Dynamic Programming

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Dynamic Programming

What is DP?

DP $\approx$ " brute force"

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DP $\approx$ "smarter brute force"

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DP $\approx$ "smart scheduling of subproblems"

What is DP?

DP $\approx$ "smarter brute force"

DP $\approx$ "smart scheduling of subproblems"

DP $\approx$ "shortest/longest paths in some DAG"

What is not DP?

Programming = Planning

What is not DP?

$$\text{Programming} = \text{Planning}$$

$$\text{Programming} \neq \text{Coding}$$

(Richard Bellman, 1940s)

Steps for applying DP

1. Define subproblems
   - # of subproblems
2. Set the goal
3. Define the recurrence
   - larger subproblem $\leftarrow$ # smaller subproblems
   - init. conditions

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1. Define subproblems
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3. Define the recurrence
   - larger subproblem $\leftarrow$ # smaller subproblems
   - init. conditions
4. Write pseudo-code: fill "table" in topo. order
5. Analyze the time complexity
6. Extract the optimal sulution

Common subproblems in DP

1D subproblems:

Input: $x_1, x_2, \ldots, x_n$ (array, sequence, string)

Subproblems: $x_1, x_2, \ldots, x_i$ (prefix/suffix)

#: $\Theta(n)$

Examples: Maximum-sum subarray, Longest increasing subsequence, Text justification (LaTeX)

Common subproblems in DP

2D subproblems:

1. Input: $x_1, x_2, \ldots, x_m$; $y_1, y_2, \ldots, y_n$

   Subproblems: $x_1, x_2, \ldots, x_i$; $y_1, y_2, \ldots, y_j$

   #: $\Theta(mn)$

   Examples: Edit distance, Longest common subsequence

Common subproblems in DP

2D subproblems:

1. Input: $x_1, x_2, \ldots, x_m$; $y_1, y_2, \ldots, y_n$

   Subproblems: $x_1, x_2, \ldots, x_i$; $y_1, y_2, \ldots, y_j$

   #: $\Theta(mn)$

   Examples: Edit distance, Longest common subsequence

2. Input: $x_1, x_2, \ldots, x_n$

   Subproblems: $x_i, \ldots, x_j$

   #: $\Theta(n^2)$

   Examples: Matrix chain multiplication, Optimal BST

Common subproblems in DP

3D subproblems:

- Floyd-Warshall algorithm

$$\mathsf{d}(i,j,k) = \min\{\mathsf{d}(i,j,k-1), \mathsf{d}(i,k,k-1) + \mathsf{d}(k,j,k-1)\}$$

Common subproblems in DP

3D subproblems:

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DP on graphs:

1. On rooted tree

   Subproblems: rooted subtrees

2. On DAG

   Subproblems: nodes after/before in the topo. order

Common subproblems in DP

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DP on graphs:

1. On rooted tree

   Subproblems: rooted subtrees

2. On DAG

   Subproblems: nodes after/before in the topo. order

Knapsack problem:

- Subset sum problem, change-making problem

Common subproblems in DP

And Others . . .

Recurrences in DP

Make choices by asking yourself the right question:

1. Binary choice
   - whether ...
2. Multi-way choices
   - where to ...
   - which one ...

Dynamic Programming

$f^{(S(n))} = 1$

$f^{(S(n))} = 1$ (Problem 7.2)

$$f(n) = \begin{cases} n-1 & \text{if } n \in \mathbb{Z}^+ \\ n/2 & \text{if } n\%2 = 0 \\ n/3 & \text{if } n\%3 = 0 \end{cases}$$

$S(n)$: minimum number of steps taking n to 1.

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$S(i)$: minimum number of steps taking i to 1

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$S(n)$: minimum number of steps taking n to 1.

$S(i)$: minimum number of steps taking i to 1

$$S(i) = 1 + \min\{S(i-1), S(i/2)(\text{if } n\%2 = 0), S(i/3)(\text{if } n\%3 = 0)\}$$

$$S(1) = 0$$

$f^{(S(n))} = 1$

Collatz $(3n+1)$ conjecture:

$$f(n) = \begin{cases} n/2 & \text{if } n\%2 = 0 \\ 3n+1 & \text{if } n\%2 = 1 \end{cases}$$

$$f^*(n) = 1?$$

$f^{(S(n))} = 1$

Collatz $(3n+1)$ conjecture:

$$f(n) = \begin{cases} n/2 & \text{if } n\%2 = 0 \\ 3n+1 & \text{if } n\%2 = 1 \end{cases}$$

$$f^*(n) = 1?$$

"Mathematics may not be ready for such problems."

— Paul Erdős

Longest increasing subsequence

Longest increasing subsequence (Problem 7.3)

- Given an integer array $A[1 \ldots n]$
- To find (the length of) a longest increasing subseqence.

$$5, 2, 8, 6, 3, 6, 9, 7 \implies 2, 3, 6, 9$$

Longest increasing subsequence

Subproblem: $L(i)$: the length of the LIS of $A[1 \dots i]$

Goal: $L(n)$

Longest increasing subsequence

Subproblem: $L(i)$: the length of the LIS of $A[1 \dots i]$

Goal: $L(n)$

Make choice: whether $A[i] \in LIS[1 \dots i]$?

Recurrence:

$$L(i) = \max\{L(i-1), 1 + \max_{j<i \wedge A[j] \leq A[i]} L(j)\}$$

Longest increasing subsequence

Subproblem: $L(i)$: the length of the LIS of $A[1 \dots i]$

Goal: $L(n)$

Make choice: whether $A[i] \in LIS[1 \dots i]$?

Recurrence:

$$L(i) = \max\{L(i-1), 1 + \max_{j<i \wedge A[j] \le A[i]} L(j)\}$$

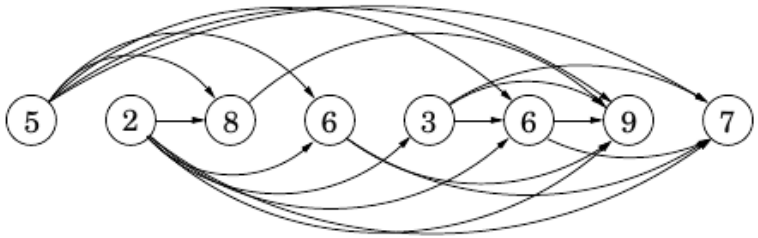
Init:

$$L(0) = 0$$

Time:

$$O(n^2) = \Theta(n) \cdot O(n)$$

Longest increasing subsequence



Longest path distance in the DAG!

Maximum-sum subarray

Maximum-sum subarray (Google Interview)

- Array $A[1 \cdots n], a_i >=< 0$
- To find (the sum of) a maximum-sum subarray of A

$$A[-2, 1, -3, 4, -1, 2, 1, -5, 4] \implies [4, -1, 2, 1]$$

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Subproblem: MSS$[i]$: sum of an MS$[i]$ of $A[1 \cdots i]$

Goal: mss $=$ MSS$[n]$

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Subproblem: MSS$[i]$: sum of an MS$[i]$ of $A[1 \cdots i]$

Goal: mss $=$ MSS$[n]$

Make choice: Is $a_i \in$ MS$[i]$?

Recurrence:

$$\text{MSS}[i] = \max\{\text{MSS}[i-1], ???\}$$

Maximum-sum subarray

Subproblem: MSS$[i]$: sum of an MS$[i]$ *ending with* a_i

Goal: mss $= \max_{1 \le i \le n}$ MSS$[i]$

Maximum-sum subarray

Subproblem: MSS$[i]$: sum of an MS$[i]$ *ending with* a_i

Goal: $\text{mss} = \max_{1 \le i \le n} \text{MSS}[i]$

Make choice: Where does the MS$[i]$ start?

Recurrence:

$$\text{MSS}[i] = \max\{\text{MSS}[i-1] + a_i, a_i\} \text{ (Proof!)}$$

Maximum-sum subarray

Subproblem: MSS$[i]$: sum of an MS$[i]$ *ending with* a_i

Goal: $\text{mss} = \max_{1 \le i \le n} \text{MSS}[i]$

Make choice: Where does the MS$[i]$ start?

