Minimum Spanning Tree (MST)

Hengfeng Wei

hfwei@nju.edu.cn

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Cut Property

$$G = (V, E, w)$$

Cut Property (I)

X : A part of some MST T of G

$(S, V \setminus S)$: A *cut* such that X does *not* cross $(S, V \setminus S)$ Âŋ

e : *A* lightest edge across $(S, V \setminus S)$

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Correctness of Prim's and Kruskal's algorithms.

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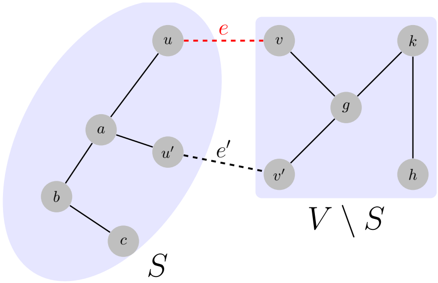

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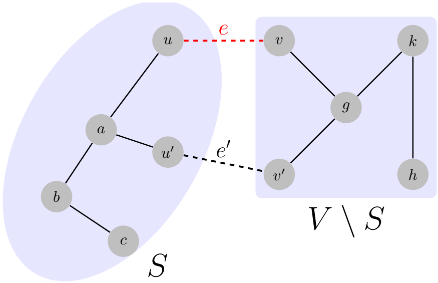

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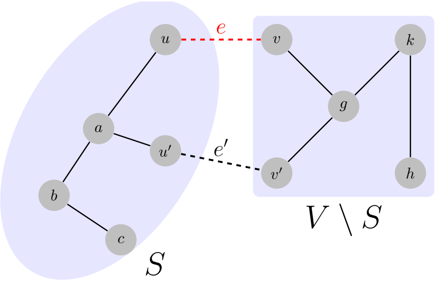

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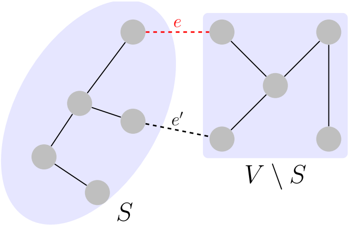
"a" → "the" $\Longrightarrow$ "some" → "all"

Cut Property (II)

A cut $(S, V \setminus S)$

Let $e = (u, v)$ be *a* lightest edge across $(S, V \setminus S)$

$\exists$ MST T of G : $e \in T$



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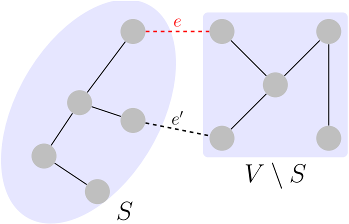

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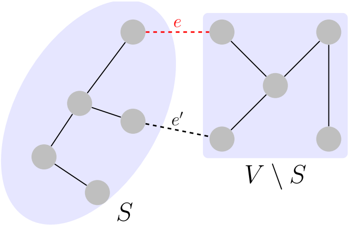

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"a" → "the" $\Longrightarrow$ "$\exists$" → "$\forall$"

Application of Cut Property [Problem: 10.15 (3)]

$e = (u, v) \in G$ is a lightest edge $\implies$ $e \in \exists$ MST of G

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Wrong Divide&Conquer Algorithm for MST [Problem: 10.21]

$$(V_1, V_2) : \Big| |V_1| - |V_2| \Big| \le 1$$

$T_1 + T_2 + \{e\} : e$ is a lightest edge across (V_1, V_2)

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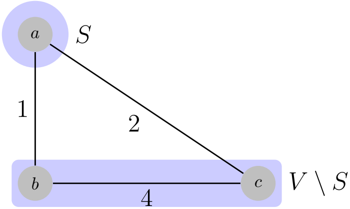
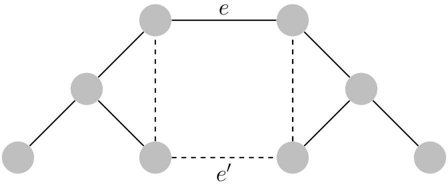


Cycle Property

Cycle Property [Problem: $10.19(b)$]

- Let C be any cycle in G
- Let $e = (u, v)$ be *a* maximum-weight edge in C

Then $\exists$ MST T of $G : e \notin T$.



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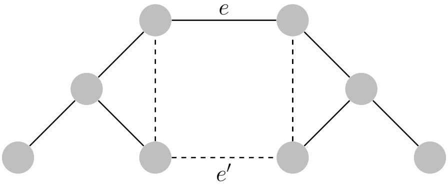

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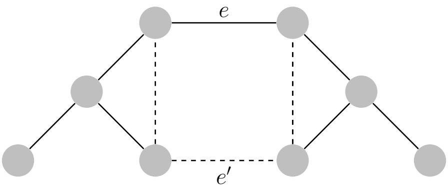

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"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.

