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THE k-CORES OF A GRAPH

by

Allan Bickle

A Dissertation

Submitted to the

Faculty of the Graduate College

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THE k -CORES OF A GRAPH

Allan Bickle, Ph.D.

Western Michigan University, 2010

The k -core of a graph is the maximal subgraph with minimum degree at least k . It is easily shown that this subgraph is unique, the cores of a graph are nested, and that it can be found by iteratively deleting vertices with degree less than k . The maximum k such that G has a k -core is the maximum core number of G , $\widehat{C}(G)$, and if $\widehat{C}(G) = \delta(G)$, we say G is k -monocore. Many common graph classes including trees and regular graphs are monocore. A deletion sequence is formed by iteratively deleting a vertex of smallest degree, and a construction sequence reverses a deletion sequence.

Following these basic results, chapter one defines the k -shell of a graph as the subgraph induced by edges in the k -core and not in the $k + 1$ -core. The 1-shell is a forest with no trivial components. The structure of 2-cores and 3-cores is analyzed and an operation characterization of 2-monocore graphs is presented.

Chapter two examines the extremal classes of k -cores. Maximal k -degenerate graphs are the upper extremal graphs. Results on their size, degree sequence, diameter, and more are presented. Labeled maximal k -degenerate graphs are shown to correspond bijectively to a certain type of sequences. The k -trees, a special type of maximal k -degenerate graph, are characterized.

The degree sequences of k -monocore graphs are characterized. Collapsible and core-critical graphs, classes of lower extremal graphs, are defined and analyzed. How graphs collapse is analyzed.

In Chapter three, the structure of the k -core of a line graph or Cartesian product or join of graphs is characterized. Ramsey core numbers, a new variation of Ramsey numbers, are defined and an exact formula is proven.

Chapter four considers applications of cores to problems in graph theory. The core number bound for chromatic number, $\chi(G) \leq 1 + \widehat{C}(G)$, is proved using construction sequences. It leads to short proofs of Brooks' Theorem and the Nordhaus-Gaddum Theorem. Extremal decompositions attaining Plesnik's Conjecture for $k = 2$ and 3 are characterized. Similar coloring techniques are discussed for edge coloring, list coloring, $L(2, 1)$ coloring, arboricity, vertex arboricity, and point partition numbers. Applications of cores to problems in planarity, integer embeddings, domination, total domination, and the Reconstruction Conjecture are discussed.

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1 Introduction

This dissertation will study k-cores of graphs. We will see that cores are a natural and useful means of simplifying the structure of graphs. They have interesting structure in their own right, and they are a useful means of attacking many other problems in graph theory, particularly graph coloring. This section will be devoted to laying out the basic definitions and results needed throughout the sections that follow.

1.1 Basics

Definition 1. The k -core of a graph G is the maximal induced subgraph $H \subseteq G$ such that $\delta(H) \geq k$.

Thus all vertices of H are adjacent to at least k other vertices in H .

The k -core was introduced by Steven B. Seidman in a 1983 paper entitled *Network structure and minimum degree*. Note that in that paper, [53] Seidman defined the k -core to also be connected. However, this condition has been omitted in other papers, and we will find it more convenient to omit it here.

We will denote the k -core of graph G by $C_k(G)$.

The definition of k -core presumes that it is unique, and hence well-defined.

Proposition 2. *The k -core is well-defined.*

Proof. Let H_1, H_2 both be maximal subgraphs of a graph G with $\delta(H_1) \geq k, \delta(H_2) \geq k$. Let $H = H_1 \cup H_2$. Then $H_1 \subseteq H, H_2 \subseteq H$, and $\delta(H) \geq k$. Then $H = H_1 = H_2$, so the k -core is unique.

□

Figure 1 shows the successive cores of a particular graph G .

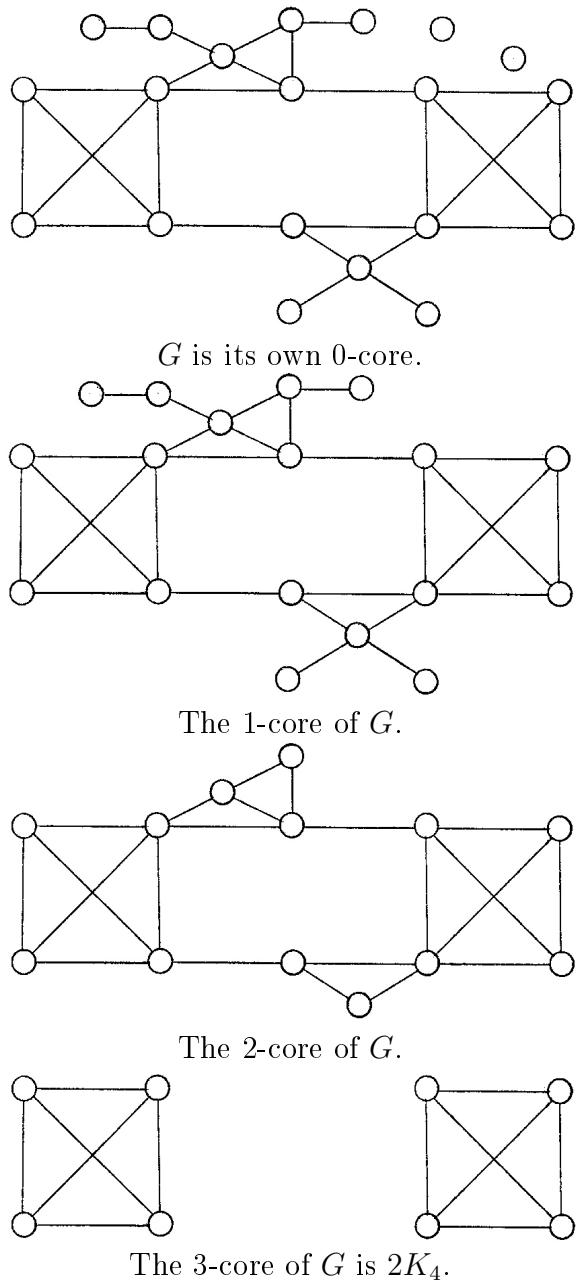


Figure 1: An example of the cores of a graph.

Here are some basic results on k -cores.

For any graph G , $C_0(G) = G$. Indeed, for all $k < 0$, $C_k(G) = G$. Even though the minimum degree of a graph is never negative, we will find it convenient to allow negative integer cores. The 1-core is the result of deleting all isolated vertices from G .

Any graph G with $\delta(G) \geq k$ can be a k -core. In particular, it is its own k -core. In this case, we say G is a k -core.

For a specific value of k , G may not have a k -core. In this case, we say its k -core is null ($C_k(G) = \emptyset$) or does not exist. Alternatively, we say G is k -core-free. If not, we say its k -core exists. For every (finite) graph, the k -core will be null for a large enough value of k .

Proposition 3. *The cores of a graph are nested. That is, if $k > j$, then $C_k(G) \subseteq C_j(G)$.*

Proof. Let $k > j$, and $v \in C_k(G)$. Then v is an element of a set of vertices that are each adjacent to at least k (and hence j) elements of the set. Thus $\delta(C_k(G)) \geq j$, so $C_k(G) \subseteq C_j(G)$.

□

The fact that the cores are nested implies that for a given vertex v there exists an integer N such that v is contained in every k -core for $k \leq N$ and is contained in no k -core for $k > N$.

Definition 4. The core number of a vertex, $C(v)$, is the largest value for k such that $v \in C_k(G)$. (This has also been named the coreness of v .) The maximum core number of a graph, $\widehat{C}(G)$, is the maximum of the core numbers of the vertices of G . Given $k = \widehat{C}(G)$, the maximum core of G is $C_k(G)$.

It is immediate that $\delta(G) \leq \widehat{C}(G) \leq \Delta(G)$. We can characterize the extremal graphs for the upper bound. For simplicity, we restrict the statement to connected graphs. (see also West p. 199 [60])

Proposition 5. *Let G be a connected graph. Then $\widehat{C}(G) = \Delta(G) \iff G$ is regular.*

Proof. If G is regular, then its maximum and minimum degrees are equal, so the result is obvious.

For the converse, let $\widehat{C}(G) = \Delta(G) = k$. Then G has a subgraph H with $\delta(H) = \Delta(G) \geq \Delta(H)$, so H is k -regular. If H were not all of G , then since G is connected, some vertex of H would have a neighbor not in H , implying that $\Delta(G) > \Delta(H) = \delta(H) = \Delta(G)$. But this is not the case, so $G = H$, and G is regular. \square

We could similarly define the minimum core number of G to be the largest k such that $C_k(G) = G$. But this is just $\delta(G)$.

Definition 6. If the maximum and minimum core numbers of G are equal, $\widehat{C}(G) = \delta(G)$, we say G is k -monocore.

Monocore graphs are the extremal graphs for the lower bound $\delta(G) \leq \widehat{C}(G)$.

We need a way to determine the k -core of a graph. Examining all 2^n induced subgraphs of a graph G of order n is impractical. Fortunately, there is an efficient algorithm to determine the k -core of a graph. This algorithm will be presented as a sketch, and more details will be presented as part of a more general algorithm below.

Algorithm 7. *The k -core algorithm (sketch)*

Input: graph G with adjacency matrix A , integer k , degree array D

Recursion: Delete all vertices with degree less than k from G . (That is, make a list of such vertices, zero out their degrees, and decrement the degrees of their neighbors.)

Result: The vertices that have not been deleted induce the k -core.

The fact that this algorithm works can be said to be the fundamental result on k -cores.

Theorem 8. Applying the k -core algorithm to graph G yields the k -core of G , provided it exists.

Proof. Let G be a graph and H be the result of the algorithm.

Let $v \in H$. Then v has at least k neighbors in H . Then $\delta(H) \geq k$. Then $H \subseteq C_k(G)$.

Let $v \in C_k(G)$. Then v is an element of a set of vertices, each of which has at least k neighbors in the set. None of these vertices will be deleted in the first iteration. If none have been deleted by the n^{th} iteration, none will be deleted by the $n + 1^{st}$ iteration. Thus none will ever be deleted. Thus $v \in H$. Thus $C_k(G) \subseteq H$.

Thus $H = C_k(G)$, so the algorithm yields the k -core.

□

If G does not have a k -core, the algorithm will delete all the vertices.

This algorithm can be generalized to determine the core structure of a graph G .

Algorithm 9. The Core Number Algorithm

Input adjacency matrix A

Set $\delta = \infty$.

For each vertex

Compute its degree by summing its row in A ; store in degree array

If $d(v) < \delta$, set $\delta = d(v)$

Set $k = \delta$.

While there is a vertex left to be deleted

For each vertex

If its degree is at most k

```

Record its core number in the core number array
Mark it in the delete array, indicate a vertex to be deleted
If there is no vertex to delete, increase k by 1
For each vertex marked in the delete array
    Make its degree 0
    For each vertex it is adjacent to
        If its degree is not 0, reduce it by 1
    Clear it from the delete array

```

The core number algorithm is a polynomial time algorithm.

Theorem 10. *The core number algorithm has efficiency $O(n^2)$.*

Proof. The row corresponding to a vertex will be run through exactly twice. Once when its degree is computed, and again when it is 'deleted'. There are n rows and n columns in the adjacency matrix. The other steps of the algorithm all run in linear time. Thus the algorithm has efficiency $O(n^2)$.

□

If we use an edge list instead of an adjacency matrix to describe the graph, it is possible to use essentially the same algorithm to determine the core numbers in $O(m)$ time. [5] This is better for sparse graphs.

The core number algorithm successively deletes vertices of relatively small degree in a graph until none remain. We can define a sequence that orders the vertices of a graph based on this process.

Definition 11. A vertex deletion sequence of a graph G is a sequence that contains each of its vertices exactly once and is formed by successively deleting a vertex of smallest degree. The graph deletion sequence of G is the corresponding sequence of subgraphs of G . The degree deletion sequence is the corresponding sequence of degrees of deleted vertices.

We may wish to construct a graph by successively adding vertices of relatively small degree. This motivates our next definition.

Definition 12. The vertex, graph, and degree construction sequences of a graph are the reversals of the corresponding deletion sequences.

By the k -core algorithm, the maximum value of the degree deletion sequence is the maximum core number of the graph in question.

Definition 13. A graph is k -degenerate if its vertices can be successively deleted so that when deleted, each has degree at most k . The degeneracy of a graph is the smallest k such that it is k -degenerate.

Thus the k -core algorithm implies a natural min-max relationship.

Corollary 14. *For any graph, its maximum core number is equal to its degeneracy.*

Proof. Let G be a graph with degeneracy d and $k = \widehat{C}(G)$. Since G has a k -core, it is not $k - 1$ -degenerate, so $k \leq d$. Since G has no $k + 1$ -core, running the k -core algorithm for the value $k + 1$ destroys the graph, so G is k -degenerate, and $k = d$.

□

There are at most $n!$ vertex deletion sequences for a labeled graph G of order n . This is exact if and only if G is complete or empty. The vertex deletion sequence of a graph is never unique for a nontrivial graph since the next-to-last graph must be K_2 . The vertex and graph deletion sequences of a graph determine each other, and both determine the degree deletion sequence, but the converse is not true. In Figure 2 below, vertex deletion sequences ABCDE and EDABC both give degree deletion sequence 11110 but are not distinct even unlabeled.

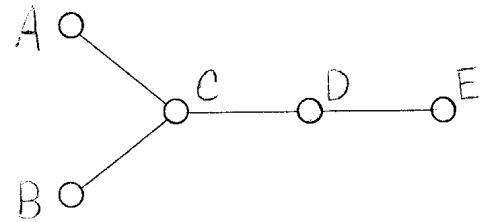


Figure 2: Degree deletion sequence does not determine vertex deletion sequence.

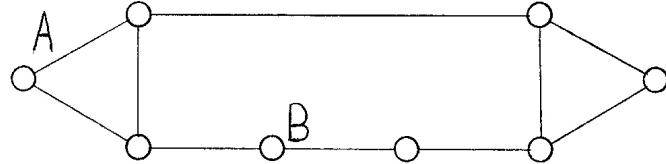


Figure 3: A graph need not have a unique degree deletion sequence.

The degree deletion sequence need not be unique. In the Figure 3, starting with A can produce a degree deletion sequence of 22111210 or 22211110 while starting with B produces the degree deletion sequence 21211210.

We now determine the cores of some special classes of graphs. First, we have the following.

Proposition 15. *Let G be a graph and G_i be the components of G , so $G = \cup G_i$. Then $C_k(G) = \cup C_k(G_i)$.*

Proof. Let $v \in C_k(G)$. Then $v \in G_i$ for some i and $v \in H \subseteq G_i$, $\delta(H) \geq k$, so $v \in C_k(G_i) \subseteq \cup C_k(G_i)$. Thus $C_k(G) \subseteq \cup C_k(G_i)$.

Let $v \in \cup C_k(G_i)$. Then $v \in C_k(G_i)$ for some i . Then $v \in H \subseteq G_i$, $\delta(H) \geq k$, so $v \in C_k(G)$. Then $\cup C_k(G_i) \subseteq C_k(G)$.

□

Class of Graphs	Maximum Core Number
r -regular	r
nontrivial trees	1
forests (no trivial components)	1
complete bipartite $K_{a,b}$, $a \leq b$	a
K_{a_1, \dots, a_n} , $a_1 \leq a_2 \leq \dots \leq a_n$	$a_1 + \dots + a_{n-1}$
wheels	3
maximal outerplanar, $n \geq 3$	2

Table 1: Classes of monocore graphs.

Hence we consider the least-connected connected graphs, trees.

Proposition 16. *Let T be a nontrivial tree. Then $C_k(T) = \begin{cases} T & k = 0, 1 \\ \emptyset & k > 1 \end{cases}$.*

Proof. Every nontrivial tree has $\delta(T) = 1$. Suppose there were a counterexample for $k \geq 2$, and let T be one with minimum order n . Now T has an end-vertex v . Now $v \notin C_k(T)$, so $T - v$ is a tree with order $n - 1$, and $C_k(T - v) \subseteq C_k(T)$, which is a contradiction. Thus $C_k(T) = \emptyset$ for $k \geq 2$.

□

Thus all trees are monocore. Indeed, many important classes of graphs are monocore. Table 1 summarizes some of the most common. The verification of their core structure is straightforward.

One class of graphs that are usually not monocore is the unicyclic graphs.

Proposition 17. Let T be a tree with order $n \geq 3$, $e = uv$ an edge not in T . Then $T + e$ is unicyclic with cycle H induced by e and the unique $u - v$ path in T . Then

$$C_k(T + e) = \begin{cases} T + e & k = 0, 1 \\ H & k = 2 \\ \emptyset & k > 2 \end{cases}.$$

Indeed, adding any two edges to a tree produces a 2-core induced by the edges and the unique paths in the tree between each pair of vertices that define the edges. Adding three edges may produce a 3-core.

For more general classes of graphs, we may only be able to bound the maximum core number.

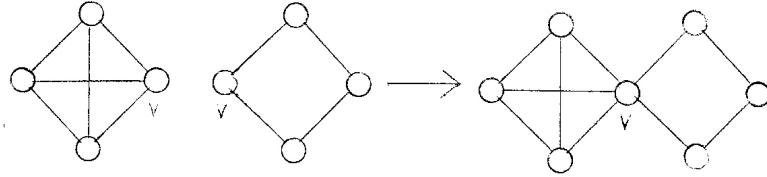
Proposition 18. If G is planar, $\widehat{C}(G) \leq 5$. If G also has order $n < 12$, then $\widehat{C}(G) \leq 4$.

Proof. If there were a planar 6-core, it would have $2m = \sum d(v_i) \geq 6n$, that is, $m \geq 3n$. But every planar graph has $m \leq 3n - 6$.

Let planar graph G have a 5-core H , where H has order n , size m . Then $2m = \sum d(v_i) \geq 5n$, so $m \geq \frac{5}{2}n$. Since H is planar, $m \leq 3n - 6$. Thus $\frac{5}{2}n \leq 3n - 6$, so $n(G) \geq n(H) \geq 12$. □

This bound is sharp. Indeed, the unique planar graph of order 12 with maximum core number 5 is the icosahedron. On the other hand, even if we restrict ourselves to maximal planar graphs, we cannot guarantee a minimum degree greater than three.

This dissertation will follow the notation primarily of the following books.



The overlap of K_4 and C_4 on a vertex, $K_4 \underset{v}{\cup} C_4$.

Figure 4: The overlap of two graphs.

Introduction to Graph Theory, 2nd edition by Douglas West [60]

Undefined terminology can be found in those books. In addition, we will use the following graph operation called the overlap of two graphs.

We say an identification of two ordered sets of equal cardinality (a_1, \dots, a_r) and (b_1, \dots, b_r) is the pairwise equation of equal-indexed elements $a_1 = b_1, \dots, a_r = b_r$.

Definition 19. Let G_1, G_2 be graphs with $H_1 \subseteq V(G_1), H_2 \subseteq V(G_2)$ ordered sets of equal cardinality. The overlap of G_1 and G_2 ,

$$G_1 \underset{H_1=H_2}{\cup} G_2$$

with identification $H_1 = H_2$ has vertex set $V(G_1) \underset{H_1=H_2}{\cup} V(G_2)$ and edge set $E(G_1) \underset{H_1=H_2}{\cup} E(G_2)$.

When H_1 and H_2 each induce graph F , we write $G_1 \underset{F}{\cup} G_2$. When $F = K_1 = \{v\}$, we write $G_1 \underset{v}{\cup} G_2$. When $F = K_2 = \langle e \rangle$, we write $G_1 \underset{e}{\cup} G_2$.

It is easily seen that the overlap operation is associative and commutative. Any decomposition of a graph G into subgraphs G_1, \dots, G_n can be expressed as an overlap of these subgraphs with each pair of identified subgraphs edge-disjoint.

1.2 k-Shells

We have seen that the cores of a graph are nested. This in turn can be used to define a decomposition of a graph into subgraphs defined based on those parts of the graph contained in one core and not in the next higher number core.

Definition 20. For $k > 0$, the k -shell of a graph G , $S_k(G)$, is the subgraph of G induced by the edges contained in the k -core and not contained in the $k+1$ -core. For $k = 0$, the 0-shell of G is the vertices of the 0-core not contained in the 1-core.

Thus the 0-shell is simply the set of isolated vertices of G . If $M = \widehat{C}(G)$, the M -shell of G is just the maximum core of G . The k -shells of G form a decomposition of G , indeed they were defined to do so.

If G has no k -shell, we say G is k -shell-free. G is k -shell-free exactly when its k -core equals its $k+1$ -core. This includes when it is k -core-free.

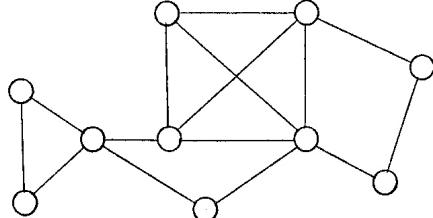
Unless each k -shell is a separate component or components of G , the shells of G will have some vertices in common.

Definition 21. The k -boundary of G , $B_k(G)$, is the set of vertices contained in both the k -shell and the $k+1$ -core.

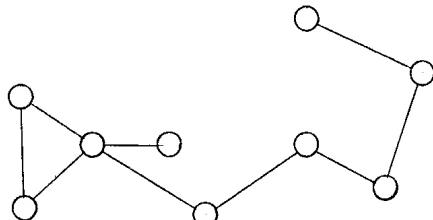
Thus a vertex is contained in the k -boundary exactly when it is contained in the $k+1$ -core and adjacent to a vertex in the k -core.

Sometimes it is convenient to exclude the boundary when considering the shell.

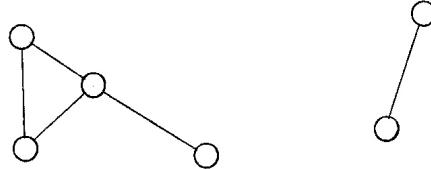
Definition 22. The proper k -shell of G , $S'_k(G)$, is the subgraph of G induced by the non-boundary vertices of the k -shell. The order of the k -shell of G is defined to be the order of the proper k -shell.



A graph G .



The 2-shell of G .



The proper 2-shell of G .

Figure 5: The 2-shell and proper 2-shell of G .

Thus the vertices of the proper k -shells partition the vertex set of G . A vertex has core number k if and only if it is contained in the proper k -shell of G . Thus the proper k -shell is induced by the vertices with core number k .

Figure 5 shows a graph G and its 2-shell and proper 2-shell.

Note that the proper k -shell was called the k -remainder of G by Seidman [53] in the 1983 paper that introduced k -cores. That term does not appear to have been used since.

We would like to know which graphs can be k -shells.

Theorem 23. A graph F with vertex subset B can be a k -shell of a graph with boundary set B if and only if no component of F has vertices entirely in B , $\delta(B) \geq 1$, $\delta_F(V(F) - B) = k$, and F contains no subgraph H with $\delta_H(V(H) - B) \geq k + 1$.

Proof. (\Rightarrow) Let F be a k -shell of graph G with boundary set B . If any component of F had all vertices in B , it would be contained in the $k + 1$ -core of G . F is induced by edges, so $\delta(B) \geq 1$. If a vertex v in F and not in B had $d(v) < k$, it would not be in the k -core of G . If F had such a subgraph H , it would be contained in the $k + 1$ -core of G .

(\Leftarrow) Let F be a graph satisfying these conditions. Overlap each vertex in B with a distinct vertex of a $k + 1$ -core G with sufficiently large order. Then F is the k -shell of the resulting graph.

□

The 1-shell of a graph can be characterized in terms of a familiar class of graphs.

Corollary 24. The 1-shell of G , if it exists, is a forest with no trivial components and at most one boundary vertex per component.

Proof. F is acyclic, $\delta(F) = 1$, and two boundary vertices in a tree are connected by a path, which would be in the 2-core.

□

We can also characterize graphs that can be proper k -shells. Certainly such a graph cannot contain a $k + 1$ -core. This obvious necessary condition is also sufficient.

Theorem 25. A graph F can be a proper k -shell if and only if F does not contain a $k + 1$ -core.

Proof. The forward direction is obvious.

Let F be a graph that does not contain a $k + 1$ -core. Let M be a $k + 1$ -core. For each vertex v in F , let $a(v) = \max\{k - d(v), 0\}$. For each vertex v , take $a(v)$ copies of M and link each to v by an edge between v and a vertex in M . The resulting graph G has F as its proper k -shell.

□

The construction used in this proof is by no means unique.

Corollary 26. *A graph F can be a proper 1-shell if and only if F is a forest.*

We can determine sharp bounds for the size of a k -shell.

Proposition 27. *The size m of a k -shell with order n satisfies $\lceil \frac{k \cdot n}{2} \rceil \leq m \leq k \cdot n$.*

Proof. The non-boundary vertices of the k -shell of G can be successively deleted so that when deleted, they have degree at most k . Thus $m \leq k \cdot n$.

The non-boundary vertices have degree at least k , so there are at least $\frac{k \cdot n}{2}$ edges.

□

The lower bound is sharp for all k . For k or n even, the extremal graphs have every component k -regular, and no vertices adjacent to the $k + 1$ -core, if it exists. For k and n both odd, the extremal graphs have a single component with one vertex of degree $k + 1$ and all others of degree k , and no vertices adjacent to the $k + 1$ -core, if it exists.

The upper bound is sharp for all $k < \hat{C}(G)$. The extremal graphs have vertices having degree exactly k when deleted, so they can be constructed by reversing this process. Thus they must have at least k boundary vertices. For $k = \hat{C}(G)$, we will see in Theorem 58 that the maximum core can have size at most $k \cdot n - \binom{k+1}{2}$, since

deleting the vertices eventually restricts the number of available neighbors. Thus we have the following corollary.

Corollary 28. *Let s_k be the order of the k -shell of G , $0 \leq k \leq r = \widehat{C}(G)$. Then the size m of G satisfies*

$$\sum_{k=1}^r \left\lceil \frac{k \cdot s_k}{2} \right\rceil \leq m \leq \sum_{k=1}^r k \cdot s_k - \binom{k+1}{2}.$$

Proof. Sum the lower and upper bounds over all k . The 0-shell must have size 0, and the maximum core has the previously stated upper bound. The result follows. \square

Both upper and lower bounds are sharp. The extremal graphs have each k -shell extremal, as above.

The bound on the size of a k -shell can be improved by considering the number of boundary vertices.

Proposition 29. *The size m of a k -shell with order n and b boundary vertices satisfies*

$$\left\lceil \frac{k \cdot n + b}{2} \right\rceil \leq m \leq k \cdot n - \binom{k-b+1}{2}.$$

Proof. When deleted, the i^{th} to last vertex can have degree at most $b+i-1$. Thus the upper bound must be reduced by $\sum_{i=1}^{k-b} i = \frac{(k-b)(k-b+1)}{2} = \binom{k-b+1}{2}$. The boundary vertices each contribute degree at least one to the lower bound. The result follows. \square

The lower bound is sharp for all k . If $k \cdot n + b$ is even, then every component of the extremal graphs connected to the $k+1$ -shell can be formed in the following way. Start

with a connected k -regular graph with an edge cut of b_i edges. Take the half of the graph on one side of the edge cut and add edges joining the vertices adjacent to the cut to b_i boundary vertices. Add enough components so that the number of boundary vertices and non-boundary vertices sum to the appropriate values. If $k \cdot n + b$ is odd, the construction is similar, but there must be one vertex with degree $k + 1$.

The upper bound is sharp for all k , and the extremal graphs are graphs whose vertices have the maximum possible degree for deletion at each step.

Corollary 30. *Let s_k be the order of the k -shell of G and b_k be the order of the k -boundary of G , $0 \leq k \leq r = \widehat{C}(G)$. Then the size m of G satisfies*

$$\sum_{k=1}^r \left\lceil \frac{k \cdot s_k + b_k}{2} \right\rceil \leq m \leq \sum_{k=1}^r \left(k \cdot s_k - \binom{k - b_k + 1}{2} \right).$$

Proof. This is similar to the previous corollary. □

Note that the maximum core must have no boundary vertices.

1.3 The Structure of k-Cores

We would like to understand the structure of k -cores. We have already seen several structural results. The k -core of G is the union of the k -cores of the components of G . In particular, if G is a k -core, then every component of G is also a k -core. Hence we can naturally restrict our attention to connected graphs.

Each component of $C_k(G)$ has order at least $k + 1$, and any order $p \geq k + 1$ can be achieved. For example, $G = K_p$ is a k -core. The unique k -core of order $k + 1$ is K_{k+1} .

We now consider the relationship between subgraphs and k -cores.

Proposition 31. *If $H \subseteq G$, then $C_k(H) \subseteq C_k(G)$. This is an equality exactly when the k -core of G is contained in H .*

This follows immediately from the definition of k -core.

Corollary 32. *Let G and H be graphs which may overlap. Then*

$$C_k(G \cap H) \subseteq C_k(G) \cap C_k(H)$$

$$C_k(G \cup H) \supseteq C_k(G) \cup C_k(H).$$

We would like to characterize the structure of k -cores. We have already seen that G is its own 0-core. The 1-core of G is formed by deleting any isolated vertices of G .

1.3.1 2-Cores

The structure of the 2-core of a graph is less trivial. The following result was observed

by Seidman [53].

Proposition 33. *If G is connected, then its 2-core is connected.*

Proof. Let G be connected, and $u, v \in C_2(G)$. Then there is a $u - v$ path in G . The vertices on the path all have degree at least two, and all are adjacent to at least two vertices in a set with minimum degree two, since u and v are in the 2-core of G . Thus the $u - v$ path is in the 2-core of G , so it is connected.

□

The corresponding results are trivial for the 0-core and 1-core. They are false for $k \geq 3$. For example, joining two vertices of two $k + 1$ -cliques by a path of length at least two yields a connected graph with a disconnected k -core for $k \geq 3$.

One way to characterize 2-cores is with a local characterization. That is, describing the structure of G 'near' an arbitrary vertex v .

Theorem 34. *A vertex v of G is contained in the 2-core of G if and only if v is on a cycle or v is on a path between vertices of distinct cycles.*

Proof. (\Leftarrow) Let v be on a cycle or a path between vertices of distinct cycles. Both such graphs are themselves 2-cores, so v is in the 2-core of G .

(\Rightarrow) Let v be in the 2-core of G . If v is on a cycle, we are done. If not, then consider a longest path P in the 2-core through v . All the edges incident with v must be bridges, so v is in the interior of P . An end-vertex u of P must have another neighbor, which cannot be a new vertex, so it must be on P . If its neighbor were on the opposite side of v , then v would be on a cycle. Thus its neighbor must be between u and v on P . Repeating this argument for the other end of P shows that v is on a path between vertices on cycles. (See Figure 6.)

□



Figure 6: Vertex v is on a path between cycles.

This characterization does not extend easily to higher values of k . The key to the local characterization for the 2-core is the fact that every 2-core contains a simple subgraph that is itself a 2-core. But as we shall see later on, there are arbitrarily large k -cores that do not contain any proper subgraph which is a k -core for $k \geq 3$.

It is also possible to offer a more global characterization of the structure of 2-cores.

Corollary 35. *A graph G is a 2-core \iff every end-block of G is 2-connected.*

Proof. If every end-block of G is 2-connected, then every vertex of G is either on a cycle or a path between cycles. Thus G is a 2-core. If some end-block of G is not 2-connected, then it is K_2 , so G has a vertex of degree one and is not a 2-core. \square

This leads to another corollary.

Definition 36. A block-tree decomposition of a 2-core G is a decomposition of G into 2-connected blocks and trees so that if T is nontrivial, each end-vertex of T is shared with a distinct 2-connected block, if T is trivial, it is a cut-vertex of at least two 2-connected blocks, and there are no two disjoint paths between two distinct blocks.

Corollary 37. *Every 2-core has a unique block-tree decomposition.*

Proof. Let F be the subgraph of a 2-core G induced by the bridges and cut-vertices of G . Then F is acyclic, so it is a forest. Break each component of F into branches at any vertex contained in a component of $G - F$. Also break $G - F$ into blocks, which must be 2-connected. By the previous corollary, each end-vertex of each of the trees must overlap a 2-connected block. If any block contained two end-vertices of the same tree, then there would be a cycle containing edges from the tree. If there were two disjoint paths between two blocks, they would not be distinct. This decomposition is unique because the block decomposition of a graph is unique and any blocks that are K_2 and on a path between 2-connected blocks that does not go through any other 2-connected blocks must be in the same tree.

□

These corollaries provide no help when the 2-core in question is itself 2-connected. But there is a well-established description of the structure of 2-connected graphs. An ear of G is a maximal path of G whose internal vertices have degree two. An ear decomposition of G is a decomposition of G into ears and one cycle.

Theorem 38. [Whitney, see West p. 163] [60] A graph is 2-connected \iff it has an ear decomposition. Every cycle is the cycle in some ear decomposition.

There is also a description of the structure of minimally 2-connected graphs.

Theorem 39. [Bollobas p. 15] [7] Let G be a minimally 2-connected graph that is not a cycle. Let $D \subset V(G)$ be the set of vertices of degree two. Then $F = G - D$ is a forest with at least two components. Each component P of $G[D]$ is a path and the end-vertices of P are not joined to the same tree of the forest F .

Corollary 40. A graph G which is not a cycle is minimally 2-connected \iff it has an

ear decomposition with each path of length at least 2, no ear joined to vertices in a single component of F , and no ear connects or creates a cycle in F .

Proof. (\Rightarrow) Let G be minimally 2-connected. Then G has an ear decomposition. A path of length one in the ear decomposition would be an essential edge. So would an edge between vertices in a component of F that are the ends of an ear. The final condition is implied by the second theorem.

(\Leftarrow) Assume the hypothesis. The ear decomposition implies that G is 2-connected. Adding the first ear makes F disconnected, and adding subsequent ears keep it a forest. The ears must connect different components of F . By the previous theorem, G is minimal.

□

We can state an operation characterization of 2-cores. An operation characterization is a rule or rules that can be used to construct all graphs in some class of graphs.

Theorem 41. *A graph G is a connected 2-core \iff it is contained in the set S whose members can be constructed by the following rules.*

1. All cycles are in S .
2. Given one or two graphs in S , the result of joining the ends of a (possibly trivial) path to it or them is in S .

Proof. (\Leftarrow) A cycle has minimum degree 2, and applying step 2 does not create any vertices of lower degree, so a graph in S is a 2-core.

(\Rightarrow) This is clearly true if G has order 3. Assume the result holds for orders up to r , and let G have order $r + 1$. Let P be an ear or cut-vertex of G . Making $P = K_2$ is only necessary when G has minimum degree at least 3 and is 2-connected. In this case, edges can be deleted until one of these conditions fails to hold. Then if P has

internal vertices, deleting them results in a component or components with order at most r . The same is true if P is a cut-vertex, and G is split into blocks. Then the result follows by induction.

□

We can also describe 2-monocore graphs by an operation characterization.

Theorem 42. *The set of connected 2-monocore graphs is equivalent to the set S of graphs that can be constructed using the following rules.*

1. All cycles are in S .
2. Given one or two graphs in S , the graph H formed by identifying the ends of a path of length at least two with vertices of the graph or graphs is in S .
3. Given a graph G in S , form H by taking a cycle and either identifying a vertex of the cycle with a vertex of G or adding an edge between one vertex in each.

Proof. (\Leftarrow) We first show that if G is in S , then G is 2-monocore. Certainly cycles are 2-monocore. Let H be formed from G in S by applying rule 2. Then H has minimum degree 2 and since G is 3-core-free and internal vertices of the path have degree 2, H is also 3-core-free. Thus H is 2-monocore. The same argument works for adding a path between two graphs. Let H be formed from G in S by applying rule 3. Then H has minimum degree 2 and since G is 3-core-free and all but one vertex of the cycle have degree 2, H is also 3-core-free. Thus H is 2-monocore.

(\Rightarrow) We now show that if G is 2-monocore, it is in S . This clearly holds for all cycles, including C_3 , so assume it holds for all 2-monocore graphs of order up to r . Let G be 2-monocore of order $r + 1$ and not a cycle. Then G has minimum degree 2, so it has a vertex v of degree 2. Then v is contained in P , an ear of length at least 2, or C , a cycle which has all but one vertex of degree 2.

Case 1. G has an ear P . If $G - P$ is disconnected, then the components of G are 2-monocore, and hence in S . Then G can be formed from them using rule 2, so G is

in S . If $G - P$ is connected, then it is 2-monocore, and hence in S . Then G can be formed from $G - P$ using rule 2, so G is in S .

Case 2. We may assume that G has no such ear P . Then G has a cycle C with all but one vertex of degree 2, and one vertex u of degree more than 2. If u has degree at least 4 in G , then let H be formed by deleting all the vertices of C except u . Then H is 2-monocore, and G can be formed from it using rule 3. If $d(u) = 3$, then its neighbor not in the cycle has degree at least three, so $G - C$ is 2-monocore, and G can be formed from it by using rule 3.

□

Many 2-monocore graphs have a nice decomposition.

Corollary 43. *Every 2-monocore graph for which the latter operation in rule 3 is not needed for its construction can be decomposed into induced cycles and paths of length at least 2 which are induced except possibly for an edge joining end-vertices.*

Proof. The construction yields a decomposition into cycles and paths. It is not possible to add a chord to a cycle or join the two vertices on a path.

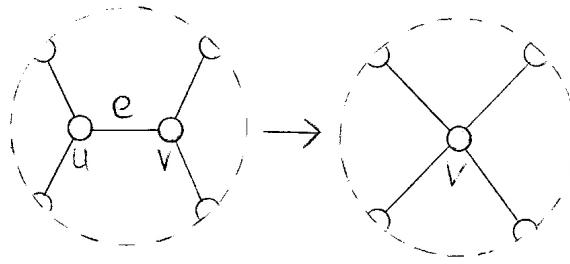
□

The converse to this corollary does not hold, since, for example, $K_{1,4}$ and W_4 can both be decomposed into such paths and cycles, but neither is 2-monocore.

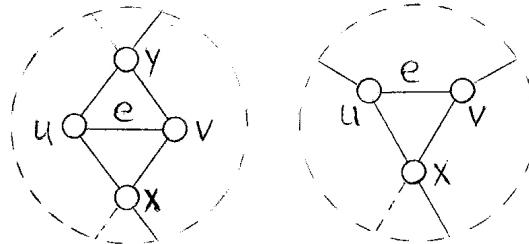
We can similarly describe the structure of 2-shells.

Corollary 44. *The set of 2-shells is equivalent to the set S' of graphs constructed using the following rules.*

1. All graphs in set S from Theorem 42 and all 3-cores are in S' .
2. Given one or two graphs in S' , the graph H formed by identifying the ends of a path of length at least two with vertices of the graph or graphs is in S' .



Case 1.



Cases 2 and 3.

Figure 7: The cases of Theorem 45.

3. Given a graph G in S' , form H by taking a cycle and either identifying a vertex of the cycle with a vertex of G or adding an edge between one vertex in each.

Finally, delete the 3-cores (keeping boundary vertices) last.

The proof is essentially the same as that of the previous theorem.

1.3.2 3-Cores

Describing the structure of 3-cores is more difficult, but we do have the following theorem, which has been observed by Dirac (see [West p. 218] [60]). A minor of a graph is a graph that can be formed by contracting edges and deleting edges or vertices.

Theorem 45. Every 3-core has K_4 as a minor.

Proof. The 3-core of smallest order is K_4 , for which the theorem certainly holds. Assume the theorem holds for all 3-cores of order between 4 and r , and let G be a 3-core

of order $r + 1$. Further, every 3-core contains a subgraph which has no edge join vertices of degree more than 3, so let this be true of G as well.

Case 1. G has an edge $e = uv$ so that u and v have no common neighbors. In this case, contracting e yields a vertex of degree at least 4, so G/e is a 3-core. (See Figure 7.)

Case 2. Vertices u and v have two common neighbors x and y . If G has an $x - y$ path not through u or v , it has K_4 as a minor. If not, then if one of x or y has degree at least 4 then contract $\{u, v, x, y\}$ to a single vertex. Otherwise, let z be the other neighbor of x , and contract $\{u, v, x, y, z\}$ to a single vertex. The result is a 3-core.

