

Dynamic Programming

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Dynamic Programming

1 Overview

2 1D DP

3 2D DP

4 3D DP

5 DP on Graphs

6 The Knapsack Problem

What is DP?

DP \approx “brute force”

DP \approx “smart scheduling of subproblems”

DP \approx “shortest/longest paths in some DAG”

What is DP?

DP \approx “smarter brute force”

DP \approx “smart scheduling of subproblems”

DP \approx “shortest/longest paths in some DAG”

What is not DP?

Programming = Planning

What is not DP?

Programming = Planning

Programming \neq Coding

(Richard Bellman, 1940s)

Steps for applying DP

1. Define subproblems
 - ▶ # of subproblems
2. Set the goal
3. Define the recurrence
 - ▶ larger subproblem \leftarrow # smaller subproblems
 - ▶ init. conditions
4. Write pseudo-code: fill “table” in topo. order
5. Analyze Time/Space complexity
6. Extract the optimal solution

Common subproblems in DP

1D subproblems:

Input: x_1, x_2, \dots, x_n (array, sequence, string)

Subproblems: x_1, x_2, \dots, x_i (prefix/suffix)

#: $\Theta(n)$

Common subproblems in DP

1D subproblems:

Input: x_1, x_2, \dots, x_n (array, sequence, string)

Subproblems: x_1, x_2, \dots, x_i (prefix/suffix)

#: $\Theta(n)$

Examples: Fib, Maximum-sum subarray, Longest increasing subsequence, Highway restaurants, Text justification

Common subproblems in DP

2D subproblems:

1. Input: $x_1, x_2, \dots, x_m; \quad y_1, y_2, \dots, y_n$

Subproblems: $x_1, x_2, \dots, x_i; \quad y_1, y_2, \dots, y_j$

#: $\Theta(mn)$

Common subproblems in DP

2D subproblems:

1. Input: $x_1, x_2, \dots, x_m; \quad y_1, y_2, \dots, y_n$

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Examples: Edit distance, Longest common subsequence

Common subproblems in DP

2D subproblems:

1. Input: $x_1, x_2, \dots, x_m; \quad y_1, y_2, \dots, y_n$

Subproblems: $x_1, x_2, \dots, x_i; \quad y_1, y_2, \dots, y_j$

#: $\Theta(mn)$

Examples: Edit distance, Longest common subsequence

2. Input: x_1, x_2, \dots, x_n

Subproblems: x_i, \dots, x_j

#: $\Theta(n^2)$

Common subproblems in DP

2D subproblems:

1. Input: $x_1, x_2, \dots, x_m; \quad y_1, y_2, \dots, y_n$

Subproblems: $x_1, x_2, \dots, x_i; \quad y_1, y_2, \dots, y_j$

#: $\Theta(mn)$

Examples: Edit distance, Longest common subsequence

2. Input: x_1, x_2, \dots, x_n

Subproblems: x_i, \dots, x_j

#: $\Theta(n^2)$

Examples: Matrix chain multiplication, Optimal BST

Common subproblems in DP

3D subproblems:

- ▶ Floyd-Warshall algorithm

$$d(i, j, k) = \min\{d(i, j, k - 1), d(i, k, k - 1) + d(k, j, k - 1)\}$$

Common subproblems in DP

3D subproblems:

- ▶ Floyd-Warshall algorithm

$$d(i, j, k) = \min\{d(i, j, k-1), d(i, k, k-1) + d(k, j, k-1)\}$$

DP on graphs:

1. On rooted tree

Subproblems: rooted subtrees

2. On DAG

Subproblems: nodes after/before in the topo. order

Common subproblems in DP

3D subproblems:

- ▶ Floyd-Warshall algorithm

$$d(i, j, k) = \min\{d(i, j, k-1), d(i, k, k-1) + d(k, j, k-1)\}$$

DP on graphs:

1. On rooted tree

Subproblems: rooted subtrees

2. On DAG

Subproblems: nodes after/before in the topo. order

Knapsack problem:

- ▶ Subset sum problem, change-making problem

Common subproblems in DP

And Others . . .

Recurrences in DP

Make choices by asking yourself the right question:

1. Binary choice
 - ▶ whether ...
2. Multi-way choices
 - ▶ where to ...
 - ▶ which one ...

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6 The Knapsack Problem

$$f(S(n)) = 1$$

$$f(S(n)) = 1 \text{ (Problem 7.2)}$$

$$f(n) = \begin{cases} n - 1 & \text{if } n \in \mathbb{Z}^+ \\ n/2 & \text{if } n \% 2 = 0 \\ n/3 & \text{if } n \% 3 = 0 \end{cases}$$

$S(n)$: minimum number of steps taking n to 1.

