Dynamic Programming

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我走过最长的路就是你的套路

Steps for applying DP:

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(1) Define subproblems
   - # of subproblems

(2) Set the goal

(3) Define the recurrence
   - larger subproblem $\leftarrow$ # smaller subproblems
   - init. conditions

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(4) Write pseudo-code
- fill "table" in some order

(5) Analyze the time complexity

(6) Extract the optimal solution (optionally)

Common subproblems in DP: 1D subproblems

Input: $x_1, x_2, \ldots, x_n$ (array, sequence, string)

Subproblems: $x_1, x_2, \ldots, x_i$ (prefix/suffix)

#: $\Theta(n)$

Examples: Maximum-sum subarray, Longest increasing subsequence, Text justification (LaTeX)

Common subproblems in DP: 2D subproblems

1. Input: $x_1, x_2, \ldots, x_m; \quad y_1, y_2, \ldots, y_n$

   Subproblems: $x_1, x_2, \ldots, x_i; \quad y_1, y_2, \ldots, y_j$

   #: $\Theta(mn)$

   Examples: Edit distance, Longest common subsequence

Common subproblems in DP: 2D subproblems

1. Input: $x_1, x_2, \ldots, x_m; \quad y_1, y_2, \ldots, y_n$
   Subproblems: $x_1, x_2, \ldots, x_i; \quad y_1, y_2, \ldots, y_j$
   #: $\Theta(mn)$
   Examples: Edit distance, Longest common subsequence

2. Input: $x_1, x_2, \ldots, x_n$
   Subproblems: $x_i, \ldots, x_j$
   #: $\Theta(n^2)$
   Examples: Matrix chain multiplication, Optimal BST

Common subproblems in DP: 3D subproblems

- Floyd-Warshall algorithm

$$\mathsf{d}(i,j,k) = \min\Big(\mathsf{d}(i,j,k-1), \mathsf{d}(i,k,k-1) + \mathsf{d}(k,j,k-1)\Big)$$

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DP on graphs

1. On rooted tree

   Subproblems: rooted subtrees

2. On DAG

   Subproblems: nodes after/before in the topo. order

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2. On DAG

   Subproblems: nodes after/before in the topo. order

Knapsack problem

Subset sum problem, Change-making problem

And Others . . .

Recurrences in DP: Make choices by asking yourself the right question

(1) Binary choice
    - whether ...

(2) Multi-way choices
    - where to ...
    - which one ...

1D DP

$f^{(S(n))} = 1$ (Problem 14.3)

$$f(n) = \begin{cases} n - 1 & \text{if } n \in \mathbb{Z}^+ \\ n/2 & \text{if } n \% 2 = 0 \\ n/3 & \text{if } n \% 3 = 0 \end{cases}$$

$S(n)$: minimum number of steps taking n to 1.

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$S(n)$: minimum number of steps taking n to 1.

$S(i)$: minimum number of steps taking i to 1

$$\begin{aligned} S(i) = 1 + \min\{S(i-1), & \\ & S(i/2) \quad (\text{if } n \,\%\, 2 = 0), \\ & S(i/3) \quad (\text{if } n \,\%\, 3 = 0)\} \end{aligned}$$

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$$S(1) = 0$$

Longest Increasing Subsequence (Problem 14.4)

- Given an integer array $A[1 \dots n]$
- To find (the length of) a longest increasing (non-decreasing) subseqence.

$$5, 2, 8, 6, 3, 6, 9, 7 \implies 2, 3, 6, 9$$

Subproblem: $L(i)$: the length of the LIS ending with $A[i]$

Goal: $\max_i L(i)$

Subproblem: $L(i)$: the length of the LIS ending with $A[i]$

Goal: $\max_i L(i)$

Make choice: What is the previous element?

Recurrence:

$$L(i) = 1 + \max_{j<i \wedge A[j] \leq A[i]} L(j)$$

Subproblem: $L(i)$: the length of the LIS ending with $A[i]$

Goal: $\max_i L(i)$

Make choice: What is the previous element?

