Minimum Spanning Tree (MST)

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Cut Property

$$G = (V, E, w)$$

Cut Property (Strong)

- X is some part of an MST T of G
- Any *cut* $(S, V \setminus S)$ *s.t.* X does *not* cross $(S, V \setminus S)$ Âŋ
- Let e be *a* lightest edge across $(S, V \setminus S)$

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Correctness of Prim's and Kruskal's algorithms.

By Exchange Argument.

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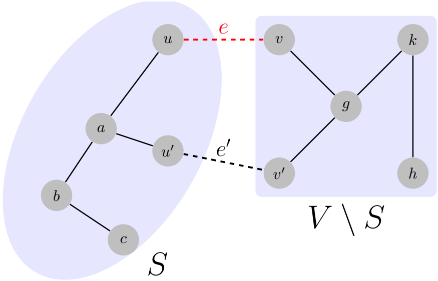

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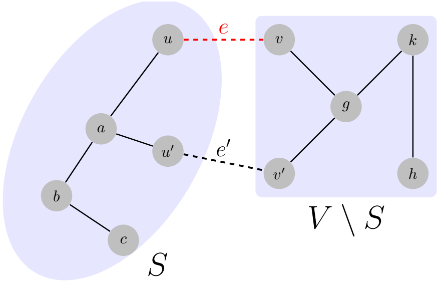

$$T + \{e\} - \{e'\}$$

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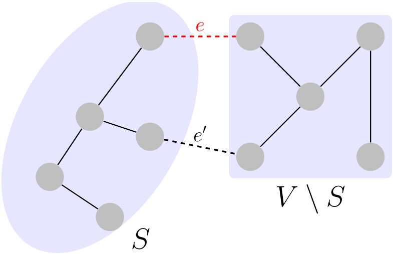

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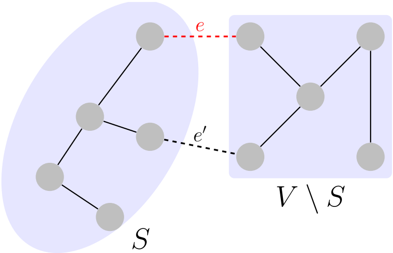

"a" → "the" $\Longrightarrow$ "some" → "any"

Converse of Cut Property (Weak)

$$e = (u, v) \in \exists \text{ MST } T \text{ of } G$$

$$\Longrightarrow$$

$$e \text{ is a lightest edge across some cut } (S, V \setminus S)$$

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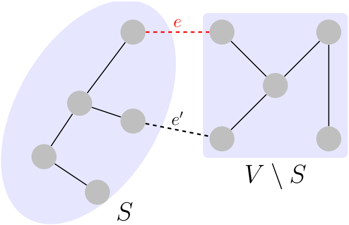

$$T' = \underbrace{T - \{e\}}_{\text{to find } (S, V\setminus S)} + \underbrace{\{e'\}}_{\exists ?}$$

Application of Cut Property [Problem: 10.15 (3)]

$e \in G$ is a lightest edge $\implies$ $e \in \exists$ MST of G

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$$(S = \{u\}, V \setminus S)$$

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$$e \in G \text{ is the unique lightest edge} \Rightarrow e \in \forall \text{ MST}$$

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By contradiction.

$$e \notin T : T' = T + \{e\} - \{e'\} \implies w(T') < w(T)$$

Wrong divide-and-conquer algorithm for MST [Problem: 10.21]

$$(V_1, V_2) : ||V_1| - |V_2|| \leq 1$$

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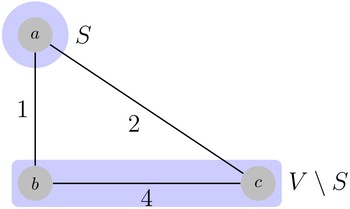

Cycle Property

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Cycle property [Problem: $10.19(b)$]

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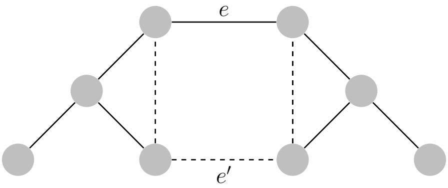
Then $\exists$ MST T of $G : e \notin T$.



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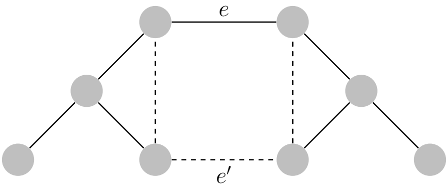

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Reverse-delete algorithm (wiki)

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"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem" — Kruskal, 1956.

Application of Cycle Property [Problem: 10.15 (1)]

$$G = (V, E), |E| > |V| - 1, \ e \text{ unique maximum-weighted edge}$$

$$\overset{?}{\Longrightarrow}$$

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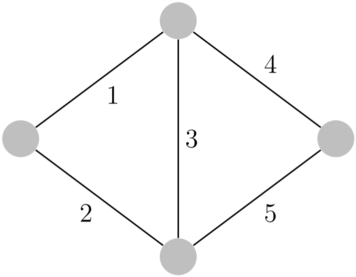

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Distinct weights $\implies$ unique MST.