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$S(i)$: minimum number of steps taking i to 1

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$S(n)$: minimum number of steps taking n to 1.

$S(i)$: minimum number of steps taking i to 1

$$S(i) = 1 + \min\{N(i-1), N(i/2)(\text{if } n \% 2 = 0), N(i/3)(\text{if } n \% 3 = 0)\}$$

$$S(1) = 0$$

$$f(S(n)) = 1$$

Collatz ($3n + 1$) conjecture:

$$f(n) = \begin{cases} n/2 & \text{if } n \% 2 = 0 \\ 3n + 1 & \text{if } n \% 2 = 1 \end{cases}$$

$$f^*(n) = 1?$$

$$f(S(n)) = 1$$

Collatz ($3n + 1$) conjecture:

$$f(n) = \begin{cases} n/2 & \text{if } n \% 2 = 0 \\ 3n + 1 & \text{if } n \% 2 = 1 \end{cases}$$

$$f^*(n) = 1?$$

“Mathematics may not be ready for such problems.”

— Paul Erdős

Longest increasing subsequence

Longest increasing subsequence (Problem 7.3)

- ▶ Given an integer array $A[1 \dots n]$
- ▶ To find (the length of) a longest increasing subsequence.

$$5, 2, 8, 6, 3, 6, 9, 7 \implies 2, 3, 6, 9$$

Longest increasing subsequence

Subproblem: $L(i)$: the length of the LIS of $A[1 \dots i]$

Goal: $L(n)$

Longest increasing subsequence

Subproblem: $L(i)$: the length of the LIS of $A[1 \dots i]$

Goal: $L(n)$

Make choice: whether $A[i] \in LIS[1 \dots i]$?

Recurrence:

$$L(i) = \max\{L(i-1), 1 + \max_{j < i \wedge A[j] \leq A[i]} L(j)\}$$

Longest increasing subsequence

Subproblem: $L(i)$: the length of the LIS of $A[1 \dots i]$

Goal: $L(n)$

Make choice: whether $A[i] \in LIS[1 \dots i]$?

Recurrence:

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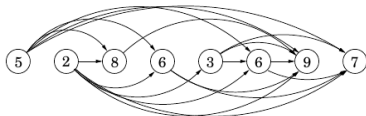
Init:

$$L(0) = 0$$

Time:

$$O(n^2) = \Theta(n) \cdot O(n)$$

Longest increasing subsequence



Longest path distance in the DAG!

Maximum-sum subarray

Maximum-sum subarray (Google Interview)

- ▶ Array $A[1 \cdots n]$, $a_i \geq 0$
- ▶ To find (the sum of) a maximum-sum subarray of A
 - ▶ $mss = 0$ if all negative

$$A[-2, 1, -3, 4, -1, 2, 1, -5, 4] \implies [4, -1, 2, 1]$$

Maximum-sum subarray

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$$A[-2, 1, -3, 4, -1, 2, 1, -5, 4] \implies [4, -1, 2, 1]$$

Subproblem: $MSS[i]$: the sum of the MS[i] of $A[1 \cdots i]$

Goal: $mss = MSS[n]$

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Subproblem: $MSS[i]$: the sum of the MS[i] of $A[1 \cdots i]$

Goal: $mss = MSS[n]$

Make choice: Is $a_i \in MS[i]$?

Recurrence:

$$MSS[i] = \max\{MSS[i-1], ???\}$$

Maximum-sum subarray

Subproblem: $MSS[i]$: the sum of the MS[i] *ending with* a_i or 0

Goal: $mss = \max_{1 \leq i \leq n} MSS[i]$

Maximum-sum subarray

Subproblem: $MSS[i]$: the sum of the MS[i] *ending with* a_i or 0

Goal: $mss = \max_{1 \leq i \leq n} MSS[i]$

Make choice: where does the MS[i] start?

Recurrence:

$$MSS[i] = \max\{MSS[i - 1] + a_i, 0\} \text{ (prove it!)}$$

Maximum-sum subarray

Subproblem: $MSS[i]$: the sum of the MS[i] *ending with* a_i or 0

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Maximum-sum subarray

Subproblem: $MSS[i]$: the sum of the MS[i] *ending with* a_i or 0

Goal: $mss = \max_{1 \leq i \leq n} MSS[i]$

Make choice: where does the MS[i] start?