Case 3. Vertices u and v have exactly one common neighbor x , and no pair of vertices has more than one common neighbor. Then vertices u, v, x , have at least three distinct neighbors and no common neighbors. Contracting $\{u, v, x\}$ to a single vertex results in a 3-core.

Thus G has a minor with smaller order which is a 3-core, so by induction it has K_4 as a minor.

□

This theorem cannot be extended to larger values of k . For the octahedron, $K_{2,2,2}$, is a 4-core but it does not have K_5 as a minor, since contracting any edge yields $K_5 - e$. It is not clear whether there is a larger finite set of minors such that any 4-core must contain at least one of them.

However, it can be extended as follows. The proof of the following theorem follows that for Wagner's Theorem, which states that a graph is planar if and only if it does not contain a K_5 or $K_{3,3}$ minor.

Theorem 46. *Let H be a graph with $\Delta(H) \leq 3$. Then G has H as a minor $\iff G$ has a subdivision of H .*

Proof. Certainly if G has a subdivision of a graph, it has that graph as a minor, since

the subdivided edges can be contracted. Suppose G has H as a minor. Then G contains sets of vertices S_i which contract to the vertices v_i of H . Thus $G[S_i]$ is connected, and there is an edge between vertices in S_i and S_j if there is an edge $v_i v_j$ in H .

If $d(v_i) = 1$, there is nothing to prove. If $d(v_i) = 2$, then S_i has two vertices x and y , possibly the same, on edges to other sets S . Since $G[S_i]$ is connected, there is an $x - y$ path in it (possibly trivial), so pick x to stand for v_i in H . If $d(v_i) = 3$, then S_i has three vertices x, y and z , not necessarily distinct, on edges to other sets S . Since $G[S_i]$ is connected, there is an $x - y$ path in it, and a path between z and some vertex u on the first path. Define u to be v_i in H .

Thus G has a subdivision of H .

□

This cannot be extended to maximum degree 4. For example, the Petersen graph has K_5 as a minor, which can be formed by contracting any perfect matching, but it has no K_5 subdivision. If G has a minor H with a vertex of degree 4, there are two distinct trees (up to subdivision) in G that could produce it.

Corollary 47. *Every 3-core contains a subdivision of K_4 .*

This can be pushed a bit further.

Corollary 48. *Every end-block of a 3-core contains a subdivision of K_4 .*

Proof. A subdivision of K_4 cannot contain a cut-vertex, so it must be contained in some block of a 3-core. Form a graph with two copies of an end-block of a 3-core by identifying their unique cut-vertices. The graph that results is a 3-core, so it has a subdivision of K_4 in a block.

□

1.3.3 Parameters of k-Cores

Graphs with minimum degree at least k have been studied in relation to a variety of graphical properties. Some of these results are of interest.

The connectivity of a k -core can be related to its order.

Proposition 49. [Chartrand/Lesniak p. 72] [15] Let G be a k -core of order n and $1 \leq l \leq n - 1$. If $k \geq \lceil \frac{n+l-2}{2} \rceil$, then G is l -connected.

Proof. Assume the hypothesis. Then the sum of the degrees of any two nonadjacent vertices of G is at least $n + l - 2$, so they have at least l common neighbors. Thus G is l -connected.

□

The result's hypothesis is equivalent to $n \leq 2k - l + 2$.

Corollary 50. Let G be a k -core with order n . If $k+1 < n < 2k+2$, then $\text{diam}(H) = 2$.

Proof. Assume the hypothesis. Since $k < n - 1$, G is not complete, so its diameter is at least 2. By the previous result, any pair of nonadjacent vertices has a common neighbor since $n \leq 2k - 1 + 2$. Thus $\text{diam}(G) = 2$.

□

We can also examine when a k -core contains a clique. The following theorem is closely related to Turan's theorem, which characterizes the extremal graph of a certain order that does not contain a clique of some size.

Theorem 51. [Chartrand/Lesniak page 294] [15] Let $n \geq r \geq 2$. Then every graph of order n and size at least $\lfloor (\frac{r-2}{2r-2}) n^2 \rfloor + 1$ contains K_r as a subgraph.

Corollary 52. [Seidman 1983] [53] A k -core with order n must contain a clique K_r as a subgraph if $n < \left(\frac{r-1}{r-2}\right)k$.

Proof. Let H be a k -core with order $n < \left(\frac{r-1}{r-2}\right)k$. Then $k > \left(\frac{r-2}{r-1}\right)n$, so H has size m with

$$m \geq \frac{n \cdot k}{2} > \left(\frac{r-2}{2r-2}\right)n^2 \geq \left\lfloor \left(\frac{r-2}{2r-2}\right)n^2 \right\rfloor.$$

Thus $m \geq \left\lfloor \left(\frac{r-2}{2r-2}\right)n^2 \right\rfloor + 1$, so by the previous theorem, H contains K_r as a subgraph.

□

It is easy to show algebraically that the bound in the previous theorem is equivalent to $r < \frac{k}{n-k} + 2$.

In the original paper of k -cores, Seidman's [53] most difficult result is this theorem relating diameters and connectivity of k -cores.

Theorem 53. Let H be a connected k -core with order $n \geq 2k + 2$ and connectivity l , then

$$\text{diam}(H) \leq 3 \left\lfloor \frac{p - 2k - 2}{\beta} \right\rfloor + b(n, k, l) + 3$$

where $\beta = \max\{k + 1, 3l\}$ and r is the element of $\{0, \dots, \beta - 1\}$ such that $r \equiv n - 2k - 2 \pmod{\beta}$ and

$$b(n, k, l) = \begin{cases} 0 & 0 \leq r < l \\ 1 & l \leq r < 2l \\ 2 & 2l \leq r \end{cases}.$$

This has the following corollary for when the connectivity is unknown.

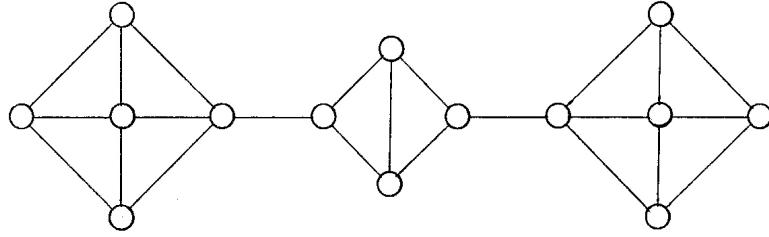


Figure 8: A 3-core with order 14 and diameter 8.

Corollary 54. [Moon 1965] [45] If H has order $n \geq 2k + 2$, then

$$\text{diam}(H) \leq 3 \left\lfloor \frac{n}{k+1} \right\rfloor + a(p, k) - 3,$$

where

$$a(p, k) = \begin{cases} 0 & p \equiv 0 \pmod{k+1} \\ 1 & p \equiv 1 \pmod{k+1} \\ 2 & \text{else} \end{cases}.$$

This bound is essentially the best possible. The graphs that achieve the bound can be described as follows. They have an odd number of blocks arranged to form a path. Alternating blocks forming the smaller partite set of the path are bridges. The other internal blocks are $K_{k+1} - e$, where $e = uv$ and u, v are the cut-vertices of G . The two end-blocks are $K_{k+2} - \lfloor \frac{k+2}{2} \rfloor K_2$. (See Figure 8.)

It is possible to describe the subtrees contained in a k -core.

Theorem 55. [West p. 70] [60] All trees of size k are contained as a subgraphs in any k -core.

The proof of this result follows by an easy induction argument.

2 Extremal Graph Classes

2.1 Maximal k-Degenerate Graphs

Recall that a k -core-free graph is a graph that contains no k -core. This section will study the properties of such graphs. In particular, we will study graphs that are maximal k -core-free.

Definition 56. A maximal k -core-free graph G is a graph that is k -core-free and is maximal with respect to this property. That is, no more edges can be added to G without creating a k -core.

Recall that a graph is k -degenerate if its vertices can be successively deleted so that when deleted, each has degree at most k , and the degeneracy of a graph G is the smallest k such that G is k -degenerate. We have seen that the degeneracy of G is equivalent to the maximum core number of G . By the k -core algorithm, we have the following.

Corollary 57. *A graph G is k -degenerate if and only if G is $k+1$ -core-free. Maximal k -degenerate graphs are equivalent to maximal $k+1$ -core-free graphs.*

The term k -degenerate was introduced in 1970 by Lick and White [38]; the concept has been introduced under other names both before and since. We will prefer the term k -degenerate since it is reasonably standard and helps to simplify formulas, while also using k -core-free when convenient.

2.1.1 Basic Properties

Our examination of k -degenerate graphs will focus on maximal k -degenerate graphs.

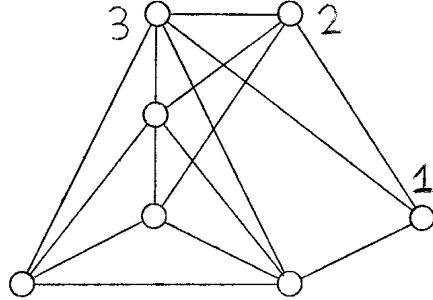


Figure 9: A maximal 3-degenerate graph, which can be seen by successively deleting 1, 2, and 3.

Most of the properties given can be generalized with appropriate modification to all k -degenerate graphs. Our first result is the size of a maximal k -degenerate graph.

Theorem 58. *The size of a maximal k -degenerate with order n is $k \cdot n - \binom{k+1}{2}$.*

Proof. If G is k -degenerate, then its vertices can be successively deleted so that when deleted they have degree at most k . Since G is maximal, the degrees of the deleted vertices will be exactly k until the number of vertices remaining is at most k . After that, the $n - j^{th}$ vertex deleted will have degree j . Thus the size m of G is

$$m = \sum_{i=0}^{k-1} i + \sum_{i=k}^{n-1} k = \frac{k(k-1)}{2} + k(n-k) = k \cdot n + \frac{k(k-1)}{2} - \frac{2k^2}{2} = k \cdot n - \binom{k+1}{2}$$

□

Thus for k -core-free graphs, maximal and maximum are equivalent.

Corollary 59. *Every graph with order n , size $m \geq (k-1)n - \binom{k}{2} + 1$, $1 \leq k \leq n-1$, has a k -core.*

The basic properties of maximal k -core-free graphs were established by Lick and White [38] and Mitchem [1977] [43].

Theorem 60. *Let G be a maximal k -degenerate graph of order n , $1 \leq k \leq n - 1$. Then*

- a. *G contains a $k + 1$ -clique and for $n \geq k + 2$, G contains $K_{k+2} - e$ as a subgraph.*
- b. *For $n \geq k + 2$, G has $\delta(G) = k$, and no two vertices of degree k are adjacent.*
- c. *G has connectivity $\kappa(G) = k$.*
- d. *Given r , $1 \leq r \leq n$, G contains a maximal k -degenerate graph of order r as an induced subgraph. For $n \geq k + 2$, if $d(v) = k$, then G is maximal k -degenerate if and only if $G - v$ is maximal k -degenerate.*
- e. *G is maximal 1-degenerate if and only if G is a tree.*

In fact, maximal k -degenerate graphs are one type of generalization of trees.

Several corollaries follow immediately from these basic results. A trivial edge cut is an edge cut such that all the edges are incident with one vertex.

Corollary 61. *Let G be a maximal k -degenerate graph of order n , $1 \leq k \leq n - 1$. Then*

- a. *For $k \geq 2$, the number of nonisomorphic maximal k -degenerate graphs of order $k + 3$ is 3.*
- b. *G is k -monocore.*
- c. *G has edge-connectivity $\kappa'(G) = k$, and for $k \geq 2$, an edge set is a minimum edge cut if and only if it is a trivial edge cut.*
- d. *The number of maximal k -degenerate subgraphs of order $n - 1$ is equal to the number of vertices of degree k in G that are in distinct automorphism classes.*

Proof. a. $K_{k+2} - e$ is the unique maximal k -degenerate graph of order $k + 2$. It has two automorphism classes of vertices, one with two, one with k . Thus there are three possibilities for order $k + 3$.

- b. G has minimum degree k , and is $k + 1$ -core-free.
- c. First, $k = \kappa(G) \leq \kappa'(G) \leq \delta(G) = k$. Certainly the edges incident with a vertex of minimum degree form a minimum edge cut. The result holds for K_{k+1} . Assume the result holds for all maximal k -core-free graphs of order r , and let G have order $r + 1$, $v \in G$, $d(v) = k$, $H = G - v$. Let F be a minimum edge cut of G . If $F \subset E(H)$, the result holds. If F is a trivial edge cut for v , the result holds. If F contained edges both from H and incident with v , it would not disconnect H and would not disconnect v from H .
- d. Deleting any minimum degree vertex yields such a subgraph, and deleting any other vertex destroys maximality. The subgraphs will be distinct unless two minimum degree vertices are in the same automorphism class.

□

2.1.2 Degree Sequences

We can characterize the degree sequences of maximal k -degenerate graphs. A different and somewhat inelegant characterization with a longer proof was offered by Borowiecki, Ivanco, Mihok, and Semanisin [1995] [10].

Lemma 62. *Let G be maximal k -degenerate with order n and nonincreasing degree sequence d_1, \dots, d_n . Then $d_i \leq k + n - i$.*

Proof. Assume to the contrary that $d_i > k + n - i$ for some i . Let H be the graph formed by deleting the $n - i$ vertices of smallest degree. Then $\delta(H) > k$, so G has a $k + 1$ -core.

□

Lemma 63. *Let G be maximal k -degenerate with degree sequence $d_1 \geq \dots \geq d_n = k$. Then G has at most $k + 1$ vertices whose degrees are equal to the upper bound $\min\{n - 1, k + n - i\}$, one of which is v_n , and has exactly $k + 1$ such vertices if and*

only if v_n has the other k as its neighborhood.

Proof. If G had more than $k + 1$ such vertices, then $H = G - v_n$ would have a vertex with degree more than than the maximum possible. If G has exactly $k + 1$, then all but v_n must have degree reduced by exactly one in H when v_n is deleted.

□

Theorem 64. *A nonincreasing sequence of integers d_1, \dots, d_n is the degree sequence of a maximal k -degenerate graph G if and only if $k \leq d_i \leq \min\{n - 1, k + n - i\}$ and $\sum d_i = 2[k \cdot n - \binom{k+1}{2}]$ for $0 \leq k \leq n - 1$.*

Proof. Let d_1, \dots, d_n be such a sequence.

(\Rightarrow) Certainly $\Delta(G) \leq n - 1$. The other three conditions have already been shown.
 (\Leftarrow) For $n = k + 1$, the result holds for $G = K_{k+1}$. Assume the result holds for order r . Let d_1, \dots, d_{r+1} be a nonincreasing sequence that satisfies the given properties. Let d'_1, \dots, d'_r be the sequence formed by deleting d_{r+1} and decreasing k other numbers greater than k by one, including any that achieve the maximum. (There are at most k by the preceding lemma.) Then the new sequence satisfies all the hypotheses and has length r , so it is the degree sequence for some maximal k -degenerate graph H . Add vertex v_{r+1} to H , making it adjacent to the vertices with degrees that were decreased for the new sequence. Then the resulting graph G has the original degree sequence and is maximal k -degenerate.

□

The numbers of vertices of different degrees are related.

Proposition 65. *Let G be maximal k -degenerate with $\Delta(G) = r$, $n \geq k + 1$, and n_i the number of vertices of degree i , $k \leq i \leq r$. Then*

$$k \cdot n_k + (k - 1) n_{k+1} + \dots + 2n_{2k-2} + n_{2k-1} = n_{2k+1} + 2n_{2k+2} + \dots + (r - 2k) n_r + k(k + 1).$$

Proof. Assume the hypothesis, and let G have order n , size m . Then $\sum_{i=1}^r n_i = n$ and

$$\sum_{i=k}^r i \cdot n_i = 2m = 2 \left[k \cdot n - \binom{k+1}{2} \right] = (2k) \sum_{i=k}^r n_i - k(k+1).$$

Thus

$$\sum_{i=k}^r (i - 2k) n_i + k(k+1) = 0.$$

□

Corollary 66. *If T is a tree,*

$$n_1 = n_3 + 2n_4 + 3n_5 + \dots + (r-2)n_r + 2.$$

For $k = 2$,

$$2n_2 + n_3 = n_5 + 2n_6 + 3n_7 + \dots + (r-4)n_r + 6.$$

We can bound the maximum degree of a maximal k -degenerate graph. Intuitively, since there are approximately $k \cdot n$ edges in G , its maximum degree should be at least $2k$, provided that G has order large enough to overcome the constant $\binom{k+1}{2}$ subtracted from the size.

The following result was first proven by Filakova, Mihok, and Semanisin [1997] [25] using contradiction. We present a direct proof.

Theorem 67. *If G is maximal k -degenerate with $n \geq \binom{k+2}{2}$, then $\Delta(G) \geq 2k$.*

Proof. Certainly the maximum degree of a graph must be at least as large as the average of any of the degrees. By Lemma 62, the i^{th} smallest degree is at most $k + n - i$. We want the average of the largest $n - k$ degrees to be larger than $2k - 1$. In that case,

$$\begin{aligned}
& \frac{1}{n-k} \left[2 \left(k \cdot n - \binom{k+1}{2} \right) - \sum_{i=k}^{2k-1} i \right] > 2k - 1 \\
& \frac{1}{n-k} \left[(2nk - k^2 - k) - \left(\frac{1}{2} (2k-1)(2k) - \frac{1}{2} k(k-1) \right) \right] > 2k - 1 \\
& 2(2nk - k^2 - k) - (4k^2 - 2k) + (k^2 - k) > 2(2k-1)(n-k) \\
& 4nk - 2k^2 - 2k - 4k^2 + 2k + k^2 - k > 4nk - 4k^2 - 2n + 2k
\end{aligned}$$

$$2n > k^2 + 3k$$

Thus $n \geq \binom{k+2}{2}$. In each step, the logical implication is reversible, so the result follows from this condition.

□

This result is the best possible in two senses. First, no larger lower bound for the minimum degree can be guaranteed, regardless how large the order is. Second, the hypothesis on n is the smallest that guarantees the result. This can be seen by constructing a maximal k -degenerate graph so that when added, each new vertex is made adjacent to the k vertices of smallest degree.

2.1.3 Further Structural Results

We can bound the diameter of a maximal k -degenerate graph.

Theorem 68. *A maximal k -degenerate graph G with $n \geq k+2$ has $2 \leq \text{diam}(G) \leq \frac{n-2}{k} + 1$.*

If the upper bound is an equality, then G has exactly two vertices of degree k and every diameter path has them as its endpoints.

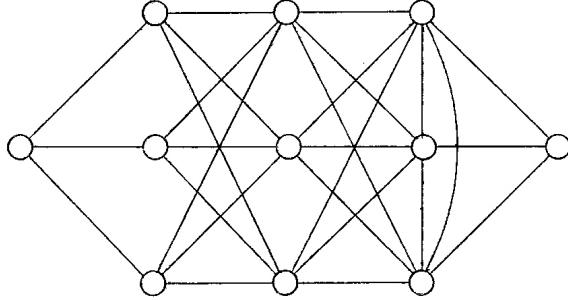


Figure 10: A maximal 3-degenerate graph with diameter 4 and order 11.

Proof. Let G be maximal k -degenerate with $r = \text{diam}(G)$. For $n \geq k+2$, G is not complete, so $\text{diam}(G) \geq 2$. Now G contains u, v with $d(u, v) = r$. Now G is k -connected, so by Menger's Theorem there are at least k independent paths of length at least r between u and v . Thus $n \geq k(r-1)+2$, so $r \leq \frac{n-2}{k} + 1$.

Let the upper bound be an equality, and $d(u, v) = r$. Then $n = k(r-1)+2$, and since there are k independent paths between u and v , all the vertices are on these paths. Thus $d(u) = d(v) = k$. If another vertex w had degree k , then $G - w$ would be maximal k -degenerate with $\kappa(G - w) = k - 1$, which is impossible. Thus any other pair of vertices has distance less than r .

□

The lower bound is sharp. For example, the graph $K_{k-1} + \overline{K}_{n-k+1}$ has diameter 2. The upper bound is sharp for all k . For $k = 1$, the unique extremal graph is P_{d+1} . In general, form a graph as follows. Establish a $k \times r-1$ grid of vertices. Add the edges between vertices $v_{i,j}$ and $v_{s,t}$ if $t = j+1$ or $t = j-1$. (Thus we have a graph that decomposes into $r-2$ copies of $K_{k,k}$ and one K_k .) Finally, add a vertex u adjacent to $v_{i,1}$ for all i and a vertex v adjacent to $v_{i,r-1}$ for all i . It is easily checked that this graph is maximal k -degenerate. (See Figure 10.)

Maximal k -core-free graphs have some interesting decompositions.

Theorem 69. *Let t_1, \dots, t_r be r positive integers which sum to t . Then a maximal t -degenerate graph can be decomposed into r graphs with degeneracies at most t_1, \dots, t_r , respectively.*

Proof. Consider a deletion sequence of a maximal t -degenerate graph G . When a vertex is deleted, the edges incident with it can be allocated to r subgraphs with at most t_1, \dots, t_r edges going to the respective subgraphs. Thus the subgraphs have at most the stated degeneracies. □

In particular, a k -degenerate graph decomposes into k forests. These can be almost trees, except for the initial k -clique. The graph G/H is formed by contracting the subgraph H of G to a single vertex.

Corollary 70. *A maximal k -degenerate graph G can be decomposed into K_k and k trees of order $n - k + 1$, which span G/K_k .*

Proof. If $n = k$, $G = K_k$, so let the k trees be k distinct isolated vertices. Build G by successively adding vertices of degree k . Allocate one edge to each of the k trees in such a way that each is connected. To do this, assign an edge incident with a vertex of the original clique to the unique tree containing that vertex. Any other edges can be assigned to any remaining tree, since every tree contains every vertex not in the original clique. □

Corollary 71. *If k is odd, a maximal k -degenerate graph decomposes into k trees of order $n - \frac{k-1}{2}$.*

Proof. Let $k = 2r - 1$. Then K_{2r} can be decomposed into k trees of order $r + 1$. □

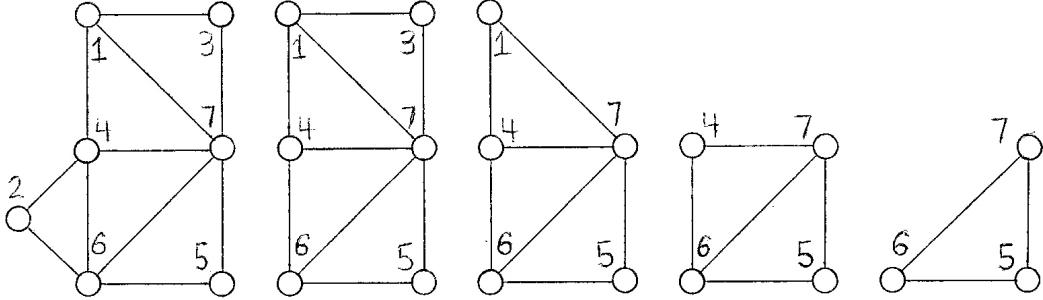


Figure 11: The Prüfer 2-code of this graph is $\{(4, 6), (1, 7), (4, 7), (6, 7)\}$.

Corollary 72. *A maximal 2-degenerate graph has two spanning trees that contain all its edges and overlap on exactly one edge. This 'overlap edge' can be any edge that is the last to be deleted by the k -core algorithm.*

2.1.4 Enumeration

We now consider counting maximal k -degenerate graphs. We first consider labeled graphs. Cayley's tree formula states that the number of labeled trees on n vertices is n^{n-2} . Perhaps the easiest way to prove this is show that there is a bijection between such trees and a certain type of sequences called Prüfer sequences. We seek to generalize this approach to maximal k -degenerate graphs. Borowiecki and Patil [1988] [11] used a somewhat similar approach to generate sequences for rooted maximal k -degenerate graphs. We first state an algorithm for how to generate such a sequence.

Algorithm 73. (Prufer k -code)

Input: Labeled maximal k -degenerate graph of order n

Iteration: While more than $k + 1$ vertices remain, delete the least-labeled vertex v of degree k , and let A_i be the unordered set of k neighbors of v .

Definition 74. A Prufer k -code is a sequence of $n-k-1$ sets of k unordered elements of $[n]$, (A_1, \dots, A_{n-k-1}) , satisfying the following condition: There exists an $(n-k-1)$ -tuple of distinct elements of $[n]$, $B = (b_1, \dots, b_{n-k-1})$ such that $b_i \notin A_j$, $i \leq j \leq n-k-1$.

As we would hope, the algorithm produces a Prufer k -code.

Lemma 75. *The Prufer k -code algorithm produces a unique Prufer k -code.*

Proof. It clearly produces a list of $n-k-1$ $(k-1)$ -tuples. The i^{th} vertex deleted can't appear in the j^{th} $(k-1)$ -tuple, $i \leq j \leq n-k-1$, since it is not its own neighbor and after the i^{th} step it has been deleted. Since there are no choices in the algorithm, it cannot produce more than one code.

□

Theorem 76. *There is a bijection between labeled maximal k -degenerate graphs and Prufer k -codes.*

Proof. In light of the previous lemma, we need only prove that the k -code algorithm produces each k -code uniquely. This is true for $n = k+1$. Assume the result holds for order $r \geq k+1$, and let G be maximal k -degenerate with order $r+1$, with $A = (A_1, \dots, A_{r-k})$ the corresponding code. No leaf label appears in A and every nonleaf label appears. Thus the first vertex deleted has the least label not in A . Call it x , and let its neighbors define A_1 .

Thus every maximal k -degenerate graph producing A has least leaf x with neighbors A_1 since there is a label not appearing in A . Now $G-x$ is maximal k -degenerate with order r and code $A' = (A_2, \dots, A_{r-k})$. By induction, there is exactly one maximal k -degenerate graph G' with this code. Adding x to G' shows that exactly one maximal k -degenerate graph corresponds to A .

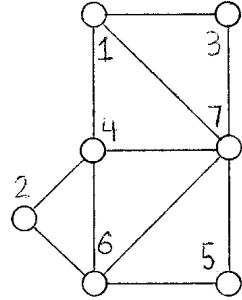


Figure 12: G is maximal 2-degenerate with 2-code $\{(4, 6), (1, 7), (4, 7), (6, 7)\}$.

□

This suggests an algorithm for generating a maximal k -degenerate graph from its k -code.

Algorithm 77. (*Uncoding Prüfer k -code*)

Input: Prüfer k -code (length $n - k - 1$, sets of k)

Initialization: Start with n labeled vertices and a list of labels $1 - n$.

Iteration: While there is a k -tuple remaining, determine the least remaining label not appearing in the k -tuples. Make the corresponding vertex adjacent to the vertices corresponding to the next k -tuple. Delete that label and k -tuple.

Conclusion: Add a clique induced by the $k + 1$ remaining labels.

As an example, consider uncoding the code $\{(4, 6), (1, 7), (4, 7), (6, 7)\}$, which was produced in the earlier example. Of the numbers 1-7, 2, 3, and 5 do not appear, and 2 is the smallest of these, so vertex 2 is adjacent to 4 and 6. Next, we find 3 is adjacent to 1 and 7. Since 1 does not appear in the final two pairs, 1 is adjacent to 4 and 7. We continue in this manner until finally adding a clique induced by vertices 5, 6, and 7. The result is the original graph. (See Figure 12.)

Proposition 78. *The uncoding algorithm produces a unique maximal k -degenerate graph.*

Order	3	4	5	6
Number	1	6	100	3285

Table 2: The number of labeled maximal 2-degenerate graphs.

Proof. This is true for $n = k + 1$, given the empty code. Assume the result holds for $r > k + 1$, and let A be a k -code of length $r - k$. Let a be the smallest element not appearing in A . Now there is a unique maximal k -degenerate graph G' corresponding to $A' = (A_2, \dots, A_{r-k})$. Then making a adjacent to A_1 yields a unique maximal k -degenerate graph G of order $r + 1$.

□

These results suggest an approach for counting labeled maximal k -degenerate graphs. We need only count the corresponding sequences.

Proposition 79. *There are at most $\binom{n}{k}^{n-k-1}$ labeled maximal k -degenerate graphs of order n , with equality exactly when $k = 1$ or $n < \frac{k(k+1)}{k-1}$.*

Proof. There are $\binom{n}{k}$ different possible sets, and the code contains $n - k - 1$ such sets. If there are fewer elements in the sets than vertices, than any code with $n - k - 1$ sets of k elements from $[n]$ will yield a graph since there will always be some element that does not appear in any of the sets. This is equivalent to $k(n - k - 1) < n$, which gives $k = 1$ or $n < \frac{k(k+1)}{k-1}$. However, if $k(n - k - 1) \geq n$, any sequence that contains every element of $[n]$ will not yield a graph.

□

It seems promising that the number of such sequences can be determined exactly, but this has not been accomplished yet. For $k = 2$, the numbers of labeled maximal k -degenerate graphs for small orders are given in Table 2.

Order	4	5	6	7
Number	1	3	11	62

Table 3: The number of unlabeled maximal 2-degenerate graphs.

The upper bound for order 6 is 3375, but there are $90 = \binom{6}{2} \binom{4}{2} \binom{2}{2}$ sequences that don't produce graphs because they contain all six numbers. We can generalize this.

Corollary 80. *There are exactly $\binom{n}{k}^{n-k-1} - \prod_{t=1}^{n-k-1} \binom{t \cdot k}{k}$ labeled maximal k -degenerate graphs of order n when $n = \frac{k(k+1)}{k-1}$.*

Proof. If $n = \frac{k(k+1)}{k-1}$, then $k(n - k - 1) = n$. The only way that the uncoding algorithm could fail is if all n numbers occur exactly once in the code. This can occur in exactly $\prod_{t=1}^{n-k-1} \binom{t \cdot k}{k}$ ways. □

We can also consider enumeration of unlabeled maximal k -degenerate graphs. This has been accomplished for trees via generating functions, but this does not appear to generalize easily to larger k . We have already seen that for $k \geq 2$, the number of nonisomorphic maximal k -degenerate graphs of order $k + 3$ is 3. By tedious examination of cases, I determined the following numbers of maximal 2-degenerate graphs of small orders, given in Table 3.

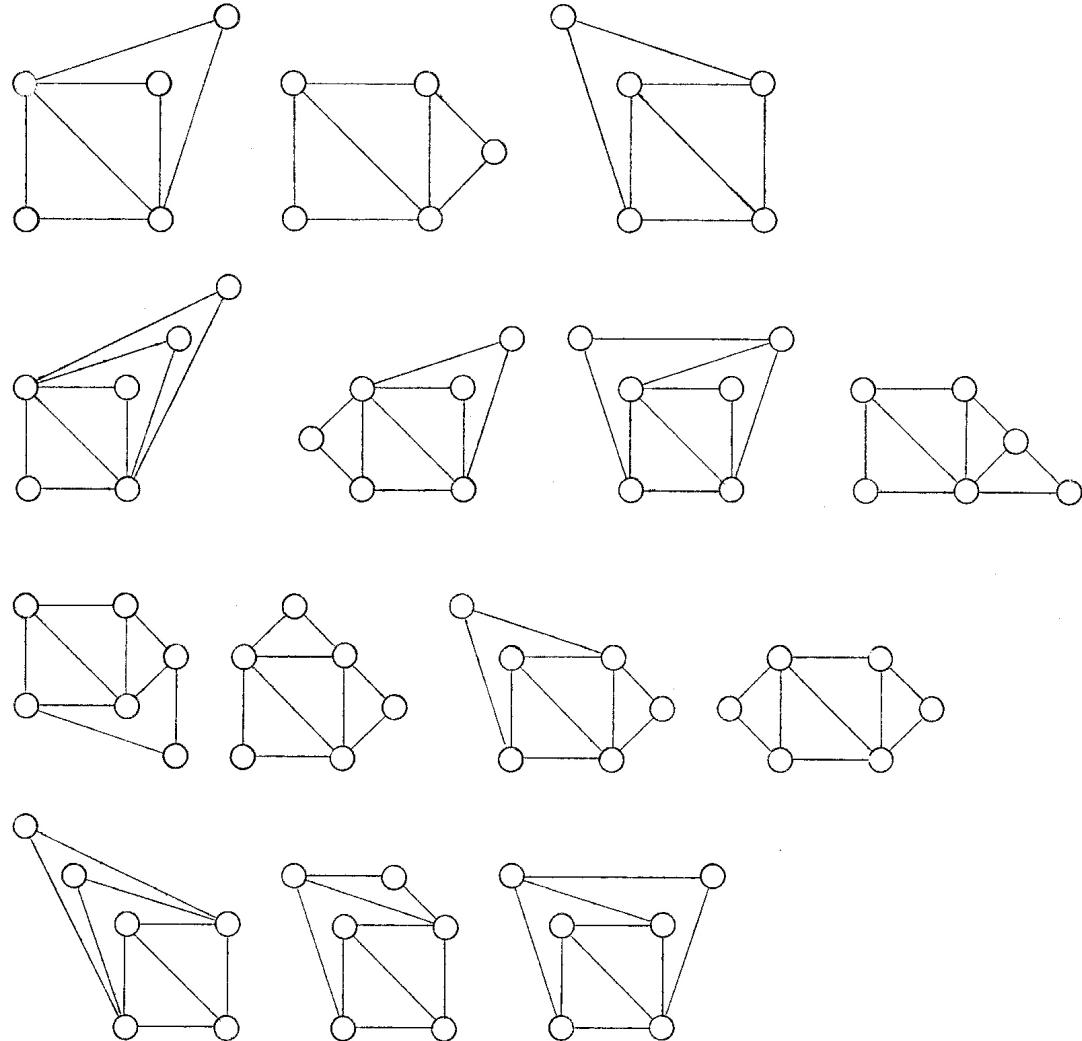


Figure 13: The maximal 2-degenerate graphs of orders 5 and 6.

2.1.5 k-Trees

There is one particular subclass of maximal k -degenerate graphs that is of interest.

Definition 81. A k -tree is a graph that can be formed by starting with K_{k+1} and iterating the operation of making a new vertex adjacent to all the vertices of a k -clique of the existing graph. The clique used to start the construction is called the root of the k -tree.

It is easy to see that a k -tree is maximal k -degenerate. A 1-tree is just a tree. However, k -trees and maximal k -degenerate graphs are not equivalent for $k \geq 2$.

In fact, every maximal k -degenerate graph contains an induced k -tree. For $n \geq k + 2$, $K_{k+2} - e$ must occur. No larger k -tree can be guaranteed. For example, let U be a set of vertices of $K_{k+2} - e$ containing both vertices of degree k , and let V be the partite set of order k of $K_{k,r}$. Then $(K_{k+2} - e) \underset{U=V}{\cup} K_{k,r}$ has order $n \geq k + 3$ and no larger induced k -tree.

Theorem 82. Every maximal k -degenerate graph G contains a unique k -tree of largest possible order containing a $k + 1$ -clique that can be used to begin the construction of G .

Proof. It is obvious that every maximal k -degenerate graph can be constructed beginning with a maximal k -tree. We prove uniqueness. Suppose to the contrary that there is a maximal k -degenerate graph containing two distinct maximal k -trees either of which can be used to begin its construction. Let G be a counterexample of minimum order $n \geq k + 3$ containing k -trees T_1 and T_2 . Divide the vertices of G into $V(T_1)$, $V(T_2)$, and $S = V(G) - V(T_1) - V(T_2)$. Now G has at least one vertex v of degree k . If $v \in S$, then $G - v$ can be constructed starting with either k -tree, so there is a

smaller counterexample. If $v \in V(T_1)$ and $n(T_1) \geq k+2$, then $G - v$ can be still be constructed starting with some other vertex of T_1 , so there is a smaller counterexample. If $v \in V(T_i)$, $i \in \{1, 2\}$, and $T_i = K_{k+1}$, then G cannot be constructed starting with T_i since any maximal k -tree that can be used to begin construction of G must contain $K_{k+2} - e$. Thus in any case we have a contradiction.

□

We offer two characterizations of k -trees as maximal k -degenerate graphs. A graph is chordal if every cycle of length more than three has a chord, that is, it contains no induced cycle other than C_3 .

Theorem 83. *A graph G is a k -tree $\iff G$ is maximal k -degenerate and G is chordal with $n \geq k+1$.*

Proof. (\Rightarrow) Let G be a k -tree. G is clearly maximal k -degenerate, since vertices of degree k can be successively deleted until K_{k+1} remains. The construction implies that G has a simplicial elimination ordering, so it is chordal.

(\Leftarrow) Assume G is maximal k -degenerate and chordal. If $n = k+1$, it is certainly a k -tree. Assume the result holds for order r , and let G have order $r+1$. Then G has a vertex v of degree k . The neighbors of v must induce a clique since if v had two nonadjacent neighbors x and y , an $x - y$ path of shortest length in $G - v$ together with yv and vx would produce a cycle with no chord. Thus $G - v$ is a k -tree, hence so is G .

□

The second characterization of k -trees as maximal k -degenerate graphs involves subdivisions.

Theorem 84. *A maximal k -degenerate graph is a k -tree if and only if it contains no subdivision of K_{k+2} .*

Proof. (\Rightarrow) Let G be a k -tree. Certainly K_{k+1} contains no subdivision of K_{k+2} . Suppose G is a counterexample of minimum order with a vertex v of degree k . Then $G - v$ is a k -tree with no subdivision of K_{k+2} , so the subdivision in G contains v . But then v is not one of the $k + 2$ vertices of degree $k + 1$ in the subdivision, so it is on a path P between two such vertices. Let its neighbors on P be u and w . But since the neighbors of v form a clique, $uw \in G - v$, so P can avoid v , implying $G - v$ has a subdivision of K_{k+2} . This is a contradiction.

(\Leftarrow) Let G be maximal k -degenerate and not a k -tree. Since G is constructed beginning with a k -tree, for a given construction sequence there is a first vertex in the sequence that makes G not a k -tree. Let v be this vertex, and H be the maximal k -degenerate subgraph induced by the vertices of the construction sequence up to v . Then $n(H) \geq k + 3$, $d_H(v) = k$, v has nonadjacent neighbors u and w , and $H - v$ is a k -tree. Now there is a sequence of at least two $k + 1$ -cliques starting with one containing u and ending with one containing w , such that each pair of consecutive $k + 1$ -cliques in the sequence overlap on a k -clique. Then two of these cliques and a path through v produces a subdivision of K_{k+2} .

□

Dirac [1964] [20] determined the minimum size of a graph G of order n that will guarantee that G contains a subdivision of K_4 . We can prove this simply and determine the extremal graphs.

Corollary 85. *If G has $m \geq 2n - 2$, then G contains a subdivision of K_4 , and the graphs of size $2n - 3$ that fail to contain a subdivision of K_4 are exactly the 2-trees.*

Proof. Let G have $m \geq 2n - 2 = (3 - 1)n - \binom{3}{2} + 1$. By Corollary 59, G contains a 3-core. By Corollary 47, it contains a subdivision of K_4 . If a graph of size $2n - 3$ has no 3-core, it is maximal 2-degenerate. By the previous theorem, exactly the 2-trees do not contain a subdivision of K_4 .

□

A natural generalization of this result is that if $m \geq 3n-5$, G contains a subdivision of K_5 . This was conjectured by Dirac and proved by Mader [1998] [41] using a much more intricate argument.

2.2 Monocore and Collapsible Graphs

2.2.1 k-Monocore Graphs

We have previously defined a monocore graph to be a graph G with $\delta(G) = \widehat{C}(G)$. Thus a graph is k -monocore if and only if $C_i(G) = \begin{cases} G & i \leq k \\ \emptyset & i > k \end{cases}$.

We have seen that many important classes of graphs are monocore, and determined a few properties of such graphs. In this section we will extensively examine the properties of such graphs.

One common technique for understanding a class of graphs is examining its extremal graphs. That is, graphs in the class which either cannot have any edges added or cannot have any edges deleted without ceasing to be in that class. We will call these two types of extremal graphs maximal and minimal extremal graphs, respectively.

We will first examine the maximal extremal k -monocore graphs. In fact, these are just maximal k -degenerate graphs. We have already seen that maximal k -degenerate graphs are k -monocore. A partial converse to this result is true.

