Minimum Spanning Tree (MST)

Hengfeng Wei

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Cut Property

$$G = (V, E, w)$$

Cut Property (I)

X : A part of some MST T of G

$(S, V \setminus S)$: A *cut* such that X does *not* cross $(S, V \setminus S)$ Âŋ

e : *A* lightest edge across $(S, V \setminus S)$

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Correctness of Prim's and Kruskal's algorithms.

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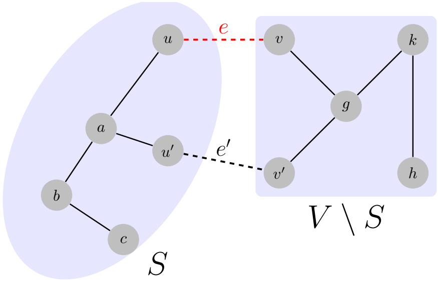

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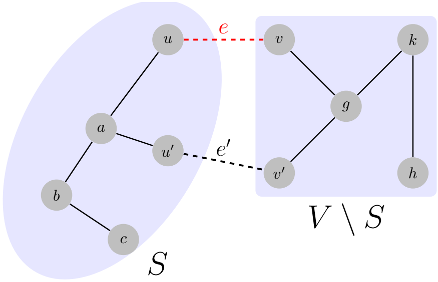

$$T' = \underbrace{T + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

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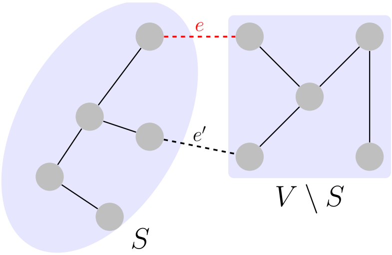

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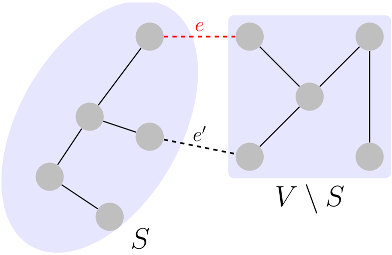

"a" $\rightarrow$ "the" $\Longrightarrow$ "$\exists$" $\rightarrow$ "$\forall$"

Application of Cut Property [Problem: 10.15 (4)]

$e \in G$ is the unique lightest edge $\Rightarrow e \in \forall$ MST

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$$\Big(S = \{u\}, V \setminus S\Big)$$

Wrong Divide&Conquer Algorithm for MST [Problem: 10.21]

$$(V_1, V_2) : ||V_1| - |V_2|| \le 1$$

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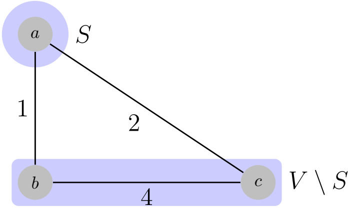

Cycle Property

Cycle property [Problem: $10.19(b)$]

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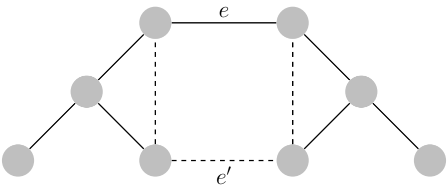

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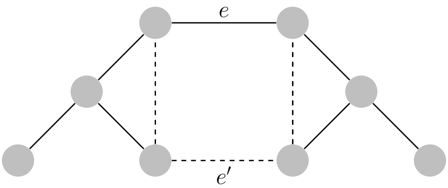

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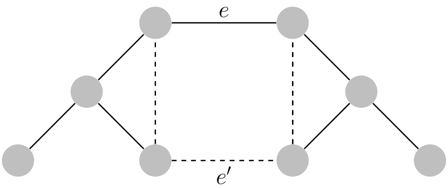

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"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.

