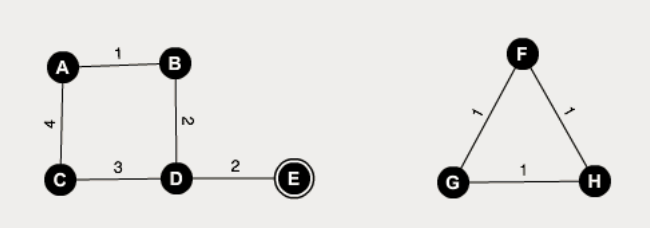
When is the minimum spanning tree for a graph not unique

Given a weighted, undirected graph G: Which **conditions** must hold true so that there are **multiple** minimum spanning trees for G?

I know that the MST is unique when all of the weights are distinct, but you can't reverse this statement. If there are muliple edges with the same weight in the graph, there *may be* multiple MSTs but there might also be just one:

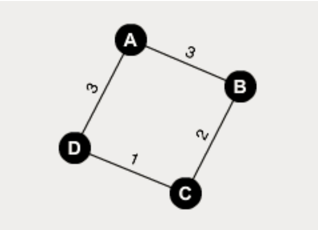


In this example, the graph on the left has a unique MST but the right one does not.

The closest I could get to finding conditions for non-uniqueness of the MST was this:

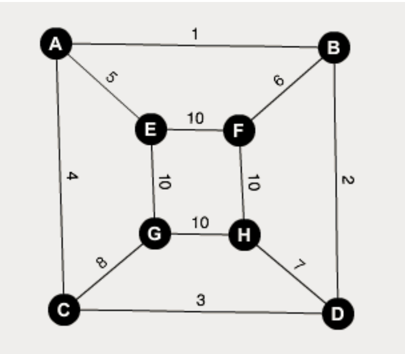
Consider all of the *chordless* cycles (cycles that don't contain other cycles) in the graph G. If in any of these cycles the maximum weighted edge exists multiple times, then the graph does not have a unique minimum spanning tree.

My idea was that for a cycle like this

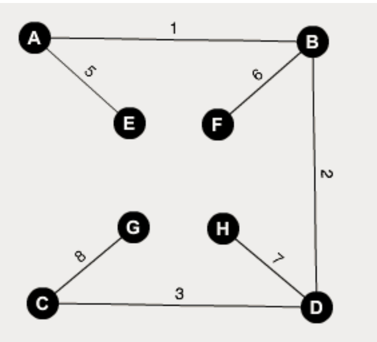


with n vertices, you can leave out exactly one of the edges and still have all of the vertices be connected. Therefore, you have multiple choices to remove the edge with the highest weight to get a MST, so the MST is not unique.

However, I then came up with this example:



You can see that this graph does have a cycle that fits my condition: **(E,F,G,H)** but as far as I can see, the minimum spanning tree is unique:



So it seems like my condition isn't correct (or maybe just not completely correct). I'd greaty appreciate any help on finding the necessary and sufficient conditions for the non uniqueness of the minimum spanning tree.

graphs graph-theory weighted-graphs spanning-trees minimum-spanning-tree

edited Jan 27 at 13:01 asked Jul 10 '16 at 9:26

Keiwan
207 1 9

1 Your smallest cycles are known as *chordless cycles* (more or less). – Yuval Filmus Jul 10 '16 at 11:22

1 Answer

~~in the first picture: the right graph has a unique MST, by taking edges (F, H) and (F, G) with total weight of 2.~~

Given a graph $G = (V, E)$ and let $M = (V, F)$ be a minimum spanning tree (MST) in G.

If there exists an edge $e = \{v, w\} \in E \setminus F$ with weight $w(e) = m$ such that adding e to our MST yields a cycle C, and let m also be the lowest edge-weight from $F \cap C$, then we can create a second MST by swapping an edge from $F \cap C$ with edge-weight m with e. Thus we do not have uniqueness.

edited Jul 10 '16 at 12:21 answered Jul 10 '16 at 12:10
HueHang
352 1 10

You're right, I corrected that graph in the question now. Do you know if this is the most general condition so that the MST is not unique? Or can it also somehow be determined without the need to first find a MST? – Keiwan Jul 10 '16 at 12:21

1 @Keiwan I believe that if you take into account this question then the condition outlined in this answeris also a *necessary* condition for having multiple MSTs. In other words: a graph G has multiple MSTs if and only if the construction outlined by HueHang can be carried out. – Bakuriu Jul 10 '16 at 16:45

m needn't be the lowest edge weight from F∩C. In fact, it can only be the highest edge weight, otherwise M would not have been minimal in the first place. Suppose there were an edge e' with w(e') = m' > m = w(e) in F∩C. Then swapping e for e' would leave a spanning tree with total weight less than M's, contradicting the minimality of M. – Chad May 4 at 3:21