What You Should Know About Algorithm Design and Analysis . . . But (Probably) Don't

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When You Design Algorithms:

Design Faster Algorithms

When to Stop?

The Complexity of Problems

Problem P Algorithm A

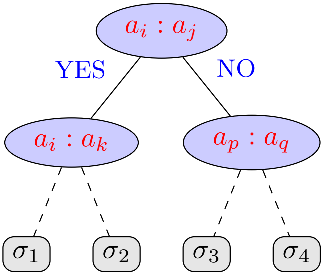
Inputs: $\mathcal{X}_n$ of size n

$$W_A(n) = \max_{x \in \mathcal{X}_n} T_A(x)$$

$$B_A(n) = \min_{x \in \mathcal{X}_n} T_A(x)$$

$$A_A(n) = \sum_{x \in \mathcal{X}_n} T_A(x) \cdot P(x) = \mathbb{E}[T_A] = \sum_{t \in T_A(\mathcal{X}_n)} t \cdot P(T = t)$$

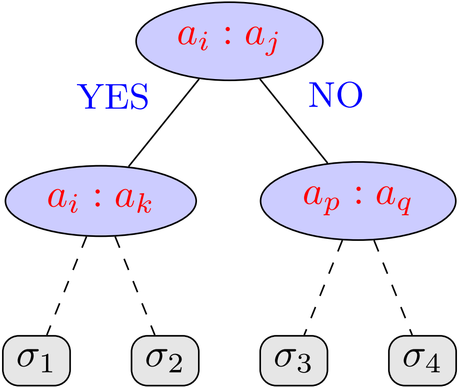
$$\boxed{T_P(n) = \min_{A \text{ solves } P} W_A(n) = \min_{A \text{ solves } P} \max_{x \in \mathcal{X}_n} T_A(x)}$$



Decision Tree

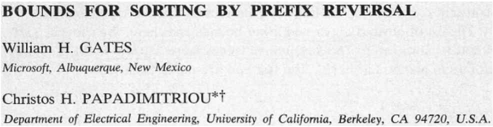
Adversary Argument

Decision Tree



Lower Bound for Comparison-based Sorting

Prove a lower bound of $\Omega(n \log n)$ on the time complexity of any **comparison-based** sorting algorithm on inputs of size n.



BOUNDS FOR SORTING BY PREFIX REVERSAL

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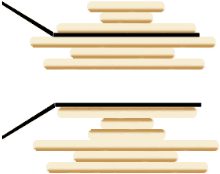
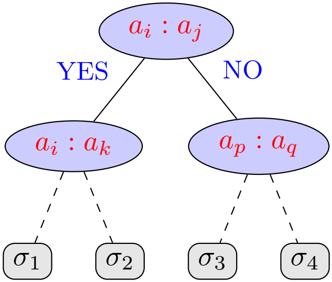


Decision Tree Model



Nodes: comparisions $a_i : a_j$

$$<, \leq, =, \geq, >$$

Edges: two-way decisions (Y/N)

Leaves: possible permutations

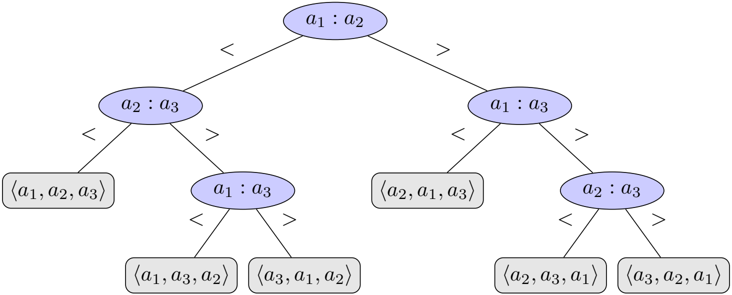
Assumption:

All the input elements are **distinct**.

$$a_i < a_j$$

$$\boxed{\text{Any Comparison-based Sorting Algorithm} \xrightarrow{\text{modeled by}} \text{A Decision Tree}}$$

Any Comparison-based Sorting Algorithm $\xRightarrow{\text{modeled by}}$ A Decision Tree

