

Searching and Selection

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April 8 ~ April 11, 2017



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1 Selection

2 Searching

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The 3rd largest element (Problem 3.1)

“ Q_1 : What is the exact value of $V_3(n)$?”

Theorem ($V_3(n)$)

$n \geq 6, n = 2^k + r (0 \leq r < 2^k)$:

$$V_3(n) = \begin{cases} (n-3) + 2k & r = 0, 1 \\ (n-3) + 2k + 1 & 2 \leq r \leq 2^{k-2} + 1 \\ (n-3) + 2k + 2 & o.w. \end{cases}$$

References

“Selecting the Top Three Elements” by Aigner, 1982.

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“YES!”

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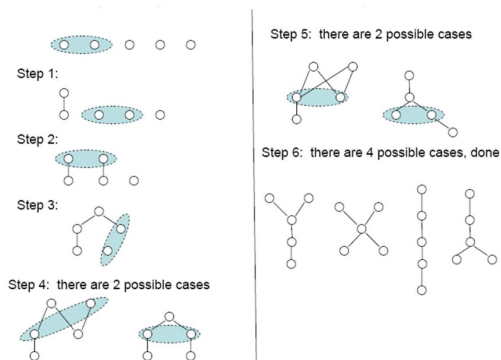
“NO!”

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Selection with minimum #comparisons (Problem 3.2)

Selecting the median of 5 elements using 6 comparisons.



Sorting with minimum #comparisons (Problem 2.4)

Sorting 5 elements using 7 comparisons.

$$S(5) = 7$$

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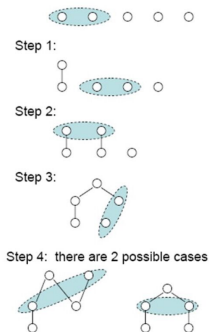
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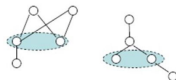
$$S(21) = 66$$

Sorting with minimum #comparisons (Problem 2.4)

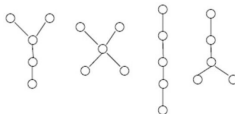
Sorting 5 elements using 7 comparisons.



Step 5: there are 2 possible cases



Step 6: there are 4 possible cases, done



Medians of sorted arrays (Problem 3.7)

<http://cs.stackexchange.com/a/33129/4911>

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max / min differences (Problem 4.5)

- (a) unsorted; $\max |x - y|$; $O(n)$
- (b) sorted; $\max |x - y|$; $O(1)$
- (c) unsorted; $\min |x - y|$; $O(n \log n)$
- (d) sorted; $\min |x - y|$; $O(n)$

Searching in matrix (Problem 4.6)

- ▶ M : matrix $m \times n$
- ▶ row: increasing from left to right
- ▶ col: increasing from top to down
- ▶ Is $x \in M$?

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$$T(m, n) = 3T\left(\frac{m}{2}, \frac{n}{2}\right) + 1$$

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$$T(m, n) = m + n - 1$$

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Assume $M : n \times n$

$$W(n) \leq 2n - 1$$

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$W(n) \geq 2n - 1$ by adversary argument!

$$i + j \leq n - 1 \implies x > M_{ij}$$

$$i + j > n - 1 \implies x < M_{ij}$$

Local minimum (Problem 4.11)

- ▶ Array $A[0 \dots n]$
- ▶ Boundary conditions:

$$\begin{aligned}A[0] &\geq A[1] \\ A[n-2] &\leq A[n-1]\end{aligned}$$

- ▶ Local minimum $A[i]$:

$$A[i-1] \geq A[i] \leq A[i+1]$$

- ▶ Goal: Find *any* local minimum.

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1. Checking each element:

$$T(n) = O(n)$$

2. min A :

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2D local minimum:

- ▶ Matrix $M : n \times n$
- ▶ Boundary conditions:

∞

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$$A[i, j - 1] \geq A[i, j] \leq A[i, j + 1]$$

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$a_i = i$ (Problem 4.2)

- ▶ Sorted integer sequence $\{a_1, a_2, \dots, a_n\}$:

$$\forall i \neq j : a_i \neq a_j$$

- ▶ Goal:

$$\exists i : a_i = i$$

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Smallest missing positive integer (Problem 4.3)

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Computing $\lceil \sqrt{N} \rceil$ (Problem 4.12)

