MST and Shortest Paths

Hengfeng Wei

hengxin0912@gmail.com

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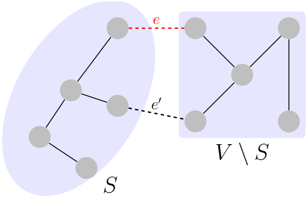
MST and Shortest Paths

Cut property of MST

Cut Property [Problem: 3.6.18 (a)]

- Graph $G = (V, E)$ (undirected, connected, weighted)
- Weights are distinct
- A cut $(S, V \setminus S)$ where $S, V - S \neq \emptyset$
- Let $e = (u, v)$ be *a* minimum-weight edge across $(S, V \setminus S)$

Then e must be in *some* MST of G.



Cut property of MST

Cut Property [Problem: 3.6.18 (a)]

Proof.

Basic idea: T is an MST of G.

- $e \in T$
- $e \notin T \Rightarrow e \in T'$
  - $T + \{e\}$ to construct a cycle C
  - $\exists e' = (u', v') \in C$ ($e' \in P_{u,v}$), e' crosses $(S, V \setminus S)$
  - $T' = T + \{e\} - \{e'\}$: spanning tree (connected, acyclic)
  - $w(e') \geq w(e) \Rightarrow w(T') \leq w(T) \Rightarrow w(T') = w(T)$

$\square$

Remark.

- a minimum-weight edge; $\in$ some MST
- exchange argument

Cut property of MST

Application of cycle property [Problem: 3.6.15]

- [Problem: 3.6.15 (3)]: $e \in G$ is a lightest edge $\Rightarrow e \in \exists$ MST of G
- [Problem: 3.6.15 (4)]: $e \in G$ is the unique lightest edge $\Rightarrow e \in \forall$ MST
- [Problem: 3.6.15 (9)]: $e = (u, v) \in \exists$ MST T of $G \Rightarrow e$ is a lightest edge across some cut $(S, V \setminus S)$ *(converse of cut property)*

Solution.

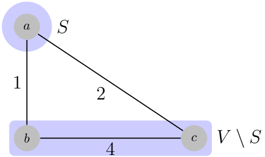
- [Problem: 3.6.15 (3)]: $(S = \{u\}, V \setminus S)$
- [Problem: 3.6.15 (4)]: By contradiction. $e \notin T$; $T' = T + \{e\} - \{e'\} \Rightarrow w(T') < w(T)$
- [Problem: 3.6.15 (9)]:
  1. to find the cut $(S, V \setminus S)$
     - $T - \{e\}$
  2. to prove that e is a lightest edge across $(S, V \setminus S)$
     - by contradiction: $T' = T - \{e\} + \{e'\}$

Cut property of MST

Wrong divide-and-conquer algorithm for MST [Problem: 3.6.29]

- $G = (V, E, w)$
- $(V_1, V_2) : ||V_1| - |V_2|| \leq 1$
- $T_1 + T_2 + \{e\}$: e is a lightest edge across (V_1, V_2)

Solution.

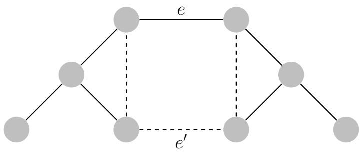


Cycle property of MST

Cycle property [Problem: 3.6.18 (b)]

- $G = (V, E, w)$
- Let C be any cycle in G
- $e = (u, v)$ is *a* maximum-weight edge in C

Then $\exists$ MST T of $G : e \notin T$.



Cycle property of MST

Cycle property [Problem: 3.6.18 (b)]

Proof.

Basic idea: pick any MST T of G

- $e \notin T$
- $e \in T \Rightarrow e \notin T'$
  - $T - \{e\} \Rightarrow (S, V \setminus S)$
  - $\exists e' = (u', v') \in C \; (e' \in P_{u,v})$ across the cut
  - $T' = T - \{e\} + \{e'\}$: spanning tree
  - $w(e') \leq w(e) \Rightarrow w(T') \leq w(T) \Rightarrow w(T') = w(T)$

$\square$

Remark.

- Why don't we pick any $e' \in C$?
- "Anti-Kruskal" (reverse-delete; also by Kruskal) [Problem: 3.6.20 (c)]

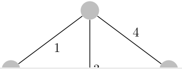
Applications of cycle property

Applications of cycle property [Problem: 3.6.15]

- [Problem: 3.6.15 (2)]: $C \subseteq G, e \in C$, e is the unique maximum-weighted edge of $G \Rightarrow e \notin$ any MST of G
- [Problem: 3.6.15 (5); 3.6.18 (c)]: $C \subseteq G, e \in C$, e is the unique lightest edge of $C \Rightarrow^{?} e \in \forall$ MST
- [Problem: 3.6.15 (1)]: $G = (V, E), |E| > |V| - 1$, e unique maximum-weighted edge $\Rightarrow^{?} e \notin$ any MST
- [Problem: 3.6.20 (a)]: e does not belong to any cycle $\Rightarrow e \in \forall$ MST

Solution.

