v) ∈ E ∧ v ∉ Q do
        if dist[v] > dist[u] + l(u, v) then
            dist[v] ← dist[u] + l(u, v)
            DecreaseKey(Q, v)
```



Min-Max Path (Problem 11.12)

$$G = (V, E): \text{ network of highways}$$

$$l_e: \text{ road length} \quad L: \text{ tank capacity}$$

Given L, $\exists ? s \rightsquigarrow t$ in $O(n + m)$.

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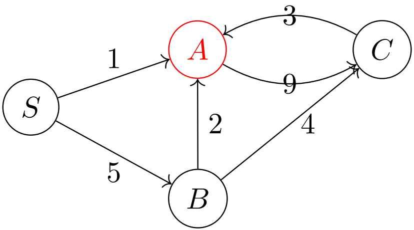

$$R \triangleq \{u \mid s \rightsquigarrow u \text{ is known}\}$$

for all $v \in V$ **do**
  $L[v] \leftarrow \infty$
$L[s] \leftarrow 0$

if $L[v] > \max(L[u], l(u, v))$ **then**
  $L[v] \leftarrow \max(L[u], l(u, v))$

Max-Min Path (Problem 13.2 (1))

$$G = (V, E) : \text{ network of oil pipelines}$$

$$c(u, v) : \text{ capacity of } (u, v)$$

$$\mathsf{cap}(s, t) : \max \min s \leadsto t$$

Given s, to compute cap(s, v).

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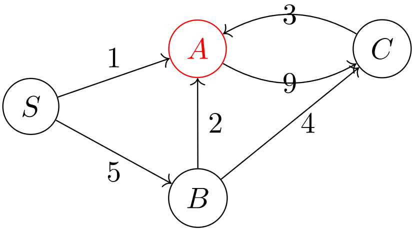

$$R \triangleq \{u \mid s \rightsquigarrow u \text{ is known}\}$$

$$Q \leftarrow \text{MaxPQ}(V)$$

for all $v \in V$ **do**
    $\text{cap}[v] \leftarrow -\infty$
$\text{cap}[s] \leftarrow 0$

if $\text{cap}[v] < \min(\text{cap}[u], c(u,v))$ **then**
    $\text{cap}[v] \leftarrow \min(\text{cap}[u], c(u,v))$

Max-Min Path (Problem 13.2 (2))

$$G = (V, E): \text{ network of oil pipelines}$$

$$c(u, v): \text{ capacity of } (u, v)$$

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Compute all-pair $\mathsf{cap}(u, v)$.

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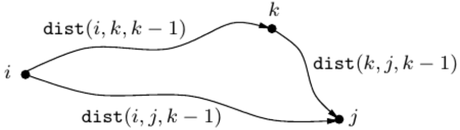
Compute all-pair $\textsf{cap}(u, v)$.

Floyd-Warshall algorithm

$$\text{dist}[i,j,k] = \min\Big(\text{dist}[i,j,k-1], \text{dist}[i,k,k-1] + \text{dist}[k,j,k-1]\Big)$$



$$\text{dist}[i,j,0] = \begin{cases} {\color{red}0} & i = j \\ w(i,j) & (i,j) \in E \\ \infty & \text{o.w.} \end{cases}$$

for $k \leftarrow 1$ **to** n **do**
  for $i \leftarrow 1$ **to** n **do**
    for $j \leftarrow 1$ **to** n **do**
      $\mathsf{dist}(i,j,k) = \min\Big(\mathsf{dist}(i,j,k-1), \mathsf{dist}(i,k,k-1) + \mathsf{dist}(k,j,k-1)\Big)$

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$$\mathsf{cap}(i, j, k) = \max\Big(\mathsf{cap}(i, j, k-1), \min(\mathsf{cap}(i, k, k-1), \mathsf{cap}(k, j, k-1))\Big)$$

Routing table (Problem 13.1)

$$\mathsf{Go}(i,j) = k \implies v_i \to v_k \rightsquigarrow v_j$$

Contruct routing table and extract shortest paths from it.

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      if $\text{dist}[i,j,k-1] > \text{dist}[i,k,k-1] + \text{dist}[k,j,k-1]$ **then**
        $\text{dist}[i,j,k] \leftarrow \text{dist}[i,k,k-1] + \text{dist}[k,j,k-1]$

      else
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for all $i \leftarrow 1 \dots n$ **do**
  for all $j \leftarrow 1 \dots n$ **do**
    $\mathsf{dist}[i,j] \leftarrow \infty$
    $\mathsf{Go}[i,j] \leftarrow \mathsf{Nil}$

for all $(i,j) \in E$ **do**
  $\mathsf{dist}[i,j] \leftarrow w(i,j)$
  $\mathsf{Go}[i,j] \leftarrow j$

for all $i \leftarrow 1 \dots n$ **do**
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  Go$[i, j] \leftarrow j$

for all $i \leftarrow 1 \dots n$ **do**
  dist$[i, i] \leftarrow 0$
  Go$[i, i] \leftarrow$ Nil

procedure PATH(i, j)
  if Go$[i, j]$ = Nil **then**
    Output "No Path."

  Output "i"
  while $i \neq j$ **do**
    $i \leftarrow$ Go$[i, j]$
  Output "i"

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Find shortest cycle in digraph $G = (V, E), \quad w(e) > 0$.

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$$\forall v : \textsf{dist}[v][v] = \infty$$

$$\exists v : \textsf{dist}[v][v] < \infty \implies \min_i \textsf{dist}[v][v]$$

Eulerian Circuit (Problem 13.5)

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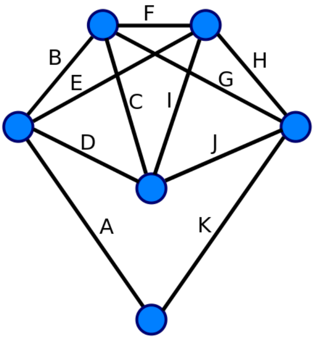

$VE \leftarrow \emptyset$ ▷ Visited Edges
$C \leftarrow \emptyset$ ▷ Current Circuit

while $VE \neq E$ **do**
  $u \leftarrow \textsc{Choose}(u : (u \to v) \notin VE)$
  $C' \leftarrow \textsc{Circuit}(u, E \setminus VE)$
  $VE \leftarrow VE \cup C'$
  $C \leftarrow C \cup C'$

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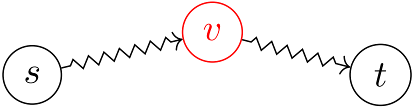
Data Structures?

Bitonic Shortest Path (Problem 11.7)



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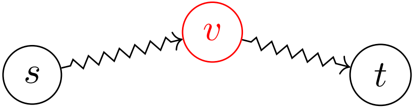


THANK YOU