Decompositions of Graphs

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June 12, 2018

John Hopcroft

Robert Tarjan

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Robert Tarjan

"For fundamental achievements in the design and analysis of algorithms and data structures."

— *Turing Award, 1986*

SIAM J. COMPUT.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

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Key words. Algorithm, backtracking, biconnectivity, connectivity, depth-first, graph, search, spanning tree, strong-connectivity.

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"DFS is a powerful technique with many applications."

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Power of DFS:

Graph Traversal $\Longrightarrow$ Graph Decomposition

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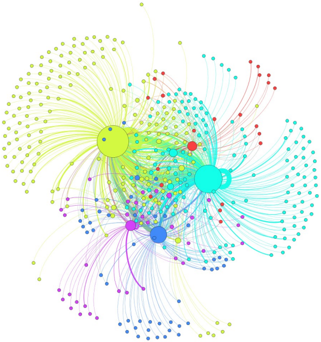
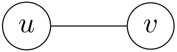
Graph Traversal $\Longrightarrow$ Graph Decomposition

Structure! Structure! Structure!



Graph *structure* induced by DFS:

states of v

types of 

Graph *structure* induced by DFS:

states of v

types of u — v

life time of v:

v :$\mathsf{d}[v], \mathsf{f}[v]$

$\mathsf{d}[v]$: BICOMP

$\mathsf{f}[v]$: TOPOSORT, SCC

Definition (Classifying edges)

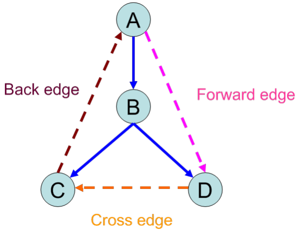
Given a DFS traversal $\implies$ DFS tree:

Tree edge: $\to$ child

Back edge: $\to$ ancestor

Forward edge: $\to$ *nonchild* descendant

Cross edge: $\to$ $(\neg\text{ancestor}) \wedge (\neg\text{descendant})$



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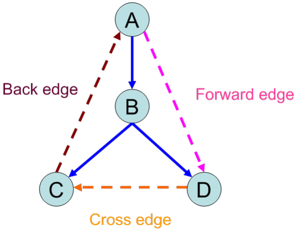
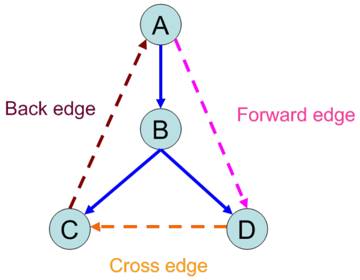
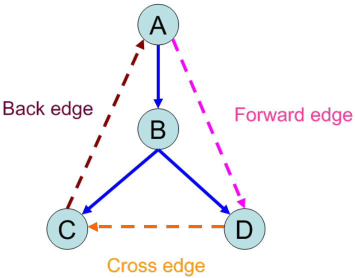


- Also applicable to BFS
- w.r.t. DFS/BFS trees



DFS on directed graph



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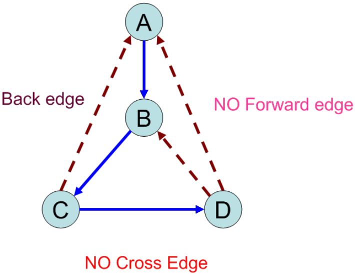
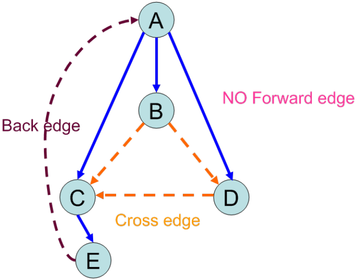
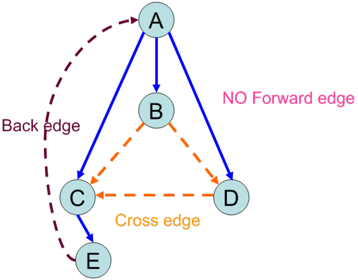


