Asymptotics, Recurrences, and Divide and Conquer

Hengfeng Wei

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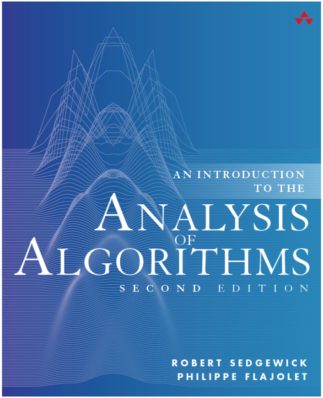
April 17, 2018


AN INTRODUCTION
TO THE
ANALYSIS
OF
ALGORITHMS
SECOND EDITION
ROBERT SEDGEWICK
PHILIPPE FLAJOLET

Problem P Algorithm A

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Average-case Time Complexity (Problem 1.8)

$$\text{Input}: r \in [1, n], \quad r \in \mathbb{Z}^+$$

$$P\{r = i\} = \begin{cases} \frac{1}{n}, & 1 \le i \le \frac{n}{4} \\ \frac{2}{n}, & \frac{n}{4} < i \le \frac{n}{2} \\ \frac{1}{2n}, & \frac{n}{2} < i \le n \end{cases}$$

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$$\begin{aligned} A &= \sum_{X \in \mathcal{X}} T(X) \cdot P(X) \\ &= T(1)P(1) + T(2)P(2) + \cdots + T(n)P(n) \\ &= \frac{n}{4} \times 10 \times \frac{1}{n} + \frac{n}{4} \times 20 \times \frac{2}{n} + \frac{n}{4} \times 30 \times \frac{1}{2n} + \frac{n}{4} \times n \times \frac{1}{2n} \\ &= \frac{1}{8}n + \frac{65}{4} \end{aligned}$$

Average-case Analysis of Quicksort

$$A(n) = n - 1 + \frac{1}{n} \sum_{i=0}^{i=n-1} \big(A(i) + A(n-i-1)\big)$$

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$$\begin{aligned} A(n) &= \mathbb{E}[T(X)] \\ &= \mathbb{E}[\mathbb{E}[T(X)|I]] \\ &= \sum_{i=0}^{i=n-1} P(I=i)\, \mathbb{E}[T(X) \mid I = i] \\ &= \sum_{i=0}^{i=n-1} \frac{1}{n} [n - 1 + A(i) + A(n-i-1)] \end{aligned}$$

3-element Sorting (Problem 1.1)

(1) Design an algorithm for sorting 3 distinct elements.

(2) Worst-case and average-case time complexity.

(3) Worst-case lower bound.

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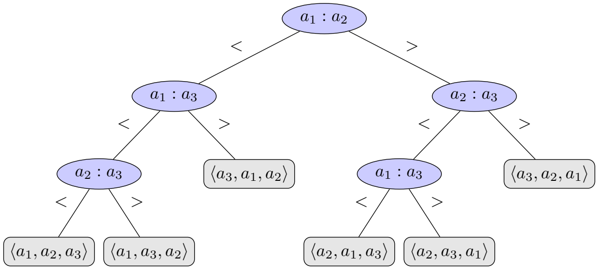

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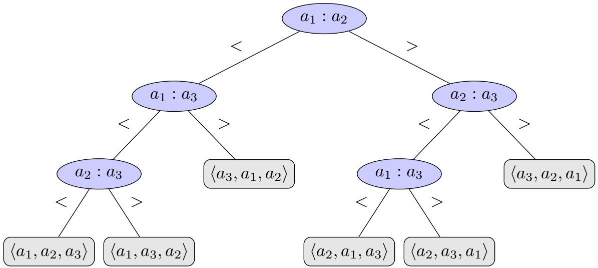

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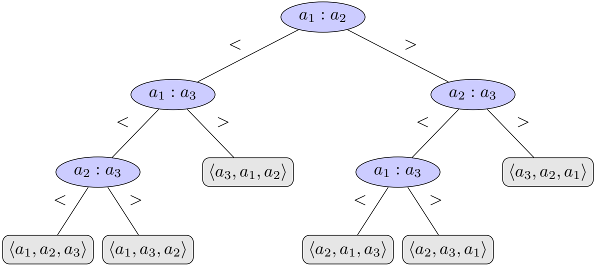