Recurrence:

$$\text{MSS}[i] = \max\{\text{MSS}[i-1] + a_i, a_i\} \text{ (Proof!)}$$

Init:

$$\text{MSS}[0] = 0$$

Maximum-sum subarray

Subproblem: MSS$[i]$: sum of an MS$[i]$ *ending with* a_i

Goal: $\text{mss} = \max_{1 \le i \le n} \text{MSS}[i]$

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Recurrence:

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Init:

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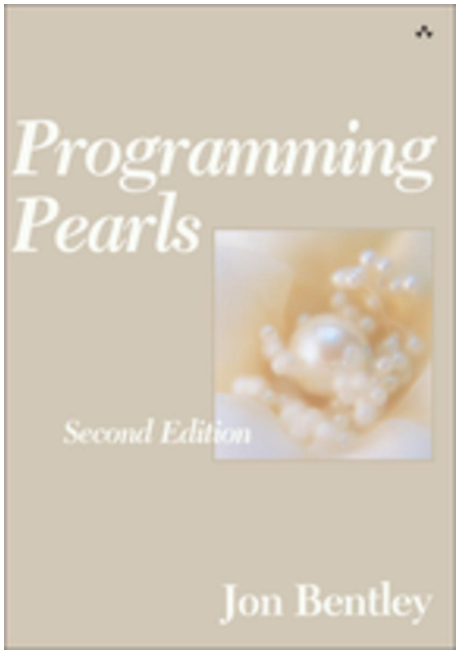
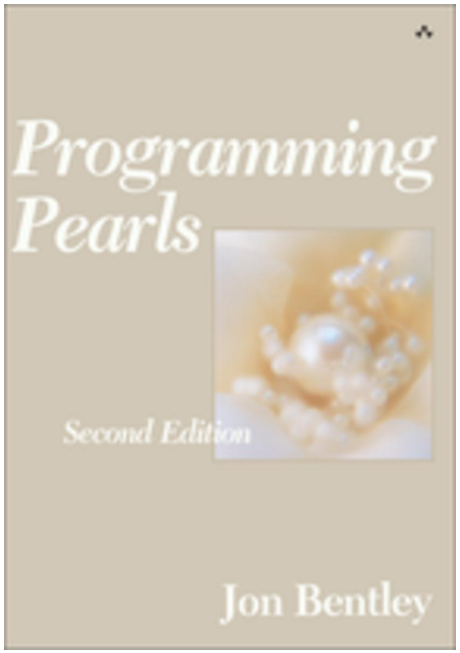
Time: $\Theta(n)$

Maximum-sum subarray



Ulf Grenander $O(n^3) \Longrightarrow O(n^2)$

Maximum-sum subarray



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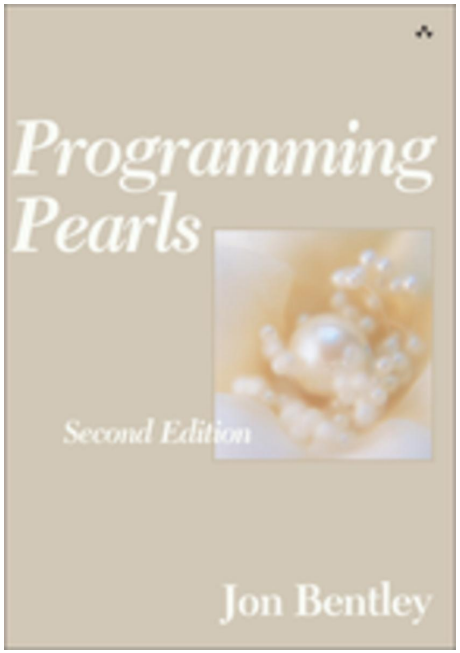
Michael Shamos $O(n \log n)$, onenight

Maximum-sum subarray



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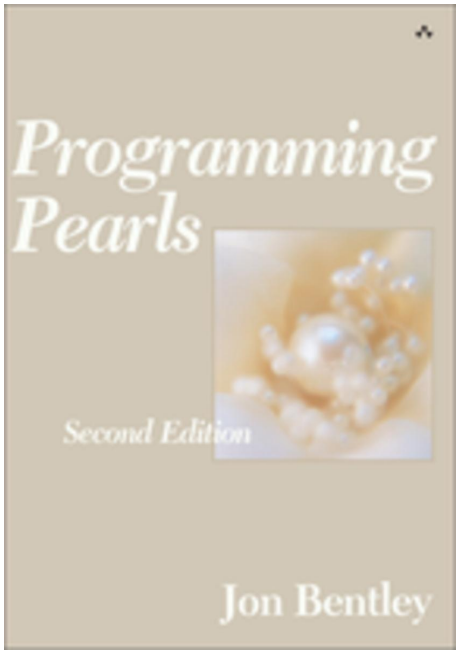
Jon Bentley Conjecture: $\Omega(n \log n)$

Maximum-sum subarray



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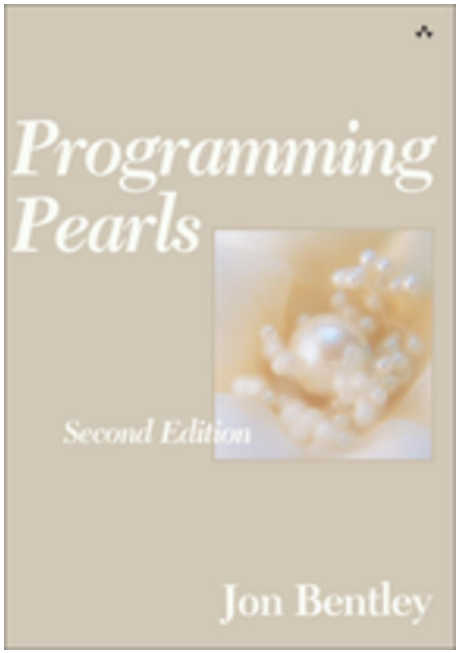
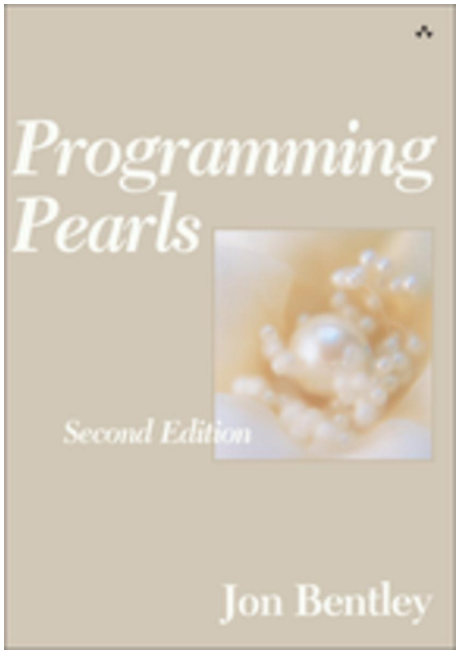
Michael Shamos Carnegie Mellon seminar

Maximum-sum subarray



Ulf Grenander	$O(n^3) \Longrightarrow O(n^2)$
Michael Shamos	$O(n \log n)$, onenight
Jon Bentley	Conjecture: $\Omega(n \log n)$
Michael Shamos	Carnegie Mellon seminar
Jay Kadane	$O(n)$,

Maximum-sum subarray



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Michael Shamos	$O(n \log n)$, onenight
Jon Bentley	Conjecture: $\Omega(n \log n)$
Michael Shamos	Carnegie Mellon seminar
Jay Kadane	$O(n)$, ≤ 1 minute

Maximum-product subarray

Maximum-product subarray (Problem 7.4)

- Array $A[1 \ldots n]$
- Find maximum-product subarray of A

(1) $a_i \in \mathbb{N}$

(2) $a_i \in \mathbb{Z}$

(3) $a_i \in \mathbb{R}$

Maximum-product subarray

Maximum-product subarray (Problem 7.4)

- Array $A[1 \ldots n]$
- Find maximum-product subarray of A

(1) $a_i \in \mathbb{N}$

(2) $a_i \in \mathbb{Z}$

(3) $a_i \in \mathbb{R}$

sum *vs.* product

Maximum-product subarray

Subproblem: $\mathsf{MaxP}[i], \mathsf{MinP}[i]$

		$\frac{1}{2}$	4	-2	5	$-\frac{1}{5}$	8
$\mathsf{MaxP}[i]$	1	$\frac{1}{2}$	4	-2	5	8	64
$\mathsf{MinP}[i]$	1	$\frac{1}{2}$	2	-8	-40	-1	-8

$$\mathsf{MaxP}[i] = \max\{\mathsf{MaxP}[i-1] \cdot a_i, \mathsf{MinP}[i-1] \cdot a_i, a_i\}$$
$$\mathsf{MinP}[i] = \min\{\mathsf{MaxP}[i-1] \cdot a_i, \mathsf{MinP}[i-1] \cdot a_i, a_i\}$$

Reconstructing string

Reconstructing string (Problem 7.9)

- String $S[1 \cdots n]$
- Dict for *lookup*:

$$\text{dict}(w) = \begin{cases} \text{true} & \text{if } w \text{ is a valid word} \\ \text{false} & \text{o.w.} \end{cases}$$

- Is $S[1 \cdots n]$ valid (reconstructed as a sequence of valid words)?

