Decompositions of Graphs

— DFS/BFS, Cycle, DAG, Toposort, SCC, Bicomp

Hengfeng Wei

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June 12, 2018

John Hopcroft

Robert Tarjan

"For fundamental achievements in the design and analysis of algorithms and data structures."

— *Turing Award*, 1986

SIAM J. Comput.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

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"DFS is a powerful technique with many applications."

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Power of DFS:

Graph Traversal $\Longrightarrow$ Graph Decomposition

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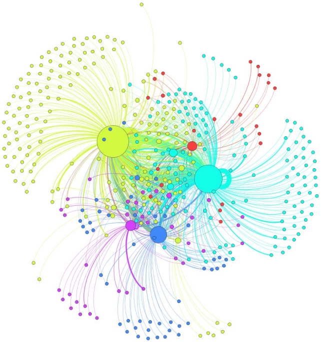
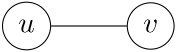
Structure! Structure! Structure!



Graph *structure* induced by DFS:

states of 

types of 

Graph *structure* induced by DFS:

states of v

types of u —— v

life time of v:

v :d$[v]$, f$[v]$

d$[v]$: Bicomp

f$[v]$: Toposort, SCC

Definition (Classifying edges)

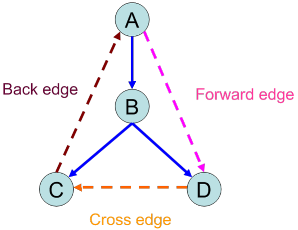
Given a DFS traversal $\implies$ DFS tree:

Tree edge: $\rightarrow$ child

Back edge: $\rightarrow$ ancestor

Forward edge: $\rightarrow$ *nonchild* descendant

Cross edge: $\rightarrow$ $(\neg\text{ancestor}) \wedge (\neg\text{descendant})$



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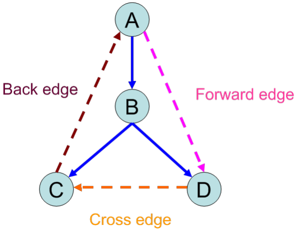
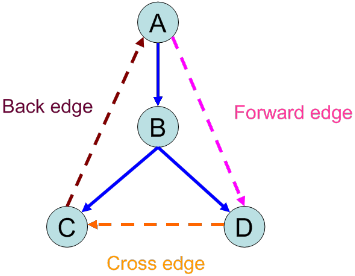
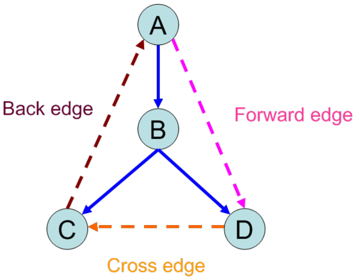


- Also applicable to BFS
- w.r.t. DFS/BFS trees



DFS on directed graph



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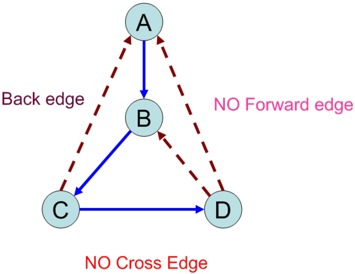
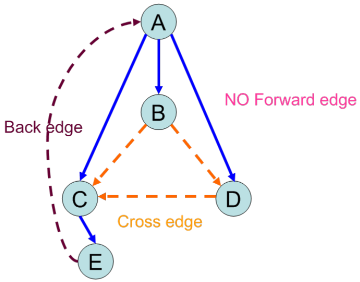
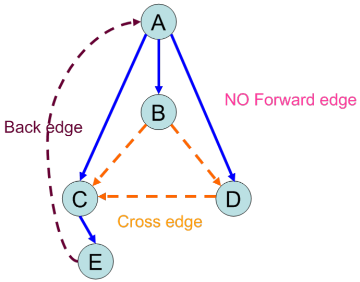


