CMSC 451: Matroids, When Greed Works

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Matroid Introduction

- There's a general argument that can be used to show why many greedy algorithms work.
- We'll describe a very general framework.
- If you can write your problem in that framework,
- and prove one simple property about your specific problem, then the greedy algorithm works.

Hereditary Subset System Examples

Definition

A hereditary subset system is a pair $S = (E, I)$, where E is a finite set and I is a collection of subsets of E closed under inclusion.

"Closed under inclusion" means that if $A \in I$ then all of the subsets of A are also in I.

Example 1: If $G = (V, E)$ is a graph and I is the collection of acyclic subgraphs of G, then (E, I) is a hereditary subset system.

Example 2: Let E be a set of intervals and I be all subsets of compatible intervals. Then (E, I) is a hereditary subset system.

Maximum Weight Problem

Maximum Weight Problem

Given a hereditary subset system (E, I) and a weight $w(e)$ for every $e \in E$, find the subset $A \in I$ so that $\sum_{e \in A} w(e)$ is maximized.

Example 1: *Maximum Spanning Tree*: Let E be edges of a graph G, let I be the collection of acyclic subgraphs of G, and $w(e) =$ a weight on each edge.

Example 2: *Interval Scheduling*: Let E be intervals, I be all subsets of compatible intervals, and $w(e) = 1$.

Greedy Algorithm

```
Greedy(E, I, w):
    S = ∅
    A = E
    While |A| > 0:
        Let e ∈ A be the element of largest weight w(e)
        A = A - e
        If S ∪ e ∈ I then S = S ∪ e
    Return S
```

Similar idea as TreeGrowing:

- S is the current set,
- A is the remaining elements to consider, and
- I determines which sets are allowed.

Augmentation Property

Augmentation Property: If $A, B \in I$ and $|A| < |B|$, then there is an element $e \in B - A$ such that $A \cup \{e\} \in I$.

Given any two sets in I, one smaller than the other, there is some element in the larger one that we can add to the smaller one to form a set that is also in I.

Definition (Matroid)

A hereditary subset system (E, I) is a matroid if it satisfies the augmentation property.

Why Matroids Are Interesting

Theorem

`Greedy` will solve every instance of a maximum weight problem associated with (E, I) if and only if (E, I) is a matroid.

Minimum Weight Problem

There is a minimization version of this as well:

Minimum Weight Problem

Given a hereditary subset system (E, I) and a weight $w(e)$ for $e \in E$, find the *maximal* subset $S \subseteq E$ that *minimizes* $\sum_{e \in S} w(e)$.

- We change `Greedy` to choose the element of A that has the *smallest* weight.
- This version of `Greedy` solves the Minimum Weight Problem if and only if it is applied to a matroid.