Reconstructing string

Subproblem: $V[i]$: Is $S[1 \cdots i]$ valid?

Goal: $V[n]$

Reconstructing string

Subproblem: $V[i]$: Is $S[1 \cdots i]$ valid?

Goal: $V[n]$

Make choice: Where does the last word start?

Recurrence:

$$V[i] = \bigvee_{j=1 \ldots i} (V[j-1] \wedge \text{dict}(S[j \cdots i]))$$

Reconstructing string

Subproblem: $V[i]$: Is $S[1 \cdots i]$ valid?

Goal: $V[n]$

Make choice: Where does the last word start?

Recurrence:

$$V[i] = \bigvee_{j=1 \ldots i} (V[j-1] \wedge \text{dict}(S[j \cdots i]))$$

Init:

$$V[0] = \text{true}$$

Time:

$$O(n^2) = \Theta(n) \cdot O(n)$$

Hotel along a trip

Hotel along a trip (Problem 7.15)

- Hotel sequence (distance): $a_0 = 0, a_1, \cdots, a_n$
- $a_0 \rightsquigarrow a_n$
- Stop at only hotels
- Cost: $(200 - x)^2$
- To minimize overall cost

Hotel along a trip

Subproblem: $C[i]$: minimum cost when the destination is a_i

Goal: $C[n]$

Hotel along a trip

Subproblem: $C[i]$: minimum cost when the destination is a_i

Goal: $C[n]$

Make choice: What is the last but one hotel a_j to stop?

Recurrence:

$$C[i] = \min_{0 \le j < i} \{C[j] + (200 - (a_i - a_j))^2\}$$

Hotel along a trip

Subproblem: $C[i]$: minimum cost when the destination is a_i

Goal: $C[n]$

Make choice: What is the last but one hotel a_j to stop?

Recurrence:

$$C[i] = \min_{0 \le j < i} \{C[j] + (200 - (a_i - a_j))^2\}$$

Init:

$$C[0] = 0$$

Time:

$$O(n^2) = \Theta(n) \cdot O(n)$$

Highway restaurants

Highway restaurants (Problem 7.16)

- Locations: $L[1 \ldots n]$
- Profits: $P[1 \ldots n]$
- Any two hotels should be $\geq k$ miles apart
- To maximize the total profit

Subproblem: $T[i]$: max profit achievable using only $L[1 \ldots i]$

Goal: $T[n]$

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Subproblem: $T[i]$: max profit achievable using only $L[1 \ldots i]$

Goal: $T[n]$

Make choice: Whether to open a restaurant at L_i?

Recurrence:

$$T[i] = \max\{T[i-1], P_i + T[\text{prev}(i)]\}$$

$$\text{prev}(i) = \max\{j \mid j < i \wedge L_i - L_j \geq k\}$$

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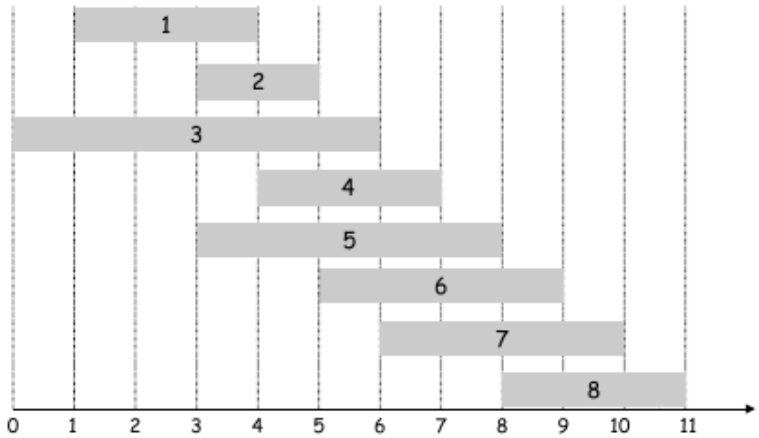
Init: $T[0] = 0$

Weighted interval/class scheduling

Weighted interval/class scheduling (Problem 7.14)

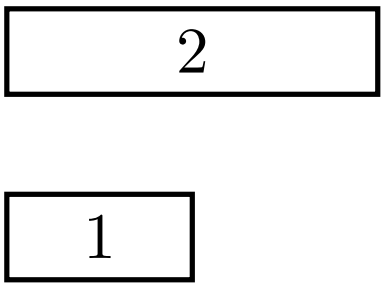
- Classes: $\mathcal{C} = \{c_1, c_2, \cdots, c_n\}$ $\quad c_i \triangleq \langle g_i, s_i, f_i \rangle$
- Choosing non-conflicting classes to maximize your grades



sort $\mathcal{C}$ by finishing time.

Weighted interval/class scheduling

Greedy algorithms fail:



By finishing time.

Weighted interval/class scheduling

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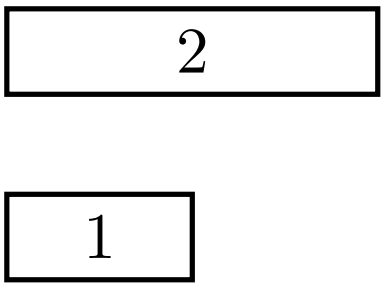
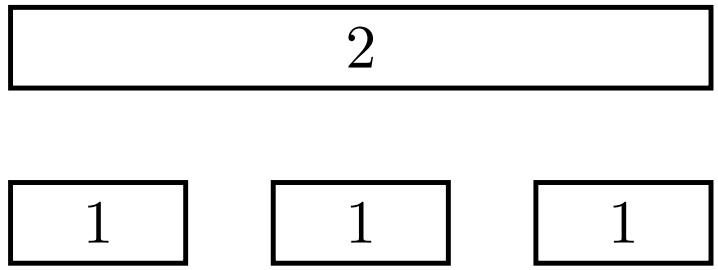


By finishing time.



By weights.

Weighted interval/class scheduling

Subproblem: $G[i]$: the maximal grades obtained from $\{c_1, c_2, \cdots, c_i\}$

Goal: $G[n]$

Weighted interval/class scheduling

Subproblem: $G[i]$: the maximal grades obtained from $\{c_1, c_2, \cdots, c_i\}$

Goal: $G[n]$

Make choice: Choose c_i or not?

