# Decompositions of Graphs

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June 12, 2018

John Hopcroft

Robert Tarjan

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*"For fundamental achievements in the design and analysis of algorithms and data structures."*

— *Turing Award, 1986*

SIAM J. COMPUT.
Vol. 1, No. 2, June 1972

## DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

**Abstract.** The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an undirect graph are presented. The space and time requirements of both algorithms are bounded by $k_1V + k_2E + k_3$ for some constants $k_1$, $k_2$, and $k_3$, where $V$ is the number of vertices and $E$ is the number of edges of the graph being examined.

**Key words.** Algorithm, backtracking, biconnectivity, connectivity, depth-first, graph, search, spanning tree, strong-connectivity.

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*"DFS is a powerful technique with many applications."*

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Power of DFS:

Graph Traversal $\Longrightarrow$ Graph Decomposition

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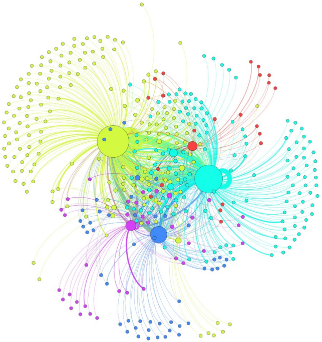
Structure! Structure! Structure!



Graph *structure* induced by DFS:

states of $v$

types of $u$ — $v$

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types of $u$ — $v$

life time of $v$:

$v$ : d$[v]$, f$[v]$

f$[v]$: DAG, SCC

d$[v]$: biconnectivity

## Definition (Classifying edges)

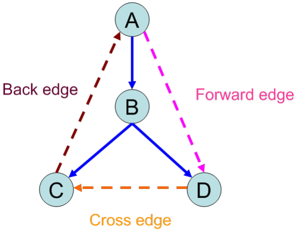
Given a DFS traversal $\Longrightarrow$ DFS tree:

Tree edge: $\rightarrow$ child

Back edge: $\rightarrow$ ancestor

Forward edge: $\rightarrow$ *nonchild* descendant

Cross edge: $\rightarrow$ $(\neg\text{ancestor}) \wedge (\neg\text{descendant})$



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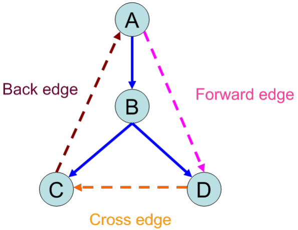
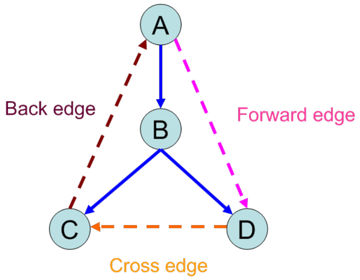
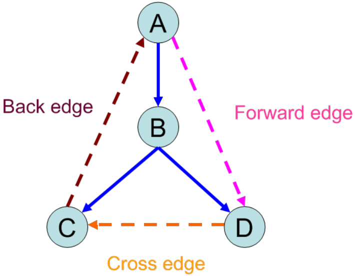



- also applicable to BFS
- w.r.t. DFS/BFS trees




DFS on directed graph

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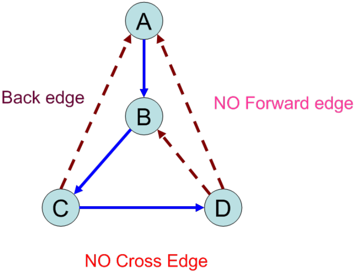
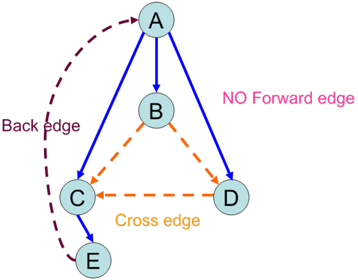
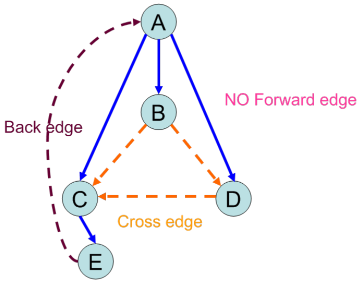



