Paths in Graphs

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Paths in Graphs
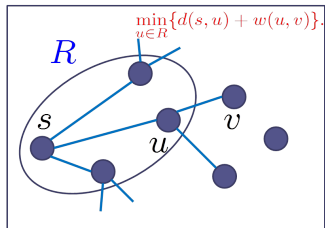
Dijkstra's algorithm for SSSP

$$R \triangleq \{v \mid s \rightsquigarrow v \text{ is known}\}$$

Finding shortest paths from s to other nodes t
in increasing order of $\text{dist}(s, t)$.

Theorem (Invariant)

$$\exists d : \begin{cases} \textit{dist}(s, v) \leq l, & \forall v \in R, \\ \textit{dist}(s, v) > l, & \forall v \notin R \end{cases}$$
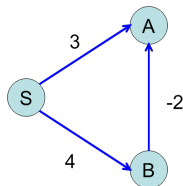
Dijkstra's algorithm for SSSP

Negative edges

Negative edges (Problem 6.16)

Dijkstra's algorithm may fail if $w(e) < 0$.

Negative edges

Negative edges from s (Problem 6.21)

All negative edges are from s.

$$\arg\min_{(s,v)} w(s, v)$$

Generalized shortest path problem

Generalized shortest path problem (Problem 6.20)
- digraph $G = (V, E), l_e > 0, c_v > 0, s \in V$
- shortest paths from s

$$l'(u, v) = l(u, v) + c_v$$

Shortest paths among nodes

Shortest paths among nodes (Problem 6.27)

- $G = (V, E), w(e) \geq 0$
- $S, T \subseteq V$
- compute $\min\{\mathsf{d}(s, t) : s \in S, t \in T\}$

$$V' = V + \{s_0, t_0\}$$

Shortest path through v_0

Shortest paths through v_0 (Problem 6.28)

- strongly connected digraph $G = (V, E), w(e) > 0$
- $v_0 \in V$
- find shortest paths $s \rightsquigarrow^{\mathsf{SP}} t$ through v_0

$$s \rightsquigarrow^{\mathsf{SP}} v_0 \rightsquigarrow^{\mathsf{SP}} t$$

$$\#v_0's = 1$$

Shortest path in maze

Shortest paths in maze (Problem 6.24)

1. $w(e) = c > 0$ without obstacles
2. $w(e) = c > 0$ with obstacles
3. $w(e) > 0$
   - 3.1 $\rightarrow, \downarrow$; in $O(n + m)$
   - 3.2 $\uparrow, \downarrow, \leftarrow, \uparrow$
4. $\exists e : w(e) < 0$ without negative cycles

$$(3.1) \; \mathsf{d}[v] = \min_{u \rightarrow v} \mathsf{d}[u] + w(u \rightarrow v)$$

Compute dist$[v]$ in topo. order.

Paths in Graphs

Dijkstra's algorithm

$$\mathsf{dist}(v) = \min_{u \in N(v)} \{\mathsf{dist}(u) + l(u, v)\}$$

for all $v \in V$ **do**
    $\mathsf{dist}[v] \leftarrow \infty$
$\mathsf{dist}[s] \leftarrow 0$

$Q \leftarrow \mathsf{MinPQ}(V)$

while $Q \neq \emptyset$ **do**
    $u \leftarrow \mathsf{deleteMin}(Q)$
    for all $(u, v) \in E \wedge v \in Q$ **do**
        if $\mathsf{dist}[v] > \mathsf{dist}[u] + l(u, v)$ **then**
            $\mathsf{dist}[v] \leftarrow \mathsf{dist}[u] + l(u, v)$
            $\mathsf{decreaseKey}(Q, v)$

$$O(n + (n + m) \log n) \implies O((n + m) \log n) \implies O(m \log n)$$

Dijkstra's algorithm

Prim's algorithm for MST:
    if $\text{cost}[u] > w(u, v)$ **then**
        $\text{cost}[u] \leftarrow w(u, v)$

BFS:
    $Q \leftarrow \text{FIFO-Q}(s)$

    if $\text{dist}[v] = \infty$ **then**
        $\text{dist}[v] \leftarrow \text{dist}[u] + 1$

Unique shortest paths

Unique shortest paths (Problem 6.18)

Undirected graph $G = (V, E), w(e) > 0, s \in V$:

$$\mathsf{usp}[v] = T \iff \exists! s \rightsquigarrow^{\mathsf{SP}} v$$

for all $v \in V$ **do**
    $\mathsf{usp}[v] \leftarrow F$
$\mathsf{usp}[s] \leftarrow T$

if $\mathsf{dist}[v] > \mathsf{dist}[u] + l(u, v)$ **then**
    $\mathsf{dist}[v] \leftarrow \mathsf{dist}[u] + l(u, v)$
    $\mathsf{usp}[v] \leftarrow \mathsf{usp}[u]$
else if $\mathsf{dist}[v] = \mathsf{dist}[u] + l(u, v)$ **then**
    $\mathsf{usp}[v] \leftarrow F$