DFS on undirected graph



BFS on directed graph



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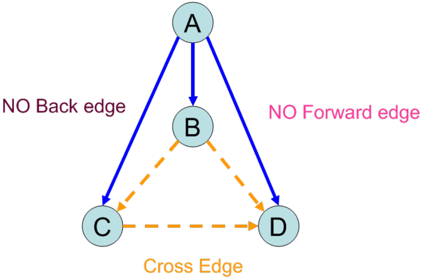

BFS on undirected graph (Problem 5.1)

DFS tree and BFS tree coincide (Problem 5.7)

Undirected connected graph $G = (V, E), v \in V$

DFS tree T from $v \equiv$ BFS tree T' from v

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Proof.

G_{DFS}: tree + back *vs.* G_{BFS}: tree + cross

□

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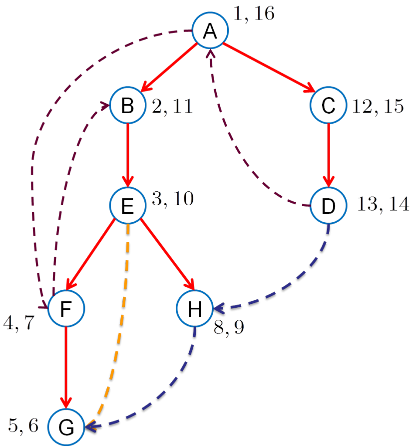
Q : What if G is a digraph?

Lift time of vertices in DFS



Theorem (Disjoint or Contained (Problem $4.2 : (1)\&(2)$))

$$\forall u, v : [_u\ ]_u \cap [_v\ ]_v = \emptyset \bigvee \Big([_u\ ]_u \subset [_v\ ]_v \vee [_v\ ]_v \subset [_u\ ]_u \Big)$$

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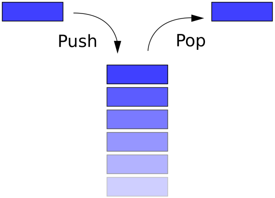
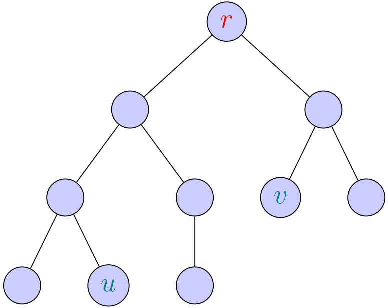


□

Preprocessing for ancestor/descendant relation (Problem 5.6)



Q : Is u an ancestor of v? $O(1)$

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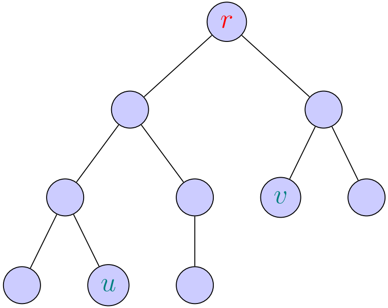

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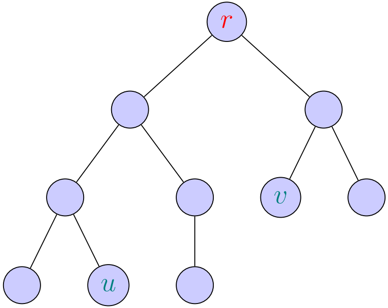

Q : *Is u an ancestor of v?* $O(1)$

$$v : \mathsf{d}[v], \mathsf{f}[v]$$

Q : $\#$ of descendants of any v?