Application of Cycle Property [Problem: $10.15\ (1)$]

$$G = (V, E), \quad |E| > |V| - 1$$

e : the unique maximum-weighted edge of G

$$\overset{?}{\Longrightarrow}$$

$e \notin$ any MST

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Application of Cycle Property [Problem: 10.15 (2)]

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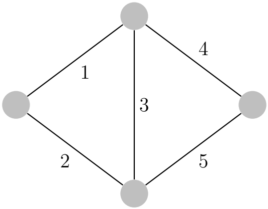

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$e \in T_1 \setminus T_2$ *(w.l.o.g)*

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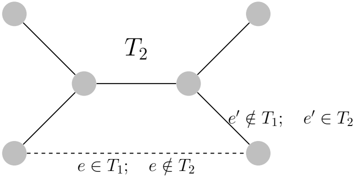
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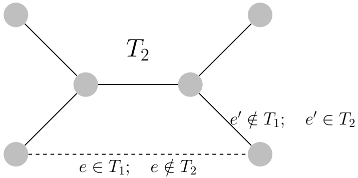
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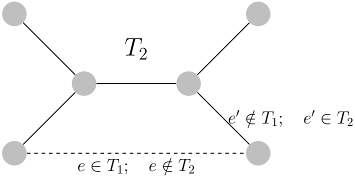
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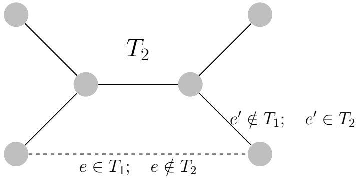
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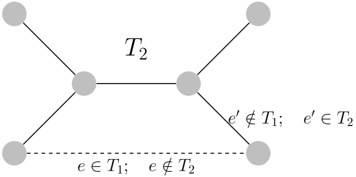
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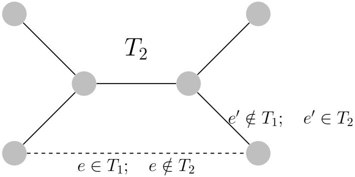
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Condition for Uniqueness of MST [Problem: 10.18 (2)]

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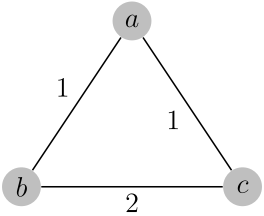

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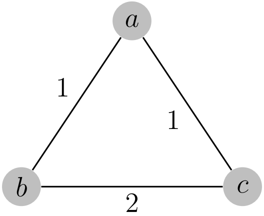

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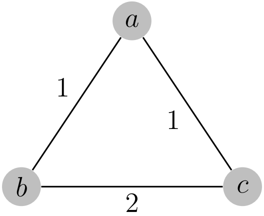

Theorem (After-class Exercise)

Minimum-weight edge across any cut is unique $\Longrightarrow$ *Unique MST.*

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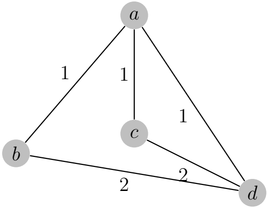

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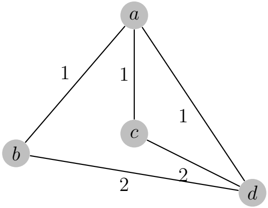

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To decide whether a graph has a unique MST.

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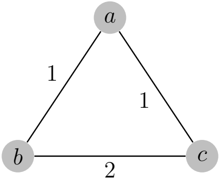

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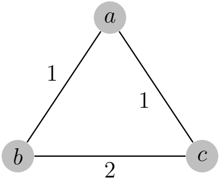

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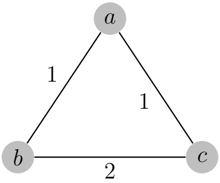

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$$G \to T, \qquad G' \triangleq G \setminus \{e\} \to T'$$

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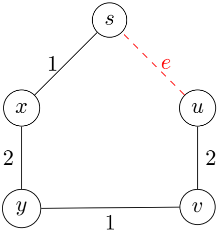

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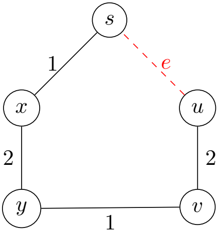

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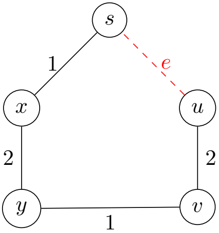

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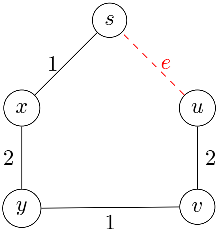

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Variants of MST

Adding a Vertex v to MST T [Problem: 10.7]

$$G' = (V', E') : V' = V + \{v\}, E' = E + E_v$$

To find an MST T' of G'.

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There exists an MST of G' that includes no edges in $G \setminus T$.

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"On Finding and Updating Spanning Tress and Shortest Paths", 1975

"Algorithms for Updating Minimum Spanning Trees", 1978

Feedback Edge Set (FES): [Problem: 10.8]

$$\text{FES} \subseteq E : G' = (V, E \setminus \text{FES}) \text{ is acyclic}$$

To find a minimum FES.

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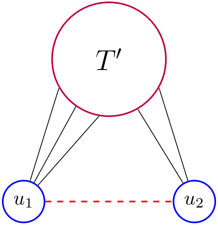

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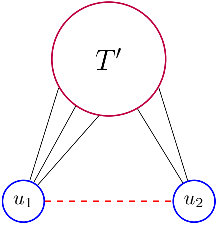

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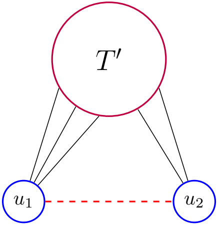

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Attach $\forall u \in U$ to T' (with lightest edge)

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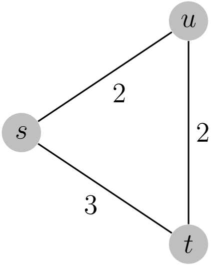
Compute MST on G'

MST *v.s.* Shortest Path

MST vs. Shortest Paths [Problem: 10.15 (6)]

✗ The shortest path between s and t is necessarily part of some MST.



Sharing Edges [Problem: 10.9]

$$G = (V, E, w), \quad w(e) > 0, \quad s \in V$$

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$\forall$ MST T of $G : T \cap E' \neq \emptyset$

THANK YOU