

Paths in Graphs

Hengfeng Wei

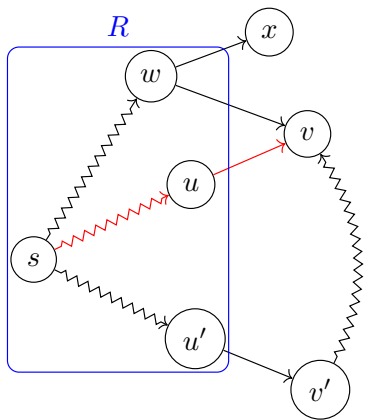
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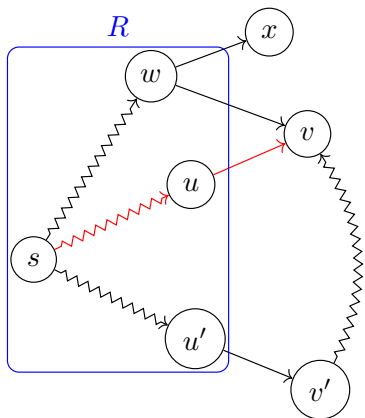
Dijkstra's Algorithm for SSSP

Finding shortest paths from s to other nodes t
in non-decreasing order of $\text{dist}(s, t)$.



$$R \triangleq \{u \mid s \rightsquigarrow u \text{ is known}\}$$

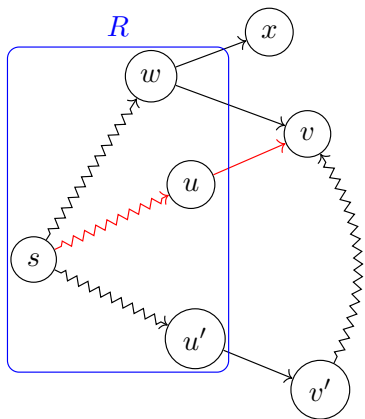
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$$l(v' \rightsquigarrow v) \geq 0$$

for all $v \in V$ **do**

$\text{dist}[v] \leftarrow \infty$

$\text{dist}[s] \leftarrow 0$

$Q \leftarrow \text{MINPQ}(V)$

while $Q \neq \emptyset$ **do**

$u \leftarrow \text{DELETETMIN}(Q)$

for all $(u, v) \in E \wedge v \notin Q$ **do**

if $\text{dist}[v] > \text{dist}[u] + l(u, v)$ **then**

$\text{dist}[v] \leftarrow \text{dist}[u] + l(u, v)$

$\text{DECREASEKEY}(Q, v)$

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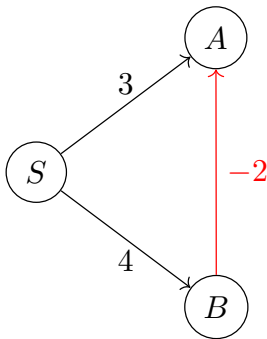
$\text{dist}[v] \leftarrow \text{dist}[u] + l(u, v)$

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$$O(n + (n + m) \log n) \implies O((n + m) \log n) \implies O(m \log n)$$

Negative Edges (Problem 11.1)

Dijkstra's algorithm may fail if $w(e) < 0$.

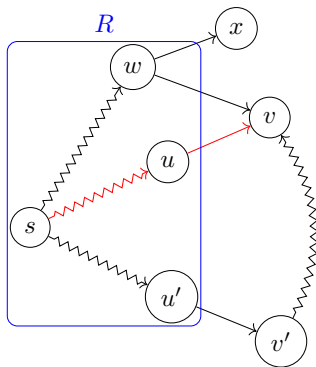


Negative Edges from s (Problem 11.9)

All negative edges are from s .

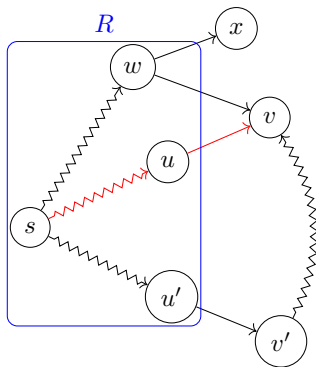
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Generalized Shortest Path (Problem 11.8)

Digraph $G = (V, E)$, $l_e > 0$, $c_v > 0$, $s \in V$

Shortest paths from s

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$$\forall u \rightarrow v : l'(u, v) = l(u, v) + c_v$$

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Shortest paths from s

$$\forall u \rightarrow v : l'(u, v) = l(u, v) + c_v$$

$$+ c_s$$

Shortest Paths Through v_0 (Problem 13.7)

Strongly connected digraph $G = (V, E)$, $w(e) > 0$

$$v_0 \in V$$

Find shortest paths $s \rightsquigarrow^{\text{SP}} t$ through v_0 .

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$$s \rightsquigarrow^{\text{SP}} v_0 \rightsquigarrow^{\text{SP}} t$$

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$$s \rightsquigarrow^{\text{SP}} v_0 \rightsquigarrow^{\text{SP}} t$$

$$\forall v : v_0 \rightsquigarrow^{\text{SP}} v$$

Dijkstra's Algorithm as a Framework

Min-Max Path (Problem 11.12)

$G = (V, E)$: network of highways

l_e : road length L : tank capacity

Given L , $\exists? s \rightsquigarrow t$ in $O(n + m)$.

