Minimum Spanning Tree (MST)

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Cut Property

$$G = (V, E, w)$$

Cut Property (I)

$$X: \text{ A part of some MST } T \text{ of } G$$

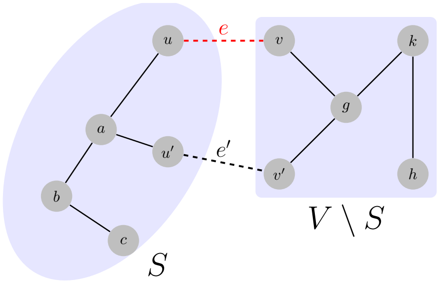
$$(S, V \setminus S): \text{ A } \textit{cut} \text{ such that } X \text{ does } \textit{not} \text{ cross } (S, V \setminus S)$$

$$e: \textit{ A} \text{ lightest edge across } (S, V \setminus S)$$

Then $X \cup \{e\}$ is a part of some MST T' of G.

Correctness of Prim's and Kruskal's algorithms.

By Exchange Argument.



$$T' = \underbrace{T + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

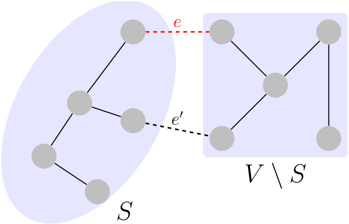
"a" $\to$ "the" $\Longrightarrow$ "some" $\to$ "all"

Cut Property (II)

A cut $(S, V \setminus S)$

Let $e = (u, v)$ be *a* lightest edge across $(S, V \setminus S)$

$\exists$ MST T of $G : e \in T$



$$T' = \underbrace{T + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

"a" $\to$ "the" $\Longrightarrow$ "$\exists$" $\to$ "$\forall$"

Application of Cut Property [Problem: 10.15 (3)]

$$e = (u, v) \in G \text{ is a lightest edge } \implies e \in \exists \text{ MST of } G$$

$$\Big(S = \{u\}, V \setminus S\Big)$$

Application of Cut Property [Problem: 10.15 (4)]

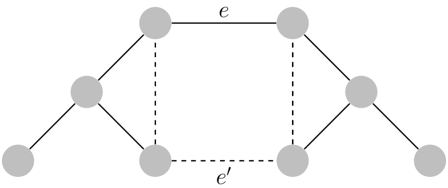
$$e = (u, v) \in G \text{ is the unique lightest edge } \implies e \in \forall \text{ MST}$$

Cycle Property

Cycle Property [Problem: 10.19(b)]

- Let C be any cycle in G
- Let $e = (u, v)$ be *a* maximum-weight edge in C

Then $\exists$ MST T of $G : e \notin T$.



$$T' = \underbrace{T - \{e\}}_{\text{if } e \in T} + \{e'\}$$

"a" $\to$ "the" $\Longrightarrow$ "$\exists$" $\to$ "$\forall$"

Anti-Kruskal algorithm [Problem: 10.19 (c)]

Reverse-delete algorithm (wiki; clickable)

Delete an edge if this does not disconnect the graph.

$$O\Big(m \log n \; (\log \log n)^3\Big)$$

Proof.

Cycle Property

$$T \subseteq F \implies \exists\, T' : T' \subseteq F - \{e\}$$

□

"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.

Application of Cycle Property [Problem: 10.15 (1)]

$$G = (V, E), \quad |E| > |V| - 1$$

e : the unique maximum-weighted edge of G

$$\overset{?}{\Longrightarrow}$$

$$e \notin \text{ any MST}$$

Bridge

Application of Cycle Property [Problem: 10.15 (2)]

$$C \subseteq G, \quad e \in C$$

$$e: \text{ the unique maximum-weighted edge of } G$$

$$\Longrightarrow$$

$$e \notin \text{ any MST}$$

Cycle Property

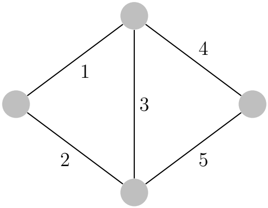
Application of Cycle Property [Problem: 10.15 (5)]

$$C \subseteq G, \quad e \in C$$

$$e: \text{ the unique lightest edge of } C$$

$$\overset{?}{\Longrightarrow}$$

$$e \in \forall \text{ MST}$$



Uniqueness of MST

Uniqueness of MST [Problem: 10.18 (1)]

Distinct weights $\implies$ Unique MST.

