What You Should Know About Algorithm Design and Analysis . . . But (Probably) Don't

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When You Design Algorithms:

Design Faster Algorithms

When to Stop?

The Complexity of Problems

Problem P Algorithm A

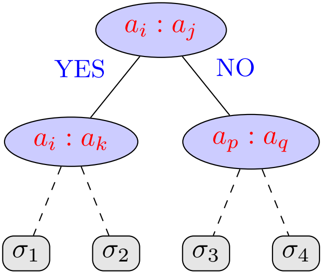
Inputs: $\mathcal{X}_n$ of size n

$$W_A(n) = \max_{x \in \mathcal{X}_n} T_A(x)$$

$$B_A(n) = \min_{x \in \mathcal{X}_n} T_A(x)$$

$$A_A(n) = \sum_{x \in \mathcal{X}_n} T_A(x) \cdot P(x) = \mathbb{E}[T_A] = \sum_{t \in T_A(\mathcal{X}_n)} t \cdot P(T = t)$$

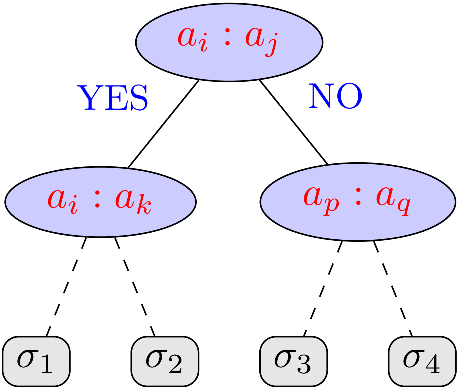
$$\boxed{T_P(n) = \min_{A \text{ solves } P} W_A(n) = \min_{A \text{ solves } P} \max_{x \in \mathcal{X}_n} T_A(x)}$$



Decision Tree

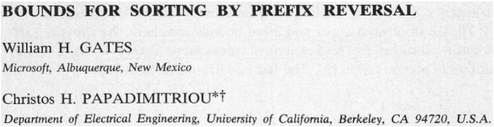
Adversary Argument

Decision Tree



Lower Bound for Comparison-based Sorting

Prove a lower bound of $\Omega(n \log n)$ on the time complexity of any **comparison-based** sorting algorithm on inputs of size n.


BOUNDS FOR SORTING BY PREFIX REVERSAL

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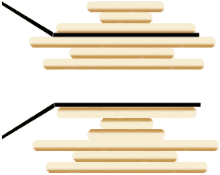
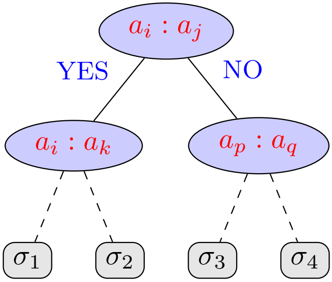


Decision Tree Model



Nodes: comparisions $a_i : a_j$

$$<, \leq, =, \geq, >$$

Edges: two-way decisions (Y/N)

Leaves: possible permutations

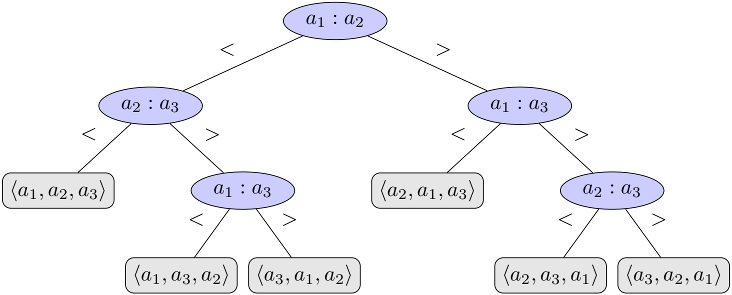
Assumption:

All the input elements are **distinct**.

$$a_i < a_j$$

$$\boxed{\text{Any Comparison-based Sorting Algorithm} \xRightarrow{\text{modeled by}} \text{A Decision Tree}}$$

Any Comparison-based Sorting Algorithm $\xRightarrow{\text{modeled by}}$ A Decision Tree

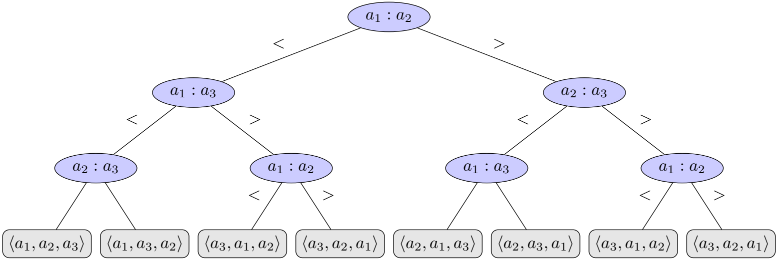


The decision tree for insertion sort on three elements.