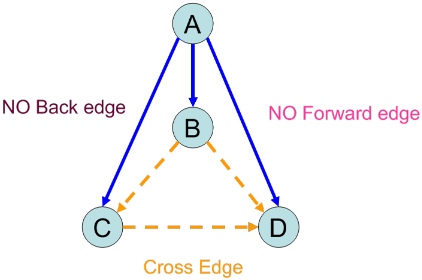
DFS on undirected graph



BFS on directed graph



BFS on directed graph



BFS on undirected graph (Problem 5.1)

DFS tree and BFS tree coincide (Problem 5.7)

Undirected connected graph $G = (V, E), v \in V$

DFS tree T from v $\equiv$ BFS tree T' from v

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Proof.

G_{DFS}: tree + back *vs.* G_{BFS}: tree + cross

□

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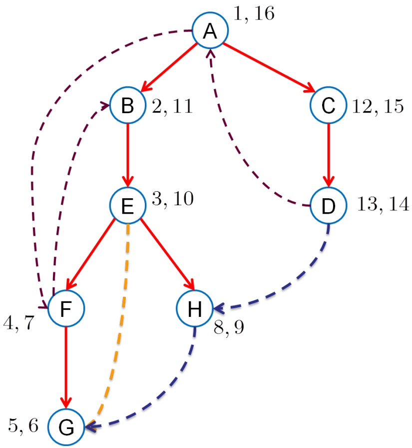
Q : What if G is a digraph?

Life time of vertices in DFS



Theorem (Disjoint or Contained (Problem $4.2 : (1)\&(2)$))

$$\forall u, v : [_u\ ]_u \cap [_v\ ]_v = \emptyset \bigvee \Big([_u\ ]_u \subset [_v\ ]_v \vee [_v\ ]_v \subset [_u\ ]_u \Big)$$

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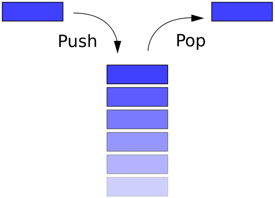
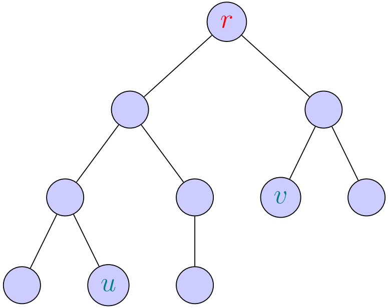


□

Preprocessing for ancestor/descendant relation (Problem 5.6)



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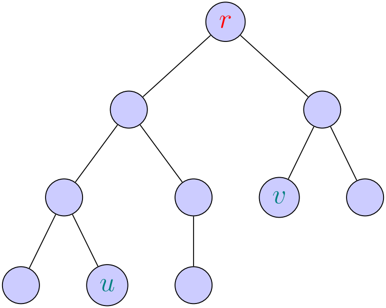

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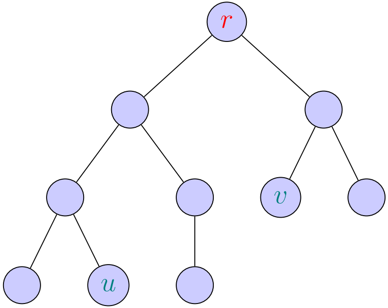

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$$v : \mathsf{d}[v], \mathsf{f}[v]$$

Q : # of descendants of any v?

Edge types and life time of vertices in DFS (Problem 4.5)

$$\forall u \to v :$$

- tree/forward edge: $[_u \ [_v \ ]_v \ ]_u$
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$$\mathsf{f}[v] < \mathsf{d}[u] \iff \qquad \text{edge}$$

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$$\nexists \text{ cycle} \implies \boxed{u \to v \iff \text{f}[v] < \text{f}[u]}$$

DFS from the perspective of a single node:

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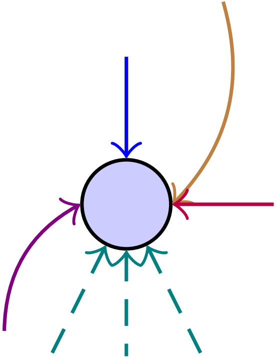