Recurrence:

$$L(i) = 1 + \max_{j<i \,\wedge\, A[j] \le A[i]} L(j)$$

Init:

$$L(0) = 0$$

Time:

$$O(n^2) = \Theta(n) \cdot O(n)$$

2D DP

LCS: Longest Common Subsequence (Problem 14.6 (1))

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

(1) Find (the length of) an LCS of X and Y

$$X = \langle A, B, C, B, D, A, B \rangle$$
$$Y = \langle B, D, C, A, B, A \rangle$$
$$Z = \langle B, C, B, A \rangle$$

Subproblem: $L[i,j]$: the length of an LCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$

Goal: $L[m,n]$

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Goal: $L[m, n]$

Make choice: Is $X_i = Y_j$?

Recurrence: (Proof!)

$$L[i, j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1, j], L[i, j-1]\} & \text{if } X_i \neq Y_j \end{cases}$$

Subproblem: $L[i, j]$: the length of an LCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$

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Init:

$$L[0, j] = 0, \ 0 \leq j \leq n$$
$$L[i, 0] = 0, \ 0 \leq i \leq m$$

Time: $\Theta(mn)$

Longest Common Subsequence (Problem 14.6 $(2)\&(3)$)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

(2) Allowing repetition of X

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(3) Allowing repetition $\leq k$ of X

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$$X \implies X^{(k)} \triangleq X_1^{(k)} \cdots X_m^{(k)}$$

Longest Contiguous Substring Both Forward and Backward (Problem 14.7)

- String $T[1 \cdots n]$
- Find a longest contiguous substring (LCS) both forward and backward

dynamicprogrammingmanytimes

- Subproblem $L[i]$: the length of an LCS in $T[1 \cdots i]$
- Subproblem $L[i, j]$: the length of an LCS in $T[i \cdots j]$

Subproblem: $L[i, j]$: the length of an LCS starting with T_i and ending with T_j

Goal: $\max_{1 \le i \le j \le n} L[i, j]$

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Goal: $\max\limits_{1 \leq i \leq j \leq n} L[i,j]$

Make choice: Is $T_i = T_j$?

Recurrence:

$$L[i,j] = \begin{cases} 0 & \text{if } T_i \neq T_j \\ L[i+1, j-1] + 1 & \text{if } T_i = T_j \end{cases}$$

Subproblem: $L[i,j]$: the length of an LCS starting with T_i and ending with T_j

Goal: $\max_{1 \le i \le j \le n} L[i,j]$

Make choice: Is $T_i = T_j$?

Recurrence:

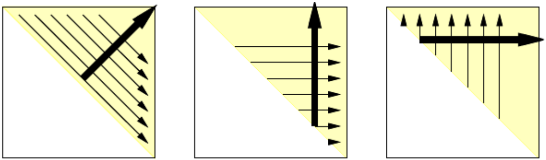
$$L[i,j] = \begin{cases} 0 & \text{if } T_i \neq T_j \\ L[i+1, j-1] + 1 & \text{if } T_i = T_j \end{cases}$$

Init:

$$L[i,i] = 0, \; 0 \le i \le n$$

$$L[i,i+1] = \begin{cases} 1 & \text{if } T_i = T_{i+1} \\ 0 & \text{if } T_i \neq T_{i+1} \end{cases}$$

Three ways of filling the table:



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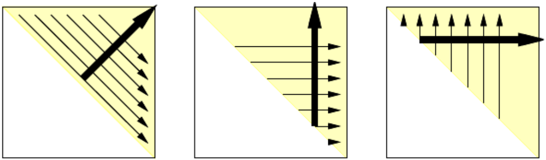

for all $d \leftarrow 2 \ldots n-1$ **do**
  for all $i \leftarrow 1 \ldots n-d$ **do**
    $j \leftarrow i + d$
    $\ldots$
return $\max\limits_{1 \le i \le j \le n} L[i,j]$

Longest Palindrome Subsequence (Problem 14.11 (1))

(1) Find (the length of) a longest palindrome subsequence of $S[1 \cdots n]$

Subproblem: $L[i, j]$: the length of an LSP of $S[i \cdots j]$

Goal: $L[1, n]$

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Goal: $L[1, n]$

Make choice: Is $S[i] = S[j]$?

