A Little Mathematics for Algorithm Analysis

Hengfeng Wei

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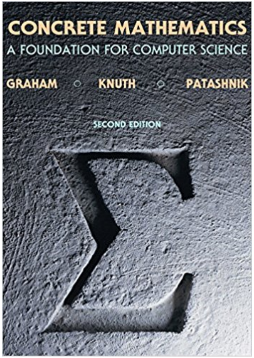
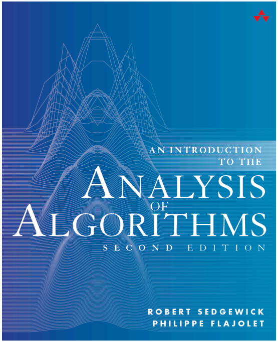
April 12, 2019


CONCRETE MATHEMATICS
A FOUNDATION FOR COMPUTER SCIENCE
GRAHAM KNUTH PATASHNIK
SECOND EDITION


AN INTRODUCTION TO THE
ANALYSIS OF ALGORITHMS
SECOND EDITION
ROBERT SEDGEWICK
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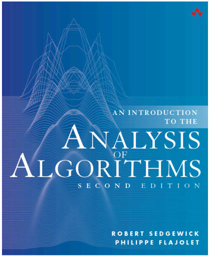
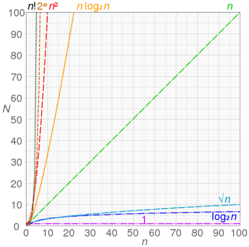
80% of the people are not good at math. I guess I belong to the other 25%

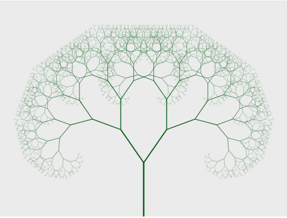


$A(n)$

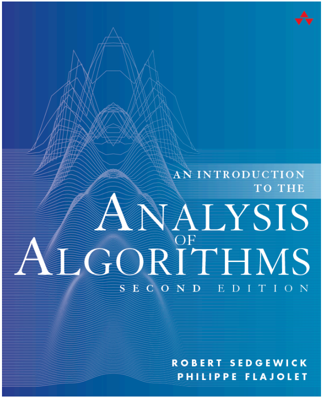
Mathematical Induction



Ω, Θ, O



$T(n) = aT(n/b) + f(n)$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Problem P Algorithm A

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Inputs: $\mathcal{X}_n$ of size n

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$$W(n) = \max_{x \in \mathcal{X}_n} T(x)$$

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Average-case Time Complexity (Problem 1.7)

$$r \in [1, n],\ r \in \mathbb{Z}^+$$

$$P\{r = i\} = \begin{cases} \frac{1}{n}, & 1 \le i \le \frac{n}{4} \\ \frac{2}{n}, & \frac{n}{4} < i \le \frac{n}{2} \\ \frac{1}{2n}, & \frac{n}{2} < i \le n \end{cases} \qquad T(r) = \begin{cases} 10, & r \le \frac{n}{4} \\ 20, & \frac{n}{4} < r \le \frac{n}{2} \\ 30, & \frac{n}{2} < r \le \frac{3n}{4} \\ n, & \frac{3n}{4} < r \le n \end{cases}$$

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Mathematical Induction

Horner's rule (Problem 1.6)

