# Dynamic Programming

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# Taolu

## Steps for Applying DP:

# Steps for Applying DP:

(I) Define subproblems

(II) Set the goal

(III) Identify the recurrence
- larger subproblem $\leftarrow$ # smaller subproblems
- init. conditions

(IV) Write pseudo-code: filling in "tables" in some order

(V) Analyze the time complexity

(VI) Extract the optimal solution (optionally)

# 1D Subproblems

Input: $x_1, x_2, \ldots, x_n$ (array, sequence, string)

Subproblems: $x_1, x_2, \ldots, x_i$ (prefix/suffix)

#: $\Theta(n)$

Examples:

- Rod cutting
- Maximum-sum subarray
- Longest increasing subsequence
- Printing neatly

## 2D Subproblems

(I) Input: $x_1, x_2, \ldots, x_m; \quad y_1, y_2, \ldots, y_n$

Subproblems: $x_1, x_2, \ldots, x_i; \quad y_1, y_2, \ldots, y_j$

#: $\Theta(mn)$

Examples: Edit distance, Longest common subsequence

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#: $\Theta(mn)$

Examples: Edit distance, Longest common subsequence

(II) Input: $x_1, x_2, \ldots, x_n$

Subproblems: $x_i, \ldots, x_j$

#: $\Theta(n^2)$

Examples: Matrix chain multiplication, Optimal BST

# 3D Subproblems

- Floyd-Warshall algorithm

$$\mathrm{d}(i, j, k) = \min\left(\mathrm{d}(i, j, k-1), \mathrm{d}(i, k, k-1) + \mathrm{d}(k, j, k-1)\right)$$

## DP on Graphs

(I) On rooted tree

Subproblems: rooted subtrees

(II) On DAG

Subproblems: nodes after/before in the topo. order

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## Knapsack Problem

Subset sum problem, Change-making problem

# And Others . . .

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How to identify the recurrence?

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# GUESS

Make Choices by asking yourself the right question

## Make Choices by asking yourself the right question

(I) Binary choice
- whether ...

(II) Multi-way choices
- where to ...
- which one ...

LCS: Longest Common Subsequence (Problem 14.6 (1))

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

(1) Find (the length of) an LCS of $X$ and $Y$

$$X = \langle A, B, C, B, D, A, B \rangle$$
$$Y = \langle B, D, C, A, B, A \rangle$$
$$Z = \langle B, C, B, A \rangle$$

Subproblem: $L[i, j]$: the length of an LCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$

Goal: $L[m, n]$

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Goal: $L[m,n]$

Make choice: Is $X_i = Y_j$?

Recurrence: (Proof!)

$$L[i,j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1,j], L[i,j-1]\} & \text{if } X_i \neq Y_j \end{cases}$$

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Init:

$$L[0,j] = 0,\ 0 \le j \le n$$
$$L[i,0] = 0,\ 0 \le i \le m$$

Time: $\Theta(mn)$

## Longest Common Subsequence (Problem 14.6 (2)&(3))

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

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$$X \implies X^{(k)} \triangleq X_1^{(k)} \cdots X_m^{(k)}$$

## Longest Contiguous Substring Both Forward and Backward (Problem 14.7)

- String $T[1 \cdots n]$
- Find a longest contiguous substring (LCS) both forward and backward

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- Subproblem $L[i]$: the length of an LCS in $T[1 \cdots i]$
- Subproblem $L[i, j]$: the length of an LCS in $T[i \cdots j]$

Subproblem: $L[i, j]$: the length of an LCS starting with $T_i$ and ending with $T_j$

Goal: $\max_{1 \le i \le j \le n} L[i, j]$

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Make choice: Is $T_i = T_j$?

Recurrence:

$$L[i, j] = \begin{cases} 0 & \text{if } T_i \neq T_j \\ L[i+1, j-1] + 1 & \text{if } T_i = T_j \end{cases}$$

Subproblem: $L[i, j]$: the length of an LCS starting with $T_i$ and ending with $T_j$

Goal: $\max_{1 \le i \le j \le n} L[i, j]$

Make choice: Is $T_i = T_j$?

