Searching and Selection

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April 8 $\sim$ April 9, 2017

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1 Selection

2 Searching

The 3rd largest element (Problem 3.1)

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"Q_1: What is the exact value of $V_3(n)$?"

Theorem ($V_3(n)$)

$n \geq 6, n = 2^k + r (0 \leq r < 2^k)$:

$$V_3(n) = \begin{cases} (n-3) + 2k & r = 0, 1 \\ (n-3) + 2k + 1 & 2 \leq r \leq 2^{k-2} + 1 \\ (n-3) + 2k + 2 & \text{o.w.} \end{cases}$$

References

"Selecting the Top Three Elements" by Aigner, 1982.

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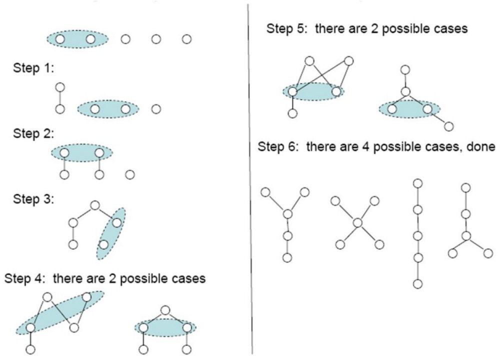
Selection with minimum #comparisons (Problem 3.2)

Selecting the median of 5 elements using 6 comparisons.


Step 1:
Step 2:
Step 3:
Step 4: there are 2 possible cases
Step 5: there are 2 possible cases
Step 6: there are 4 possible cases, done

Sorting with minimum #comparisons (Problem 2.4)

Sorting 5 elements using 7 comparisons.

$$S(5) = 7$$

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$$S(21) = 66$$

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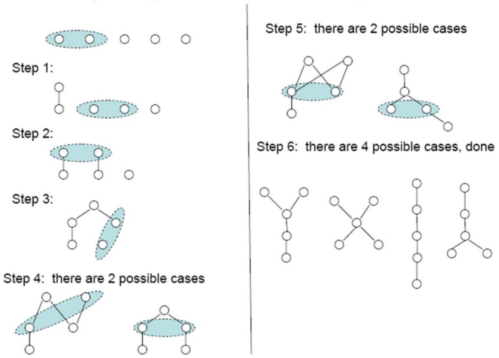
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Medians of sorted arrays (Problem 3.7)

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1. Selection
2. Searching

$\max$ / $\min$ differences (Problem 4.5)

(a) unsorted; $\max |x - y|$; $O(n)$

(b) sorted; $\max |x - y|$; $O(1)$

(c) unsorted; $\min |x - y|$; $O(n \log n)$

(d) sorted; $\min |x - y|$; $O(n)$

Searching in matrix (Problem 4.6)

- M: matrix $m \times n$
- row: increasing from left to right
- col: increasing from top to down
- Is $x \in M$?

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$$T(m,n) = m + n - 1$$

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Assume $M : n \times n$

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$$i + j \le n - 1 \implies x > M_{ij}$$
$$i + j > n - 1 \implies x < M_{ij}$$

Local minimum (Problem 4.11)

- Array $A[0 \ldots n]$
- Boundary conditions:

$$A[0] \geq A[1]$$
$$A[n-2] \leq A[n-1]$$

- Local minimum $A[i]$:

$$A[i-1] \geq A[i] \leq A[i+1]$$

- Goal: Find *any* local minimum.

Local minimum (Problem 4.11)

1. Checking each element:

$$T(n) = O(n)$$

2. $\min A$:

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2D local minimum:

- Matrix $M : n \times n$
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$$O(n^2) \implies O(n \log n) \implies O(n) \implies O(\log n)$$

$a_i = i$ (Problem 4.2)

- Sorted integer sequence $\{a_1, a_2, \dots, a_n\}$:

$$\forall i \neq j : a_i \neq a_j$$

- Goal:

$$\exists ? i : a_i = i$$

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Smallest missing positive integer (Problem 4.3)

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Computing $\lceil \sqrt{N} \rceil$ (Problem 4.12)

- Given an n-bit natural number N $(0 \le N < 2^n - 1)$
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- n-bit $+$ n-bit: $O(1)$
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- $\Rightarrow x^2 : O(n)$