DFS on undirected graph




BFS on directed graph

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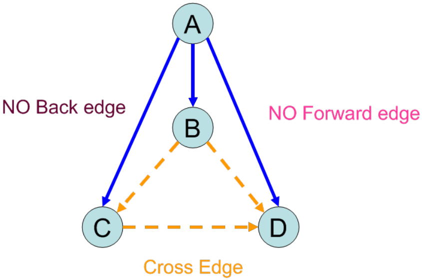


BFS on undirected graph

## DFS tree and BFS tree coincide (Problem $5.7$)

Undirected connected graph $G = (V, E), v \in V$

DFS tree $T$ from $v \equiv$ BFS tree $T'$ from $v$

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Proof.

$G_{\text{DFS}}$: tree + back *vs.* $G_{\text{BFS}}$: tree + cross

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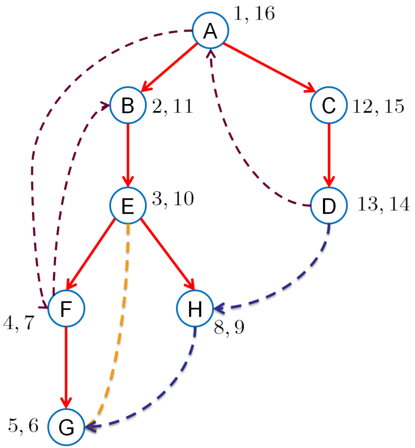
$Q$ : What if $G$ is a digraph?

# Lift time of vertices in DFS



Theorem (Disjoint or Contained (Problem $4.2$: (1) & (2)))

$$\forall u, v : [_u\ ]_u \cap [_v\ ]_v = \emptyset \bigvee \left( [_u\ ]_u \subsetneqq [_v\ ]_v \vee [_v\ ]_v \subsetneqq [_u\ ]_u \right)$$

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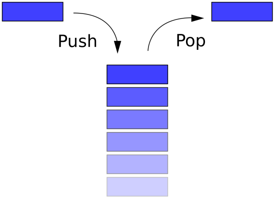
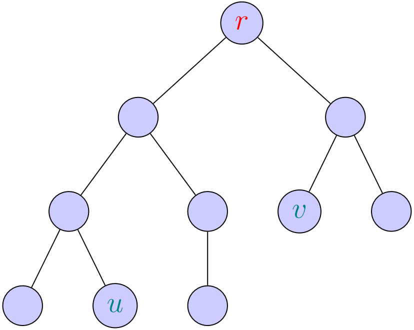



$\square$

## Preprocessing for ancestor/descendant relation (Problem 5.23)




*$Q$ : Is $u$ an ancestor of $v$? $O(1)$*

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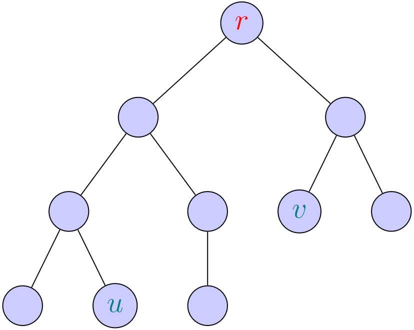


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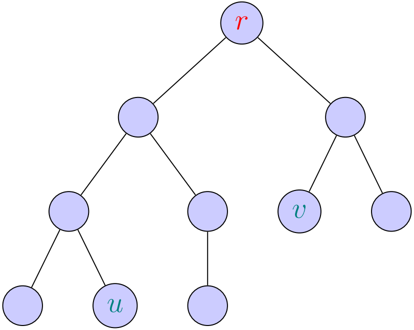


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$Q$ : # of descendants of any $v$?

## Edge types and life time of vertices in DFS (Problem $4.5$)

$$\forall u \to v :$$

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$$\nexists \text{ cycle} \implies \boxed{u \to v \iff \mathsf{f}[v] < \mathsf{f}[u]}$$

### Height and diameter of tree (Problem $5.4$)

Binary tree $T = (V, E)$ with $|V| = n$ and the root $r$:

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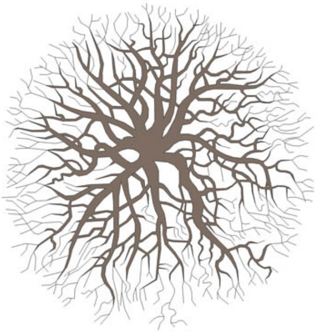

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Your Job: Prove it!