Recurrence:

$$L[i, j] = \begin{cases} 0 & \text{if } T_i \neq T_j \\ L[i+1, j-1] + 1 & \text{if } T_i = T_j \end{cases}$$

Init:

$$L[i, i] = 0, \ 0 \le i \le n$$

$$L[i, i+1] = \begin{cases} 1 & \text{if } T_i = T_{i+1} \\ 0 & \text{if } T_i \neq T_{i+1} \end{cases}$$

## Longest Palindrome Subsequence (Problem 14.11 (1))

(1) Find (the length of) a longest palindrome subsequence of $S[1 \cdots n]$

Subproblem: $L[i,j]$: the length of an LSP of $S[i \cdots j]$

Goal: $L[1,n]$

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Goal: $L[1,n]$

Make choice: Is $S[i] = S[j]$?

Recurrence:

$$L[i,j] = \begin{cases} L[i+1, j-1] + 2 & \text{if } S[i] = S[j] \\ \max\{L[i+1,j], L[i,j-1]\} & \text{if } S[i] \neq S[j] \end{cases}$$

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Init:

$$L[i,i] = 1, \ \forall 1 \le i \le n$$

$$L[i,i+1] = \begin{cases} 2 & \text{if } S[i] = S[i+1] \\ 0 & \text{if } S[i] \neq S[i+1] \end{cases}$$

## Palindrome Splitting (Problem 14.11 (2))

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes (# cuts)

Subproblem: $C[i, j]$: minimum number of cuts for string $S[i \dots j]$

Goal: $C[1, n] + 1$

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**Goal:** $C[1,n]+1$

**Make choice:** Where is the first cut?

**Recurrence:**

$$C[i,j] = \begin{cases} 0 \;\; \text{if } S[i \dots j] \text{ is a palindrome} \\ \displaystyle\min_{i+1 \le k \le j-1} C[i,k-1] + 1 + C[k,j] & \text{o.w.} \end{cases}$$

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Init: $C[i, i] = 0$

Time: $O(n^3)$

## Palindrome Splitting (Problem 14.11 (2))

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes

Subproblem: $P[i]$: minimum number of palindromes for $S[1 \cdots i]$

Goal: $P[n]$

## Palindrome Splitting (Problem 14.11 (2))

(2) Split a string $S[1 \ldots n]$ into minimum number of palindromes

Subproblem: $P[i]$: minimum number of palindromes for $S[1 \cdots i]$

Goal: $P[n]$

Make choice: Where does the last palindrome start from?

Recurrence:

$$P[i] = \min_{\substack{1 \le k \le i \\ S[k \ldots i] \text{ is a palindrome}}} P[k-1] + 1$$

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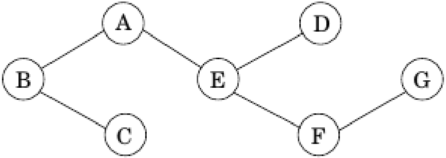
Init: $P[0] = 1$

Time: $O(n^3)$ *vs.* $O(n^2)$

## Minimum Vertex Cover on Trees (Problem 14.14)

- Undirected tree $T = (V, E)$; No designated root!
- Compute (the size of) a minimum vertex cover of $T$



Rooted $T$ at any node $r$.

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Subproblem: $I(u)$: the size of an MVC of subtree $T_u$ rooted at $u$

Goal: $I(r)$

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Goal: $I(r)$

Make choice: Is $u$ in MVC[$u$]?

Recurrence:

$$I(u) = \min\{\#\text{ children of } u + \sum_{v:\text{ grandchildren of } u} I(v), \quad 1 + \sum_{v:\text{ children of } u} I(v)\}$$

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Init: $I(u) = 0$, if $u$ is a leave

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DFS from root $r$.


#10
IS GREED
GOOD?


*There is an MVC which contains no leaves.*

The Change-making Problem (Problem 14.13)

- Coins values: $x_1 \dots x_n$
- Amount: $v$
- Is it possible to make change for $v$?

## The Change-making Problem (Problem 14.13 (2))

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Subproblem: $C[i, w]$: Make change for $w$ using only values of $x_1 \ldots x_i$?

Goal: $C[n, v]$

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Subproblem: $C[i, w]$: Make change for $w$ using only values of $x_1 \ldots x_i$?

Goal: $C[n, v]$

Make choice: Using value $x_i$ or not?

Recurrence:

$$C[i, w] = C[i-1, w] \lor (C[i-1, w-x_i] \land w \geq x_i)$$

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Recurrence:

$$C[i, w] = C[i-1, w] \lor (C[i-1, w-x_i] \land w \geq x_i)$$

Init:

$$C[i, 0] = \text{true}, \ \forall i = 0 \ldots n$$
$$C[0, w] = \text{false}, \ \text{if } w > 0$$

Time: $O(nv)$

## The Change-making Problem (Problem 14.13 (1))

(1) Unbounded repetition ($\infty$)

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(1) Unbounded repetition ($\infty$)

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Goal: $C[n, v]$

Make choice: Using value $x_i$ or not?

