Decompositions of Graphs

(DFS/BFS, DAG, SCC, Bicomp)

Hengfeng Wei

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June 12, 2019

John Hopcroft

Robert Tarjan

"For fundamental achievements in the design and analysis of algorithms and data structures."

— *Turing Award*, 1986

SIAM J. COMPUT.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

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Power of DFS:

Graph Traversal $\Longrightarrow$ Graph Decomposition

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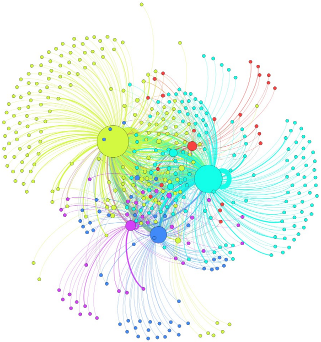
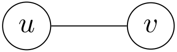
Structure! Structure! Structure!



Graph *structure* induced by DFS:

states of 

types of 

Graph *structure* induced by DFS:

states of v

types of u — v

life time of v:

v : d$[v]$, f$[v]$

d$[v]$: Bicomp

f$[v]$: Toposort, SCC

Definition (Classifying edges)

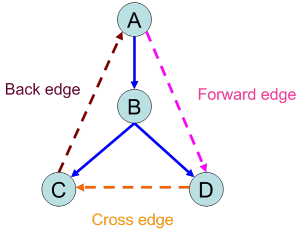
Given a DFS traversal $\implies$ DFS tree:

Tree edge: $\rightarrow$ child

Back edge: $\rightarrow$ ancestor

Forward edge: $\rightarrow$ *nonchild* descendant

Cross edge: $\rightarrow$ $(\neg\text{ancestor}) \wedge (\neg\text{descendant})$



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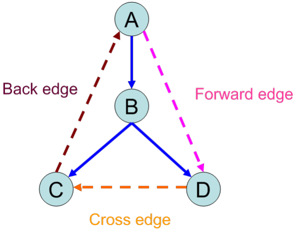
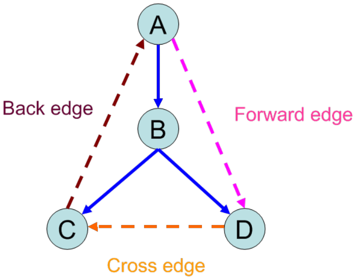
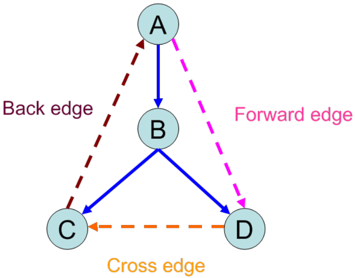


Also applicable to BFS



DFS on directed graph



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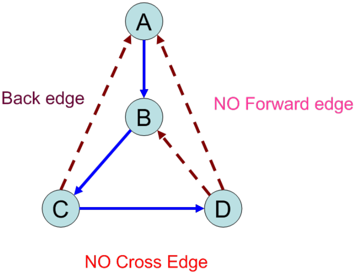
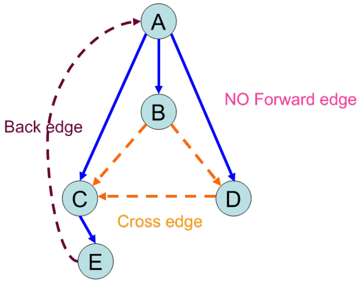
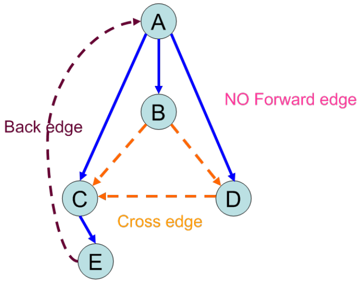


DFS on undirected graph



BFS on directed graph



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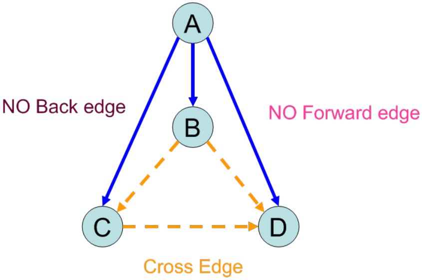

BFS on undirected graph

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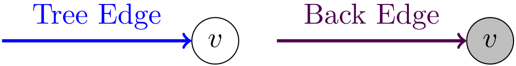

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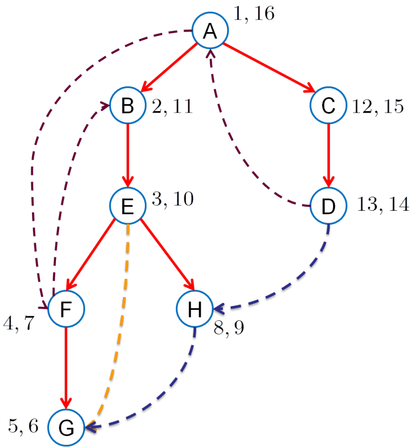
Life time of vertices in DFS



Theorem (Disjoint or Contained (Problem 4.2 : (1)&(2)))

$$\forall u, v : [_u\ ]_u \cap [_v\ ]_v = \emptyset \bigvee \left([_u\ ]_u \subset [_v\ ]_v \lor [_v\ ]_v \subset [_u\ ]_u\right)$$

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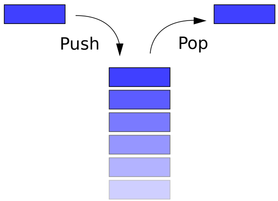