Lemma 86. *Every k -monocore graph is contained in a maximal k -degenerate graph.*

Proof. Let G be k -monocore. Determine a deletion sequence for G , and reverse it to obtain a construction sequence. Now construct graph G' by adding not only the edges of G , but enough additional edges so that $\min\{k, i - 1\}$ edges are added when the i^{th} vertex is added. The resulting graph is maximal k -degenerate.

□

Adding an edge to a maximal k -degenerate graph creates a $k + 1$ -core, so maximal k -monocore graphs are maximal k -degenerate.

We can use this lemma to determine sharp bounds on the size of a k -monocore graph.

Proposition 87. *The size m of a k -monocore graph G of order n satisfies*

$$\left\lceil \frac{k \cdot n}{2} \right\rceil \leq m \leq k \cdot n - \binom{k+1}{2}.$$

Proof. The sum of the degrees of G is at least $k \cdot n$, so $m \geq \lceil \frac{k \cdot n}{2} \rceil$. The upper bound follows from the previous lemma. \square

Both bounds are sharp. The graphs achieving the upper bound are maximal k -degenerate graphs. For n or k even, the graphs achieving the lower bound are just regular graphs, and for n and k both odd, they are graphs with exactly one vertex of degree $k+1$, and all others of degree k .

We can make some observations about the degree sequences of k -monocore graphs.

Lemma 88. *If a nonincreasing sequence of integers d_1, \dots, d_n is the degree sequence of a k -monocore graph G , then $k \leq d_i \leq \min \{n-1, k+n-i\}$ and $\sum d_i = 2m$, where m satisfies the bounds of the previous theorem for $0 \leq k \leq n-1$.*

Proof. For the first inequalities, the lower bound is obvious, and the upper bound follows from the corresponding result for maximal k -degenerate graphs. The latter equation follows from the first theorem of graph theory and the previous result. \square

We will prove that the converse to this result holds. We need another lemma to reduce the set of sequences that we need to consider.

Lemma 89. *The converse to Lemma 88 holds if and only if every nonincreasing sequence of integers d_1, \dots, d_n satisfying $d_1 \leq n-1$, $d_k = d_n = k$, and $\sum d_i = 2m$ is graphical.*

Proof. (\Rightarrow) The latter set of sequences is a subset of the former.

(\Leftarrow) Assume the hypothesis and that the conjecture holds for order $r \geq k+2$. Let $D : d_1, \dots, d_{r+1}$ be a nonincreasing sequence that satisfies the given properties. If D has fewer than k integers larger than k , then it is graphical by assumption. Any graph G satisfying D is a k -core since $d_n = \delta(G) = k$ and cannot have a $k+1$ -core, so it is k -monocore.

Hence we assume additionally that D has at least k integers larger than k . Let $D' : d'_1, \dots, d'_r$ be the sequence formed by deleting $d_{r+1} = k$ and decreasing k other numbers greater than k by one, including any that achieve the maximum. (There are at most k by Lemma 63.) Then D' satisfies all the hypotheses and has length r , so it is the degree sequence for some k -monocore graph H . Add vertex v_{r+1} to H , making it adjacent to the k vertices with degrees that were decreased to form D' . Then the resulting graph G has degree sequence D and is k -monocore.

□

In light of the lemma, we need only consider sequences that end with many k 's. We use the following operations to limit the number of k 's at the end of the sequence that we must consider.

Definition 90. Operation. [Add a vertex of degree $k = 2r$]

Subdivide r independent edges and identify the r new vertices. This produces a graph with all the same degrees as before plus one more vertex of degree k .

Operation. [Add two vertices of degree $k = 2r + 1$]

Delete $2r$ edges which use each vertex at most twice, add two adjacent vertices, and make each of them adjacent to $2r$ of the neighbors of the deleted edges. This produces a graph with a degree sequence that adds two k 's to the degree sequence of the original graph.

It is easily verified that the required sets of edges exist.

Proposition 91. *The edge independence number $\beta(G)$ of a k -core G satisfies $\beta(G) \geq \lceil \frac{k}{2} \rceil$.*

Proof. We use induction on k . The result is obvious when k is 0 or 1. Assume it holds for all k with $0 \leq k \leq r$ and let G be an $r + 2$ -core. Let $e = uv$ be an edge of G . Then $G - u - v$ is an r -core, so $\beta(G - u - v) \geq \lceil \frac{r}{2} \rceil$, and $\beta(G) \geq \lceil \frac{r}{2} \rceil + 1 = \lceil \frac{r+2}{2} \rceil$.

□

The bound is sharp for empty graphs, stars, and complete graphs. Note that the second operation above can work by using two independent sets of edges.

Now we can prove the theorem.

Theorem 92. *A nonincreasing sequence of integers d_1, \dots, d_n is the degree sequence of some k -monocore graph G if and only if $k \leq d_i \leq \min\{n-1, k+n-i\}$ and $\sum d_i = 2m$, where m satisfies the bounds of the previous theorem for $0 \leq k \leq n-1$.*

Proof. (\Rightarrow) The forward direction is just Lemma 88.

(\Leftarrow) We use induction on k . For $k = 0$, it is obvious. Assume the conjecture holds for $k \geq 1$. By Lemma 89, the conjecture will hold if it holds for sequences with at most $k - 1$ integers larger than k . Let D be such a sequence of length n . We may assume that d_1 is $n - 1$ or $n - 2$, since otherwise we may delete some k 's so that this holds, obtain a graph for this shorter sequence, and use the above operations to obtain a graph with the longer sequence.

If $d_1 = n - 1$, then the sequence D' formed by deleting v_1 and reducing every other element by one has at most $k - 2$ integers larger than $k - 1$. Thus it is the degree sequence of a $k - 1$ -monocore graph H by the induction hypothesis, and $G = H + v$ is k -monocore. If $d_1 = n - 2$, then the sequence D' formed by deleting v_1 and reducing all integers but one of the k 's by one has at most $k - 1$ integers larger than $k - 1$. Thus it is the degree sequence of a $k - 1$ -monocore graph H by the induction hypothesis,

and the graph G formed by joining a vertex to the vertices of H with degrees that had been reduced is k -monocore. Thus the conjecture holds for k -monocore graphs by induction.

□

Thus Theorem 92 can be proven as a corollary of this theorem.

A related question is when a given graphical sequence must be the degree sequence of a monocore graph. For example, the sequence 2,2,2,1,1 could represent P_5 , which is 1-monocore, or $K_3 \cup K_2$, which is not.

2.2.2 k -Collapsible Graphs

We will next consider a variation on the lower extremal k -monocore graphs, where instead of considering deleting an edge, we consider deleting vertices.

Definition 93. A graph G is k -collapsible if it is k -monocore and has no proper induced k -core.

This immediately implies that a k -monocore graph is k -collapsible if and only if for every vertex v in G , $G - v$ has no k -core.

For small values of k , we can characterize k -collapsible graphs.

Proposition 94. Let G be a graph.

G is 0-collapsible $\iff G = K_1$.

G is 1-collapsible $\iff G = K_2$.

G is 2-collapsible $\iff G$ is a cycle.

Proof. Certainly the given graphs are all collapsible. Every 0-monocore graph contains

a vertex, and every 1-monocore graph contains an edge. We have seen that every 2-monocore graph contains a cycle.

□

The structure of k -collapsible graphs is considerably more complicated for $k > 2$.

Collapsible graphs are interesting in part because every k -monocore graph contains one.

Proposition 95. *Every k -monocore graph G contains a k -collapsible graph as an induced subgraph. Indeed, every component of G contains such a subgraph.*

Proof. If G has order $n = k + 1$, then the unique k -monocore graph of that order, $G = K_{k+1}$ is k -collapsible. Assume the result holds for all k -monocore graphs with order up to r , and let G have order $r + 1$. If $G - v$ has no k -core for all v in G , then G is k -collapsible. If not, then there is some vertex v in G so that $G - v$ has a k -core. Let H be the k -core of $G - v$. Then H is an induced subgraph of G with order at most r , so by induction it contains a k -collapsible subgraph. The final statement holds since every component of a k -monocore graph is k -monocore.

□

We can offer a characterization of sorts for k -collapsible graphs.

Definition 96. A barrier in a k -monocore graph is a minimal cutset $S \subset V(G)$ such that for some component H of $G - S$, every vertex v of S has $d_{G-H}(v) \geq k$.

Note that every vertex in a barrier of a k -monocore graph G necessarily has degree greater than k in G .

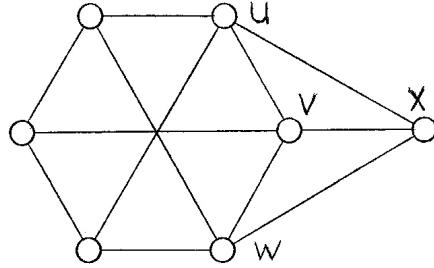


Figure 14: The set $S = \{u, v, w\}$ is a barrier of G since $G - \{x\}$ is a 3-core.

Theorem 97. *A k -monocore graph G is collapsible \iff it does not have a barrier.*

Proof. If G has a barrier S and corresponding component H , then the vertices of $G - H$ all have degree at least k . Thus G has a proper k -core, so it is not collapsible.

If G is not collapsible, then it has a proper induced subgraph F such that is a k -core. Then the vertices of F adjacent to vertices of $G - F$ must be a barrier.

□

Checking every set of vertices, or even every cutset of vertices of degree greater than k is not very practical. It is easier to determine whether G has a barrier by running the k -core algorithm on $G - v$ for all v .

A barrier need not be a large set or have large degrees. Indeed, for all $k \geq 3$, there is a k -monocore graph G with a barrier of one vertex of degree $k + 1$ or $k + 2$. If k is odd, a barrier need only have one vertex of degree $k + 1$, while if k is even, it need have no more than two.

For k even, such a graph can be constructed by taking two copies of K_{k+1} and adding an edge between them. Two vertices of degree $k + 1$ is the least possible by the First Theorem of graph theory.

For k odd, $k = 2r + 1$, G can be constructed by taking K_{k+1} , deleting r independent edges, and making a new vertex v adjacent to the vertices they were incident with. Finally, add an edge between v and a new copy of K_{k+1} . The graph G has the desired properties. (See Figure 15.)

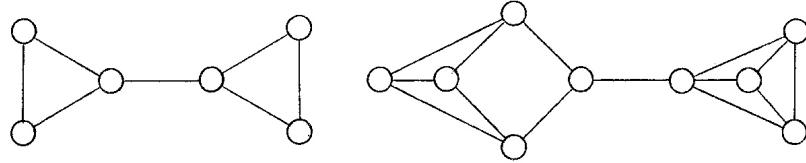


Figure 15: Examples of non-collapsible graphs.

Collapsible graphs include a well-known family of graphs.

Corollary 98. *A regular graph is collapsible \iff it is connected.*

We can use the previous theorem to find information on the structure of k -collapsible graphs.

Corollary 99. *Let G be a k -core with vertex v . Then $C_k(G - v) = C_k(G) - v \iff$ every neighbor of v has degree at least $k + 1$.*

The next corollary follows from the previous one.

Corollary 100. *If G is k -collapsible, then every vertex of G is adjacent to a vertex of degree k . This implies that the vertices of degree k of a k -collapsible graph G form a total dominating set of G .*

The converse is false, as can be seen for example in the graphs with small barriers constructed above. Even if the dominating set is connected, it is still false. For example, the graph formed by adding a perfect matching between the vertices of C_n and K_n , $n > 3$, is not 3-collapsible.

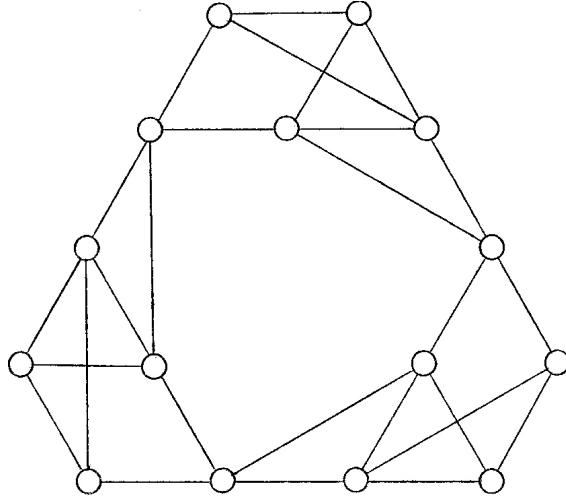


Figure 16: A 3-collapsible graph with $2/5$ of its vertices having degree 3.

Conjecture 101. *Every k -collapsible graph G of order n has at least $\frac{2}{2k-1}n$ vertices of degree k .*

For $k = 3$ and every order $n = 5r$, $r \geq 2$, this conjectured bound is achieved. Let the vertices of degree 4 form a cycle of $3r$ vertices numbered $0, \dots, 3r - 1$, and add edges between vertices $3i + 2$ and $3i + 4 \pmod{3r}$. Then for each i , $0 \leq i \leq r - 1$, add a pair of adjacent vertices and make both adjacent to vertex $3i$ and one each adjacent to vertices $3i + 1$ and $3i + 2$. This graph has $3r$ vertices of degree 4 and $2r$ vertices of degree 3 and is 3-collapsible. (See Figure 16.)

We can bound the size of a k -collapsible graph.

Theorem 102. *For $k \geq 1$, the size m of a k -collapsible graph G of order n satisfies*

$$\left\lceil \frac{k \cdot n}{2} \right\rceil \leq m \leq (k - 1) \cdot n - \binom{k}{2} + 1.$$

Proof. The lower bound is the same as that for k -monocore graphs.

For the upper bound, let G be k -collapsible and e an edge of G incident with a vertex v of degree k . Then $G - e$ is k -core free, since $d_{G-e}(v) < k$ and G has no proper induced k -core. Then $G - e$ is contained in a maximal k -core free graph H , and $G \subseteq H + e$. Thus $m \leq (k-1) \cdot n - \binom{k}{2} + 1$.

□

The relationship between k -collapsible graphs and maximal k -core free graphs seen in the previous proof is a pleasant surprise.

It is not hard to show that if G is maximal k -core-free and $G + e$ is k -monocore, it need not be k -collapsible.

The upper bound of this theorem is sharp. For example, for $k \geq 3$ the graph $G = C_{n-k+2} + K_{k-2}$ achieves the upper bound and is k -collapsible since every vertex on the cycle has degree k and G has no k -core not containing one of them. These are not the only graphs achieving the upper bound, as we will soon see. This example also shows that for all k, n with $3 \leq k \leq n-1$ there is a k -collapsible graph with maximum degree $n-1$. Along with the cycles, this shows that for all k, n with $2 \leq k \leq n-1$ there is a k -collapsible graph of order n .

2.2.3 k -Core-critical Graphs

While collapsible graphs are a sort of extremal graph for k -monocore graphs, they are also a class of graphs that have their own extremal graphs. We first consider minimal k -collapsible graphs.

Definition 103. A k -collapsible graph G is k -core-critical if no proper subgraph of G is a k -core. A graph is core-critical if it is k -core-critical for some k .

This immediately implies that a k -collapsible graph is k -core-critical exactly when

it has no edge e such that $G - e$ is a k -core. This implies the following result.

Proposition 104. *A k -collapsible graph G is k -core-critical \iff no edge of G joins vertices of degree more than k .*

For all k, n with $3 \leq k \leq n - 1$ there is a k -core-critical graph. For example, for $k \geq 3$ the graph $G = C_{n-k+2} + \overline{K}_{k-2}$ is k -core-critical since every vertex on the cycle has degree k and no others are adjacent. These are not the only graphs achieving the upper bound. Along with the cycles, this shows that for all k, n with $2 \leq k \leq n - 1$ there is a k -core-critical graph of order n .

The previous result allows us to bound the number of vertices of minimum degree in a k -core-critical graph.

Theorem 105. *A k -core-critical graph G with order n has at least $\lceil (\frac{k+1}{2k}) n \rceil$ vertices of degree k and hence at most $\lfloor (\frac{k-1}{2k}) n \rfloor$ vertices of degree more than k .*

Proof. Let A be the set of vertices of G of degree k and B the set of vertices of degree more than k , and let their sizes be a and b , respectively. By the previous result, G has no BB edges. Since each vertex of degree k is adjacent to another such vertex, G has at least $\lceil \frac{a}{2} \rceil AA$ edges. The sum of the degrees of the A vertices is $k \cdot a$, so G has at most $k \cdot a - 2 \lceil \frac{a}{2} \rceil AB$ edges. Thus G has size at most $k \cdot a - \lceil \frac{a}{2} \rceil = \lfloor (\frac{2k-1}{2}) a \rfloor$.

Since the B vertices have degree at least $k + 1$, $b \leq \lfloor \frac{1}{k+1} (k \cdot a - 2 \lceil \frac{a}{2} \rceil) \rfloor \leq \lfloor (\frac{k-1}{k+1}) a \rfloor$. Then $n = a + b \leq a + \lfloor (\frac{k-1}{k+1}) a \rfloor \leq \lfloor (\frac{2k}{k+1}) a \rfloor$. Thus $a \geq \lceil (\frac{k+1}{2k}) n \rceil$ and $b \leq \lfloor (\frac{k-1}{2k}) n \rfloor$.

□

For $k = 3$, this bound gives $a \geq \lceil (\frac{2}{3}) n \rceil$ and $b \leq \lfloor (\frac{1}{3}) n \rfloor$. The bound is sharp for $k = 3$.

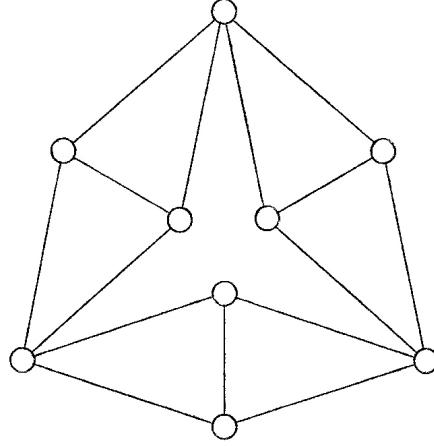


Figure 17: The 3-necklace.

Definition 106. The r -necklace is the graph formed from the multigraph C_r by replacing each edge by $K_4 - e$, where the nonadjacent vertices overlap the vertices of the edge they replace.

The r -necklace has order $3r$, $2r$ vertices of degree 3, r vertices of degree 4, and is 3-core-critical. Thus the bound is sharp for this family of graphs. Consider replacing one $K_4 - e$ in the necklace by $W_4 - e$, where e is incident with the center vertex. This yields a graph that has order $3r + 1$, includes $2r + 1$ vertices of degree 3, r vertices of degree at least 4, and is 3-core-critical. Allowing two $W_4 - e$'s in the construction yields a graph that has order $3r + 2$, $2r + 2$ vertices of degree 3, r vertices of degree at least 4, and is 3-core-critical. The bound is also sharp for $n = 5$, but it is one off for $n = 4$, since K_4 is the only possibility in that case.

It is not immediately clear whether this bound is sharp for $k > 3$. It is certainly not sharp for $n = k + 1$.

Conjecture 107. *The bound of Theorem 105 is sharp for infinitely many orders n for all $k > 3$.*

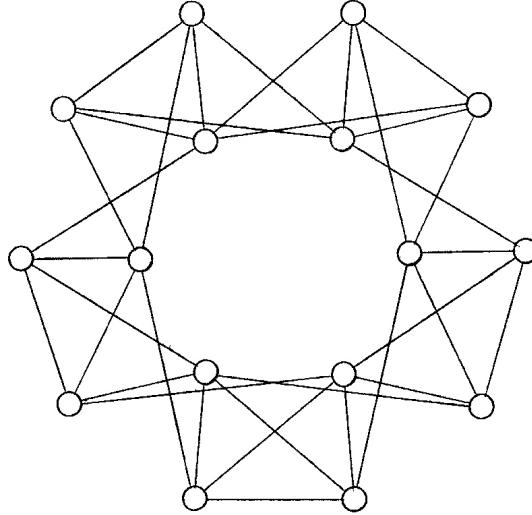


Figure 18: A 4-core-critical graph with $5/8$ of its vertices having degree 4.

We now describe an infinite class of 4-core-critical graphs that have $5/8$ of their vertices with degree 4. These graphs have two vertex classes U and W . U has $10r$ vertices numbered 0 to $10r-1$ and W has $6r$ vertices numbered 0 to $6r-1$. The graph has the following edges. Vertex u_{5i} is adjacent to $w_{\frac{3}{5}i}$, $w_{\frac{3}{5}i-1}$, and $w_{\frac{3}{5}i+1}$. Vertices u_{5i+1} and u_{5i+2} are both adjacent to $w_{\frac{3}{5}i}$, $w_{\frac{3}{5}i+1}$, and $w_{\frac{3}{5}i+2}$. Vertices u_{5i+3} and u_{5i+4} are both adjacent to $w_{\frac{3}{5}i+1}$, $w_{\frac{3}{5}i+2}$, and $w_{\frac{3}{5}i+3}$. Finally, u_{2i} and u_{2i+1} are adjacent. (Indices for U and W are taken mod $10r$ and $6r$, respectively.)

This construction makes all vertices in U have degree 4 and all in W have degree 5. Each vertex in W has all its neighbors in U . Each vertex in U has one neighbor in U , and this pair has at least two common neighbors in W . These vertices in W have neighbors in U on either 'side' of the original pair. Thus when any vertex is deleted, the graph collapses. This construction is illustrated for order 16 in Figure 18.

Let k be odd. Then the graph $\left(\frac{k+1}{2}\right)K_2 + \overline{K}_{k-1}$ is k -core-critical and achieves the bound of Theorem 105 since it has $k+1$ vertices of degree k and $k-1$ vertices of degree $k+1$.

2.2.4 Enumeration of k -Collapsible Graphs

Since we do not have a general description of k -collapsible graphs for $k \geq 3$, we may examine the classes of k -collapsible graphs with additional specified properties. Certainly the unique k -collapsible graph of order $k + 1$ is K_{k+1} . We can describe the collapsible graphs of order $k + 2$.

Proposition 108. *Let $l = \lfloor \frac{k+2}{2} \rfloor$, and let $G_r = K_{k+2} - rK_2$, $2 \leq r \leq l$. Then G_r is k -collapsible, so there are $l - 1 = \lfloor \frac{k}{2} \rfloor$ k -collapsible graphs of order $k + 2$. G_l is the unique k -core-critical graph of order $k + 2$.*

This follows since the complement of a k -collapsible graph of order $k + 2$ must have maximum degree 1. We can generalize this idea.

Proposition 109. *Let $k \leq n - 1$. Then the number of k -core-critical graphs of order n equals the number of graphs that are maximal with respect to the property of having maximum degree $n - k - 1$.*

Proof. The complement of a k -core-critical graph G must have maximum degree $n - k - 1$. If the complement were not maximal with respect to this property, G would not be minimal.

□

Corollary 110. *A generating function for the number of k -core-critical graphs of order $n = k + 3$, $k \geq 3$, is*

$$G(x) = (1 + x + x^2) \prod_{n=3}^{\infty} \frac{1}{1 - x^n}.$$

Proof. The graphs that are maximal with respect to the property of having maximum degree 2 have at most one component that is K_1 or K_2 . Every other vertex in the

complement must have degree 2, so the other components must be cycles. Graphs that are 2-regular correspond to partitions of the integer n into integers that are at least three. Now $\frac{1}{1-x^n} = 1 + x^n + x^{2n} + \dots$, so this product counts such partitions. Multiplying by $1 + x + x^2$ adds the n , $n - 1$, and $n - 2$ terms of the sequence together. Thus the generating function is as stated.

□

The first few terms of this sequence are 4, 5, 7, 9, 12, 15,

We can also examine the number of k -collapsible graphs with a given maximum degree.

Proposition 111. *The number of k -collapsible graphs G of order n with $\Delta(G) = n - 1$ equals the number of $k - 1$ -collapsible graphs with order $n - 1$.*

This follows since deleting a vertex adjacent to all other vertices of a k -collapsible graph results in a $k - 1$ -collapsible graph with order $n - 1$.

Corollary 112. *The number of k -core-critical graphs G of order n with $\Delta(G) = n - 1$ equals the number of connected regular graphs with order $n - 1$.*

2.2.5 3-Collapsible Graphs

Since 3-collapsible graphs are the smallest undecided case, it makes sense to examine them in more detail. In addition to connected cubic graphs, another well-known family of graphs that are 3-collapsible are the wheels, which are the only ones with maximum degree $n - 1$. Examining small orders, we see K_4 and W_4 are the unique 3-core-critical graphs of order 4 and 5. For order 6, by the earlier result we have four 3-core-critical graphs of order six. They are W_5 , $2K_2 + 2K_1$, $K_3 \times K_2$, and $K_{3,3}$. The first two achieve the maximum possible size, while the latter two do not. Both

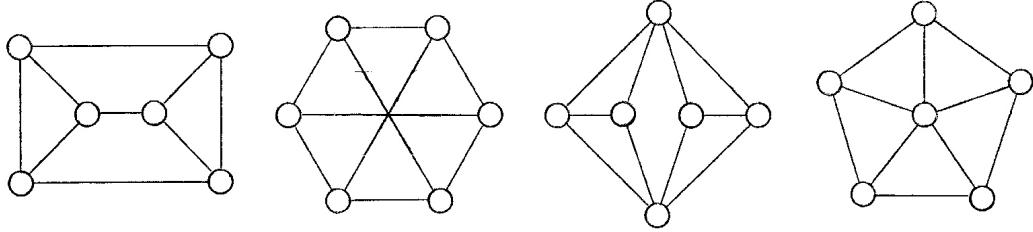


Figure 19: The four 3-core-critical graphs of order 6.

$\Delta \setminus n$	4	5	6	7	8	9
3	1	0	2	0	5	0
4		1	1	4	6	28
5			1	1	6	27
6				1	2	11
7					1	2
8						1
Total	1	1	4	6	20	69

Table 4: The number of 3-core-critical graphs with order n and maximum degree Δ .

can have a single edge added, uniquely up to isomorphism in both cases, and still be 3-collapsible.

By exhaustive (and exhausting) examination of cases, I determined the following numbers of 3-core-critical graphs with order n and maximum degree Δ , given in Table 4.

Justification of the larger numbers in this table is too tedious to include. As an example, see Figure 20 for the four 3-core-critical graphs with order seven and a single vertex of degree four.

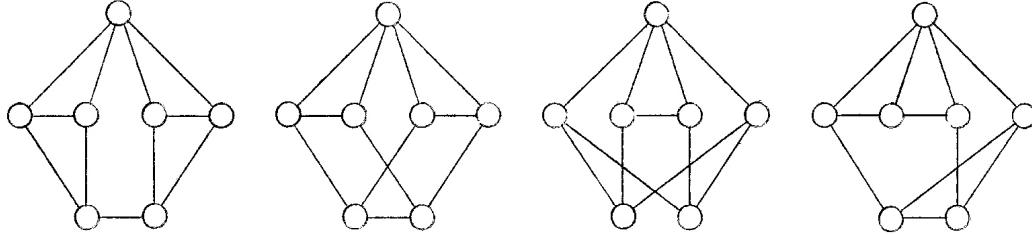


Figure 20: The four 3-core-critical graph with order 7 and $\Delta = 4$.

It is not too hard to determine the number of minimal 3-collapsible graphs with order n , $\Delta = n - 2$.

Proposition 113. *The number of 3-core-critical graphs with order n , $\Delta = n - 2$, is $\lfloor \frac{n-4}{2} \rfloor$.*

Proof. Let v be the vertex of G with degree $n - 2$. Then G has one vertex u not adjacent to v . Then $G - v$ has all vertices of degree 2 except u . Thus $G - v - u$ is a linear forest with nontrivial paths, and the endvertices of the paths are adjacent to u . The linear forest has at least two components, since u has degree at least 3, and it has at most two since the graph induced by u, v , and exactly two of them is a 3-core. Thus the number of such graphs is the number of integer solutions to $x + y = n - 2$, $2 \leq x \leq y \leq n - 4$, which is $\lfloor \frac{n-4}{2} \rfloor$.

□

We might hope to determine an operation characterization of 3-collapsible graphs. Toward that end, we have the following partial results. We say that a graph operation preserves a given class of graphs if applying it to any graph or graphs in that class results in a graph in that class.

Definition 114. The operation splitting a vertex v of degree at least 4 replaces v by

two adjacent vertices x and y joins each neighbor of v with exactly one of them so that both x and y have degree at least 3.

Proposition 115. *Splitting a vertex preserves 3-cores and 3-monocore graphs. It preserves 3-collapseable graphs with the added condition that each of the vertices resulting from the split has a neighbor of degree 3. It preserves 3-core-critical graphs with the added condition that at most one of the vertices resulting from the split has degree more than 3.*

Proof. If G is a 3-core, then splitting a vertex results in a graph with minimum degree at least 3. Let G be 3-monocore and H be the result of splitting a vertex v . Then H has no 4-core not containing v , and if it had one containing v then so would G .

Let G be 3-collapseable and H be the result of splitting a vertex v into x and y . Then $G - v$ has no 3-core and with the added restriction neither do $H - x$ or $H - y$. Thus H is 3-collapseable.

Let G be 3-core-critical and H be the result of splitting a vertex v into x and y . Then v has degree at least 4 in G , so its neighbors have degree 3. With the added restriction, no neighboring vertices of H have degree more than 3. As before, H is 3-core-critical.

□

We have the following theorem due to Tutte [Bollobas [7] p.16] that provides an operation characterization of 3-connected graphs.

Theorem 116. *[Tutte] A graph is 3-connected \iff it is a wheel or can be constructed from one by repeatedly applying the operations:*

1. Add an edge.
2. Split a vertex.

The proof of this theorem is fairly difficult.

2.2.6 Maximal k -Collapsible Graphs

Having considered k -core-critical graphs, we can also consider the other extremal class of k -collapsible graphs.

Definition 117. A k -collapsible graph G is maximal k -collapsible if the addition of any edge makes it cease to be k -collapsible.

Every k -collapsible graph is necessarily contained in a maximal k -collapsible graph. Since adding an edge to a maximal k -collapsible graph makes it cease to be collapsible, it either has a $k + 1$ -core or a proper induced k -core. In fact, it must be the latter.

Proposition 118. *Adding an edge to a maximal k -collapsible graph G produces a proper induced k -core.*

Proof. Suppose instead it produces a $k + 1$ -core. Then G contains a subgraph H with all but at most two vertices having degree at least $k + 1$ in H and either one vertex of degree k or two nonadjacent vertices of degree k in H . But deleting a vertex with degree k in H leaves G with a k -core, so it is not collapsible.

□

We have seen that a sharp upper bound for the size of a k -collapsible graph is $m \leq (k - 1) \cdot n - \binom{k}{2} + 1$, so any graph achieving this bound is necessarily maximal k -collapsible. In fact, there are graphs that are both maximal and minimal 3-collapsible at the same time.

Definition 119. A k -collapsible graph G is k -fragile if it is both maximal and minimal.

The wheels are 3-fragile, as are several other 3-collapsible graphs considered previously that achieve the upper bound. Perhaps surprisingly, there are graphs that are 3-fragile without achieving the upper bound.

Proposition 120. *The r -necklace is 3-fragile and for $r \geq 3$ does not achieve the upper bound for the size of a k -collapsible graph.*

Proof. We have seen that it is minimal 3-collapsible, and it has order $3r$ and size $5r$, so it does not achieve the upper bound except when $r = 2$. Consider adding an edge to the r -necklace. If we add an edge incident with a vertex of degree 3, then its neighbor of degree 3 could be deleted to produce a smaller 3-core. If the edge is added between two vertices of degree 4, deleting one part of the 'cycle' between these vertices yields a smaller 3-core.

□

It is unknown whether k -fragile graphs exist for larger values of k .

It is unclear how small the size of a maximal 3-collapsible graph can be. We offer the following conjecture, which is based on the size of the r -necklace.

Conjecture 121. *The size m of a maximal 3-collapsible graph of order n satisfies $m \geq \lceil \frac{5}{3}n \rceil$.*

2.2.7 Connectivity of k -Cores

We can also analyze other structural properties of monocore and collapsible graphs. One example is connectivity. Certainly if G is k -connected, then it is a k -core. This holds for minimally connected graphs as well, but in this case much more is also true.

Theorem 122. *Let G be a minimally k -connected graph. Then the vertices of degree*

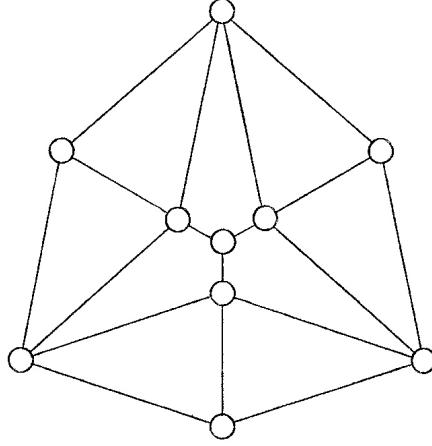


Figure 21: A minimally 3-connected graph that is not 3-collapsible.

more than k induce a (possibly empty) forest.

This theorem is due to Mader and appears in [Bollobas [7] p. 21-24]. The proof is quite difficult.

Corollary 123. *If a graph is minimally k -connected, then it is k -monocore.*

A minimally k -connected graph need not be k -collapsible. For example, in the graph in Figure 21, deleting the center vertex does not destroy the 3-core, but leaves a graph with connectivity 2.

Conjecture 124. *If a graph is minimally k -edge-connected, then it is k -monocore.*

This conjecture holds for minimally 0-edge-connected graphs (empty graphs) and minimally 1-edge-connected graphs (trees). Mader [40] proved [see West [60] p.175] that if G is minimally k -edge-connected, then $\delta(G) = k$.

A k -core need not be k -connected.

Proposition 125. *There is a k -monocore graph that is l -connected for each $0 \leq l \leq k$ and the smallest possible order of such a graph is $2k + 2 - l$.*

Proof. Let G be k -monocore and l -connected. If G is disconnected, then every component is a k -core with order at least $k + 1$. The graph $2K_{k+1}$ uniquely achieves the minimum. If G is connected, then it has a minimal cutset S of size l , and each component of $G - S$ has order at least $k + 1 - l$, so $2k + 2 - l$ is the smallest possible order. The graph $K_{k+1} \cup_{K_l} K_{k+1}$ achieves the bound.

□

Certainly a k -collapsible graph must be connected. The same bound as before holds for them except when $l = 1$.

Proposition 126. *The minimum order n of an l -connected k -collapsible graph is given by*

$$n = \begin{cases} 2k + 3 & l = 1 \\ 2k + 2 - l & \text{else} \end{cases}.$$

Proof. For $l \geq 2$, use a construction similar to before but modify the intersection of the cliques. Let $a = 2l - k - 2$. If $a \leq 0$, use $K_{k+1} \cup_{\overline{K}_l} K_{k+1}$. If $a = 1$, use $K_{k+1} \cup_{\frac{l}{2}K_2} K_{k+1}$ if l is even and given $H = (\frac{l-3}{2})K_2 \cup P_3$, use $K_{k+1} \cup_H K_{k+1}$ if l is odd. If $a > 1$, let H be a -core-critical and use $K_{k+1} \cup_H K_{k+1}$. In each case, G is k -collapsible.

If $l = 1$, the earlier construction produces $K_{k+1} \cup_v K_{k+1}$, which is not collapsible, so each block must have order at least $k + 2$. Instead, form G by subdividing $\lceil \frac{k}{4} \rceil$ independent edges of each copy of K_{k+1} and identifying all the subdivision vertices together. Then G has order $2k + 3$ and is k -collapsible.

□

Note that this construction produces k -core-critical graphs.

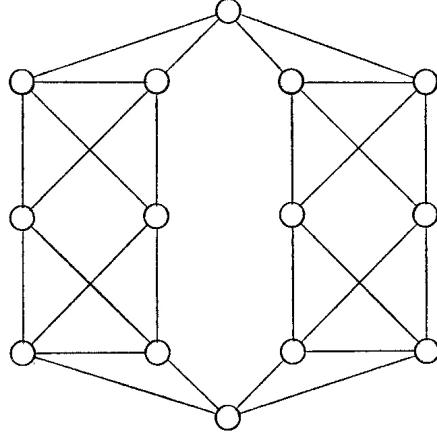


Figure 22: A 4-regular graph with no 3-connected subgraph.

We have seen that a k -core need not be k -connected. But perhaps a k -core has a subgraph with high connectivity. For $0 \leq k \leq 2$, a k -core contains a k -connected subgraph. But it is easy to find 3-cores with no 3-connected subgraph. Extending this, we have the following result.

Proposition 127. *There exists a $2k$ -regular graph with no $k+1$ -connected subgraph.*

Proof. Let S be a set of k independent vertices of $K_k + \overline{K}_{k+1}$. Then the graph $H = (K_k + \overline{K}_{k+1}) \underset{S=S}{\cup} (K_k + \overline{K}_{k+1})$ has a set U of two vertices of degree k . Then $H \underset{U=U}{\cup} H$ is $2k$ -regular with no $k+1$ -connected subgraph.

□

Conjecture 128. *For $k \geq 3$, every k -core contains a $\lceil \frac{k}{2} \rceil$ -connected subgraph.*

The smallest undecided case is $k = 5$.

2.2.8 Collapse and Lobe Graphs

We have seen that each k -monocore graph contains a k -collapsible graph. We now

turn to the structure of collapsible subgraphs within monocore graphs.

We have seen that when a vertex is deleted from a k -monocore graph, some portion of the resulting graph may not be in its k -core. This motivates our next definitions.

Definition 129. Let G be a graph, $H \subseteq G$, $F = G - H$, $v \in H$. If G is a k -core and $C_k(G - v) = F$, we say G collapses to F with the deletion of v . We also say H collapses with the deletion of v . The analogous definitions apply for $e \in H$.

A k -lobe of a k -core G is a proper subgraph $H \subset G$ such that for all $v \in H$, $C_k(G - v) = G - H$. (We let H be induced by the edges deleted.)

The fact that some vertices can cause others to collapse suggests defining a relation.

Definition 130. The one-step collapse relation $cl_k^1(v, u)$ is defined as vRu if u and v are vertices of a k -core G and u is deleted in one iteration of the k -core algorithm on $G - v$. The r -step and (eventual) collapse relations are defined similarly.

The one-step collapse relation is easy to characterize.

Proposition 131. Let G be a k -core. Then $cl_k^1(v, u)$ is true $\iff d(u) = k$ and $uv \in G$.

The most commonly studied properties of a relation are whether it is reflexive, symmetric, and transitive.

Proposition 132. The r -step collapse relation is reflexive, but need not be symmetric or transitive.

Proof. It is obvious that this relation is reflexive. Consider $G = K_{k+1} \cup \bigcup_v K_{k+1}$. Then $cl_k^r(v, u)$ is true, but $cl_k^r(u, v)$ is not. Thus the relation need not be symmetric.

If u is deleted exactly r steps after the deletion of v , and w is deleted exactly r steps after the deletion of u , w need not be deleted within r steps of v . If w is deleted s steps from v , $r < s \leq 2r$ there exists some u such that transitivity fails.

□

We can guarantee transitivity in the r -step collapse relation by bounding the order of the graph.

Proposition 133. *If $r \geq n - k$, then cl_k^r is transitive for a k -core of order n .*

Proof. At least one vertex must be deleted at each step or the algorithm terminates. After $n - k$ steps, at most k vertices remain and so are deleted.

□

Corollary 134. *The eventual collapse relation is transitive.*

Proof. If v causes u to be deleted eventually, and u does so for v , then v does so for w .

□

Any relation can be represented as a digraph.

Definition 135. The (one-step) collapse digraph $CL^1(G)$ of a graph G is defined by the same vertex set and $vu \in E(CL^1(G)) \iff cl_k^1(v, u)$ is true. The n -step collapse digraph and eventual collapse digraph $CL(G)$ are defined similarly.

The one-step collapse digraph of G has a natural relationship with G .

Proposition 136. $CL^1(G)$ is a subdigraph of G , with equality $\iff G$ is regular.

Collapsible graphs have a natural relationship with the collapse digraph.

Proposition 137. *G is collapsible $\iff CL^1(G)$ is strongly connected.*

The action of the k -core algorithm can be described using the one-step collapse relation.

Proposition 138. *The n^{th} power of the matrix of $CL^1(G)$ tells whether u would be deleted following the deletion of v after n steps, if not sooner.*

Proof. The n^{th} power of the adjacency matrix of a digraph counts the number of (directed) walks from v to u . Thus if $CL(G)_{vu}^n$ is positive, u will be deleted unless it already has been.