Recurrence:

$$C[i, w] = C[i-1, w] \lor (C[i, w - x_i] \land w \geq x_i)$$

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Time: $O(nv)$

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(3) Unbounded repetition with $\leq k$ coins

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Subproblem: $C[i, w, l]$: Possible to make change for $w$ with $\leq l$ coins of values of $x_1 \ldots x_i$?

Goal: $C[n, v, k]$

## The Change-making Problem (Problem 14.13 (3))

(3) Unbounded repetition with $\leq k$ coins

Subproblem: $C[i, w, l]$: Possible to make change for $w$ with $\leq l$ coins of values of $x_1 \ldots x_i$?

Goal: $C[n, v, k]$

Make choice: Using value $x_i$ or not?

Recurrence:

$$C[i, w, l] = C[i-1, w, l] \lor (C[i, w - x_i, l - 1] \land w \geq x_i)$$

## The Change-making Problem (Problem 14.13 (3))

(3) Unbounded repetition with $\le k$ coins

Subproblem: $C[i, w, l]$: Possible to make change for $w$ with $\le l$ coins of values of $x_1 \dots x_i$?

Goal: $C[n, v, k]$

Make choice: Using value $x_i$ or not?

Recurrence:

$$C[i, w, l] = C[i-1, w, l] \lor (C[i, w - x_i, l-1] \land w \ge x_i)$$

Init:

$$C[0, 0, l] = \text{true}, \quad C[0, w, l] = \text{false}, \text{if } w > 0$$
$$C[i, 0, l] = \text{true}, \quad C[i, w, 0] = \text{false}, \text{if } w > 0$$

## Algorithms that use dynamic programming [ edit | edit source ]

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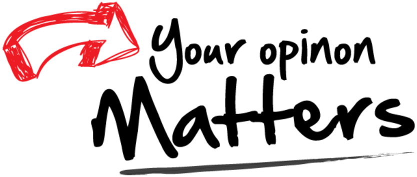
- Recurrent solutions to lattice models for protein-DNA binding
- Backward induction as a solution method for finite-horizon discrete-time dynamic optimization problems
- Method of undetermined coefficients can be used to solve the Bellman equation in infinite-horizon, discrete-time, discounted, time-invariant dynamic optimization problems
- Many string algorithms including longest common subsequence, longest increasing subsequence, longest common substring, Levenshtein distance (edit distance)
- Many algorithmic problems on graphs can be solved efficiently for graphs of bounded treewidth or bounded clique-width by using dynamic programming on a tree decomposition of the graph.
- The Cocke–Younger–Kasami (CYK) algorithm which determines whether and how a given string can be generated by a given context-free grammar
- Knuth's word wrapping algorithm that minimizes raggedness when word wrapping text
- The use of transposition tables and refutation tables in computer chess
- The Viterbi algorithm (used for hidden Markov models)
- The Earley algorithm (a type of chart parser)
- The Needleman–Wunsch algorithm and other algorithms used in bioinformatics, including sequence alignment, structural alignment, RNA structure prediction
- Floyd's all-pairs shortest path algorithm
- Optimizing the order for chain matrix multiplication
- Pseudo-polynomial time algorithms for the subset sum, knapsack and partition problems
- The dynamic time warping algorithm for computing the global distance between two time series
- The Selinger (a.k.a. System R) algorithm for relational database query optimization
- De Boor algorithm for evaluating B-spline curves
- Duckworth–Lewis method for resolving the problem when games of cricket are interrupted
- The value iteration method for solving Markov decision processes
- Some graphic image edge following selection methods such as the "magnet" selection tool in Photoshop
- Some methods for solving interval scheduling problems
- Some methods for solving the travelling salesman problem, either exactly (in exponential time) or approximately (e.g. via the bitonic tour)
- Recursive least squares method
- Beat tracking in music information retrieval
- Adaptive-critic training strategy for artificial neural networks
- Stereo algorithms for solving the correspondence problem used in stereo vision
- Seam carving (content-aware image resizing)
- The Bellman–Ford algorithm for finding the shortest distance in a graph
- Some approximate solution methods for the linear search problem
- Kadane's algorithm for the maximum subarray problem


THANK YOU

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