Paths in Graphs

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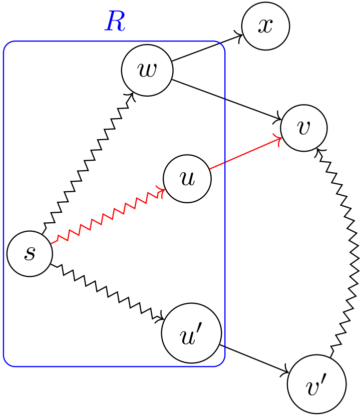
Dijkstra's Algorithm for SSSP

Finding shortest paths from s to other nodes t
in non-decreasing order of $\text{dist}(s, t)$.



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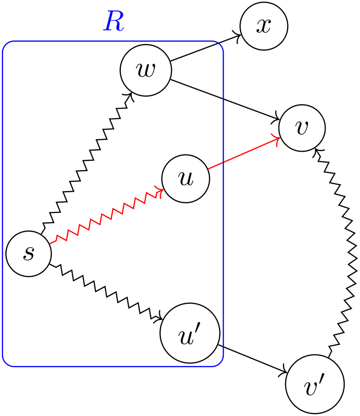

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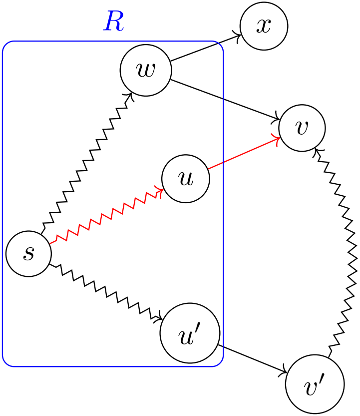

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$l(v' \rightsquigarrow v) \geq 0$

for all $v \in V$ **do**
    $\text{dist}[v] \leftarrow \infty$
$\text{dist}[s] \leftarrow 0$

$Q \leftarrow \textsc{MinPQ}(V)$
while $Q \neq \emptyset$ **do**
    $u \leftarrow \textsc{DeleteMin}(Q)$
    for all $(u, v) \in E \land v \notin Q$ **do**
        if $\text{dist}[v] > \text{dist}[u] + l(u, v)$ **then**
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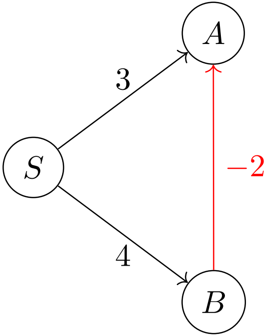
$$O(n + (n + m) \log n) \implies O((n + m) \log n) \implies O(m \log n)$$

Negative Edges (Problem 11.1)

Dijkstra's algorithm may fail if $w(e) < 0$.



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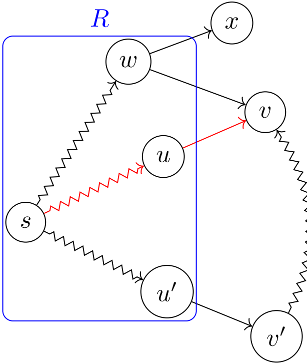

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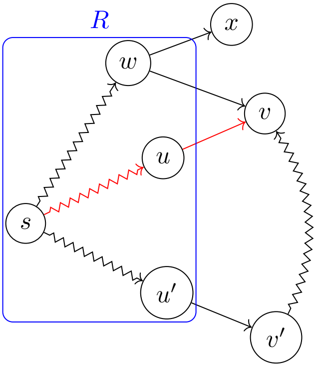

$$l(v' \rightsquigarrow v) \geq 0$$

Generalized Shortest Path (Problem 11.8)

$$\text{Digraph } G = (V, E), \quad l_e > 0, \quad c_v > 0, \quad s \in V$$

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$$+ \, c_s$$

Shortest Paths Through v_0 (Problem 13.7)

Strongly connected digraph $G = (V, E), \quad w(e) > 0$

$$v_0 \in V$$

Find shortest paths $s \leadsto^{\mathsf{SP}} t$ through v_0.

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$$s \rightsquigarrow^{\textsf{SP}} v_0 \rightsquigarrow^{\textsf{SP}} t$$

$$\forall v : v_0 \rightsquigarrow^{\textsf{SP}} v$$

Dijkstra's Algorithm as a Framework

Min-Max Path (Problem 11.12)

$$G = (V, E) : \text{ network of highways}$$

$$l_e : \text{ road length} \quad L : \text{ tank capacity}$$

Given L, $\exists ? s \rightsquigarrow t$ in $O(n + m)$.