## Cycle detection (Problem $5.8 - 1$)

| | Digraph | Undirected graph |
|---|---|---|
| DFS | | |
| BFS | | |

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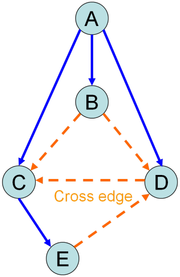

## Evasiveness of acyclicity of undirected graphs (Problem $5.8 - 2$)

$$\text{Evasiveness} \triangleq \text{check} \binom{n}{2} \text{ edges (adjacency matrix)}$$

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By Adversary Argument.

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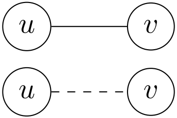


Algorithm $\mathbb{A}$:

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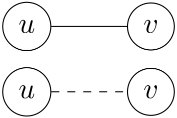


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Hint: Kruskal

$$\textsc{CheckEdge}(u, v) \leftarrow u \text{ —— } v$$

$$\Longleftrightarrow$$

$$\color{red}{\nexists \text{ cycle}} \in G + u \text{ —— } v$$

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$\Longleftrightarrow$

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$Q$ : Why adjacency matrix?

## After-class Exercise: Evasiveness of connectivity of undirected graphs

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Hint: Anti-Kruskal

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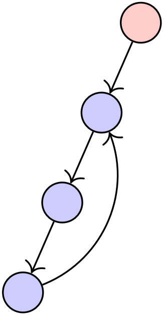

## Shortest cycle of undirected graph (Problem $4.12$)

A WRONG DFS-based algorithm:

$$\forall v : \textsf{level}[v]$$

$$\text{Back edge } u \to v : \textsf{level}[u] - \textsf{level}[v] + 1$$

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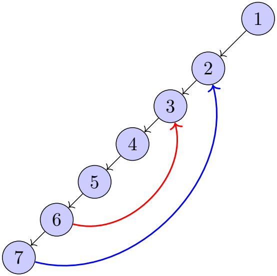

## Shortest cycle of digraph (Problem $4.12$)

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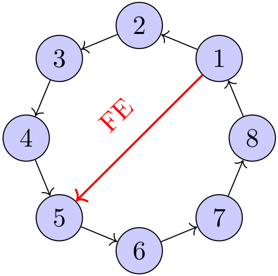

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TOPOSORT by Tarjan (probably), 1976

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Sort vertices in *decreasing* order of their *finish* times.

Kahn's $\textsc{Toposort}$ algorithm (1962; Problem 4.16)

- Queue $Q$ for source vertices ($\text{in}[v] = 0$)
- Repeat: $\textsc{Dequeue}(u \in Q)$, delete $u$ and $u \to v$ from $Q$, output $u$, $\textsc{Enqueue}(v)$ if $\text{in}[v] = 0$

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**Lemma (Correctness of Kahn's TOPOSORT)**

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**Lemma (Correctness of Kahn's $\textsc{Toposort}$)**

*Every DAG has at least one source (and at least one sink vertex).*

$Q$ : What if $G$ is *not* a DAG?

## Taking courses in few semesters (Problem 5.14)

- $n$ courses
- $m$ of $c_1 \to c_2$: prerequisite
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Critical path *OR* Longest path using DFS in $O(n + m)$

For general digraph, LONGEST-PATH is NP-hard.

## Line up (Problem $4.22$)

1. $i$ hates $j$: $i \succ j$
2. $i$ hates $j$: $\#i < \#j$

TOPOSORT

Critical path

## Hamiltonian path in DAG (Problem $4.14$)

HP: path visiting each vertex once

$Q$ : $\exists$ HP in a DAG in $O(n+m)$

Hamiltonian path in DAG (Problem $4.14$)

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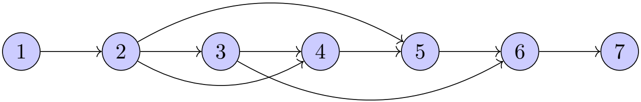