Height and diameter of tree (Problem 5.4)

Binary tree $T = (V, E)$ with $|V| = n$ and the root r:

(I) Height $H(T)$ in $O(n)$

(II) Diameter $D(T)$ in $O(n)$

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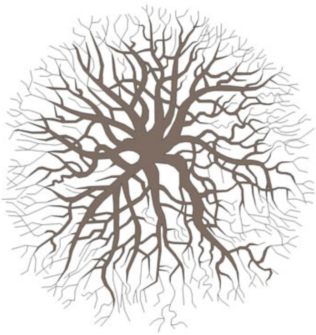

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A beautiful algorithm:

- Pick any u
- Run BFS from u, obtain the farthest v
- Run BFS from v, obtain the farthest w

$$(v, w)$$

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Your Job: Prove it!

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Back to our original problem with $u \leftarrow r$.

Cycle detection (Problem $5.8 - 1$)

	Digraph	Undirected graph
DFS		
BFS		

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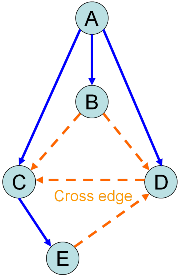

Evasiveness of acyclicity of undirected graphs (Problem $5.8 - 2$)

$$\text{Evasiveness} \triangleq \text{check} \binom{n}{2} \text{ edges (adjacency matrix)}$$

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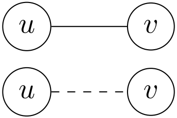

Algorithm $\mathbb{A}$:

$\textsc{CheckEdge}(u, v)$

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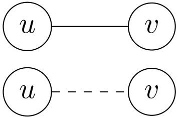

Algorithm $\mathbb{A}$:

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Hint: Kruskal

$$\mathbb{A} : \textsc{CheckEdge}(u, v) \leftarrow \mathcal{A} : \; (u) \text{———} (v)$$

$$\Longleftrightarrow$$

$$\mathcal{A} : \nexists \; \text{cycle} \; \in G + (u) \text{———} (v)$$

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$$\iff$$

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Q : Why adjacency matrix?

After-class Exercise: Evasiveness of connectivity of undirected graphs

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Hint: Anti-Kruskal

Orientation of undirected graph (Problem 4.13)

- undirected (connected) graph G
- edges oriented *s.t.*

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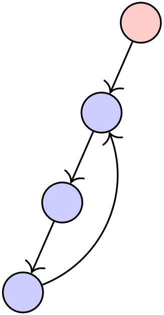

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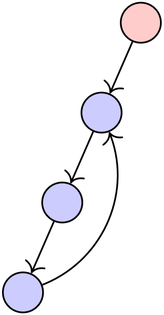
Q : BFS?



Shortest cycle of undirected graph (Problem 4.12)

A WRONG DFS-based algorithm:

$$\forall v : \text{level}[v]$$

$$\text{Back edge } u \to v : \text{level}[u] - \text{level}[v] + 1$$

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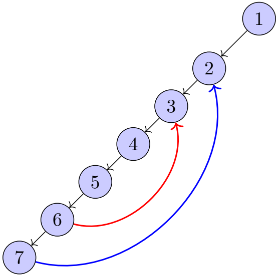

Shortest cycle of digraph (Problem 4.12)

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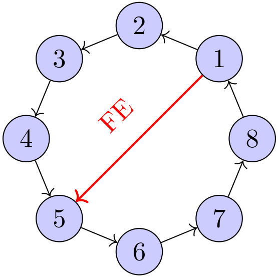

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TOPOSORT by Tarjan (probably), 1976

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Sort vertices in *decreasing* order of their *finish* times.