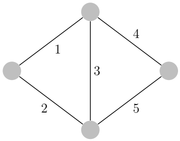
- [Problem: 3.6.15 (2)]: By contradiction. $T' = T - \{e\} + \{e'\}$
- [Problem: 3.6.15 (5); 3.6.18 (c)]



Properties of MST

✓ or ✗ [Problem: 3.6.15]

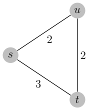
1. ✗ $|E| > |V| - 1$, e is the unique maximum edge $\Rightarrow$ e does not belong to any MST.
2. ✓ If G has a cycle with a unique maximum edge e, then e cannot be part of any MST. (Prove: Cycle property)
3. ✓ Let e be any edge of minimum edge in G. Then e belongs to some MST. (Prove: Cut property)
4. ✓ If the minimum edge is unique, then it belongs to every MST.
5. ✗ If G has a cycle with a unique minimum edge e, then e belongs to every MST.



Properties of MST

✓or ✗[Problem: 3.6.15]

6. ✗The shortest-path tree computed by Dijkstra's algorithm is necessarily an MST.
7. ✗The shortest path between two nodes is necessarily part of some MST.



8. ✓Prim's algorithm works correctly when there are negative edges.
9. ✓If e belongs to some MST, then e is a minimum edge across some cut.
10. ✓$w > 0$; Vertex s; shortest-path tree of s and some MST share a common edge [Problem: 6.1.5]
11. ✓$w'(e) = (w(e))^2$ [Problem: 6.2.2]

Uniqueness of MST

Uniqueness of MST [Problem: 3.6.21]

Distinct weights $\Rightarrow$ unique MST.

Solution.

Proof.

By contradiction: two MSTs $T_1 \neq T_2$.

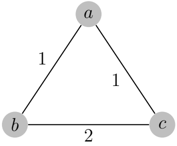
- $\Delta E = \{e \mid e \in T_1 \setminus T_2 \vee e \in T_2 \setminus T_1\}$
- $e = \min \Delta E$. Suppose $e \in T_1 \setminus T_2$
- $T_2 + \{e\} \Rightarrow C$
- $\exists (e' \in C) \notin T_1 \Rightarrow e' \in T_2 \setminus T_1 \Rightarrow e' \in \Delta E \Rightarrow w(e') > w(e)$
- $e' \in T_2 \setminus T_1 \Rightarrow e' \in \Delta E$
- $T' = T_2 + \{e\} - \{e'\} \Rightarrow w(T') < w(T_2)$

□

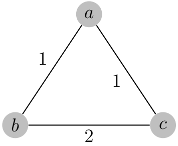
Uniqueness of MST

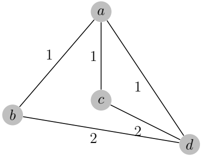
Conditions for Uniqueness of MST [Problem: 3.6.19]

- [Problem: 3.6.19 (a)]: unique MST $\not\Rightarrow$ equal weights



- [Problem: 3.6.19 (c)]: Counterexamples
  - ✗cut: minimum-weight edge across any cut is unique
  - ✗cycle: maximum-weight edge in any cycle is unique





Updating MST

Decreasing/increasing edge weight [Problem: 3.6.6]

G and an MST T

1. $w(e)$ is decreased: $w'(e) = w(e) - k$
2. $w(e)$ is increased

Solution for (1).

- $e \in T$: no need to update $T' = T$.
  $w'(T') = w(T) - k \Rightarrow w'(T') < w(T)$.
  To prove that T' is an MST of G':
  Suppose $\exists T'' : T''$ is an ST of G' and $w'(T'') < w'(T')$.
  - $e \notin T''$: $w(T'') = w'(T'') < w'(T') < w(T)$
  - $e \in T''$: $w(T'') = w'(T'') + k < w'(T') + k = w(T)$
- $e \notin T$: $T' = T + \{e\} - \{e'\}$; e' is the maximum-weight edge in cycle and $w(e') > w(e)$
  - $e \notin T''$: $w(T'') = w'(T'') < w'(T') < w(T)$

Updating MST

Adding vertex to MST [Problem: 3.6.2]

- $G = (V, E)$; an MST T
- $G' = (V', E')$: $V' = V + \{X\}, E' = E + E_X$; E_X: incident edges to X
- To find an MST T' of G'

Solution.