Application of Cycle Property [Problem: $10.15\ (1)$]

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$$\overset{?}{\Longrightarrow}$$

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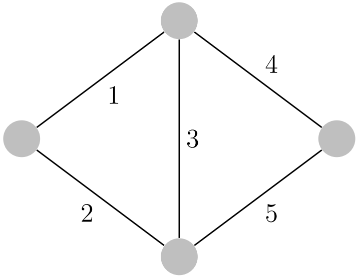

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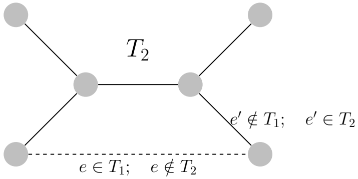

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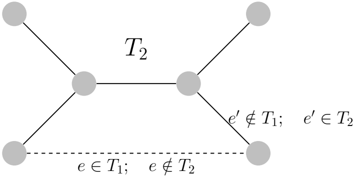

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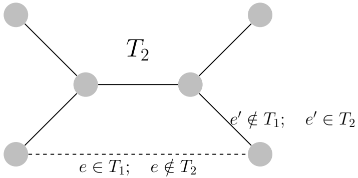

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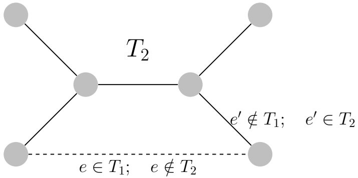

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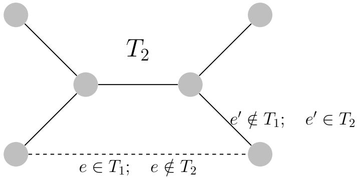

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Condition for Uniqueness of MST [Problem: 10.18 (2)]

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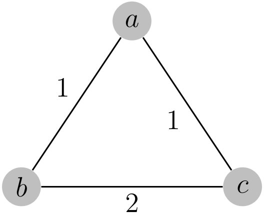

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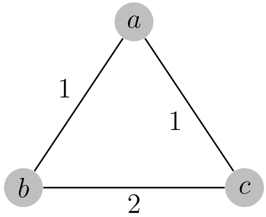

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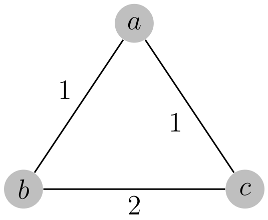

Theorem

Minimum-weight edge across any cut is unique $\Longrightarrow$ *Unique MST.*

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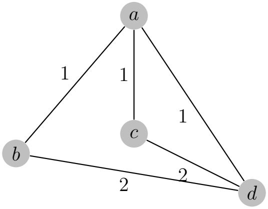

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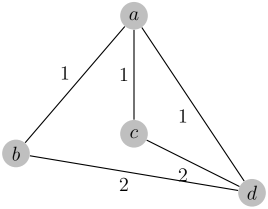

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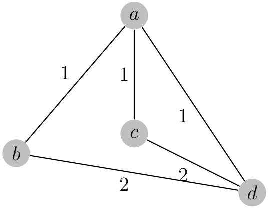

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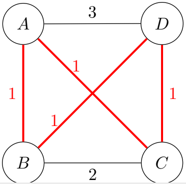

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Proof.

Cut property and Cycle property.

□

Variants of MST

Adding a Vertex v to MST T [Problem: 10.7]

$$G' = (V', E') : V' = V + \{v\}, E' = E + E_v$$

To find an MST T' of G'.

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"On Finding and Updating Spanning Tress and Shortest Paths", 1975
"Algorithms for Updating Minimum Spanning Trees", 1978

Feedback Edge Set (FES): [Problem: 10.8]

$$\text{FES} \subseteq E : G' = (V, E \setminus \text{FES}) \text{ is acyclic}$$

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$$\text{FES} \iff G \setminus \text{Max-ST}$$

MST with Specified Leaves: [Problem: 10.11]

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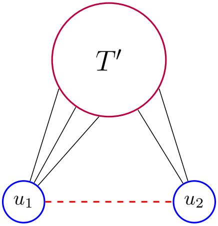

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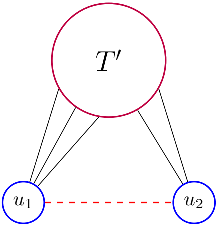

MST T' of $G' = G \setminus U$

Attach $\forall u \in U$ to T' (with lightest edge)

MST with Specified Edges: [Problem: 10.13]

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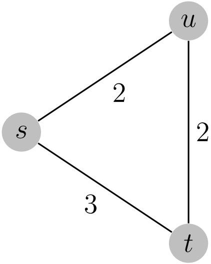
To find an MST with S as edges.

$G \to G'$: contract each component of S to a vertex

MST *v.s.* Shortest Path

MST vs. Shortest Paths [Problem: 10.15 (6)]

✗The shortest path between two nodes is necessarily part of some MST.



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$E' \subseteq \forall$ sssp trees from s

$\forall$ MST T of $G : T \cap E' \neq \emptyset$


THANK YOU