□

Edge types and life time of vertices in DFS (Problem 4.5)

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$$\nexists \text{ cycle} \implies \boxed{u \to v \iff \mathrm{f}[v] < \mathrm{f}[u]}$$

Cycle detection (Problem 5.8 – 1)

	Digraph	Undirected graph
DFS		
BFS		

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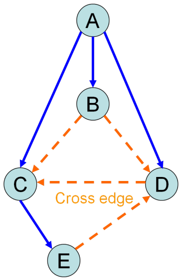

Evasiveness of acyclicity of undirected graphs (Problem $5.8 - 2$)

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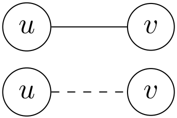

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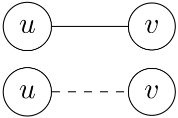

Hint: Kruskal

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Q : Why adjacency matrix?

On digraphs:

$$\nexists \text{ back edge} \iff \text{DAG} \iff \exists \text{ topo. ordering}$$

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TOPOSORT by Tarjan (probably), 1976

$$\nexists \text{ cycle} \implies \boxed{u \to v \iff \mathrm{f}[v] < \mathrm{f}[u]}$$

Sort vertices in *decreasing* order of their *finish* times.

Hamiltonian path in DAG (Problem 4.14)

HP: path visiting each vertex once

Q : $\exists$ HP in a DAG in $O(n + m)$

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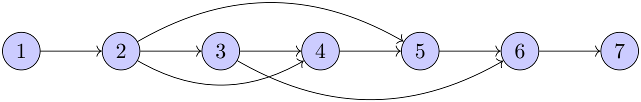

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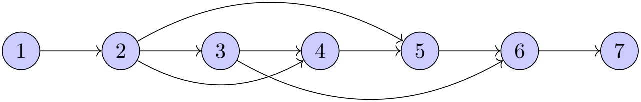

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Tarjan's TOPOSORT + Check edges (v_i, v_{i+1})

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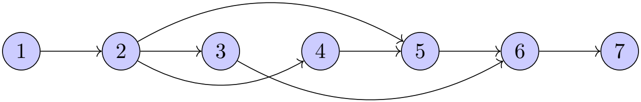

Theorem (Digraph as DAG (Problem 4.6))

Every digraph is a dag of its SCCs.

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Two tiered structure of digraphs:

$$\text{digraph} \equiv \text{a dag of SCCs}$$

SCC: equivalence class over reachability

One-to-all reachability in a digraph (Problem 4.17)

$$v : v \leadsto^{?} \forall u$$

$$\exists ?\ v : v \leadsto \forall u$$

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$$\Longleftarrow : \exists !\ \text{source}$$

$$\Longrightarrow : \text{By contradiction.}$$

$$\exists u : v \not\leadsto u \land \text{in}[u] > 0 \implies \exists \text{ cycle}$$

Impacts of vertices in a digraph (Problem 4.18)

$$\mathrm{impact}(v) = |\{w \neq v : v \rightsquigarrow w\}|$$

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$$\boxed{Q : \forall v, \text{computing impact}(v)}$$

2SAT (Problem 4.23)

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Implication graph G_I.

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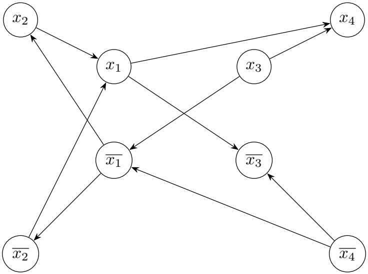
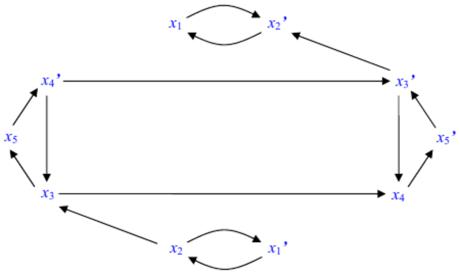
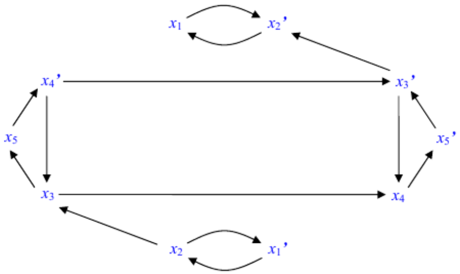




Theorem (2SAT)

$$\exists\ SCC\ \exists x : v_x \in SCC \wedge v_{\overline{x}} \in SCC \iff I \text{ is not satisfiable.}$$



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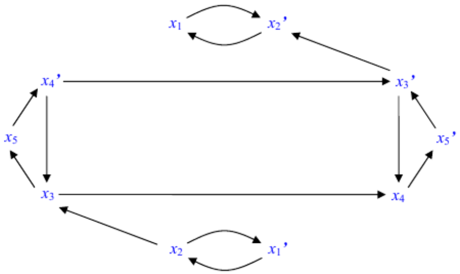
$\perp$ for source SCC & $\top$ for sink SCC



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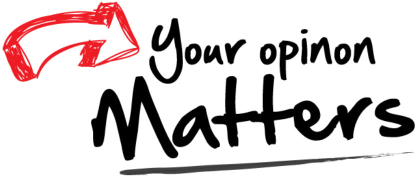
"A Linear-time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas", Bengt Aspvall, Michael Plass, Robert Tarjan, 1979


THANK YOU



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