Kahn's $\textsc{Toposort}$ algorithm (1962; Problem 4.16)

- Queue Q for source vertices (in$[v] = 0$)
- Repeat: $\textsc{Dequeue}(\exists u \in Q)$, output u

  delete u and $u \to v$ from Q,

  $\textsc{Enqueue}(v)$ if in$[v] = 0$

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$$O(m + n)$$

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  $\text{ENQUEUE}(v)$ if in$[v] = 0$

$$O(m+n)$$

Lemma (Correctness of Kahn's TOPOSORT)

Every DAG has at least one source (and at least one sink vertex).

Kahn's TOPOSORT algorithm (1962; Problem 4.16)

- Queue Q for source vertices ($\text{in}[v] = 0$)
- Repeat: $\textsc{Dequeue}(\exists u \in Q)$, output u

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Q : What if G is *not* a DAG?

Taking courses in few semesters (Problem 4.20)

- n courses
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For general digraph, LONGEST-PATH is NP-hard.

Line up (Problem 4.22)

1. i hates j: $i \succ j$
2. i hates j: $\#i < \#j$

TOPOSORT

Critical path *OR* Longest path

Hamiltonian path in DAG (Problem 4.14)

HP: path visiting each vertex once

Q : $\exists$ HP in a DAG in $O(n+m)$

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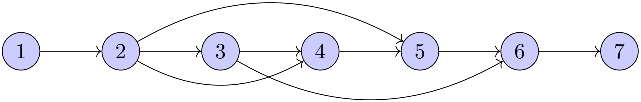

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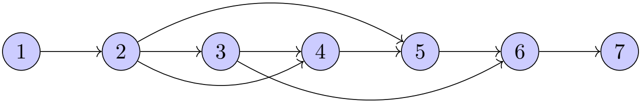

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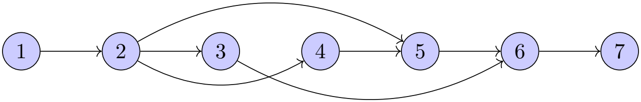

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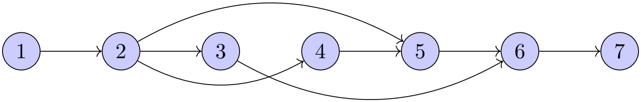

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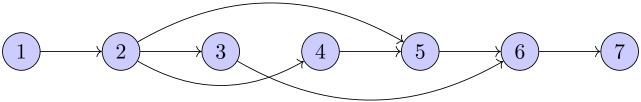

Kahn's $\textsc{Toposort}$ (Problem 4.16)

$$|Q| \leq 1$$

Theorem (Digraph as DAG (Problem 4.6))

Every digraph is a dag of its SCCs.

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Two tiered structure of digraphs:

digraph $\equiv$ a dag of SCCs

SCC: equivalence class over reachability

$\boxed{\text{digraph} \equiv \text{a dag of SCCs}}$

Kosaraju's SCC algorithm, 1978

"SCCs can be topo-sorted

in decreasing order of their highest finish time."

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$$\text{2nd DFS} \stackrel{?}{\Longrightarrow} \text{BFS}$$

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1st DFS: toposort between SCCs

2nd DFS: reachability within an SCC

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One-to-all reachability in a digraph (Problem 5.12)

$$v : v \rightsquigarrow^{?} \forall u$$

$$\exists ?\ v : v \rightsquigarrow \forall u$$

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$$\Longleftarrow : \exists !\ \text{source}$$

$$\Longrightarrow : \text{By contradiction.}$$

$$\exists u : v \not\rightsquigarrow u \land \text{in}[u] > 0 \implies \exists \text{ cycle}$$

Impacts of vertices in a digraph (Problem 4.18)

$$\mathsf{impact}(v) = |\{w \neq v : v \rightsquigarrow w\}|$$

- $\arg\min_v \mathsf{impact}(v)$
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Q : $\forall v$, computing impact(v)

BICOMP: Back!