1. Recomputing $O((m+n)\log n)$
2.
   - There *exists* an MST of G' that includes no edges in $G \setminus T$
   - Run MST alg. on $G'' = (V + \{X\}, T + E_X)$
   - $O(n \log n)$
3. $O(n)$
   - "On Finding and Updating Spanning Tress and Shortest Paths", 1975
   - "Algorithms for Updating Minimum Spanning Trees", 1978

Variants of MST

Feedback edge set: [Problem: 3.6.4]

1. maximum spanning tree
2. (minimum) feedback edge set:
   - a set of edges which, when removed from the graph, leave an acyclic graph G'
   - assuming G is connected $\Rightarrow$ G' is connected
   - feedback *arc* set: "cycle" $\Rightarrow$ circular dependency

Solution.

- G' is connected + acyclic $\Rightarrow$ G' is an ST
- $\text{FES} \Leftrightarrow G \setminus \text{Max-ST}$

Variants of MST

Edge weights [Problem: 3.6.15 (8); 3.6.16]

- [Problem: 3.6.15 (8)]: negative edges for Prim algorithm
- [Problem: 3.6.16]: $w'(e) = (w(e))^2$

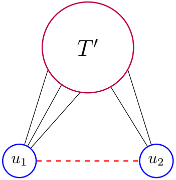
Variants of MST

MST with specified leaves: [Problem: 3.6.7]

- $G = (V, E), U \subset V$
- finding an MST with U as leaves

Solution.

- $G' = G \setminus U$
- MST T' of G'
- attach $\forall u \in U$ to T' (lightest edge)



Variants of MST

ST with specified edges: [Problem: 3.6.10]

- $G = (V, E), S \subset E$ (no cycle in S)
- finding an MST with E as edges

Solution.

- contract each isolated component of S to a *super-vertex*
- $G \to G'$
- find MST of G'

MST vs. shortest paths

[Problem: 3.6.15]

- [Problem: 3.6.15 (6)]: Dijkstra $\Rightarrow$ SSSP tree $\Rightarrow^{?}$ MST
- [Problem: 3.6.15 (7)]: $s \to t$ shortest path $\Rightarrow^{?} \subseteq \exists$ MST

MST vs. shortest paths

Sharing edges [Problem: 3.6.5]

- $G = (V, E), w(e) > 0$
- Given s: all sssp trees from s must share some edge with all (some) MSTs of G

Solution

E': lightest edges leaving s

- any MST T of G: $T \cap E' \neq \emptyset$
- $E' \subset \forall$ sssp trees

MST and Shortest Paths

1. Minimum Spanning Tree

2. Shortest Paths
   - Dijkstra's algorithm for SSSP
   - Cycles

Dijkstra's algorithm

- maintain $R \subseteq V$: $\forall u \in R : s \rightsquigarrow u$ is known
- choose the next v and (u, v):

$$\min_{u \in R} \mathsf{dist}(s, u) + w(u, v)$$

- key points for the correctness proof
  1. $u_1 \to v$
  2. $u_1 \to x \rightsquigarrow v$

Dijkstra' algorithm

Negative edges [Problem: 3.7.9]

Dijkstra's algorithm on graphs with negative edges

Dijkstra' algorithm

Negative edges leaving s [Problem: 3.7.17]

- digraph $G = (V, E, w)$
- all negative edges are from s

Solution.

Dijkstra's algorithm works.

Dijkstra's algorithm

Uniqueness of shortest path [Problem: 3.7.7]

Dijkstra's algorithm

Uniqueness of shortest path [Problem: 3.7.16]

- digraph $G = (V, E, w)$
- $w(e) \geq 0$
- $S \cap T = \emptyset$
- to compute $\forall s \in S, \forall t \in T, s \rightsquigarrow t$ shortest paths

Solution.

- adding s_0
- $s_0 \to s \in S$
- $w(s_0 \to s) = 0$

Cycles

4-Cycle in undirected graph [Problem: 3.7.1]

Cycles

Shortest cycle in digraph [Problem: 3.7.4]

Solution.

Floyd-Warshall: $W^{(0)}[i][i] = \infty$

Cycles

Shortest cycle in undirected graph [Problem: 3.7.14]

Solution.

Cycles

Shortest cycle containing a specific edge [Problem: 3.7.5]

- undirected edge $G = (V, E)$

Solution.

$P_{u,v} + (u, v)$

Cycles

Hamiltonian path in tournament graph [Problem: 3.7.18]

- digraph $G = (V, E)$
- $\forall u, v : (u \to v \vee v \to u) \wedge \neg(u \to v \wedge v \to u)$
- hamiltonial path

Solution.

- existence
- algorithm $O(n^2)$


THANK YOU