Edge types and life time of vertices in DFS (Problem 4.5)

$$\forall u \to v :$$

- tree/forward edge: $[_u \; [_v \;]_v \;]_u$
- back edge: $[_v \; [_u \;]_u \;]_v$
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$$\mathsf{f}[v] < \mathsf{d}[u] \iff \qquad \text{edge}$$

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$$\nexists \text{ cycle} \implies \boxed{u \to v \iff \mathsf{f}[v] < \mathsf{f}[u]}$$

DFS from the perspective of a single node:

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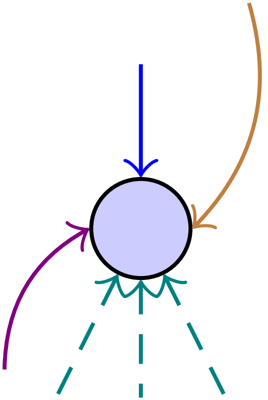

Height and diameter of tree (Problem 5.4)

Binary tree $T = (V, E)$ with $|V| = n$ and the root r:

(I) Height $H(T)$ in $O(n)$

(II) Diameter $D(T)$ in $O(n)$

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A beautiful algorithm:

- Pick any u
- Run BFS from u, obtain the farthest v
- Run BFS from v, obtain the farthest w

$$(v, w)$$

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Your Job: Prove it!

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Back to our original problem with $u \leftarrow r$.

Cycle detection (Problem $5.8-1$)

	Digraph	Undirected graph
DFS		
BFS		

Cycle detection (Problem $5.8 - 1$)

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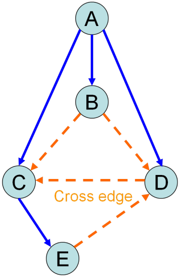

Evasiveness of acyclicity of undirected graphs (Problem $5.8-2$)

$$\text{Evasiveness} \triangleq \text{check} \binom{n}{2} \text{ edges (adjacency matrix)}$$

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By Adversary Argument.

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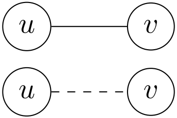

Algorithm $\mathbb{A}$:

$\textsc{CheckEdge}(u, v)$

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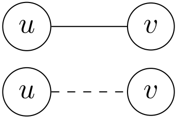

Algorithm $\mathbb{A}$:

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Hint: Kruskal

$$\mathbb{A} : \textsc{CheckEdge}(u, v) \leftarrow \mathcal{A} : (u) \text{———} (v)$$

$$\iff$$

$$\mathcal{A} : \nexists \text{ cycle } \in G + (u) \text{———} (v)$$

$$\mathbb{A} : \textsc{CheckEdge}(u, v) \leftarrow \mathcal{A} : u \text{ --- } v$$

$$\Longleftrightarrow$$

$$\mathcal{A} : \nexists \text{ cycle } \in G + u \text{ --- } v$$

Q : Why adjacency matrix?

After-class Exercise: Evasiveness of connectivity of undirected graphs

$$\text{Evasiveness} \triangleq \text{check } \binom{n}{2} \text{ edges (adjacency matrix)}$$

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Hint: Anti-Kruskal

Orientation of undirected graph (Problem 4.13)

- undirected (connected) graph G
- edges oriented *s.t.*

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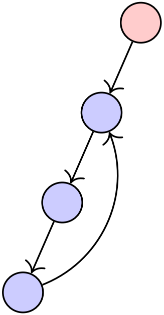

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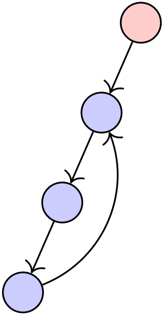

Shortest cycle of undirected graph (Problem 4.12)

A WRONG DFS-based algorithm:

$$\forall v : \text{level}[v]$$

$$\text{Back edge } u \to v : \text{level}[u] - \text{level}[v] + 1$$

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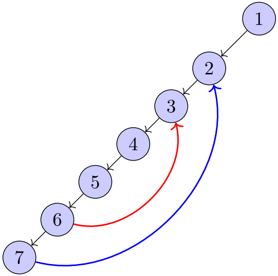

Shortest cycle of digraph (Problem 4.12)

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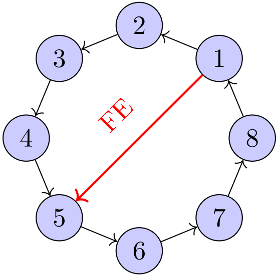

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TOPOSORT by Tarjan (probably), 1976

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Sort vertices in *decreasing* order of their *finish* times.