Recurrence:

$$MSS[i] = \max\{MSS[i-1] + a_i, 0\} \text{ (prove it!)}$$

Init:

$$MSS[0] = 0$$

Time: $\Theta(n)$

Maximum-sum subarray

Maximum-product subarray

Maximum-product subarray (Problem 7.4)

Reconstructing string

Reconstructing string (Problem 7.9)

- ▶ String $S[1 \cdots n]$
- ▶ Dict for *lookup*:

$$\text{dict}(w) = \begin{cases} \text{true} & \text{if } w \text{ is a valid word} \\ \text{false} & \text{o.w.} \end{cases}$$

- ▶ Is $S[1 \cdots n]$ valid (reconstructed as a sequence of valid words)?

Reconstructing string

Subproblem: $V[i]$: is $S[1 \cdots i]$ valid?

Goal: $V[n]$

Reconstructing string

Subproblem: $V[i]$: is $S[1 \cdots i]$ valid?

Goal: $V[n]$

Make choice: where does the last word start?

Recurrence:

$$V[i] = \bigvee_{j=1 \dots i} (V[j-1] \wedge \text{dict}(S[j \cdots i]))$$

Reconstructing string

Subproblem: $V[i]$: is $S[1 \cdots i]$ valid?

Goal: $V[n]$

Make choice: where does the last word start?

Recurrence:

$$V[i] = \bigvee_{j=1 \dots i} (V[j-1] \wedge \text{dict}(S[j \cdots i]))$$

Init:

$$V[0] = \text{true}$$

Time: $O(n^2)$

Hotel along a trip

Hotel along a trip (Problem 7.15)

- ▶ Hotel sequence (distance): $a_0 = 0, a_1, \dots, a_n$
- ▶ $a_0 \rightsquigarrow a_n$
- ▶ Stop at only hotels
- ▶ Cost: $(200 - x)^2$
- ▶ To minimize overall cost

Hotel along a trip

Subproblem: $C[i]$: minimum cost when the destination is a_i

Goal: $C[n]$

Hotel along a trip

Subproblem: $C[i]$: minimum cost when the destination is a_i

Goal: $C[n]$

Make choice: what is the last but one hotel a_j to stop?

Recurrence:

$$C[i] = \min_{0 \leq j < i} \{C[j] + (200 - (a_i - a_j))^2\}$$

Hotel along a trip

Subproblem: $C[i]$: minimum cost when the destination is a_i

Goal: $C[n]$

Make choice: what is the last but one hotel a_j to stop?

Recurrence:

$$C[i] = \min_{0 \leq j < i} \{C[j] + (200 - (a_i - a_j))^2\}$$

Init:

$$C[0] = 0$$

Time:

$$O(n^2) = \Theta(n) \cdot O(n)$$

Highway restaurants

Highway restaurants (Problem 7.16)

- ▶ Locations: $L[1 \dots n]$
- ▶ Profits: $P[1 \dots n]$
- ▶ Any two hotels should be $\geq k$ miles apart
- ▶ To maximize the total profit

Subproblem:

Goal:

Make choice:

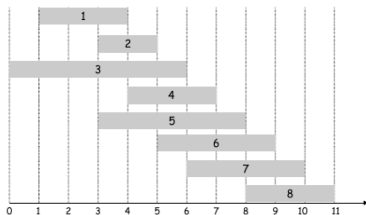
Recurrence:

Time:

Weighted interval/class scheduling

Weighted interval/class scheduling (Problem 7.14)

- ▶ Classes: $\mathcal{C} = \{c_1, c_2, \dots, c_n\}$ $c_i \triangleq \langle g_i, s_i, f_i \rangle$
- ▶ Choosing non-conflicting classes to maximize your grades



sort \mathcal{C} by finishing time.

Weighted interval/class scheduling

Greedy algorithms by finishing time or weights fail.

Weighted interval/class scheduling

Subproblem: $G[i]$: the maximal grades obtained from $\{c_1, c_2, \dots, c_i\}$

Goal: $G[n]$

Weighted interval/class scheduling

Subproblem: $G[i]$: the maximal grades obtained from $\{c_1, c_2, \dots, c_i\}$

Goal: $G[n]$

Make choice: choose c_i or not in $G[i]$?