By Contradiction.

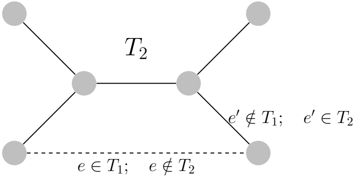
$$\exists \text{ MSTs } T_1 \neq T_2$$

$$\Delta E = \Big\{ e \mid e \in T_1 \setminus T_2 \lor e \in T_2 \setminus T_1 \Big\}$$

$$e = \min \Delta E$$

$$e \in T_1 \setminus T_2 \;\textit{(w.l.o.g)}$$

$$e \in T_1 \setminus T_2$$



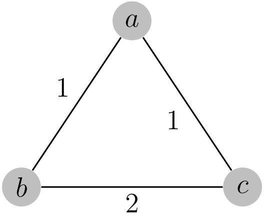
$$T_2 + \{e\} \implies C$$

$$\exists (e' \in C) \notin T_1 \implies e' \in T_2 \setminus T_1 \implies e' \in \Delta E \implies w(e') > w(e)$$

$$T' = T_2 + \{e\} - \{e'\} \implies w(T') < w(T_2)$$

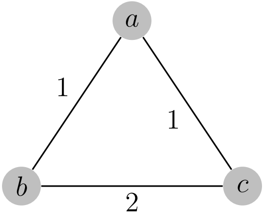
Condition for Uniqueness of MST [Problem: 10.18 (2)]

$$\text{Unique MST} \not\Longrightarrow \text{Distinct weights.}$$



Unique MST [Problem: 10.18 (3)]

Unique MST $\not\Rightarrow$ Minimum-weight edge across any cut is unique.



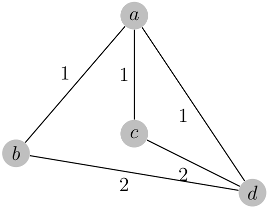
Theorem

Minimum-weight edge across any cut is unique $\Longrightarrow$ *Unique MST.*

Construct T by adding all such edges.

Unique MST [Problem: 10.18 (3)]

Unique MST $\not\Rightarrow$ Maximum-weight edge in any cycle is unique.



Theorem

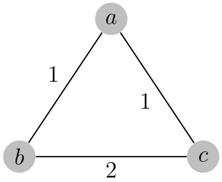
Maximum-weight edge in any cycle is unique $\implies$ *Unique MST.*

Construct T by deleting all such edges.

Unique MST [Problem: 10.18 (4)]

To decide whether a graph has a unique MST.

Ties in Prim's and Kruskal's algorithms



$$\underbrace{T}_{\text{Any MST}} + \underbrace{\{e\}, \forall e \notin T}_{\text{Cycle}}$$

By Kruskal Algorithm.

Variants of MST

Adding a Vertex v to MST T [Problem: 10.7]

$$G' = (V', E') : V' = V + \{v\}, E' = E + E_v$$

To find an MST T' of G'.

$$O\Big((m+n)\log n\Big) \quad \text{(recompute on } G'\text{)}$$

Theorem

There exists an MST of G' that includes no edges in $G \setminus T$.

$$O(n \log n) \quad \text{(recompute on } G'' = (V + \{v\}, T + E_v)\text{)}$$

$$O(n)$$

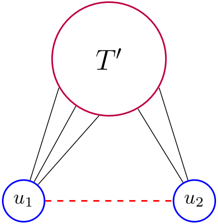
"On Finding and Updating Spanning Tress and Shortest Paths", 1975

"Algorithms for Updating Minimum Spanning Trees", 1978

MST with Specified Leaves: [Problem: 10.11]

$$G = (V, E), \quad U \subset V$$

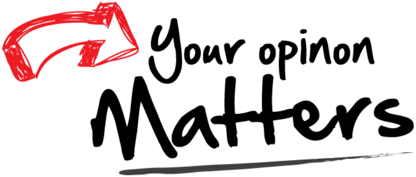
To find an MST with U as leaves.



MST T' of $G' = G \setminus U$

Attach $\forall u \in U$ to T' (with lightest edge)


THANK YOU



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