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

```
procedure HORNER(A[0...n], x)        ▷ A : {a_0 ... a_n}
    p ← A[n]
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Loop invariant (after the k-th loop):

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$$\boxed{\mathcal{I}: \ p = \sum_{j=n}^{j=n-k} a_j x^{k-(n-j)}}$$

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When you are in an exam:

20% : Finding $\mathcal{I}$

80% : Proving $\mathcal{I}$ by PMI

$\boxed{\text{Prove by mathematical induction on the number } k \text{ of loops.}}$

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Base Step: $k = 0$.

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$$\boxed{\left(\sum_{j=n}^{j=n-k} a_j x^{k-(n-j)} \right) \cdot x + A[n-k-1] = \sum_{j=n}^{j=n-(k+1)} a_j x^{(k+1)-(n-j)}}$$

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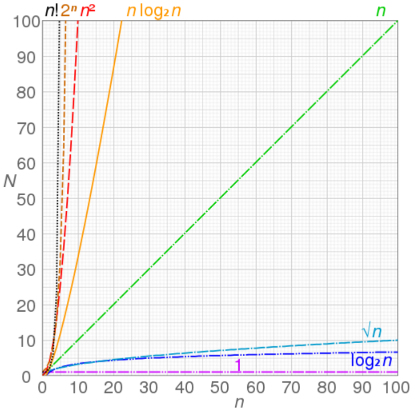
Termination

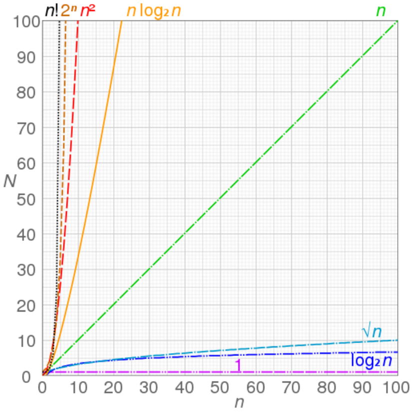
(a) $i \leftarrow n-1 \Downarrow 0$

(b) $k = n \implies p = \sum_{i=0}^{i=n} a_i x^i$

Asymptotics



Asymptotics



$$Q : \theta(f) \ ?$$

$$O(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \right\}$$

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$$f(n) \sim g(n) \iff \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$$

Asymptotics (Problem 2.6 (4))

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \land f(n) = \Omega(g(n))$$

Asymptotics (Problem 2.6 (4))

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$$\Theta(g(n)) \cap o(g(n)) = \emptyset$$

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$$f(n) = n, \quad g(n) = n^{1+\sin n}$$

Asymptotics (Problem 2.7 (2))

$$(\log n)^2 \quad vs. \quad \sqrt{n}$$

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$$(\log n)^2 \quad vs. \quad \sqrt{n}$$

$$(\log n)^{c_1} = O(n^{c_2}) \quad c_1, c_2 > 0$$

Summation (Problem 2.20)