Kahn's TOPOSORT algorithm (1962; Problem 4.16)

- Queue Q for source vertices (in$[v] = 0$)
- Repeat: $\textsc{Dequeue}(\exists u \in Q)$, output u

  delete u and $u \to v$ from Q,

  $\textsc{Enqueue}(v)$ if in$[v] = 0$

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Lemma (Correctness of Kahn's TOPOSORT)

Every DAG has at least one source (and at least one sink vertex).

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- Repeat: $\textsc{Dequeue}(\exists u \in Q)$, output u

  delete u and $u \to v$ from Q,

  $\textsc{Enqueue}(v)$ if in$[v] = 0$

$$O(m+n)$$

Lemma (Correctness of Kahn's $\textsc{Toposort}$)

Every DAG has at least one source (and at least one sink vertex).

Q : What if G is *not* a DAG?

Taking courses in few semesters (Problem 5.14)

- n courses
- m of $c_1 \to c_2$: prerequisite
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For general digraph, LONGEST-PATH is NP-hard.

Line up (Problem 4.22)

1. i hates j: $i \succ j$
2. i hates j: $\#i < \#j$

Toposort

Critical path *OR* Longest path

Hamiltonian path in DAG (Problem 4.14)

HP: path visiting each vertex once

Q : $\exists$ HP in a DAG in $O(n+m)$

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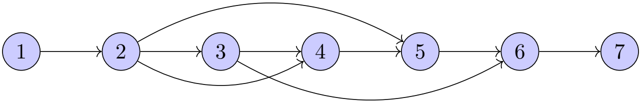

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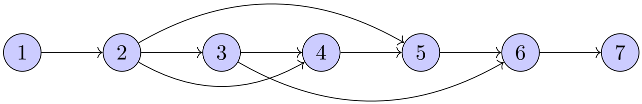

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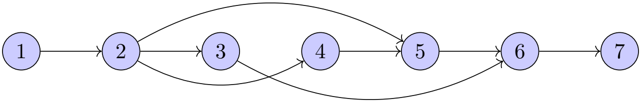

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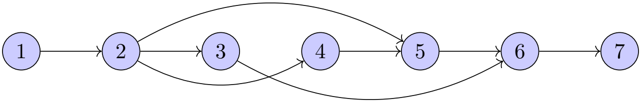

Kahn's TOPOSORT (Problem 4.16)

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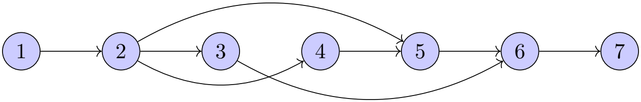

Kahn's TOPOSORT (Problem 4.16)

$$|Q| \leq 1$$

Theorem (Digraph as DAG (Problem 4.6))

Every digraph is a dag of its SCCs.

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Two tiered structure of digraphs:

digraph $\equiv$ a dag of SCCs

SCC: equivalence class over reachability

$\boxed{\text{digraph} \equiv \text{a dag of SCCs}}$

Kosaraju's SCC algorithm, 1978

"SCCs can be topo-sorted

in decreasing order of their highest finish time."

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One-to-all reachability in a digraph (Problem 5.12)

$$v : v \leadsto^{?} \forall u$$

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$\Longleftarrow$: $\exists!$ source

$\Longrightarrow$: By contradiction.

$$\exists u : v \not\leadsto u \land \text{in}[u] > 0 \implies \exists \text{ cycle}$$

Impacts of vertices in a digraph (Problem 4.18)

$$\mathsf{impact}(v) = |\{w \neq v : v \rightsquigarrow w\}|$$

- $\arg\min_v \mathsf{impact}(v)$
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Q : $\forall v$, computing $\text{impact}(v)$

Bicomp: Back!

