Decompositions of Graphs

— DFS/BFS, Cycle, DAG, Toposort, SCC, Bicomp

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John Hopcroft

Robert Tarjan

"For fundamental achievements in the design and analysis of algorithms and data structures."

— *Turing Award*, 1986

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DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

Abstract. The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an undirect graph are presented. The space and time requirements of both algorithms are bounded by $k_1V + k_2E + k_3$ for some constants k_1, k_2, and k_3, where V is the number of vertices and E is the number of edges of the graph being examined.

Key words. Algorithm, backtracking, biconnectivity, connectivity, depth-first, graph, search, spanning tree, strong-connectivity.

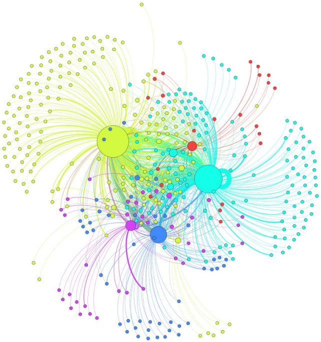
"DFS is a powerful technique with many applications."

- "Depth-First Search And Linear Graph Algorithms" by Robert Tarjan.

Power of DFS:

Graph Traversal $\Longrightarrow$ Graph Decomposition

Structure! Structure! Structure!



Graph *structure* induced by DFS:

states of v

types of u — v

life time of v:

v : $\mathsf{d}[v], \mathsf{f}[v]$

$\mathsf{d}[v]$: $\textsc{Bicomp}$

$\mathsf{f}[v]$: $\textsc{Toposort}$, SCC

Definition (Classifying edges)

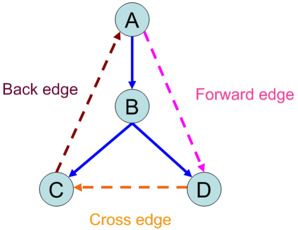
Given a DFS traversal $\implies$ DFS tree:

Tree edge: $\to$ child

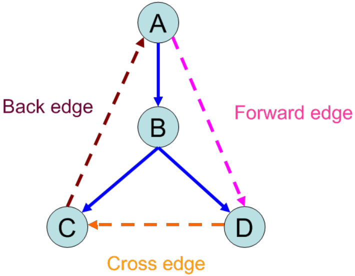
Back edge: $\to$ ancestor

Forward edge: $\to$ *nonchild* descendant

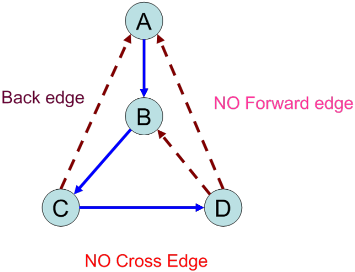
Cross edge: $\to$ $(\neg\text{ancestor}) \land (\neg\text{descendant})$



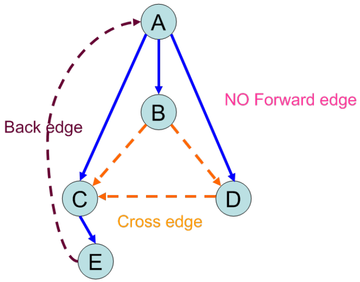
- Also applicable to BFS
- w.r.t. DFS/BFS trees



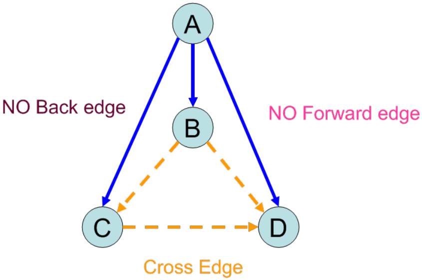
DFS on directed graph



DFS on undirected graph



BFS on directed graph



BFS on undirected graph (Problem 5.1)

DFS tree and BFS tree coincide (Problem 5.7)

Undirected connected graph $G = (V, E), v \in V$

DFS tree T from $v \equiv$ BFS tree T' from v

$$G \equiv T$$

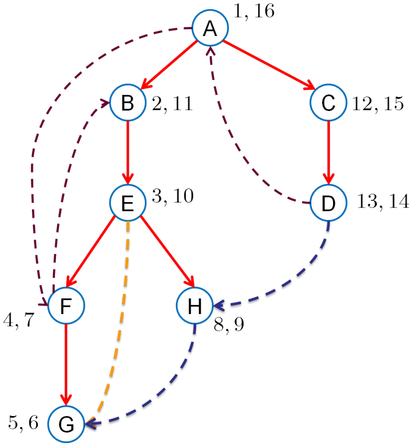
Proof.