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1. Naïve search: $O(2^n \cdot n)$
2. Binary search: $O(n \cdot n)$
3. Binary search in range:

$$2^{\lfloor \frac{n-1}{2} \rfloor} \leq \lceil \sqrt{N} \rceil \leq 2^{\lceil \frac{n}{2} \rceil}$$

$$\lg \left(2^{\lceil \frac{n}{2} \rceil} - 2^{\lfloor \frac{n-1}{2} \rfloor}\right) = O(n)$$

$$O(n \cdot n)$$

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A Little History:

2007: Mid-term problem

$O(n)$ required; NO $O(n)$ solutions, however

~ 2013: $O(n^2)$

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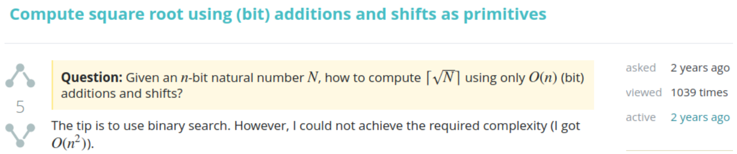

Compute square root using (bit) additions and shifts as primitives

5

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The tip is to use binary search. However, I could not achieve the required complexity (I got $O(n^2)$).

asked 2 years ago

viewed 1039 times

active 2 years ago

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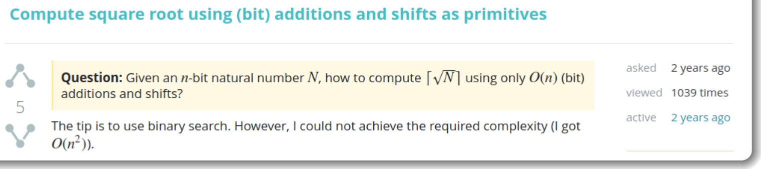

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$$x^2 : O(n) \to O(1)$$

Computing $\lceil \sqrt{N} \rceil$ (Problem 4.12)

Given

$$M = \lfloor N/4 \rfloor$$

$$x = \lceil \sqrt{M} \rceil \text{ and } (x, x^2),$$

what is

$$y = \lceil \sqrt{N} \rceil \text{ and } (y, y^2)?$$

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An Example:

$$N = 280 \qquad y = \lceil\sqrt{280}\rceil = 17 \quad y^2 = 289$$

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An Example:

$$\begin{array}{lll} N = 280 & y = \lceil \sqrt{280} \rceil = 17 & y^2 = 289 \\ M = \lfloor 280/4 \rfloor = 70 & x = \lceil \sqrt{70} \rceil = 9 & x^2 = 81 \end{array}$$

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N = 280 & y = \lceil\sqrt{280}\rceil = 17 & y^2 = 289 \\
M = \lfloor 280/4 \rfloor = 70 & x = \lceil\sqrt{70}\rceil = 9 & x^2 = 81 \\
M = \lfloor 70/4 \rfloor = 17 & x = \lceil\sqrt{17}\rceil = 5 & x^2 = 25 \\
M = \lfloor 17/4 \rfloor = 4 & x = \lceil\sqrt{4}\rceil = 2 & x^2 = 4 \\
M = \lfloor 4/4 \rfloor = 1 & x = \lceil\sqrt{1}\rceil = 1 & x^2 = 1
\end{array}$$

Computing $\lceil \sqrt{N} \rceil$ (Problem 4.12)

Algorithm 1 Computing $\lceil \sqrt{N} \rceil$.

```
procedure SQRT-ROOT($N$)
    if $N < 3$ then
        return $1 \Rightarrow (1,1); 2 \Rightarrow (2,4); 3 \Rightarrow (2,4)$
    $M \leftarrow \lfloor N/4 \rfloor$
    $(x, x^2) \leftarrow$ SQRT-ROOT($M$)
    return the $(y, y^2)$ with $y^2 \sim N$:
```

$$(y, y^2) = \begin{cases} y = 2x & y^2 = 4x^2 \\ y = 2x+1 & y^2 = 4x^2 + 4x + 1 \\ y = 2x-1 & y^2 = 4x^2 - 4x + 1 \end{cases}$$

Computing $\lceil \sqrt{N} \rceil$

~~$T(n) = T(n/4) + O(1) = \Theta(\lg n)$~~

$$T(n) = T(n-2) + O(1) = \Theta(n)$$

Space for hashing (Problem 4.4)

Key:	x
Node:	y
Load factor:	α

Space for hashing (Problem 4.4)

Key: x
Node: y
Load factor: α

- Closed-address hashing

$$h_c + \alpha y h_c$$

- Open-address hashing

$$\frac{\alpha h_c}{\frac{h_c + \alpha y h_c}{x}} = \frac{\alpha x}{1 + \alpha y}$$

THANK YOU