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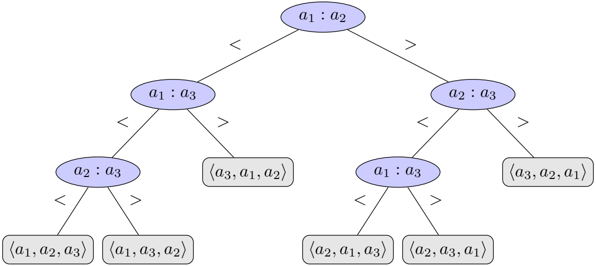

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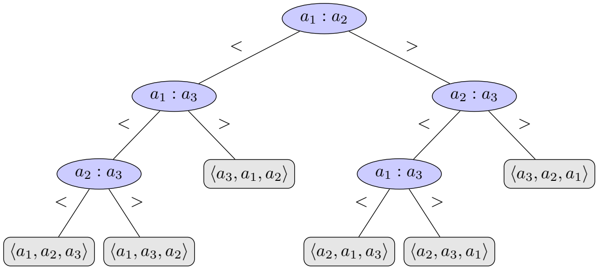

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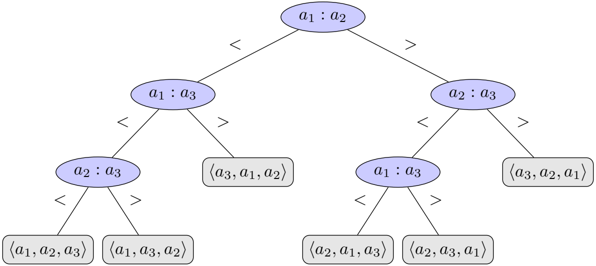

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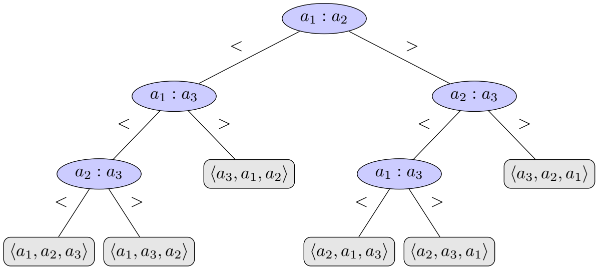

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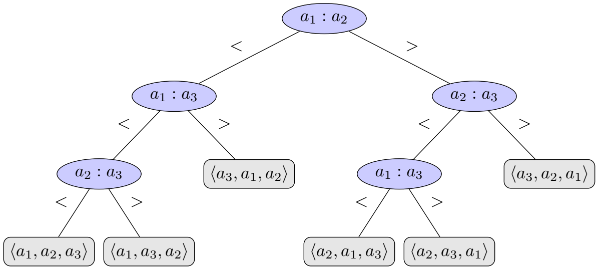

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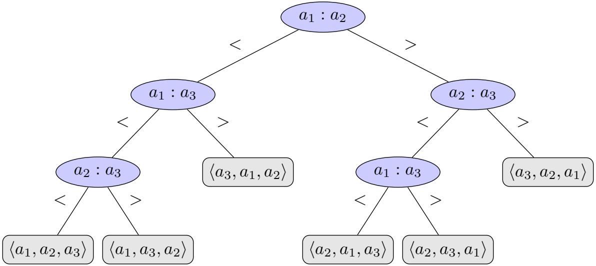

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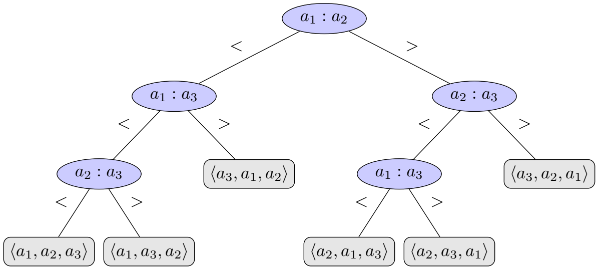

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$$\mathsf{LB}(3) = 3 \qquad (\mathsf{LB}(3) \geq \log 3!)$$

3-element Median Seletion (Problem 1.2)

(1) Design an algorithm for selecting the median of 3 distinct elements.