Number of shortest paths

Number of shortest paths (Problem 6.31, 5.26)

$$\# s \rightsquigarrow^{\mathsf{SP}} v$$

$$O(n + m) \text{ if } w(e) = 1$$

$$\mathsf{Num}[s] = 1 \text{ vs. } \mathsf{Num}[s] = 0$$

Min-max path problem

Min-max path problem (Problem 6.23)

- $G = (V, E)$: network of highways
- l_e: road length; L: tank capacity
- (1) Given L, $\exists? s \rightsquigarrow t$.
- (2) Given G, compute $\min L$.

$$L[v] = \min_{u \in N(v)} \max\{L[u], l(u, v)\}$$

for all $v \in V$ **do**
$\quad L[v] \leftarrow \infty$
$L[s] \leftarrow 0$

if $L[v] > \max(L[u], l(u, v))$ **then**
$\quad L[v] \leftarrow \max(L[u], l(u, v))$

Min-max path problem

Min-max path problem (Problem 6.23)

- $G = (V, E)$: network of highways
- l_e: road length; L: tank capacity
- (1) Given L, $\exists ? s \rightsquigarrow t$.
- (2) Given G, compute $\min L$.

$O(\log m)$ binary searches for $\min L$

Max-min path problem

Max-min path problem (Problem 6.26)

- $G = (V, E)$: network of oil pipelines
- $c(u, v)$: capacity of (u, v)
- (1) Given s, compute $\mathsf{cap}(s, v)$.
- (2) Compute all-pair $\mathsf{cap}(u, v)$.

$$\mathsf{cap}[v] = \max_{u \in N(v)} \min(\mathsf{cap}[u], c(u, v))$$

$$Q \leftarrow \mathsf{MaxPQ}(V)$$

Paths in Graphs

Bellman-Ford algorithm

Bellman-Ford algorithm (Problem 6.30)

- digraph $G = (V, E), \exists e : w(e) < 0$
- $\forall s, t : |s \rightsquigarrow^{\mathsf{SP}} t| \leq k$
- Given s, t, find $s \rightsquigarrow^{\mathsf{SP}} t$.

$\mathsf{d}(v, k)$: shortest path distance from s to v using $\leq k$ edges

$$s \rightsquigarrow^{\mathsf{SP}; \leq k-1} u \rightarrow^{=1} v$$

$$\mathsf{d}(v, k) = \min_{u \in N(v)} \mathsf{d}(u, k - 1) + w(u, v)$$

$$\textbf{for all } i = 1 \rightarrow n - 1 \textbf{ do}$$
$$\textbf{for all } e \in E \textbf{ do}$$
$$\text{update } e$$

Floyd-Warshall algorithm

$$\mathsf{dist}(i, j, k) = \min(\mathsf{dist}(i, j, k - 1), \mathsf{dist}(i, k, k - 1) + \mathsf{dist}(k, j, k - 1))$$

$$\#k's = 1 \implies \mathsf{dist}(i, k, k - 1)$$

Routing table

Routing table (Problem 6.25)

Contruct routing table and extract shortest paths from it.

Init: $\mathsf{Go}(i,j) \leftarrow \mathsf{Null}$

$\forall (i,j) \in E : \mathsf{Go}(i,j) \leftarrow j$

if ... **then**
   $\mathsf{Go}(i,j) \leftarrow \mathsf{Go}(i,k)$

if $Go(i,j) = \mathsf{Null}$ **then**
   . . .

while $i \neq j$ **do**
   $i \leftarrow \mathsf{Go}(i,j)$

$\mathsf{Prev}(i,j) \leftarrow \mathsf{Prev}(k,j)$

$\mathsf{Intermediate}(i,j) \leftarrow k$

Shortest cycle in digraph

Shortest cycle in digraph

Find shortest cycle in digraph $G = (V, E), w(e) > 0$.

Initialize dist$[v][v] \leftarrow \infty$ in Floyd-Warshall algorithm

$\exists v : \text{dist}[v][v] < 0$ *vs.* $\forall v : \text{dist}[v][v] = \infty$

Max-min path problem

Max-min path problem (Problem 6.26)

- $G = (V, E)$: network of oil pipelines
- $c(u, v)$: capacity of (u, v)
- (2) Compute all-pair $\mathsf{cap}(u, v)$.

$$\mathsf{cap}(u, v, k) = \max(\mathsf{cap}(u, v, k - 1), \min(\mathsf{cap}(u, k, k - 1), \mathsf{cap}(k, v, k - 1)))$$

Paths in Graphs

Hamiltonian path in Tournament graph

Hamiltonian path in Tournament graph (Problem 6.22)

$$\forall u, v : (u \rightarrow v \lor v \rightarrow u)$$
$$\land \neg(u \rightarrow v \land v \rightarrow u)$$

By mathematical induction on n.