Decompositions of Graphs

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June 12, 2018

John Hopcroft

Robert Tarjan

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Robert Tarjan

"For fundamental achievements in the design and analysis of algorithms and data structures."

— *Turing Award, 1986*

SIAM J. COMPUT.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

Abstract. The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an undirect graph are presented. The space and time requirements of both algorithms are bounded by $k_1V + k_2E + k_3$ for some constants k_1, k_2, and k_3, where V is the number of vertices and E is the number of edges of the graph being examined.

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"DFS is a powerful technique with many applications."

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Power of DFS:

Graph Traversal $\Longrightarrow$ Graph Decomposition

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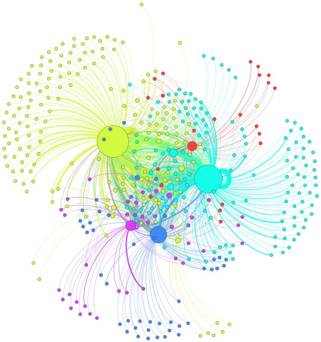
Structure! Structure! Structure!



Graph *structure* induced by DFS:

states of v

types of $u - v$

Graph *structure* induced by DFS:

states of v

types of u — v

life time of v:

v : $\mathsf{d}[v], \mathsf{f}[v]$

$\mathsf{f}[v]$: DAG, SCC

$\mathsf{d}[v]$: biconnectivity

Definition (Classifying edges)

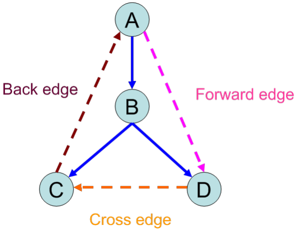
Given a DFS traversal $\Longrightarrow$ DFS tree:

Tree edge: $\rightarrow$ child

Back edge: $\rightarrow$ ancestor

Forward edge: $\rightarrow$ *nonchild* descendant

Cross edge: $\rightarrow$ $(\neg\text{ancestor}) \wedge (\neg\text{descendant})$



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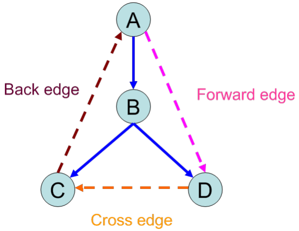
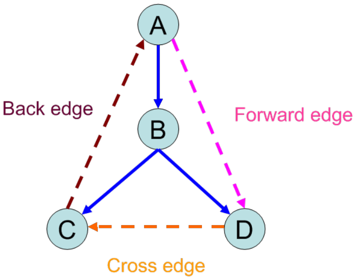
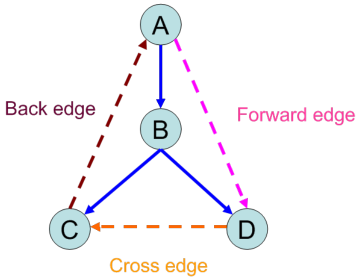


- also applicable to BFS
- w.r.t. DFS/BFS trees



DFS on directed graph



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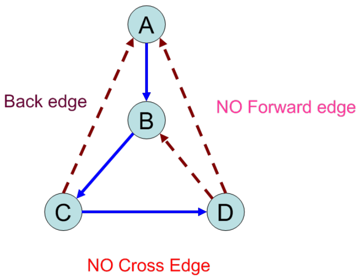
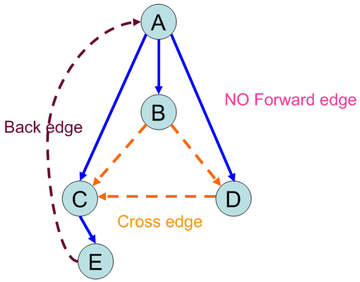
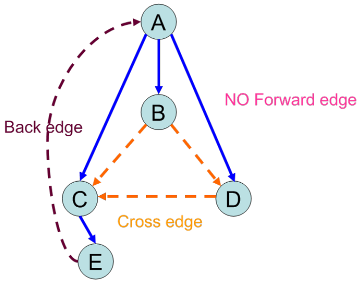


DFS on undirected graph



BFS on directed graph



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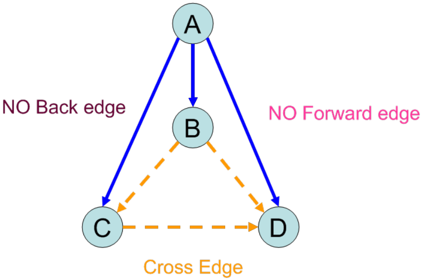

BFS on undirected graph

DFS tree and BFS tree coincide (Problem 5.7)

Undirected connected graph $G = (V, E), v \in V$

DFS tree T from $v \equiv$ BFS tree T' from v

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Proof.