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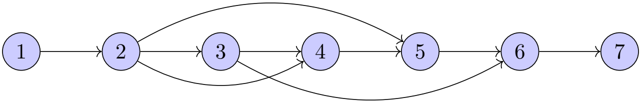


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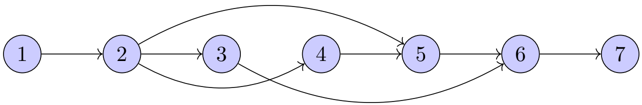

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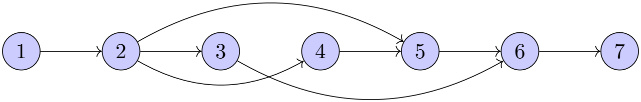


Kahn's TOPOSORT (Problem 4.16)

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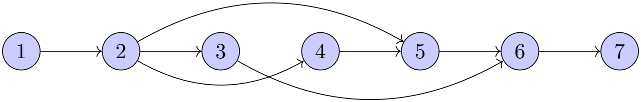


Kahn's $\textsc{Toposort}$ (Problem 4.16)

$$|Q| \leq 1$$

**Theorem (Digraph as DAG (Problem $4.6$))**

*Every digraph is a dag of its SCCs.*

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Two tiered structure of digraphs:

digraph $\equiv$ a dag of SCCs

SCC: equivalence class over reachability

$\boxed{\text{digraph} \equiv \text{a dag of SCCs}}$

Kosaraju SCC algorithm, 1978

*"SCCs can be topo-sorted*
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(I) DFS on $G$; DFS/BFS on $G^T$

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## One-to-all reachability in a digraph (Problem 5.12)

$$v : v \rightsquigarrow^{?} \forall u$$

$$\exists ?\ v : v \rightsquigarrow \forall u$$

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$$\Longleftarrow : \exists! \text{ source}$$

$$\Longrightarrow : \text{ By contradiction.}$$

$$\exists u : v \not\leadsto u \land \text{in}[u] > 0 \implies \exists \text{ cycle}$$

## Impacts of vertices in a digraph (Problem 4.18)

$$\mathsf{impact}(v) = |\{w \neq v : v \rightsquigarrow w\}|$$

- $\arg\min_v \mathsf{impact}(v)$
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$$Q : \forall v : \text{computing impact}(v)$$

## 2SAT (Problem 4.23)

$$I : (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (\overline{x_3} \vee x_4) \wedge (\overline{x_1} \vee x_4)$$

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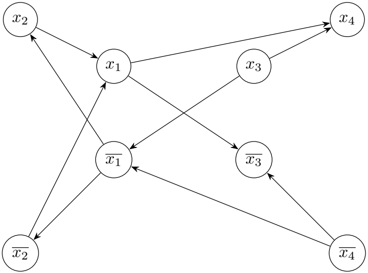
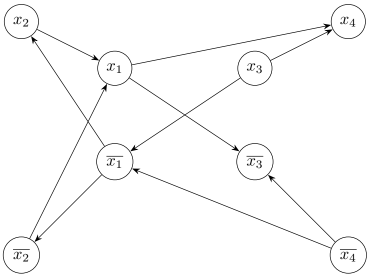
*$\exists$ SCC $\exists x : v_x \in$ SCC $\wedge\, v_{\overline{x}} \in$ SCC $\iff I$ is not satisfiable.*

## Reference:

- *"A Linear-time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas"* by Bengt Aspvall, Michael Plass, and Robert Tarjan, $1979$.



$x_2$
$x_4$
$x_1$
$x_3$
$\overline{x_1}$
$\overline{x_3}$
$\overline{x_2}$
$\overline{x_4}$



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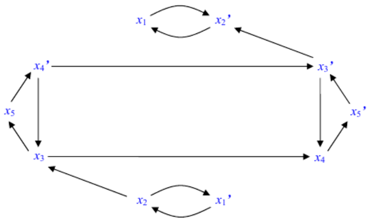

