An example of a NP-completeness proof

In class I stress the intuitive aspects of the reductions and skim over the proof. It would be good if you had an example of how a reduction is written up in detail so that you can use it as a template for your own solutions. Here is one, for the set-cover problem. We begin with relevant definitions.

The SET-COVER problem is:

Input: $\langle U, \mathcal{A}, k\rangle$ where $\mathcal{A}$ is a set whose members are subsets of U, and $k \in \mathbb{N}$
Question: Does U have an $\mathcal{A}$-cover of size k?

An $\mathcal{A}$ cover of U is a subset $\mathcal{B}$ of $\mathcal{A}$ such that for every point $u \in U$ there exists a set $B \in \mathcal{B}$ such that $u \in B$.

The VERTEX-COVER problem is:

Input: $\langle G, K\rangle$ where G is a graph and $K \in \mathbb{N}$
Question: Does G have a vertex cover of size K?

A vertex cover of $G = (V, E)$ is a subset W of V such that for every edge $\{i, j\} \in E$ either $i \in W$ or $j \in W$.

Claim 1: SET-COVER is in **NP**.

Proof: The following verifier for SET-COVER runs in time polynomial in the length of its first input:

Verifier $V(\langle U, \mathcal{A}, k\rangle, \langle \mathcal{B}\rangle)$
If all the following are all true then accept else reject:
- $\mathcal{B}$ is a subset of $\mathcal{A}$
- $|\mathcal{B}| \le k$
- $\forall u \in U\ \exists B \in \mathcal{B}\, [u \in B]$ ▮

The **NP**-hardness of SET-COVER is implied by the following.

Claim 2: VERTEX-COVER $\le_p$ SET-COVER.

Proof: Define function f to take input $\langle G, K\rangle$, where $G = (V, E)$, and output $\langle E, \mathcal{A}, K\rangle$ where $\mathcal{A} = \{\, A_w : w \in V \,\}$ and $A_w = \{\, e \in E : w \in e \,\}$ for all $w \in V$. We now show that f is a poly-time reduction of VERTEX-COVER to SET-COVER.

Suppose G has a vertex cover W of size K, and let $\mathcal{B} = \{\, A_w : w \in W \,\}$. Then $|\mathcal{B}| = K$. Furthermore $\mathcal{B}$ is a $\mathcal{A}$-cover of E. To justify this claim, suppose $e = \{u, v\} \in E$. Then e is in both

A_u and A_v. But since W is a vertex cover, at least one of u or v is in W and hence at least one of the sets A_u or A_v is in $\mathcal{B}$.

Conversely, suppose E has a $\mathcal{A}$-cover $\mathcal{B}$ of size K, and let $W = \{w \in V : A_w \in \mathcal{B}\}$. Then $|W| = K$. Furthermore W is a vertex cover of G. To justify this claim, suppose $e \in E$. Since $\mathcal{B}$ is a $\mathcal{A}$-cover of E, there is a set $A_w \in \mathcal{B}$ such that $e \in A_w$. The definition of W then says that $w \in W$ and the definition of A_w says that $w \in e$. ▌