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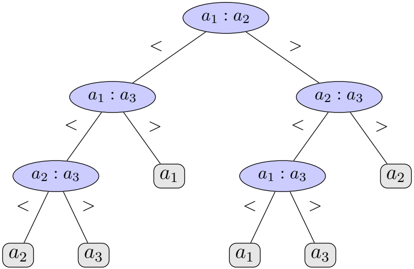

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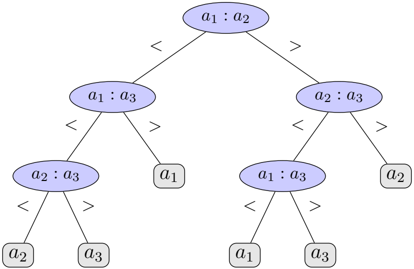

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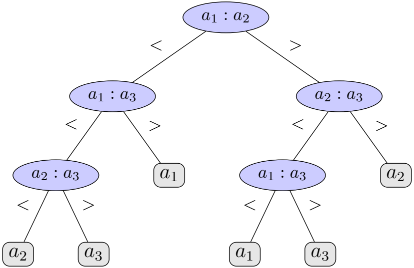

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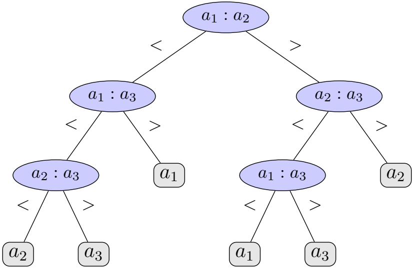

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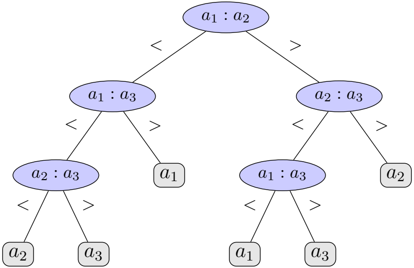

$$W(3) = 3 \qquad B(3) = 2 \qquad A(3) = \frac{8}{3}$$

$$\mathsf{LB}(3) = 3 \qquad (\mathsf{LB}(3) \geq \frac{3n}{2} - \frac{3}{2})$$

$\text{LB} = 2$

LB $= 2$

---

procedure $\textsc{Median}(a, b, c)$
    if $(a-b)(a-c) < 0$ **then**
        return a
    if $(b-a)(b-c) < 0$ **then**
        return b
    return c

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LB $= 2$

Not comparison-based!

Exercise

$$n = 5$$

Exercise

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Reference

"*The Art of Computer Programming, Vol 3: Sorting and Searching (Section 5.3.1)*" by Donald E. Knuth

$$S(21) = 66$$

Mathematical Induction

Horner's rule (Problem 1.5)

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

```
procedure HORNER(A[0...n], x)        ▷ A : {a_0 ... a_n}
    p ← A[n]
    for i ← n − 1 downto 0 do
        p ← px + A[i]
    return p
```

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procedure $\textsc{Horner}(A[0 \ldots n], x)$ $\quad\quad \triangleright\ A : \{a_0 \ldots a_n\}$

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  for $i \leftarrow n - 1$ **downto** 0 **do**

    $p \leftarrow px + A[i]$

  return p

Loop invariant (after the k-th loop):

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Loop invariant (after the k-th loop):

$$\boxed{\mathcal{I}: \; p = \sum_{i=n}^{i=n-k} a_i x^{k-(n-i)}}$$

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When you are in an exam:

20% : Finding $\mathcal{I}$

80% : Proving $\mathcal{I}$ by PMI

$$\boxed{\mathcal{I}: \ p = \sum_{i=n}^{i=n-k} a_i x^{k-(n-i)}}$$

Proof.