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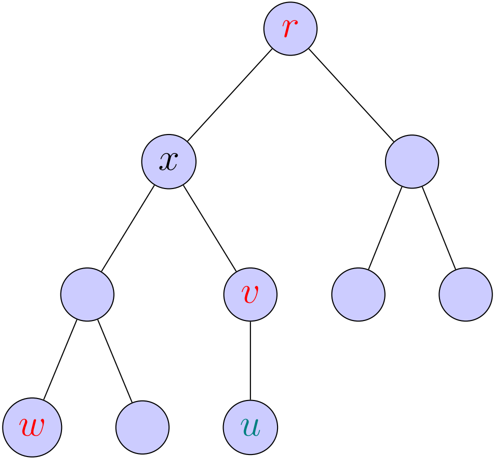

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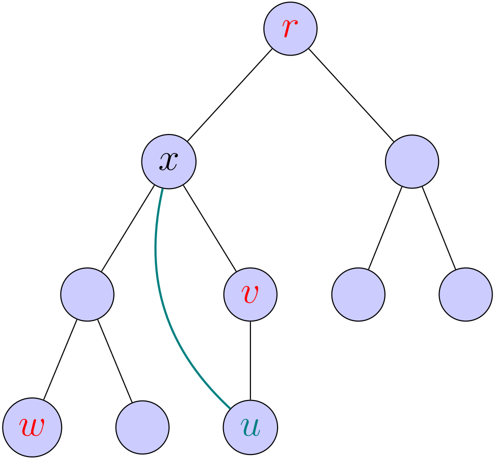

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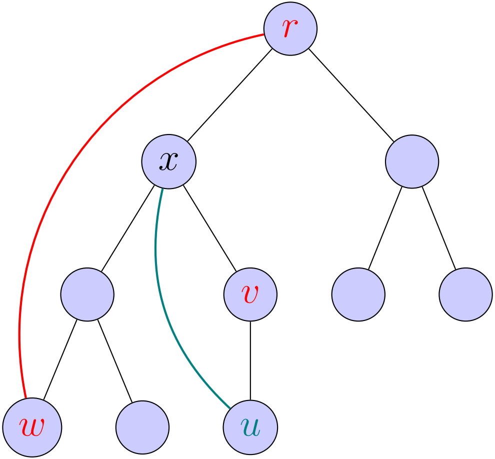

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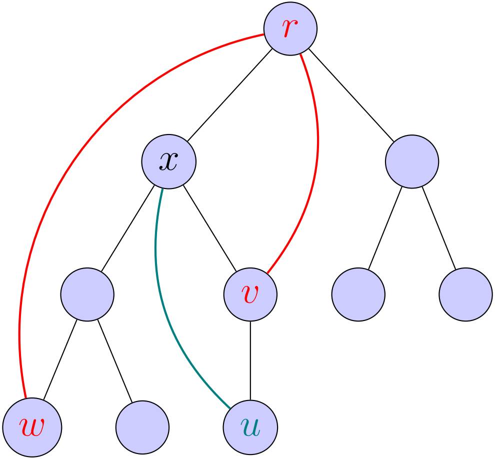

back[v]: the earliest reachable ancestor of v

(I) When and how to update back[v]?

(II) When and how to identify a bicomponent?

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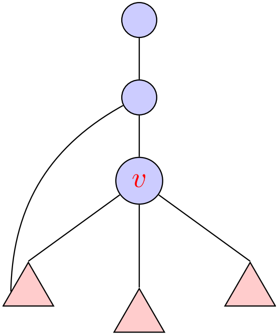

Initialization of back$[v]$ (Problem 4.9)

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$\square$

Root cutnode v (Problem 4.8)

$$v \text{ is a cutnode } \iff \text{out}[v] \geq 2$$

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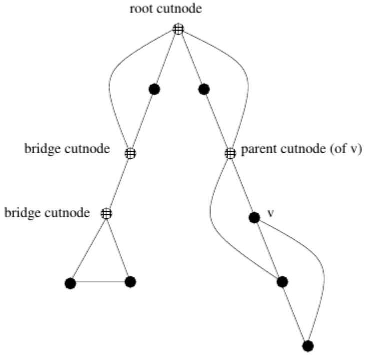


THANK YOU