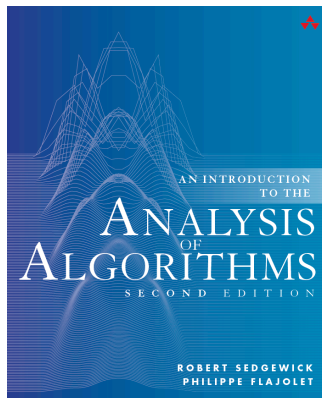# Asymptotics, Recurrences, and Divide and Conquer

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Problem P Algorithm A

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Average-case Time Complexity (Problem 1.8)

$$\mathsf{Input}: r \in [1, n], \;\; r \in \mathbb{Z}^+$$

$$P\{r = i\} = \begin{cases} \frac{1}{n}, & 1 \le i \le \frac{n}{4} \\[2mm] \frac{2}{n}, & \frac{n}{4} < i \le \frac{n}{2} \\[2mm] \frac{1}{2n}, & \frac{n}{2} < i \le n \end{cases}$$

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$$\begin{aligned} A &= \sum_{X \in \mathcal{X}} T(X) \cdot P(X) \\ &= T(1)P(1) + T(2)P(2) + \cdots + T(n)P(n) \\ &= \frac{n}{4} \times 10 \times \frac{1}{n} + \frac{n}{4} \times 20 \times \frac{2}{n} + \frac{n}{4} \times 30 \times \frac{1}{2n} + \frac{n}{4} \times n \times \frac{1}{2n} \\ &= \frac{1}{8}n + \frac{55}{4} \end{aligned}$$

Average-case Analysis of Quicksort

$$A(n) = n - 1 + \frac{1}{n} \sum_{i=0}^{i=n-1} \left(A(i) + A(n-i-1) \right)$$

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$$
\begin{aligned}
A(n) &= \mathbb{E}[T(X)] \\
&= \mathbb{E}[\mathbb{E}[T(X)|I]] \\
&= \sum_{i=0}^{i=n-1} P(I = i) \, \mathbb{E}[T(X) \mid I = i] \\
&= \sum_{i=0}^{i=n-1} \frac{1}{n} [n - 1 + A(i) + A(n - i - 1)]
\end{aligned}
$$