```
procedure Conundrum(n)
    r ← 0
    for i ← 1 to n do
        for j ← i + 1 to n do
            for k ← i + j − 1 to n do
                r ← r + 1
    return r
```

Summation (Problem 2.20)

procedure CONUNDRUM(n)
  $r \leftarrow 0$
  for $i \leftarrow 1$ **to** n **do**
    for $j \leftarrow i + 1$ **to** n **do**
      for $k \leftarrow i + j - 1$ **to** n **do**
        $r \leftarrow r + 1$
  return r

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 =$$

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$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \frac{n^2 - n}{2} = \Theta(n^2)$$

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$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \frac{1}{48}\Big(3\big(-1 + (-1)^n\big) + 2n(n+2)(2n-1)\Big) = \Theta(n^3)$$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1$$

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$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} (n - i - j + 2)$$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} (n - i - j + 2) \left[j \le n - i + 1, i \le \frac{n}{2}\right]$$

$$
\begin{aligned}
&\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 \\
&= \sum_{i=1}^{n} \sum_{j=i+1}^{n} (n-i-j+2) \; [j \le n-i+1, i \le \frac{n}{2}] \\
&= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=i+1}^{n-i+1} (n-i-j+2)
\end{aligned}
$$

$$\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=i+j-1}^{n}1$$

$$=\sum_{i=1}^{n}\sum_{j=i+1}^{n}(n-(i+j-1)+1)\,[i+j-1\leq n]$$

$$=\sum_{i=1}^{n}\sum_{j=i+1}^{n}(n-i-j+2)\,[j\leq n-i+1] \qquad n-i+1\geq i+1 \Rightarrow i\geq\frac{n}{2}$$

$$=\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=i+1}^{n-i+1}(n-i-j+2)$$

$$=\frac{1}{2}\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}(n-2i+1)(n-2i+2)$$

当n为偶时 $=\frac{1}{2}\left[\sum_{i=1}^{\frac{n}{2}}(n^2+3n+2)+4\sum_{i=1}^{\frac{n}{2}}i^2-\sum_{i=1}^{\frac{1}{2}}(4n+6)i\right)$

$$=\frac{1}{2}\times\left(\frac{1}{2}(n^3+3n^2+2n)+\frac{n(\frac{n}{2}+1)(n+1)}{3}-\frac{(2n+3)(\frac{n}{2}+1)n}{2}\right)$$

$=$ 化简 $=\frac{2n^3+3n^2-2n}{24}=\frac{1}{48}\left(0+2n(2+n)(2n-1)\right)$

当n为奇时，$\lfloor\frac{n}{2}\rfloor=\frac{n-1}{2}$，代入，可化得 $=\frac{1}{48}\left(-6+2n(2+n)(2n-1)\right)$

(这个我懒的化了，谁有兴趣化一下，多个常数项).

通解 $\frac{1}{48}\left(3(-1+(-1)^n)+2n(2+n)(2n-1)\right)$

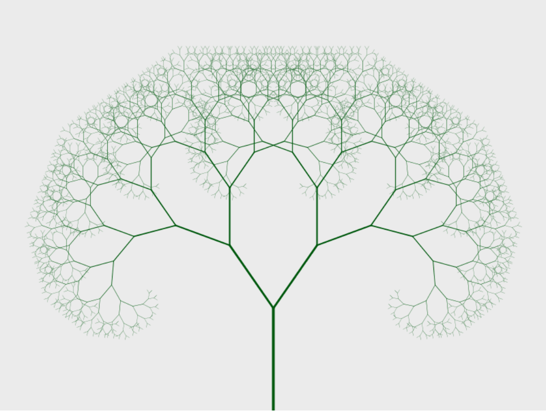
* $\lfloor\frac{n}{2}\rfloor=\frac{n+\frac{(-1)^n+(-1)}{2}}{2}$，代入理应直接得结果，太繁

From Zheng (171860658)

Reference:

"Big Omicron and Big Omega and Big Theta" by Donald E. Knuth, 1976.

Recurrences



$$T(n) = aT(n/b) + f(n) \quad (a > 0, b > 1)$$

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$$\left.\begin{array}{r} f(n) \\ af(\frac{n}{b}) \\ a^2 f(\frac{n}{b^2}) \\ \vdots \\ a^{\log_b n} T(1) = \Theta(n^{\log_b a}) \end{array}\right\}$$

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$$T(n) = aT(n/b) + f(n) \quad (a > 0, b > 1)$$

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$$T(n) = aT(n/b) + f(n) \quad (a > 0, b > 1)$$

Assume that $T(n)$ is constant for sufficiently small n.

