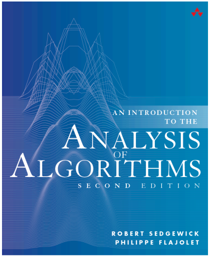
A Little Mathematics for Computer Science

Hengfeng Wei

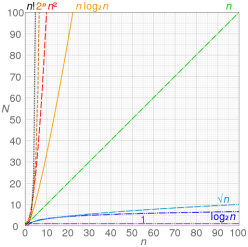
hfwei@nju.edu.cn

April 12, 2019

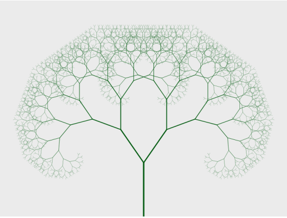


$A(n)$

Mathematical Induction

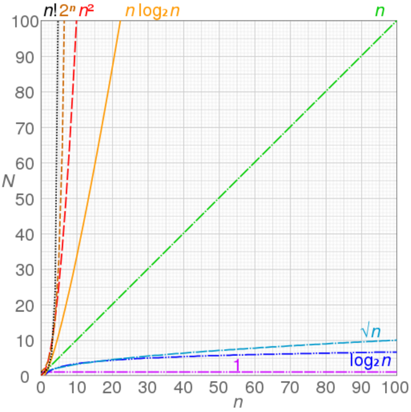


Ω, Θ, O

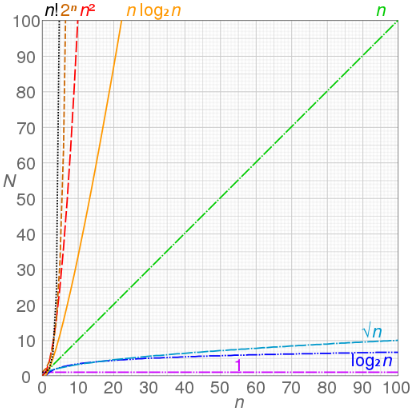


$T(n) = aT(n/b) + f(n)$

Asymptotics



Asymptotics



$Q : \theta(f)$?

$$O(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \Big\}$$

$$O(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \Big\}$$

$$\Big\{ \quad \Big\}$$

$$O(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \Big\}$$

$$\Big\{ \quad \Big\}$$

$$\exists n_0 > 0, \forall n \geq n_0$$

$$O(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \Big\}$$

$$\Big\{ \quad \Big\}$$

$$\exists n_0 > 0, \forall n \geq n_0$$

$$\exists c > 0$$

$$O(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \ge n_0 : 0 \le f(n) \le cg(n) \Big\}$$

$$\Omega(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \ge n_0 : 0 \le cg(n) \le f(n) \Big\}$$

$$O(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \ge n_0 : 0 \le f(n) \le cg(n) \Big\}$$

$$\Omega(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \ge n_0 : 0 \le cg(n) \le f(n) \Big\}$$

$$\begin{aligned} \Theta(g(n)) = \Big\{ f(n) \mid & \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \ge n_0 : \\ & 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \Big\} \end{aligned}$$

$$O(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \right\}$$

$$\Omega(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) \leq f(n) \right\}$$

$$\Theta(g(n)) = \Big\{ f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \geq n_0 : \\ 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \Big\}$$

$$o(g(n)) = \left\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < cg(n) \right\}$$

$$\omega(g(n)) = \left\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) < f(n) \right\}$$

$$O(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \Big\}$$

$$\Omega(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) \leq f(n) \Big\}$$

$$\Theta(g(n)) = \Big\{ f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \Big\}$$

$$o(g(n)) = \Big\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < cg(n) \Big\}$$

$$\omega(g(n)) = \Big\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) < f(n) \Big\}$$

$$f(n) \sim g(n) \iff \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$$

Asymptotics (Problem 2.6 (4))

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \land f(n) = \Omega(g(n))$$

Asymptotics (Problem 2.6 (4))

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \wedge f(n) = \Omega(g(n))$$

Asymptotics (Problem 2.6 (6))

$$\Theta(g(n)) \cap o(g(n)) = \emptyset$$

Asymptotics (Problem 2.6 (4))

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \land f(n) = \Omega(g(n))$$

Asymptotics (Problem 2.6 (6))

$$\Theta(g(n)) \cap o(g(n)) = \emptyset$$

$$Q : f(n) = O(g(n)) \lor g(n) = \Omega(f(n)) \ ?$$

Asymptotics (Problem 2.6 (4))

