Paths in Graphs

Hengfeng Wei

hfwei@nju.edu.cn

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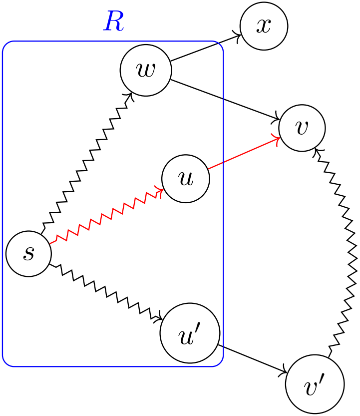
Dijkstra's Algorithm for SSSP

Finding shortest paths from s to other nodes t in non-decreasing order of $\text{dist}(s, t)$.



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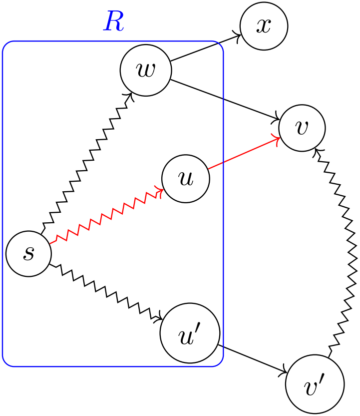

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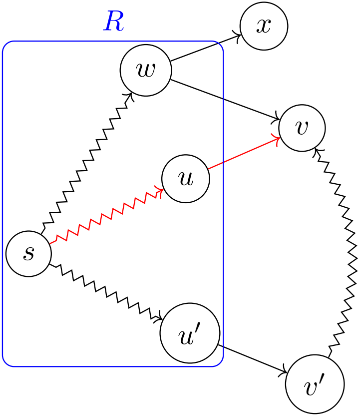

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$l(v' \rightsquigarrow v) \geq 0$

for all $v \in V$ **do**
  $\text{dist}[v] \leftarrow \infty$
$\text{dist}[s] \leftarrow 0$

$Q \leftarrow \textsc{MinPQ}(V)$
while $Q \neq \emptyset$ **do**
  $u \leftarrow \textsc{DeleteMin}(Q)$

  for all $(u, v) \in E \wedge v \notin Q$ **do**
    if $\text{dist}[v] > \text{dist}[u] + l(u, v)$ **then**
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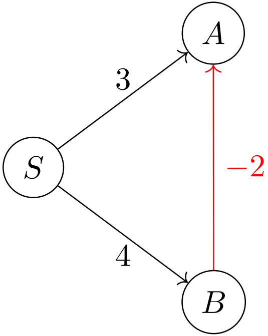
$$O(n + (n + m) \log n) \implies O((n + m) \log n) \implies O(m \log n)$$

Negative Edges (Problem 11.1)

Dijkstra's algorithm may fail if $w(e) < 0$.



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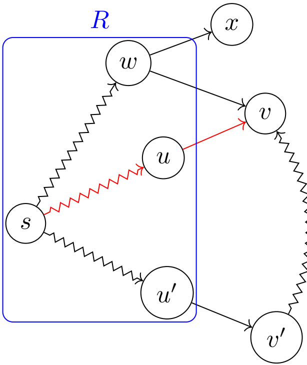

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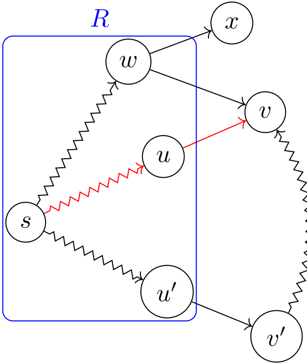

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Generalized Shortest Path (Problem 11.8)

$$\text{Digraph } G = (V, E), \quad l_e > 0, \quad c_v > 0, \quad s \in V$$

Shortest paths from s

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$$+ c_s$$

Shortest Paths Through v_0 (Problem 13.7)

Strongly connected digraph $G = (V, E), \quad w(e) > 0$

$$v_0 \in V$$

Find shortest paths $s \overset{\mathsf{SP}}{\rightsquigarrow} t$ through v_0.

