What You Should Know About Algorithm Design and Analysis . . . But (Probably) Don't

Hengfeng Wei

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When You Design Algorithms:

Design Faster Algorithms

Design Faster Algorithms

When to Stop?

Design Faster Algorithms

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The Complexity of Problems

Problem P Algorithm A

Inputs: $\mathcal{X}_n$ of size n

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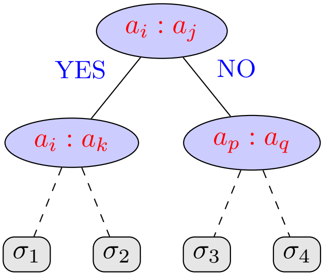
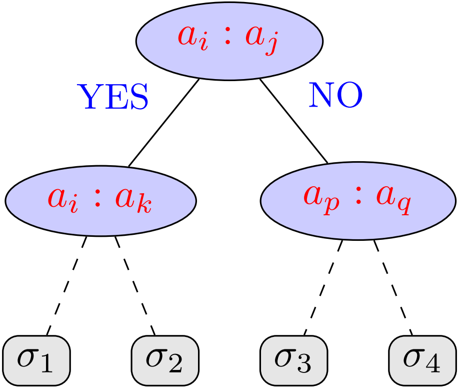


Decision Tree

Adversary Argument

Decision Tree



Lower Bound for Comparison-based Sorting

Prove a lower bound of $\Omega(n \log n)$ on the time complexity of any **comparison-based** sorting algorithm on inputs of size n.

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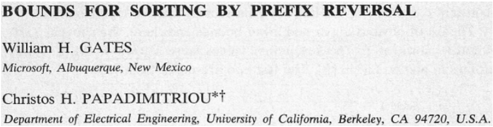
BOUNDS FOR SORTING BY PREFIX REVERSAL

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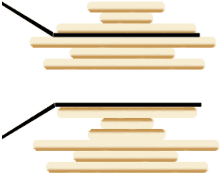
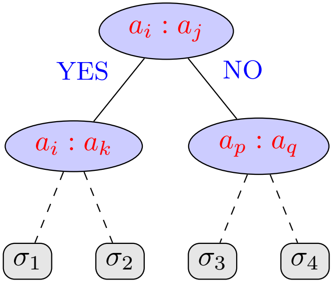


Decision Tree Model

Decision Tree Model



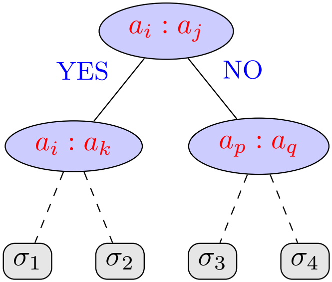
Nodes: comparisions $a_i : a_j$

$$<,\ \leq,\ =,\ \geq,\ >$$

Edges: two-way decisions (Y/N)

Leaves: possible permutations

Decision Tree Model



Nodes: comparisions $a_i : a_j$

$$<, \leq, =, \geq, >$$

Edges: two-way decisions (Y/N)

Leaves: possible permutations

Assumption:

All the input elements are **distinct**.

$$a_i < a_j$$

$$\boxed{\text{Any Comparison-based Sorting Algorithm} \xRightarrow{\text{modeled by}} \text{A Decision Tree}}$$

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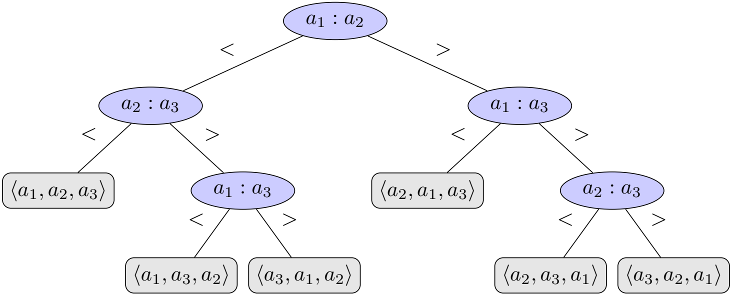
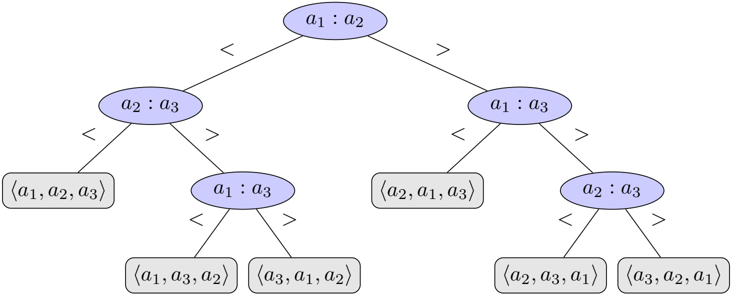