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \land f(n) = \Omega(g(n))$$

Asymptotics (Problem 2.6 (6))

$$\Theta(g(n)) \cap o(g(n)) = \emptyset$$

$$Q : f(n) = O(g(n)) \lor g(n) = \Omega(f(n)) \ ?$$

$$f(n) = n, \quad g(n) = n^{1+\sin n}$$

Asymptotics (Problem 2.7 (2))

$$(\log n)^2 \quad vs. \quad \sqrt{n}$$

Asymptotics (Problem 2.7 (2))

$$(\log n)^2 \quad vs. \quad \sqrt{n}$$

$$(\log n)^{c_1} = O(n^{c_2}) \quad c_1, c_2 > 0$$

Asymptotics (Problem 2.10)

$$\log(n!) = \Theta(n \log n)$$

Asymptotics (Problem 2.10)

$$\log(n!) = \Theta(n \log n)$$

Stirling Formula (by *James Stirling*):

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Asymptotics (Problem 2.10)

$$\log(n!) = \Theta(n \log n)$$

Stirling Formula (by *James Stirling*):

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\log(n!) = \log 1 + \log 2 + \cdots + \log n$$

Asymptotics (Problem 2.10)

$$\log(n!) = \Theta(n \log n)$$

Stirling Formula (by *James Stirling*):

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\log(n!) = \log 1 + \log 2 + \cdots + \log n$$

$$\log(n!) \leq n \log n$$

Asymptotics (Problem 2.10)

$$\log(n!) = \Theta(n \log n)$$

Stirling Formula (by *James Stirling*):

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\log(n!) = \log 1 + \log 2 + \cdots + \log n$$

$$\log(n!) \le n \log n \qquad \log(n!) \ge \frac{n}{2} \log \frac{n}{2}$$

Summation (Problem 2.20)

```
procedure CONUNDRUM(n)
    r ← 0
    for i ← 1 to n do
        for j ← i + 1 to n do
            for k ← i + j − 1 to n do
                r ← r + 1
    return r
```

Summation (Problem 2.20)

```
procedure CONUNDRUM(n)
    r ← 0
    for i ← 1 to n do
        for j ← i + 1 to n do
            for k ← i + j − 1 to n do
                r ← r + 1
    return r
```

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 =$$

Summation (Problem 2.20)

```
procedure CONUNDRUM(n)
    r ← 0
    for i ← 1 to n do
        for j ← i + 1 to n do
            for k ← i + j − 1 to n do
                r ← r + 1
    return r
```

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \frac{n^2 - n}{2} = \Theta(n^2)$$

Summation (Problem 2.20)

```
procedure Conundrum(n)
    r ← 0
    for i ← 1 to n do
        for j ← i + 1 to n do
            for k ← i + j − 1 to n do
                r ← r + 1
    return r
```

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \frac{n^2 - n}{2} = \Theta(n^2)$$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \frac{1}{48}\Big(3\big(-1 + (-1)^n\big) + 2n(n+2)(2n-1)\Big) = \Theta(n^3)$$

$$\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=i+j-1}^{n} 1$$

$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n} \left(n-(i+j-1)+1\right)[i+j-1 \le n]$$

$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n} (n-i-j+2)[j \le n-i+1] \qquad n-i+1 \ge i+1 \Rightarrow i \le \frac{n}{2}$$

$$= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor}\sum_{j=i+1}^{n-i+1} (n-i-j+2)$$

$$= \frac{1}{2}\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (n-2i+1)(n-2i+2)$$

当n为偶时 $= \frac{1}{2}\left[\sum_{i=1}^{\frac{n}{2}} (n^2+3n+2) + 4\sum_{i=1}^{\frac{n}{2}} i^2 - \sum_{i=1}^{\frac{1}{2}} (4n+6)i\right]$

$$= \frac{1}{2}\times\left(\frac{1}{2}(n^3+3n^2+2n) + \frac{n(\frac{n}{2}+1)(n+1)}{3} - \frac{(2n+3)(\frac{n}{2}+1)n}{2}\right)$$

$$= \text{化简} = \frac{2n^3+3n^2-2n}{24} = \frac{1}{48}\left(0+2n(2+n)(2n-1)\right)$$

当n为奇时，$\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$，代入，可化得 $= \frac{1}{48}\left(-6+2n(2+n)(2n-1)\right)$

（这个我懒的化了，谁有兴趣化一下，多个常数项）.

*通解 $\frac{1}{48}\left(3(-1+(-1)^n)+2n(2+n)(2n-1)\right)$

* $\lfloor \frac{n}{2} \rfloor = \frac{n+\frac{(-1)^n+(-1)}{2}}{2}$，代入理应可直接得结果，太繁

From Zheng (171860658)

Reference:

"Big Omicron and Big Omega and Big Theta" by Donald E. Knuth, 1976.


Thank
You!

Your opinon Matters

Office 302

Mailbox: H016

hfwei@nju.edu.cn