G_{DFS}: tree + back *vs.* G_{BFS}: tree + cross

$\square$

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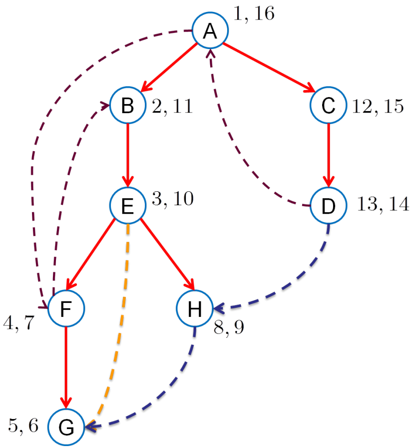
□

Q : What if G is a digraph?

Lift time of vertices in DFS



Theorem (Disjoint or Contained (Problem 4.2: (1) & (2)))

$$\forall u, v : [_u\ ]_u \cap [_v\ ]_v = \emptyset \bigvee \left([_u\ ]_u \subsetneq [_v\ ]_v \vee [_v\ ]_v \subsetneq [_u\ ]_u \right)$$

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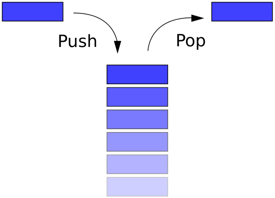
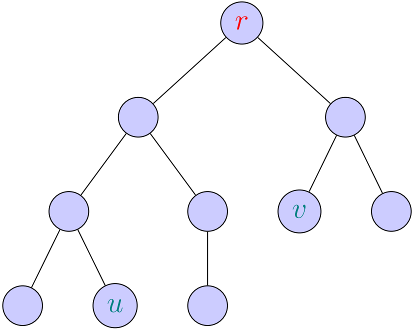


□

Preprocessing for ancestor/descendant relation (Problem 5.23)



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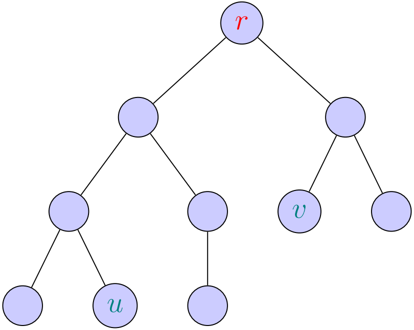

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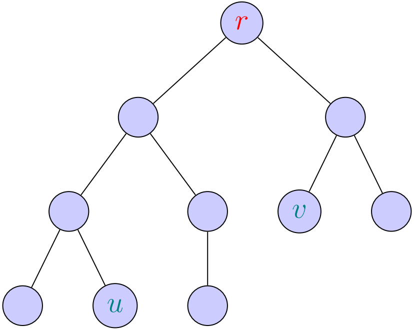

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Q : # of descendants of any v?

Edge types and life time of vertices in DFS (Problem 4.5)

$$\forall u \to v :$$

- tree/forward edge: $[_u \; [_v \;]_v \;]_u$
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$$\mathsf{f}[u] < \mathsf{f}[v] \iff \text{back edge}$$

$$\nexists \text{ cycle} \implies \boxed{u \to v \iff \mathsf{f}[v] < \mathsf{f}[u]}$$

Height and diameter of tree (Problem 5.4)

Binary tree $T = (V, E)$ with $|V| = n$ and the root r:

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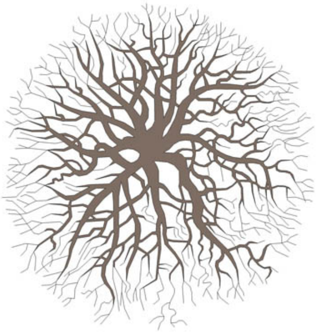

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Your Job: Prove it!