Any Comparison-based Sorting Algorithm $\xRightarrow{\text{modeled by}}$ A Decision Tree

procedure SELECTION-SORT(A, n)
    for $i \leftarrow 1$ **to** $n - 1$ **do**
        for $j \leftarrow i + 1$ **to** n **do**
            if $A[j] < A[i]$ **then**
                SWAP$(A[j], A[i])$

Any Comparison-based Sorting Algorithm $\xRightarrow{\text{modeled by}}$ A Decision Tree



The decision tree for selection sort on three elements.

$$\boxed{\text{Any Comparison-based Sorting Algorithm } \mathcal{A} \xRightarrow{\text{modeled by}} \text{A Decision Tree } \mathcal{T}}$$

$$\text{Algorithm } \mathcal{A} \text{ on a specific input of size } n \xRightarrow{\text{modeled by}} \text{A path through } \mathcal{T}$$

$$\text{Worst-case time complexity of } \mathcal{A} \xRightarrow{\text{modeled by}} \text{The height of } \mathcal{T}$$

$$\boxed{\begin{array}{c}\text{Worst-case Lower Bound of Comparison-based Sorting} \\ \text{(on inputs of size } n\text{)} \\ \xRightarrow{\text{modeled by}} \\ \text{The Minimum Height of All } \mathcal{T}\text{s}\end{array}}$$

Worst-case Lower Bound of Comparison-based Sorting
(on inputs of size n)

$$\xRightarrow{\text{modeled by}}$$

The Minimum Height of All $\mathcal{T}$s

To be a correct sorting algorithm:

$$L = \#\text{ of leaves} \geq n!$$

To be a full binary tree:

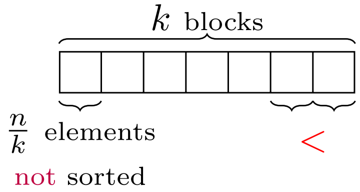
$$L = \#\text{ of leaves} \leq 2^h$$

$$\boxed{n! \leq L = \#\text{ of leaves} \leq 2^h}$$

$$h \geq \log n! = \Omega(n \log n)$$

K-sorted Array (Problem 6.8)

$$A[1 \cdots n]$$



$$O(n \log k)$$

$$n = 16, \quad k = 4, \quad \frac{n}{k} = 4$$

$$1,\ 2,\ 4,\ 3; \quad 7,\ 6,\ 8,\ 5; \quad 10,\ 11,\ 9,\ 12; \quad 15,\ 13,\ 16,\ 14$$

k-sorted

$$1\text{-sorted} \to 2\text{-sorted} \to 4\text{-sorted} \to \cdots \to n\text{-sorted}$$

Quicksort (with median as pivot) stops after the $\log k$ recursions.

$$\boxed{\Theta(n \log k)}$$

$$\Omega(n \log k)$$

$$L \geq \binom{n}{n/k}\binom{n - n/k}{n/k} \cdots \binom{n/k}{n/k} = \binom{n}{n/k, \ldots, n/k} = \frac{n!}{\left((\frac{n}{k})!\right)^k}$$

$$H \geq \log\left(\frac{n!}{\left((\frac{n}{k})!\right)^k}\right) = \Omega(n \log k)$$

$$\boxed{n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \implies \log n! \sim n \log n}$$

Bolts and Nuts (Problem 6.9)

Quicksort

$$A(n) = O(n \log n)$$

In the worst case:

- "Matching Nuts and Bolts" by Alon *et al.*, $\Theta(n \log^4 n)$
- "Matching Nuts and Bolts Optimality" by Bradford, 1995, $\Theta(n \log n)$

$$\Omega(n \log n)$$

$$3^H \geq L \geq n! \implies H \geq \log n! \implies H = \Omega(n \log n)$$

Repeated Elements (Problem 6.13)

$$R[1 \ldots n]$$

$$\# > \lfloor \frac{n}{13} \rfloor$$

$$\text{To find all } \frac{n}{13}\text{-repeated elements}$$

$$\textsc{Check}(R[i], R[j])$$

$$\# > \lfloor \frac{n}{k} \rfloor$$

Whether there are any elements that occur $> \lfloor \frac{n}{k} \rfloor$ times

$$\Omega(n \log k)$$

Adversary Argument

Searching in Matrix (Problem 9.8)

$$M : m \times n$$

Row: Increasing from left to right

Col: Increasing from top to down

$$x \in M?$$

$$\textsc{Compare}(x, M[i][j])$$

$$\text{Divide \& Conquer} : T(m,n) = 3T(\frac{m}{2}, \frac{n}{2}) + 1$$

Always checking the lower left corner.

$$T(m,n) = m + n - 1$$

Assume $M : n \times n$

$$W(n) \leq 2n - 1$$

$$\boxed{W(n) \geq 2n - 1}$$

By Adversary Argument!

$$\boxed{W(n) \geq 2n-1}$$

Adversary $\mathcal{A}$:

$x > M[i][j]$

$x = M[i][j]$

$x < M[i][j]$

Algorithm $\mathbb{A}$:

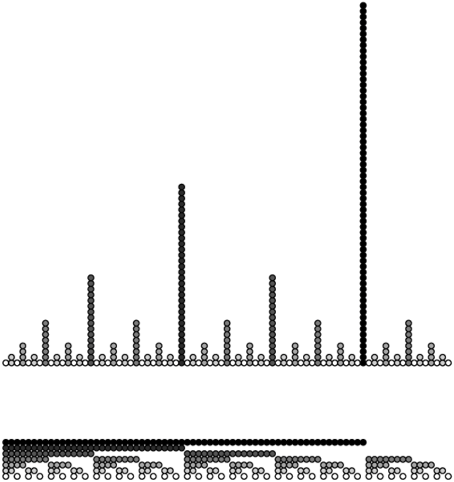
$\textsc{Compare}(x, M[i][j])$

Diagonals: $i + j = n - 1 \quad \& \quad i + j = n$

No particular ordering requirements on these two diagonals!

$$i + j \leq n - 1 \implies x > M_{ij}$$
$$i + j > n - 1 \implies x < M_{ij}$$

Amortized Analysis



Amortized analysis is

an algorithm analysis technique for

analyzing a sequence of operations

irrespective of the input to show that

the average cost per operation is small, even though

a single operation within the sequence might be expensive.

Accounting Method

Summation Method

Potential Method

Amortized Analysis

The Summation Method

$$\Sigma$$

$$o_1, o_2, \ldots, o_n$$

$$c_1, c_2, \ldots, c_n$$

$$\boxed{\forall i,\ \hat{c}_i = \frac{\left(\sum_{i=1}^{n} c_i\right)}{n}}$$

The Summation Method for Array Doubling

On any sequence of n INSERTs on an initially empty array.

o_i :	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9	o_{10}
c_i :	1	2	3	1	5	1	1	1	9	1

$$c_i = \begin{cases} (i-1)+1 = i & \text{if } i-1 \text{ is an exact power of 2} \\ 1 & \text{o.w.} \end{cases}$$

$$\sum_{i=1}^{n} c_i \le n + \sum_{j=0}^{\lceil \lg n \rceil - 1} 2^j = n + (2^{\lceil \lg n \rceil} - 1) \le n + 2n = 3n$$

$$\boxed{\forall i,\ \hat{c}_i = 3}$$

The Accounting Method

$$o_1, o_2, \ldots, o_n$$

$$c_1, c_2, \ldots, c_n$$

$$a_1, a_2, \ldots, a_n$$

$$\boxed{\hat{c}_i = c_i + a_i,\ a_i >=< 0}$$

Amortized Cost = Actual Cost + Accounting Cost

$$\forall n,\ \sum_{i=1}^{n} c_i \le \sum_{i=1}^{n} \hat{c}_i \implies \boxed{\forall n,\ \sum_{i=1}^{n} a_i \ge 0}$$

Key Point: Put the accounting cost on specific objects.