□

We can describe the structure of the eventual collapse digraph.

Theorem 139. *The eventual collapse digraph can be partitioned into maximal complete subgraphs with either no edges between cliques or all possible edges having the same direction.*

Proof. Let u and v be vertices in the eventual collapse digraph of G and suppose that each is eventually deleted following the deletion of the other. Then uv and vu are in $CL(G)$. Since $CL(G)$ is transitive, u and v are contained in a maximal clique. Now suppose $cl(u, v)$ and $cl(v, u)$ are false. Then uv and vu are not in $CL(G)$ and xy is not in $CL(G)$ for all x and y in the same cliques as u and v , respectively.

Finally suppose that $cl(v, u)$ is true and $cl(u, v)$ is false. Then vu is in $CL(G)$ and uv is not in $CL(G)$. By transitivity, for all x, y in the same cliques as u and v , respectively, yx is in $CL(G)$ and xy is not in $CL(G)$.

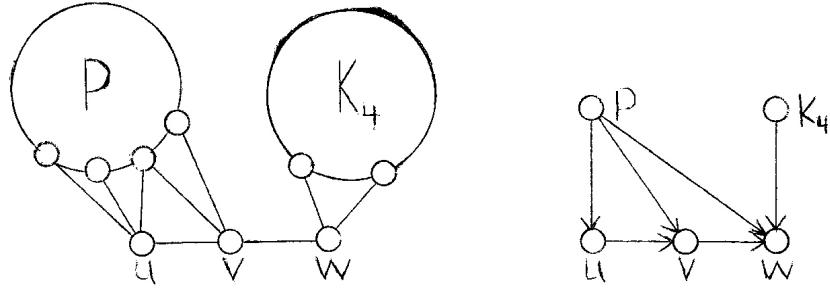


Figure 23: A 3-monocore graph G , where P represents the Petersen graph, and its lobe graph.

□

Corollary 140. *Every lobe of G generates a maximal clique in $CL(G)$, with no edges out of it.*

□

Proof. The deletion of any vertex in a lobe collapses the lobe, and nothing else.

This suggests a simpler way of describing the structure of a monocore graph.

Definition 141. The lobe graph $Lobe(G)$ is the digraph whose vertices are the maximal cliques of $CL(G)$ and whose edges follow the directions of the edges between the cliques of $CL(G)$.

Figure 23 gives an example of a graph and its lobe graph.

We are interested in the structure of lobe graphs.

Proposition 142. *Every lobe graph is acyclic.*

Proof. Suppose $Lobe(G)$ had a directed cycle. Then the deletion of any vertex of any clique corresponding to a vertex of the cycle would cause the collapse of every vertex on the cycle. This would yield a single clique, not a cycle. This is a contradiction.

□

Proposition 143. *Every vertex of $Lobe(G)$ with outdegree 0 corresponds to a lobe of G for a connected graph G .*

Proof. Every lobe of G generates a clique of $CL(G)$ with no outbound edges. Every such vertex corresponds to a clique of $CL(G)$. Thus every vertex causes the collapse of the subgraph in G and nothing else, so it is a lobe.

□

We would like to characterize which digraphs can be lobe graphs, but this remains an unsolved problem.

3 Cores and Graph Operations

We can study the interactions between cores and graph operations such as the Cartesian product, join, line graph, and complement.

3.1 Cartesian Products

Graph products offer a way to combine existing graphs to form new graphs. They have interesting interactions with k -cores. We consider the Cartesian product of two or more graphs. The structure of the k -core of a Cartesian product has a simple description.

Theorem 144. *Let G and H be graphs. Then*

$$C_k(G \times H) = \bigcup_{i+j=k} [C_i(G) \times C_j(H)].$$

Proof. Let $v = (u, w) \in \bigcup_{i+j=k} (C_i(G) \times C_j(H))$. Then there exist i, j with $i + j = k$ and $v \in C_i(G) \times C_j(H)$. Then $u \in C_i(G)$ and $w \in C_j(H)$. Then v is adjacent to at least $i + j = k$ vertices in $C_i(G) \times C_j(H)$. Since this holds for any vertex in $C_i(G) \times C_j(H)$, $v \in C_k(G \times H)$. Thus $\bigcup_{i+j=k} (C_i(G) \times C_j(H)) \subseteq C_k(G \times H)$.

Let $v = (u, w) \in C_l(G \times H)$, $l = C(v) \geq k$. Now there exists $v' = (u', w') \in C_l(G \times H)$ with $d(v') = l$, since otherwise the l -core is also the $l+1$ -core, and $C(v) \geq l+1$. Denote the graph induced by the vertices of $C_l(G \times H)$ contained in the same copy of G as v by $G(l, v)$. Then there exist i, j with $i + j = l$, $d(u') = i$ in $G(l, v')$, and $d(w') = j$ in $H(l, v')$. Let $x \in G(l, v')$. Then $d(x) \geq l - j = i$ in $G(l, v')$. Thus $x \in C_i(G)$, so $u' \in C_i(G)$. Similarly, $w' \in C_j(H)$. Thus $v' \in C_i(G) \times C_j(H)$.

If $v \in C_i(G) \times C_j(H)$, we are almost done. Suppose not. WLOG, say $u \notin C_i(G)$. Then there exists $t > 0$ such that u is adjacent to $i-t$ vertices in $C_i(G)$. Thus $C_{i-t}(G)$. Then v is adjacent to at least $j+t$ vertices in $H(l, v)$. The same is true for any x in

3	4	5	6
2	3	4	5
1	2	3	4
0	1	2	3



Figure 24: Cores of a Cartesian product.

$H(l, v)$. Thus $w \in C_{j+t}(H)$. Thus $v \in C_{i-t}(G) \times C_{j+t}(H) \subseteq C_r(G) \times C_s(H)$ for some r, s with $r + s = k$, $i - t \geq r$, $j + t \geq s$. Thus $C_k(G \times H) \subseteq \bigcup_{i+j=k} (C_i(G) \times C_j(H))$. Thus $C_k(G \times H) = \bigcup_{i+j=k} (C_i(G) \times C_j(H))$.

□

Consider the following example shown in Figure 24. Take two graphs G and H and put all their vertices on one line each, one horizontal and one vertical. Arrange the vertices by core number. Then their product will have the form shown in the figure below, where the vertices of the Cartesian product graph contained in each box have the core number that appears in the box. So as the theorem states, the 2-core of the product is $(C_0(G) \times C_2(H)) \cup (C_1(G) \times C_1(H)) \cup (C_2(G) \times C_0(H))$.

The proof of this theorem is unexpectedly long. This theorem can be generalized to more than two factors.

Corollary 145. Let G_i be graphs, $1 \leq i \leq n$. Then

$$C_k(G_1 \times \dots \times G_n) = \bigcup_{\sum i_r=k} [C_{i_1}(G_1) \times \dots \times C_{i_n}(G_n)].$$

Proof. The result is obvious for $n = 1$ and has been proven for $n = 2$. Assume the result holds for $n = p$. Then

$$C_k(G_1 \times \dots \times G_{p+1}) = C_k((G_1 \times G_p) \times G_{p+1})$$

$$\begin{aligned} &= \bigcup_{i+j=k} [C_i(G_1 \times \dots \times G_p) \times C_j(G_{p+1})] \\ &= \bigcup_{i+j=k} \left[\bigcup_{\sum i_r=i} [C_{i_1}(G_1) \times \dots \times C_{i_p}(G_p)] \times C_j(G_{p+1}) \right] \\ &= \bigcup_{\sum i_r=k} [C_{i_1}(G_1) \times \dots \times C_{i_{p+1}}(G_{p+1})]. \end{aligned}$$

Thus the result holds for all n .

□

Theorem 144 implies a simple formula for the core number of a vertex of a Cartesian product.

Corollary 146. Let $v = (u, w) \in G \times H$. Then $C(v) = C(u) + C(w)$.

Proof. Let $v = (u, w) \in G \times H$ with $C(v) = k$. Then there exist i, j with $i + j = k$, so that $v \in C_i(G) \times C_j(H)$. Now $u \notin C_{i+1}(G)$ and $w \notin C_{j+1}(H)$ since otherwise v would be in $C_{k+1}(G \times H)$. Thus $C(u) = i$ and $C(w) = j$, so $C(v) = C(u) + C(w)$.

□

This result can be generalized to more dimensions.

Corollary 147. Let $v = (v_1, \dots, v_n) \in G_1 \times \dots \times G_n$. Then $C(v) = \sum_{i=1}^n C(v_i)$.

Proof. The result holds for $n = 1$. Assume the result holds for r dimensions, and let $v = (v_1, \dots, v_{r+1}) \in G_1 \times \dots \times G_{r+1}$. Then

$$C(v) = C((v_1, \dots, v_r)) + C(v_{r+1}) = \sum_{i=1}^r C(v_i) + C(v_{r+1}) = \sum_{i=1}^{r+1} C(v_i).$$

□

These results have an immediate corollary on the maximum core numbers.

Corollary 148. Let $v = (v_1, \dots, v_n) \in G_1 \times \dots \times G_n$. Then $\widehat{C}(v) = \sum_{i=1}^n \widehat{C}(v_i)$. In particular, for $n = 2$, $\widehat{C}(v) = \widehat{C}(v_1) + \widehat{C}(v_2)$.

Proof. It is purely a set theory result that $\widehat{C}(v) \leq \sum_{i=1}^n \widehat{C}(v_i)$. Let $k_i = \widehat{C}(G_i)$, and $v_i \in C_{k_i}(G_i)$ for all i , $1 \leq i \leq n$. Then $v = (v_1, \dots, v_n)$ has maximum core number $\sum_{i=1}^n \widehat{C}(v_i)$, so the upper bound is achieved. The latter statement is simply a special case of the former.

□

Using Corollary 146, we can characterize Cartesian products that are monocore.

Corollary 149. $G \times H$ is monocore \iff G and H are both monocore. Further, if G is k -monocore and H is l -monocore, then $G \times H$ is $k + l$ -monocore.

Proof. If G is k -monocore and H is l -monocore, then every vertex in $G \times H$ has core number $k + l$, so $G \times H$ is $k + l$ -monocore.

Suppose without loss of generality that G is not monocore. Let $u_1, u_2 \in G$, $w \in H$, $v_1 = (u_1, w)$, $v_2 = (u_2, w)$, $C(u_1) = k_1 \neq k_2 = C(u_2)$, $C(w) = l$. Then $C(v_1) = k_1 + l \neq k_2 + l = C(v_2)$, so $G \times H$ is not monocore.

□

This can be extended to higher dimensions immediately by induction.

Corollary 150. $G_1 \times \dots \times G_n$ is monocore \iff Every factor G_i is monocore. Further, if G_i is k_i -monocore for all i , then $G_1 \times \dots \times G_n$ is $(\sum k_i)$ -monocore.

We can similarly characterize the structure of the proper k -shell of a Cartesian product in terms of the k -shells of its factors.

Theorem 151. Let G and H be graphs. Then

$$S'_k(G \times H) = \bigcup_{i+j=k} [S'_i(G) \times S'_j(H)].$$

Proof. Let $v = (u, w) \in \bigcup_{i+j=k} (S'_i(G) \times S'_j(H))$. Then there exist i, j with $i + j = k$ and $v \in S'_i(G) \times S'_j(H)$. Then $u \in S'_i(G)$ and $w \in S'_j(H)$. Then $C(u) = i$ and $C(w) = j$. Then $C(v) = C(u) + C(w) = i + j = k$, so $v \in S'_k(G \times H)$. Thus $\bigcup_{i+j=k} (S'_i(G) \times S'_j(H)) \subseteq S'_k(G \times H)$.

Let $v = (u, w) \in S'_k(G \times H)$. Then $C(v) = k$, so $C(v) = C(u) + C(w)$. Then there exist nonnegative integers i, j such that $C(u) = i$ and $C(w) = j$, and $i + j = k$. Thus $u \in S'_i(G)$ and $w \in S'_j(H)$. Thus $v \in S'_i(G) \times S'_j(H)$, so $S'_k(G \times H) \subseteq \bigcup_{i+j=k} (S'_i(G) \times S'_j(H))$.

Thus

$$S'_k(G \times H) = \bigcup_{i+j=k} [S'_i(G) \times S'_j(H)].$$

□

This can be generalized to higher dimensions.

Corollary 152. Let G_i be graphs, $1 \leq i \leq n$. Then

$$S'_k(G_1 \times \dots \times G_n) = \bigcup_{\sum i_r=k} [S'_{i_1}(G_1) \times \dots \times S'_{i_n}(G_n)].$$

The proof is essentially the same as for the corresponding result on cores.

It is straightforward to verify the following description of the k -boundary of a Cartesian product.

Corollary 153. Let G and H be graphs. Then

$$B_k(G \times H) = \bigcup_{i+j=k} [S'_i(G) \times B_j(H)] \cup \bigcup_{i+j=k} [B_i(G) \times S'_j(H)].$$

The k -shell of a Cartesian product is induced by the edges of the proper k -shell and the edges between vertices in the proper k -shell and the k -boundary. It does not seem to have as convenient a description as the proper k -shell.

We can describe collapsible Cartesian products.

Theorem 154. $G \times H$ is collapsible \iff G and H are both collapsible and G or H is regular. Further, if G is k -collapsible and H is l -collapsible, then $G \times H$ is $k+l$ -collapsible.

Proof. (\Rightarrow) WLOG, let G be k -monocore and not collapsible, and H be l -monocore. Then G contains a proper subgraph G' with $\delta(G') = k$. Then $G' \times H \subset G \times H$ and $\delta(G' \times H) = k + l$. Thus $G \times H$ is not collapsible.

Now let G and H be nonregular. Then there exists $u \in G$, $w \in H$ so that $d(u) > k$ and $d(w) > l$. Let $v = (u, w) \in G \times H$. Now if $v' = (u', w')$ is adjacent to v , then

$u = u'$ or $w = w'$. Then $d(v') > k + l$. Thus $\delta(G \times H - v) \geq k + l$, so $G \times H$ is not collapsible.

(\Leftarrow) Let G, H be collapsible and WLOG, G be regular. Let $v = (v_1, v_2) \in G \times H$. Then v is adjacent to $u = (v_1, u_2)$, $d(u_2) = l = \delta(H)$. Then u is not in $C_{k+l}(G \times H - v)$. Then neither are (x, u_2) for all x in G . Then neither are (x, y) for all x in G and y in H . Thus $G \times H - v$ has no $k + l$ -core, so $G \times H$ is collapsible.

The final statement follows immediately from the minimum degrees.

□

Corollary 155. $G_1 \times \dots \times G_n$ is collapsible \iff All graphs G_i are collapsible and at most one G_i is nonregular.

We have the following corollary on core-critical products.

Corollary 156. $G \times H$ is core-critical \iff G and H are both core-critical and G or H is regular.

Proof. (\Rightarrow) If both graphs are nonregular, then $G \times H$ is not collapsible. If one of them is monocore and not core-critical, then it has an edge that joins vertices of more than minimum degree. Hence so does $G \times H$, so it is not core-critical.

(\Leftarrow) If these conditions hold, then $G \times H$ is monocore and has no adjacent vertices with larger than minimum degree, so it is core-critical.

□

An area for future research is the structure of cores of other products such as the tensor product, lexicographic product, and offspring product.

3.2 Graph Joins

We now describe the k -core of the join of two graphs.

Theorem 157. *Let G and H be graphs. Then*

$$C_k(G + H) = \min_{i,j} \{C_i(G) + C_j(H) \mid i + |C_j(H)| \geq k \text{ and } j + |C_i(G)| \geq k\}$$

where the minimum is unique over both i and j .

Proof. Let property $P(i, j)$ be that $i + |C_j(H)| \geq k$ and $j + |C_i(G)| \geq k$, and let $C_i(G) + C_j(H)$ satisfy $P(i, j)$. Let $i' \leq i$, $j' \leq j$, and $P(i', j)$ and $P(i, j')$ be true. Then $P(i', j')$ is also true since $i' + |C_j(H)| \geq k$ and $C_j(H) \subseteq C_{j'}(H)$ imply $i' + |C_{j'}(H)| \geq k$, and similarly for the other case. Thus the minimum is well-defined.

Let $v \in C_i(G) + C_j(H)$, $P(i, j)$ true for i, j minimum. If $v \in G$, then $d(v) \geq i + |C_j(H)| \geq k$, and if $v \in H$, $d(v) \geq j + |C_i(G)| \geq k$. Thus $v \in C_k(G + H)$, so $C_i(G) + C_j(H) \subseteq C_k(G + H)$.

Let $G' = G \cap C_k(G + H)$ and $H' = H \cap C_k(G + H)$. Then $C_k(G + H) = G' + H'$. Let $\delta(G') = i$ and $\delta(H') = j$. Then $G' \subseteq C_i(G)$ and $H' \subseteq C_j(H)$. If $v \in G'$, then $i + |H'| \geq k$ and if $v \in H'$, then $j + |G'| \geq k$. Now $|C_i(G)| \geq |G'|$ and $|C_j(H)| \geq |H'|$, so $P(i, j)$ is true. Thus $C_k(G + H) \subseteq C_i(G) + C_j(H) \subseteq C_{i'}(G) + C_{j'}(H)$, where i', j' are the guaranteed minimums.

Thus $C_k(G + H) = \min_{i,j} \{C_i(G) + C_j(H) \mid P(i, j)\}$.

□

Consider the following example, shown in Figure 25. Let G be the graph on the left and H be on the right. Both $C_2(G) + C_2(H)$ and $C_3(G) + C_2(H)$ are 6-cores, but $C_2(G) + C_2(H)$ is the 6-core of $G + H$, as guaranteed in the theorem.

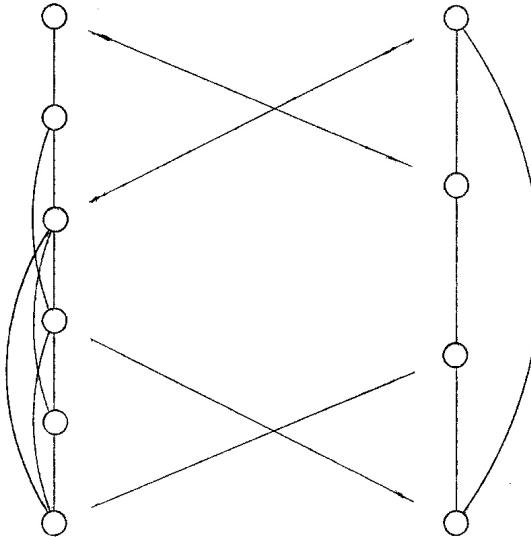


Figure 25: The cores of a join of two graphs. (Not all edges are shown.)

This can be generalized to more than two graphs.

Corollary 158. *Let G_j , $1 \leq j \leq n$ be graphs. Then*

$$C_k(G_1 + \dots + G_n) = \min_{i_1, \dots, i_n} \left\{ C_{i_1}(G_1) + \dots + C_{i_n}(G_n) \mid i + \sum_{j=1, j \neq i}^n |C_{i_j}(G_j)| \geq k \forall i \right\}.$$

We have the following corollary for the maximum core number.

Corollary 159. *We have $\widehat{C}(G + H) = \max_{i,j} \min(i + |C_j(H)|, j + |C_i(G)|)$.*

The following examples follow from the previous theorem. We have $C_k(G + v) = C_{k-1}(G) + v$. More generally, $C_k(G + K_n) = C_{k-n}(G) + K_n$, and $C_k(G + \overline{K}_n) = C_{k-n}(G) + \overline{K}_n$ provided $|C_{k-n}(G)| \geq k$. This implies that given $a \leq b$, $K_{a,b}$ is a -monocore, justifying our earlier assertion in the first section. Similarly, given $a_1 \leq \dots \leq a_n$, K_{a_1, \dots, a_n} is $(a_1 + \dots + a_{n-1})$ -monocore. Also, the wheel W_n is 3-monocore.

We have the following for proper k -shells.

Corollary 160. *Let i_k and j_k be the minimums for the k -core of $G + H$ guaranteed by the previous theorem. Then*

$$S'_k(G + H) = (C_{i_{k+1}}(G) - C_{i_k}(G)) + (C_{j_{k+1}}(H) - C_{j_k}(H)).$$

No complete characterization of monocore joins is known. It is immediate that the join of two monocore graphs is monocore, but the converse is false. For example, $(K_2 \cup K_1) + (K_2 \cup K_1)$ is 3-monocore without either factor being monocore.

For monocore collapsible graphs, we have the following partial results.

Definition 161. The score of G in $G + H$ is $\delta(G) + n(H)$. G has minimum score in $G + H$ if $\text{score}(G) \leq \text{score}(H)$.

Theorem 162. *Let $\text{score}(G) = k$, $\text{score}(H) = l \geq k$. If every component of G is k -collapsible with order at least $l - k + 1$, then $G + H$ is collapsible.*

Proof. Assume the hypothesis. Deletion of any vertex in the copy of G in $G + H$ collapses a component with order at least $l - k + 1$. This makes some vertex in the copy of H in $G + H$ have degree less than k , so it also collapses. Its deletion causes a vertex in each component of G to have degree less than k , so all of G collapses, and so does all of H . Starting with a vertex in the copy of H similarly collapses $G + H$.

□

Note that order $l - k + 1$ is best possible in this result since $6K_1 + 3K_2$ is 6-monocore but not 6-collapsible.

Corollary 163. *If G has minimum score in $G + H$ and G is collapsible, then $G + H$ is collapsible.*

Proof. We have $l - k = (\delta(H) + n(G)) - (\delta(G) + n(H)) \leq n(G) - \delta(G) - 1 \leq n - 1$, so the result follows immediately from the theorem. \square

Proposition 164. *If G and H are monocore with equal score, then $G + H$ is collapsible.*

Proof. $G + H$ is collapsible unless it has a proper subgraph with both factors having the same scores as the whole graph. But at least one of its components must have fewer vertices, and the other has no larger core to compensate for this. \square

We have a good characterization for core-critical joins.

Theorem 165. *Given $G + H$, let $\text{score}(G) \geq \text{score}(H) = k$ and $r = \text{score}(G) - \text{score}(H)$. Then $G + H$ is k -core-critical if and only if either*

1. *G and H have the same score k , one of them is regular, and the other has no adjacent vertices of more than minimum degree*
2. *$r > 0$, G is empty and each component of H is j -regular for some j and the order of the smallest component of H is more than r .*

Proof. (\Leftarrow) It is straightforward to verify that in both cases, the graphs described are k -core-critical.

(\Rightarrow) If $G + H$ is k -core-critical, then at least one of G or H is regular. If G and H have the same score, then certainly if H is nonregular, it has no adjacent vertices of more than minimum degree. Now if G has larger score than H , G has no adjacent vertices, so it is empty. Then H must be regular, and each component must be large enough so that if it is deleted, the score of G becomes less than k . \square

3.3 Edge Cores

One well-known graph operation is forming the line graph of a graph. The line graph $L(G)$ of a nonempty graph G has vertices that correspond to the edges of G , with vertices of $L(G)$ adjacent when the corresponding edges of G are adjacent. There are many corresponding properties of graphs for vertices and edges. These can often be related via line graphs. For example, an independent set of edges of G corresponds to an independent set of vertices of $L(G)$. For background on line graphs see [West [60] p. 279-286]

While the cores studied in this dissertation are defined in terms of minimum degree, analogous concepts can be defined in terms of other graph parameters. In particular, in light of what we know about line graphs, the following definition is natural.

Definition 166. The k -edge-core of G , $EC_k(G)$, is the maximal edge-induced subgraph H of G such that each edge of H is adjacent to at least k edges of H .

It is easily checked that, analogous with k -cores, the k -edge-core is well-defined and the k -edge-cores are nested. As expected, k -edge-cores are connected with line graphs.

Theorem 167. We have $C_k(L(G)) = L(EC_k(G))$.

Proof. We have $EC_k(G) \subseteq G$, so $L(EC_k(G)) \subseteq L(G)$. Each edge in $EC_k(G)$ is adjacent to at least k edges, so $\delta(L(EC_k(G))) \geq k$, so $L(EC_k(G)) \subseteq C_k(L(G))$.

Let $v \in C_k(L(G))$. Then $d_{L(G)}(v) \geq k$, and v corresponds to an edge e of G , which is adjacent to at least k edges also corresponding to vertices of $C_k(L(G))$. This is true for all $v \in C_k(L(G))$, so $e \in EC_k(G)$. Thus $v \in L(EC_k(G))$. Thus $C_k(L(G)) \subseteq L(EC_k(G))$.

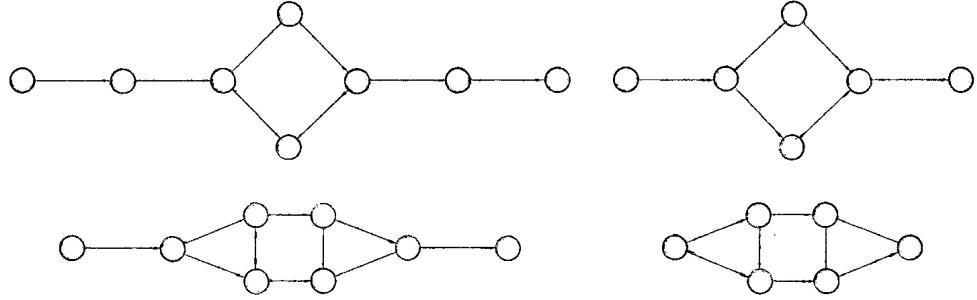


Figure 26: A graph G , $EC_2(G)$, $L(G)$, and $C_2(L(G)) = L(EC_2(G))$.

□

Figure 26 illustrates the theorem.

We can describe the edge cores for small k . Clearly $C_0(L(G)) = L(G)$. The only way to get an isolated vertex in a line graph is from a component that is K_2 , so this corollary follows immediately from the theorem.

Corollary 168. *Let H be the graph formed from G by deleting any component that is K_2 . Then $C_1(L(G)) = L(H)$.*

The next corollary is similar.

Corollary 169. *Let H be the graph formed from G by iteratively deleting each vertex of degree 1 adjacent to a vertex of degree 1 or 2 in G . Then $C_2(L(G)) = L(H)$.*

Proof. The only way an end-vertex can occur in $L(G)$ is if G has an edge incident with exactly one edge, and hence an end-vertex adjacent to a vertex of degree 2. Hence H is exactly the 2-edge-core of G . By the previous theorem, the result follows.

□

We now define an analogous concept to the core number.

Definition 170. The edge core number of a vertex, $EC(v)$, is the largest value for k such that $v \in EC_k(G)$. The maximum edge core number of a graph, $\widehat{EC}(G)$, is the maximum of the edge core numbers of the vertices of G .

We have the following corollary to the initial theorem.

Corollary 171. We have $\widehat{C}(L(G)) = \widehat{EC}(G)$.

We will find the following result useful.

Theorem 172. [50] For a connected graph G , $L(G)$ is regular if and only if G is regular or G is bipartite with vertices of the same partite set having the same degrees.

We can provide bounds for $\widehat{C}(L(G))$ in terms of $\Delta(G)$.

Theorem 173. Let $\Delta = \Delta(G)$. Then $\Delta - 1 \leq \widehat{C}(L(G)) \leq 2(\Delta - 1)$, and the latter is an equality for connected graphs if and only if G is regular.

Proof. Let v have degree Δ . Then $L(G)$ contains K_Δ , so it has a $\Delta - 1$ -core.

For the upper bound, we have $\widehat{C}(L(G)) \leq \Delta(L(G)) \leq 2(\Delta - 1)$ since an edge of G is adjacent to at most $\Delta - 1$ edges on each side. By Proposition 5, The first inequality in this chain is an equality exactly when $L(G)$ is regular, which occurs exactly when G is regular or G is bipartite with vertices of the same partite set having the same degrees. The second inequality is an equality exactly when G has

two adjacent vertices of maximum degree, which is true for nonempty regular graphs, but false for the latter class of graphs.

□

The lower bound is sharp, being exact for trees, but it is unclear what all the extremal graphs are.

The operation of forming a line graph can be applied repeatedly. The iterated line graph $L^k(G)$ is defined recursively by $L^0(G) = G$, $L^k(G) = L(L^{k-1}(G))$. We consider the maximum core number for iterated line graphs.

For regular graphs, we have the following recursion. Denote $\Delta_k = \Delta(L^k(G))$. Then $\Delta_0 = \Delta$ and $\Delta_k = 2\Delta_{k-1} - 2$. This immediately yields $\Delta_{k+1} = 2\Delta_k - 2$, and subtracting the former from the latter, we find $\Delta_{k+1} = 3\Delta_k - 2\Delta_{k-1}$. This recurrence relation must have a solution of the form $\Delta_k = r^k$. Substituting, we find $r^{k+1} = 3r^k - 2r^{k-1}$, so $r^2 - 3r + 2 = 0$, which has roots 1 and 2. The general solution to the recurrence relation must have the form $\Delta_k = C \cdot 2^k + D \cdot 1^k$. The initial conditions imply $\Delta = C + D$ and $2\Delta - 2 = 2C + D$, so $C = \Delta - 2$ and $D = 2$.

Corollary 174. *We have $(\frac{\Delta-3}{2})2^k + 2 \leq \widehat{C}(L^k(G)) \leq (\Delta - 2)2^k + 2$, $k \geq 1$, with equality for the upper bound over connected graphs exactly for regular graphs.*

Proof. The formula for the upper bound is derived in the previous discussion and can easily be verified by induction. The extremal graphs follow from the previous theorem. For the lower bound, we have $\Delta_1 \geq \Delta - 1$ and $\Delta_2 \geq 2\Delta - 4$, with equality for stars. Solving for the coefficients in this case, we have $\Delta - 1 = 2C + D$ and $2\Delta - 4 = 4C + D$, so $C = \frac{\Delta-3}{2}$ and $D = 2$.

□

It is unclear what are the extremal graphs for the lower bound. It is also unknown whether all values in the range may be obtained.

We can examine the relationships between cores and edge cores.

Proposition 175. *We have $C_k(G) \subseteq EC_{2k-2}(G)$.*

Proof. Every edge of the k -core is adjacent to at least $k - 1$ other edges on each end. \square

This is an equality for regular graphs, but the complete list of extremal graphs is unknown.

Corollary 176. *We have $\widehat{C}(L(G)) \geq 2\widehat{C}(G) - 2$.*

For example, we have $\widehat{C}(L(K_{m,n})) = m+n-2$. Perhaps equality always eventually holds.

Conjecture 177. *For each graph G , there exists K such that for all $k \geq K$, $\widehat{C}(L^{k+1}(G)) = 2 \cdot \widehat{C}(L^k(G)) - 2$.*

We now examine analogues for monocore and collapsible graphs.

Definition 178. Let the edge degree of e , $ed(e)$, be the number of edges adjacent to edge e in G . Let $e\delta(G) = \min_{e \in G} ed(e)$.

G is k -edge-monocore if $e\delta(G) = \widehat{EC}(G)$.

G is k -edge-collapsible if it is k -edge-monocore and has no proper k -edge-monocore subgraph.

Proposition 179. *G is 0-edge-collapsible if and only if $G = K_2$.*

G is 1-edge-collapsible if and only if $G = P_3$.

G is 2-edge-collapsible if and only if $G = K_{1,3}$ or C_n .

Proof. The first two are obvious. $K_{1,3}$ and C_n are certainly both 2-edge-collapsible. If G is 2-edge-collapsible, then either it has a vertex of degree 3, and so contains $K_{1,3}$, or every vertex has degree 2, so it contains a cycle.

□

Proposition 180. G is 0-edge-monocore if and only if every component of G is K_2 .

G is 1-edge-monocore if and only if every component of G is a path of length at least two.

We can characterize edge-monocore trees.

Proposition 181. A tree T is k -edge-monocore if and only if $\Delta(T) = k + 1$ and if $e = uv \in T$ with $i = d(u)$, $1 \leq i \leq k$, then $k - i + 1 \leq d(v) \leq k + 1$. (Hence every vertex adjacent to an end-vertex has degree $k + 1$.)

Proof. Every block of $L(T)$ is a clique. For $L(T)$ to be k -monocore, every end-block of $L(T)$ must be a k -clique, and $L(T)$ has no $k + 1$ -clique, so $\Delta(T) = k + 1$. For $e = uv$, $ed(e) \geq k \iff d(u) + d(v) \geq k + 2$. The converse is easy.

□

Corollary 182. A tree T is k -edge-collapsible if and only if $T = K_{1,k+1}$.

We can characterize 3-edge-collapsible graphs.

Proposition 183. A graph G is 3-edge-collapsible if and only if $G = K_{1,4}$ or G is connected with the property that $\Delta(G) = 3$, $\delta(G) = 2$, and G has no adjacent vertices of degree 2, and has no proper subgraph with this property.

Proof. The graphs in question are clearly 3-edge-collapsible. If G is 3-edge-collapsible with a vertex of degree more than 3, it contains $K_{1,4}$, and hence is $K_{1,4}$. Now any adjacent vertices must have degrees sum to at least 5 and some pair must sum to exactly 5, so $\Delta(G) = 3$, $\delta(G) = 2$, and G has no adjacent vertices of degree 2. If G had a proper subgraph with this property, it would not be 3-edge-collapsible.

□

3.4 Ramsey Core Numbers

The problem of Ramsey numbers is one of the major problems of extremal graph theory. Given positive integers t_1, t_2, \dots, t_k , the classical Ramsey number $r(t_1, \dots, t_k)$ is the smallest integer n such that for any decomposition of K_n into k factors, for some i , the i^{th} factor has a t_i -clique. This problem can be modified to require the existence of other classes of graphs. Since classical Ramsey numbers are defined, which is not trivial to show, such modifications are also defined, since every finite graph is a subgraph of some clique. When considering cores, the following modified problem arises naturally.

Definition 184. Given nonnegative integers t_1, t_2, \dots, t_k , the Ramsey core number $rc(t_1, t_2, \dots, t_k)$ is the smallest n such that for all edge colorings of K_n with k colors, there exists an index i such that the subgraph induced by the i^{th} color, H_i , has a t_i -core.

Several basic results can be obtained immediately.

Proposition 185. 1. $rc(t_1, t_2, \dots, t_k) \leq r(t_1 + 1, \dots, t_k + 1)$, the classical multidimensional Ramsey number.

2. For any permutation σ of $[k]$, $rc(t_1, t_2, \dots, t_k) = rc(t_{\sigma(1)}, t_{\sigma(2)}, \dots, t_{\sigma(k)})$. Thus we need only consider nondecreasing orderings of the numbers.

3. $rc(0, t_2, \dots, t_k) = 1$

4. $rc(1, t_2, \dots, t_k) = rc(t_2, \dots, t_k)$.

We can easily determine some classes of multidimensional Ramsey core numbers.

Proposition 186. For k dimensions, $rc(2, 2, \dots, 2) = 2k + 1$.

Proof. It is well known that the complete graph K_{2k} can be decomposed into k spanning paths, each of which has no 2-core. Thus $rc(2, 2, \dots, 2) \geq 2k + 1$. K_{2k+1} has size $\binom{2k+1}{2} = k(2k+1)$, so if it decomposes into k graphs, one of them has at least $2k+1$ edges, and hence contains a cycle. Thus $rc(2, 2, \dots, 2) = 2k + 1$.

□

The technique of this proof suggests a general upper bound for Ramsey core numbers.

Definition 187. The multidimensional upper bound for the Ramsey core number $rc(t_1, t_2, \dots, t_k)$ is the function $B(t_1, t_2, \dots, t_k)$, where $T = \sum t_i$ and

$$B(t_1, \dots, t_k) = \left\lceil \frac{1}{2} - k + T + \sqrt{T^2 - \sum t_i^2 + (2 - 2k)T + k^2 - k + \frac{9}{4}} \right\rceil.$$

With a definition like that, this had better actually be an upper bound.

Theorem 188. [The Upper Bound] $rc(t_1, t_2, \dots, t_k) \leq B(t_1, \dots, t_k)$.

Proof. The size of a maximal k -core-free graph of order n is $(k-1)n - \binom{k}{2}$. Now by the Pigeonhole Principle, some H_i has a t_i -core when

$$\binom{n}{2} \geq \sum_{i=1}^k \left((t_i - 1)n - \binom{t_i}{2} \right) + 1.$$

This leads to

$$n^2 - n \geq 2n \sum_{i=1}^k (t_i - 1) - \sum_{i=1}^k \left(t_i^2 - t_i \right) + 2.$$

Thus we obtain a quadratic inequality $n^2 - bn + c \geq 0$ with $b = 1 + 2 \sum t_i - 2k$ and $c = \sum (t_i^2 - t_i) - 2$. By the quadratic formula,

$$n \geq \frac{1}{2} \left(b + \sqrt{b^2 - 4c} \right)$$

and

$$\begin{aligned} b^2 - 4c &= (1 + 4T - 4k + 4T^2 - 8kT + 4k^2) - \left(4 \sum t_i^2 - 4T - 8 \right) \\ &= 4 \left(T^2 - \sum t_i^2 + (2 - 2k)T + k^2 - k + \frac{9}{4} \right) \end{aligned}$$

Thus

$$n \geq \left\lceil \frac{1}{2} - k + T + \sqrt{T^2 - \sum t_i^2 + (2 - 2k)T + k^2 - k + \frac{9}{4}} \right\rceil = B(t_1, \dots, t_k).$$

Now $rc(t_1, \dots, t_k) \leq \min \{n \mid n \geq B(t_1, \dots, t_k)\} = B(t_1, \dots, t_k)$.

□

Thus to show that a Ramsey core number achieves the upper bound, we must find a decomposition of the complete graph of order $B(t_1, \dots, t_k) - 1$ for which none of the factors contain the stated cores. For example, the decomposition in Figure 27 shows that $rc(3, 3) = 8$.

When first studying this problem in late 2008 I made the following conjecture, initially restricted to two dimensions.

Conjecture 189. *The upper bound is exact. That is, $rc(t_1, t_2, \dots, t_k) = B(t_1, \dots, t_k)$.*

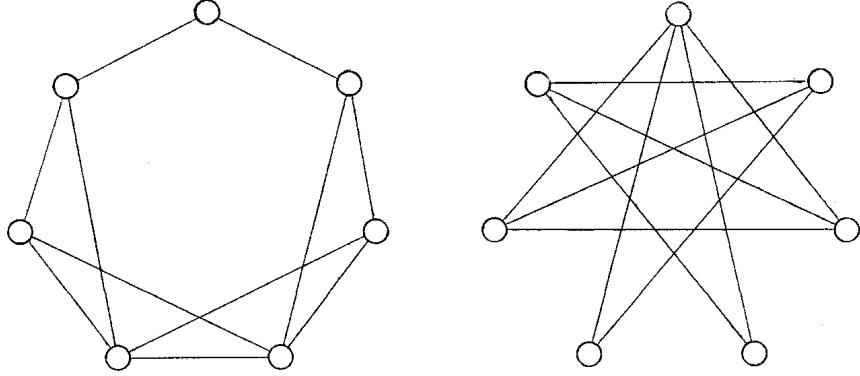


Figure 27: A decomposition that shows that $rc(3, 3) = 8$.

To prove this, we state the following theorem due to R. Klien and J. Schonheim from 1992 [37], which I became aware of in summer 2010.

Theorem 190. *Any complete graph with order $n < B(t_1, \dots, t_k)$ has a decomposition into k subgraphs with degeneracies at most $t_1 - 1, \dots, t_k - 1$.*

The proof of this theorem is long and difficult. It uses a complicated algorithm to construct a decomposition of a complete graph with order satisfying the inequality into k subgraphs given a decomposition of a smaller complete graph into $k - 1$ subgraphs without the first $k - 1$ cores, a copy of K_{t_k} , and some extra vertices. Thus the proof that the algorithm works uses induction on the number of subgraphs.

Using this theorem, proving the conjecture is not hard.

Theorem 191. *We have $rc(t_1, t_2, \dots, t_k) = B(t_1, \dots, t_k)$.*

Proof. We know that $B(t_1, \dots, t_k)$ is an upper bound. By the previous theorem, there exists a decomposition of the complete graph of order $B(t_1, \dots, t_k) - 1$ such that subgraph H_i has degeneracy $t_i - 1$, and hence has no t_i -core. Thus $rc(t_1, t_2, \dots, t_k) > B(t_1, \dots, t_k) - 1$, so $rc(t_1, t_2, \dots, t_k) = B(t_1, \dots, t_k)$. □

Since the exact answer depends on a complicated construction, some simpler constructions remain of interest.