Recurrence:

$$G[i] = \max\{G[i-1], G[\text{prev}(i)] + g_i\}$$
$$\text{prev}(i) = \max\{j \mid j < i \wedge c_i \cap c_j = \emptyset\}$$

Weighted interval/class scheduling

Subproblem: $G[i]$: the maximal grades obtained from $\{c_1, c_2, \cdots, c_i\}$

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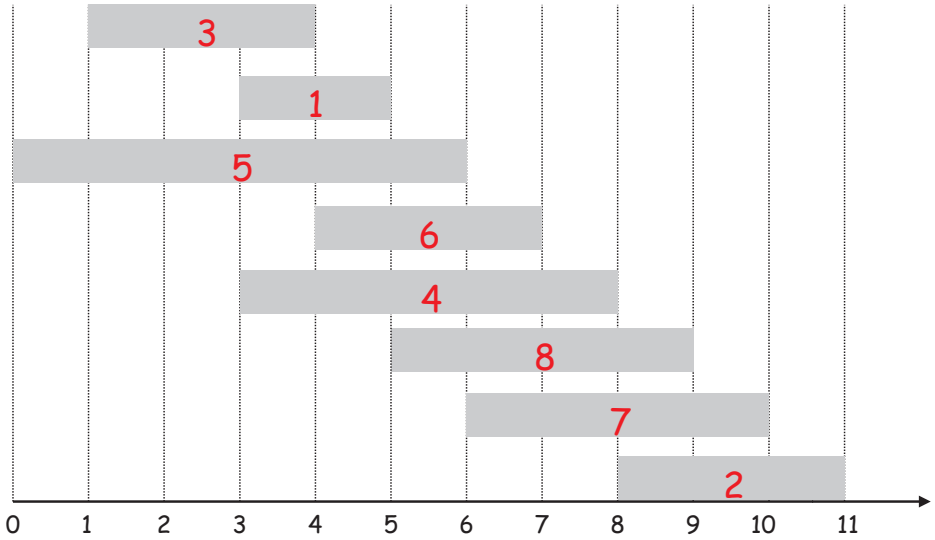
Init: $G[0] = 0$

Time: $O(n \log n) + T(\mathsf{prev}(i)) + O(n) \cdot O(1)$

Weighted interval/class scheduling

Why is ordering necessary?

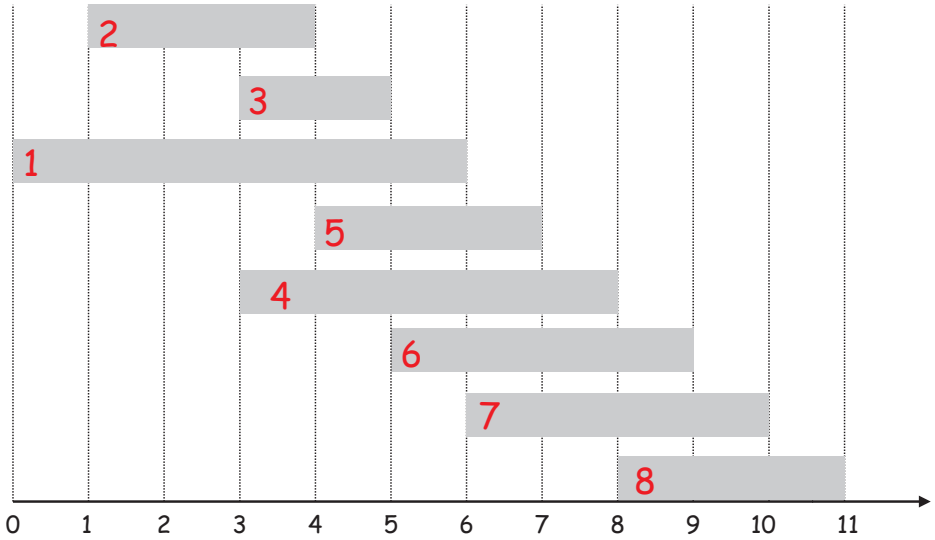


$$G[7] = \max\{G[6], G[\{1,3,5\}] + g_7\}$$

subproblems changed: all $O(2^n)$ subsets

Weighted interval/class scheduling

What about sorting by starting time?



$$G[6] = \max\{G[5], G[\{2,3\}] + g_6\}$$

subproblems changed: all $O(2^n)$ subsets

Dynamic Programming

Longest common subsequence

LCS: longest common subsequence (Problem 7.5)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

(1) Find (the length of) an LCS of X and Y

$$X = \langle A, B, C, B, D, A, B \rangle$$
$$Y = \langle B, D, C, A, B, A \rangle$$
$$Z = \langle B, C, B, A \rangle$$

Longest common subsequence

Subproblem: $L[i, j]$: the length of an LCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$

Goal: $L[m, n]$

Longest common subsequence

Subproblem: $L[i, j]$: the length of an LCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$

Goal: $L[m, n]$

Make choice: Is $X_i = Y_j$?

Recurrence: (Proof!)

$$L[i,j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1, j], L[i, j-1]\} & \text{if } X_i \neq Y_j \end{cases}$$

Longest common subsequence

Subproblem: $L[i, j]$: the length of an LCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$

Goal: $L[m, n]$

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Init:

$$L[0, j] = 0, \ 0 \leq j \leq n$$
$$L[i, 0] = 0, \ 0 \leq i \leq m$$

Time: $\Theta(mn)$

Longest common subsequence

Longest common subsequence (Problem 7.5)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

(2) Allowing repetition of X

(3) Allowing repetition $\leq k$ of X

Longest common subsequence

Longest common subsequence (Problem 7.5)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

(2) Allowing repetition of X

(3) Allowing repetition $\leq k$ of X

$$L[i,j] = \begin{cases} L[i, j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1, j], L[i, j-1]\} & \text{if } X_i \neq Y_j \end{cases}$$

Longest common subsequence

Longest common subsequence (Problem 7.5)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

(2) Allowing repetition of X

(3) Allowing repetition $\leq k$ of X

$$L[i,j] = \begin{cases} L[i, j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1, j], L[i, j-1]\} & \text{if } X_i \neq Y_j \end{cases}$$

$$X \implies X^{(k)} \triangleq X_1^{(k)} \cdots X_m^{(k)}$$

Longest common substring

What about longest common *substring*?

Shortest common supersequence

Shortest common supersequence (Problem 7.6)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

- Find (the length of) a shortest common subsequence of X and Y

Shortest common supersequence

Subproblem: $L[i, j]$: the length of an SCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$

Goal: $L[m, n]$

Shortest common supersequence

Subproblem: $L[i, j]$: the length of an SCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$

Goal: $L[m, n]$

Make choice: Is $X_i = Y_j$?

Recurrence:

$$L[i,j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1, j] + 1, L[i, j-1] + 1\} & \text{if } X_i \neq Y_j \end{cases}$$

Shortest common supersequence

Subproblem: $L[i, j]$: the length of an SCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$

Goal: $L[m, n]$

Make choice: Is $X_i = Y_j$?

Recurrence:

$$L[i,j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1, j] + 1, L[i, j-1] + 1\} & \text{if } X_i \neq Y_j \end{cases}$$

Init:

$$L[0, j] = j,\ 0 \leq j \leq n$$
$$L[i, 0] = i,\ 0 \leq i \leq m$$

Shortest common supersequence

Subproblem: $L[i,j]$: the length of an SCS of $X[1\cdots i]$ and $Y[1\cdots j]$

Goal: $L[m,n]$

Make choice: Is $X_i = Y_j$?

Recurrence:

$$L[i,j] = \begin{cases} L[i-1,j-1]+1 & \text{if } X_i = Y_j \\ \max\{L[i-1,j]+1, L[i,j-1]+1\} & \text{if } X_i \neq Y_j \end{cases}$$

Init:

$$L[0,j] = j,\ 0 \leq j \leq n$$
$$L[i,0] = i,\ 0 \leq i \leq m$$

Remark

$$\max(m,n) \leq L(m,n) \leq m+n$$

Shortest common supersequence

$$X[1 \ldots m] \qquad Y[1 \ldots n]$$

$$r = |\mathsf{LCS}(X, Y)|$$
$$t = |\mathsf{SCS}(X, Y)|$$

Shortest common supersequence

$$X[1 \ldots m] \qquad Y[1 \ldots n]$$

$$r = |\mathsf{LCS}(X, Y)|$$
$$t = |\mathsf{SCS}(X, Y)|$$

$$X \cap Y = \emptyset \implies r = 0 \quad t = m + n$$

Shortest common supersequence

$$X[1 \ldots m] \qquad Y[1 \ldots n]$$

$$r = |\mathsf{LCS}(X, Y)|$$
$$t = |\mathsf{SCS}(X, Y)|$$

$$X \cap Y = \emptyset \implies r = 0 \quad t = m + n$$

$$X \prec Y \implies r = m \quad t = n$$

Shortest common supersequence

$$X[1 \ldots m] \qquad Y[1 \ldots n]$$

$$r = |\mathsf{LCS}(X, Y)|$$
$$t = |\mathsf{SCS}(X, Y)|$$

$$X \cap Y = \emptyset \implies r = 0 \quad t = m + n$$

$$X \prec Y \implies r = m \quad t = n$$

$$r + t = m + n$$

String shuffling

String shuffling (Problem 7.7)

- Strings $X[1 \ldots m], Y[1 \ldots n], Z[1 \ldots k]$
- Is $X \oplus Y = Z$?