Recurrence:

$$G[i] = \max\{G[i-1], G[p(i)] + g_i\}$$

$p(i)$: the largest index $j < i$ s.t. c_i and c_j are disjoint

Weighted interval/class scheduling

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Init:

$$G[0] = 0$$

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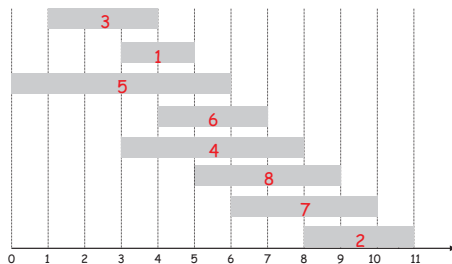
Init:

$$G[0] = 0$$

Time: $O(n \log n) + T(p(i)) + O(n) \cdot O(1)$

Weighted interval/class scheduling

Why is ordering necessary?

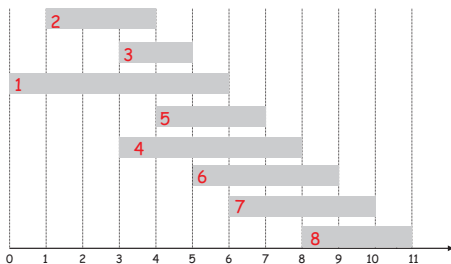


$$G[7] = \max\{G[6], G[\{1, 3, 5\}] + g_7\}$$

subproblems changed: all $O(2^n)$ subsets

Weighted interval/class scheduling

What about sorting by starting time?



$$G[6] = \max\{G[5], G[\{2, 3\}] + g_6\}$$

subproblems changed: all $O(2^n)$ subsets

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Longest common subsequence

LCS: longest common subsequence (Problem 7.5)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

(1) Find (the length of) an LCS of X and Y

$$X = \langle A, B, C, B, D, A, B \rangle$$

$$Y = \langle B, D, C, A, B, A \rangle$$

$$Z = \langle B, C, B, A \rangle$$

Longest common subsequence

Subproblem: $L[i, j]$: the length of an LCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$

Goal: $L[m, n]$

Longest common subsequence

Subproblem: $L[i, j]$: the length of an LCS of $X[1 \dots i]$ and $Y[1 \dots j]$

Goal: $L[m, n]$

Make choice: Is $X_i = Y_j$?

Recurrence: (Proof!)

$$L[i, j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1, j], L[i, j-1]\} & \text{if } X_i \neq Y_j \end{cases}$$

Longest common subsequence

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Init:

$$L[0, j] = 0, \quad 0 \leq j \leq n$$

$$L[i, 0] = 0, \quad 0 \leq i \leq m$$

Time: $\Theta(mn)$

Longest common subsequence

Longest common subsequence (Problem 7.5)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

- (2) Allowing repetition of X
- (3) Allowing repetition $\leq k$ of X

Longest common subsequence

Longest common subsequence (Problem 7.5)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

- (2) Allowing repetition of X
- (3) Allowing repetition $\leq k$ of X

$$L[i, j] = \begin{cases} L[\textcolor{red}{i}, j - 1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i - 1, j], L[i, j - 1]\} & \text{if } X_i \neq Y_j \end{cases}$$

Longest common subsequence

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$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

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$$X \implies X^{(k)} \triangleq X_1^{(k)} \cdots X_m^{(k)}$$

Longest common substring

What about longest common *substring*?

Shortest common supersequence

Shortest common supersequence (Problem 7.6)

$$X = X_1 \cdots X_m \quad Y = Y_1 \cdots Y_n$$

- Find (the length of) a shortest common subsequence of X and Y

Shortest common supersequence

Subproblem: $L[i, j]$: the length of an SCS of $X[1 \dots i]$ and $Y[1 \dots j]$

Goal: $L[m, n]$

Shortest common supersequence

Subproblem: $L[i, j]$: the length of an SCS of $X[1 \dots i]$ and $Y[1 \dots j]$

Goal: $L[m, n]$

Make choice: Is $X_i = Y_j$?

Recurrence:

$$L[i, j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1, j] + 1, L[i, j-1] + 1\} & \text{if } X_i \neq Y_j \end{cases}$$

Shortest common supersequence

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$$L[i, j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1, j] + 1, L[i, j-1] + 1\} & \text{if } X_i \neq Y_j \end{cases}$$

Init:

$$L[0, j] = j, \quad 0 \leq j \leq n$$

$$L[i, 0] = i, \quad 0 \leq i \leq m$$

Shortest common supersequence

Subproblem: $L[i, j]$: the length of an SCS of $X[1 \cdots i]$ and $Y[1 \cdots j]$

Goal: $L[m, n]$

Make choice: Is $X_i = Y_j$?