Proposition 192. *[The Lower Bound]* We have $rc(t_1 + 1, t_2, \dots, t_k) \geq rc(t_1, \dots, t_k) + 1$.

Proof. Let $n = rc(t_1 + 1, t_2, \dots, t_k)$. Then there exists a decomposition of K_{n-1} with each factor having no t_i -core for all i . Let $H = G + v$. Consider the decomposition of K_n formed from the previous decomposition by joining a vertex to the first factor. Then the first factor has no $t_1 + 1$ -core. Thus $rc(t_1 + 1, t_2, \dots, t_k) \geq rc(t_1, \dots, t_k) + 1$.

□

The next corollary follows easily.

Corollary 193. Let $t_i \geq 2$ for all i , and $T = \sum t_i$. Then $rc(t_1, t_2, \dots, t_k) \geq T + 1$.

Proof. We have $rc(2, 2, \dots, 2) = 2k + 1$. The result follows by induction on the lower bound.

□

Along with the upper bound, this implies that for k dimensions, $rc(2, 2, \dots, 2, 3) = 2k + 2$.

There is one more lower bound to consider. This one varies the number of dimensions.

Lemma 194. Let $T = \sum t_i$, with $t_i \geq 2$ and $X = rc(t_1, t_2, \dots, t_k)$. Then $2(T - k) \geq X - 1$.

Proof. We have $\sum t_i^2 \geq T + 2$. We have the following implications.

$$T^2 - 2Tk + T + k^2 - k + \frac{1}{4} \geq T^2 - \sum t_i^2 + 2T - 2Tk + k^2 - k + \frac{9}{4}$$

$$T - k + \frac{1}{2} \geq \sqrt{T^2 - \sum t_i^2 + (2 - 2k)T + k^2 - k + \frac{9}{4}}$$

$$2(T - k) \geq B(t_1, \dots, t_k) - 1 \geq X - 1.$$

□

Theorem 195. *We have $rc(2, t_1, t_2, \dots, t_k) \geq 2 + rc(t_1, \dots, t_k)$.*

Proof. Let $X = rc(t_1, t_2, \dots, t_k)$. Then there exists an edge coloring of K_{X-1} so that for all i , H_i has no t_i -core. Add two vertices u, v to this graph. Each subgraph H_i can be extended by $t_i - 1$ edges to each new vertex. Together, they contribute $\sum(t_i - 1) = T - k$ edges. Now by the previous lemma, $2(T - k) \geq X - 1$, so the remaining edges form a tree. This gives an edge-coloring of K_{k+1} , with none of the given cores, so $rc(2, t_1, t_2, \dots, t_k) \geq 2 + X$.

□

Corollary 196. *For k dimensions and $t \geq 3$,*

$$2(k-2) + \left\lceil t + \frac{1}{2} + \sqrt{2t + \frac{1}{4}} \right\rceil \leq rc(2, 2, \dots, 2, t) \leq 2k + 2t - 4.$$

Proof. The formula for $B(2, t)$ and the theorem imply the lower bound. Setting $t = 2 + r$ gives $T = 2k + t$, and $\sum t_i^2 = 4k + r^2 + 2r$. Plugging this into the upper bound, we find $rc(2, 2, \dots, 2, 2 + r) \leq k + r + \frac{1}{2} + \sqrt{(k + (r - \frac{1}{2}))^2 + (-r^2 + r + 2)} \leq 2k + 2r$.

□

The upper bound in the proof is slightly better when t is large relative to k . This corollary implies that $rc(2, 2, \dots, 2, 4) = 2k + 4$. We can similarly show that $rc(2, 2, \dots, 2, 3, 3) = 2k + 4$ (k dimensions).

For two dimensions, the upper bound simplifies to where

$$B(s, t) = \left\lceil s + t - \frac{3}{2} + \sqrt{2(s-1)(t-1) + \frac{9}{4}} \right\rceil.$$

The formula for $rc(2, t)$ can be expressed in another form, and proven using a simple construction.

Theorem 197. Let $t = \binom{r}{2} + q$, $1 \leq q \leq r$. Then $rc(2, t) = \binom{r}{2} + r + q + 1 = t + r + 1 = B(2, t)$.

Proof. We first show that the Upper Bound for $rc(2, t)$ can be expressed as a piecewise linear function with each piece having slope one and breaks at the triangular numbers. Let $t = \binom{r}{2}$. Let

$$B'(s, t) = s + t - \frac{3}{2} + \sqrt{2(s-1)(t-1) + \frac{9}{4}}.$$

Then $B(s, t) = \lceil B'(s, t) \rceil$. Now

$$B'(2, t) = 2 + t - \frac{3}{2} + \sqrt{2 \cdot 1(t-1) + \frac{9}{4}} = t + \frac{1}{2} + \sqrt{2 \frac{r(r-1)}{2} + \frac{1}{4}} = t + \frac{1}{2} + \sqrt{\left(r - \frac{1}{2}\right)^2}$$

$= t + r$, which is an integer. Now $B'(2, t+1) > t + r + 1$, so $B(2, t+1) \geq t + r + 2$. Then $B(2, t+q) \geq t + r + 1 + q$ for $q \geq 1$ by the Lower Bound. Now $B'(2, t+r) = B'(2, \binom{r+1}{2}) = t+r+r+1$, an integer. Thus $B(2, t+r) = t+r+r+1$, so $B(2, t+q) \leq t + r + 1 + q$ for $1 \leq q \leq r$ by the Lower Bound. Thus $B(2, t+q) = t + r + 1 + q$, $1 \leq q \leq r$, so $rc(2, t) \leq t + r + 1$ for $t = \binom{r}{2} + q$.

We next show that the upper bound is attained with an explicit construction. Let T be a caterpillar whose spine with length r is

$$r - r - (r-1) - (r-2) - \dots - 4 - 3 - 2,$$

s \ t	1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10	11	12
2	3	5	6	8	9	10	12	13	14	15	17
3	4	6	8	10	11	13	14	15	17	18	19
4	5	8	10	11	13	15	16	18	19	20	22
5	6	9	11	13	15	16	18	20	21	23	24
6	7	10	13	15	16	18	20	21	23	25	26
7	8	12	14	16	18	20	22	23	25	26	28

Table 5: Values of some 2-dimensional Ramsey core numbers.

where a number is the degree of a vertex and end-vertices are not shown. Now T has

$$[(r-1) + (r-2) + (r-3) + \dots + 2 + 1] + 1 = \binom{r}{2} + 1$$

end-vertices, so it has order $n = \binom{r}{2} + r + 1$. The degrees of corresponding vertices in T and \bar{T} must add up to $n - 1 = \binom{r}{2} + r$. Then the degrees of corresponding vertices in \bar{T} are

$$\binom{r}{2}, \binom{r}{2}, \binom{r}{2} + 1, \binom{r}{2} + 2, \dots, \binom{r}{2} + r - 3, \binom{r}{2} + r - 2.$$

Take the $(\binom{r}{2} + 1)$ -core of \bar{T} . The first two vertices will be deleted by the k -core algorithm. The p^{th} vertex will be deleted because it has degree $\binom{r}{2} + p - 2$ and is adjacent to the first $p - 2$ vertices, which were already deleted. Thus all the spine vertices will be deleted, leaving $\binom{r}{2} + 1$ vertices, which must also be deleted. Thus \bar{T} has no $(\binom{r}{2} + 1)$ -core, and T has no 2-core. Thus $rc(2, \binom{r}{2} + 1) \geq \binom{r}{2} + r + 1 + 1$. Thus $rc(2, \binom{r}{2} + q) \geq \binom{r}{2} + r + 1 + q$ by the Lower Bound.

Thus $rc(2, t) = t + r + 1$ for $t = \binom{r}{2} + q$, $1 \leq q \leq r$.

□

4 Applications of Cores in Graph Theory

4.1 Proper Vertex Coloring

Vertex coloring is perhaps the most popular topic in graph theory. Cores have natural applications to vertex coloring problems. The chromatic number of a graph, $\chi(G)$, is the smallest number of subsets into which the vertices of a graph can be partitioned so that no two vertices in the same subset are adjacent. See the following book for background on graph coloring.

Chromatic Graph Theory by Chartrand and Zhang [16]

Determining the chromatic number when it is larger than two is an NP-complete problem, meaning that it is difficult to efficiently determine in general. Because of this, a major issue in chromatic graph theory is to determine upper and lower bounds for the chromatic number in terms of graph parameters that can be calculated efficiently. If the bounds are good, they may be equal for a graph in question, solving the problem in that case.

Many, but not all upper bounds for $\chi(G)$ are based on algorithms for coloring graphs. The most naive coloring scheme simply puts the vertices in any order and assigns the least available color to a vertex that has not already been used on its neighbor. This leads to the upper bound $\chi(G) \leq 1 + \Delta(G)$. This is not a very good upper bound, as it is easy to construct graphs that have maximum degree much larger than their chromatic number.

4.1.1 The Core Number Bound

Using cores, we can prove a better upper bound. We can establish a deletion sequence for a graph by successively deleting vertices of smallest degree. This orders the vertices in terms of core number. Reverse this sequence to obtain a construction sequence for

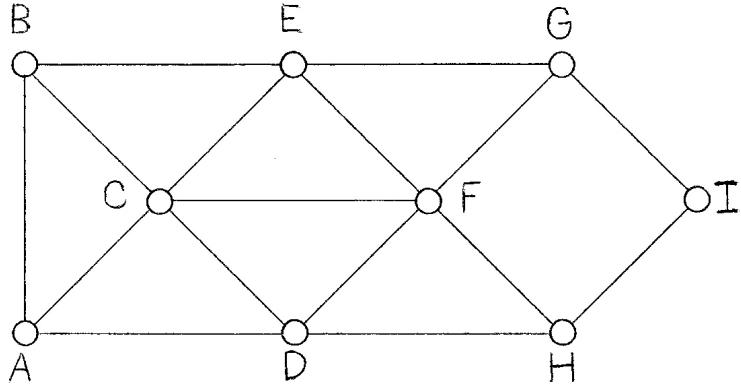


Figure 28: Coloring this graph G greedily using construction sequence ABCDEFGHI produces the 4-coloring 1,2,3,2,1,4,2,1,3; and $\chi(G) = 4 = 1 + \widehat{C}(G)$.

the graph. Color the graph using this sequence.

We obtain a bound first proved by Szekeres and Wilf in 1968 [57], restated in terms of cores.

Theorem 198. [*The core number bound*] For any graph G , $\chi(G) \leq 1 + \widehat{C}(G)$.

Proof. Establish a construction sequence for G . Each vertex has degree at most equal to its core number when colored. Coloring it uses at most one more color. Thus $\chi(G) \leq 1 + \widehat{C}(G)$. □

This bound was originally stated as $\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$, with the maximum over all subgraphs of G . Of course, checking every subgraph of a graph is not realistic. This can easily be restricted to induced subgraphs, and the theorem is sometimes stated with this condition. But even so, there are 2^n possibilities to check. Of course, we don't really need to check all of them. In fact, we only need to check one, the maximum core. It is almost immediate that $\widehat{C}(G) = \max_{H \subseteq G} \delta(H)$. The reader may note that this upper bound, $1 + \max_{H \subseteq G} \delta(H)$, has been named the Szekeres-Wilf number after its discoverers.

This upper bound is essentially as easy to compute as $1 + \Delta(G)$. Computing the maximum core number has complexity $O(m)$. Computing the maximum degree also has complexity $O(n^2)$, though it might be improved to $O(m)$ by using an edge list, which is more efficient for sparse graphs.

Certainly we have $\widehat{C}(G) \leq \Delta(G)$ for all graphs. Thus the core number bound is always at least as good as the maximum degree bound. We have seen in Proposition 5 that for connected graphs, this is an equality exactly for regular graphs.

Now almost all graphs are not regular, so the core number bound is better for almost all graphs. We should note however that Brooks' Theorem states that the maximum degree bound is attained only for complete graphs and odd cycles, so it can be improved by one to $\chi(G) \leq \Delta(G)$ by excluding those cases. The core number bound can be employed to give a reasonably short proof of Brooks' Theorem. We employ the following lemma based on the proof of Lovasz, modified by West (see p.198 [60]).

Lemma 199. *Given $r \geq 3$, if G is an r -regular 2-connected noncomplete graph, then G has a vertex v with two nonadjacent neighbors x and y such that $G - x - y$ is connected.*

Proof. Let G satisfy the hypothesis. Let u be a vertex of G . If $G - u$ is 2-connected, let u be x and y be a vertex at distance two from u , which exists because G is regular and not complete. Let v be their common neighbor.

If $G - u$ has connectivity one, then let v be u . Then G has at least two end-blocks, and u has neighbors in all of them. Let x, y be two such neighbors. They must be nonadjacent, and $G - x - y$ is connected since blocks have no cut-vertices and $r \geq 3$.

□

Theorem 200. *[Brooks' Theorem] If G is connected, then $\chi(G) = 1 + \Delta(G) \iff G$ is complete or an odd cycle.*

Proof. Equality certainly holds for cliques and odd cycles. Let G satisfy the hypotheses. Then by the previous result, G is r -regular. The result certainly holds for $r \leq 2$, so we may assume $r \geq 3$. If G had a cut-vertex, each block could be colored with fewer than $r + 1$ colors to agree on that vertex, so we may assume G is 2-connected and to the contrary not a clique.

By the lemma, we can establish a deletion sequence for G starting with some vertex v and ending with its nonadjacent neighbors x and y so that all vertices but v have at most $r - 1$ neighbors when deleted. Reversing this yields a construction sequence and coloring greedily gives x and y the same color, so G needs at most r colors.

□

Even though the core number bound is strictly better in almost all cases, it seems not to have attained the same level of prominence as the maximum degree bound, perhaps due to the fact that without the concept of cores, it may initially appear difficult to compute.

We can get a sense of how much better this bound is by examining some special classes of graphs. For the stars, $K_{1,s}$, the maximum degree bound gives $1+s$, while the core number bound gives two, which is exact. Indeed, the difference between the two bounds for trees (more generally, forests) can be arbitrarily large. For the wheel W_n , the maximum degree bound gives $1+n$, while the core number bound gives four, which is exact for odd wheels and one off for even wheels.

Unfortunately, the core number bound can still be arbitrarily far from exact. For example, the complete bipartite graph $K_{n,n}$ has chromatic number 2, but the core number bound gives $n+1$.

The core number bound has the following corollary on coloring k -shells.

Corollary 201. *If G has a k -shell, then $\chi(S_k(G)) \leq k+1$.*

Proof. A k -shell is $k+1$ -core-free, so its maximum core number is k . The corollary

follows from the theorem.

□

One way to understand the properties of a class of graphs is to study the extremal graphs with that property. A graph is critically k -chromatic if it has chromatic number k but no proper subgraph has chromatic number k . Our next corollary follows from the previous one.

Corollary 202. *A critically k -chromatic graph is a $k - 1$ -core.*

Note however that a critically k -chromatic graph need not have minimum degree $k - 1$. For example, the Chvatal graph is critically 4-chromatic and 4-regular. In fact, critically k -chromatic graphs need not be collapsible, nor even monocore. To show this, we need to describe Mycielski's construction.

Mycielski's construction starts with a graph G and defines a new graph $M(G)$ as follows. For each vertex of G , add a corresponding vertex adjacent to its neighbors. Finally, add one more vertex adjacent to all the new vertices. If G has order n , $M(G)$ has order $2n + 1$. The original vertices have their degrees doubled, the corresponding new vertices have degree one larger than their original copies, and the new vertex has degree equal to the order of the original graph. Thus for a nonempty graph G , $\delta(M(G)) = \delta(G) + 1$, and $\Delta(M(G)) = \max\{2\Delta(G), n(G)\}$. It is well-known that Mycielski's construction does not create any new triangles, $\chi(M(G)) = \chi(G) + 1$, and if G is k -critical, $M(G)$ is $k + 1$ -critical.

Applying Mycielski's construction to the smallest 2-critical graph, K_2 , yields the smallest triangle-free 3-critical graph, C_5 . Applying it to C_5 yields the Grotzsch graph, which at order 11 is the smallest triangle-free 4-critical graph. It is 3-monocore, but not 3-collapsible because no two vertices of degree three are neighbors. Applying Mycielski's construction to the Grotzsch graph yields a 5-critical graph G of order 23 with $\delta(G) = 4$ and $\widehat{C}(G) = 5$. It is easily seen that $\widehat{C}(M(G)) \geq \widehat{C}(G) + 1$, but

this need not be an equality as in the previous example. No exact description of the maximum core number of the result of applying Mycielski's construction to a graph is known.

We would like a way to possibly improve on the core number bound. If it is not exact, then the maximum core can be colored with fewer than $1 + \hat{C}(G)$ colors. This forms the rationale for our next bound.

Theorem 203. *For all graphs G ,*

$$\chi(G) \leq \max_k \{\min \{k + 1, \chi(C_k(G))\}\}.$$

Proof. Establish a construction sequence for G . The cores are colored from largest to smallest core number. Thus so are the shells. The k -core can be colored with $\chi(C_k(G))$ colors, and the k -shell can be colored with at most $k + 1$ colors. At each stage, the minimum of these two can be used, and the maximum of these minimums over all k is required.

□

This bound requires some explanation. If $\chi(C_k(G))$ is known for all k , then the bound reduces to the trivial $\chi(G) = \chi(G)$. This bound may be useful if some of the chromatic numbers of the cores are known, but not all, or they can at least be bounded by some other means. Every core is an induced subgraph of the entire graph, so coloring them is no harder than coloring the entire graph, and may be easier. However, if G is monocore, then this bound provides no additional information.

This bound has several corollaries.

Corollary 204. *If G has a 2-core, then $\chi(G) = \chi(C_2(G))$.*

Proof. Any nonempty graph requires at least two colors. The 1-shell of a graph is a forest, and requires at most two colors.

□

One of the basic results on vertex coloring is that the 2-colorable graphs are exactly the bipartite graphs.

Corollary 205. *If G has a 3-core which is not bipartite, then $\chi(G) = \chi(C_3(G))$.*

Proof. If the hypothesis holds, then coloring G requires at least three colors. Coloring its 2-shell requires at most three colors.

□

Thus the problem of optimally coloring a graph can be readily reduced to coloring its 3-core.

We may possibly be able to extend this analysis further. Certainly any graph requires at least as many colors as any of its subgraphs require. The simplest graph requiring k colors is a complete graph with k vertices. The clique number of G , $\omega(G)$, is the largest order of a clique contained in G . The clique number is a lower bound for the chromatic number, $\omega(G) \leq \chi(G)$. However, it may also be used to obtain a possibly improved upper bound. We first state an obvious result.

Proposition 206. *If $\omega(G) = k$, then $\widehat{C}(G) \geq k - 1$.*

Corollary 207. *Let $k = \omega(G)$, then $\chi(G) = \chi(C_{k-1}(G))$.*

4.1.2 The Coloring Chain

Thus we have the following chain of inequalities.

Definition 208. The coloring chain is the following chain of inequalities.

$$\omega(G) \leq \chi(G) \leq 1 + \widehat{C}(G) \leq 1 + \Delta(G)$$

We have seen that the uppermost inequality is an equality for connected graphs exactly when G is regular. We have mentioned Brooks' Theorem, that the chromatic number equals the maximum degree bound for connected graphs exactly when G is complete or an odd cycle.

A graph is defined to be perfect if the lower bound is an equality for every induced subgraph of G . The problem of which graphs are perfect has been solved within the last decade. The Strong Perfect Graph Theorem states that a graph is perfect if and only if it does not contain any odd cycle other than K_3 , or the complement of any such odd cycle, as an induced subgraph. (see [16] p.170)

The clique number and the maximum degree bound are equal exactly for the complete graphs. This is because by Brooks' Theorem, complete graphs and odd cycles are the only possibilities, and we have equality for complete graphs but not odd cycles other than K_3 .

How about the clique number and the maximum core bound? Define a graph to be chordal if every cycle has a chord. That is, it does not contain any induced cycle other than K_3 . We have the following theorem due to Voloshin [1982] [58].

Theorem 209. A graph G is chordal $\iff \omega(H) = 1 + \widehat{C}(H)$ for all induced subgraphs H in G .

Proof. (\Rightarrow) The result holds for order $n = 1$. Assume it holds for order r , and let G be chordal with order $r + 1$. Every chordal graph has a simplicial vertex, that is a vertex whose neighbors induce a clique. Let v be simplicial, and $H = G - v$. Then H

has order r , so $\omega(H) = 1 + \widehat{C}(H)$. Now v 's neighborhood must be a clique, so it has degree at most $\omega(H)$. Thus adding v to H can increase the maximum core number by at most one, and if it does, then it must also increase the clique number by one.

(\Leftarrow) Let G be not chordal. Then G contains a cycle C_n , $n \geq 4$, as an induced subgraph. Then $\omega(C_n) = 2 < 3 = 1 + \widehat{C}(C_n)$.

□

Of the six pairs of expressions in the coloring chain, this leaves only the chromatic number and the maximum core bound.

Definition 210. A graph G is core perfect if $\chi(G) = 1 + \widehat{C}(G)$.

We can state a few basic facts about core perfect graphs. A graph G is core perfect if and only if its maximum core is core perfect. Hence, we can restrict the question of determining which graphs are core perfect to monocore graphs. All 0-monocore and 1-monocore graphs (empty graphs and forests without trivial components, respectively) are core perfect, and a 2-monocore graph is core perfect if and only if it contains an odd cycle.

A k -monocore graph is core perfect if and only if it has a component that is core perfect. A connected k -monocore graph is core perfect if and only if it has a block which is a k -core and core perfect. By a result of David Matula, [Chartrand/Zhang p.177] [16] a connected k -monocore graph is core perfect only if it has edge-connectivity $k - 1$. By Brooks' Theorem, the only core perfect connected regular graphs are cliques and odd cycles.

Characterizing core perfect graphs appears to be a difficult problem in general.

On the other hand, it is possible for all three inequalities to be strict simultaneously. For example, the graph $C_5 \times P_3$ has 2, 3, 4, 5 for the respective values of the four quantities in question.

4-tuple	Must have component	May have component
(1, 1, 1, 1)	K_1	K_1
(2, 2, 2, 2)	K_2	K_1
(2, 2, 2, 3)	$P_n, n \geq 3$	path
(2, 2, 3, 3)	even cycle	path
(2, 3, 3, 3)	odd cycle (not K_3)	cycle, path (not K_3)
(k, k, k, k)	K_k	H with $\Delta(H) \leq k - 1$
(2, 2, 2, k)	tree T with $\Delta(T) = k - 1$	tree T with $\Delta(T) \leq k - 1$

Table 6: Some 4-tuples characterize graph classes.

This raises the question of which 4-tuples (a, b, c, d) are possible, such that there is a graph G with $a = \omega(G)$, $b = \chi(G)$, $c = 1 + \hat{C}(G)$, $d = 1 + \Delta(G)$. We call such a 4-tuple valid. Graphs with some 4-tuples can be characterized, as in Table 6.

Certainly a 4-tuple must satisfy $a \leq b \leq c \leq d$, and for any nontrivial graph, $a \geq 2$. We have the following helpful result.

Proposition 211. *Let (a, b, c, d) be a valid 4-tuple, $a \geq 2$. Then (a, b, e, f) is a valid 4-tuple provided $c \leq e \leq f$ and $d \leq f$.*

Proof. Let G be a graph corresponding to (a, b, c, d) . Construct a graph H as follows. While the graph obtained is not regular, create two copies of the graph and join the vertices of smallest degree. This process will eventually terminate with a regular graph H . Now $H \subseteq (G \times Q_{e-c})$, where Q_n is the n-dimensional hypercube. H has chromatic number and clique number equal to those of G , since the chromatic number of a Cartesian product is the maximum of those of its factors. Then $H \cup K_{1,f-1}$ has the 4-tuple (a, b, e, f) since the product increases the maximum core number by $e - c$,

and the star increases the maximum degree to $f - 1$.

□

It is known that there are graphs with any clique number and larger chromatic number. These can be generated by repeatedly applying Mycielski's construction to a clique, for example. The difficult case in determining which 4-tuples are valid is finding such graphs with relatively low maximum core numbers and maximum degrees, or showing that no such graphs exist.

The Grotzsch graph has 4-tuple $(2, 4, 4, 6)$, and the Chvatal graph has 4-tuple $(2, 4, 5, 5)$. By Brooks' Theorem, $(a, b, b, b) \neq (2, 3, 3, 3)$, $a < b$, is not a valid 4-tuple. It is not hard to see that $(2, 4, 4, 5)$ is the smallest 4-tuple with respect to lexicographic order for which validity is unclear.

4.1.3 Eigenvalue and Independence Bounds

There is another upper bound for the chromatic number worth considering. It involves the eigenvalues of a graph. Any graph can be represented by its adjacency matrix, which is square. The eigenvalues of this matrix can be computed. The spectrum of a graph is defined to be the sequence of eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, so that $\lambda_1 = \lambda_1(G)$ is the largest eigenvalue of the graph. The following Theorem is due to Wilf [1967] [62].

Theorem 212. *[The eigenvalue bound] Let G be a connected graph. Then $\chi(G) \leq 1 + \lambda_1$, with equality exactly for complete graphs and odd cycles.*

The proof of this result uses linear algebra and will be omitted. We also have the following results on eigenvalues of graphs, which will be stated without proof. See Schwenk/Wilson [52] for background on eigenvalues of graphs.

Theorem 213. [Properties of eigenvalues of graphs]

- a. The eigenvalues of a graph are all real algebraic numbers, but need not be rational.
- b. The spectrum of a graph is the union of the spectra of its components. (Thus we may restrict our attention to connected graphs. This result follows from the fact that the adjacency matrix of a disconnected graph may be partitioned into blocks.)
- c. Let \bar{d} be the average degree of G and Δ its maximum degree. Then $\bar{d} \leq \lambda_1 \leq \Delta$ with equality in both cases exactly when G is regular.
- d. If G is connected, $\lambda_2 < \lambda_1$. If H is an induced subgraph of G , then $\lambda_1(H) < \lambda_1(G)$.

Since the eigenvalue bound need not be an integer, it could be stated as $\chi(G) \leq 1 + \lfloor \lambda_1 \rfloor$. However, this form turns out to be harder to work with, and is equivalent in any case, so we will not use it.

We want to know how the eigenvalue bound relates to the other bounds we have discussed. As an immediate corollary of part c, we have the following.

Corollary 214. The eigenvalue bound is better than the maximum degree bound. That is, $1 + \lambda_1 \leq 1 + \Delta$, with equality exactly for regular graphs.

We can also compare the eigenvalue bound to the core number bound.

Theorem 215. A connected graph G has $1 + \widehat{C}(G) \leq 1 + \lambda_1(G)$, with equality exactly for regular graphs.

Proof. Let $k = \widehat{C}(G)$, $H = C_k(G)$. Then $\widehat{C}(G) = \widehat{C}(H) = \delta(H) \leq \lambda_1(H) \leq \lambda_1(G)$.

If G is regular, then $\widehat{C}(G) = \delta(G) = \lambda_1(G) = \Delta(G)$. Assume G is nonregular. Suppose first that G is monocore. Then $\widehat{C}(G) = \delta(G) < \bar{d}(G) < \lambda_1(G)$. Next

suppose that G is not monocore. Then $\widehat{C}(G) = \widehat{C}(H) = \delta(H) \leq \lambda_1(H) < \lambda_1(G)$, so the result holds in either case.

□

Thus the eigenvalue bound is better than the maximum degree bound, but weaker than the core number bound. We could extend the coloring chain to the following.

$$\omega(G) \leq \chi(G) \leq 1 + \widehat{C}(G) \leq 1 + \lambda_1(G) \leq 1 + \Delta(G)$$

The only case in which we have not yet determined equality with the eigenvalue bound is the clique number.

Corollary 216. *For connected graphs G , $\omega(G) \leq 1 + \lambda_1(G)$, with equality exactly for complete graphs.*

Proof. We have seen that $\chi(G) \leq 1 + \lambda_1$, with equality exactly for complete graphs and odd cycles. Now $\omega(G) \leq \chi(G)$, with equality for complete graphs and not for odd cycles beyond C_3 .

□

Determining the corresponding equalities for the floor of the eigenvalue bound is unsolved and appears difficult.

There is another bound that may be of interest. It involves the independence number $\alpha(G)$. It is not hard to see that $\chi(G) \leq n - \alpha(G) + 1$, since G can be colored with $n - \alpha(G) + 1$ colors by coloring a maximum independent set with one color and all other vertices with distinct colors. Determining the independence number of a graph is NP-complete, so this bound is not easy to calculate in general. It is also no better than the core number bound.

Proposition 217. *We have $1 + \widehat{C}(G) \leq n - \alpha(G) + 1$.*

Proof. Let G have independence number $\alpha = \alpha(G)$, and S be an independent set of α vertices. Construct a graph H by adding all possible edges to G that don't have both ends in S . Then $H = K_{n-\alpha} + \alpha K_1$. Then $\widehat{C}(G) \leq \widehat{C}(H) = n - \alpha$, so $1 + \widehat{C}(G) \leq n - \alpha(G) + 1$.

□

It is not immediately obvious which graphs produce equality in this result. However, it is not difficult to determine which graphs produce equality for the original bound.

Corollary 218. *We have $\chi(G) = n - \alpha + 1$ exactly for $G = K_{n-\alpha} + \alpha K_1$.*

Proof. G has an independent set S of size α . If some vertex of G not in S was not adjacent to some other vertex of G , then either S could be enlarged, or some other color could be used more than once, which is impossible.

□

4.1.4 The Order and Size Bound

The core number bound is useful in proving an upper bound for the chromatic number in terms of order and size only. This theorem is due to Coffman, Hakimi, and Schmeichel [2003] [19]. The proof of part a below is a simplification of the original proof and the version appearing in Chartrand/Zhang [p. 183-184] [16].

Theorem 219. *Let G be connected with a 2-core and order n , size m .*

a. *If the 2-core of G is not a clique or an odd cycle, then*

$$\chi(G) \leq \left\lfloor \frac{3 + \sqrt{1 + 8(m-n)}}{2} \right\rfloor$$

b. *For every pair (n, m) with either $n = m$ or $n < m < \binom{n}{2}$, the bound is sharp.*

Proof. Let $d = \widehat{C}(G)$. Deleting the 1-shell of G leaves $m - n$ unchanged, so we may assume G is a 2-core. If G is an even cycle then the bound is 2, which is exact. If G is any other 2-monocore graph, then $m \geq n + 1$ and the bound is at least 3, so it holds. Now we may assume $d \geq 3$, and we wish to show that $m \geq n + \binom{d}{2}$. Since G is not complete, $d \leq n - 2$.

Let H be the maximum core of G with order $r \geq d + 1$ and size at least $\frac{r \cdot d}{2}$. If G is d -monocore, then

$$m \geq \left\lceil \frac{n \cdot d}{2} \right\rceil = n + \left\lceil \frac{n(d-2)}{2} \right\rceil \geq n + \left\lceil \frac{(d+2)(d-2)}{2} \right\rceil \geq n + \binom{d}{2}.$$

If G is not monocore, the size of $G - H$ is at least $n - r + 1$ by Proposition 29. Then

$$m \geq n - r + 1 + \frac{r \cdot d}{2} \geq n + 1 + \frac{r(d-2)}{2} \geq n + 1 + \frac{(d+1)(d-2)}{2} = n + \binom{d}{2}.$$

Then $d^2 - d - 2(m-n) \leq 0$, so by the core number bound,

$$\chi(G) \leq 1 + d \leq \frac{3 + \sqrt{1 + 8(m-n)}}{2}.$$

□

Using the 2-core improves what can be proven without it, which follows as a corollary.

Corollary 220. *Let G by connected with order n , size m . Then*

$$\chi(G) \leq \left\lfloor \frac{3 + \sqrt{9 + 8(m-n)}}{2} \right\rfloor$$

and the bound is sharp for every pair (n, m) with $n \geq 2$ and $n - 1 \leq m \leq \binom{n}{2}$.

4.1.5 Graph Classes Related to Cores

We can study coloring of graphs that are related to cores. One class we have studied is core perfect.

Proposition 221. *If G is maximal k -degenerate, then $\chi(G) = k + 1$.*

Proof. G contains a $k + 1$ -clique, and has maximum core number k . □

It would seem to be an achievable goal to determine the chromatic numbers of all 3-core-critical graphs. Since it is easy to determine when a graph is bipartite, the question comes down to distinguishing between 4-chromatic and 3-colorable graphs. Since deleting any edge of a 3-core-critical graph destroys its 3-core, any such 4-chromatic graph is 4-critical.

We consider some special classes of 3-core-critical graphs. Note that odd wheels are 4-chromatic. We can generate many more 4-chromatic graphs using the following operation. The Hajos sum of two graphs G and H is defined by identifying two vertices of the graphs as v . Given edges uv and vw in G and H respectively, the Hajos sum of G and H is the graph $G \cup_H v - uv - vw + uw$. It is not hard to show that the Hajos sum of two k -critical graphs is k -critical. Provided that the vertex v is not adjacent to a vertex of degree more than three except possibly one of u and w , the Hajos sum preserves 3-core-critical graphs.

On the other hand, it appears that many more minimally 3-collapseable graphs are 3-colorable. By Brooks' Theorem, this includes all regular graphs except K_4 . It also includes any with a cut-vertex or a 2-edge-cut. Even wheels are certainly 3-colorable, which takes care of all such graphs with maximum degree $\Delta = n - 1$. All such graphs

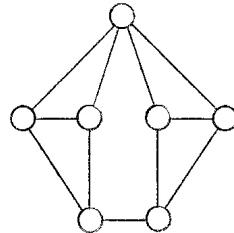


Figure 29: The Hajos sum of K_4 and K_4 is 4-critical and 3-core-critical.

with $\Delta = n - 2$ are 3-colorable since we have seen that they have two nonadjacent vertices of degree more than three and all the other vertices induce a linear forest. Finally we note that an r -necklace is 3-colorable since its vertices of degree 4 form an independent set and its vertices of degree 3 induce a matching.

We offer the following conjecture.

Conjecture 222. *All 4-chromatic 3-core-critical graphs can be formed from odd wheels using the Hajos sum.*

4.1.6 The Greedy Core Algorithm

The maximum core bound need not produce the best possible bound for large classes of graphs. For planar graphs, it produces an upper bound of six, as we will see in the section on planarity.

We might hope that the maximum core bound were exact for almost all graphs, but this is not the case. Michael Molloy [1996] [44] has shown using random graph theory that for $k \geq 4$, a k -core appears in the random graph before a $k + 1$ -chromatic subgraph. It would be interesting to know the values of the maximum core number and chromatic number for almost all graphs in terms of n .

We may hope to improve on the maximum core bound by a shrewd coloring of the cores. We say a coloring algorithm is adjacency-based if every vertex that is colored must be adjacent to a previously colored vertex, if one can exist. This leads us to the

following algorithm.

Algorithm 223. [*Greedy Core Coloring*]

Input: a graph G

Initialization: Establish a construction sequence for G .

Iteration: Color G according to the construction sequence. Assign each vertex the least color not used by any of its neighbors.

It is possible that starting with a vertex of large degree might improve the greedy core coloring algorithm, but this is unknown.

This algorithm can improve on the maximum core bound. For example, it will be exact for all 2-chromatic graphs, that is nontrivial bipartite graphs, since there is no point at which a vertex of a bipartite graph could be adjacent to vertices of more than one color class.

How does this algorithm fare on 3-core-critical graphs of small order? It is exact for K_4 and W_4 , as well as three of the four 3-core-critical graphs of order six. However, it need not be exact for $K_3 \times K_2$. Whether it is exact will depend on the particular construction sequence used. By exhaustively checking all possibilities, I determined that the algorithm fails to be exact $\frac{25}{54}$ of the time, and so is exact $\frac{29}{54}$ of the time.

I will nonetheless make the following somewhat audacious conjecture.

Conjecture 224. *The Greedy Core Algorithm yields the chromatic number exactly for almost all graphs and construction sequences.*

The following conjecture seems likely, but has yet to be proven.

Conjecture 225. *Every graph has a construction sequence which yields an exact coloring.*

4.1.7 Chromatic Polynomials

A more general problem than determining the chromatic number of a graph is to determine the number of distinct colorings of a graph using k colors, for any k . Specifically, the problem is to determine a function that gives the number of distinct k -colorings of G in terms of k . It turns out that such a function must be a polynomial, so it is called the chromatic polynomial of G . (See [16] p. 211-216 for background.)

Algorithms exist to determine this polynomial, but they are not efficient, so determining the chromatic polynomial is difficult for large graphs. Using cores can simplify this problem somewhat.

Certainly any isolated vertex of a graph can be colored with any of k colors independent of the coloring of the rest of the graph.

Proposition 226. *Let n_0 be the order of the 0-shell of G . Then the chromatic polynomial $\chi(G, k) = k^{n_0} \chi(C_1(G), k)$.*

Proof. There are k choices for the color of each isolated vertex, independent of any choices for the coloring elsewhere.

□

For each vertex of degree one, one choice is excluded, so there are $k - 1$ colors available for it. This leads to the next theorem.

Theorem 227. *Let G be a connected 1-core containing a 2-core, n_1 the order of its 1-shell. Then*

$$\chi(G, k) = (k - 1)^{n_1} \chi(C_2(G), k).$$

Proof. If $n_1 = 0$, then

$$\chi(G, k) = \chi(C_2(G), k) = (k-1)^0 \chi(C_2(G), k).$$

Assume the result holds for order $n_1 = r$ and let G have a 1-shell of order $r+1$. Then G has an end-vertex vertex v . Let $H = G - v$ and $e = uv$, the edge incident with v . By the chromatic recurrence,

$$\begin{aligned} \chi(G, k) &= \chi(G - e, k) - \chi(G \cdot e, k) \\ &= k \cdot \chi(H, k) - \chi(H, k) \\ &= (k-1) \cdot \chi(H, k) \\ &= (k-1)(k-1)^r \cdot \chi(C_2(G), k) \\ &= (k-1)^{r+1} \cdot \chi(C_2(G), k) \end{aligned}$$

□

4.2 Nordhaus-Gaddum Class Theorems

4.2.1 The Nordhaus-Gaddum Theorem

One common way to study a graph parameter $p(G)$ is to examine the sum $p(G) + p(\overline{G})$ and product $p(G) \cdot p(\overline{G})$. A theorem providing sharp upper and lower bounds for this sum and product is known as a theorem of the Nordhaus-Gaddum class. Of the four possible bounds, the sum upper bound has attracted the most attention. We will examine results of this type for maximum core number and chromatic number.

Theorem 228. *We have $\widehat{C}(G) + \widehat{C}(\overline{G}) \leq n - 1$. The graphs for which $\widehat{C}(G) + \widehat{C}(\overline{G}) = n - 1$ are exactly the graphs constructed by starting with a regular graph and iterating the following operation.*

Given $k = \widehat{C}(G)$, H a k -monocore subgraph of G , add a vertex adjacent to at least $k + 1$ vertices of H , and all vertices of degree k in H (or similarly for \overline{G}).

Proof. Let $p = \widehat{C}(G)$ and suppose \overline{G} has an $n - p$ -core. These cores use at least $(p + 1) + (n - p + 1) = n + 2$ vertices, and hence share a common vertex v . But then $d_G(v) + d_{\overline{G}}(v) \geq p + (n - p) = n$, a contradiction.

If G is regular with $k = \widehat{C}(G)$, then \overline{G} is $n - k - 1$ -regular, so $\widehat{C}(G) + \widehat{C}(\overline{G}) = n - 1$. If a vertex v is added as in the operation, producing a graph H , a $k + 1$ -core is produced, so $\widehat{C}(H) + \widehat{C}(\overline{H}) = (n + 1) - 1$.