The Accounting Method for Array Doubling

$$Q : \hat{c}_i = 3 \ vs. \ \hat{c}_i = 2$$

$$\hat{c}_i = 3 = \underbrace{1}_{\text{insert}} + \underbrace{1}_{\text{move itself}} + \underbrace{1}_{\text{help move another}}$$

	$\hat{c}_i$	c_i *(actual cost)*	a_i *(accounting cost)*
INSERT *(normal)*	3	1	2
INSERT *(expansion)*	3	$1+t$	$-t+2$

Simulating a queue Q using two stacks S_1, S_2 (Problem $\mathbb{E}$3)

procedure $\textsc{Enq}(x)$
    $Push(S_1, x)$

procedure $\textsc{Deq}()$
    if $S_2 = \emptyset$ **then**
        while $S_1 \neq \emptyset$ **do**
            $Push(S_2,\ Pop(S_1))$
    $Pop(S_2)$

The Summation Method for Queue Simulation

$$\frac{\left(\sum_{i=1}^{n} c_i\right)}{n}$$

The operation sequence is *NOT* known.

The Accounting Method for Queue Simulation

item:	Push into S_1	Pop from S_1	Push into S_2	Pop from S_2
	1	1	1	1

$$\hat{c}_{\text{ENQ}} = 3$$
$$\hat{c}_{\text{DEQ}} = 1$$

$$\sum_{i=1}^{n} a_i \geq 0 \Longleftarrow \sum_{i=1}^{n} a_i = \#S_1 \times 2$$

The Accounting Method for Queue Simulation

$$\hat{c}_{\text{ENQ}} = 3$$
$$\hat{c}_{\text{DEQ}} = 1$$

$$\#S_1 = t$$

	$\hat{c}_i$	c_i *(actual cost)*	a_i *(accounting cost)*
Enqueue	3	1	2
Dequeue ($S_2 \neq \emptyset$)	1	1	0
Dequeue ($S_2 = \emptyset$)	1	$1 + 2t$	$-2t$

Array Merging Dictionary (Problem $\mathbb{E}$ 2)

i	$s_i = 2^i$
A_0	1
A_1	2
A_2	4
A_3	8
$\vdots$	$\ldots$
A_i	2^i

$$11 = 2^0 + 2^1 + 2^3$$

i	e_i
A_0	$[5]$
A_1	$[4, 8]$
A_2	$[\,]$
A_3	$[2, 6, 9, 12, 13, 16, 20, 25]$

$\textsc{Create} : 1 \quad \textsc{Merge}(A_i, A_i) : 2 \cdot 2^i$

$\textsc{Insert}() : 1 + 2 + 4; \quad \textsc{Insert}() : 1; \quad \textsc{Insert}() : 1 + 2$

The Summation Method for "Array Merging Dictionary"

$$\textsc{Create}: 1 \quad \textsc{Merge}(A_i, A_i): 2 \cdot 2^i$$

i	c_i
1	1
2	$1+2$
3	1
4	$1+2+4$
5	1
6	$1+2$
7	1
8	$1+2+4+8$
$\vdots$	$\ldots$

$$\sum_{i=1}^{n} c_i = \sum_{j=0}^{\lfloor \log n \rfloor} \lfloor \frac{n}{2^j} \rfloor 2^j \le n(\lfloor \log n \rfloor + 1)$$

$$\forall i,\ \hat{c}_i = 1 + \lfloor \log n \rfloor$$

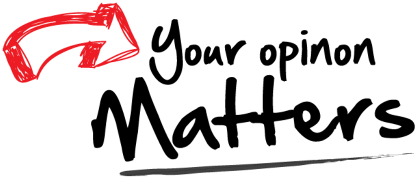
The Accounting Method for "Array Merging Dictionary"

$$\textsc{Create} : 1 \quad \textsc{Merge}(A_i, A_i) : 2 \cdot 2^i$$

$$\hat{c}_i = 1 + \lfloor \log n \rfloor$$

$$\forall n, \ \sum_{i=1}^{n} a_i \geq 0$$

Thank You!



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