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$$s \rightsquigarrow? t$$

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for all $v \in V$ **do**

$L[v] \leftarrow \infty$

$L[s] \leftarrow 0$

if $L[v] > \max(L[u], l(u, v))$ **then**

$L[v] \leftarrow \max(L[u], l(u, v))$

Max-Min Path (Problem 13.2 (1))

$G = (V, E)$: network of oil pipelines

$c(u, v)$: capacity of (u, v)

$\text{cap}(s, t)$: $\max \min s \rightsquigarrow t$

Given s , to compute $\text{cap}(s, v)$.

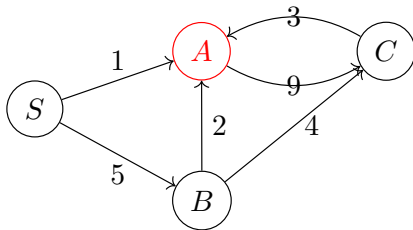
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for all $v \in V$ **do**

$\text{cap}[v] \leftarrow -\infty$

$\text{cap}[s] \leftarrow 0$

if $\text{cap}[v] < \min(\text{cap}[u], c(u, v))$ **then**

$\text{cap}[v] \leftarrow \min(\text{cap}[u], c(u, v))$

Max-Min Path (Problem 13.2 (2))

$G = (V, E)$: network of oil pipelines

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$\text{cap}(s, t)$: $\max \min s \rightsquigarrow t$

Compute all-pair $\text{cap}(u, v)$.

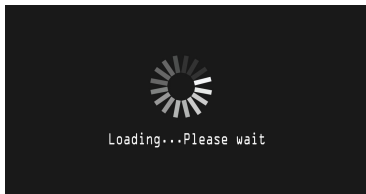
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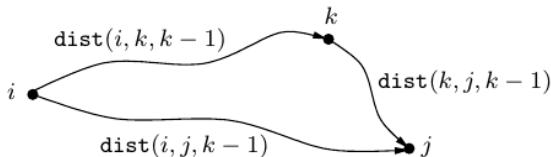
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$$\text{dist}(i, j, k) = \min \left(\text{dist}(i, j, k-1), \text{dist}(i, k, k-1) + \text{dist}(k, j, k-1) \right)$$

$$\#k's = 1$$

for all (i, j) do

if $(i, j) \in E$ then

$\text{dist}(i, j, 0) \leftarrow l(i, j)$

else

$\text{dist}(i, j, 0) \leftarrow \infty$

for $k \leftarrow 1$ to n do

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for $j \leftarrow 1$ to n do

$\text{dist}(i, j, k) = \min\left(\text{dist}(i, j, k - 1), \text{dist}(i, k, k - 1) + \text{dist}(k, j, k - 1)\right)$

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$$\text{cap}(u, v, k) = \max\left(\text{cap}(u, v, k-1), \min(\text{cap}(u, k, k-1), \text{cap}(k, v, k-1))\right)$$

Routing table (Problem 13.1)

$$\text{Go}(i, j) = k \implies v_i \rightarrow v_k \rightsquigarrow v_j$$

Construct routing table and extract shortest paths from it.

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$\text{Go}(i, j) \leftarrow \text{Nil}$

$\forall (i, j) \in E : \text{Go}(i, j) \leftarrow j$

if ... then

$\text{Go}(i, j) \leftarrow \text{Go}(i, k)$

if $\text{Go}(i, j) = \text{Nil}$ then

...

while $i \neq j$ do

$i \leftarrow \text{Go}(i, j)$

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$\forall (i, j) \in E : \text{Go}(i, j) \leftarrow j$	\dots
if \dots then	while $i \neq j$ do
$\text{Go}(i, j) \leftarrow \text{Go}(i, k)$	$i \leftarrow \text{Go}(i, j)$
$\text{Prev}(i, j) \leftarrow \text{Prev}(k, j)$	

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Construct routing table and extract shortest paths from it.

```
Go(i, j) ← Nil                                if Go(i, j) = Nil then
    ∀(i, j) ∈ E : Go(i, j) ← j                ...
if ... then                                    while i ≠ j do
    Go(i, j) ← Go(i, k)                        i ← Go(i, j)

Prev(i, j) ← Prev(k, j)

Intermediate(i, j) ← k
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Shortest Cycle in Digraph (Problem 13.9)

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$$\exists v : \text{dist}[v][v] < \infty \implies \min_i \text{dist}[v][v]$$

$$\forall v : \text{dist}[v][v] = \infty$$

Hamiltonian Path in Tournament Graph (Problem 11.10)

$$\forall u, v : (u \rightarrow v \vee v \rightarrow u) \\ \wedge \neg(u \rightarrow v \wedge v \rightarrow u)$$

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By mathematical induction on n .

Eulerian Circuit (Problem 13.5)

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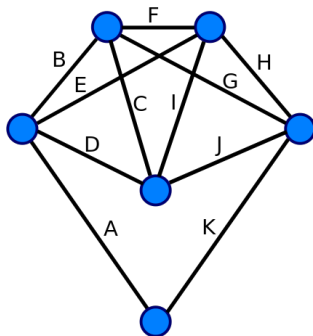
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$$VE \leftarrow \emptyset$$
$$C \leftarrow \emptyset$$

while $VE \neq E$ **do**

$$u \leftarrow \text{CHOOSE}(u : (u \rightarrow v) \notin VE)$$
$$C' \leftarrow \text{CIRCUIT}(u, E \setminus VE)$$
$$VE \leftarrow VE \cup C'$$
$$C \leftarrow C \cup C'$$

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Data Structures?

Most Critical Edge (Problem 11.3)

$$s, t \in V$$

$$e : E \setminus \{e\} \implies \text{dist}(s, t) \text{ increases most}$$

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$$O(n (m \log n)) = O(mn \log n)$$

Bitonic Shortest Path (Problem 11.7)



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