Suppose that for a graph G , $\widehat{C}(G) + \widehat{C}(\overline{G}) = n - 1$. If G and \overline{G} are both monocore, then they must be regular. If G has a vertex v that is not contained in the maximum cores of both G and \overline{G} , then $\widehat{C}(G - v) + \widehat{C}(\overline{G} - v) = (n - 1) - 1$. Then v is contained in the maximum core of one of them, say G . Further, given $k = \widehat{C}(G)$, v is contained in a k -monocore subgraph H of G , and $H - v$ must be $k - 1$ -monocore. Then v must have been adjacent to all vertices of degree $k - 1$ in $H - v$. Thus G can be constructed as described using the operation.

□

The original Nordhaus-Gaddum Theorem deals with the chromatic number (see Chartrand/Zhang p. 185-186 [16]). Its proof follows as a corollary, using the core number bound.

Corollary 229. *[Nordhaus-Gaddum] We have $\chi(G) + \chi(\overline{G}) \leq n + 1$.*

Proof. We have $\chi(G) + \chi(\overline{G}) \leq 1 + \hat{C}(G) + 1 + \hat{C}(\overline{G}) \leq n - 1 + 2 = n + 1$.

□

It is convenient to consider a graph and its complement as a decomposition of a complete graph. This makes it possible to generalize the problem to more than two factors.

Definition 230. A k -decomposition of a graph G is a decomposition of G into k subgraphs. For a graph parameter p , let $p(k; G)$ denote the maximum of $\sum_{i=1}^k p(G_i)$ over all k -decompositions of G .

It may be that it is possible to delete some edges from one of the subgraphs so that it still has the same chromatic number. Thus we are most interested in the color-critical subgraphs of the subgraphs of the decomposition. Conversely, given the critical subgraphs, we can distribute the extra edges arbitrarily. This final step is uninteresting, so we will tend to describe a k -decomposition as $\{H_1, \dots, H_k\}$, where each H_i is a color-critical subgraph.

We would like to characterize the extremal decompositions for the Nordhaus-Gaddum Theorem. Note that if a 2-decomposition of K_n achieves $\chi(G) + \chi(\overline{G}) = n + 1$, then we can easily construct a 2-decomposition of K_{n+1} with $\chi(G) + \chi(\overline{G}) = n + 2$,

by joining a vertex to all the vertices of a color-critical subgraph of G or \overline{G} , and allocating the extra edges arbitrarily. Conversely, we may be able to delete some vertex v of K_n so that $\chi(G) + \chi(\overline{G}) = n$. If this is impossible, we say that an extremal decomposition is fundamental.

Definition 231. A k -decomposition of K_n with $K = \sum_{i=1}^k \chi(G_i)$ achieving the maximum possible such that no vertex v of K_n can be deleted so that $\sum_{i=1}^k \chi(G_i - v) = K - 1$ is called a fundamental decomposition.

Theorem 232. For $k = 2$, the fundamental decompositions that attain $\chi(2; K_n) = n+1$ are $\{K_1, K_1\}$ and $\{C_5, C_5\}$.

Proof. It is easily seen that $\chi(K_1) + \chi(K_1) = 2$ and $\chi(C_5) + \chi(C_5) = 6$, so these decomposition satisfy the equation. They are fundamental since no vertex can be deleted from the first, and deleting a vertex from the second produces two copies of P_4 , and $\chi(P_4) + \chi(P_4) = 4$.

Consider a fundamental 2-decomposition $\{G, \overline{G}\}$. Then both graphs are connected. Let $\chi(G) = r$, so that $\chi(\overline{G}) = n + 1 - r$. Then G is an $r - 1$ -core and \overline{G} is an $n - r$ -core. But then G and \overline{G} must be regular, since $n - 1 = d_G(v) + d_{\overline{G}}(v) \leq \widehat{C}(G) + \widehat{C}(\overline{G}) \leq n - 1$. Now by Brooks' Theorem, the only connected regular graphs achieving $\chi(G) = 1 + \widehat{C}(G)$ are cliques and odd cycles. The only such graphs whose complements are connected and also achieve the upper bound are K_1 and C_5 . Thus the fundamental 2-decompositions are as stated.

□

We can now describe all extremal 2-decompositions for the upper bound of the Nordhaus-Gaddum theorem.

Corollary 233. *The extremal 2-decompositions for the upper bound of the Nordhaus-Gaddum theorem are exactly $\{K_p, K_{n-p+1}\}$ and $\{C_5 + K_p, C_5 + K_{n-p-5}\}$.*

Proof. It is immediate that these are extremal 2-decompositions. Assume that we have an extremal 2-decomposition $\{G, \overline{G}\}$ with order n and let G be r -critical. If the critical subgraphs overlap on a single vertex and $G = K_r$, then $\overline{G} = K_{n-r} + rK_1$, which is uniquely $n - r + 1$ -colorable. Deleting any edge of the copy of K_{n-r} would reduce the chromatic number, so K_{n-r+1} is the only possible $n - r + 1$ -critical subgraph. If $G \neq K_r$ has order $p \geq r+2$, then the critical subgraph of \overline{G} is contained in $K_{n-p} + pK_1$, which is impossible. If the critical subgraphs overlap on C_5 , the argument is similar.

□

In 1968, H. J. Finck [26] determined a similar but inelegant characterization whose proof is more than three pages long. In 2008, Starr and Turner [56] determined the following alternative characterization.

Theorem 234. *Let G and \overline{G} be complementary graphs on n vertices. Then $\chi(G) + \chi(\overline{G}) = n + 1$ if and only if $V(G)$ can be partitioned into three sets S , T , and $\{x\}$ such that $G[S] = K_{\chi(G)-1}$ and $G[T] = K_{\chi(\overline{G})-1}$.*

The proof of this result is almost three pages. This characterization leaves something to be desired since it is not obvious which graphs satisfy the condition given in the theorem. However, this result follows immediately as a corollary to the previous theorem. Hence the proof of Theorem 232 is significantly shorter than either of the previous characterizations.

Having characterized the graphs for which $\chi(G) + \chi(\overline{G}) = n + 1$, we may consider when $\chi(G) + \chi(\overline{G}) = n$. Restricted to regular graphs, this is not a difficult problem.

Proposition 235. *If G is regular and $\chi(G) + \chi(\overline{G}) = n$, then the 2-decompositions that satisfy this equation are $\{C_7, \overline{C}_7\}$ and $\{C_4, 2K_2\}$.*

Proof. Assume the hypothesis. Then $n = \chi(G) + \chi(\overline{G}) \leq 1 + \widehat{C}(G) + 1 + \widehat{C}(\overline{G}) = n + 1$, so exactly one of G or \overline{G} achieves the core number bound, say G . If G is connected, then by Brooks' Theorem, G is a complete graph or odd cycle. But the complement of a complete graph also achieves the upper bound. If G is an odd cycle of length at least 5, then $\chi(\overline{C}_n) = \frac{n+1}{2}$. But \overline{C}_n is $n - 3$ -regular, so $\frac{n+1}{2} = n - 3$ implies $n = 7$.

If G is disconnected, then it is a union of r -regular components, at least one of which is a clique or an odd cycle. Consider starting with only this component and adding another component with order k . This increases $\chi(G) + \chi(\overline{G})$ by at most $k - r$. Thus to satisfy the equation we want $r = 1$, so the new component is K_2 , and no other component can be added. Thus only the 2-decomposition $\{C_4, 2K_2\}$ works.

□

We can also consider products of the maximum core numbers of a graph and its complement.

Corollary 236. *We have $0 \leq \widehat{C}(G) \cdot \widehat{C}(\overline{G}) \leq \left(\frac{n-1}{2}\right)^2$. The lower bound is an equality exactly for $\{K_n, \overline{K}_n\}$. The upper bound is an equality exactly when the sum bound is attained and $\widehat{C}(G) = \widehat{C}(\overline{G})$.*

Proof. The lower bound is obvious. Equality occurs exactly when one of the factors is 0, implying one is empty and the other is complete. Now since $\sqrt{xy} \leq \frac{x+y}{2}$ with equality exactly when $x = y$, $\widehat{C}(G) \cdot \widehat{C}(\overline{G}) \leq \left(\frac{n-1}{2}\right)^2$ with equality exactly when the sum bound is attained and $\widehat{C}(G) = \widehat{C}(\overline{G})$.

□

Corollary 237. [Nordhaus-Gaddum] *We have $\chi(G) \cdot \chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2$. The bound is*

attained exactly for $\left\{K_{\frac{n+1}{2}}, K_{\frac{n+1}{2}}\right\}$ and $\left\{C_5 + K_{\frac{n-5}{2}}, C_5 + K_{\frac{n-5}{2}}\right\}$.

This follows using the same argument as in the previous corollary. This bound is attained exactly when the sum bound is attained and $\chi(G) = \chi(\overline{G})$.

4.2.2 Plesnik's Conjecture

We now consider decompositions of graphs into more than two parts. Jan Plesnik [49] studied $\chi(k; K_n)$ and in 1978 made the following conjecture.

Conjecture 238. *[Plesnik's Conjecture] For $n \geq \binom{k}{2}$, $\chi(k; K_n) = n + \binom{k}{2}$.*

For $k = 2$, this is just the Nordhaus-Gaddum theorem. Plesnik proved the conjecture for $k = 3$ and determined an upper bound for $\chi(k; K_n)$.

There is a simple construction that shows $\chi(k; K_n)$ is at least $n + \binom{k}{2}$. Take the line graph $L(K_k)$ with order $\binom{k}{2}$ and decompose it into k copies of K_{k-1} . For any additional vertex, make it adjacent to all the vertices of one of the cliques in the decomposition and allocate any extra edges arbitrarily.

Plesnik proved a recursive upper bound of $\chi(k; K_n) \leq n + t(k)$, where $t(2) = 1$ and $t(k) = \sum_{i=2}^{k-1} \binom{k}{i} t(i)$. Thus $t(3) = 3$ and $t(4) = 18$. This implies a worse explicit bound of $\chi(k; K_n) = n + 2^{\binom{k+1}{2}}$. In 1985, Timothy Watkinson [59] improved this upper bound to $\chi(k; K_n) = n + \frac{k!}{2}$. In 2005, Furedi, Kostochka, Stiebitz, Skrekovski, and West [27] proved an improved upper bound for large k of $\chi(k; K_n) \leq n + 7^k$. All of these bounds remain far from Plesnik's conjecture, however.

We can describe many fundamental decompositions for $k \geq 3$ using the following construction.

Algorithm 239. [Construction of fundamental k -decompositions] For $k \geq 3$ and $n \geq \binom{k}{2}$, construct a decomposition of K_n as follows.

1. Start with the line graph $L(K_k)$ decomposed into k copies of K_{k-1} .
2. Replace each vertex by either K_1 decomposed into $\{K_1, K_1\}$ or K_5 decomposed into $\{C_5, C_5\}$.
3. Join each factor to the other factors corresponding to the same copy of K_{k-1} in the decomposition of $L(K_k)$.
4. Allocate any remaining edges arbitrarily.

We will see below that the graphs produced by this algorithm attain the bound of Plesnik's conjecture. This algorithm produces all such graphs for $k = 2$ but not all for $k = 3$.

Lemma 240. 1. For $k \geq 3$, let D be a k -decomposition with every vertex contained in exactly two color-critical subgraphs of the decomposition that maximizes $\sum_{i=1}^k \chi(G_i)$. Then $\sum_{i=1}^k \chi(G_i) = n + \binom{k}{2}$.

2. The k -decompositions produced by the preceding algorithm satisfy $\sum_{i=1}^k \chi(G_i) = n + \binom{k}{2}$.

Proof. Assume the hypothesis and let H_i be the critical subgraphs of the k graphs. Thus we can partition the n vertices into $\binom{k}{2}$ classes: $V_{ij} = V(H_i) \cap V(H_j)$. Now the edges between V_{ij} and V_{il} may as well be in H_i since this is the only critical subgraph with vertices in both classes. Similarly, if V_{ij} and V_{lm} have no common indices, then no edges between them are contained in a critical subgraph. Then $\chi(H_i) \leq \sum_j \chi(H_i[V_{ij}])$, where $1 \leq j \leq k$, $i \neq j$. Then $n + \binom{k}{2} \leq \sum \chi(H_i) \leq \sum_{i,j} \chi(H_i[V_{ij}]) \leq \sum (n(V_{ij}) + 1) = n + \binom{k}{2}$, with the last inequality following from the Nordhaus-Gaddum theorem. But then we have equalities, which implies that $\sum_{i=1}^k \chi(G_i) = n + \binom{k}{2}$, and the two graphs that decompose $K_n[V_{ij}]$ form an extremal

2-decomposition. Since $\{K_1, K_1\}$ and $\{C_5, C_5\}$ are fundamental 2-decompositions, Algorithm 239 produces fundamental k -decompositions.

□

Now we can prove Plesnik's conjecture for $k = 3$.

Theorem 241. *For $k = 3$ and $n \geq 3$, $\chi(3; K_n) = n + 3$.*

Proof. Assume that some fundamental decomposition of K_n into three factors yields $\chi(G_1) = a + 1$, $\chi(G_2) = b + 1$, and $\chi(G_3) = c + 1$, with $a + b + c = n$. We may consider the critical subgraphs H_i of the three graphs, which are a , b , and c -cores, respectively. Now no vertex of K_n can be contained in all three of the H_i 's, since this would imply that K_n has at least $a + b + c + 1 = n + 1$ vertices.

Since deleting a vertex from a k -critical graph produces a $k - 1$ -chromatic graph and the decomposition is fundamental, every vertex is contained in exactly two of the three critical subgraphs. Then by Lemma 240, $\chi(3; K_n) = n + 3$.

□

The fundamental 3-decompositions produced by Algorithm 239 are $\{K_2, K_2, K_2\}$, $\{W_5, W_5, K_2\}$, $\{W_5, W_5, C_5 + C_5\}$, and $\{C_5 + C_5, C_5 + C_5, C_5 + C_5\}$. However, these are not all the fundamental 3-decompositions. This is because the extremal 2-decompositions produced in the next-to-last sentence of the proof of Lemma 240 need not be fundamental, as can be seen in Figure 30.

We can determine all fundamental 3-decompositions. We need the following lemma.

Lemma 242. *There are exactly six 4-critical subgraphs of $G = C_5 + \overline{K}_{n-5}$.*

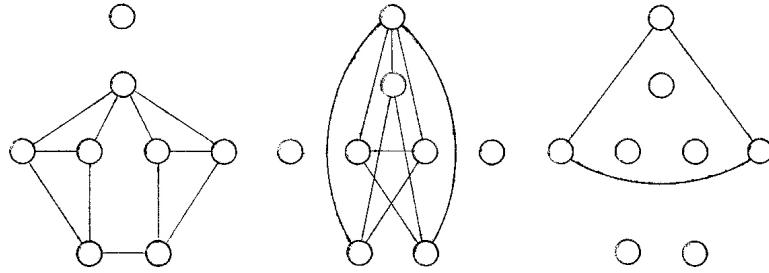


Figure 30: A fundamental 3-decomposition with 2-decompositions $\{K_1, K_1\}, \{C_5, C_5\}$ and $\{K_2, \overline{K}_2\}$.

Proof. Clearly any such subgraph must contain the copy of C_5 and at least one more vertex. If $n = 6$, $W_5 = C_5 + K_1$ is clearly the only possibility. Let S be the set of vertices not on C_5 . Since we want a 4-critical subgraph, consider adding each vertex in S one at a time. Then each of them must successively restrict the possible colorings of G , since otherwise it could be deleted and there would be a smaller 4-critical subgraph.

Suppose a vertex in S has degree 4. Note that any vertex of degree 3 either (A) neighbors three consecutive vertices on the cycle or (B) exactly two consecutive vertices on the cycle and a third not adjacent to either of them. Adding a vertex of degree 4 or a degree 3 vertex of type A adjacent to the remaining vertex on the cycle produces a 4-chromatic graph, but at least one edge can be deleted to obtain a 4-critical graph G_1 . Checking the possible placements of a degree 3 vertex of type B, only one possibility produces a 4-critical graph. (This graph is $M(K_3)$, the result of applying Mycielski's construction to K_3 .)

Now suppose that we start with a vertex v of type A. Adding a vertex u of type A having one common neighbor with v produces a 4-critical graph G_2 (which is the result of applying the Hajos construction to two copies of K_4). If instead u has two common neighbors with v , checking cases shows that there is only one way to add a vertex w (of type B) to produce a 4-critical graph G_3 .

Now suppose that we allow exactly one vertex v of type A. Checking cases shows

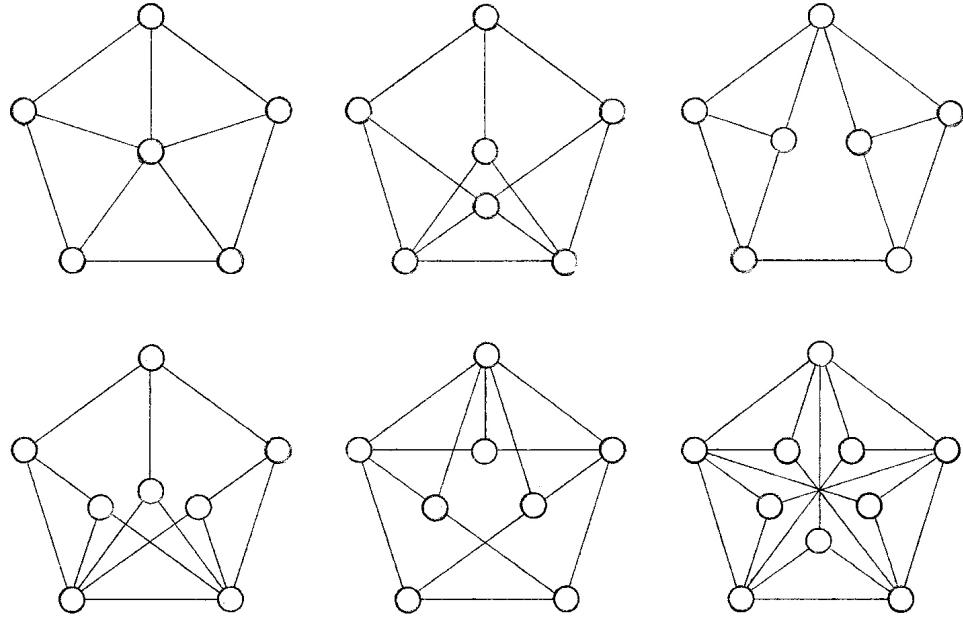


Figure 31: The 4-critical graphs W_5 , $G_1 = M(K_3)$, G_2 , G_3 , G_4 , G_5 .

that there is exactly one way to produce a 4-critical graph G_4 , by adding two type B vertices, each having two consecutive vertices of the cycle as common neighbors with v .

Finally, suppose that we allow only vertices of type B. There are five possible placements of a type B vertex. Adding all five of them produces a 4-chromatic graph G_5 , but deleting one produces a 3-chromatic graph. Thus G_5 must be 4-critical.

□

We denote the generalized wheel $W_{p,q} = C_p + K_q$. It is $3+q$ -critical if p is odd.

Theorem 243. *There are exactly 29 fundamental 3-decompositions. These are given in Table 7, where we let $C_{5,5} = C_5 + C_5$.*

$\{K_2, K_2, K_2\}$	$\{W_5, W_5, K_2\}$	$\{W_5, W_5, C_{5,5}\}$	$\{C_{5,5}, C_{5,5}, C_{5,5}\}$	
$\{W_5, K_3, G_1\}$	$\{W_5, K_3, G_2\}$	$\{W_5, K_4, G_3\}$	$\{W_5, K_4, G_4\}$	$\{W_5, K_6, G_5\}$
$\{C_{5,5}, W_{5,2}, G_1\}$	$\{C_{5,5}, W_{5,2}, G_2\}$	$\{C_{5,5}, W_{5,3}, G_3\}$	$\{C_{5,5}, W_{5,3}, G_4\}$	$\{C_{5,5}, W_{5,5}, G_5\}$
$\{G_1, G_1, K_4\}$				
$\{G_1, G_2, K_4\}$	$\{G_2, G_2, K_4\}$			
$\{G_1, G_3, K_5\}$	$\{G_2, G_3, K_5\}$	$\{G_3, G_3, K_6\}$		
$\{G_1, G_4, K_5\}$	$\{G_2, G_4, K_5\}$	$\{G_3, G_4, K_6\}$	$\{G_4, G_4, K_6\}$	
$\{G_1, G_5, K_7\}$	$\{G_2, G_5, K_7\}$	$\{G_3, G_5, K_8\}$	$\{G_4, G_5, K_8\}$	$\{G_5, G_5, K_{10}\}$

Table 7: The 29 fundamental 3-decompositions.

Proof. By the lemma, there are exactly five extremal 2-decompositions that can appear in a fundamental 3-decomposition: $\{C_5, C_5\}$, $\{K_1, K_1\}$, $\{K_2, \overline{K}_2\}$, $\{K_3, \overline{K}_3\}$, and $\{K_5, \overline{K}_5\}$. Denote the first two as symmetric and the last three nonsymmetric. One of these five must be chosen for each of the three overlap sets of a fundamental 3-decomposition, but this choice is not independent. If a nonsymmetric 2-decomposition appears, then $\{C_5, C_5\}$ must also appear. Joining a pair of graphs from the 2-decompositions produces a color-critical graph except in the case $G = C_5 + \overline{K}_{n-5}$, $n \geq 7$, for which the lemma provides five possible color-critical subgraphs.

If all three 2-decompositions are symmetric, there are four possibilities, as given in the first row of the table.

Suppose exactly one nonsymmetric 2-decomposition appears. Then $\{C_5, C_5\}$ must also appear, and the third 2-decomposition can be either $\{C_5, C_5\}$ or $\{K_1, K_1\}$. Thus there are $5 \cdot 2 = 10$ possibilities, which are given in the second and third rows of the table.

Suppose exactly two nonsymmetric 2-decompositions appear, so $\{C_5, C_5\}$ is the third. Then we must choose two of the five color-critical subgraphs as factors, and the third must be a clique. Thus there are $\binom{5}{2} + 5 = 15$ possibilities, which are given in last five rows of the table.

□

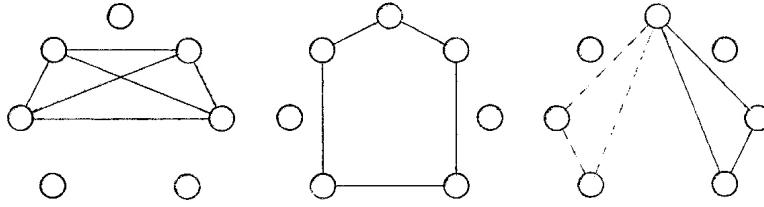


Figure 32: A 4-decomposition of K_7 into $\{K_4, K_3, K_3, C_5\}$ with a vertex contained in three factors.

We would like to determine all extremal 3-decompositions. Examples of some that are not fundamental include $\{K_p, K_p, C_{2p-1}\}$ or $\{K_p + C_5, K_p + C_5, C_{2p-1}\}$.

Not all fundamental k -decompositions are produced by Algorithm 239 for $k \geq 4$. Watkinson [59] describes a decomposition of K_7 into $\{K_4, K_3, K_3, C_5\}$, though his presentation of this example contains an error. This example has a vertex contained in three critical subgraphs. The decomposition nonetheless has $\sum_{i=1}^4 \chi(G_i) = n + \binom{4}{2} = 7 + 6 = 13$.

4.2.3 Related Results

Furedi et al [27] studied related problems and proved the following.

Theorem 244. *For all positive integers n and k ,*

- a. *If $n \geq \binom{k}{2}$, $\omega(k; K_n) = n + \binom{k}{2}$.*
- b. *We have $\chi(k; K_n) \leq n + 7^k$.*
- c. *We have $\widehat{C}(k; K_n) \leq \sqrt{k} \cdot n$.*
- d. *We have $\widehat{C}(2; K_n) = n - 1$, $\widehat{C}(3; K_n) = \lfloor \frac{3}{2}(n - 1) \rfloor$, and $\widehat{C}(4; K_n) = \lfloor \frac{5}{3}(n - 1) \rfloor$.*

The decomposition of the line graph $L(K_k)$ into k copies of K_{k-1} achieves $\omega(k; K_n) = n + \binom{k}{2}$. Given the simplicity of this decomposition, it is surprising that the bound in part b remains far from Plesnik's conjecture.

We can determine an analogous results for independence number and maximum degree.

Proposition 245. *We have $\alpha(k; K_n) = (k - 1)n + 1$.*

Proof. Consider the decomposition $\{K_n, \overline{K}_n, \dots, \overline{K}_n\}$. Then $\sum \alpha(G_i) = (k - 1)n + 1$.

We use induction on order. Certainly $\alpha(k; K_1) = k$. Assume $\alpha(k; K_r) = (k - 1)r + 1$, and let D be a decomposition of $G = K_{r+1}$. Consider the decomposition D' of $G - v$ formed by deleting v from each subgraph of D . If $\sum_{D'} \alpha(G_i) < (k - 1)r + 1$, then $\sum_D \alpha(G_i) \leq (k - 1)r + k = (k - 1)(r + 1) + 1$. If $\sum_{D'} \alpha(G_i) = (k - 1)r + 1$, then by the pigeonhole principle, some vertex of K_r is contained in all k independent sets. Then v is contained in at most $k - 1$ independent sets, so $\sum_D \alpha(G_i) \leq (k - 1)r + 1 + (k - 1) = (k - 1)(r + 1) + 1$. In either case, the result holds by induction.

□

Note that the case $k = 2$, $\alpha(2; K_n) = n + 1$ is essentially the Nordhaus-Gaddum theorem due to the symmetry of complementation.

Proposition 246. *We have $\Delta(k; K_n) = \binom{n}{2} - \binom{n-k}{2}$.*

Proof. Consider the decomposition with $G_i = K_{1,\max(n-i,0)}$ and any extra edges distributed arbitrarily. Then $\sum \Delta(G_i) = \sum_{i=\max(n-k,0)}^{n-1} i = \binom{n}{2} - \binom{n-k}{2}$.

We use induction on order. If $k \geq n$, then $\sum_D \Delta(G_i) \leq \sum_D m(G_i) = \binom{n}{2}$. If $k < n$, assume $\Delta(k; K_r) = \binom{r}{2} - \binom{r-k}{2}$, and let D be a decomposition of $G = K_{r+1}$. Let v be a vertex that does not uniquely have maximum degree in any of the k subgraphs. Consider the decomposition D' of $G - v$ formed by deleting v from each subgraph of D . Then adding v to the subgraphs of D' increases each maximum degree

by at most one. Then $\sum_D \Delta(G_i) \leq \binom{r}{2} - \binom{r-k}{2} + k = \binom{r+1}{2} - \binom{r+1-k}{2}$. The result holds by induction.

□

4.3 Other Forms of Coloring

4.3.1 Edge Coloring

While proper vertex coloring is the best-known type of graph coloring, it is far from the only kind. Edge coloring is similar to vertex coloring, except that the edges are colored. Clearly the edge chromatic number, $\chi_1(G)$, is at least as large as the maximum degree. Vizing showed that it is never more than $\Delta(G) + 1$. A graph is called class one if $\chi(G) = \Delta(G)$, and class two if $\chi(G) = \Delta(G) + 1$. Determining which of the two is the case is in general a difficult problem, but cores can help somewhat.

Theorem 247. *Let G be a graph with D the maximum degree in G of the vertices in the 1-shell of G . Then*

$$\chi_1(G) = \max \{D, \chi_1(C_2(G))\}.$$

Proof. Certainly $\chi_1(G) \geq D$ and since the 2-core of G is contained in G , $\chi_1(G) \geq \chi_1(C_2(G))$.

To show equality, color the 2-core of G with $\chi_1(C_2(G))$ colors. Now color the 1-shell using a construction sequence. Adding an edge adjacent to a boundary vertex will require an additional color if and only if edges of every color used up to that point are incident with v . This holds for adding any edge of the 1-shell. Thus we have equality.

□

It is not hard to see that every tree is class one. Since the 1-shell is a forest, we have the following corollary.

Corollary 248. G is class two \iff the 2-core of G is class two and $\Delta(G) = \Delta(C_2(G))$.

Zhou Goufei proved the following result on edge coloring of k -degenerate graphs. Its proof uses Vizing's adjacency lemma.

Theorem 249. [Goufei 2003 [30]] Every k -degenerate graph with $\Delta \geq 2k$ is class one.

This theorem and Theorem 67 produce the following corollary.

Corollary 250. If G is maximal k -degenerate with $n \geq \binom{k+2}{2}$, then G is class one.

This implies that almost all maximal k -degenerate graphs are class one. In particular, this theorem implies that if G is 2-degenerate and $\Delta(G) \geq 4$, then G is class one. This raises the question of determining the class of all 2-monocore graphs.

A graph G is overfull if n is odd and $m > \frac{n-1}{2}\Delta(G)$. It is easily seen that an overfull graph is class two. This result and the preceding theorem imply that the only maximal 2-degenerate graphs of class two are K_3 and K_4 with a subdivided edge.

Conjecture 251. A maximal k -degenerate graph is class two if and only if it is overfull.

In consideration of 2-monocore graphs, we can readily limit ourselves to connected graphs. We saw in Theorem 42 that all 2-monocore graphs can be constructed by a sequence of operations starting with a cycle. For convenience, we summarize those operations here.

- 1a. Add a path of length at least two to a 2-monocore graph.
- 1b. Add a path of length at least two between two 2-monocore graphs.

2a. Add a cycle identified at one vertex of a 2-monocore graph.

2b. Add an edge joining a vertex of a cycle and a vertex of a 2-monocore graph.

It is easily seen that even cycles are class one and odd cycles are class two. Any other 2-monocore graph has maximum degree at least three. If a 2-monocore graph has maximum degree three and is class two then some operation in its construction makes it class two and it cannot become class one again. Thus we analyze each of the operations to determine their effect on edge chromatic number. Note that the vertices where new edges are added must have degree two.

Operation 2a would create a vertex of degree four and so cannot be employed. Note though that this operation cannot move a graph from class one to class two since the two edges adjacent to the identified vertex can be colored with whatever colors are available, and the rest of the cycle needs at most one more color.

Operation 2b could not move a graph from class one to class two since one color is required for the special edge and at most three are required for the cycle. Operation 1b also cannot move a graph from class one to class two since only two colors are required for the path and only one each for the edges adjacent to the 2-monocore graphs.

The most difficult case is operation 1a. Now if the path added has length at least three, then whatever colors are available can be used on the edges adjacent to the existing graph and there will be colors available for the other edges. But if the path has length two, there may be a conflict. A conflict will occur exactly when any coloring of a class one 2-monocore graph G requires that two vertices of degree two have the edges incident with them colored with the same two colors. Note that this means that these two vertices cannot be on a triangle. It may be difficult to determine when this last condition occurs, however.

We summarize the above discussion with the following theorem.

Theorem 252. *Let G be 2-monocore. If $\Delta(G) \geq 4$, then G is class one. If $\Delta(G) = 3$,*

then G is class two if and only if any construction using the above operations requires adding a path of length two between two vertices which must have edges incident with them colored using the same two colors.

The next corollary follows from the above discussion.

Corollary 253. Any 4-edge-critical graph cannot have adjacent vertices of degree two.

4.3.2 List Coloring

Cores are also useful for other more obscure forms of coloring.

Definition 254. A list coloring of a graph begins with lists of length k assigned to each vertex and chooses a color from each list to obtain a proper vertex coloring. A graph G is k -choosable if any assignment of lists to the vertices permits a proper coloring. The list chromatic number $\chi_l(G)$, is the smallest k such that G is k -choosable.

We can provide upper and lower bounds for the list chromatic number. (see also West p. 408, 423 [60])

Theorem 255. We have $\chi(G) \leq \chi_l(G) \leq 1 + \widehat{C}(G)$.

Proof. The lower bound holds since the lists could be identical.

For the upper bound, establish a construction sequence for G . If a vertex v has degree $d(v)$, a list of $1 + d(v)$ colors guarantees v can be colored. Thus $\chi_l(G) \leq 1 + \widehat{C}(G)$.

□

While the list chromatic number is technically an upper bound for the chromatic number, it is not useful because it is harder to calculate than the chromatic number. The theorem implies that for core perfect graphs, the list chromatic number is the same as the chromatic number.

Corollary 256. *Every tree is 2-choosable.*

The next corollary follows immediately.

Corollary 257. *If G has a 2-core, then $\chi_l(G) = \chi_l(C_2(G))$.*

Erdos, Rubin, and Taylor [1979] [23] characterized 2-choosable graphs. Define the θ -graph $\theta_{i,j,k}$ to be the graph formed by identifying the endpoints of three paths of lengths i , j , and k .

Theorem 258. *A connected graph G is 2-choosable \iff it is a tree or its 2-core is an even cycle or $\theta_{2,2,2k}$ for $k \geq 1$.*

Thus every 2-monocore graph G that is not an even cycle or $\theta_{2,2,2k}$, $k \geq 1$, has $\chi_l(G) = 3$. Note the theorem implies that every 2-choosable graph has no 3-core. This leads to the following corollary.

Corollary 259. *If G has a 3-core, then $\chi_l(G) = \chi_l(C_3(G))$.*

This cannot directly be improved, as there is a 4-core with list chromatic number 3. For example, $\chi_l(K_{4,4}) = 3$. However, it can be improved if we know something about the clique number.

Corollary 260. Let $k = \omega(G)$. Then $\chi_l(G) = \chi_l(C_{k-1}(G))$.

4.3.3 L(2,1) Coloring

Definition 261. For nonnegative integers h and k , an $L(h, k)$ coloring c of a graph G is an assignment of colors (nonnegative integers) to the vertices of G such that if u and w are adjacent vertices of G , then $|c(u) - c(w)| \geq h$ while if $d(u, w) = 2$, then $|c(u) - c(w)| \geq k$.

Hence an $L(1, 0)$ coloring is just a proper vertex coloring. Some effort has gone into the study of $L(2, 1)$ colorings (see Chartrand/Zhang p. 403-410 [16]). Determining the minimum number of colors required in an $L(h, k)$ coloring is not an interesting problem, since for h and k both positive, this is just $\chi(G^2)$. Since this type of coloring is concerned with the distance between colors, another parameter is of interest.

Definition 262. For an $L(h, k)$ coloring c of a graph G , the c -span of G is $\lambda_{h,k}(c) = \max_{u,w \in G} |c(u) - c(w)|$. The L-span of G is $\lambda_{h,k}(G) = \min \{\lambda_{h,k}(c)\}$. We shall denote $\lambda_{2,1}(G)$ by $\lambda(G)$.

Thus we are interested in the minimum length of an interval containing the colors used for this type of coloring.

In 1992, Griggs and Yeh showed that if G has maximum degree Δ , $\lambda(G) \leq \Delta^2 + 2\Delta$. They further conjectured that $\lambda(G) \leq \Delta^2$ for all graphs, and proved this for graphs of diameter 2. We can prove an upper bound that is generally better.

Theorem 263. If G is a graph with $k = \widehat{C}(G)$, then $\lambda(G) \leq \widehat{C}(G^2) + 2\widehat{C}(G)$.

Proof. Color G using a construction sequence. Assign the first vertex color 0. Each vertex v added has at most $\widehat{C}(G)$ neighbors and at most $\widehat{C}(G^2) - \widehat{C}(G)$ vertices distance two away. Thus when assigning a color to v , we must avoid three colors for its neighbors and one color for each vertex distance two away. Thus we must avoid at most $\widehat{C}(G^2) + 2\widehat{C}(G)$ colors for v , so at least one of the $\widehat{C}(G^2) + 2\widehat{C}(G) + 1$ colors between 0 and $\widehat{C}(G^2) + 2\widehat{C}(G)$ is available, so $\lambda(G) \leq \widehat{C}(G^2) + 2\widehat{C}(G)$.

□

Note that it is possible to show by an argument similar to that in the theorem that $\lambda(G) \leq \max_{H \subseteq G} (\delta(H^2) + 2\delta(H))$. This is may be better than the previous result since $\lambda(G) \leq \max_{H \subseteq G} (\delta(H^2) + 2\delta(H)) \leq \widehat{C}(G^2) + 2\widehat{C}(G)$. For example, for $G = K_3 \cup_{v \in v} K_2$, $\lambda(G) = 4$, the smaller upper bound is 6, the larger upper bound is 7, and $\Delta^2 = 9$.

The previous theorem is better than the result of Griggs and Yeh.

Corollary 264. *We have $\widehat{C}(G^2) + 2\widehat{C}(G) \leq \Delta^2 + 2\Delta$. If G is connected with $\Delta \geq 2$, then equality holds exactly for regular graphs with girth at least five.*

Proof. We have $\widehat{C}(G^2) \leq \Delta(G^2) \leq \Delta^2$, so the inequality follows. By Proposition [#5, Chapter One], if it is an equality, then G is regular. If G has a cycle of length less than five, then some vertex of G has less than Δ^2 neighbors. If G is regular with girth at least five, then every vertex is within distance two of Δ^2 vertices.

□

We can prove Griggs and Yeh's conjecture for a large class of graphs.

Corollary 265. *If $\widehat{C}(G) \leq \Delta - 2$, then $\lambda(G) \leq \Delta^2$.*

Proof. Let v have $d_G(v) = \widehat{C}(G) \leq \Delta - 2$. Then $d_{G^2}(v) \leq \Delta(\Delta - 2) = \Delta^2 - 2\Delta$. Then $\lambda(G) \leq \Delta^2 - 2\Delta + 2(\Delta - 2) = \Delta^2 - 4 < \Delta^2$.

□

Note also that if $\widehat{C}(G) = \Delta - 1$, then $\lambda(G) \leq \Delta^2 + \Delta - 2$. Thus Griggs and Yeh's conjecture need only be proved for graphs that are regular or 'almost regular'.

The previous theorem is applicable to trees.

Corollary 266. *Let T be a tree. Then $\Delta + 1 \leq \lambda(G) \leq \Delta + 2$.*

Proof. It is easily seen that then $\lambda(K_{1,\Delta}) = \Delta + 1$ since every leaf must have a distinct label and the center vertex must be two away from all of them. Now $\widehat{C}(T^2) = \Delta$, so T can be colored with a span of at most $\Delta + 2$ colors.

□

Cores are also useful for still other types of coloring such as harmonious coloring [22] and 2-tone coloring [6].

4.4 Arboricity

One common theme in graph theory is splitting a graph into pieces that satisfy some rule. One example of this is arboricity.

4.4.1 Arboricity

Definition 267. The vertex-arboricity of a graph $a(G)$ is the minimum number of subsets that the vertices of G can be partitioned so that the subgraph induced by each set of vertices is a forest.

This can be seen as a generalization of proper vertex coloring, in which each subgraph induced by a color class must be empty. It is immediate that $a(G) \leq \chi(G)$. We seek better bounds for vertex-arboricity.

The following theorem is a restatement of a result of Chartrand and Kronk with a different proof. See [Chartrand/Lesniak p. 67] [15].

Theorem 268. *The vertex-arboricity of a graph G satisfies $a(G) \leq 1 + \left\lfloor \frac{1}{2} \widehat{C}(G) \right\rfloor$.*

Proof. Let $k = \widehat{C}(G)$, and consider a construction sequence for G . The result is obvious for the trivial graph. Assume the result holds for the first r vertices. That is, the graph so far constructed has a vertex partition that induces at most $1 + \left\lfloor \frac{1}{2}k \right\rfloor \leq \frac{k+2}{2}$ forests. The next vertex v added is adjacent to at most k existing vertices, so by the Pigeonhole Principle, there is some (possibly empty) set S of vertices, $G[S]$ a forest, with v adjacent to at most $\left\lfloor k / \left(\frac{k+2}{2} \right) \right\rfloor = \left\lfloor \frac{2k}{k+2} \right\rfloor \leq 1$ vertex of S . Then $G[S \cup v]$ is a forest. Thus the result holds for G by induction.

□

This bound is computationally easy to compute. This bound is exact whenever

$0 \leq \widehat{C}(G) \leq 3$ since any 2-core contains a cycle. The proof of this bound also implies an algorithm for determining a partition of the vertices of G into at most $1 + \left\lfloor \frac{1}{2} \widehat{C}(G) \right\rfloor$ sets that induce forests.

Algorithm 269. [*Vertex-arboricity Algorithm*]

Initialization: Graph G with $k = \widehat{C}(G)$, a construction sequence, and $1 + \left\lfloor \frac{1}{2} \widehat{C}(G) \right\rfloor$ sets.