- ▶ Given an n -bit natural number N ($0 \leq N < 2^n - 1$)
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Elementary operations:

- ▶ n -bit $+$ n -bit: $O(1)$
- ▶ n -bit shifted by 1-bit: $O(1)$
- ▶ $\Rightarrow x^2 : O(n)$

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2. Binary search: $O(n \cdot n)$

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1. Naïve search: $O(2^n \cdot n)$
2. Binary search: $O(n \cdot n)$
3. Binary search in range:

$$2^{\lfloor \frac{n-1}{2} \rfloor} \leq \lceil \sqrt{N} \rceil \leq 2^{\lceil \frac{n}{2} \rceil}$$

$$\lg(2^{\lceil \frac{n}{2} \rceil} - 2^{\lfloor \frac{n-1}{2} \rfloor}) = O(n)$$

$$O(n \cdot n)$$

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A Little History:

2007: Mid-term problem

$O(n)$ required; NO $O(n)$ solutions, however

~ 2013: $O(n^2)$

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Compute square root using (bit) additions and shifts as primitives



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Question: Given an n -bit natural number N , how to compute $\lceil \sqrt{N} \rceil$ using only $O(n)$ (bit) additions and shifts?

The tip is to use binary search. However, I could not achieve the required complexity (I got $O(n^2)$).

asked 2 years ago

viewed 1039 times

active 2 years ago

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$$x^2 : O(n) \rightarrow O(1)$$

Computing $\lceil \sqrt{N} \rceil$ (Problem 4.12)

Given

$$M = \lfloor N/4 \rfloor$$

$$x = \lceil \sqrt{M} \rceil \text{ and } (x, x^2),$$

what is

$$y = \lceil \sqrt{N} \rceil \text{ and } (y, y^2)?$$

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An Example:

$$N = 280$$

$$y = \lceil \sqrt{280} \rceil = 17 \quad y^2 = 289$$

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An Example:

$$\begin{array}{lll} N = 280 & y = \lceil \sqrt{280} \rceil = 17 & y^2 = 289 \\ M = \lfloor 280/4 \rfloor = 70 & x = \lceil \sqrt{70} \rceil = 9 & x^2 = 81 \end{array}$$

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An Example:

$N = 280$	$y = \lceil \sqrt{280} \rceil = 17$	$y^2 = 289$
$M = \lfloor 280/4 \rfloor = 70$	$x = \lceil \sqrt{70} \rceil = 9$	$x^2 = 81$
$M = \lfloor 70/4 \rfloor = 17$	$x = \lceil \sqrt{17} \rceil = 5$	$x^2 = 25$
$M = \lfloor 17/4 \rfloor = 4$	$x = \lceil \sqrt{4} \rceil = 2$	$x^2 = 4$
$M = \lfloor 4/4 \rfloor = 1$	$x = \lceil \sqrt{1} \rceil = 1$	$x^2 = 1$

Computing $\lceil \sqrt{N} \rceil$ (Problem 4.12)

Algorithm 1 Computing $\lceil \sqrt{N} \rceil$.

procedure SQRT-ROOT(N)

if $N < 3$ **then**

return $1 \Rightarrow (1, 1); 2 \Rightarrow (2, 4); 3 \Rightarrow (2, 4)$

$M \leftarrow \lfloor N/4 \rfloor$

$(x, x^2) \leftarrow \text{SQRT-ROOT}(M)$

return the (y, y^2) with $y^2 \sim N$:

$$(y, y^2) = \begin{cases} y = 2x & y^2 = 4x^2 \\ y = 2x + 1 & y^2 = 4x^2 + 4x + 1 \\ y = 2x - 1 & y^2 = 4x^2 - 4x + 1 \end{cases}$$

Computing $\lceil \sqrt{N} \rceil$

$$T(n) = T(n/4) + O(1) = \Theta(\lg n)$$

$$T(n) = T(n-2) + O(1) = \Theta(n)$$

Space for hashing (Problem 4.4)

Key: x

Node: y

Load factor: α

Space for hashing (Problem 4.4)

Key: x

Node: y

Load factor: α

- ▶ Closed-address hashing

$$h_c + \alpha y h_c$$

- ▶ Open-address hashing

$$\frac{\alpha h_c}{\frac{h_c + \alpha y h_c}{x}} = \frac{\alpha x}{1 + \alpha y}$$