Recurrence:

$$L[i, j] = \begin{cases} L[i+1, j-1] + 2 & \text{if } S[i] = S[j] \\ \max\{L[i+1, j], L[i, j-1]\} & \text{if } S[i] \neq S[j] \end{cases}$$

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Init:

$$L[i,i] = 1, \ \forall 1 \le i \le n$$

$$L[i, i+1] = \begin{cases} 2 & \text{if } S[i] = S[i+1] \\ 0 & \text{if } S[i] \neq S[i+1] \end{cases}$$

Palindrome Splitting (Problem 14.11 (2))

(2) Split a string $S[1 \ldots n]$ into minimum number of palindromes ($\#$ cuts)

Subproblem: $C[i, j]$: minimum number of cuts for string $S[i \ldots j]$

Goal: $C[1, n] + 1$

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Make choice: Where is the first cut?

Recurrence:

$$C[i,j] = \begin{cases} 0 & \text{if } S[i \dots j] \text{ is a palindrome} \\ \min\limits_{i+1 \le k \le j-1} C[i,k-1] + 1 + C[k,j] & \text{o.w.} \end{cases}$$

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Init: $C[i,i] = 0$

Time: $O(n^3)$

Palindrome Splitting (Problem 14.11 (2))

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Goal: $P[n]$

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Make choice: Where does the last palindrome start from?

Recurrence:

$$P[i] = \min_{\substack{1 \le k \le i \\ S[k \ldots i] \text{ is a palindrome}}} P[k-1] + 1$$

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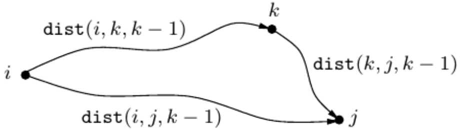
Init: $P[0] = 1$

Time: $O(n^3)$ *vs.* $O(n^2)$

3D DP

Floyd-Warshall algorithm

$$\text{dist}[i,j,k] = \min\{\text{dist}[i,j,k-1], \text{dist}[i,k,k-1] + \text{dist}[k,j,k-1]\}$$

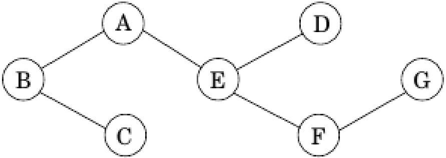


$$\text{dist}[i,j,0] = \begin{cases} 0 & i = j \\ w(i,j) & (i,j) \in E \\ \infty & \text{o.w.} \end{cases}$$

DP on Graphs

Minimum Vertex Cover on Trees (Problem 14.14)

- Undirected tree $T = (V, E)$; No designated root!
- Compute (the size of) a minimum vertex cover of T



Rooted T at any node r.

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Subproblem: $I(u)$: the size of an MVC of subtree T_u rooted at u

Goal: $I(r)$

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Goal: $I(r)$

Make choice: Is u in MVC$[u]$?

Recurrence:

$$I(u) = \min\{\#\text{ children of } u + \sum_{v\text{: grandchildren of } u} I(v), \quad 1 + \sum_{v\text{: children of } u} I(v)\}$$

Rooted T at any node r.

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Init: $I(u) = 0$, if u is a leave

Rooted T at any node r.

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Goal: $I(r)$

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$$I(u) = \min\{\#\text{ children of } u + \sum_{v:\text{ grandchildren of } u} I(v), \quad 1 + \sum_{v:\text{ children of } u} I(v)\}$$

Init: $I(u) = 0$, if u is a leave

DFS from root r.

The Knapsack Problem

The Change-making Problem (Problem 14.13)

- Coins values: $x_1 \dots x_n$
- Amount: v
- Is it possible to make change for v?

The Change-making Problem (Problem 14.13 (2), Problem 14.2 (Subset sum))

(2) Without repetition $(0/1)$

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Subproblem: $C[i, w]$: Make change for w using only values of $x_1 \dots x_i$?

Goal: $C[n, v]$

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Subproblem: $C[i, w]$: Make change for w using only values of $x_1 \ldots x_i$?