Iteration: Add the next vertex v in the sequence and add it to the smallest set in which it has at most one neighbor.

Result: The sets partition the vertices of G and induce forests.

It is immediate that $a(K_n) = \lceil \frac{n}{2} \rceil$. Since $\widehat{C}(K_n) = n - 1$, this bound is sharp for arbitrarily large maximum core numbers. This also implies the following.

Proposition 270. Let $k = \omega(G)$. Then $a(G) \geq \lceil \frac{k}{2} \rceil$.

As with proper vertex coloring, we may ask what part of a subgraph is essential to determining its arboricity.

Definition 271. A graph is critical with respect to vertex-arboricity if $a(G - v) < a(G)$ for all vertices of G . When the context is clear, we will simply use k -critical to refer to vertex-arboricity in this section.

Corollary 272. If G has $a(G) = k \geq 2$ and is critical with respect to vertex-arboricity, then G is a $(2k - 2)$ -core.

Proof. If G had no $2k - 2$ -core, then by the previous theorem, $a(G) \leq k - 1$. Suppose that $d(v) \leq 2k - 3$. Then $G - v$ has a vertex partition inducing $k - 1$ forests, so by

the Pigeonhole Principle v is adjacent to at most one vertex of one of them. But then $a(G) = k - 1$, a contradiction.

□

Corollary 273. *Let $k = \widehat{C}(G)$. If $0 \leq k \leq 4$, then $a(G) = a(C_k(G))$.*

Proof. A 2-core has arboricity at least 2, and a 3-shell has arboricity at most 2.

□

The upper bound in the previous theorem need not be exact. If it is not, then the maximum core of G must have arboricity smaller than necessitated by the theorem. For example, if G is complete k -partite, $a(G) \leq k$. Thus we may be able to improve on the previous upper bound if we can determine or at least otherwise bound the arboricity of the higher cores of G . This corollary is analogous to Theorem 203 on proper vertex coloring.

Corollary 274. *The vertex-arboricity of G satisfies $a(G) \leq \max_k \min\left(\left\lfloor \frac{k+2}{2} \right\rfloor, a(C_k(G))\right)$.*

Proof. Each vertex in core i is adjacent to at most one vertex in one of $\left\lfloor \frac{i+2}{2} \right\rfloor$ vertex sets, and it can be put into one of $a(C_k(G))$ sets.

□

We can also consider the analogous question for edge partitions.

Definition 275. The edge-arboricity, or simply arboricity $a_1(G)$ is the minimum number of forests into which G can be decomposed.

It is inherent in the definitions that $a(G) \leq a_1(G)$, since for any decomposition of G into forests, there is a collection of subforests for which each vertex is used exactly

once. These parameters must be equal when $0 \leq a_1(G) \leq 2$. The smallest graph for which they are unequal is $G = K_5 - e$, for which $a(G) = 2$ and $a_1(G) = 3$.

We can determine upper and lower bounds for edge-arboricity which are easy to calculate. The lower bound follows Chartrand and Kronk. See [Chartrand/Lesniak p. 68] [15].

Proposition 276. *The edge-arboricity of G satisfies $\left\lceil \frac{1}{2} \left(1 + \widehat{C}(G) \right) \right\rceil \leq a_1(G) \leq \widehat{C}(G)$.*

Proof. Let $k = \widehat{C}(G)$. Now G has a construction sequence so that each vertex has degree at most k when added. Then at most k edges can be distributed amongst at most k forests that decompose the graph up to that point. Thus $a_1(G) \leq \widehat{C}(G)$.

Let H be the maximum core of G with order n , size m . Then H can be decomposed into at most $a_1(G)$ forests each with size at most $n - 1$. Then $\widehat{C}(G) \leq \frac{2m}{n} \leq \frac{2(n-1)a_1(G)}{n} < 2a_1(G)$. Thus $2a_1(G) \geq \widehat{C}(G) + 1$, so $a_1(G) \geq \left\lceil \frac{1}{2} \left(1 + \widehat{C}(G) \right) \right\rceil$.

□

While this bound is computationally simple, it is not the best possible. The proof of the previous proposition suggests a stronger result.

Proposition 277. *For every nonempty graph G , $a_1(G) \geq \max_{H \subseteq G} \left\lceil \frac{m(H)}{n(H)-1} \right\rceil$, where the maximum is taken over all induced subgraphs of G .*

Proof. If G decomposes into k forests, then $m(G) \leq k \cdot (n(G) - 1)$. Hence this condition applies to any subgraph of G .

□

In fact, this is an equality.

Theorem 278. [Nash-Williams 1964] [46] For every nonempty graph G , $a_1(G) = \max_{H \subseteq G} \left\lceil \frac{m(H)}{n(H)-1} \right\rceil$, where the maximum is taken over all induced subgraphs of G .

This is a difficult theorem to prove, though a relatively short proof appears in [Chen, Matsumoto, Wang, Zhang, and Zhang 1994] [17].

While this is in some sense an exact answer to the problem of edge-arboricity, it requires determining a maximum over all induced subgraphs of a graph, which is impractical for all but the smallest graphs. We would like to simplify this computation by reducing the number of subgraphs which must be checked.

Definition 279. The density of a graph is $\frac{m}{n}$. The a-density of a nontrivial graph G is $\frac{m}{n-1}$.

Thus our goal is to find the subgraph of a graph with maximum a-density.

Theorem 280. Let $H \subseteq G$ be a subgraph of G with maximum a-density, and $k = \left\lceil \frac{1}{2} \widehat{C}(G) \right\rceil$. Then $H \subseteq C_k(G)$.

Proof. Assume the hypothesis, and to the contrary that $H \not\subseteq C_k(G)$. Let $l < k$ be the smallest integer such that H has an l -shell. Let n be the order of H , $n_l = |S_l(H)|$, and $r = \widehat{C}(G)$. Now the a-density of $C_r(G)$ is at least $\frac{1}{n_r-1} \cdot \frac{r \cdot n_r}{2} > \frac{r}{2}$. But then

$$\begin{aligned} \frac{m_H}{n_H-1} &\geq \frac{m_H-m_l}{n_H-n_l-1} \\ m_H(n_H - n_l - 1) &\geq (m_H - m_l)(n_H - 1) \\ m_l(n_H - 1) &\geq m_H n_l \\ l \cdot n_l \cdot (n_H - 1) &\geq \frac{r}{2} (n_H - 1) n_l \\ l &\geq \left\lceil \frac{r}{2} \right\rceil = k \end{aligned}$$

But this is a contradiction. □

Virtually the same argument, replacing $n_H - 1$ with n_H , proves the following.

Corollary 281. *Let $H \subseteq G$ be a subgraph of G with maximum density, and $k = \left\lceil \frac{1}{2} \widehat{C}(G) \right\rceil$. Then $H \subseteq C_k(G)$.*

There may be more than one distinct subgraph with maximum density. If we are trying to determine the arboricity of a graph, we may only need to find some subgraph with maximum a-density. Hence if $\widehat{C}(G)$ is even, the following slight improvement is possible.

Corollary 282. *Let $j = \left\lceil \frac{1}{2} (\widehat{C}(G) + 1) \right\rceil$. Then there exists a subgraph $H \subseteq G$ of maximum a-density with $H \subseteq C_j(G)$.*

Proof. Let H be a subgraph of maximum a-density, $k = \left\lceil \frac{1}{2} \widehat{C}(G) \right\rceil$, and $r = \widehat{C}(G)$. If $H \subseteq C_j(G)$, we are done, and if r is odd, the result is immediate. Suppose that $H \not\subseteq C_j(G)$, and r is even. Then $H \subseteq C_k(G)$. Now

$$\frac{m_H - m_k}{n_H - n_k - 1} \geq \frac{m_H}{n_H - 1} \iff m_H n_k \geq m_k (n_H - 1) \iff \frac{m_H}{n_H - 1} \geq \frac{m_k}{n_k}.$$

But the a-density of H is $\frac{m_H}{n_H - 1} \geq \frac{r}{2} = \left\lceil \frac{r}{2} \right\rceil = k$, which is at least the density of the k -shell of G , so the result holds. □

Again, virtually the same argument proves the following.

Corollary 283. *Let $j = \left\lceil \frac{1}{2} (\widehat{C}(G) + 1) \right\rceil$. Then there exists a subgraph $H \subseteq G$ of maximum density with $H \subseteq C_j(G)$.*

Thus in search of a subgraph of maximum density, we may strip away the relatively low-numbered shells, since some such subgraph contains none of their vertices. We may be able to improve on this if we have more information on the density of some subgraphs of G .

Corollary 284. *Let G have a subgraph with a-density r , $k = \lceil r \rceil$, and $j = \lfloor r + 1 \rfloor$. Then for any subgraph H of maximum a-density, $H \subseteq C_k(G)$, and there exists some subgraph H' of maximum a-density with $H' \subseteq C_j(G)$. Corresponding results hold for density.*

These results are best possible. For example, consider the graph $G = K_{2r+1} \cup_{K_r=rK_1} K_{r,s} = K_r + (K_{r+1} \cup sK_1)$. Then $\delta(G) = r$, $\widehat{C}(G) = 2r$, the density of G is $\frac{\binom{2r+1}{2} + rs}{2r+1+s} = r$, and the density of its maximum core is $\binom{2r+1}{2} \frac{1}{2r+1} = r$. Furthermore, if $G = K_{2r} \cup_{K_r=rK_1} K_{r,s} = K_r + (K_r \cup sK_1)$, then $\delta(G) = r$, $\widehat{C}(G) = 2r - 1$, the density of G is $\frac{\binom{2r}{2} + rs}{2r+s} = r \left(\frac{2r+s-1}{2r+s} \right) > r - \frac{1}{2}$ for $r, s \geq 1$, and the density of its maximum core is $\binom{2r}{2} \frac{1}{2r} = r - \frac{1}{2}$.

This latter graph has a-density r for both itself and its maximum core. However, for $G = K_{2r-1} \cup_{K_r=rK_1} K_{1,r} = K_r + (K_{r-1} \cup K_1)$, $\delta(G) = r$, $\widehat{C}(G) = 2r - 2$, the a-density of G is $\frac{\binom{2r-1}{2} + r}{2r-1} = (r-1) + \frac{r}{2r-1} > r - \frac{1}{2}$ for $r \geq 1$, and the a-density of its maximum core is $\binom{2r-1}{2} \frac{1}{2r-1} = r - 1$.

Beyond these results, we cannot say in general where a subgraph of maximum density lies. It might be the maximum core, be inside the maximum core, properly contain the maximum core, or partially overlap the maximum core.

We now consider the arboricity of maximal k -degenerate graphs. This question has been previously considered by [Patil 1984] [47]. He used Nash-Williams' theorem to give an existential proof. We provide a much shorter proof. Note that it follows immediately from Theorem 69 that if G is maximal k -degenerate, then $a_1(G) \leq k$.

The arboricity may be smaller if n is small relative to k .

Theorem 285. *Let G be maximal k -degenerate. Then $a_1(G) = \lceil k - \binom{k}{2} \frac{1}{n-1} \rceil$.*

Proof. A maximal k -degenerate graph of order n has size $m = k \cdot n - \binom{k+1}{2}$. Then its a-density is $\frac{m}{n-1} = [k \cdot n - \binom{k+1}{2}] \frac{1}{n-1} = k + [k - \binom{k+1}{2}] \frac{1}{n-1} = k - \binom{k}{2} \frac{1}{n-1}$. Note that this function is monotone with respect to n . Now any subgraph of a k -degenerate graph is also k -degenerate, so this implies that any proper subgraph of G has smaller a-density. Then by Nash-Williams' theorem, $a_1(G) = \lceil k - \binom{k}{2} \frac{1}{n-1} \rceil$.

□

It may be possible to provide a constructive proof of this theorem that avoids use of Nash-Williams' theorem.

Since any graph with $\widehat{C}(G) = k$ is contained in a maximal k -degenerate graph, this theorem implies that the bound $a_1(G) \leq \widehat{C}(G)$ is sharp for all k . More specifically, for a given k , it is sharp for all $n \geq \binom{k}{2} + 2$. For $n \leq \binom{k}{2} + 1$, it is not sharp. But this theorem implies the following easy-to-calculate upper bound.

Corollary 286. *Let $k = \widehat{C}(G)$. Then $a_1(G) \leq \lceil k - \binom{k}{2} \frac{1}{n-1} \rceil$.*

For k -monocore graphs, we have the following.

Corollary 287. *Let G be k -monocore. Then $\lceil \frac{k+1}{2} \rceil \leq a_1(G) \leq \lceil k - \binom{k}{2} \frac{1}{n-1} \rceil$.*

We can use the bounds on the size of k -collapsible graphs to show the following.

Corollary 288. *Let G be k -collapsible. Then $\lceil \frac{k+1}{2} \rceil \leq a_1(G) \leq \lceil k - 1 - ((\binom{k-1}{2} + 1) \frac{1}{n-1}) \rceil$.*

Proof. Let G be k -collapsible. The lower bound is the same as in Proposition 276. A k -collapsible graph of order n has size at most $m = (k - 1) \cdot n - \binom{k}{2} + 1$. Then its a-density is at most $\frac{m}{n-1} = [(k - 1) \cdot n - \binom{k}{2} + 1] \frac{1}{n-1} = k - 1 + [k - 1 - \binom{k}{2} + 1] \frac{1}{n-1} = k - 1 - (\binom{k-1}{2} + 1) \frac{1}{n-1}$. Now any proper induced subgraph of G is $k - 1$ -degenerate, so no subgraph has larger a-density. Thus by Nash-Williams' theorem, the upper bound holds. \square

In our efforts to reduce the number of subgraphs of a graph G that must be checked to determine its arboricity, we can also bound the orders of the subgraphs. Clearly a very small subgraph has no chance of achieving the maximum.

Corollary 289. *Let G be a graph with $k = \hat{C}(G)$ and some subgraph with a-density $d < k$. Then any subgraph of maximum a-density has order at least $n \geq \binom{k}{2} \frac{1}{k-d} + 1$.*

Proof. A subgraph H with maximum core number k has maximum a-density when it is maximal k -degenerate. Thus the order n of H must satisfy $k - \binom{k}{2} \frac{1}{n-1} \geq d$. This is equivalent to $k - d \geq \binom{k}{2} \frac{1}{n-1}$, and $n \geq \binom{k}{2} \frac{1}{k-d} + 1$, so the result follows. \square

Thus determining the arboricity can be simplified by stripping away small shells and checking subgraphs of relatively large order. This still appears to be a difficult problem, however.

4.4.2 Generalizations to k -Degenerate Graphs

Proper vertex coloring studies partitioning the vertex set of a graph into independent sets, while vertex arboricity studies partitions that induce forests. We have seen that 0-degenerate graphs are exactly empty graphs, while 1-degenerate graphs are exactly

forests. Thus it is a natural question to consider partitions of the vertices of a graph into classes that induce k -degenerate graphs.

Definition 290. The point partition number $\rho_k(G)$ is the minimum number of sets into which the vertices of G can be partitioned so that each set induces a k -degenerate graph.

Point partition numbers were first introduced in 1970 by Lick and White [38] in the same paper that introduced k -degenerate graphs. It is immediate that $\chi(G) = \rho_0(G) \leq a(G) = \rho_1(G) \leq \rho_2(G) \leq \dots \leq \rho_k(G) \leq \dots$

There was a flurry of research on these numbers in the 1970s. A survey of results up to its writing appears in [Simoes-Pereira 1976] [54]. In their original paper, Lick and White determined these numbers exactly for complete multipartite graphs. They also determined an upper bound which we can state in terms of cores, and will prove by a different means.

Theorem 291. *The point partition number $\rho_k(G)$ of a graph G satisfies $\rho_k(G) \leq 1 + \left\lfloor \frac{1}{k+1} \widehat{C}(G) \right\rfloor$.*

Proof. Let $d = \widehat{C}(G)$, and consider a construction sequence for G . The result is obvious for the trivial graph. Assume the result holds for the first r vertices. That is, the graph so far constructed has a vertex partition that induces at most $1 + \left\lfloor \frac{d}{k+1} \right\rfloor \leq \frac{d+k+1}{k+1}$ k -degenerate graphs. The next vertex v added is adjacent to at most d existing vertices, so by the Pigeonhole Principle, there is some (possibly empty) set S of vertices, $G[S]$ k -degenerate, with v adjacent to at most $\lfloor d / (\frac{d+k+1}{k+1}) \rfloor = \left\lfloor \frac{d(k+1)}{d+k+1} \right\rfloor \leq k$ vertices of S . Then $G[S \cup v]$ is k -degenerate. Thus the result holds for G by induction. \square

Thus this upper bound implies both the core number bound and the corresponding bound for vertex-arboricity in Theorem 268. Once again, this bound is computationally easy to compute. This bound is exact whenever $0 \leq \widehat{C}(G) \leq 2k + 1$ since any graph with $k + 1 \leq \widehat{C}(G) \leq 2k + 1$ is not k -degenerate, but the upper bound is two. The proof of this bound also implies an algorithm for determining a partition of the vertices of G into at most $1 + \left\lfloor \frac{1}{k+1} \widehat{C}(G) \right\rfloor$ sets that induce k -degenerate graphs.

Algorithm 292. [Point Partition Algorithm]

Initialization: Graph G with $d = \widehat{C}(G)$, a construction sequence, and $1 + \left\lfloor \frac{1}{k+1} \widehat{C}(G) \right\rfloor$ sets.

Iteration: Add the next vertex v in the sequence and add it to the smallest set in which it has at most k neighbors.

Result: The sets partition the vertices of G and induce k -degenerate graphs.

The following result is immediate.

Proposition 293. We have $\rho_k(G) \geq \left\lceil \frac{\omega(G)}{k+1} \right\rceil$.

As with proper vertex coloring and vertex-arboricity, we may ask what part of a subgraph is essential to determining its point partition number.

Definition 294. A graph is d -critical with respect to ρ_k if $\rho_k(G - v) < \rho_k(G) = d$ for all vertices of G .

Corollary 295. If G has $\rho_k(G) = d$ and is critical with respect to vertex-arboricity, then G is a $(k + 1)(d - 1)$ -core.

Proof. If G had no $(k+1)(d-1)$ -core, then by the previous theorem, $\rho_k(G) \leq d-1$. Suppose that $d(v) \leq (k+1)(d-1)-1$. Then $G-v$ has a vertex partition inducing $d-1$ k -degenerate graphs, so by the Pigeonhole Principle v is adjacent to at most k vertices of one of them. But then $\rho_k(G) = d-1$, a contradiction. \square

Corollary 296. Let $d = \widehat{C}(G)$. If $0 \leq d \leq 2(k+1)$, then $\rho_k(G) = \rho_k(C_d(G))$.

Proof. A $k+1$ -core has ρ_k at least 2, and a $2k+1$ -shell has ρ_k at most 2. \square

The upper bound in the previous theorem need not be exact. If it is not, then the maximum core of G must have ρ_k smaller than necessitated by the theorem. Thus we may be able to improve on the previous upper bound if we can determine or at least otherwise bound ρ_k for the higher cores of G . This corollary is analogous to Theorem 203 on proper vertex coloring.

Corollary 297. The number ρ_k for G satisfies $\rho_k(G) \leq \max_i \min\left(\left\lfloor \frac{d+i+1}{i+1} \right\rfloor, \rho_k(C_i(G))\right)$.

Proof. Each vertex in core i is adjacent to at most k vertices in one of $\left\lfloor \frac{d+i+1}{i+1} \right\rfloor$ vertex sets, and it can be put into one of $\rho_k(C_i(G))$ sets. \square

We can also consider the analogous question for edge partitions.

Definition 298. The edge-partition number, $\rho'_k(G)$ is the minimum number of k -degenerate graphs into which G can be decomposed.

This definition appears in [Simoes-Pereira 1976] [54], which describes it as “little-investigated”. So far as I can tell, this has not changed. It is inherent in the definitions

that $\rho_k(G) \leq \rho'_k(G)$, since for any decomposition of G into k -degenerate graphs, there is a collection of k -degenerate subgraphs for which each vertex is used exactly once.

These parameters must be equal when $0 \leq \rho'_k(G) \leq 2$.

It is also immediate that $a_1(G) = \rho_1(G) \leq \rho_2(G) \leq \dots \leq \rho_k(G) \leq \dots$

We can determine upper and lower bounds for edge-partition numbers which are easy to calculate.

Theorem 299. *The edge-partition number, $\rho'_k(G)$ satisfies $\left\lceil \frac{1}{2k} \left(1 + \widehat{C}(G) \right) \right\rceil \leq \rho'_k(G) \leq \left\lceil \frac{1}{k} \widehat{C}(G) \right\rceil$.*

Proof. Let $d = \widehat{C}(G)$. G has a construction sequence so that each vertex has degree at most d when added. Then at most d edges can be distributed amongst at most $\left\lceil \frac{d}{k} \right\rceil$ k -degenerate graphs that decompose the graph up to that point. Thus $\rho'_k(G) \leq \left\lceil \frac{1}{k} \widehat{C}(G) \right\rceil$.

Let H be the maximum core of G with order n , size m . Then H can be decomposed into at most $\rho'_k(G)$ k -degenerate graphs each with size at most $k \cdot n - \binom{k}{2}$. Then $\widehat{C}(G) \leq \frac{2m}{n} \leq \frac{2}{n} \left(k \cdot n - \binom{k+1}{2} \right) \rho'_k(G) < 2k \cdot \rho'_k(G)$. Thus $2k \cdot \rho'_k(G) \geq \widehat{C}(G) + 1$, so $\rho'_k(G) \geq \left\lceil \frac{1}{2k} \left(1 + \widehat{C}(G) \right) \right\rceil$.

□

While this bound is computationally simple, it may not be the best possible. The proof of the previous result suggests a stronger result.

Proposition 300. *For every nonempty graph G , $\rho'_k(G) \geq \max_{H \subseteq G} \left\lceil \frac{m(H)}{k \cdot n(H) - \binom{k+1}{2}} \right\rceil$, where the maximum is taken over all induced subgraphs of G .*

Proof. If G decomposes into d k -degenerate graphs, then $m(G) \leq d \cdot (k \cdot n(G) - \binom{k+1}{2})$. Hence this condition applies to any subgraph of G .

□

It would be a natural generalization of Nash-Williams' theorem if this were an equality.

Conjecture 301. [Degenerate Covering Conjecture] Let G be a nonempty graph. Then $\rho'_k(G) = \max_{H \subseteq G} \left\lceil \frac{m(H)}{k \cdot n(H) - \binom{k+1}{2}} \right\rceil$, where the maximum is taken over all induced subgraphs of G .

It may be possible to generalize the approach of Chen et al [17] to prove this conjecture.

Definition 302. A graph is d -critical with respect to ρ'_k if $\rho'_k(G - e) < \rho'_k(G) = d$ for all edges of G .

The following conjectured lemma is analogous to that used by Chen et al.

Conjecture 303. Let G be connected and d -critical with respect to ρ'_k , $n > 1$. Then for any edge e of G , any decomposition of $G - e$ into $d - 1$ k -degenerate graphs is a decomposition into $d - 1$ maximal k -degenerate graphs of order n .

In fact, these two conjectures are equivalent.

Theorem 304. The degenerate covering conjecture holds if and only if the previous conjectured lemma holds.

Proof. Assume the Degenerate Covering Conjecture holds, and let G be connected and d -critical with respect to ρ'_k , $n > 1$. Then $G - e$ decomposes into $d - 1$ k -degenerate graphs. So $m(G - e) \leq (d - 1)(k \cdot n - \binom{k+1}{2})$ and $m(G) \geq (d - 1)(k \cdot n - \binom{k+1}{2}) +$

1. Then both inequalities are equalities, so $G - e$ decomposes into $d - 1$ maximal k -degenerate graphs of order n , and the lemma holds.

Assume the conjectured lemma holds and let G be a counterexample to the Degenerate Covering Conjecture that minimizes $n + m$. Now the left-hand side is greater than the right in the conjecture. Then G is connected with $n > 1$, and d -critical with respect to ρ'_k . Then $m - 1 = m(G - e) = (k \cdot n - \binom{k+1}{2})(d - 1)$. But then $d > \left\lceil \frac{m}{k \cdot n - \binom{k+1}{2}} \right\rceil = \lceil d - 1 + 1 / (k \cdot n - \binom{k+1}{2}) \rceil = d$, which is a contradiction.

□

Theorem 190 can be reinterpreted to state that the Degenerate Covering Conjecture is true for complete graphs. Thus this conjecture can be considered a potential generalization of Theorem 190.

4.5 Planarity

Cores have natural relationships to planar graphs and concepts related to planarity. We have already seen that if G is planar, $\widehat{C}(G) \leq 5$, and if it has order $n < 12$, then $\widehat{C}(G) \leq 4$.

4.5.1 Trees and Triangulations

It is obvious that all trees are planar. There is a related class of graphs for which planarity is somewhat less obvious. Recall that a k -tree is formed iteratively by starting with K_{k+1} and adding each new vertex adjacent to an existing k -clique.

Proposition 305. *Every 2-tree is planar.*

Proof. First note that K_3 is planar. Each subsequent vertex is added adjacent to two mutually adjacent vertices, so it can be inserted inside a region that they are on. □

It is not the case that every 2-degenerate graph is planar. The 2-degenerate nonplanar graph of smallest order and size is $K_{3,3}$ with a subdivided edge. The two maximal 2-degenerate nonplanar graphs of smallest order are both formed from $K_{3,3}$ by subdividing one edge and adding another edge.

Not every 3-tree is planar. We can characterize exactly which 3-trees are planar.

Proposition 306. *A 3-tree G is planar if and only if each triangle of G is used at most once as the root of a new vertex in some construction sequence of G .*

Proof. A 3-tree is constructed beginning with K_4 . For any triangle, either inside or outside it there is a vertex adjacent to its three vertices, while the other is a region

in which a vertex can be placed and made adjacent to the vertices of the triangle. If G is constructed using each triangle at most once as the root of a new vertex in its construction sequence, this fact will continue to hold, so G will be planar. If the same triangle is used twice as the root of two vertices then these vertices, the edges incident with them, and the vertex that must already be adjacent to the vertices of this triangle produce a copy of $K_{3,3}$, so G is nonplanar.

□

For planar 3-trees, we can state more information.

Proposition 307. *Let G be a graph with order $n \geq 4$. Then G is a 4-core-free maximal planar graph \iff it is a planar 3-tree.*

Proof. If G is a planar 3-tree, then it is 3-degenerate and so 4-core-free. Now G has size $3n - 6$ and is planar, so it is maximal planar.

The result holds for K_4 . Let G be a 4-core free maximal planar graph of order more than 4. Then it has a vertex v of degree 3, and $G - v$ is also maximal planar and a 3-tree. The result holds by induction.

□

When proving results about planar graphs, proofs are often restricted to maximal planar graphs. This is because these are a special category of planar graphs, and any planar graph is a subgraph of a maximal planar graph. Thus for any graph property unaffected by adding edges, it suffices to prove it holds in this case.

A maximal planar graph is also called a triangulation. We say that we triangulate a planar graph when we add edges so that it becomes a triangulation. In general, there are many ways to do this. We would prefer to triangulate a graph so that it does not create any relatively large cores, as these create restrictions that are harder to work with.

Definition 308. A planar graph G has a k -core-free triangulation if there exists a graph H with the same vertex set, $G \subseteq H$, and H is a triangulation. G has a k -monocore triangulation if its triangulation is k -monocore.

Any triangulation is a 3-core, while any planar graph is 6-core-free. Thus we will be interested in 3, 4, and 5-cores.

Proposition 309. *Every tree with order at least four has a 3-monocore triangulation.*

Proof. Both trees of order four are subgraphs of K_4 . Every time an edge is added, make the vertex adjacent to all three vertices of the region that it is inside. The result is a 3-tree, and hence a triangulation.

□

This can be extended to larger classes of graphs using an analogous argument.

Corollary 310. *Every 2-tree has a 3-monocore triangulation. A graph has a 3-monocore triangulation \iff its 2-core does.*

The following conjecture seems reasonable.

Conjecture 311. *Every planar 3-core-free graph and planar 3-collapsible graph with order at least four has a 3-monocore triangulation.*

In light of this conjecture, we might wonder if every 4-core-free planar graph has a 3-monocore triangulation. We know that every 4-core-free graph is contained in both a maximal planar graph, and in a maximal 3-degenerate graph. Furthermore, both a triangulation and a 3-tree have size $3n - 6$. But these need not be the same graph. For

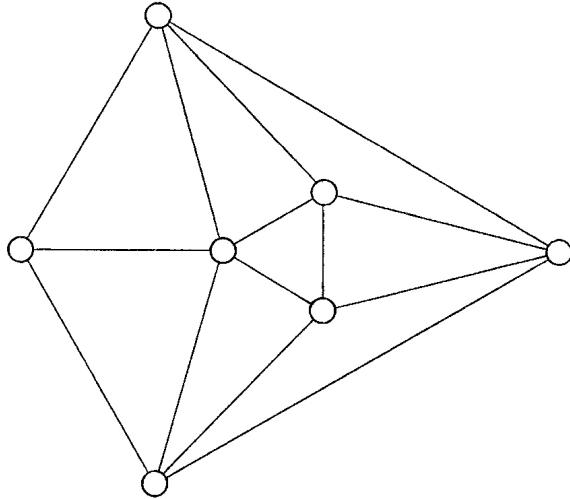


Figure 33: A 4-core-free planar graph with no 3-monocore triangulation.

example, consider the graph formed by starting with an octahedron with triangular region R . Add a vertex inside R adjacent to its boundary vertices and then delete an edge of the region. The resulting graph has no 4-core, but has no 3-monocore triangulation.

We also note the following result.

Proposition 312. *If a triangulation has a 4-core, its 4-core is a triangulation.*

Proof. If a planar 4-core has a region which is not a triangle, adding a vertex of degree less than four cannot triangulate the region, so the new graph must still have another non-triangular region. □

4.5.2 Dual Graphs

The dual graph G^* of a planar graph G has vertices representing the regions of G and edges of G^* between regions of G that share an edge. In general, the dual graph

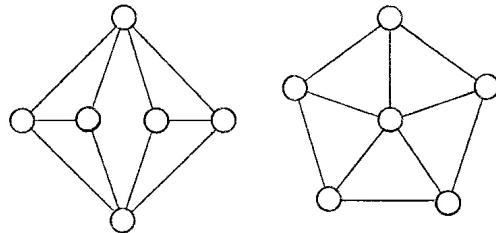


Figure 34: The self-dual graphs $2K_2 + 2K_1$ and W_5 .

of a graph is a multigraph, not necessarily a graph. A bridge and minimal 2-edge-cut in G produce a loop and parallel edges in G^* . To avoid this, we will assume that G 3-edge-connected, and hence a 3-core. In general, the dual graph is not unique, it depends on the particular embedding of G . However, if G is 3-connected, it has a unique dual graph.

Proposition 313. *At most one of G and G^* is a 4-core.*

Proof. Assume G is a 4-core. Then in G^* , every region has length at least four. Then $4r \leq 2m$, so using Euler's Identity, $8 = 4n - 4m + 4r \leq 4n - 4m + 2m$, so $m \leq 2n - 4$. But any 4-core has $m \geq 2n$, so G^* is not a 4-core.

□

A graph is self-dual if $G \cong G^*$.

Proposition 314. *If G is a nontrivial self-dual graph, then $m = 2n - 2$, G has at least four vertices of degree three and at least four triangles.*

Proof. If G is self-dual, then $n = r$, so $m = 2n - 2$ follows from Euler's formula. This implies that G has at least four vertices of degree three since G is a 3-core, and these map to triangles.

□

The smallest nontrivial self-dual graph is K_4 . Indeed, any wheel is self-dual. The graph $2K_2 + 2K_1$ is self-dual. Perhaps surprisingly, a self-dual graph can have a cut-vertex. The smallest such graph is $K_4 \cup_v K_4 = 2K_3 + K_1$. In this case, the two blocks are both self-dual, but this need not be the case either. In light of the previous result, we might suspect that a self-dual graph must be 3-monocore. But this is not the case.

Proposition 315. *The smallest order of a self-dual graph with a 4-core is 13, and there are exactly three such graphs of that order.*

Proof. The smallest planar 4-core is the octahedron, $K_{2,2,2}$, which has order 6. In order to obtain $m = 2n - 2$, we need four vertices of degree three. But since G is 3-edge-connected, the sum of the degrees in G of the vertices in the 4-core will be at least three larger than this sum in the subgraph, so we need at least seven vertices of degree 3. If G has the octahedron as its 4-core, it must also contain $Q_3 - v$, which is the dual of its interior regions. This cannot have any vertices in common with the 4-core. Finally, G must have three edges to connect these two subgraphs, corresponding to the three exterior edges of the octahedron. There are three possibilities. All three edges can go to one vertex of the octahedron, all three can go to distinct exterior vertices, or two can go to one and one to another. All three possibilities in fact produce self-dual graphs.

□

These three graphs have connectivities 1, 2, and 3. At this point, it isn't hard to guess that a self-dual graph can contain a 5-core.

Corollary 316. *The smallest order of a self-dual graph with a 5-core is 31, and there are exactly three such graphs of that order.*

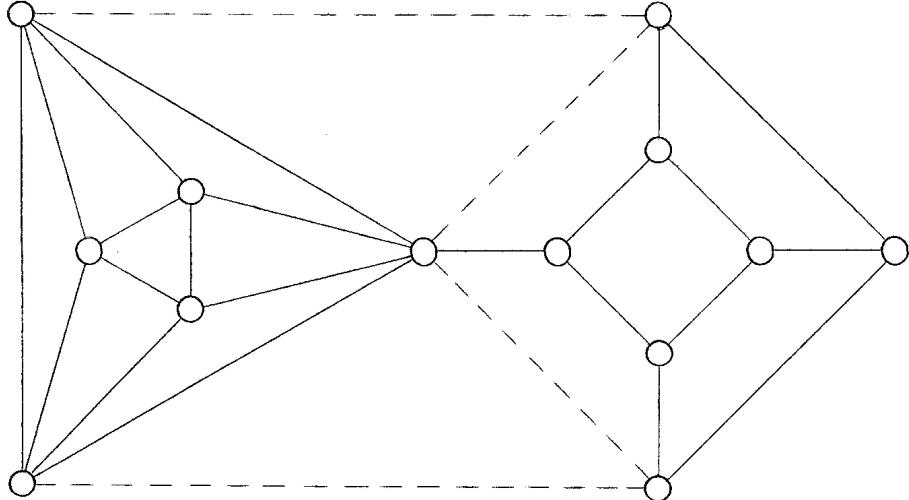


Figure 35: Construction of the three self-dual graphs with a 4-core and order 13.

Proof. The smallest planar 5-core is the icosahedron (IC), which has order 12. As before, in order to obtain $m = 2n - 2$, we need 19 vertices of degree three. The dual of the icosahedron is the dodecahedron (DD). If G has IC as its 5-core, it must also contain $DD - v$, which is the dual of its interior regions. This cannot have any vertices in common with the 5-core. Finally, G must have three edges to connect these two subgraphs, corresponding to the three exterior edges of the octahedron. As before, there are three possibilities, all of which in fact produce self-dual graphs.

□

In the examples we have seen, the vertices of the 4-core of a self-dual graph map to the regions outside the 4-core. It is unclear whether this always happens.

Note also that it is possible to take the dual of a graph on other topological surfaces. For example, the product of cycles $C_m \times C_n$ is 4-regular and it is self-dual on the torus.

4.5.3 Measures of Nonplanarity

For graphs that are not planar, we have several measures of how nonplanar they

are.

We first note that since the 1-shell of a graph is a forest, we have the following trivial result.

Proposition 317. *A graph G with a 2-core is planar \iff its 2-core is planar.*

Since planar graphs have been characterized in Kuratowski's Theorem, this does not improve on existing results.

The crossing number $v(G)$ of a graph G is the smallest number of crossings with which it can be drawn in the plane. The genus $\gamma(G)$ of a graph G is the smallest genus of a surface on which it can be embedded without crossings. The thickness $\theta_1(G)$ of a graph G is the smallest number of planar graphs into which it can be decomposed.

The next corollary follows the same reasoning as the previous result.

Corollary 318. *If G has a 2-core, $v(G) = v(C_2(G))$, $\gamma(G) = \gamma(C_2(G))$, and $\theta_1(G) = \theta_1(C_2(G))$.*

While still easy, this result is not completely trivial since there is no known characterization of graphs with a specific positive crossing number or genus or thickness larger than one.

For crossing number and genus, this cannot be improved, since deleting a vertex of degree two may eliminate a crossing or reduce the genus. For thickness, we can push this a bit further.

Proposition 319. *If G has thickness at least k , then $\theta_1(G) = \theta_1(C_{k+1}(G))$.*

Proof. The l -shell, $l \leq k$ can be decomposed into l forests, which are planar.

□

The thickness of complete graphs is known, so determining the clique number of a graph can help to strip away the outer shells that don't affect the thickness. Then, a lower bound such as $\theta_1(G) \geq \lceil \frac{m}{3n-6} \rceil$ can be employed on the denser part of the graph.

Conversely, the maximum core number can be used to bound the thickness.

Proposition 320. *For a nontrivial graph G , we have $\theta_1(G) \leq \widehat{C}(G)$.*

This does not appear to be a particularly good bound. For example, the smallest clique with thickness 3 is K_9 , while the smallest complete bipartite graph with thickness 3 is $K_{7,7}$. We can improve this result for some graphs.

Theorem 321. *Let G be a k -tree. Then $\lceil \frac{1}{3}k \rceil \leq \theta_1(G) \leq \lceil \frac{1}{2}k \rceil$.*

Proof. We have $\theta_1(G) \geq \lceil \frac{m}{3n-6} \rceil = \left\lceil \frac{k \cdot n - \binom{k+1}{2}}{3n-6} \right\rceil = \lceil \frac{k}{3} \rceil$.

If k is odd, G is contained in a $k+1$ -tree, so we assume k is even. K_{k+1} can be decomposed into $\frac{k}{2}$ unions of cycles, which are planar. Further, given any k vertices of this clique, it is possible to select $\frac{k}{2}$ edges, one from each of the cycles (for example, take parallel chords, one of each possible length). Use a construction sequence for G and divide the k edges added with a new vertex into $\frac{k}{2}$ classes of two edges so that each pair of edges are adjacent to neighboring vertices of some distinct subgraph in the decomposition. Now this produces a new $k+1$ -clique.

Consider a k -clique using the new vertex w and $k-1$ vertices from the previous clique. Those vertices must have $\frac{k-2}{2}$ edges from distinct subgraphs in the decomposition and one vertex v whose mate u was not chosen for the new clique. Then the edge vw was assigned to the subgraph containing uv , so the new clique contains a

similar selection of edges. Thus the process of adding a vertex and assigning edges incident with it to distinct planar subgraphs can continue. This produces subgraphs that are 'almost' 2-trees (except for the root), and so they are planar analogously.

□

The upper bound applies to any graph that is contained in a k -tree, such as a chordal graph with $\omega(G) = k + 1$. This result is best possible for $1 \leq k \leq 4$, but it is unclear how good it is for larger values of k . It does not hold for all k -degenerate graphs, since there are nonplanar 2-degenerate graphs.

Analogous results do not hold for crossing number and genus since given any 2-core with large crossing number or genus, the graph formed by subdividing every edge is 2-monocore with the same crossing number and genus.

On the other hand, we can bound the maximum core number of graphs of a given genus, extending the result that if G is planar, $\widehat{C}(G) \leq 5$. The following is adapted from [White 2001 p. 94] [61].

Proposition 322. *Let G have genus $\gamma(G) = k$. Then $\widehat{C}(G) \leq 6 + \frac{12(k-1)}{n}$.*

Proof. Let $a = \frac{2m}{n}$ be the average degree of vertices of G . We have $3r \leq 2m$, so $m \leq 3(m - r)$. Now by the generalized Euler Identity, $2 - 2k = n - m + r$, so $m \leq 3(m - r) = 3(n + 2k - 2)$. If G embeds on S_k , then so does its maximum core H , so $\widehat{C}(G) = \widehat{C}(H) = \delta(H) \leq a = \frac{2m}{n} \leq 6 + \frac{12(k-1)}{n}$.

□

This implies that if G is planar, $\widehat{C}(G) \leq 5$, while if G embeds on the torus, $\widehat{C}(G) \leq 6$.