Recurrence:

$$L[i, j] = \begin{cases} L[i-1, j-1] + 1 & \text{if } X_i = Y_j \\ \max\{L[i-1, j] + 1, L[i, j-1] + 1\} & \text{if } X_i \neq Y_j \end{cases}$$

Init:

$$L[0, j] = j, \quad 0 \leq j \leq n$$

$$L[i, 0] = i, \quad 0 \leq i \leq m$$

Remark

$$\max(m, n) \leq L(m, n) \leq m + n$$

Variants of LCS

Variants of LCS (Problem 7.7)

Longest contiguous substring both forward and backward

Longest contiguous substring both forward and backward (Problem 7.8)

- ▶ String $T[1 \cdots n]$
- ▶ Find a longest contiguous substring (LCS) both forward and backward

dynamicprogrammingmanytimes

- ▶ try subproblem $L[i]$: the length of an LCS in $T[1 \cdots i]$
- ▶ try subproblem $L[i, j]$: the length of an LCS in $T[i \cdots j]$

Longest contiguous substring both forward and backward

Subproblem: $L[i, j]$: the length of an LCS starting with T_i and ending with T_j

Goal: $\max_{1 \leq i \leq j \leq n} L[i, j]$

Longest contiguous substring both forward and backward

Subproblem: $L[i, j]$: the length of an LCS starting with T_i and ending with T_j

Goal: $\max_{1 \leq i \leq j \leq n} L[i, j]$

Make choice: Is $T_i = T_j$?

Recurrence:

$$L[i, j] = \begin{cases} 0 & \text{if } T_i \neq T_j \\ L[i + 1, j - 1] + 1 & \text{if } T_i = T_j \end{cases}$$

Longest contiguous substring both forward and backward

Subproblem: $L[i, j]$: the length of an LCS **starting with T_i and ending with T_j**

Goal: $\max_{1 \leq i \leq j \leq n} L[i, j]$

Make choice: Is $T_i = T_j$?

Recurrence:

$$L[i, j] = \begin{cases} 0 & \text{if } T_i \neq T_j \\ L[i + 1, j - 1] + 1 & \text{if } T_i = T_j \end{cases}$$

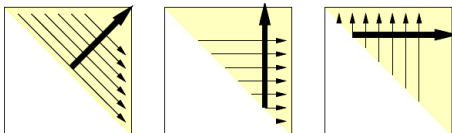
Init:

$$L[i, i] = 0, 0 \leq i \leq n$$

$$L[i, i + 1] = \begin{cases} 1 & \text{if } T_i = T_{i+1} \\ 0 & \text{if } T_i \neq T_{i+1} \end{cases}$$

Longest contiguous substring both forward and backward

Code: three ways of filling the table



```

for d = 2 to n-1
  for i = 1 to n-d
    j = i + d
    ...
return max_{1 <= i <= j <= n} L[i,j]

```

Longest palindrome subsequence

Longest palindrome subsequence (Problem 7.10)

(1) Find (the length of) a longest palindrome subsequence of $S[1 \cdots n]$

Subproblem: $L[i, j]$: the length of an LSP of $S[i \cdots j]$

Goal: $L[1, n]$

Longest palindrome subsequence

Longest palindrome subsequence (Problem 7.10)

(1) Find (the length of) a longest palindrome subsequence of $S[1 \cdots n]$

Subproblem: $L[i, j]$: the length of an LSP of $S[i \cdots j]$

Goal: $L[1, n]$

Make choice: Is $S[i] = S[j]$?

Recurrence:

$$L[i, j] = \begin{cases} L[i + 1, j - 1] + 2 & \text{if } S[i] = S[j] \\ \max L[i + 1, j], L[i, j - 1] & \text{if } S[i] \neq S[j] \end{cases}$$

Longest palindrome subsequence

Longest palindrome subsequence (Problem 7.10)

(1) Find (the length of) a longest palindrome subsequence of $S[1 \cdots n]$

Subproblem: $L[i, j]$: the length of an LSP of $S[i \cdots j]$

Goal: $L[1, n]$

Make choice: Is $S[i] = S[j]$?

Recurrence:

$$L[i, j] = \begin{cases} L[i + 1, j - 1] + 2 & \text{if } S[i] = S[j] \\ \max L[i + 1, j], L[i, j - 1] & \text{if } S[i] \neq S[j] \end{cases}$$

Init:

$$L[i, i] = 1, \forall 1 \leq i \leq n$$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes (# cuts)

Subproblem: $C[i, j]$: minimum number of cuts for string $S[i \dots j]$

Goal: $C[1, n] + 1$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes (# cuts)

Subproblem: $C[i, j]$: minimum number of cuts for string $S[i \dots j]$

Goal: $C[1, n] + 1$

Make choice: Where is the first cut?