G_{DFS}: tree + back *vs.* G_{BFS}: tree + cross

□

Q : What if G is a digraph?

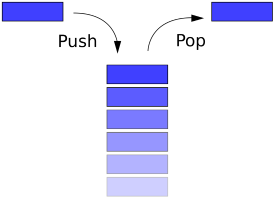
Life time of vertices in DFS



Theorem (Disjoint or Contained (Problem $4.2 : (1)\&(2)$))

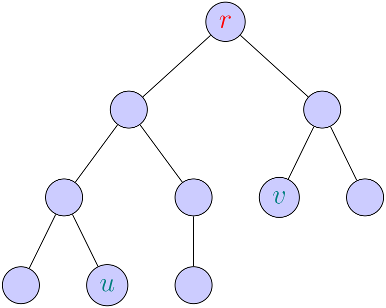
$$\forall u, v : [_u\ ]_u \cap [_v\ ]_v = \emptyset \bigvee \left([_u\ ]_u \subset [_v\ ]_v \vee [_v\ ]_v \subset [_u\ ]_u \right)$$

Proof.



$\square$

Preprocessing for ancestor/descendant relation (Problem 5.6)



Q : Is u an ancestor of v? $O(1)$

$$v : \mathsf{d}[v], \mathsf{f}[v]$$

Q : $\#$ of descendants of any v?

Edge types and life time of vertices in DFS (Problem 4.5)

$$\forall u \to v :$$

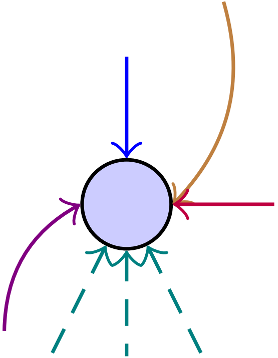
- tree/forward edge: $[_u \; [_v \;]_v \;]_u$
- back edge: $[_v \; [_u \;]_u \;]_v$
- cross edge: $[_v \;]_v \; [_u \;]_u$

$$\mathsf{f}[v] < \mathsf{d}[u] \iff \text{cross edge}$$

$$\mathsf{f}[u] < \mathsf{f}[v] \iff \text{back edge}$$

$$\nexists \text{ cycle} \implies \boxed{u \to v \iff \mathsf{f}[v] < \mathsf{f}[u]}$$

DFS from the perspective of a single node:



Height and diameter of tree (Problem 5.4)

Binary tree $T = (V, E)$ with $|V| = n$ and the root r:

(I) Height $H(T)$ in $O(n)$

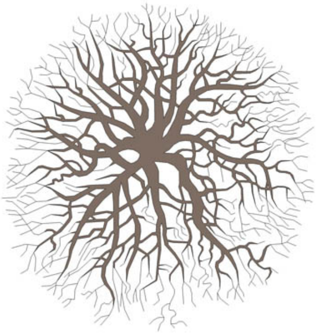
(II) Diameter $D(T)$ in $O(n)$

$$\begin{cases} H(T) = 0, & T \text{ is a leave} \\ H(T) = \max\Big(H(L_T), H(R_T)\Big) + 1, & \text{o.w.} \end{cases}$$

$$\begin{cases} D(T) = 0, & T \text{ is a leave} \\ D(T) = \max\Big(D(L_T), D(R_T), \underbrace{H(L_T) + H(R_T) + 2}_{\text{through the root}}\Big), & \text{o.w.} \end{cases}$$

Binary tree $T = (V, E)$ with $|V| = n$ and the root r

Q : Diameter of a *tree* *without* a designated root



Q : Diameter of a tree *without* a designated root

A beautiful algorithm:

- Pick any u
- Run BFS from u, obtain the farthest v
- Run BFS from v, obtain the farthest w

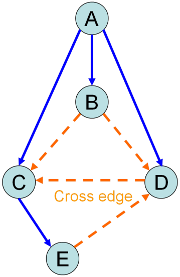
$$(v, w)$$

Your Job: Prove it!

Back to our original problem with $u \leftarrow r$.

Cycle detection (Problem $5.8-1$)

	Digraph	Undirected graph
DFS	back edge $\iff$ cycle	back edge $\iff$ cycle
BFS	back edge $\implies$ cycle cycle $\not\Longrightarrow$ back edge	cross edge $\iff$ cycle



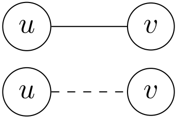
Evasiveness of acyclicity of undirected graphs (Problem $5.8-2$)

$$\text{Evasiveness} \triangleq \text{check} \binom{n}{2} \text{ edges (adjacency matrix)}$$

Q : Is acyclicity evasive?