(1) Reduced to LCS:

$$(Z' \triangleq Z \setminus \mathsf{LCS}(X, Z)) = Y$$

(2) $O(mn)$

(3) Minimum # deleted characters to ensure $X \oplus Y = Z$ in $O(mnk)$

String shuffling

$$(Z' \triangleq Z \setminus \mathsf{LCS}(X, Z)) = Y$$

"YES" $\Longrightarrow$ $X \oplus Y = Z$ "NO" $\Longrightarrow$ $X \oplus Y \neq Z$

String shuffling

$$(Z' \triangleq Z \setminus \mathsf{LCS}(X, Z)) = Y$$

"YES" $\Longrightarrow$ $X \oplus Y = Z$

"NO" $\Longrightarrow$ $X \oplus Y \neq Z$

$$X = AC$$
$$Y = CB$$
$$Z = CBA$$

String shuffling

$$(Z' \triangleq Z \setminus \mathsf{LCS}(X, Z)) = Y$$

"YES" $\Longrightarrow$ $X \oplus Y = Z$

"NO" $\Longrightarrow$ $X \oplus Y \neq Z$

$$X = AC$$
$$Y = CB$$
$$Z = CBA$$

$$X = A$$
$$Y = AB$$
$$Z = ABA$$

String shuffling

Subproblem: $S[i, j]$: Is $X[1 \ldots i] \oplus Y[1 \ldots j] = Z[1 \ldots r \triangleq i + j]$?

Goal: $S[m, n]$

String shuffling

Subproblem: $S[i,j]$: Is $X[1\ldots i] \oplus Y[1\ldots j] = Z[1\ldots r \triangleq i+j]$?

Goal: $S[m,n]$

Make choice: Is $Z[r] = X[i] \vee Z[r] = Y[j]$?

Recurrence:

$$\begin{aligned} S[i,j] =& (Z[r] = X[i] \wedge S[i-1,j]) \vee \\ & (Z[r] = Y[j] \wedge S[i,j-1]) \end{aligned}$$

String shuffling

Subproblem: $S[i,j]$: Is $X[1\ldots i] \oplus Y[1\ldots j] = Z[1\ldots r \triangleq i+j]$?

Goal: $S[m,n]$

Make choice: Is $Z[r] = X[i] \vee Z[r] = Y[j]$?

Recurrence:

$$\begin{aligned} S[i,j] =& (Z[r] = X[i] \wedge S[i-1,j]) \vee \\ & (Z[r] = Y[j] \wedge S[i,j-1]) \end{aligned}$$

Init:

$$\begin{aligned} S[0,j] &= (Z = Y),\ \forall 0 \le j \le n \\ S[i,0] &= (Z = X),\ \forall 0 \le i \le m \end{aligned}$$

Time: $O(mn)$

String shuffling

Subproblem: $D[i, j, r]$: minimum # deleted characters to ensure $X[1 \ldots i] \oplus Y[1 \ldots j] = Z[1 \ldots r]$

Goal: $D[m, n, k]$

String shuffling

Subproblem: $D[i, j, r]$: minimum # deleted characters to ensure $X[1 \ldots i] \oplus Y[1 \ldots j] = Z[1 \ldots r]$

Goal: $D[m, n, k]$

Make choice: Is $Z[r] = X[i] \vee Z[r] = Y[j]$?

Recurrence:

$$D[i, j, r] = \min \begin{cases} D[i-1, j, r-1] & \text{if } Z[r] = X[i] \\ D[i, j-1, r-1] & \text{if } Z[r] = Y[j] \\ 1 + D[i-1, j, r] \\ 1 + D[i, j-1, r] \\ 1 + D[i, j, r-1] \end{cases}$$

String shuffling

Subproblem: $D[i, j, r]$: minimum # deleted characters to ensure $X[1 \ldots i] \oplus Y[1 \ldots j] = Z[1 \ldots r]$

Goal: $D[m, n, k]$

Make choice: Is $Z[r] = X[i] \vee Z[r] = Y[j]$?

Recurrence:

$$D[i, j, r] = \min \begin{cases} D[i-1, j, r-1] & \text{if } Z[r] = X[i] \\ D[i, j-1, r-1] & \text{if } Z[r] = Y[j] \\ 1 + D[i-1, j, r] \\ 1 + D[i, j-1, r] \\ 1 + D[i, j, r-1] \end{cases}$$

Init:

$$D[0, j, r] = j + r - 2|\mathsf{LCS}(Y[1 \ldots j], Z[1 \ldots r])|$$
$$D[i, 0, r] = i + r - 2|\mathsf{LCS}(X[1 \ldots i], Z[1 \ldots r])|$$
$$D[i, j, 0] = i + j$$

Longest contiguous substring both forward and backward

Longest contiguous substring both forward and backward (Problem 7.8)

- String $T[1 \cdots n]$
- Find a longest contiguous substring (LCS) both forward and backward

dynamicprogrammingmanytimes

- Subproblem $L[i]$: the length of an LCS in $T[1 \cdots i]$
- Subproblem $L[i, j]$: the length of an LCS in $T[i \cdots j]$

Longest contiguous substring both forward and backward

Subproblem: $L[i, j]$: the length of an LCS starting with T_i and ending with T_j

Goal: $\max_{1 \le i \le j \le n} L[i, j]$

Longest contiguous substring both forward and backward

Subproblem: $L[i, j]$: the length of an LCS starting with T_i and ending with T_j

Goal: $\max_{1 \le i \le j \le n} L[i, j]$

Make choice: Is $T_i = T_j$?

Recurrence:

$$L[i, j] = \begin{cases} 0 & \text{if } T_i \neq T_j \\ L[i+1, j-1] + 1 & \text{if } T_i = T_j \end{cases}$$

Longest contiguous substring both forward and backward

Subproblem: $L[i, j]$: the length of an LCS starting with T_i and ending with T_j

Goal: $\max_{1 \le i \le j \le n} L[i, j]$

Make choice: Is $T_i = T_j$?

Recurrence:

$$L[i, j] = \begin{cases} 0 & \text{if } T_i \neq T_j \\ L[i+1, j-1] + 1 & \text{if } T_i = T_j \end{cases}$$

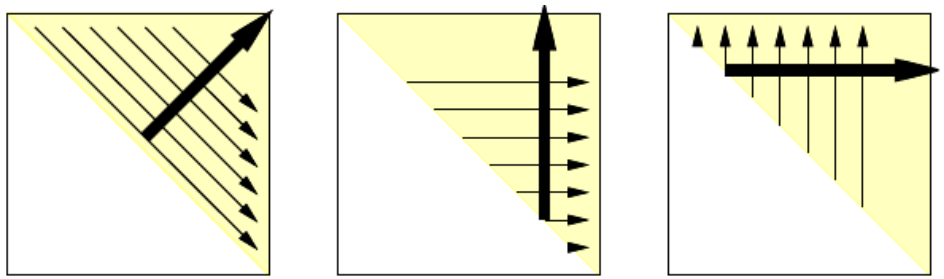
Init:

$$L[i, i] = 0, \ 0 \le i \le n$$

$$L[i, i+1] = \begin{cases} 1 & \text{if } T_i = T_{i+1} \\ 0 & \text{if } T_i \neq T_{i+1} \end{cases}$$

Longest contiguous substring both forward and backward

Code: three ways of filling the table



for all $d \leftarrow 2 \ldots n-1$ **do**
    for all $i \leftarrow 1 \ldots n-d$ **do**
        $j \leftarrow i+d$
        $\ldots$
return $\max_{1 \leq i \leq j \leq n} L[i,j]$

Longest palindrome subsequence

Longest palindrome subsequence (Problem 7.10)

(1) Find (the length of) a longest palindrome subsequence of $S[1 \cdots n]$

Subproblem: $L[i, j]$: the length of an LSP of $S[i \cdots j]$

Goal: $L[1, n]$

Longest palindrome subsequence

Longest palindrome subsequence (Problem 7.10)

(1) Find (the length of) a longest palindrome subsequence of $S[1 \cdots n]$

Subproblem: $L[i, j]$: the length of an LSP of $S[i \cdots j]$

Goal: $L[1, n]$

Make choice: Is $S[i] = S[j]$?