Recurrence:

$$C[i, j] = \begin{cases} 0 & \text{if } S[i \dots j] \text{ is a palindrome} \\ \min_{i+1 \leq k \leq j-1} C[i, k] + 1 + C[k+1, j] & \text{o.w.} \end{cases}$$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes (# cuts)

Subproblem: $C[i, j]$: minimum number of cuts for string $S[i \dots j]$

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Init: $C[i, i] = 0$

Time: $O(n^3)$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes

Subproblem: $P[i]$: minimum number of palindromes for $S[1 \dots i]$

Goal: $P[n]$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes

Subproblem: $P[i]$: minimum number of palindromes for $S[1 \dots i]$

Goal: $P[n]$

Make choice: Where does the last palindrome start from?

Recurrence:

$$P[i] = \min_{\substack{1 \leq k \leq i \\ S[k \dots i] \text{ is a palindrome}}} P[k-1] + 1$$

Palindrome splitting

Palindrome splitting (Problem 7.10)

(2) Split a string $S[1 \dots n]$ into minimum number of palindromes

Subproblem: $P[i]$: minimum number of palindromes for $S[1 \dots i]$

Goal: $P[n]$

Make choice: Where does the last palindrome start from?

Recurrence:

$$P[i] = \min_{\substack{1 \leq k \leq i \\ S[k \dots i] \text{ is a palindrome}}} P[k-1] + 1$$

Init: $P[0] = 1$

Time: $O(n^2)$

String splitting

String splitting (Problem 7.11)

- ▶ Split a string S into many pieces
- ▶ Cost $|S| = n \implies n$
- ▶ Given locations of m cuts: $C_0, C_1, \dots, C_m, C_{m+1}$
- ▶ Find the minimum cost of splitting S into $m + 1$ pieces $S_0 \cdots S_m$

String splitting

Subproblem: $C[i, j]$: the minimum cost of splitting substring $S_i \cdots S_{j-1}$
using cuts $C_{i+1} \cdots C_{j-1}$

Goal: $C[0, m + 1]$

String splitting

Subproblem: $C[i, j]$: the minimum cost of splitting substring $S_i \cdots S_{j-1}$ using cuts $C_{i+1} \cdots C_{j-1}$

Goal: $C[0, m + 1]$

Make choice: What is the first cut in $C_{i+1} \cdots C_{j-1}$?

Recurrence:

$$C[i, j] = \min_{i < k < j} (C[i, k] + C[k, j] + l(S_i \cdots S_{j-1}))$$

String splitting

Subproblem: $C[i, j]$: the minimum cost of splitting substring $S_i \cdots S_{j-1}$ using cuts $C_{i+1} \cdots C_{j-1}$

Goal: $C[0, m + 1]$

Make choice: What is the first cut in $C_{i+1} \cdots C_{j-1}$?

Recurrence:

$$C[i, j] = \min_{i < k < j} (C[i, k] + C[k, j] + l(S_i \cdots S_{j-1}))$$

Init: $C[i, i + 1] = 0$

Subproblem:

Subproblem:

Goal:

Make choice:

Subproblem:

Goal:

Make choice:

Recurrence:

Init:

Time:

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3-D DP

Floyd-Warshall algorithm

(1) DP for Floyd-Warshall algorithm for APSP on **directed** graphs

Subproblem: $\text{dist}[i, j, k]$: the length of the shortest path from i to j via only nodes in $v_1 \cdots v_k$

Goal: $\text{dist}[i, j, n], \forall i, j$

3-D DP

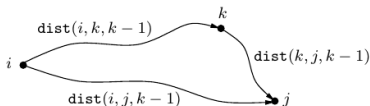
Floyd-Warshall algorithm

(1) DP for Floyd-Warshall algorithm for APSP on **directed** graphs

Make choice: Is v_k on the ShortestPath $[i, j, k]$?

Recurrence:

$$\text{dist}[i, j, k] = \min\{\text{dist}[i, j, k-1], \text{dist}[i, k, k-1] + \text{dist}[k, j, k-1]\}$$



3-D DP

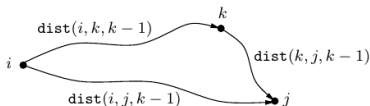
Floyd-Warshall algorithm

(1) DP for Floyd-Warshall algorithm for APSP on **directed** graphs

Make choice: Is v_k on the ShortestPath $[i, j, k]$?