Counting shortest paths (Problem 5.26)

Counting # of shortest paths in (un)directed graphs using BFS.

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Maybe in the next class...

Cycle detection (Problem $5.8 - 1$)

	Digraph	Undirected graph
DFS		
BFS		

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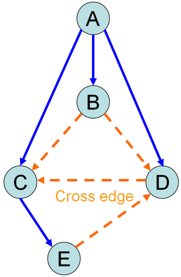

Evasiveness of acyclicity of undirected graphs (Problem $5.8 - 2$)

$$\text{Evasiveness} \triangleq \text{check} \binom{n}{2} \text{edges (adjacency matrix)}$$

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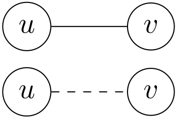

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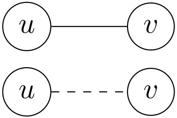

Algorithm $\mathbb{A}$:

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Hint: Kruskal

$$\textsc{CheckEdge}(u, v) \leftarrow \textcircled{u} \text{———} \textcircled{v}$$

$$\Longleftrightarrow$$

$$\nexists \ \text{cycle} \ \in G + \textcircled{u} \text{———} \textcircled{v}$$

$$\textsc{CheckEdge}(u, v) \leftarrow (u) - (v)$$

$$\iff$$

$\nexists$ cycle $\in G + (u) - (v)$

Q : Why adjacency matrix?

After-class Exercise: Evasiveness of connectivity of undirected graphs

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Hint: Anti-Kruskal

Orientation of undirected graph (Problem 4.13)

- undirected (connected) graph G
- edges oriented *s.t.*

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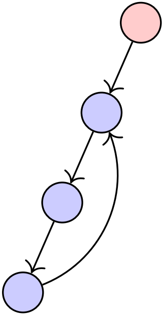

Shortest cycle of undirected graph (Problem 4.12)

A WRONG DFS-based algorithm:

$$\forall v : \mathsf{level}[v]$$

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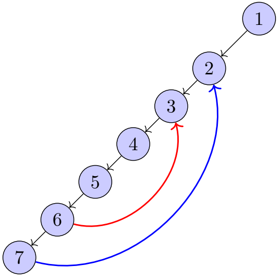

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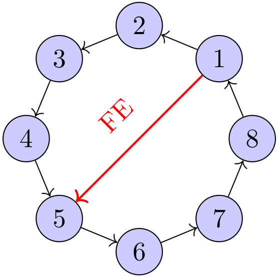

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TOPOSORT by Tarjan (probably), 1976

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Sort vertices in *decreasing* order of their *finish* times.

Kahn's TOPOSORT algorithm (1962; Problem 4.16)

- Queue Q for source vertices (in$[v] = 0$)
- Repeat: $\textsc{Dequeue}(u \in Q)$, delete u and $u \to v$ from Q,
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Lemma (Correctness of Kahn's $\textsc{Toposort}$)

Every DAG has at least one source (and at least one sink vertex).

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Q : What if G is *not* a DAG?

Taking courses in few semesters (Problem 5.14)

- n courses
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Critical path *OR* Longest path using DFS in $O(n+m)$

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Critical path *OR* Longest path using DFS in $O(n+m)$

For general digraph, LONGEST-PATH is NP-hard.

Line up (Problem 4.22)

1. i hates j: $i \succ j$
2. i hates j: $\#i < \#j$

TOPOSORT

Critical path

Hamiltonian path in DAG (Problem 4.14)

HP: path visiting each vertex once

Q : $\exists$ HP in a DAG in $O(n+m)$

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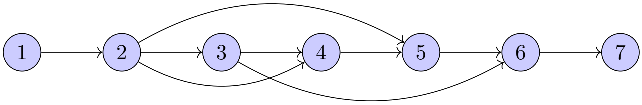

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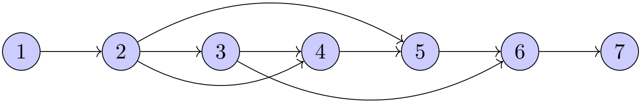
Q : $\exists$ HP in a DAG in $O(n+m)$

For general (di)graph, HP is NP-hard.