Recurrence:

$$L[i,j] = \begin{cases} L[i+1, j-1] + 2 & \text{if } S[i] = S[j] \\ \max\{L[i+1, j], L[i, j-1]\} & \text{if } S[i] \neq S[j] \end{cases}$$

Longest palindrome subsequence

Longest palindrome subsequence (Problem 7.10)

(1) Find (the length of) a longest palindrome subsequence of $S[1 \cdots n]$

Subproblem: $L[i, j]$: the length of an LSP of $S[i \cdots j]$

Goal: $L[1, n]$

Make choice: Is $S[i] = S[j]$?

Recurrence:

$$L[i,j] = \begin{cases} L[i+1, j-1] + 2 & \text{if } S[i] = S[j] \\ \max\{L[i+1, j], L[i, j-1]\} & \text{if } S[i] \neq S[j] \end{cases}$$

Init:

$$L[i, i] = 1, \ \forall 1 \leq i \leq n$$

$$L[i, i+1] = 2, \text{ if } S[i] = S[i+1], \ \forall 1 \leq i \leq n-1$$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes (# cuts)

Subproblem: $C[i, j]$: minimum number of cuts for string $S[i \dots j]$

Goal: $C[1, n] + 1$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes (# cuts)

Subproblem: $C[i, j]$: minimum number of cuts for string $S[i \dots j]$

Goal: $C[1, n] + 1$

Make choice: Where is the first cut?

Recurrence:

$$C[i, j] = \begin{cases} 0 & \text{if } S[i \dots j] \text{ is a palindrome} \\ \min\limits_{i+1 \le k \le j-1} C[i, k-1] + 1 + C[k, j] & \text{o.w.} \end{cases}$$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes (# cuts)

Subproblem: $C[i, j]$: minimum number of cuts for string $S[i \dots j]$

Goal: $C[1, n] + 1$

Make choice: Where is the first cut?

Recurrence:

$$C[i, j] = \begin{cases} 0 & \text{if } S[i \dots j] \text{ is a palindrome} \\ \min\limits_{i+1 \le k \le j-1} C[i, k-1] + 1 + C[k, j] & \text{o.w.} \end{cases}$$

Init: $C[i, i] = 0$

Time: $O(n^3)$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes

Subproblem: $P[i]$: minimum number of palindromes for $S[1 \cdots i]$

Goal: $P[n]$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes

Subproblem: $P[i]$: minimum number of palindromes for $S[1 \cdots i]$

Goal: $P[n]$

Make choice: Where does the last palindrome start from?

Recurrence:

$$P[i] = \min_{\substack{1 \le k \le i \\ S[k \dots i] \text{ is a palindrome}}} P[k-1] + 1$$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes

Subproblem: $P[i]$: minimum number of palindromes for $S[1 \cdots i]$

Goal: $P[n]$

Make choice: Where does the last palindrome start from?

Recurrence:

$$P[i] = \min_{\substack{1 \le k \le i \\ S[k \dots i] \text{ is a palindrome}}} P[k-1] + 1$$

Init: $P[0] = 1$

Time: $O(n^2)$

String splitting

String splitting (Problem 7.11)

- Split a string S into many pieces
- Cost $|S| = n \implies n$
- Given locations of m cuts: $C_0, C_1, \cdots, C_m, C_{m+1}$
- Find the minimum cost of splitting S into $m+1$ pieces $S_0 \cdots S_m$

String splitting

Subproblem: $C[i, j]$: the minimum cost of splitting substring $S_i \cdots S_{j-1}$ using cuts $C_{i+1} \cdots C_{j-1}$

Goal: $C[0, m + 1]$

String splitting

Subproblem: $C[i, j]$: the minimum cost of splitting substring $S_i \cdots S_{j-1}$ using cuts $C_{i+1} \cdots C_{j-1}$

Goal: $C[0, m+1]$

Make choice: What is the first cut in $C_{i+1} \cdots C_{j-1}$?

Recurrence:

$$C[i, j] = \min_{i<k<j} \left(C[i, k] + C[k, j] + l(S_i \cdots S_{j-1})\right)$$

String splitting

Subproblem: $C[i, j]$: the minimum cost of splitting substring $S_i \cdots S_{j-1}$ using cuts $C_{i+1} \cdots C_{j-1}$

Goal: $C[0, m+1]$

Make choice: What is the first cut in $C_{i+1} \cdots C_{j-1}$?

Recurrence:

$$C[i, j] = \min_{i<k<j} \left(C[i, k] + C[k, j] + l(S_i \cdots S_{j-1})\right)$$

Init: $C[i, i+1] = 0$

Dynamic Programming

3-D DP

Floyd-Warshall algorithm

(1) DP for Floyd-Warshall algorithm for APSP on directed graphs

Subproblem: dist$[i, j, k]$: the length of the shortest path from i to j via only nodes in $v_1 \cdots v_k$

Goal: dist$[i, j, n], \forall i, j$

3-D DP

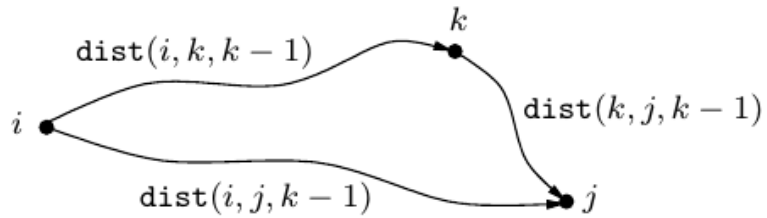
Floyd-Warshall algorithm

(1) DP for Floyd-Warshall algorithm for APSP on directed graphs

Make choice: Is v_k on the ShortestPath$[i, j, k]$?

Recurrence:

$$\text{dist}[i, j, k] = \min\{\text{dist}[i, j, k-1], \text{dist}[i, k, k-1] + \text{dist}[k, j, k-1]\}$$



3-D DP

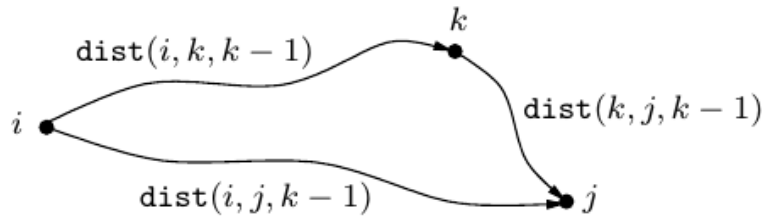
Floyd-Warshall algorithm

(1) DP for Floyd-Warshall algorithm for APSP on directed graphs

Make choice: Is v_k on the ShortestPath$[i, j, k]$?

Recurrence:

$$\text{dist}[i, j, k] = \min\{\text{dist}[i, j, k-1], \text{dist}[i, k, k-1] + \text{dist}[k, j, k-1]\}$$



Init:

$$\text{dist}[i, j, 0] = \begin{cases} 0 & i = j \\ w(i, j) & (i, j) \in E \\ \infty & \text{o.w.} \end{cases}$$

3-D DP

Floyd-Warshall algorithm (Problem 6.25)

(2) Routing table for Floyd-Warshall algorithm

for all $k \leftarrow 1 \ldots n$ **do**
  for all $i \leftarrow 1 \ldots n$ **do**
    for all $j \leftarrow 1 \ldots n$ **do**
      if $\text{dist}[i,j] > \text{dist}[i,k] + \text{dist}[k,j]$ **then**
        $\text{dist}[i,j] \leftarrow \text{dist}[i,k] + \text{dist}[k,j]$

3-D DP

Floyd-Warshall algorithm (Problem 6.25)

(2) Routing table for Floyd-Warshall algorithm

for all $k \leftarrow 1 \ldots n$ **do**
  for all $i \leftarrow 1 \ldots n$ **do**
    for all $j \leftarrow 1 \ldots n$ **do**
      if $\text{dist}[i,j] > \text{dist}[i,k] + \text{dist}[k,j]$ **then**
        $\text{dist}[i,j] \leftarrow \text{dist}[i,k] + \text{dist}[k,j]$

Time: $\Theta(n^3)$ Space: $\Theta(n^2)$

3-D DP

Floyd-Warshall algorithm (Problem 6.25)