The decision tree for sort on three elements.

Any Comparison-based Sorting Algorithm $\xRightarrow{\text{modeled by}}$ A Decision Tree



The decision tree for insertion sort on three elements.

Any Comparison-based Sorting Algorithm $\xRightarrow{\text{modeled by}}$ A Decision Tree

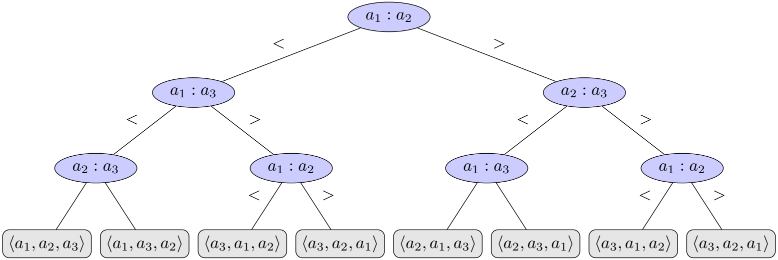
---

procedure -$\textsc{Sort}(A, n)$
  for $i \leftarrow 1$ **to** $n - 1$ **do**
    for $j \leftarrow i + 1$ **to** n **do**
      if $A[j] < A[i]$ **then**
        $\textsc{Swap}(A[j], A[i])$

Any Comparison-based Sorting Algorithm $\xRightarrow{\text{modeled by}}$ A Decision Tree

```
1: procedure SELECTION-SORT(A, n)
2:     for i ← 1 to n − 1 do
3:         for j ← i + 1 to n do
4:             if A[j] < A[i] then
5:                 SWAP(A[j], A[i])
```

Any Comparison-based Sorting Algorithm $\xrightarrow{\text{modeled by}}$ A Decision Tree



The decision tree for selection sort on three elements.

$$\boxed{\text{Any Comparison-based Sorting Algorithm } \mathcal{A} \xRightarrow{\text{modeled by}} \text{A Decision Tree } \mathcal{T}}$$

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$$\boxed{\begin{array}{c}\text{Worst-case Lower Bound of Comparison-based Sorting} \\ \text{(on inputs of size } n\text{)} \\ \xRightarrow{\text{modeled by}} \\ \text{The Minimum Height of All } \mathcal{T}\text{s}\end{array}}$$

Worst-case Lower Bound of Comparison-based Sorting
(on inputs of size n)

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The Minimum Height of All $\mathcal{T}$s

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The Minimum Height of All $\mathcal{T}$s

To be a correct sorting algorithm:

$$L = \# \text{ of leaves} \geq n!$$

Worst-case Lower Bound of Comparison-based Sorting (on inputs of size n)

$\xRightarrow{\text{modeled by}}$

The Minimum Height of All $\mathcal{T}$s

To be a correct sorting algorithm:

$$L = \# \text{ of leaves} \geq n!$$

To be a full binary tree:

$$L = \# \text{ of leaves} \leq 2^h$$

$$\boxed{n! \le L = \# \text{ of leaves} \le 2^h}$$

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$$h \geq \log n! = \Omega(n \log n)$$

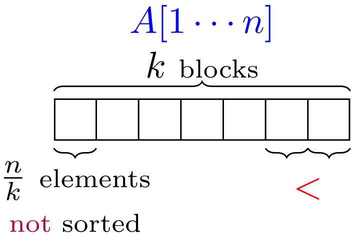
$$\boxed{n! \le L = \#\ \text{of leaves} \le 2^h}$$

$$h \ge \log n! = \Omega(n \log n)$$

Stirling Formula (by *James Stirling*):

$$n! = \Theta\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)$$

K-sorted Array (Problem 6.8)