$$\left.\begin{array}{r} f(n) \\ af(\frac{n}{b}) \\ a^2 f(\frac{n}{b^2}) \\ \vdots \\ a^{\log_b n} T(1) = \Theta(n^{\log_b a}) \end{array}\right\} \sum \overset{f(n) vs.\ n^E}{=} \begin{cases} n^{\log_b a}, & f(n) = O(n^{E-\epsilon}) \\ n^{\log_b a} \log n, & f(n) = \Theta(n^E) \\ f(n), & f(n) = \Omega(n^{E+\epsilon}) \end{cases}$$

Solving Recurrences (Problem 2.15)

(1) $\Theta(n^{\log_3 2})$

(2) $\Theta(\log^2 n)$

(3) $\Theta(n)$

(4) $\Theta(n \log n)$

(5) $\Theta(n \log^2 n)$

(6) $\Theta(n^2)$

(7) $\Theta(n^{\frac{3}{2}} \log n)$

(8) $\Theta(n)$

(9) $\Theta(n^{c+1})$

(10) $\Theta(c^{n+1})$

(11) $\cdots$

$$T(n) = T(n/2) + \log n$$

$$T(n) = 2T(n/2) + n \log n$$

Solving Recurrences (Problem 2.15)

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Reference:

$$f(n) = \Theta(n^{\log_b a} \log^k n) \implies \Theta(n^{\log_b a} \log^{k+1} n)$$

Solving Recurrences (Problem 2.15)

(1) $\Theta(n^{\log_3 2})$

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Gaps in Master Theorem (Problem 2.18)

Solving Recurrences (Problem 2.15)

(1) $\Theta(n^{\log_3 2})$

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Gaps in Master Theorem (Problem 2.18)

$$T(n) = 2T(n/2) + \frac{n}{\log n}$$

Solving Recurrences (Problem 2.15)

(1) $\Theta(n^{\log_3 2})$

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Gaps in Master Theorem (Problem 2.18)

$$T(n) = 2T(n/2) + \frac{n}{\log n} = \Theta(n \log \log n)$$

Solving Recurrences (Problem 2.15)

(1) $\Theta(n^{\log_3 2})$
(2) $\Theta(\log^2 n)$
(3) $\Theta(n)$
(4) $\Theta(n \log n)$
(5) $\Theta(n \log^2 n)$
(6) $\Theta(n^2)$
(7) $\Theta(n^{\frac{3}{2}} \log n)$
(8) $\Theta(n)$
(9) $\Theta(n^{c+1})$
(10) $\Theta(c^{n+1})$
(11) $\cdots$

$$T(n) = T(n-1) + c^n \quad c > 1$$

$$T(n) = T(n-1) + n^c \quad c \geq 1$$

Solving Recurrences (Problem 2.15)

(1) $\Theta(n^{\log_3 2})$

(2) $\Theta(\log^2 n)$

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$$\int$$

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$$T(n) = T(n-1) + c^n \quad c > 1$$

$$T(n) = T(n-1) + n^c \quad c \geq 1$$

$$\int$$

$$(\frac{n}{2}) \cdot (\frac{n}{2})^c \leq T(n) \leq n \cdot n^c$$

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8)$$

Solving Recurrences (Problem 2.15 (11))

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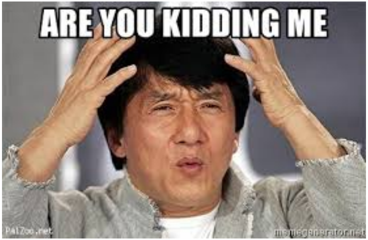

Where is $f(n)$?

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8)$$

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8)$$

$$\boxed{T(n) = \Theta(n^{0.879146})}$$

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8)$$

$$\boxed{T(n) = \Theta(n^{0.879146})}$$

$$\boxed{T(n) = \Theta(n^{\alpha})}$$

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8)$$

$$\boxed{T(n) = \Theta(n^{0.879146})}$$

$$\boxed{T(n) = \Theta(n^{\alpha})}$$

$$2^{-\alpha} + 4^{-\alpha} + 8^{-\alpha} = 1$$

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8)$$

$$\boxed{T(n) = \Theta(n^{0.879146})}$$

$$\boxed{T(n) = \Theta(n^{\alpha})}$$

$$2^{-\alpha} + 4^{-\alpha} + 8^{-\alpha} = 1$$