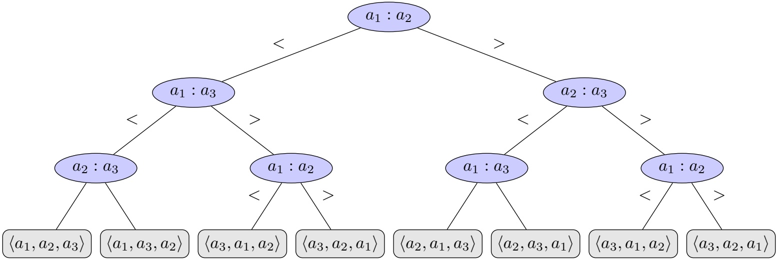


The decision tree for insertion sort on three elements.

Any Comparison-based Sorting Algorithm $\xRightarrow{\text{modeled by}}$ A Decision Tree

```
procedure SELECTION-SORT(A, n)
    for i ← 1 to n − 1 do
        for j ← i + 1 to n do
            if A[j] < A[i] then
                SWAP(A[j], A[i])
```

Any Comparison-based Sorting Algorithm $\xRightarrow{\text{modeled by}}$ A Decision Tree



The decision tree for selection sort on three elements.

$$\boxed{\text{Any Comparison-based Sorting Algorithm } \mathcal{A} \xRightarrow{\text{modeled by}} \text{A Decision Tree } \mathcal{T}}$$

$$\text{Algorithm } \mathcal{A} \text{ on a specific input of size } n \xRightarrow{\text{modeled by}} \text{A path through } \mathcal{T}$$

$$\text{Worst-case time complexity of } \mathcal{A} \xRightarrow{\text{modeled by}} \text{The height of } \mathcal{T}$$

Worst-case Lower Bound of Comparison-based Sorting
(on inputs of size n)

$$\xRightarrow{\text{modeled by}}$$

The Minimum Height of All $\mathcal{T}$s

Worst-case Lower Bound of Comparison-based Sorting
(on inputs of size n)

$$\xRightarrow{\text{modeled by}}$$

The Minimum Height of All $\mathcal{T}$s

To be a correct sorting algorithm:

$$L = \# \text{ of leaves} \geq n!$$

To be a full binary tree:

$$L = \# \text{ of leaves} \leq 2^h$$

$$\boxed{n! \leq L = \# \text{ of leaves} \leq 2^h}$$

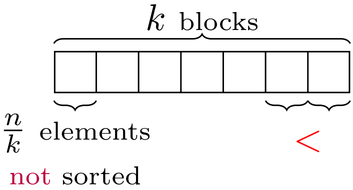
$$h \geq \log n! = \Omega(n \log n)$$

Stirling Formula (by *James Stirling*):

$$n! = \Theta\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)$$

K-sorted Array (Problem 6.8)

$$A[1 \cdots n]$$



$$O(n \log k)$$

$$n = 16, \quad k = 4, \quad \frac{n}{k} = 4$$

1, 2, 4, 3; 7, 6, 8, 5; 10, 11, 9, 12; 15, 13, 16, 14

k-sorted

1-sorted $\rightarrow$ 2-sorted $\rightarrow$ 4-sorted $\rightarrow \cdots \rightarrow$ n-sorted

Quicksort (with median as pivot) stops after the $\log k$ recursions.

$$\boxed{\Theta(n \log k)}$$

$$\Omega(n \log k)$$

$$L \geq \binom{n}{n/k}\binom{n-n/k}{n/k}\cdots\binom{n/k}{n/k} = \binom{n}{n/k,\ldots,n/k} = \frac{n!}{\left((\frac{n}{k})!\right)^k}$$

$$H \geq \log\left(\frac{n!}{\left((\frac{n}{k})!\right)^k}\right) = \Omega(n \log k)$$

$$\boxed{n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \implies \log n! \sim n \log n}$$

Bolts and Nuts (Problem 6.9)

Quicksort

$$A(n) = O(n \log n)$$

In the worst case:

- "Matching Nuts and Bolts" by Alon *et al.*, $\Theta(n \log^4 n)$
- "Matching Nuts and Bolts Optimality" by Bradford, 1995, $\Theta(n \log n)$

$$\Omega(n \log n)$$

$$3^H \geq L \geq n! \implies H \geq \log n! \implies H = \Omega(n \log n)$$

Adversary Argument

Searching in Matrix (Problem 9.8)

$$M : m \times n$$

Row: Increasing from left to right

Col: Increasing from top to down

$$x \in M?$$

$$\textsc{Compare}(x, M[i][j])$$

$$\text{Divide \& Conquer} : T(m,n) = 3T(\frac{m}{2}, \frac{n}{2}) + 1$$

Always checking the lower left corner.

$$T(m,n) = m + n - 1$$

Assume $M : n \times n$

$$W(n) \leq 2n - 1$$

$$\boxed{W(n) \geq 2n - 1}$$

By Adversary Argument!

$$\boxed{W(n) \geq 2n-1}$$

Adversary $\mathcal{A}$:

$x > M[i][j]$

$x = M[i][j]$

$x < M[i][j]$

Algorithm $\mathbb{A}$:

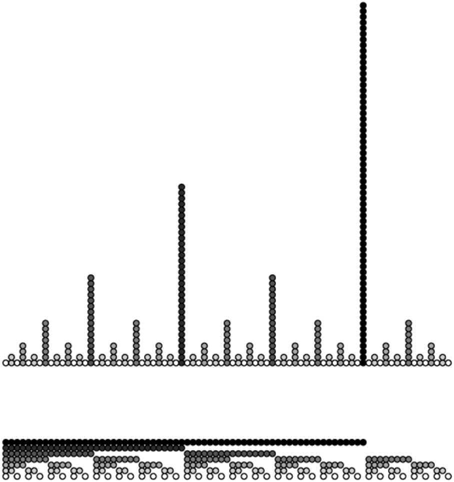
$\textsc{Compare}(x, M[i][j])$

Diagonals: $i+j=n-1 \quad \& \quad i+j=n$

No particular ordering requirements on these two diagonals!

$$i+j \leq n-1 \implies x > M_{ij}$$
$$i+j > n-1 \implies x < M_{ij}$$

Amortized Analysis



Amortized analysis is

an algorithm analysis technique for

analyzing a sequence of operations

irrespective of the input to show that

the average cost per operation is small, even though

a single operation within the sequence might be expensive.

Accounting Method

Summation Method

Potential Method

Amortized Analysis

The Summation Method

$$\sum$$

$$o_1, o_2, \ldots, o_n$$

$$c_1, c_2, \ldots, c_n$$

$$\boxed{\forall i,\ \hat{c}_i = \frac{\left(\sum_{i=1}^{n} c_i\right)}{n}}$$

The Summation Method for Array Doubling

On any sequence of n INSERTs on an initially empty array.

$$
\begin{array}{lcccccccccc}
o_i: & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} \\
c_i: & 1 & 2 & 3 & 1 & 5 & 1 & 1 & 1 & 9 & 1
\end{array}
$$

$$
c_i = \begin{cases} (i-1)+1 = i & \text{if } i-1 \text{ is an exact power of } 2 \\ 1 & \text{o.w.} \end{cases}
$$

$$
\sum_{i=1}^{n} c_i \le n + \sum_{j=0}^{\lceil \lg n \rceil - 1} 2^j = n + (2^{\lceil \lg n \rceil} - 1) \le n + 2n = 3n
$$

$$
\boxed{\forall i,\ \hat{c}_i = 3}
$$

The Accounting Method

$$o_1, o_2, \ldots, o_n$$

$$c_1, c_2, \ldots, c_n$$

$$a_1, a_2, \ldots, a_n$$

$$\boxed{\hat{c}_i = c_i + a_i,\ a_i >=< 0}$$

Amortized Cost $=$ Actual Cost $+$ Accounting Cost

$$\forall n,\ \sum_{i=1}^{n} c_i \le \sum_{i=1}^{n} \hat{c}_i \implies \boxed{\forall n,\ \sum_{i=1}^{n} a_i \ge 0}$$

Key Point: Put the accounting cost on specific objects.