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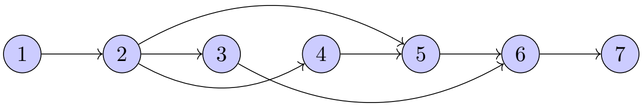

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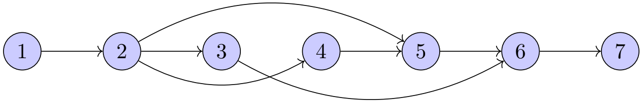

Kahn's TOPOSORT (Problem 4.16)

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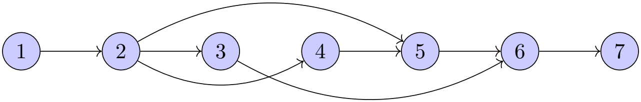

Kahn's $\textsc{Toposort}$ (Problem 4.16)

$$|Q| \leq 1$$

Theorem (Digraph as DAG (Problem 4.6))

Every digraph is a dag of its SCCs.

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Two tiered structure of digraphs:

digraph $\equiv$ a dag of SCCs

SCC: equivalence class over reachability

$\boxed{\text{digraph} \equiv \text{a dag of SCCs}}$

Kosaraju SCC algorithm, 1978

"SCCs can be topo-sorted
in decreasing order of their highest finish time."

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(I) DFS on G; DFS/BFS on G^T
(II) DFS on G^T; DFS/BFS on G

Kosaraju SCC algorithm, 1978 (Problem 4.7)

$$\text{1st DFS} \overset{?}{\Longrightarrow} \text{BFS}$$

$$\text{2nd DFS} \overset{?}{\Longrightarrow} \text{BFS}$$

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One-to-all reachability in a digraph (Problem 5.12)

$$v : v \rightsquigarrow^{?} \forall u$$

$$\exists ?\; v : v \rightsquigarrow \forall u$$

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$\Longleftarrow$: $\exists!$ source

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$$\Longleftarrow : \exists !\ \text{source}$$

$\Longrightarrow$: By contradiction.

$$\exists u : v \not\rightsquigarrow u \land \text{in}[u] > 0 \implies \exists \text{ cycle}$$

Impacts of vertices in a digraph (Problem 4.18)

$$\mathsf{impact}(v) = |\{w \neq v : v \rightsquigarrow w\}|$$

- $\arg\min_v \mathsf{impact}(v)$
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$$Q : \forall v : \text{computing impact}(v)$$

2SAT (Problem 4.23)

$$I : (x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (\overline{x_3} \vee x_4) \wedge (\overline{x_1} \vee x_4)$$

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Implication graph G_I.

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Theorem (2SAT)

$\exists$ *SCC* $\exists x : v_x \in$ *SCC* $\wedge\, v_{\overline{x}} \in$ *SCC* $\iff$ I *is not satisfiable.*

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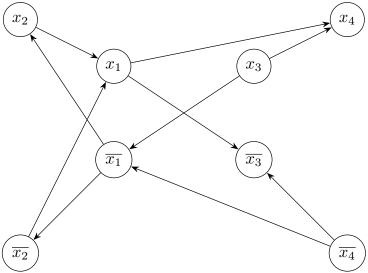
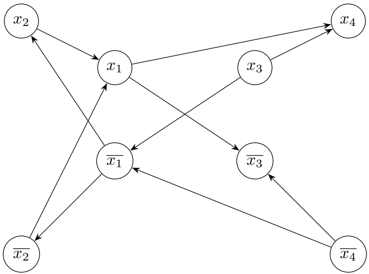
Theorem (2SAT)

$\exists$ SCC $\exists x : v_x \in$ SCC $\wedge\, v_{\overline{x}} \in$ SCC $\iff I$ is not satisfiable.

Reference:

- *"A Linear-time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas"* by Bengt Aspvall, Michael Plass, and Robert Tarjan, 1979.


x_2
x_4
x_1
x_3
$\overline{x_1}$
$\overline{x_3}$
$\overline{x_2}$
$\overline{x_4}$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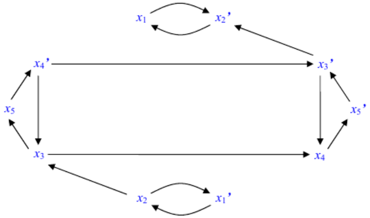
x_1
x_2'
x_4'
x_3'
x_5
x_5'
x_3
x_4
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THANK YOU