```
Solve[2^{-x} + 4^{-x} + 8^{-x} == 1, x] // N
```

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8) + n$$

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8) + n$$

By recursion-tree.

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8) + n$$

By recursion-tree.

$$\boxed{T(n) = \Theta(n)}$$

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$$T(n) = T(n/2) + T(n/4) + T(n/8) + n$$

By recursion-tree.

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Exercise: Prove it by mathematical induction.

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8) + n$$

By recursion-tree.

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Exercise: Prove it by mathematical induction.

Reference:

"On the Solution of Linear Recurrence Equations" by Akra & Bazzi, 1996.

$$T(n) = \sum_{i=1}^{k} a_i T(n/b_i) + f(n)$$

Solving Recurrences (Problem 2.17)

$$\begin{aligned}
\mathrm{T}(n) &= \sqrt{n}\ \mathrm{T}(\sqrt{n}) + n \\
&= n^{\frac{1}{2}}\ \mathrm{T}\left(n^{\frac{1}{2}}\right) + n
\end{aligned}$$

Solving Recurrences (Problem 2.17)

$$
\begin{aligned}
T(n) &= \sqrt{n}\ T(\sqrt{n}) + n \\
&= n^{\frac{1}{2}}\ T\left(n^{\frac{1}{2}}\right) + n \\
&= n^{\frac{1}{2}}\left(n^{\frac{1}{2^2}}\ T\left(n^{\frac{1}{2^2}}\right) + n^{\frac{1}{2}}\right) + n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}}\ T\left(n^{\frac{1}{2^2}}\right) + 2n
\end{aligned}
$$

Solving Recurrences (Problem 2.17)

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&= n^{\frac{1}{2}+\frac{1}{2^2}}\left(n^{\frac{1}{2^3}}\ \mathrm{T}\left(n^{\frac{1}{2^3}}\right) + n^{\frac{1}{2^2}}\right) + 2n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}}\ \mathrm{T}\left(n^{\frac{1}{2^3}}\right) + 3n
\end{aligned}
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Solving Recurrences (Problem 2.17)

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&= n^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}}\ \mathrm{T}\left(n^{\frac{1}{2^3}}\right) + 3n \\
&= \cdots \\
&= n^{\sum_{i=1}^{k}\frac{1}{2^i}}\ \mathrm{T}\left(n^{\frac{1}{2^k}}\right) + kn
\end{aligned}
$$

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$$n^{\frac{1}{2^k}} = 1$$

$$n^{\frac{1}{2^k}} = 2$$

$$n^{\frac{1}{2^k}} = 2 \implies k = \log \log n$$

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$$\begin{aligned}
\mathrm{T}(n) &= n^{\sum_{i=1}^{k} \frac{1}{2^i}}\ \mathrm{T}\left(n^{\frac{1}{2^k}}\right) + kn \\
&= n^{\sum_{i=1}^{\log\log n} \frac{1}{2^i}}\ \mathrm{T}(2) + n\log\log n
\end{aligned}$$

$$n^{\frac{1}{2^k}} = 2 \implies k = \log\log n$$

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\mathrm{T}(n) &= n^{\sum_{i=1}^{k} \frac{1}{2^i}}\ \mathrm{T}\left(n^{\frac{1}{2^k}}\right) + kn \\
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\end{aligned}$$

$$\sum_{i=1}^{\log\log n} \frac{1}{2^i} < 1 \implies T(n) = \Theta(n\log\log n)$$

$$n^{\frac{1}{2^k}} = 2 \implies k = \log\log n$$

$$\begin{aligned} \mathrm{T}(n) &= n^{\sum_{i=1}^{k} \frac{1}{2^i}}\ \mathrm{T}\left(n^{\frac{1}{2^k}}\right) + kn \\ &= n^{\sum_{i=1}^{\log\log n} \frac{1}{2^i}}\ \mathrm{T}(2) + n\log\log n \end{aligned}$$

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Exercise: Prove it by mathematical induction.

$$\boxed{T(n) = \sqrt{n}T(\sqrt{n}) + n}$$

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$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

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$$n \leftrightarrow 2^m$$

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$$S(m) = S(m/2) + 1 = \Theta(\log m)$$

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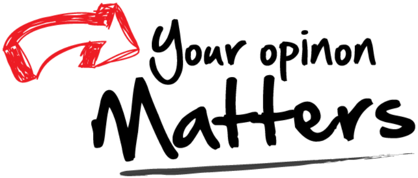
$$S(m) = S(m/2) + 1 = \Theta(\log m)$$

$$\boxed{T(n) = n \log \log n}$$


Thank
You!



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