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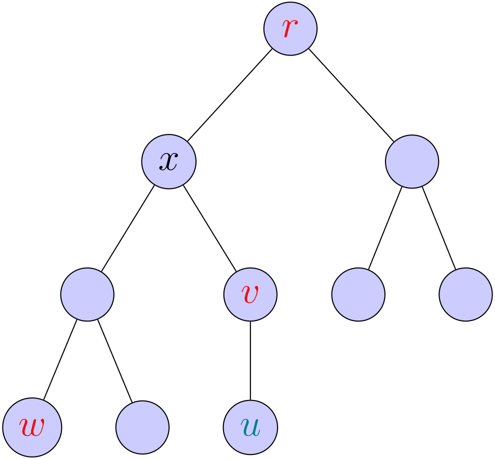

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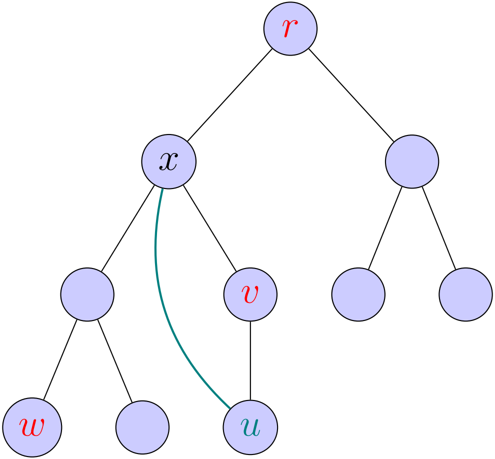

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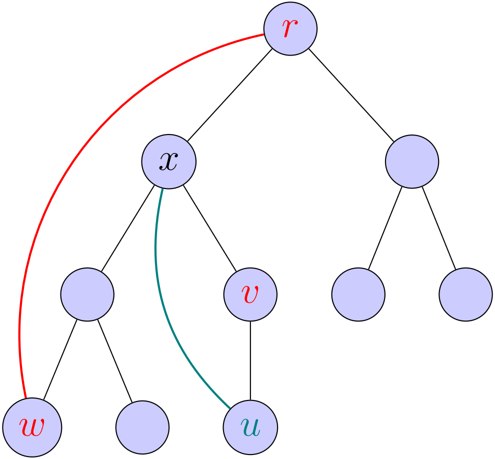

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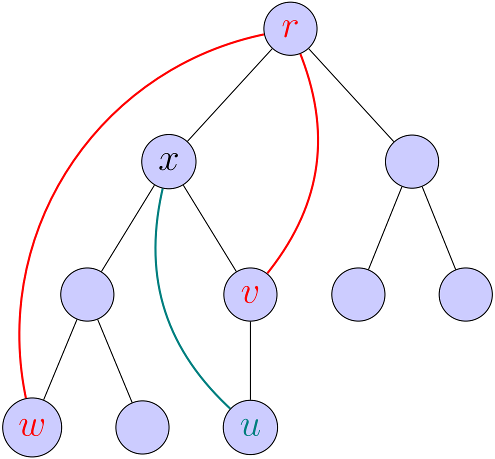

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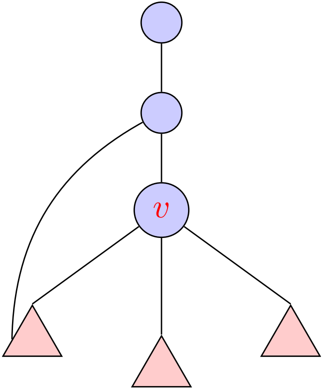

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□

Root cutnode v (Problem 4.8)

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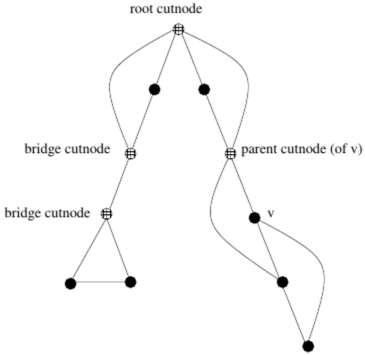


THANK YOU