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for all $v \in V$ **do**
  $L[v] \leftarrow \infty$
$L[s] \leftarrow 0$

if $L[v] > \max(L[u], l(u,v))$ **then**
  $L[v] \leftarrow \max(L[u], l(u,v))$

Max-Min Path (Problem 13.2 (1))

$$G = (V, E): \text{ network of oil pipelines}$$

$$c(u, v): \text{ capacity of } (u, v)$$

$$\mathsf{cap}(s, t): \max \min s \rightsquigarrow t$$

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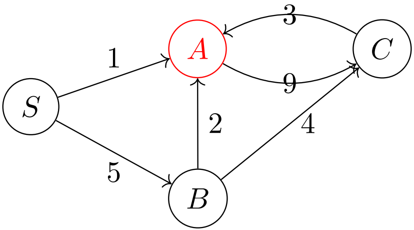

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for all $v \in V$ **do**
  $\text{cap}[v] \leftarrow -\infty$
$\text{cap}[s] \leftarrow 0$

if $\text{cap}[v] < \min(\text{cap}[u], c(u,v))$ **then**
  $\text{cap}[v] \leftarrow \min(\text{cap}[u], c(u,v))$

Max-Min Path (Problem 13.2 (2))

$$G = (V, E): \text{ network of oil pipelines}$$

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$$\mathsf{cap}(s, t): \max \min s \leadsto t$$

Compute all-pair $\mathsf{cap}(u, v)$.

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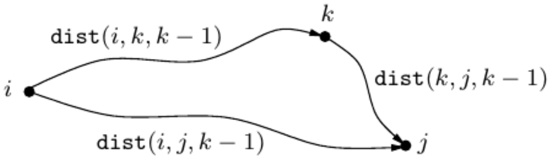

$$\mathsf{dist}(i,j,k) = \min\Big(\mathsf{dist}(i,j,k-1), \mathsf{dist}(i,k,k-1) + \mathsf{dist}(k,j,k-1)\Big)$$

$$\#k's = 1$$

```
for all (i, j) do
    if (i, j) ∈ E then
        dist(i, j, 0) ← l(i, j)
    else
        dist(i, j, 0) ← ∞

for k ← 1 to n do
    for i ← 1 to n do
        for j ← 1 to n do
            dist(i, j, k) = min(dist(i, j, k − 1), dist(i, k, k − 1) + dist(k, j, k − 1))
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$$\mathsf{cap}(u, v, k) = \max\Big(\mathsf{cap}(u, v, k-1), \min\big(\mathsf{cap}(u, k, k-1), \mathsf{cap}(k, v, k-1)\big)\Big)$$

Routing table (Problem 13.1)

$$\mathsf{Go}(i, j) = k \implies v_i \to v_k \leadsto v_j$$

Contruct routing table and extract shortest paths from it.

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$$\forall (i,j) \in E : \mathsf{Go}(i,j) \leftarrow j$$

if $\ldots$ **then**

$\quad \mathsf{Go}(i,j) \leftarrow \mathsf{Go}(i,k)$

if $Go(i,j) = \mathsf{Nil}$ **then**

$\quad \cdots$

while $i \neq j$ **do**

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while $i \neq j$ **do**
    $i \leftarrow \mathsf{Go}(i,j)$

$$\mathsf{Prev}(i,j) \leftarrow \mathsf{Prev}(k,j)$$

$$\mathsf{Intermediate}(i,j) \leftarrow k$$

Shortest Cycle in Digraph (Problem 13.9)

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$$\forall v : \mathsf{dist}[v][v] = \infty$$

Hamiltonian Path in Tournament Graph (Problem 11.10)

$$\forall u, v : (u \to v \vee v \to u) \\ \wedge \neg(u \to v \wedge v \to u)$$

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By mathematical induction on n.

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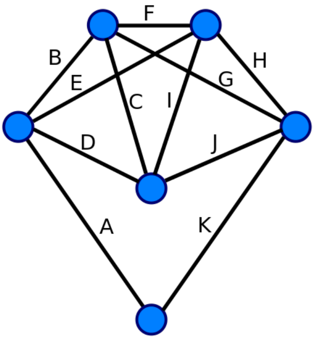

$VE \leftarrow \emptyset$
$C \leftarrow \emptyset$

while $VE \neq E$ **do**
  $u \leftarrow \textsc{Choose}(u : (u \to v) \notin VE)$
  $C' \leftarrow \textsc{Circuit}(u, E \setminus VE)$
  $VE \leftarrow VE \cup C'$
  $C \leftarrow C \cup C'$

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Data Structures?

Most Critical Edge (Problem 11.3)

$$s, t \in V$$

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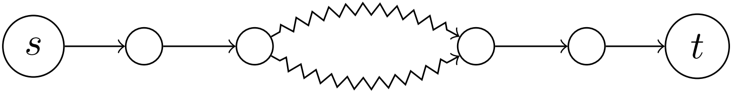

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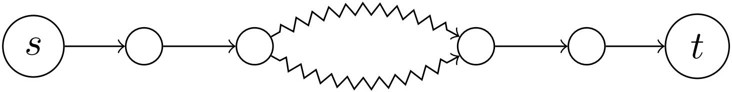

$$O(n\ (m \log n)) = O(mn \log n)$$

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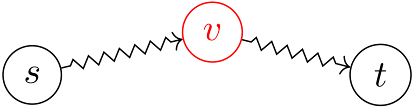

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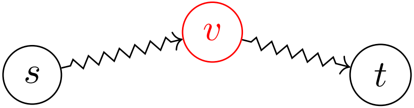


THANK YOU