A Theorem on the Uniqueness of Minimum Spanning Tree

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Theorem. *Let $G = (V, E)$ be a connected graph with unique edge weights ($w : E \to R$ such that for every $e_i, e_j \in E$ if $i \neq j$ then $w(e_i) \neq w(e_j)$). Then there is a unique minimum spanning tree.*

Proof. Consider a minimum spanning tree T of the graph G and let $L = w(e_1) < w(e_2) < \cdots < w(e_k)$ ($k = |V| - 1$) be the sorted list of edge weights of T. Assume G has a different minimum spanning tree T' and let $L' = w(e'_1) < w(e'_2) < \cdots < w(e'_k)$ be the sorted list of edge weights of T'.

Let t be the first position where L and L' are different, that is $w(e_t) \neq w(e'_t)$ and $w(e_q) = w(e'_q)$ (thus $e_q = e'_q$) for any $q < t$. Without loss of generality assume that $w(e_t) < w(e'_t)$. Let $H = T' \cup \{e_t\}$. Then H must contain a simple cycle C. Consider $S = C - E(T)$, the set of all edges on C that are not in T (and hence are in T'). The weight of every edge in S is greater than $w(e_t)$, since e_t is the lightest edge not shared by both T and T'. Therefore, by removing one of the edges in S from H we obtain a spanning tree (why?) whose weight is smaller than $w(T')$, a contradiction to T' being a minimum spanning tree. $\square$

Remark: The theorem could also be proved using the corollary from tutorial 5 saying that for every spanning tree T there is a possible run of Kruskal's algorithms that yields T: Since the weights are all different there is only one possible run of Kruskal's algorithm, thus the spanning tree is unique.