(2) Routing table for Floyd-Warshall algorithm

for all $k \leftarrow 1 \ldots n$ **do**
  for all $i \leftarrow 1 \ldots n$ **do**
    for all $j \leftarrow 1 \ldots n$ **do**
      if $\text{dist}[i,j] > \text{dist}[i,k] + \text{dist}[k,j]$ **then**
        $\text{dist}[i,j] \leftarrow \text{dist}[i,k] + \text{dist}[k,j]$
        $\text{Go}[i,j] \leftarrow \text{Go}[i,k]$

Time: $\Theta(n^3)$ Space: $\Theta(n^2)$

3-D DP

Floyd-Warshall algorithm (Problem 6.25)

(2) Routing table for Floyd-Warshall algorithm

for all $i \leftarrow 1 \ldots n$ **do**
    for all $j \leftarrow 1 \ldots n$ **do**
        $\text{dist}[i, j] \leftarrow \infty$
        $\text{Go}[i, j] \leftarrow \text{Nil}$
for all $(i, j) \in E$ **do**
    $\text{dist}[i, j] \leftarrow w(i, j)$
    $\text{Go}[i, j] \leftarrow j$
for all $i \leftarrow 1 \ldots n$ **do**
    $\text{dist}[i, i] \leftarrow 0$
    $\text{Go}[i, i] \leftarrow \text{Nil}$

3-D DP

Floyd-Warshall algorithm (Problem 6.25)

(2) Routing table for Floyd-Warshall algorithm

for all $i \leftarrow 1 \ldots n$ **do**
  for all $j \leftarrow 1 \ldots n$ **do**
    $\text{dist}[i, j] \leftarrow \infty$
    $\text{Go}[i, j] \leftarrow \text{Nil}$
for all $(i, j) \in E$ **do**
  $\text{dist}[i, j] \leftarrow w(i, j)$
  $\text{Go}[i, j] \leftarrow j$
for all $i \leftarrow 1 \ldots n$ **do**
  $\text{dist}[i, i] \leftarrow 0$
  $\text{Go}[i, i] \leftarrow \text{Nil}$

procedure $\textsc{Path}(i, j)$
  if $\text{Go}[i, j] = \text{Nil}$ **then**
    Output "No Path."

Output "i"
while $i \neq j$ **do**
  $i \leftarrow \text{Go}[i, j]$
Output "i"

3-D DP

Floyd-Warshall algorithm (Problem 6.29)

(3) Find minimum-weighted cycle of directed graph ($w(e) > 0$)

$$\text{dist}[i,i] \leftarrow 0 \implies \text{dist}[i,i] \leftarrow \infty$$

$$\forall i : \text{dist}[i,i] = \infty$$

3-D DP

Floyd-Warshall algorithm (Problem 6.29)

(3) Find minimum-weighted cycle of directed graph ($w(e) > 0$)

$$\text{dist}[i,i] \leftarrow 0 \implies \text{dist}[i,i] \leftarrow \infty$$

$$\forall i : \text{dist}[i,i] = \infty$$

Q: $\exists e : w(e) < 0$

3-D DP

Floyd-Warshall algorithm (Problem 6.29)

(3) Find minimum-weighted cycle of directed graph ($w(e) > 0$)

$$\text{dist}[i,i] \leftarrow 0 \implies \text{dist}[i,i] \leftarrow \infty$$

$$\forall i : \text{dist}[i,i] = \infty$$

Q: $\exists e : w(e) < 0$

$\exists i : \text{dist}[i,i] < 0$ $\qquad$ $\forall i : \text{dist}[i,i] \geq 0 \ (= \infty)$

Shortest paths on undirected graphs

Finding shortest paths in undirected graphs with possibly negative edge weights

2

The book "Algorithms" by Robert Sedgewick and Kevin Wayne hinted that (*see the quote below*) there are efficient algorithms for finding shortest paths in undirected graphs with possibly negative edge weights (***not*** by treating an undirected edge as two directed one which means that a single negative edge implies a negative cycle). However, no references are given in the book. Are you aware of any such algorithms?

2

Q. How can we find shortest paths in undirected (edge-weighted) graphs?

A. For positive edge weights, Dijkstra's algorithm does the job. We just build an EdgeWeightedDigraph corresponding to the given EdgeWeightedGraph (by adding two directed edges corresponding to each undirected edge, one in each direction) and then run Dijkstra's algorithm. ***If edge weights can be negative*** (***emphasis added***), efficient algorithms are available, but they are more complicated than the Bellman-Ford algorithm.

algorithms graph-theory shortest-path weighted-graphs reference-question

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edited Jun 9 at 14:15

asked Jun 9 at 13:58

hengxin

6,056 ● 11 ● 38

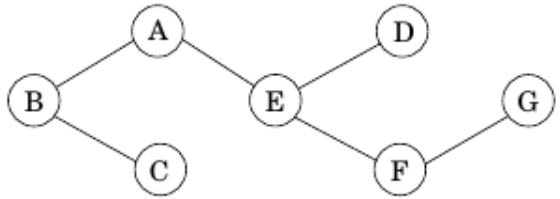
https://cs.stackexchange.com/q/76578/4911

Dynamic Programming

Minimum vertex cover on trees

Minimum vertex cover on trees [Problem: 2.2.18]

- Undirected tree $T = (V, E)$; No designated root!
- Compute (the size of) a minimum vertex cover of T



Minimum vertex cover on trees

Rooted T at any node r.

Minimum vertex cover on trees

Rooted T at any node r.

Subproblem: $I(u)$: the size of an MVC of subtree T_u rooted at u

Goal: $I(r)$

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Make choice: Is u in MVC$[u]$?

Recurrence:

$$I(u) = \min\{\#\text{ children of } u + \sum_{v\text{: grandchildren of } u} I(v),$$
$$1 + \sum_{v\text{: children of } u} I(v)\}$$

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Init: $I(u) = 0$, if u is a leave

Minimum vertex cover on trees

DFS on T from root r:

when u is "finished":
if u is a leave **then**
$I(u) \leftarrow 0$
else
$I(u) \leftarrow \ldots$

Minimum vertex cover on trees

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Greedy algorithm (Rough Proof!):

Theorem

There is an MVC which contains no leaves.

DP on DAG

Longest path in DAG (Problem 7.17)

- Direction: $\downarrow$ OR $\rightarrow$
- Score: $>=<0$

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4. $G \rightarrow G^T$

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Compute a longest path from s in DAG

DP on DAG

Subproblem: dist$[v]$: longest distance from s to v

Goal: dist$[v], \forall v \in V$

DP on DAG

Subproblem: $\text{dist}[v]$: longest distance from s to v

Goal: $\text{dist}[v], \forall v \in V$

Make choice: What is the previous node before v on the longest path?

Recurrence:

$$\text{dist}[v] = \max_{u \to v} \left(\text{dist}[u] + w(u \to v) \right)$$

DP on DAG

Subproblem: $\mathsf{dist}[v]$: longest distance from s to v

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Init: $\mathsf{dist}[s] = 0$

DP on DAG

Subproblem: $\text{dist}[v]$: longest distance from s to v

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$$\text{dist}[v] = \max_{u \to v} \left(\text{dist}[u] + w(u \to v) \right)$$

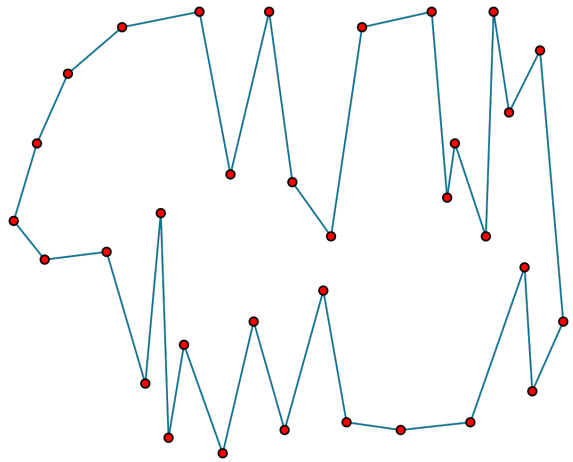
Init: $\text{dist}[s] = 0$

Compute $\text{dist}[v]$ in topo. order

Bitonic tour

Bitonic tour (Problem 7.18)

- Points: $P[1 \ldots n]$, $p_i = (x_i, y_i)$
- $x_1 < x_2 < \cdots < x_n$
- Bitonic tour: $p_1 \rightsquigarrow^{x_i < x_{i+1}} p_n \rightsquigarrow^{x_i > x_{i+1}} p_1$
- Compute a shorest bitonic tour.



