Decompositions of Graphs

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June 12, 2018

John Hopcroft

Robert Tarjan

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Robert Tarjan

"For fundamental achievements in the design and analysis of algorithms and data structures."

— *Turing Award, 1986*

SIAM J. Comput.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

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"DFS is a powerful technique with many applications."

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Power of DFS:

Graph Traversal $\Longrightarrow$ Graph Decomposition

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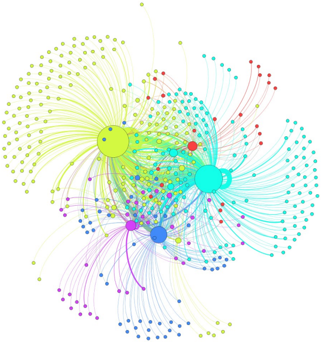
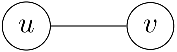
Structure! Structure! Structure!



Graph *structure* induced by DFS:

states of v

types of



Graph *structure* induced by DFS:

states of v

types of u v

life time of v:

v : $\mathsf{d}[v], \mathsf{f}[v]$

$\mathsf{f}[v]$: DAG, SCC

$\mathsf{d}[v]$: biconnectivity

Definition (Classifying edges)

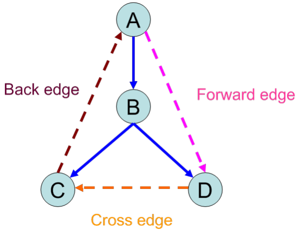
Given a DFS traversal $\Longrightarrow$ DFS tree:

Tree edge: $\rightarrow$ child

Back edge: $\rightarrow$ ancestor

Forward edge: $\rightarrow$ *nonchild* descendant

Cross edge: $\rightarrow$ $(\neg\text{ancestor}) \wedge (\neg\text{descendant})$



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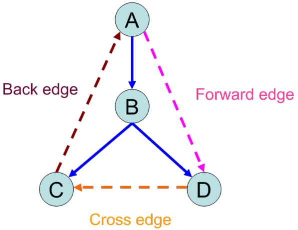
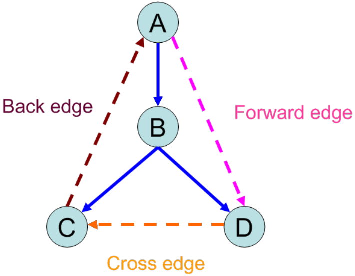
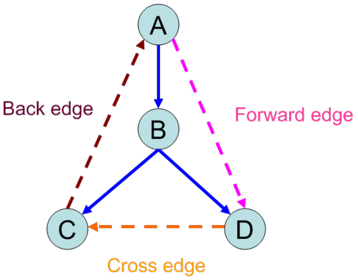


- also applicable to BFS
- w.r.t. DFS/BFS trees



DFS on directed graph



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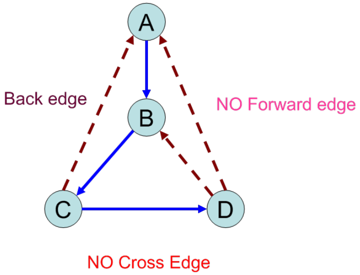
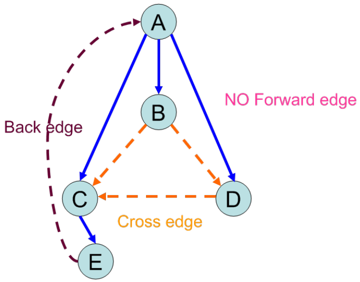
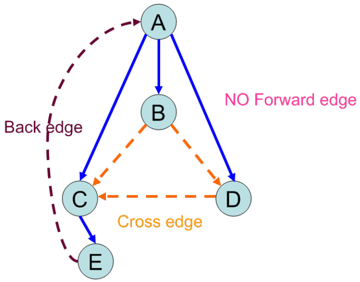


DFS on undirected graph



BFS on directed graph



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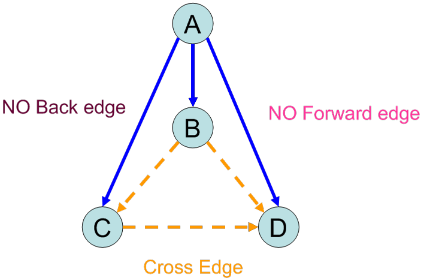

BFS on undirected graph

DFS tree and BFS tree coincide (Problem 5.7)

Undirected connected graph $G = (V, E), v \in V$

DFS tree T from $v \equiv$ BFS tree T' from v

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Proof.