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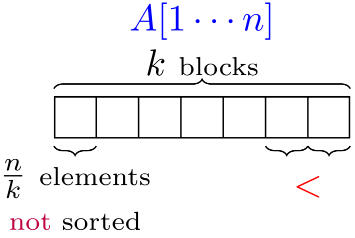
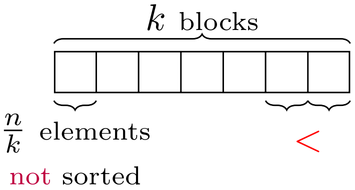


$O(n \log k)$

K-sorted Array (Problem 6.8)

$$A[1 \cdots n]$$



$$O(n \log k)$$

$$n = 16, \quad k = 4, \quad \frac{n}{k} = 4$$

1, 2, 4, 3; 7, 6, 8, 5; 10, 11, 9, 12; 15, 13, 16, 14

k-sorted

k-sorted

1-sorted

k-sorted

1-sorted $\rightarrow$ 2-sorted

k-sorted

1-sorted $\rightarrow$ 2-sorted $\rightarrow$ 4-sorted

k-sorted

$$1\text{-sorted} \rightarrow 2\text{-sorted} \rightarrow 4\text{-sorted} \rightarrow \cdots \rightarrow n\text{-sorted}$$

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Quicksort (with median as pivot) stops after the $\log k$ recursions.

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$$\boxed{\Theta(n \log k)}$$

$$\Omega(n \log k)$$

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$L \geq$

$$\Omega(n \log k)$$

$$L \geq \binom{n}{n/k}\binom{n - n/k}{n/k}\cdots\binom{n/k}{n/k}$$

$$\Omega(n \log k)$$

$$L \geq \binom{n}{n/k}\binom{n - n/k}{n/k} \cdots \binom{n/k}{n/k} = \binom{n}{n/k, \ldots, n/k}$$

$$\Omega(n \log k)$$

$$L \geq \binom{n}{n/k}\binom{n - n/k}{n/k} \cdots \binom{n/k}{n/k} = \binom{n}{n/k, \ldots, n/k} = \frac{n!}{\left((\frac{n}{k})!\right)^k}$$

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$$H \geq \log\left(\frac{n!}{\left((\frac{n}{k})!\right)^k}\right)$$

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$$H \geq \log\left(\frac{n!}{\left(\left(\frac{n}{k}\right)!\right)^k}\right) = \Omega(n \log k)$$

$$\boxed{n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \implies \log n! \sim n \log n}$$

Bolts and Nuts (Problem 6.9)

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Quicksort

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Quicksort

$$A(n) = O(n \log n)$$

Bolts and Nuts (Problem 6.9)

Quicksort

$$A(n) = O(n \log n)$$

In the worst case:

- "Matching Nuts and Bolts" by Alon *et al.*, $\Theta(n \log^4 n)$
- "Matching Nuts and Bolts Optimality" by Bradford, 1995, $\Theta(n \log n)$

$$\Omega(n \log n)$$

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$3^H \geq L \geq n!$

$$\Omega(n \log n)$$

$$3^H \geq L \geq n! \implies H \geq \log n! \implies H = \Omega(n \log n)$$

Adversary Argument

Searching in Matrix (Problem 9.8)

$$M : m \times n$$

Row: Increasing from left to right

Col: Increasing from top to down

$$x \in M?$$

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Divide & Conquer :

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Always checking the lower left corner.

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$$\text{Divide \& Conquer} : T(m, n) = 3T(\frac{m}{2}, \frac{n}{2}) + 1$$

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$$T(m, n) = m + n - 1$$

Assume $M : n \times n$

$$W(n) \leq 2n - 1$$

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By Adversary Argument!

$$\boxed{W(n) \geq 2n - 1}$$

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Adversary $\mathcal{A}$:

$x > M[i][j]$

$x = M[i][j]$

$x < M[i][j]$

Algorithm $\mathbb{A}$:

$\textsc{Compare}(x, M[i][j])$

$$\boxed{W(n) \geq 2n - 1}$$

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Diagonals: $i + j = n - 1 \quad \& \quad i + j = n$

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No particular ordering requirements on these two diagonals!

