Decompositions of Graphs

(DFS/BFS, DAG, SCC, Bicomp)

Hengfeng Wei

hfwei@nju.edu.cn

June 12, 2019

John Hopcroft

Robert Tarjan

"For fundamental achievements in the design and analysis of algorithms and data structures."

— *Turing Award*, 1986

SIAM J. COMPUT.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

Abstract. The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an undirect graph are presented. The space and time requirements of both algorithms are bounded by $k_1V + k_2E + k_3$ for some constants k_1, k_2, and k_3, where V is the number of vertices and E is the number of edges of the graph being examined.

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Power of DFS:

Graph Traversal $\Longrightarrow$ Graph Decomposition

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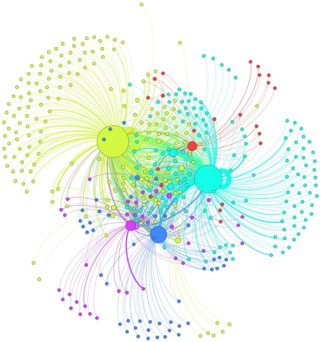
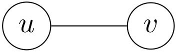
Structure! Structure! Structure!



Graph *structure* induced by DFS:

states of 

types of 

Graph *structure* induced by DFS:

states of v

types of u — v

life time of v:

v : $\mathrm{d}[v], \mathrm{f}[v]$

$\mathrm{d}[v]$: Bicomp

$\mathrm{f}[v]$: Toposort, SCC

Definition (Classifying edges)

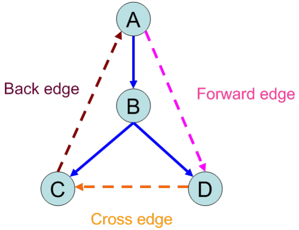
Given a DFS traversal $\implies$ DFS tree:

Tree edge: $\to$ child

Back edge: $\to$ ancestor

Forward edge: $\to$ *nonchild* descendant

Cross edge: $\to$ $(\neg$ancestor$) \wedge (\neg$descendant$)$



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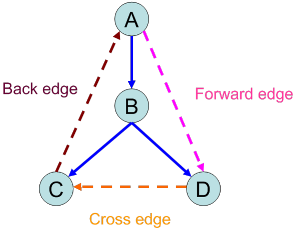
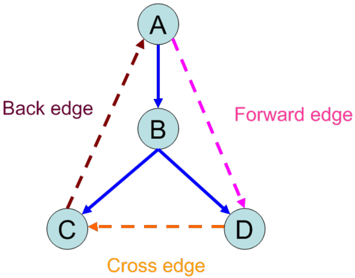
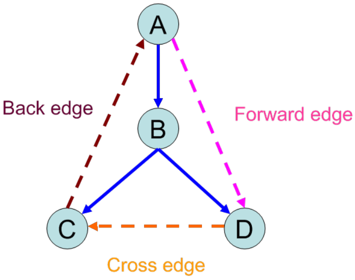


Also applicable to BFS



DFS on directed graph



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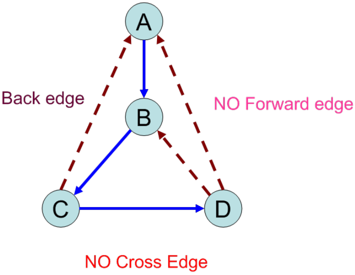
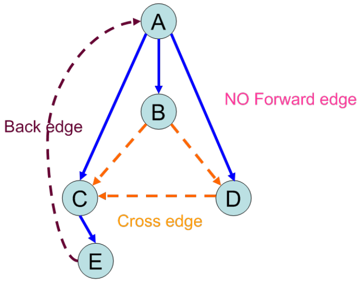
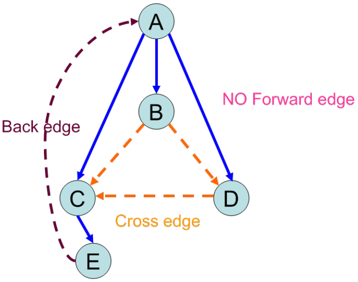


DFS on undirected graph



BFS on directed graph



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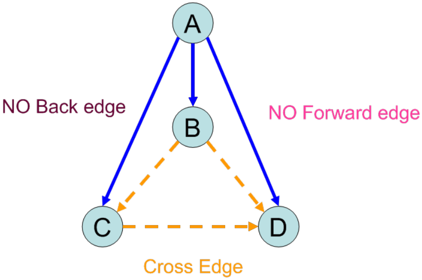

BFS on undirected graph

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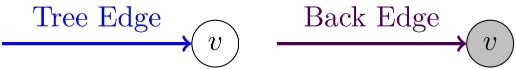

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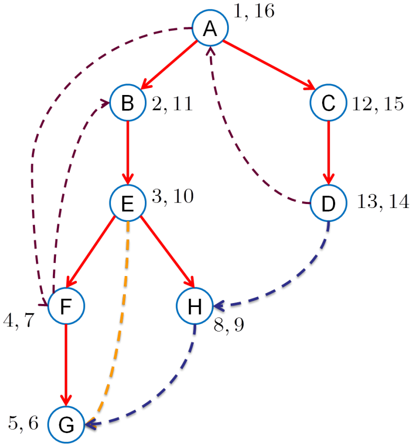
Life time of vertices in DFS