Bitonic tour

$P_{i,j} (i \leq j)$: bitonic path $p_i \rightsquigarrow^{x_i > x_{i+1}} p_1 \rightsquigarrow^{x_i < x_{i+1}} p_j$ includes all $p_1, p_2, \ldots, p_j$

Subproblem: $d[i, j]$: the length of a shortest bitonic path $P_{i,j}$

Goal: $d[n, n] = d[n-1, n] + l(p_{n-1} p_n)$

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Goal: $d[n,n] = d[n-1,n] + l(p_{n-1}p_n)$

Make choice: Is p_{j-1} on the increasing path or the decreasing path?

Recurrence:

$$d[i,j] = d[i,j-1] + l(p_{j-1}p_j) \quad \forall i < j-1$$
$$d[i,j] = \min_{1 \le k < j-1} \{d[k,j-1] + l(p_k p_j)\} \quad \forall i = j-1$$

Bitonic tour

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Init: $d[1,2] = l(p_1 p_2)$

Time:

$$O(n^2) = O(n \log n) + O(n^2) \cdot O(1) + O(n) \cdot O(n)$$

Dynamic Programming

The change-making problem

The change-making problem (Problem 7.12)

- Coins values: $x_1 \dots x_n$
- Amount: v
- Is it possible to make change for v?

The change-making problem

The change-making problem (Problem 7.12(2), Problem 7.1 (Subset sum))

(2) Without repetition $(0/1)$

The change-making problem

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Subproblem: $C[i, w]$: Make change for w using only values of $x_1 \dots x_i$?

Goal: $C[n, v]$

The change-making problem

The change-making problem (Problem 7.12(2), Problem 7.1 (Subset sum))

(2) Without repetition $(0/1)$

Subproblem: $C[i, w]$: Make change for w using only values of $x_1 \dots x_i$?

Goal: $C[n, v]$

Make choice: Using value x_i or not?

Recurrence:

$$C[i, w] = C[i-1, w] \vee (C[i-1, w - x_i] \wedge w \geq x_i)$$

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$$C[i, w] = C[i-1, w] \vee (C[i-1, w - x_i] \wedge w \geq x_i)$$

Init:

$$C[i, 0] = \text{true}$$
$$C[0, w] = \text{false, if } w > 0$$
$$C[0, 0] = \text{true}$$

Time: $O(nv)$

The change-making problem

The change-making problem (Problem 7.12(1))

(1) Unbounded repetition (∞)

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Init:

$$C[i, 0] = \text{true}, \forall i = 0 \dots n$$
$$C[0, w] = \text{false, if } w > 0$$

Time: $O(nv)$

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The change-making problem (Problem 7.12(1))

(1) Unbounded repetition (∞)

Subproblem: $C[w]$: Possible to make change for w?

Goal: $C[v]$

The change-making problem

The change-making problem (Problem 7.12(1))

(1) Unbounded repetition (∞)

Subproblem: $C[w]$: Possible to make change for w?

Goal: $C[v]$

Make choice: What is the first coin to use?

Recurrence:

$$C[w] = \bigvee_{i:\, x_i \leq w} C[w - x_i]$$

The change-making problem

The change-making problem (Problem 7.12(1))

(1) Unbounded repetition (∞)

Subproblem: $C[w]$: Possible to make change for w?

Goal: $C[v]$

Make choice: What is the first coin to use?

Recurrence:

$$C[w] = \bigvee_{i:\, x_i \le w} C[w - x_i]$$

Init: $C[0] = \text{true}$

Time: $O(nv)$

The change-making problem

The change-making problem (Problem 7.12(1))

(1) Unbounded repetition (∞)

$$C[i, w] \text{ vs. } C[w]$$

$$C[i, w] = C[i-1, w] \vee (C[i, w - x_i] \wedge w \geq x_i)$$

$$C[w] = \bigvee_{i:\, x_i \leq w} C[w - x_i]$$

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The change-making problem (Problem 7.12(3))

(3) Unbounded repetition with $\leq k$ coins

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Subproblem: $C[i, w, l]$: Possible to make change for w with $\leq l$ coins of values of $x_1 \ldots x_i$?

Goal: $C[n, v, k]$

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The change-making problem (Problem 7.12(3))

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Subproblem: $C[i, w, l]$: Possible to make change for w with $\leq l$ coins of values of $x_1 \dots x_i$?

Goal: $C[n, v, k]$

Make choice: Using value x_i or not?

Recurrence:

$$C[i, w, l] = C[i-1, w, l] \vee (C[i, w - x_i, l-1] \wedge w \geq x_i)$$

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Subproblem: $C[i, w, l]$: Possible to make change for w with $\leq l$ coins of values of $x_1 \dots x_i$?

Goal: $C[n, v, k]$

Make choice: Using value x_i or not?

Recurrence:

$$C[i, w, l] = C[i-1, w, l] \vee (C[i, w - x_i, l-1] \wedge w \geq x_i)$$

Init:

$$C[0, 0, l] = \text{true}, \quad C[0, w, l] = \text{false, if } w > 0$$
$$C[i, 0, l] = \text{true}, \quad C[i, w, 0] = \text{false, if } w > 0$$

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Subproblem: $C[w, l]$: Possible to make change for w with $\leq l$ coins?

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The change-making problem (Problem 7.12(3))

(3) Unbounded repetition with $\leq k$ coins

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Goal: $C[v, k]$

Make choice: What is the first coin to use?

Recurrence:

$$C[w, l] = \bigvee_{i:\, x_i \leq w} C[w - x_i, k - 1]$$

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Dynamic Programming

More DPs . . .

Algorithms that use dynamic programming [edit | edit source]

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- Recurrent solutions to lattice models for protein-DNA binding
- Backward induction as a solution method for finite-horizon discrete-time dynamic optimization problems
- Method of undetermined coefficients can be used to solve the Bellman equation in infinite-horizon, discrete-time, discounted, time-invariant dynamic optimization problems
- Many string algorithms including longest common subsequence, longest increasing subsequence, longest common substring, Levenshtein distance (edit distance)
- Many algorithmic problems on graphs can be solved efficiently for graphs of bounded treewidth or bounded clique-width by using dynamic programming on a tree decomposition of the graph.
- The Cocke–Younger–Kasami (CYK) algorithm which determines whether and how a given string can be generated by a given context-free grammar
- Knuth's word wrapping algorithm that minimizes raggedness when word wrapping text
- The use of transposition tables and refutation tables in computer chess
- The Viterbi algorithm (used for hidden Markov models)
- The Earley algorithm (a type of chart parser)
- The Needleman–Wunsch algorithm and other algorithms used in bioinformatics, including sequence alignment, structural alignment, RNA structure prediction
- Floyd's all-pairs shortest path algorithm
- Optimizing the order for chain matrix multiplication
- Pseudo-polynomial time algorithms for the subset sum, knapsack and partition problems
- The dynamic time warping algorithm for computing the global distance between two time series
- The Selinger (a.k.a. System R) algorithm for relational database query optimization
- De Boor algorithm for evaluating B-spline curves
- Duckworth–Lewis method for resolving the problem when games of cricket are interrupted
- The value iteration method for solving Markov decision processes
- Some graphic image edge following selection methods such as the "magnet" selection tool in Photoshop
- Some methods for solving interval scheduling problems
- Some methods for solving the travelling salesman problem, either exactly (in exponential time) or approximately (e.g. via the bitonic tour)
- Recursive least squares method
- Beat tracking in music information retrieval
- Adaptive-critic training strategy for artificial neural networks
- Stereo algorithms for solving the correspondence problem used in stereo vision
- Seam carving (content-aware image resizing)
- The Bellman–Ford algorithm for finding the shortest distance in a graph
- Some approximate solution methods for the linear search problem
- Kadane's algorithm for the maximum subarray problem


THANK YOU