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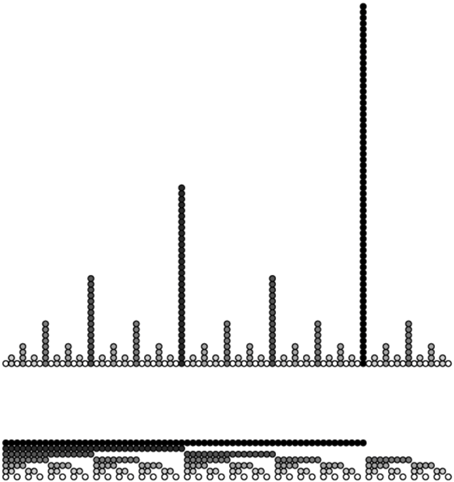
Diagonals: $i + j = n - 1 \quad \& \quad i + j = n$

No particular ordering requirements on these two diagonals!

$$i + j \leq n - 1 \implies x > M_{ij}$$
$$i + j > n - 1 \implies x < M_{ij}$$

Amortized Analysis



Amortized analysis is

an algorithm analysis technique for

analyzing a sequence of operations

irrespective of the input to show that

the average cost per operation is small, even though

a single operation within the sequence might be expensive.

Accounting Method

Summation Method

Potential Method

Amortized Analysis

The Summation Method

$$\sum$$

$$o_1, o_2, \ldots, o_n$$

$$c_1, c_2, \ldots, c_n$$

$$o_1, o_2, \ldots, o_n$$

$$c_1, c_2, \ldots, c_n$$

$$\boxed{\forall i,\ \hat{c}_i = \frac{\left(\sum_{i=1}^{n} c_i\right)}{n}}$$

The Summation Method for Array Doubling

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On any sequence of n INSERTs on an initially empty array.

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$$
\begin{array}{lcccccccccc}
o_i: & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} \\
c_i: & 1 & 2 & 3 & 1 & 5 & 1 & 1 & 1 & 9 & 1
\end{array}
$$

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\end{array}
$$

$$
c_i = \begin{cases} (i-1)+1 = i & \text{if } i-1 \text{ is an exact power of } 2 \\ 1 & \text{o.w.} \end{cases}
$$

The Summation Method for Array Doubling

On any sequence of n INSERTs on an initially empty array.

$$\begin{array}{ccccccccccc} o_i: & o_1 & o_2 & o_3 & o_4 & o_5 & o_6 & o_7 & o_8 & o_9 & o_{10} \\ c_i: & 1 & 2 & 3 & 1 & 5 & 1 & 1 & 1 & 9 & 1 \end{array}$$

$$c_i = \begin{cases} (i-1)+1 = i & \text{if } i-1 \text{ is an exact power of } 2 \\ 1 & \text{o.w.} \end{cases}$$

$$\sum_{i=1}^{n} c_i \le n + \sum_{j=0}^{\lceil \lg n \rceil - 1} 2^j = n + (2^{\lceil \lg n \rceil} - 1) \le n + 2n = 3n$$

The Summation Method for Array Doubling

On any sequence of n INSERTs on an initially empty array.

o_i :	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9	o_{10}
c_i :	1	2	3	1	5	1	1	1	9	1

$$c_i = \begin{cases} (i-1)+1 = i & \text{if } i-1 \text{ is an exact power of } 2 \\ 1 & \text{o.w.} \end{cases}$$

$$\sum_{i=1}^{n} c_i \le n + \sum_{j=0}^{\lceil \lg n \rceil - 1} 2^j = n + (2^{\lceil \lg n \rceil} - 1) \le n + 2n = 3n$$

