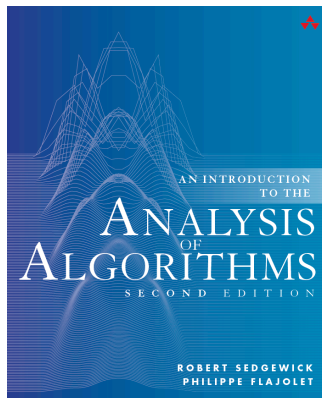


# Asymptotics, Recurrences, and Divide and Conquer

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Problem  $P$

Algorithm  $A$

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## Average-case Time Complexity (Problem 1.8)

Input :  $r \in [1, n]$ ,  $r \in \mathbb{Z}^+$

$$P\{r = i\} = \begin{cases} \frac{1}{n}, & 1 \leq i \leq \frac{n}{4} \\ \frac{2}{n}, & \frac{n}{4} < i \leq \frac{n}{2} \\ \frac{1}{2n}, & \frac{n}{2} < i \leq n \end{cases}$$

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$$\begin{aligned} A &= \sum_{X \in \mathcal{X}} T(X) \cdot P(X) \\ &= T(1)P(1) + T(2)P(2) + \cdots + T(n)P(n) \\ &= \frac{n}{4} \times 10 \times \frac{1}{n} + \frac{n}{4} \times 20 \times \frac{2}{n} + \frac{n}{4} \times 30 \times \frac{1}{2n} + \frac{n}{4} \times n \times \frac{1}{2n} \\ &= \frac{1}{8}n + \frac{55}{4} \end{aligned}$$

# Average-case Analysis of Quicksort

$$A(n) = n - 1 + \frac{1}{n} \sum_{i=0}^{n-1} (A(i) + A(n - i - 1))$$

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$$\begin{aligned} A(n) &= \mathbb{E}[T(X)] \\ &= \mathbb{E}[\mathbb{E}[T(X)|I]] \\ &= \sum_{i=0}^{i=n-1} P(I = i) \mathbb{E}[T(X) \mid I = i] \\ &= \sum_{i=0}^{i=n-1} \frac{1}{n} [n - 1 + A(i) + A(n - i - 1)] \end{aligned}$$

# Mathematical Induction



## Horner's rule (Problem 1.5)

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n$$

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1: <b>procedure</b> HORNER( $A[0 \dots n], x$ )	$\triangleright A : \{a_0 \dots a_n\}$
2: $p \leftarrow A[n]$	
3: <b>for</b> $i \leftarrow n - 1$ <b>downto</b> 0 <b>do</b>	
4: $p \leftarrow px + A[i]$	
5: <b>return</b> $p$	

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Loop invariant (after the  $k$ -th loop):

$$\sum_{i=n-k}^{i=n} a_i x^{k-(n-i)}$$

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## Integer Multiplication (Problem 1.6)

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1: procedure INT-MULT( $y, z$ )  
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