We can also consider nonorientable surfaces, for which the Euler identity is $n - m + r = 2 - k$, where k is the number of crosscaps.

Proposition 323. *Let G have nonorientable genus $\gamma(G) = k$. Then $\widehat{C}(G) \leq 6 + \frac{6(k-2)}{n}$.*

Proof. Let $a = \frac{2m}{n}$ be the average degree of vertices of G . We have $3r \leq 2m$, so $m \leq 3(m-r)$. Now by the generalized Euler Identity, $2 - k = n - m + r$, so $m \leq 3(m-r) = 3(n+k-2)$. If G embeds on N_k , then so does its maximum core H , so $\widehat{C}(G) = \widehat{C}(H) = \delta(H) \leq a = \frac{2m}{n} \leq 6 + \frac{6(k-2)}{n}$.

□

This implies that if G embeds on the projective plane, $\widehat{C}(G) \leq 5$.

It is probably possible to bound the maximum core number in terms of the crossing number, but such a formula is unknown.

4.5.4 Planarity and Coloring

The most famous problem in graph theory involves the coloring of planar graphs. The core number bound can be applied to this problem, yielding the following corollary.

Corollary 324. *[The Six Color Theorem] If G is planar, then $\chi(G) \leq 6$.*

Proof. If G is planar, $\chi(G) \leq 1 + \widehat{C}(G) \leq 1 + 5 = 6$.

□

This is not the best possible bound. This bound can be reduced to five rather easily by switching colors on chains of vertices. Of course, the famous Four Color Theorem guarantees that the best possible upper bound is four. While no simple proof of this theorem has been found in general, it can be proven rather easily for large classes of graphs. Lick and White observed [38] that using the same argument as in the Five Color Theorem, it can be shown that any planar 4-degenerate graph is 4-colorable.

Heawood proved in 1898 [33] that a maximal planar graph with order at least three is 3-chromatic if and only if every vertex has even degree. Even with this restriction, we cannot guarantee that such graphs are monocore. For order at least four, any such graph must be a 4-core. But such a graph may have a 5-core. Consider the graph whose 5-core is the graph of the Archimedean solid the snub cube. Add vertices in each of its six regions that are 4-cycles and join them to each of the vertices on the 4-cycle. The resulting graph is maximal planar, has all even degrees, and is not monocore.

A graph is outerplanar if it is planar and it has a plane embedding with all its vertices on a single region. A maximal outerplanar graph is maximal with respect to being outerplanar, so for order at least three it is a Hamiltonian cycle that has been triangulated. We can determine additional information on maximal outerplanar graphs. The following theorem is a strengthened restatement of an existing result (see [16] page 131).

Theorem 325. *Every maximal outerplanar graph with $n \geq 3$ is a 2-tree.*

Proof. Let G be maximal outerplanar, so its interior regions are triangles. If $n = 3$, then $G = K_3$, which is a 2-tree. Suppose $n \geq 4$. Take the dual graph G^* with vertex v corresponding to the exterior region and let $T = G^* - v$. Now T is a nontrivial tree (whose internal vertices have degree 3) since if T had a cycle it would have a vertex not on the outer region. Then T has at least two end-vertices. In G , these correspond to triangles with two edges on the exterior region, and hence a vertex of degree 2. Deleting one of these vertices produces a smaller maximal outerplanar graph, so the result holds by induction.

□

This formally justifies our assertion in the first section that maximal outerplanar graphs are 2-monocore.

Corollary 326. *If G is outerplanar, $\widehat{C}(G) \leq 2$, and $\chi(G) \leq 3$. If G is maximal outerplanar with order at least 3, both of these are equalities and G is 2-monocore.*

We can characterize the 2-trees that are maximal outerplanar.

Theorem 327. *The following are equivalent for a 2-tree T .*

1. *T is maximal outerplanar with $n \geq 3$.*
2. *T is Hamiltonian.*
3. *T is constructed using each K_2 as a root at most once.*

Proof. (1 \Rightarrow 2) If T is maximal outerplanar with $n \geq 3$, then the outside region forms a Hamiltonian cycle.

(2 \Rightarrow 3) Suppose T is constructed using some K_2 as a root more than once. Then deleting that K_2 produces at least three components since this is true when the vertices using this root are first added and any other vertex added cannot reduce the number of components of $T - K_2$. Thus T is not Hamiltonian.

(3 \Rightarrow 1) Suppose T is constructed using each K_2 as a root at most once. If $n = 3$, then $T = K_3$, which is maximal outerplanar, has no chords, and no edge has been used as a root. Assume that any maximal outerplanar graph with order n is constructed using each chord of the outer cycle as a root exactly once while each edge of the cycle has not been used as a root. Let T have order $n + 1$. Then T is constructed by adding a vertex v adjacent to some previously unused root of $T - v$, which is maximal outerplanar by assumption. Then the root is on the outer cycle, and v can be placed in the exterior region of $T - v$. The outer cycle of T thus contains the edges incident with v , the root for v is now a chord of this cycle, and T is maximal outerplanar.

□

We can also color graphs on other surfaces. In 1890, Heawood proved the following

upper bound for the chromatic number of graphs that embed on surface S_k . This proof follows [[16] p. 218].

Theorem 328. [Heawood Map Coloring Theorem Upper Bound] Let G be a graph that embeds on surface S_k , $k > 0$. Then

$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{1 + 48k}}{2} \right\rfloor.$$

Proof. Let G be a graph that embeds on surface S_k , $k > 0$, with maximum core H with order n and size m . Let $h = \frac{7 + \sqrt{1 + 48k}}{2}$, so that $1 + 48k = (2h - 7)^2$ and $h = 7 + \frac{12(k-1)}{h}$. Now $\chi(G) \leq 1 + \widehat{C}(H) \leq h$ if $n \leq h$, so suppose $n > h$. By Proposition 322,

$$\chi(G) \leq 1 + \widehat{C}(H) \leq 1 + 6 + \frac{12(k-1)}{n} \leq 7 + \frac{12(k-1)}{h} = h = \frac{7 + \sqrt{1 + 48k}}{2}.$$

□

Note that the final inequality does not hold for the plane ($k = 0$). Heawood's formula can also be shown to hold for nonorientable surfaces. It is much more difficult to show that it is an equality for every surface except the Klein bottle.

A graph is said to be uniquely colorable if every minimal coloring of it produces the same partition.

It is easily seen that in a uniquely colorable graph, the subgraph induced by the union of any two color classes must be connected, since otherwise the colors could be swapped on one of its components. We can easily describe a large class of uniquely colorable graphs.

Proposition 329. Let G be a graph with $k = \chi(G) = \omega(G)$, every vertex contained in a k -clique, and every pair of k -cliques connected by a chain of k -cliques which overlap on $(k-1)$ -cliques. Then G is uniquely colorable.

Proof. Any minimal coloring of G assigns exactly one vertex of a maximum clique to each color class. Then there is only one color choice for a vertex whose k -clique overlaps this one on $k - 1$ vertices. Since every vertex can be reached in this way, G is uniquely colorable.

□

Some specific graph classes satisfy these conditions.

Corollary 330. *Graphs that are k -trees, 3-colorable maximal planar graphs, and maximal outerplanar graphs are uniquely colorable.*

Proof. The k -trees satisfy the hypothesis, and maximal outerplanar graphs are 2-trees. Maximal planar graphs are triangulations, so if such a graph is 3-colorable, it satisfies the hypothesis.

□

Chartrand and Geller showed that the only uniquely 3-colorable outerplanar graphs are maximal outerplanar. [see Chartrand/Zhang p.228-9 [16]] We present a different proof of this result.

Proposition 331. *An outerplanar graph of order at least three is uniquely 3-colorable
 \iff it is maximal outerplanar.*

Proof. We have already observed the converse. If G is outerplanar but not maximal, it has an induced cycle C of length at least four. This cycle is not uniquely 3-colorable. No component of $G - C$ contains more than two consecutive vertices of C , so there is more than one distinct coloring of G .

□

We might wonder if all uniquely k -colorable graphs satisfy the hypothesis of Proposition 329. This is true when k is 1 or 2, but false in general. In fact, there is a graph that is not only uniquely 3-colorable but also triangle-free (see [31]).

Chartrand and Geller also proved the following.

Theorem 332. *Every uniquely 4-colorable planar graph is maximal planar.*

The proof of this theorem implies the following corollary.

Corollary 333. *In a uniquely 4-colorable planar graph, the subgraph induced by the union of any two color classes is a tree.*

Proof. The subgraph induced by any two color classes must be connected. The sum of the sizes of these subgraphs is at least $3n - 6$, so all of them have size as small as possible, that is one less than their order. Thus they are trees.

□

We already know a class of uniquely 4-colorable planar graphs, namely the 3-trees. A maximal planar graph with both a 3-shell and a 4-core is uniquely 4-colorable if and only if its 4-core is uniquely 4-colorable. We offer the following conjecture.

Conjecture 334. *A planar graph is uniquely 4-colorable if and only if it is a 3-tree.*

4.5.5 Integer Embeddings

Given a planar graph, we may seek embeddings with other properties. Fary [1948] [24] proved the following.

Theorem 335. [Fary's Theorem] Every planar graph has a plane embedding in which its edges are straight lines.

This can be proven by induction on maximal planar graphs using the observations that every planar graph is 5-degenerate and that each 3, 4, or 5-gon contains an interior point that 'sees' each of its vertices.

We can seek straight-line embeddings with further properties.

Definition 336. An integer embedding of a graph G is a straight-line plane embedding so that the edges of G have integer lengths. An integer graph is a graph with an integer embedding.

Kemnitz and Harborth [2001] [36] offered the following conjecture.

Conjecture 337. All planar graphs are integer graphs.

Note that for finite graphs, having an integer embedding is equivalent to having an embedding with rational edge lengths, since such lengths could be multiplied by their least common denominator.

Lemma 338. If G is an integer graph, it has an integer embedding where each edge has length at least $l \geq 1$.

Proof. Given an integer embedding of a graph, scaling it by an integer produces another integer embedding. □

It is not surprising that every tree is an integer graph. We prove a somewhat stronger result.

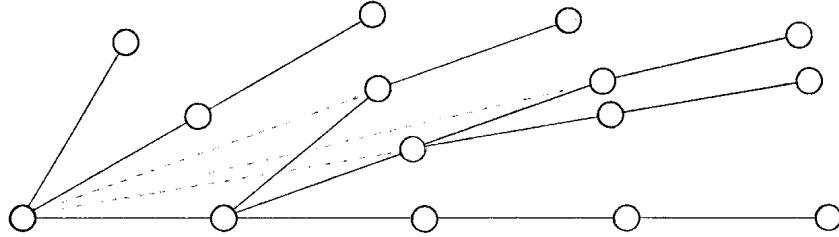


Figure 36: An integer embedding of a tree with every edge having length one.

Proposition 339. *Every tree has an integer embedding with every edge having length one.*

Proof. Let T be a tree with maximum degree $\Delta \geq 2$. Choose an angle α , $0 < \alpha < \frac{\pi}{2}$, and let $\beta = \frac{\alpha}{\Delta}$. Choose a vertex v as the root of the tree and suppose it is at the origin. Place the neighbors of v one unit from it on the lines $y = r \cdot \beta x$, where r is an integer satisfying $0 \leq r \leq d(v) - 1$. In general, suppose a vertex u is at (graph) distance d from v and has neighbor w nearer to v on line $y = q \cdot \frac{\alpha}{\Delta^{d-1}} x$, $0 \leq q \leq \Delta^{d-1} - 1$. Then place u on the point farther from v which is on the line $y = \left(q \cdot \frac{\alpha}{\Delta^{d-1}} + r \cdot \frac{\alpha}{\Delta^d}\right) x$, $0 \leq r \leq d(w) - 2$, and at distance one from w .

We show that such an intersection point must exist. Let w be at geometric distance $l \leq d$ from v . If an arc of length at most one exists between the two lines in question, then so does a line segment of length at most one. The length of such an arc is $s = l \cdot \theta \leq d \frac{\alpha}{\Delta^d} < \frac{\pi d}{2^{d+1}} < 1$, so the intersection must exist. Thus such an embedding exists.

□

Since the angle α in the previous proof can be arbitrarily small, we have the following corollary.

Corollary 340. *Let T be a tree with diameter d . Then T has an integer embedding with every side having length one inside a sector with radius d and any angle $\alpha > 0$.*

Thus we have the following.

Proposition 341. *G is an integer graph \iff its 2-core is an integer graph.*

Proof. Clearly if G has an integer embedding, any subgraph has such an embedding. The 1-shell of G is a forest, and each tree can be rooted in a sector with arbitrarily small angle. If necessary, the 2-core of G can be scaled up to allow the trees to be appended to it.

□

To go further, we need the following lemma which is reported in [29]. Its proof uses advanced techniques in number theory and will not be included. We will state the lemma somewhat informally.

Lemma 342. *If for any placement ϕ of the vertices of a graph $G - v$ in the plane, ϕ can be 'nudged' to a rational embedding, and v is made adjacent to three vertices of G , two adjacent, then G has a rational embedding.*

The following theorem follows the approach of [Geleen/Guo/McKinnon 2008] [29], extending their result.

Theorem 343. *Let G be a planar graph that can be constructed by successively adding vertices so that when added, each vertex v satisfies one of the following.*

1. $d(v) \leq 2$
2. $d(v) = 3$ and v has two adjacent neighbors
3. $d(v) = 3$ and v has a neighbor x with $d(x) \leq 2$

Then G has an integer embedding.

Proof. Since G is planar, it has a straight-line plane embedding. Construct G using a construction sequence satisfying the hypothesis. If $d(v) = 1$, it can be added as in the previous result. If $d(v) = 2$, place v at the intersection of two circles with rational radii, 'close' to its original location in the original embedding. In the second case, the lemma guarantees that v can be placed arbitrarily close to its original location with rational side lengths. In the third case, add an edge xy , where y is a neighbor of v and 'nudge' x so that xy is rational. (Note that xy is allowed to cross other edges.) Add v using the lemma, and delete xy . Finally, scale the graph so that it has integer side lengths.

□

Corollary 344. *If given a plane embedding of G , its 3-core can be 'nudged' to a rational embedding, then G has an integer embedding.*

Note the difference between this corollary and Proposition 341. In the proposition, we start with any integer embedding and append the 1-shell to it, while in this case, we start with a straight-line embedding and manipulate it to obtain an integer embedding. A vertex of degree two may not be able to be added for a given embedding. It is unknown whether there is a graph such that every rational embedding forbids such a vertex from being added.

We can describe several graph classes that have integer embeddings.

Corollary 345. *If G is 2-degenerate, 3-collapsible, a planar 3-tree, or a planar chordal graph, it is an integer graph.*

Proof. If G is 2-degenerate, every vertex of a construction sequence satisfies the first condition. If G is 3-collapsible, all but the last vertex of a construction sequence satisfy the first condition and the last satisfies the third condition. If G is a 3-tree, it is maximal planar so every vertex of a construction sequence beyond the initial

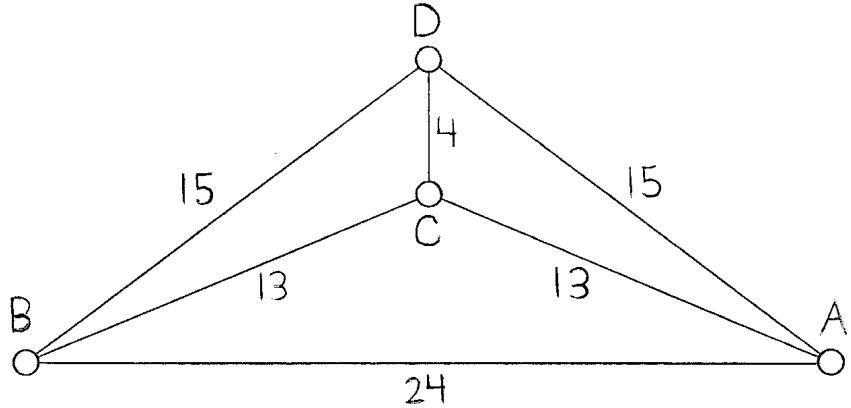


Figure 37: An integer embedding of K_4 .

K_4 satisfies the second condition. If G is a planar chordal graph, it has a simplicial elimination ordering, so it has a construction sequence where every vertex satisfies one of the three conditions.

□

Note that this theorem is purely existential. It does not provide a way to find a valid location for the vertex being added. Thus there is still a problem of determining integer embeddings for specific graphs. Pythagorean triples prove helpful in some cases.

For example, consider the smallest 3-core, K_4 . Place points $A = (12, 0)$, $B = (-12, 0)$, $C = (0, 5)$, and $D = (0, 9)$ in the plane. Then this embedding has side lengths $AB = 24$, $AC = 13$, $AD = 15$, $BC = 13$, $BD = 15$, and $CD = 4$, so it is an integer embedding.

Pythagorean triples can also be used to show that classes of 3-cores such as wheels are integer graphs.

4.6 Domination

4.6.1 Domination

A set of vertices of a graph is a dominating set if each vertex of G is either in the set or adjacent to a vertex in the set. The domination number of a graph $\gamma(G)$ is minimum size of a dominating set. See [32] for background. (Note that the notation $\gamma(G)$ was also used for genus. In this section, it refers only to domination number.)

Note that any isolated vertex must be contained in a dominating set. Hence we have the following result.

Proposition 346. *Let n_0 be the number of isolated vertices of a graph. Then $\gamma(G) = n_0 + \gamma(C_1(G))$.*

The domination number of a graph sums over its components, so we will typically assume in this section that graphs are connected. The following theorem provides bounds on the domination number for all k-cores.

Theorem 347. [13] [14] *Let G be a k -core. Then $\gamma(G) \leq \left[1 - k \left(\frac{1}{k+1}\right)^{1+\frac{1}{k}}\right]n$.*

This is slightly better than the upper bound $\gamma(G) \leq \frac{1+\ln(k+1)}{k+1}n$ (see [1]). Using L'Hôpital's rule, it can be shown that the fractions of n in the theorem go to 0 as $k \rightarrow \infty$. The bounds provided by the theorem are not sharp for positive small k . For $k = 1$, it gives $\gamma(G) \leq .75n$. Any connected graph has a spanning tree and either partite set is a dominating set. This implies the following result.

Proposition 348. *Let G be a 1-core. Then $\gamma(G) \leq \frac{1}{2}n$.*

The extremal graphs for this bound have been characterized by [48]. The corona of a graph G is formed by adding a pendant edge adjacent to each vertex of G . We present a different proof.

Theorem 349. *Let G be a connected 1-core. Then $\gamma(G) = \frac{1}{2}n$ if and only if $G = C_4$ or G is the corona of a graph.*

Proof. It is easily seen that these graphs have the given domination number, since in a corona either a vertex or its copy must be in the dominating set. Let $\gamma(G) = \frac{1}{2}n$, so n is even and the result is obvious for $n = 2$. Since domination number is not reduced by deleting edges, consider a spanning tree T attaining the bound. For $n > 2$, T has a minimum dominating set containing no leaves since any leaf could be replaced by its neighbor. If any leaves of T have a common neighbor, then deleting one produces a contradiction. If T has an internal vertex not adjacent to a leaf then deleting leaves and their neighbors produces a forest with every isolated vertex and leaf already dominated, so the bound is not attained. Thus T is the corona of some other tree. Adding an edge between a leaf and non-leaf of T produces a graph with another spanning tree that is not a corona, and so does adding an edge between two leaves unless $T = P_4$ and $G = C_4$.

□

For 2-cores, the following sharp upper bound was obtained by McQuaig and Shepherd [42]. They also characterized the extremal graphs.

Theorem 350. *Let G be a connected 2-core. With the exception of seven graphs, $\gamma(G) \leq \frac{2}{5}n$. The seven graphs are C_4 , $C_4 \cup_{v} C_4$, C_7 , and four graphs formed by adding chords to C_7 .*

It has been conjectured (see [32]) that for k -cores, $\gamma(G) \leq \frac{k}{3k-1}n$. Reed [51] proved that for 3-cores, $\gamma(G) \leq \frac{3}{8}n$. Sohn and Xudong [55] proved that for 4-cores, $\gamma(G) \leq \frac{4}{11}n$, and claim that they have proved it for 5 and 6. For $k \geq 7$, it is worse than Theorem 347. Presuming that Sohn and Xudong are correct, we have the following corollary.

Corollary 351. *Let n_k be the number of vertices in the (non-proper) k -shell of G and suppose the 2-shell of G has no component being one of the exceptional graphs in Theorem 350. Then*

$$\gamma(G) \leq n_0 + \sum_{k=1}^6 \frac{k}{3k-1} n_k + \sum_{k=7}^{\infty} \left[1 - k \left(\frac{1}{k+1} \right)^{1+\frac{1}{k}} \right] n_k.$$

How good a bound this is will depend on the graph. It depends on a construction that dominates every boundary vertex at least twice, so it will tend to be worse when the boundaries are larger.

We may improve this bound with more information on the structure of the shells. Since the 2-core and 1-shell decompose a nontrivial connected graph, we consider domination of trees. There is a straightforward algorithm to determine the domination number of a tree. We find the following definition convenient.

Definition 352. The depth of a vertex v in a tree T is the minimum distance between v and an end-vertex of T .

It is immediate that for a vertex v of a tree, $0 \leq \text{depth}(v) \leq e(v)$, the eccentricity of v . Further, $\text{depth}(v) = 0$ exactly for end-vertices, $\text{depth}(v) = 1$ exactly for neighbors of end-vertices, and $\text{depth}(v) = e(v)$ exactly when v is equidistant from each end-vertex. When this final condition holds, v is the unique central vertex of the tree,

but note that vertices with high eccentricity can have low depth. For example, in the corona of any tree, the maximum depth is 1.

It is also immediate that for any vertex v of a tree, all its neighbors have depth within one of v .

Proposition 353. *Let v be a vertex of a tree. Then $\text{depth}(v) + e(v) \leq \text{diam}(T)$.*

Proof. Let u be an end-vertex of T with minimum distance to v and P be a path of maximum length through u and v ending at end-vertex w . The length of the $v-w$ subpath is at least $e(v)$. Then $\text{depth}(v) + e(v) \leq \text{length}(P) \leq \text{diam}(T)$.

□

This bound is sharp for all vertices of paths, stars, and double stars.

Corollary 354. *We have the following chains of inequalities.*

$$0 \leq \text{depth}(v) \leq \frac{1}{2}(\text{depth}(v) + e(v)) \leq \frac{1}{2}\text{diam}(T) \leq \text{radius}(T)$$

$$0 \leq \text{depth}(v) \leq \max \text{depth}(T) \leq \frac{1}{2}\text{diam}(T) \leq \text{radius}(T)$$

Conjecture 355. *For all vertices of any tree T , $\max \text{depth}(T) \leq \frac{1}{2}(\text{depth}(v) + e(v))$.*

The algorithm for domination of a tree is based on the observation that for any tree of at least three vertices, no end-vertex need be contained in a minimal dominating set of a tree, because if it is, it could be replaced by its neighbor. The algorithm allows for rooted trees, in which case we do not consider the root to be an end-vertex.

Algorithm 356. *[Domination of a Rooted Tree]*

Input T with root v .

If T is K_1 or K_2 , assign a vertex to the dominating set D and stop.

Else

Label all depth 0 and 2 vertices as dominated.

Add all depth 1 vertices to D .

Let T' be the subtree formed by deleting all depth 0 and 1 vertices and iteratively deleting all end-vertices labeled as dominated.

If T' is not null, run the algorithm on T' .

Proposition 357. *This algorithm produces a minimum dominating set for the tree T .*

Proof. No end-vertex need be in D provided that it is adjacent to a depth 1 vertex, which may be adjacent to more vertices. No depth 2 vertex which is an end-vertex when depth 0 and 1 vertices have been deleted need be in D . Thus the algorithm makes the ideal choice at each step, and it clearly produces a dominating set.

□

The proof implies that each tree with at least three vertices has a minimum dominating set containing all depth 1 vertices. One might think that we could produce a minimum dominating set by adding all vertices of depth $3r + 1$, as well as the maximum depth if it is a multiple of three. But while this will produce a dominating set, it need not be minimum.

Combining the 2-core and 1-shell produces a good estimate of the domination number of a graph.

Corollary 358. *Let G be a graph with a 1-shell composed of rooted trees T_i with domination numbers $\gamma(T_i)$. Let r' be the number of roots or vertices adjacent to roots*

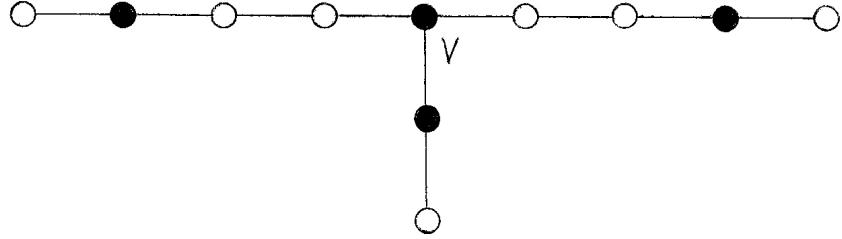


Figure 38: In this tree T , $\gamma(T) = 4$ and any minimum dominating set contains v , which has depth 2.

contained in dominating sets D_i produced by the algorithm. Then

$$\gamma(C_2(G)) + \sum \gamma(T_i) - r' \leq \gamma(G) \leq \gamma(C_2(G)) + \sum \gamma(T_i)$$

In particular, if no component of the 2-core is one of the seven graphs in Theorem 350, then $\gamma(G) \leq \frac{2}{5}|C_2(G)| + \sum \gamma(T_i)$.

Proof. The algorithm optimally dominates the 1-shell, possibly overlapping the 2-core, producing an overestimate of the domination number. Removing the vertices in or dominating part of the 2-core produces an underestimate. The final bound follows from Theorem 350. □

Note that in some cases, the process used to produce minimum dominating sets can be extended to the 2-shell. If the root must be in the dominating set, we may be able to determine other vertices in the set optimally. The same holds if neighboring roots have neighbors in the minimum dominating sets.

This points out an interesting contrast between vertex coloring and domination. In both cases, we have employed the decomposition of a graph into its 1-shell and 2-core. But when coloring, the trees of the 1-shell are simply annoying appendages to be lopped off toward determining the chromatic number. In contrast, the trees

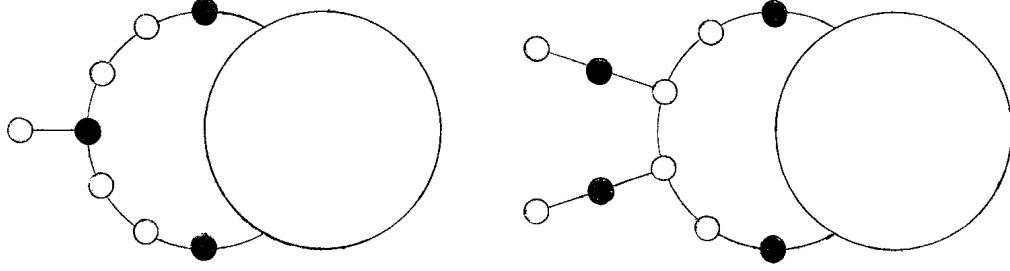


Figure 39: In these cases, we may optimally extend the minimum dominating set into the 2-shell.

provide a cornerstone upon which to build the foundation of an optimal dominating set, greatly reducing the number of possible dominating sets that need to be checked.

4.6.2 Total Domination

The techniques employed in this section can likely be applied to yield similar results for other forms of domination, such as total domination.

In a total dominating set, each vertex is adjacent to some vertex in the set. The total domination number of a graph $\gamma_t(G)$ is the minimum size of a total dominating set. The definition immediately implies that the total dominating set is a dominating set with no isolated vertices. The total domination number is defined exactly for graphs without isolated vertices.

This basic upper bound is due to Cockayne, Dawes, and Hedetniemi [18]. We present a much shorter proof.

Theorem 359. *Let G be a connected graph with $n \geq 3$. Then $\gamma_t(G) \leq \frac{2}{3}n$.*

Proof. Let T be a spanning tree of G and v be a leaf of T . Label each vertex of T with its distance from v mod 3. This produces three sets that partition the vertices of G . Then some set contains at least one third of the vertices of G , and the union S of the other two contains at most two thirds of the vertices. Each internal vertex of T is

adjacent to a vertex in each of the other sets. If S contains an isolated leaf, replace it with its neighbor. Then S is a total dominating set.

□

The graphs for which $\gamma_t(G) = \lfloor \frac{2}{3}n \rfloor$ have been characterized by [12]. We present a short proof for when $\gamma_t(G) = \frac{2}{3}n$. A brush is a graph formed by starting with some graph G and identifying a leaf of a copy of P_3 with each vertex of G .

Theorem 360. *Let G be a connected graph with $n \geq 3$. Then $\gamma_t(G) = \frac{2}{3}n$ exactly when G is C_3 , C_6 , or a brush.*

Proof. It is easily seen that the stated graphs are extremal, since in a brush each depth 1 vertex and a neighbor must be in the total dominating set. Let $\gamma_t(G) = \frac{2}{3}n$ so n is a multiple of 3 and the result is obvious for $n = 3$. Let T be a spanning tree of G . For $n \geq 6$, T has a minimum total dominating set containing no leaves since any leaf could be replaced by a corresponding depth 2 vertex. If any leaves of T have a common neighbor, then deleting one produces a contradiction.

If T has leaves v_1 and v_2 with neighbors u_1 and u_2 with common neighbor w , then u_1 , u_2 , and w are contained in a minimum total dominating set. Then deleting v_1 , u_1 , and v_2 from T and u_2 from the total dominating set produces a contradiction.

Suppose that deleting all depth 1 vertices of degree 2 and their neighbors produces a forest F . Then any isolated vertex and at least one leaf of each nontrivial component of F are already dominated, so T cannot achieve the upper bound, and so F does not exist. Thus T is a brush. Adding an edge between depth 2 vertices does not change this. But adding any other edge produces a spanning tree that is not a brush unless $T = P_6$ and $G = C_6$.

□

Upper bounds for total domination number of k -cores are summarized in Table 8 from [34].

k	Bound	Conditions
1	$\frac{2}{3}n$	G connected, $n \geq 3$
2	$\frac{4}{7}n$	G connected, not C_3, C_5, C_6, C_{10}
3	$\frac{1}{2}n$	
4	$\frac{3}{7}n$	
large	$\frac{1+\ln k}{k}n$	

Table 8: Upper bounds for total domination number.

Hence it is possible to construct results analogous to Corollaries 351 and 358 and Algorithm 356.

4.7 The Reconstruction Conjecture

The Reconstruction Conjecture states that for each graph G with order at least three, G can be determined up to isomorphism from its vertex-deleted subgraphs, the n subgraphs of G formed by deleting one vertex from G .

This is one of the most famous unsolved problems in graph theory. Toward proving this conjecture, one can prove that a particular property of a graph is recognizable, that is, whether G has its property can be determined from its vertex deleted subgraphs. One can also prove that a particular class of graphs is reconstructible, that all graphs in that class can be reconstructed.

It is straightforward to show that the degree sequence of a graph is recognizable. Several properties related to cores are recognizable.

Theorem 361. *The property of having a k -core is recognizable. In particular, G has a k -core \iff either for some vertex v $G - v$ has a k -core, or $\delta(G) \geq k$.*

Proof. (\Rightarrow) Let G have a k -core. If G is not the entire k -core, then $G - v_i$ has a k -core for some v_i . If G is the entire k -core, then $\delta(G) \geq k$, which is known to be recognizable.

(\Leftarrow) Certainly if either of these conditions hold, G has a k -core. □

Corollary 362. *The order and size of the k -core are recognizable, and if G is not k -monocore, the structure of its k -core can be exactly determined..*

Proof. We may assume G has a k -core. Let G have order n and size m . If $\delta(G) \geq k$, then G is a k -core with order n and size m . If not, then there is a vertex v of G not in the k -core of G . Then $C_k(G) \subseteq G - v$. Thus the k -core of G is the largest k -core of its vertex deleted subgraphs.

□

Corollary 363. *The order of the proper k -shell and size of the k -shell of G are recognizable.*

Conjecture 364. *The orders of the k -shell and k -boundary are recognizable.*

Corollary 365. *The property of a graph being k -monocore is recognizable.*

Proof. The properties of having a k -core and not having a $k + 1$ -core are recognizable.

□

Corollary 366. *The property of a graph being maximal k -degenerate is recognizable.*

Proof. A graph G is maximal k -degenerate if and only if it is $k + 1$ -core-free and has size $k \cdot n - \binom{k+1}{2}$. Both of these properties are recognizable. Hence so is being maximal k -degenerate.

□

Corollary 367. *The property of being k -collapsible is recognizable.*

Proof. Let G be monocore. Then G is collapsible $\iff G - v$ has no k -core for all vertices v of G .

□

It is known that regular graphs are reconstructible. In fact, we can show that a larger class of graphs containing connected regular graphs is reconstructible.

Theorem 368. *All k -core-critical graphs are reconstructible.*

Proof. Let G be k -core-critical. Then it contains a vertex v with $d(v) = l \geq k$ adjacent only to vertices of degree k . Then $G - v$ has l vertices of degree $k - 1$, and the degree of v can be recognized. Then v must be adjacent to these vertices. Adding v and making it adjacent to them reconstructs G .

□

It is unknown whether all maximal k -collapsible graphs are recognizable.

It is known that disconnected graphs are reconstructible. It is also known that trees are reconstructible. We have seen that the 1-shell of a connected graph that is not a tree is a union of trees rooted on the 2-core, and that the order of the 1-shell of a graph is recognizable. Thus the following modest conjecture seems plausible.

Conjecture 369. *If G has a 1-shell, then it is reconstructible.*

5 Conclusion

This dissertation began at the end of summer 2008. My advisor, Dr. Allen Schwenk, suggested three problems to me as potential research projects, of which studying k -cores was the first. I read the Wikipedia article that he gave to me and looked up some references. I quickly whipped up some mostly trivial results, some of which appear in the introduction, and gave two talks on the subject that fall. Over Christmas break I worked out some small values of Ramsey core numbers.

The project kept expanding from there. I went wherever my interests took me, which often meant that I was working on several disparate problems at a time. A single sentence of my early talk or a single seemingly innocuous theorem in a textbook would end up turning into a 20-page section. The project ended up going places that I never expected.

The topic of k -cores started with single paper in 1983 by Steven B. Seidman entitled *Network structure and minimum degree*. [53] He proved a few results that I have mentioned in the introduction. With one major exception, the paper didn't seem to have much of an impact in pure graph theory. A few other papers cite it, but its terminology did not become common.

The major exception mentioned above is random graph theory. The k -core was defined inconspicuously in the proof of a theorem toward the end of a 1984 paper entitled *The evolution of sparse graphs* by Bella Bollobas. [8] In 1991 it was explored on its own in a paper entitled *Size and connectivity of the k -core of a random graph* by Tomasz Łuczak [39] and in a number of subsequent papers. I have not examined this area closely as I believe that I need additional background before I can contribute to it.

While we have explored applications of cores within graph theory, they also have applications outside of mathematics. Seidman briefly explores social networks in his paper. Cores have applications in computer science to network visualization [3] [28].

They also have applications in bioinformatics [2] [4] [63].

Since cores have been little studied in pure graph theory, a major goal of this dissertation has been to develop a theory of k -cores. Sometimes this means incorporating existing results into this theory. In some cases I found an existing theorem and rewrote or reexpressed it in terms of cores. Sometimes I also rewrote or simplified the proof using cores. In other cases, I unknowingly rediscovered results that had been found earlier by others. (This is the case for most of the section on maximal k -degenerate graphs.)

But plenty of other material in this theory of k -cores is, to the best that I can determine, my original discovery. Incorporated into the theory of k -cores, the original material sheds new light on the old, previously disparate, results.

Throughout this theory, some persistent themes emerge again and again.

1. The duality between k -cores and k -degenerate graphs.

This is expressed in the fundamental result on cores, the k -core algorithm, which states that the k -core of a graph can be determined by iteratively deleting vertices with degree less than k . It implies that the maximum core number and degeneracy of a graph are equal. It also yields the equivalence of k -degenerate and $k + 1$ -core-free graphs. The solution to the problem of Ramsey core numbers follows from the fact that if a factor of a decomposition is not a k -core, then it is $k - 1$ -degenerate.

2. Determining the core structure of graphs.

In the introduction, we determined that a number of common graph classes are monocore. Along the way we encountered other classes of monocore graphs, including maximal k -degenerate graphs and minimally k -connected graphs. For some graph classes, including planar graphs and graphs with genus k , we can only bound which cores can appear. For other graph classes, such as color-critical graphs and self-dual graphs, determining information on their core structure was more difficult. We also characterized the core structure of graphs formed by using graph operations including the Cartesian product, the join, and the line graph.

3. The quantity $f(k, n) = k \cdot n - \binom{k+1}{2}$ is the size of a maximal k -degenerate graph of order n .

This quantity occurs in a number of contexts. It is the size of k -trees, a special class of maximal k -degenerate graphs. In some situations, it occurs as the size of the graph $K_k + \overline{K}_{n-k}$. A number of variations occur, including $f(k-1, n) = (k-1) \cdot n - \binom{k}{2}$, the maximum size of a k -core-free graph. There is $f(k-1, n) + 1 = (k-1) \cdot n - \binom{k}{2} + 1$, the minimum size of a graph guaranteeing a k -core, and also the maximum size of a k -collapsible graph. For specific values of k , $f(1, n) = n - 1$ is the size of a tree, $f(2, n) = 2n - 3$ is the size of a maximal outerplanar graph, a type of 2-tree, and $f(3, n) = 3n - 6$ is the size of a maximal planar graph, which can be a 3-tree but does not have to be.

4. Construction sequences and the core number bound.

A construction sequence provides an ordering of the vertices of a graph that is often useful. This is particularly so for proper vertex coloring. Greedily coloring a graph using a construction sequence implies the core number bound $\chi(G) \leq 1 + \widehat{C}(G)$. This simple but powerful bound is superior to several other common upper bounds for the chromatic number. It also can be computed efficiently (polynomial-time), while chromatic number is NP-complete.

It has a number of important implications. It helps to simplify the proof of Brooks' Theorem. It fits neatly into the coloring chain. It is essential in the proof of the order/size bound, the Nordhaus-Gaddum Theorem, and the Heawood Map Coloring Theorem upper bound. Bounds using construction sequences analogous to the core number bound exist for list coloring, $L(2, 1)$ coloring, 2-tone coloring, arboricity, point partition numbers, and thickness.

5. A graph decomposes into its 2-core and 1-shell, which is a forest with no trivial components.

This fact is useful because much is known about trees that is not known about graphs in general. In proper vertex coloring and list coloring, the 1-shell can be

easily pruned off a graph. It is easy to account for in the chromatic polynomial. In edge coloring and 2-tone coloring, it can easily be accounted for in determining the relevant chromatic number. These ideas generalize to larger shells for arboricity and point-partition numbers. The 1-shell is also easily pruned off for the parameters of planarity and integer embeddings. In contrast, the 1-shell provides a valuable approach to finding an optimal dominating set of a graph.

While this dissertation is in the neighborhood of 200 pages, in some ways I feel like I have just scratched the surface. In some important areas, only a few basic results are known so far. There are many conjectures and unsolved problems scattered throughout this dissertation that I would like to tackle in the future.

There are also several ways in which cores can be generalized. One generalization is to define analogous concepts for structures related to graphs, including multigraphs and digraphs. Another generalization is to consider graph properties other than minimum degree. For graph property p , we could define a (p, k) -core of G to be a maximal subgraph of G with $p(G) \geq k$. Unlike for minimum degree, such a subgraph may not be unique. Then (p, k) -cores could be studied both in relation to standard graph properties and in relation to (δ, k) -cores, i. e. just k -cores. One example of this that we considered briefly are edge cores, which occur naturally when considering line graphs.

I have thoroughly enjoyed researching and writing this dissertation. I studied graph theory intensely for the past two years and I know far more about it now than when I started. Even the results that gave me the most grief at the time, including Theorem 92, Theorem 144, and the still unproven Plesnik's Conjecture were well worth the effort. It is a little sad to see this process end, but I am reassured by the conviction that there is plenty more to be discovered about the cores of graphs.

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