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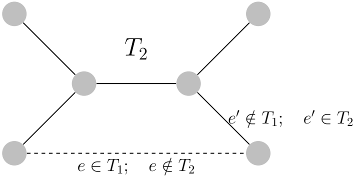
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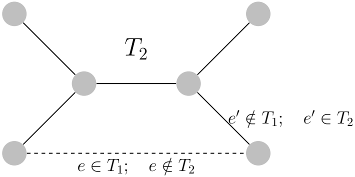
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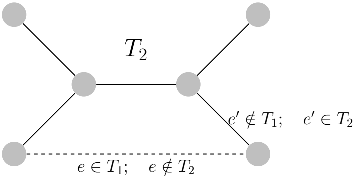
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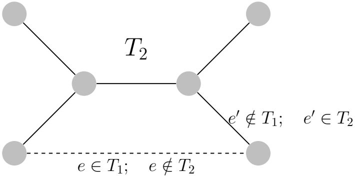
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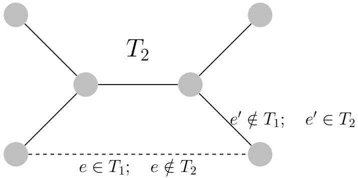
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Condition for Uniqueness of MST [Problem: 10.18 (2)]

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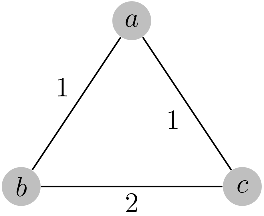

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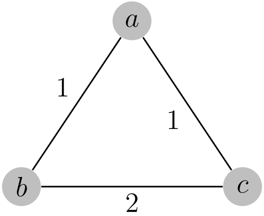

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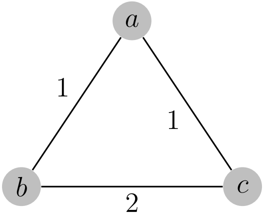

Theorem

Minimum-weight edge across any cut is unique $\Longrightarrow$ *Unique MST.*

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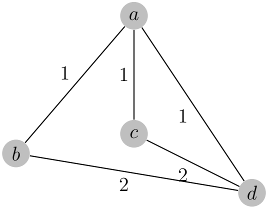

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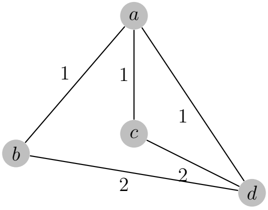

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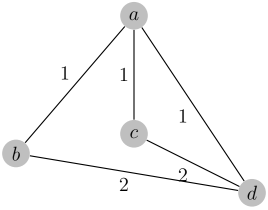

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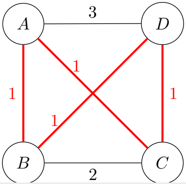

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Proof.

Cut property and Cycle property. □

Adding a Vertex v to MST T [Problem: 10.7]

$$G' = (V', E') : V' = V + \{v\}, E' = E + E_v$$

To find an MST T' of G'.

$$O\Big((m+n)\log n\Big) \quad \text{(recompute on } G'\text{)}$$

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"On Finding and Updating Spanning Tress and Shortest Paths", 1975
"Algorithms for Updating Minimum Spanning Trees", 1978

Feedback Edge Set: [Problem: 10.8]

1. Maximum spanning tree
2. (Minimum) feedback edge set:
   - a set of edges which, when removed from the graph, leave an acyclic graph G'
   - assuming G is connected $\Rightarrow$ G' is connected
   - feedback *arc* set: "cycle" $\Rightarrow$ circular dependency

- G' is connected $+$ acyclic $\Rightarrow$ G' is an ST
- $\text{FES} \Leftrightarrow G \setminus \text{Max-ST}$

Edge Weights

- [Problem: 10.15 (7)]: negative edges for Prim algorithm
- [Problem: 10.16]: $w'(e) = (w(e))^2$

MST with Specified Leaves: [Problem: 10.11]

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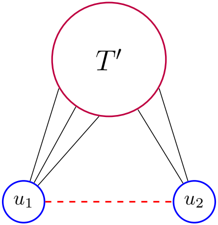

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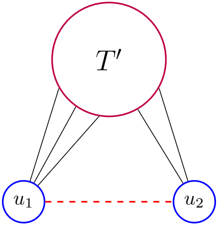

MST T' of $G' = G \setminus U$

Attach $\forall u \in U$ to T' (with lightest edge)

MST with Specified Edges: [Problem: 10.13]

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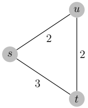
To find an MST with S as edges.

$G \to G'$: contract each component of S to a vertex

MST vs. Shortest Paths [Problem: 10.15 (6)]

✗The shortest path between two nodes is necessarily part of some MST.



Sharing edges [Problem: 3.6.5]

- $G = (V, E), w(e) > 0$
- Given s: all sssp trees from s must share some edge with all (some) MSTs of G

✓$w > 0$; Vertex s; shortest-path tree of s and some MST share a common edge [Problem: 10.9]

Solution

E': lightest edges leaving s

- any MST T of G: $T \cap E' \neq \emptyset$
- $E' \subset \forall$ sssp trees

THANK YOU