By Adversary Argument.

Adversary $\mathcal{A}$:



Hint: Kruskal

Algorithm $\mathbb{A}$:

$\textsc{CheckEdge}(u, v)$

$\mathbb{A}$: $\textsc{CheckEdge}(u, v) \leftarrow \mathcal{A}$: $u - v$

$$\iff$$

$\mathcal{A}$: $\nexists$ cycle $\in G + u - v$

Q : Why adjacency matrix?

After-class Exercise: Evasiveness of connectivity of undirected graphs

$$\text{Evasiveness} \triangleq \text{check} \binom{n}{2} \text{edges (adjacency matrix)}$$

Q : Is connectivity evasive?

Hint: Anti-Kruskal

Orientation of undirected graph (Problem 4.13)

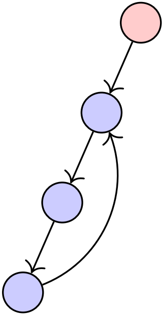
- undirected (connected) graph G
- edges oriented *s.t.*

$$\forall v, \text{in}[v] \geq 1$$

orientation $\Longleftrightarrow$ $\exists$ cycle C

DFS from $v \in C$

Q : BFS?

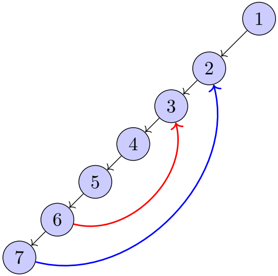


Shortest cycle of undirected graph (Problem 4.12)

A WRONG DFS-based algorithm:

$$\forall v : \mathsf{level}[v]$$

$$\text{Back edge } u \to v : \mathsf{level}[u] - \mathsf{level}[v] + 1$$

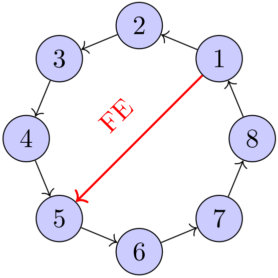


Shortest cycle of digraph (Problem 4.12)

A WRONG DFS-based algorithm:

$$\forall v : \mathsf{level}[v]$$

Back edge $u \to v$: $\mathsf{level}[u] - \mathsf{level}[v] + 1$



On digraphs:

$$\nexists \text{ back edge} \iff \text{DAG} \iff \exists \text{ topo. ordering}$$

TOPOSORT by Tarjan (probably), 1976

$$\nexists \text{ cycle} \implies \boxed{u \to v \iff \mathsf{f}[v] < \mathsf{f}[u]}$$

Sort vertices in *decreasing* order of their *finish* times.

Kahn's TOPOSORT algorithm (1962; Problem 4.16)

- Queue Q for source vertices ($\text{in}[v] = 0$)
- Repeat: $\text{DEQUEUE}(\exists u \in Q)$, output u

  delete u and $u \to v$ from Q,

  $\text{ENQUEUE}(v)$ if $\text{in}[v] = 0$

$$O(m+n)$$

Lemma (Correctness of Kahn's TOPOSORT)

Every DAG has at least one source (and at least one sink vertex).

Q : What if G is *not* a DAG?

Taking courses in few semesters (Problem 4.20)

- n courses
- m of $c_1 \to c_2$: prerequisite
- Goal: taking courses in few semesters

Critical path *OR* Longest path using DFS in $O(n + m)$

For general digraph, LONGEST-PATH is NP-hard.

Line up (Problem 4.22)

1. i hates j: $i \succ j$
2. i hates j: $\#i < \#j$

$\textsc{Toposort}$

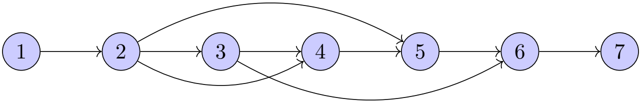
Critical path *OR* Longest path

Hamiltonian path in DAG (Problem 4.14)

HP: path visiting each vertex once

Q : $\exists$ HP in a DAG in $O(n+m)$

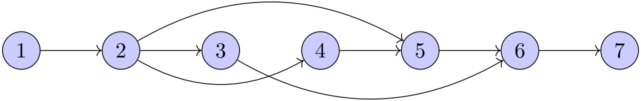
For general (di)graph, HP is NP-hard.



$\boxed{\text{DAG: } \exists \text{ HP} \iff \exists! \text{ topo. ordering}}$

$\boxed{\text{DAG: } \exists \text{ HP} \iff \exists! \text{ topo. ordering}}$

Tarjan's TOPOSORT + Check edges (v_i, v_{i+1})



Kahn's TOPOSORT (Problem 4.16)

$$|Q| \leq 1$$

Theorem (Digraph as DAG (Problem 4.6))

Every digraph is a dag of its SCCs.