G_{DFS}: tree + back *vs.* G_{BFS}: tree + cross

□

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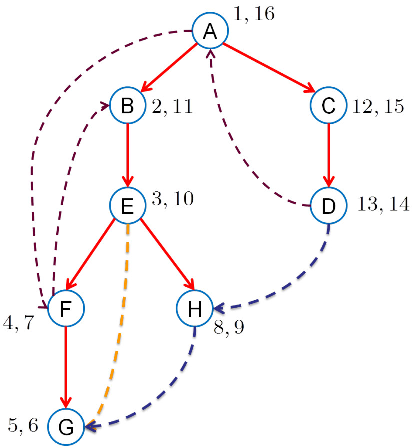
Q : What if G is a digraph?

Lift time of vertices in DFS



Theorem (Disjoint or Contained (Problem 4.2: (1) & (2)))

$$\forall u, v : [_u \ ]_u \cap [_v \ ]_v = \emptyset \bigvee \left([_u \ ]_u \subsetneqq [_v \ ]_v \vee [_v \ ]_v \subsetneqq [_u \ ]_u \right)$$

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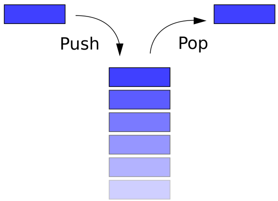
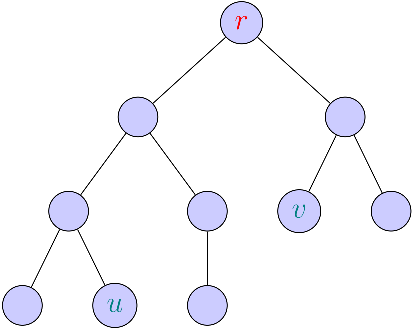


$\square$

Preprocessing for ancestor/descendant relation (Problem 5.23)



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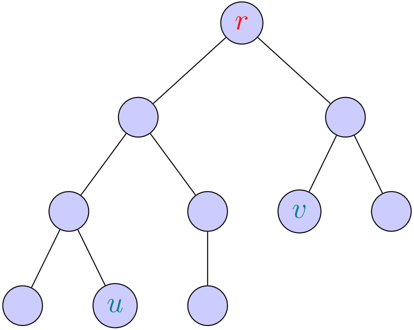

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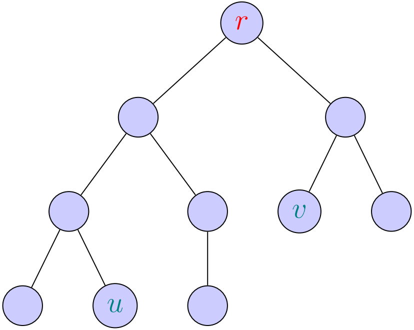

Q : *Is u an ancestor of v? $O(1)$*

$$v : \mathsf{d}[v], \mathsf{f}[v]$$

Q : # of descendants of any v?