The Accounting Method for Array Doubling

$$\color{red}{Q:}\ \hat{c}_i = 3 \ \textit{vs.}\ \hat{c}_i = 2$$

$$\hat{c}_i = 3 = \underbrace{1}_{\text{insert}} + \underbrace{1}_{\text{move itself}} + \underbrace{1}_{\color{red}{\text{help move another}}}$$

	$\hat{c}_i$	c_i *(actual cost)*	a_i *(accounting cost)*
INSERT *(normal)*	3	1	2
INSERT *(expansion)*	3	$1 + t$	$-t + 2$

Simulating a queue Q using two stacks S_1, S_2 (Problem $\mathbb{E}3$)

procedure $\textsc{Enq}(x)$
    $Push(S_1, x)$

procedure $\textsc{Deq}()$
    if $S_2 = \emptyset$ **then**
        while $S_1 \neq \emptyset$ **do**
            $Push(S_2, Pop(S_1))$
    $Pop(S_2)$

The Summation Method for Queue Simulation

$$\frac{\left(\sum_{i=1}^{n} c_i\right)}{n}$$

The operation sequence is *NOT* known.

The Accounting Method for Queue Simulation

item:	Push into S_1	Pop from S_1	Push into S_2	Pop from S_2
	1	1	1	1

$$\hat{c}_{\text{ENQ}} = 3$$

$$\hat{c}_{\text{DEQ}} = 1$$

$$\sum_{i=1}^{n} a_i \geq 0 \Longleftarrow \sum_{i=1}^{n} a_i = \#S_1 \times 2$$

The Accounting Method for Queue Simulation

$$\hat{c}_{\text{ENQ}} = 3$$
$$\hat{c}_{\text{DEQ}} = 1$$

$$\#S_1 = t$$

	$\hat{c}_i$	c_i *(actual cost)*	a_i *(accounting cost)*
Enqueue	3	1	2
Dequeue ($S_2 \neq \emptyset$)	1	1	0
Dequeue ($S_2 = \emptyset$)	1	$1 + 2t$	$-2t$

Array Merging Dictionary (Problem $\mathbb{E}$ 2)

i	$s_i = 2^i$
A_0	1
A_1	2
A_2	4
A_3	8
$\vdots$	$\dots$
A_i	2^i

$$11 = 2^0 + 2^1 + 2^3$$

i	e_i
A_0	$[5]$
A_1	$[4, 8]$
A_2	$[\,]$
A_3	$[2, 6, 9, 12, 13, 16, 20, 25]$

$\textsc{Create} : 1 \quad \textsc{Merge}(A_i, A_i) : 2 \cdot 2^i$

$\textsc{Insert}() : 1 + 2 + 4; \quad \textsc{Insert}() : 1; \quad \textsc{Insert}() : 1 + 2$

The Summation Method for "Array Merging Dictionary"

$$\textsc{Create} : 1 \quad \textsc{Merge}(A_i, A_i) : 2 \cdot 2^i$$

i	c_i
1	1
2	$1 + 2$
3	1
4	$1 + 2 + 4$
5	1
6	$1 + 2$
7	1
8	$1 + 2 + 4 + 8$
$\vdots$	$\dots$

$$\sum_{i=1}^{n} c_i = \sum_{j=0}^{\lfloor \log n \rfloor} \lfloor \frac{n}{2^j} \rfloor 2^j \le n(\lfloor \log n \rfloor + 1)$$

$$\forall i,\ \hat{c}_i = 1 + \lfloor \log n \rfloor$$

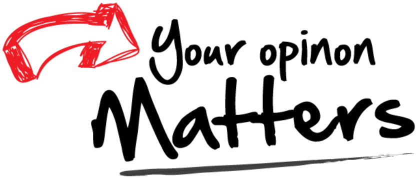
The Accounting Method for "Array Merging Dictionary"

$$\textsc{Create} : 1 \quad \textsc{Merge}(A_i, A_i) : 2 \cdot 2^i$$

$$\hat{c}_i = 1 + \lfloor \log n \rfloor$$

$$\forall n, \ \sum_{i=1}^{n} a_i \geq 0$$


Thank
You!



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