Two tiered structure of digraphs:

digraph $\equiv$ a dag of SCCs

SCC: equivalence class over reachability

$\boxed{\text{digraph} \equiv \text{a dag of SCCs}}$

Kosaraju's SCC algorithm, 1978

"SCCs can be topo-sorted

in decreasing order of their highest finish time."

The vertice with the highest finish time is in a source SCC.

(I) DFS on G; DFS/BFS on G^T

(II) DFS on G^T; DFS/BFS on G

Kosaraju's SCC algorithm, 1978 (Problem 4.7)

$$1\text{st DFS} \overset{?}{\Longrightarrow} \text{BFS}$$

$$2\text{nd DFS} \overset{?}{\Longrightarrow} \text{BFS}$$

1st DFS: toposort between SCCs

2nd DFS: reachability within an SCC

$\boxed{\text{digraph} \equiv \text{a dag of SCCs}}$

One-to-all reachability in a digraph (Problem 5.12)

$$v : v \rightsquigarrow^{?} \forall u$$

$$\exists ?\ v : v \rightsquigarrow \forall u$$

SCC

$$\exists !\ \text{source vertex } v \iff v \rightsquigarrow \forall u$$

$$\Longleftarrow :\ \exists !\ \text{source}$$

$$\Longrightarrow :\ \text{By contradiction.}$$

$$\exists u : v \not\rightsquigarrow u \land \text{in}[u] > 0 \implies \exists \text{ cycle}$$

Impacts of vertices in a digraph (Problem 4.18)

$$\text{impact}(v) = |\{w \neq v : v \leadsto w\}|$$

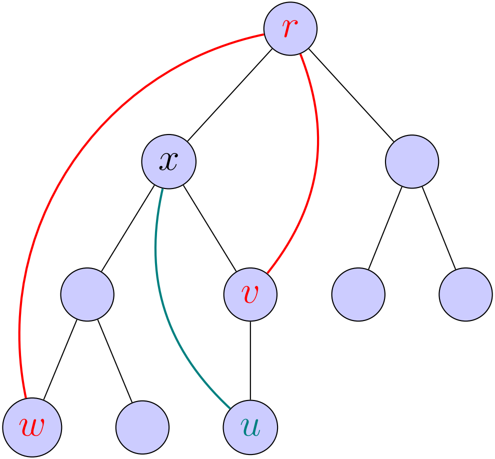
- $\arg\min_v \text{impact}(v)$
- $\arg\max_v \text{impact}(v)$

$$\arg\min_v \text{impact}(v) \in \text{sink SCC of smallest cardinality}$$

$$\arg\max_v \text{impact}(v) \in \text{source SCC}$$

Q : $\forall v$, computing impact(v)

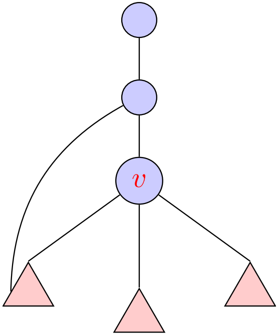
Bicomp: Back!



back$[v]$: the earliest reachable ancestor of v

(I) When and how to update back$[v]$?

(II) When and how to identify a bicomponent?



Initialization of back$[v]$ (Problem 4.9)

$$\text{back}[v] = d[v] \textit{ vs. } \text{back}[v] = \infty \vee 2(n+1)$$

tree edge $(\to v)$: $\text{back}[v] = d[v]$

back edge $(v \to w)$: $\text{back}[v] = \min\{\text{back}[v], d[w]\}$

backtracking from w: $\text{back}[v] = \min\{\text{back}[v], \text{back}[w] = \text{wBack}\}$

Proof.

if never updated:

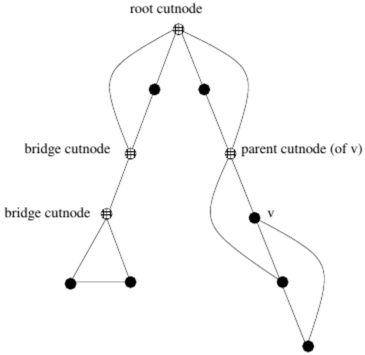
if ever updated

$$\text{wBack} = \infty > d[v] \textit{ vs. } \text{wBack} = d[w] > d[v]$$

□

Root cutnode v (Problem 4.8)

$$v \text{ is a cutnode } \iff \text{out}[v] \geq 2$$




THANK YOU