算法习题整理 (4.33 P_{213})

October 4, 2011

1 算法习题整理 (4.33 P_{213})

Problem: Give lower bound based on decision tree for the number of comparisons needed to merge two sorted segments of length k and m, where $k+m=n, k \leq m$.

1. Derive an expression that may involves sums, but is exact.

   Solution: 习题 4.25 已给出 merge sort 所能形成的排列数, 即 $\binom{n}{m}$. 所以 decision tree 的叶子数为 $\binom{n}{m}$, lower bound (denoted LB) of number of comparisons (height of decision tree) is

   $$LB = \log\binom{n}{m} = \log\frac{n!}{m!(n-m)!} = \log\frac{n!}{m!k!} = \log n! - \log m! - \log k!.$$

   你可以进一步将 $\log n!$ 改写成 $\sum$ 的形式.

2. For $k = m = \frac{n}{2}$, find an approximation in closed form that is close to, but guaranteed to be less than, your expression from part (a).

   Solution: 使用 Stirling 公式近似得,

   $$\log n! \sim n\log n, \log(\frac{n}{2})! \sim \frac{n}{2}\log\frac{n}{2}$$

   故 (when n is sufficiently large),

   $$LB = n\log n - 2\frac{n}{2}(\log n - 1) = n$$

   Notice that in Theorem 4.4 (P_{173}), the result is $(n-1)$.

3. Find an approximation in closed from for the case when $k < m$. To get good results, you might wish to consider several ranges for the relationship between k and n.

Solution: 使用 Stirling 公式:

$$n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$$

得,

$$LB = n \log n - m \log m - k \log k + \frac{1}{2} \log \frac{n}{2\pi m k}$$

设

$$k = n\alpha, m = n - k = n(1-\alpha)$$

则有

$$LB = n((\alpha - 1) \log (1-\alpha) - \alpha \log \alpha) + \frac{1}{2} \log \frac{1}{2\pi n \alpha (1-\alpha)}$$

对 α 求导, 得

$$(LB)' = n(\log (1-\alpha) - \log \alpha) + \frac{2\alpha - 1}{2\alpha(1-\alpha) \ln 2}$$