Goal: $C[n, v]$

Make choice: Using value x_i or not?

Recurrence:

$$C[i, w] = C[i-1, w] \vee (C[i-1, w - x_i] \wedge w \geq x_i)$$

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Goal: $C[n, v]$

Make choice: Using value x_i or not?

Recurrence:

$$C[i, w] = C[i-1, w] \lor (C[i-1, w-x_i] \land w \geq x_i)$$

Init:

$$C[i, 0] = \text{true } \forall i = 0 \dots n$$
$$C[0, w] = \text{false}, \text{ if } w > 0$$
$$C[0, 0] = \text{true}$$

Time: $O(nv)$

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(1) Unbounded repetition (∞)

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(3) Unbounded repetition with $\leq k$ coins

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Subproblem: $C[i, w, l]$: Possible to make change for w with $\leq l$ coins of values of $x_1 \ldots x_i$?

Goal: $C[n, v, k]$

The Change-making Problem (Problem 14.13 (3))

(3) Unbounded repetition with $\leq k$ coins

Subproblem: $C[i, w, l]$: Possible to make change for w with $\leq l$ coins of values of $x_1 \dots x_i$?

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Recurrence:

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Subproblem: $C[i, w, l]$: Possible to make change for w with $\leq l$ coins of values of $x_1 \dots x_i$?

Goal: $C[n, v, k]$

Make choice: Using value x_i or not?

Recurrence:

$$C[i, w, l] = C[i-1, w, l] \lor (C[i, w - x_i, l - 1] \land w \geq x_i)$$

Init:

$$C[0, 0, l] = \text{true}, \quad C[0, w, l] = \text{false, if } w > 0$$
$$C[i, 0, l] = \text{true}, \quad C[i, w, 0] = \text{false, if } w > 0$$

Algorithms that use dynamic programming [edit | edit source]

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how and when to remove this template message)

- Recurrent solutions to lattice models for protein-DNA binding
- Backward induction as a solution method for finite-horizon discrete-time dynamic optimization problems
- Method of undetermined coefficients can be used to solve the Bellman equation in infinite-horizon, discrete-time, discounted, time-invariant dynamic optimization problems
- Many string algorithms including longest common subsequence, longest increasing subsequence, longest common substring, Levenshtein distance (edit distance)
- Many algorithmic problems on graphs can be solved efficiently for graphs of bounded treewidth or bounded clique-width by using dynamic programming on a tree decomposition of the graph.
- The Cocke–Younger–Kasami (CYK) algorithm which determines whether and how a given string can be generated by a given context-free grammar
- Knuth's word wrapping algorithm that minimizes raggedness when word wrapping text
- The use of transposition tables and refutation tables in computer chess
- The Viterbi algorithm (used for hidden Markov models)
- The Earley algorithm (a type of chart parser)
- The Needleman–Wunsch algorithm and other algorithms used in bioinformatics, including sequence alignment, structural alignment, RNA structure prediction
- Floyd's all-pairs shortest path algorithm
- Optimizing the order for chain matrix multiplication
- Pseudo-polynomial time algorithms for the subset sum, knapsack and partition problems
- The dynamic time warping algorithm for computing the global distance between two time series
- The Selinger (a.k.a. System R) algorithm for relational database query optimization
- De Boor algorithm for evaluating B-spline curves
- Duckworth–Lewis method for resolving the problem when games of cricket are interrupted
- The value iteration method for solving Markov decision processes
- Some graphic image edge following selection methods such as the "magnet" selection tool in Photoshop
- Some methods for solving interval scheduling problems
- Some methods for solving the travelling salesman problem, either exactly (in exponential time) or approximately (e.g. via the bitonic tour)
- Recursive least squares method
- Beat tracking in music information retrieval
- Adaptive-critic training strategy for artificial neural networks
- Stereo algorithms for solving the correspondence problem used in stereo vision
- Seam carving (content-aware image resizing)
- The Bellman–Ford algorithm for finding the shortest distance in a graph
- Some approximate solution methods for the linear search problem
- Kadane's algorithm for the maximum subarray problem


THANK YOU