Recurrence:

$$\text{dist}[i, j, k] = \min\{\text{dist}[i, j, k-1], \text{dist}[i, k, k-1] + \text{dist}[k, j, k-1]\}$$



Init:

$$\text{dist}[i, j, 0] = \begin{cases} 0 & i = j \\ w(i, j) & (i, j) \in E \\ \infty & \text{o.w.} \end{cases}$$

3-D DP

Floyd-Warshall algorithm (Problem 6.25)

(2) Routing table for Floyd-Warshall algorithm

```
for all  $k \leftarrow 1 \dots n$  do  
  for all  $i \leftarrow 1 \dots n$  do  
    for all  $j \leftarrow 1 \dots n$  do  
      if  $\text{dist}[i, j] > \text{dist}[i, k] + \text{dist}[k, j]$  then  
         $\text{dist}[i, j] \leftarrow \text{dist}[i, k] + \text{dist}[k, j]$ 
```


3-D DP

Floyd-Warshall algorithm (Problem 6.25)

(2) Routing table for Floyd-Warshall algorithm

```
for all  $k \leftarrow 1 \dots n$  do
  for all  $i \leftarrow 1 \dots n$  do
    for all  $j \leftarrow 1 \dots n$  do
      if  $\text{dist}[i, j] > \text{dist}[i, k] + \text{dist}[k, j]$  then
         $\text{dist}[i, j] \leftarrow \text{dist}[i, k] + \text{dist}[k, j]$ 
         $\text{Go}[i, j] \leftarrow \text{Go}[i, k]$ 
```

3-D DP

Floyd-Warshall algorithm (Problem 6.25)

(2) Routing table

```

for all  $i \leftarrow 1 \dots n$  do
    for all  $j \leftarrow 1 \dots n$  do
         $\text{dist}[i, j] \leftarrow \infty$ 
         $\text{Go}[i, j] \leftarrow \text{Nil}$ 
for all  $(i, j) \in E$  do
     $\text{dist}[i, j] \leftarrow w(i, j)$ 
     $\text{Go}[i, j] \leftarrow j$ 
for all  $i \leftarrow 1 \dots n$  do
     $\text{dist}[i, i] \leftarrow 0$ 
     $\text{Go}[i, i] \leftarrow \text{Nil}$ 
  
```

3-D DP

Floyd-Warshall algorithm (Problem 6.25)

(2) Routing table

```

for all  $i \leftarrow 1 \dots n$  do
    for all  $j \leftarrow 1 \dots n$  do
         $\text{dist}[i, j] \leftarrow \infty$ 
         $\text{Go}[i, j] \leftarrow \text{Nil}$ 
for all  $(i, j) \in E$  do
     $\text{dist}[i, j] \leftarrow w(i, j)$ 
     $\text{Go}[i, j] \leftarrow j$ 
for all  $i \leftarrow 1 \dots n$  do
     $\text{dist}[i, i] \leftarrow 0$ 
     $\text{Go}[i, i] \leftarrow \text{Nil}$ 
  
```

```

procedure  $\text{PATH}(i, j)$ 
    if  $\text{Go}[i, j] = \text{Nil}$  then
        Output "No Path."
  
```

```

    Output " $i$ "
    while  $i \neq j$  do
         $i \leftarrow \text{Go}[i, j]$ 
    Output " $i$ "
  
```

3-D DP

Floyd-Warshall algorithm (Problem 6.29)

(3) Find minimum-weighted cycle of directed graph ($w(e) > 0$)

$$\text{dist}[i, i] \leftarrow 0 \implies \text{dist}[i, i] \leftarrow \infty$$

3-D DP

Floyd-Warshall algorithm (Problem 6.29)

(3) Find minimum-weighted cycle of directed graph ($w(e) > 0$)

$$\text{dist}[i, i] \leftarrow 0 \implies \text{dist}[i, i] \leftarrow \infty$$

$$\exists i : \text{dist}[i, i] < 0$$

$$\forall i : \text{dist}[i, i] \geq 0$$

Shortest paths on undirected graphs

Finding shortest paths in undirected graphs with possibly negative edge weights



2



2

The book "[Algorithms](#)" by Robert Sedgewick and Kevin Wayne hinted that (*see the quote below*) there are efficient algorithms for finding shortest paths in undirected graphs with possibly negative edge weights (**not** by treating an undirected edge as two directed one which means that a single negative edge implies a negative cycle). However, no references are given in the book. Are you aware of any such algorithms?