$x_1$
$x_2'$
$x_4'$
$x_3'$
$x_5$
$x_5'$
$x_3$
$x_4$
$x_2$
$x_1'$

# BICOMP: Back!

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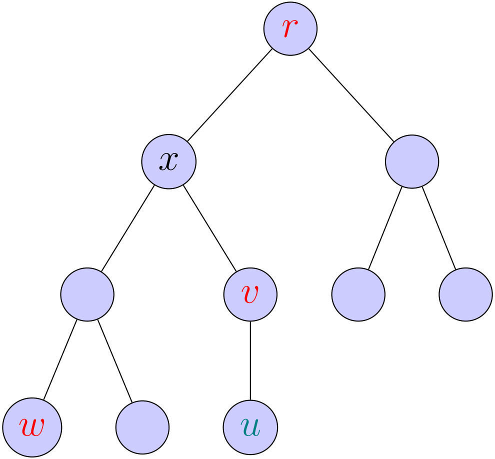

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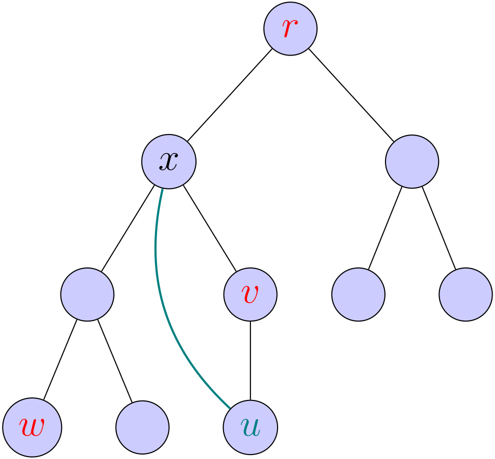

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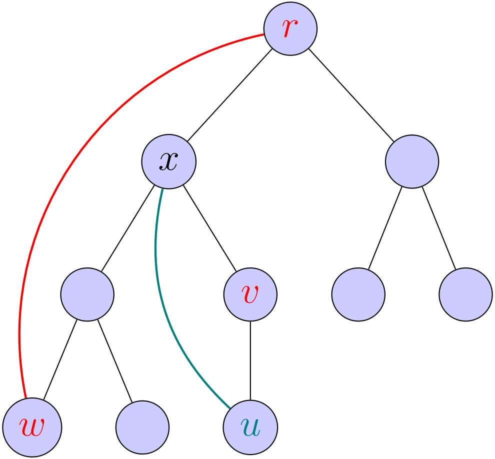

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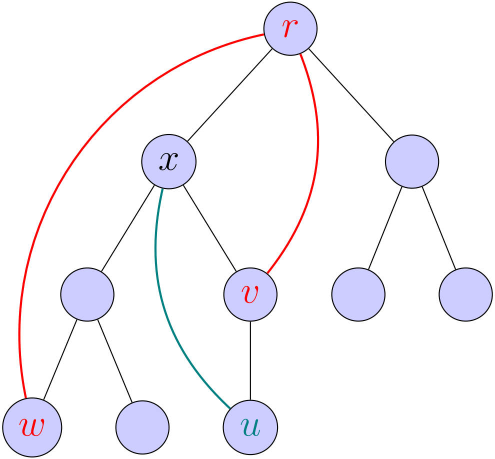

(I) When and how to update back$[v]$?

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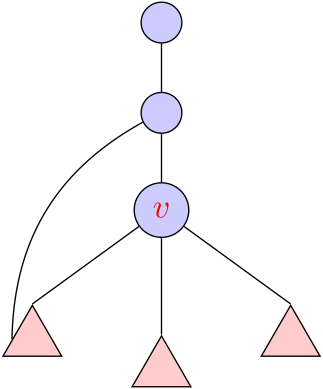

Initialization of back$[v]$ (Problem $4.9$)

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□

## Root cutnode $v$ (Problem $4.8$)

$$v \text{ is a cutnode } \iff \text{out}[v] \geq 2$$

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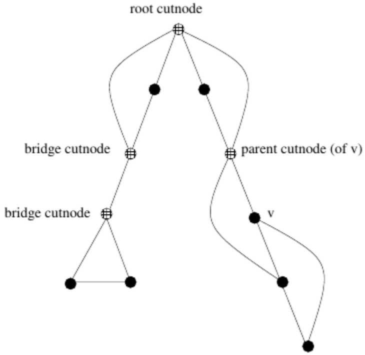


THANK YOU