Edge types and life time of vertices in DFS (Problem 4.5)

$$\forall u \to v :$$

- tree/forward edge: $[_u \; [_v \;]_v \;]_u$
- back edge: $[_v \; [_u \;]_u \;]_v$
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$$\nexists \text{ cycle} \implies \boxed{u \to v \iff \mathsf{f}[v] < \mathsf{f}[u]}$$

Height and diameter of tree (Problem 5.4)

Binary tree $T = (V, E)$ with $|V| = n$ and the root r:

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$$\begin{cases} D(T) = 0, & T \text{ is a leave} \\ D(T) = \max\Big(D(L_T), D(R_T), \underbrace{H(L_T) + H(R_T) + 2}_{\text{through the root}}\Big), & \text{o.w.} \end{cases}$$

Binary tree $T = (V, E)$ with $|V| = n$ and the root r

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Q : Diameter of a *tree* *without* a designated root

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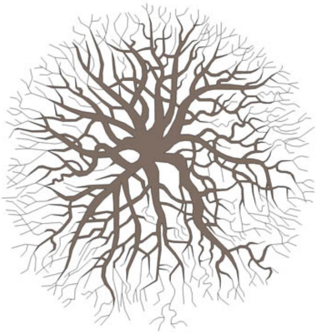

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Your Job: Prove it!

Counting shortest paths (Problem 5.26)

Counting # of shortest paths in (un)directed graphs using BFS.

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Counting # of shortest paths in (un)directed graphs using BFS.

Maybe in the next class...

Cycle detection (Problem $5.8-1$)

	Digraph	Undirected graph
DFS		
BFS		

Cycle detection (Problem $5.8-1$)

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Cycle detection (Problem $5.8 - 1$)

	Digraph	Undirected graph
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BFS		

Cycle detection (Problem $5.8 - 1$)

	Digraph	Undirected graph
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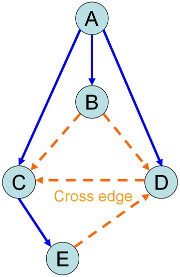

Evasiveness of acyclicity of undirected graphs (Problem $5.8-2$)

$$\text{Evasiveness} \triangleq \text{check} \binom{n}{2} \text{edges (adjacency matrix)}$$

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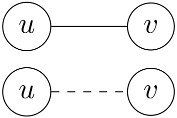

Algorithm $\mathbb{A}$:

$\textsc{CheckEdge}(u, v)$

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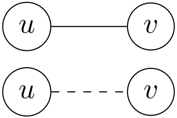

Algorithm $\mathbb{A}$:

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Hint: Kruskal

$$\textsc{CheckEdge}(u, v) \leftarrow u \text{ --- } v$$

$$\Longleftrightarrow$$

$$\color{red}{\nexists\ \text{cycle}}\ \color{black}{\in G + u \text{ --- } v}$$

$$\textsc{CheckEdge}(u, v) \leftarrow u - v$$

$$\iff$$

$$\nexists \text{ cycle } \in G + u - v$$

Q : Why adjacency matrix?

After-class Exercise: Evasiveness of connectivity of undirected graphs

$$\text{Evasiveness} \triangleq \text{check } \binom{n}{2} \text{ edges (adjacency matrix)}$$

Q : Is connectivity evasive?

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Hint: Anti-Kruskal

Orientation of undirected graph (Problem 4.13)

- undirected (connected) graph G
- edges oriented *s.t.*

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DFS from $v \in C$

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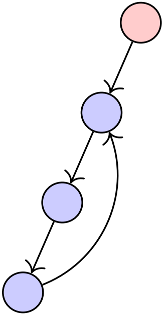

Shortest cycle of undirected graph (Problem 4.12)

A WRONG DFS-based algorithm:

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$$\text{Back edge } u \to v : \mathsf{level}[u] - \mathsf{level}[v] + 1$$

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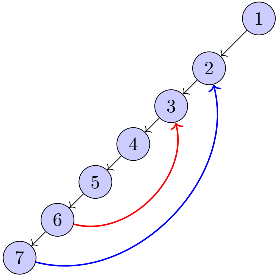

Shortest cycle of digraph (Problem 4.12)

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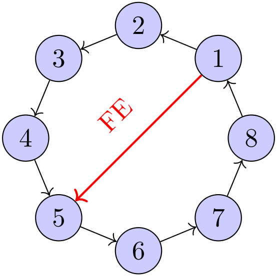


THANK YOU