Q. How can we find shortest paths in undirected (edge-weighted) graphs?

A. For positive edge weights, Dijkstra's algorithm does the job. We just build an `EdgeWeightedDigraph` corresponding to the given `EdgeWeightedGraph` (by adding two directed edges corresponding to each undirected edge, one in each direction) and then run Dijkstra's algorithm. ***If edge weights can be negative (emphasis added)***, efficient algorithms are available, but they are more complicated than the Bellman-Ford algorithm.

algorithms

graph-theory

shortest-path

weighted-graphs

reference-question

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edited Jun 9 at 14:15

asked Jun 9 at 13:58



hengxin

6,056 11 38

<https://cs.stackexchange.com/q/76578/4911>

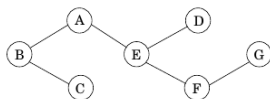
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Minimum vertex cover on trees

Minimum vertex cover on trees [Problem: 2.2.18]

- ▶ Undirected tree $T = (V, E)$; **No designated root!**
- ▶ Compute (the size of) a minimum vertex cover of T



Minimum vertex cover on trees

Rooted T at any node r .

Minimum vertex cover on trees

Rooted T at any node r .

Subproblem: $I(u)$: the size of an MVC of subtree T_u rooted at u

Goal: $I(r)$

Minimum vertex cover on trees

Rooted T at any node r .

Subproblem: $I(u)$: the size of an MVC of subtree T_u rooted at u

Goal: $I(r)$

Make choice: Is u in MVC $[u]$?

Recurrence:

$$I(u) = \min\{\# \text{ children of } u + \sum_{v: \text{ grandchildren of } u} I(v), \\ 1 + \sum_{v: \text{ children of } u} I(v)\}$$

Minimum vertex cover on trees

Rooted T at any node r .

Subproblem: $I(u)$: the size of an MVC of subtree T_u rooted at u

Goal: $I(r)$

Make choice: Is u in MVC $[u]$?

Recurrence:

$$I(u) = \min\{\# \text{ children of } u + \sum_{v: \text{ grandchildren of } u} I(v), \\ 1 + \sum_{v: \text{ children of } u} I(v)\}$$

Init: $I(u) = 0$, if u is a leaf

Minimum vertex cover on trees

DFS on T from root r :

when u is “finished”:

if u is a leaf **then**

$$I(u) \leftarrow 0$$

else

$$I(u) \leftarrow \dots$$

Minimum vertex cover on trees

DFS on T from root r :

when u is “finished”:

if u is a leaf **then**

$$I(u) \leftarrow 0$$

else

$$I(u) \leftarrow \dots$$

Greedy algorithm:

Theorem

There is an MVC which contains no leaves.

DP on DAG

Longest path in DAG (Problem 7.17)

- ▶ Direction: \downarrow OR \rightarrow
- ▶ Score: $\geq < 0$

DP on DAG

Longest path in DAG (Problem 7.17)

- ▶ Direction: \downarrow OR \rightarrow
- ▶ Score: $\geq < 0$

1. digraph G
2. node weight \rightarrow edge weight
3. adding an extra sink s
4. $G \rightarrow G^T$

DP on DAG

Longest path in DAG (Problem 7.17)

- ▶ Direction: \downarrow OR \rightarrow
- ▶ Score: \geq or ≤ 0

1. digraph G
2. node weight \rightarrow edge weight
3. adding an extra sink s
4. $G \rightarrow G^T$

Compute a longest path from s in DAG

DP on DAG

Subproblem: $\text{dist}[v]$: longest distance from s to v

Goal: $\text{dist}[v], \forall v \in V$

DP on DAG

Subproblem: $\text{dist}[v]$: longest distance from s to v

Goal: $\text{dist}[v], \forall v \in V$

Make choice:

Recurrence:

$$\text{dist}[v] = \max_{u \rightarrow v} (\text{dist}[u] + w(u \rightarrow v))$$

DP on DAG

Subproblem: $\text{dist}[v]$: longest distance from s to v

Goal: $\text{dist}[v], \forall v \in V$

Make choice:

Recurrence:

$$\text{dist}[v] = \max_{u \rightarrow v} (\text{dist}[u] + w(u \rightarrow v))$$

Init: $\text{dist}[s] = 0$

DP on DAG

Subproblem: $\text{dist}[v]$: longest distance from s to v

Goal: $\text{dist}[v], \forall v \in V$

Make choice:

Recurrence:

$$\text{dist}[v] = \max_{u \rightarrow v} (\text{dist}[u] + w(u \rightarrow v))$$

Init: $\text{dist}[s] = 0$

Compute $\text{dist}[v]$ in topo. order

Bitonic tour

Bitonic tour (Problem 7.18)

Bitonic tour

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