Prove by mathematical induction on non-negative integer k, the number of loops.

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Proof.

Prove by mathematical induction on non-negative integer k, the number of loops.

Basis:

$$k = 0 : p = a_n = \mathcal{I}_0$$

Inductive Hypothesis:

Inductive Step:

□

Integer Multiplication (Problem 1.6)

procedure $\textsc{Int-Mult}(y, z)$
  if $z = 0$ **then**
    return 0
  return $\textsc{Int-Mult}(cy, \lfloor \frac{z}{c} \rfloor) + y(z \mod c)$

Integer Multiplication (Problem 1.6)

```
procedure INT-MULT(y, z)
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```
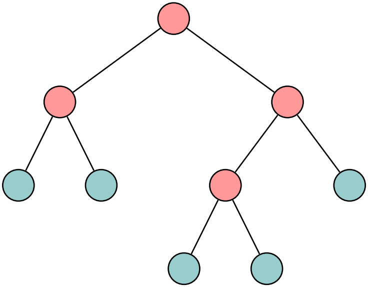

Proof.

Prove by mathematical induction on non-negative integer z. $\square$

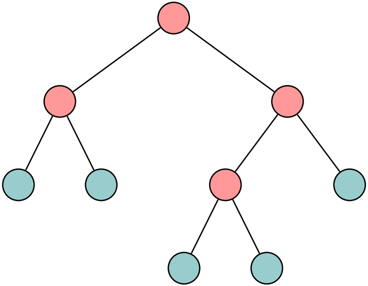
2-tree; full binary tree (Problem 2.5)



$$n_0 = n_2 + 1$$

Proof.

2-tree; full binary tree (Problem 2.5)



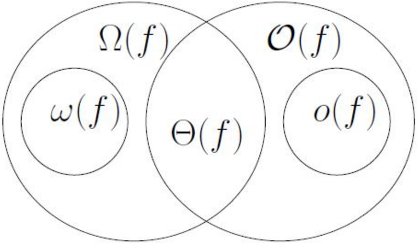
$$n_0 = n_2 + 1$$

Proof.

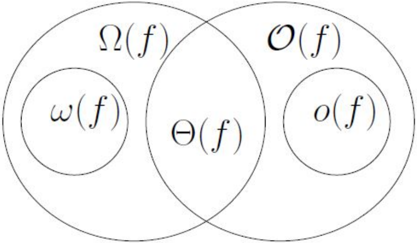
Prove by mathematical induction on *the structure of binary tree.*

$\square$

Asymptotics



Asymptotics



$$Q : \theta(f)?$$

$$O(g(n)) = \{f(n) \mid \exists c > 0, \exists n_0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n)\}$$

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$$\Theta(g(n)) = \{f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0, \forall n \geq n_0 : 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$$

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$$f(n) \sim g(n) \iff \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$$

Asymptotics (Problem 2.6 (4))

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \land f(n) = \Omega(g(n))$$

Asymptotics (Problem 2.6 (4))

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Asymptotics (Problem 2.6 (6))

$$\Theta(g(n)) \cap o(g(n)) = \emptyset$$

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$Q : f(n) = O(g(n)) \lor g(n) = \Omega(f(n))?$

$$f(n) = n, \quad g(n) = n^{1+\sin n}$$

Asymptotics (Problem 2.6 (4))

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Reference:

"Big Omicron and Big Omega and Big Theta" by Donald E. Knuth, 1976.

Asymptotics (Problem 2.7 (2))

$$(\log n)^2 \quad \textit{vs.} \quad \sqrt{n}$$

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$$(\log n)^{c_1} = O(n^{c_2}) \quad c_1, c_2 > 0$$

Asymptotics (Problem 2.10)

$$\log(n!) = \Theta(n \log n)$$

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Stirling Formula (by *James Stirling*):

$$n! = \Theta\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)$$

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$$\log(n!) \leq n \log n$$

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$$\log(n!) \le n \log n \qquad \log(n!) \ge \frac{n}{2} \log \frac{n}{2}$$