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$$s \leadsto^{\mathsf{SP}} v_0 \leadsto^{\mathsf{SP}} t$$

$$\forall v : v_0 \leadsto^{\mathsf{SP}} v$$

Dijkstra's Algorithm as a Framework

Min-Max Path (Problem 11.12)

$$G = (V, E): \text{ network of highways}$$

$$l_e: \text{ road length} \quad L: \text{ tank capacity}$$

Given L, $\exists ? s \rightsquigarrow t$ in $O(n+m)$.

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$$l_e > L \implies l_e = \infty$$

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$$s \rightsquigarrow^{?} t$$

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Given G, to compute $\min L$ in $O((n+m)\log n)$.

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$$R \triangleq \{u \mid s \rightsquigarrow u \text{ is known}\}$$

for all $v \in V$ **do**
  $L[v] \leftarrow \infty$
$L[s] \leftarrow 0$

if $L[v] > \max(L[u], l(u, v))$ **then**
  $L[v] \leftarrow \max(L[u], l(u, v))$

Max-Min Path (Problem 13.2)

$$G = (V, E): \text{ network of oil pipelines}$$

$$c(u, v): \text{ capacity of } (u, v)$$

$$\mathsf{cap}(s, t) : \max \min s \rightsquigarrow t$$

Given s, to compute $\mathsf{cap}(s, v)$.

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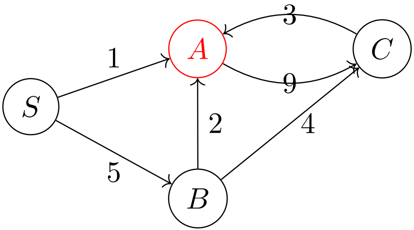

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for all $v \in V$ **do**
  cap$[v] \leftarrow -\infty$
cap$[s] \leftarrow 0$

if cap$[v] < \min(\text{cap}[u], c(u, v))$ **then**
  cap$[v] \leftarrow \min(\text{cap}[u], c(u, v))$

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Compute all-pair $\mathsf{cap}(u, v)$.

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$$\mathsf{dist}(i, j, k) = \min(\mathsf{dist}(i, j, k-1), \mathsf{dist}(i, k, k-1) + \mathsf{dist}(k, j, k-1))$$

$$\#k's = 1 \implies \mathsf{dist}(i, k, k-1)$$

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Compute all-pair $\textsf{cap}(u, v)$.

$$\textsf{cap}(u, v, k) = \max(\textsf{cap}(u, v, k-1), \min(\textsf{cap}(u, k, k-1), \textsf{cap}(k, v, k-1)))$$

Routing table (Problem 13.1)

Contruct routing table and extract shortest paths from it.

Init: $\mathsf{Go}(i,j) \leftarrow$ Null

$\forall (i,j) \in E : \mathsf{Go}(i,j) \leftarrow j$

if $\ldots$ **then**
    $\mathsf{Go}(i,j) \leftarrow \mathsf{Go}(i,k)$

if $Go(i,j) =$ Null **then**
    $\cdots$

while $i \neq j$ **do**
    $i \leftarrow \mathsf{Go}(i,j)$

$$\mathsf{Prev}(i,j) \leftarrow \mathsf{Prev}(k,j)$$

$$\mathsf{Intermediate}(i,j) \leftarrow k$$

Shortest Cycle in Digraph (Problem 13.9)

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$\exists v$: dist$[v][v] < \infty$

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Find shortest cycle in digraph $G = (V, E), w(e) > 0$.

Initialize $\text{dist}[v][v] \leftarrow \infty$ in Floyd-Warshall algorithm

$$\exists v : \text{dist}[v][v] < \infty$$

$$\forall v : \text{dist}[v][v] = \infty$$

Paths in Graphs

Hamiltonian path in Tournament graph

Hamiltonian path in Tournament graph (Problem 6.22)

$$\forall u, v : (u \to v \lor v \to u) \\ \land \neg(u \to v \land v \to u)$$

By mathematical induction on n.


THANK YOU