Edge types and life time of vertices in DFS (Problem 4.5)

$$\forall u \to v :$$

- tree/forward edge: $[_u \; [_v \;]_v \;]_u$
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$$\nexists \text{ cycle} \implies \boxed{u \to v \iff \mathrm{f}[v] < \mathrm{f}[u]}$$

Cycle detection (Problem 5.8 – 1)

	Digraph	Undirected graph
DFS		
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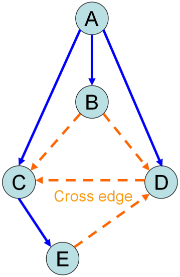

Evasiveness of acyclicity of undirected graphs (Problem 5.8 – 2)

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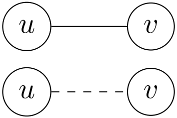

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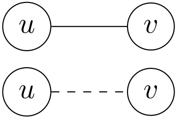

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Hint: Kruskal

$$\mathbb{A} : \textsc{CheckEdge}(u, v) \leftarrow \mathcal{A} : u \text{ --- } v$$

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Q : Why adjacency matrix?

Hamiltonian path in DAG (Problem 4.14)

HP: path visiting each vertex once

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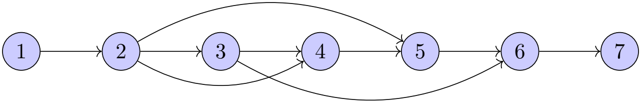

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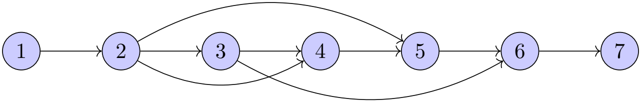

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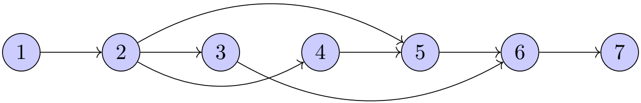
By contradiction; Perform Toposort; Swap



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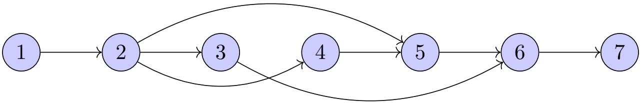

Tarjan's TOPOSORT + Check edges (v_i, v_{i+1})

Theorem (Digraph as DAG (Problem 4.6))

Every digraph is a dag of its SCCs.

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Two tiered structure of digraphs:

digraph $\equiv$ a dag of SCCs

SCC: equivalence class over reachability

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$$\Longleftarrow : \exists!\ \text{source}$$

$$\Longrightarrow : \text{ By contradiction.}$$

$$\exists u : v \not\leadsto u \land \text{in}[u] > 0 \implies \exists \text{ cycle}$$

Impacts of vertices in a digraph (Problem 4.18)

$$\text{impact}(v) = |\{w \neq v : v \leadsto w\}|$$

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$$\boxed{Q : \forall v, \text{computing impact}(v)}$$

2SAT (Problem 4.23)

$$I : (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (\overline{x_3} \vee x_4) \wedge (\overline{x_1} \vee x_4)$$

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$$\alpha \vee \beta \equiv \overline{\alpha} \to \beta \equiv \overline{\beta} \to \alpha$$

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Implication graph G_I.

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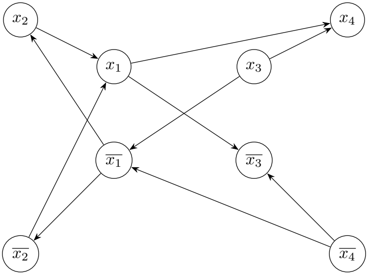
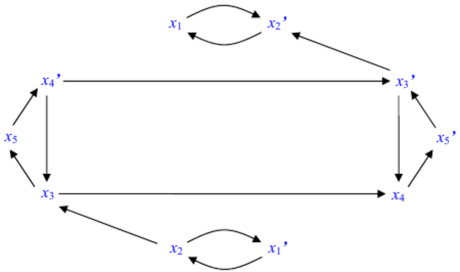
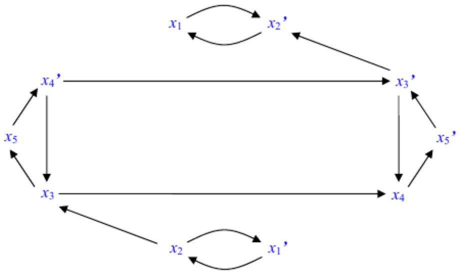




Theorem (2SAT)

$$\exists\ SCC\ \exists x : v_x \in SCC \wedge v_{\overline{x}} \in SCC \iff I \textit{ is not satisfiable.}$$



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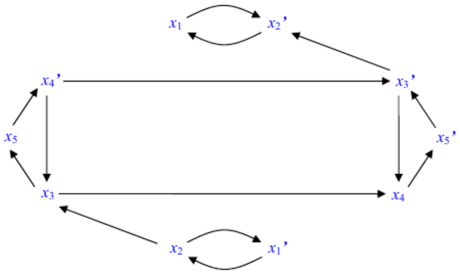
"F" for source SCC & "T" for sink SCC



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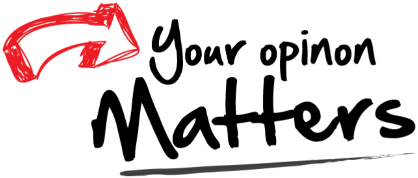
"A Linear-time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas", Bengt Aspvall, Michael Plass, Robert Tarjan, 1979


THANK YOU



Office 302

Mailbox: H016

hfwei@nju.edu.cn