$$\boxed{\forall i,\ \hat{c}_i = 3}$$

The Accounting Method

$$o_1, o_2, \ldots, o_n$$

$$c_1, c_2, \ldots, c_n$$

$$a_1, a_2, \ldots, a_n$$

$$o_1, o_2, \ldots, o_n$$

$$c_1, c_2, \ldots, c_n$$

$$a_1, a_2, \ldots, a_n$$

$$\boxed{\hat{c}_i = c_i + a_i,\ a_i >=< 0}$$

$$\text{Amortized Cost} = \text{Actual Cost} + \text{Accounting Cost}$$

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$$c_1, c_2, \ldots, c_n$$

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Amortized Cost = Actual Cost + Accounting Cost

$$\forall n, \ \sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i$$

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$$\forall n,\ \sum_{i=1}^{n} c_i \le \sum_{i=1}^{n} \hat{c}_i \implies \boxed{\forall n,\ \sum_{i=1}^{n} a_i \ge 0}$$

$$o_1, o_2, \ldots, o_n$$

$$c_1, c_2, \ldots, c_n$$

$$a_1, a_2, \ldots, a_n$$

$$\boxed{\hat{c}_i = c_i + a_i,\ a_i >=< 0}$$

$$\text{Amortized Cost} \;=\; \text{Actual Cost} \;+\; \text{Accounting Cost}$$

$$\forall n,\ \sum_{i=1}^{n} c_i \le \sum_{i=1}^{n} \hat{c}_i \implies \boxed{\forall n,\ \sum_{i=1}^{n} a_i \ge 0}$$

Key Point: Put the accounting cost on specific objects.

The Accounting Method for Array Doubling

$$Q : \hat{c}_i = 3 \ vs. \ \hat{c}_i = 2$$

The Accounting Method for Array Doubling

$$\color{red}{Q:}\ \hat{c}_i = 3\ \ vs.\ \ \hat{c}_i = 2$$

$$\hat{c}_i = 3 = \underbrace{1}_{\text{insert}} + \underbrace{1}_{\text{move itself}} + \underbrace{1}_{\color{red}{\text{help move another}}}$$

The Accounting Method for Array Doubling

$$\color{red}{Q:}\ \hat{c}_i = 3 \ \ vs. \ \ \hat{c}_i = 2$$

$$\hat{c}_i = 3 = \underbrace{1}_{\text{insert}} + \underbrace{1}_{\text{move itself}} + \underbrace{1}_{\color{red}{\text{help move another}}}$$

	$\hat{c}_i$	c_i *(actual cost)*	a_i *(accounting cost)*
INSERT *(normal)*	3	1	2
INSERT *(expansion)*	3	$1 + t$	$-t + 2$

Simulating a queue Q using two stacks S_1, S_2 (Problem $\mathbb{E}3$)

```
procedure Enq(x)
    Push(S_1, x)

procedure Deq()
    if S_2 = ∅ then
        while S_1 ≠ ∅ do
            Push(S_2, Pop(S_1))
    Pop(S_2)
```

The Summation Method for Queue Simulation

$$\frac{\left(\sum_{i=1}^{n} c_i\right)}{n}$$

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$$\frac{\left(\sum_{i=1}^{n} c_i\right)}{n}$$

The operation sequence is *NOT* known.

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item:	Push into S_1	Pop from S_1	Push into S_2	Pop from S_2
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The Accounting Method for Queue Simulation

$$\hat{c}_{\text{ENQ}} = 3$$
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$$\#S_1 = t$$

	$\hat{c}_i$	c_i *(actual cost)*	a_i *(accounting cost)*
ENQUEUE	3	1	2
DEQUEUE ($S_2 \neq \emptyset$)	1	1	0
DEQUEUE ($S_2 = \emptyset$)	1	$1 + 2t$	$-2t$

Array Merging Dictionary (Problem $\mathbb{E}$ 2)

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i	$s_i = 2^i$
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$\vdots$	$\ldots$
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$$11 = 2^0 + 2^1 + 2^3$$

i	e_i
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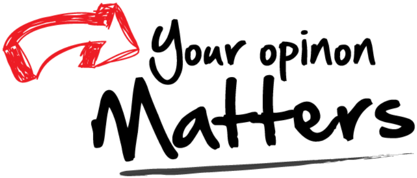
$$\forall n,\ \sum_{i=1}^{n} a_i \ge 0$$


Thank
You!



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