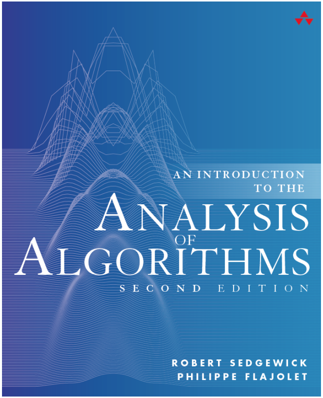
Algorithm Analysis, PMI, Asymptotics, and Recurrences

Hengfeng Wei

hfwei@nju.edu.cn

April 17, 2018


AN INTRODUCTION
TO THE
ANALYSIS
OF
ALGORITHMS
SECOND EDITION
ROBERT SEDGEWICK
PHILIPPE FLAJOLET

Problem P Algorithm A

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Average-case Time Complexity (Problem 1.8)

$$r \in [1, n],\ r \in \mathbb{Z}^+$$

$$P\{r = i\} = \begin{cases} \frac{1}{n}, & 1 \le i \le \frac{n}{4} \\ \frac{2}{n}, & \frac{n}{4} < i \le \frac{n}{2} \\ \frac{1}{2n}, & \frac{n}{2} < i \le n \end{cases} \qquad T(r) = \begin{cases} 10, & r \le \frac{n}{4} \\ 20, & \frac{n}{4} < r \le \frac{n}{2} \\ 30, & \frac{n}{2} < r \le \frac{3n}{4} \\ n, & \frac{3n}{4} < r \le n \end{cases}$$

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$$\begin{aligned} A &= \sum_{X \in \mathcal{X}} T(X) \cdot P(X) \\ &= T(1)P(1) + T(2)P(2) + \cdots + T(n)P(n) \\ &= \frac{n}{4} \times 10 \times \frac{1}{n} + \frac{n}{4} \times 20 \times \frac{2}{n} + \frac{n}{4} \times 30 \times \frac{1}{2n} + \frac{n}{4} \times n \times \frac{1}{2n} \\ &= \cdots \end{aligned}$$

3-element Sorting (Problem 1.1)

(1) Design an algorithm for sorting 3 distinct elements.

(2) Worst-case and average-case time complexity.

(3) Worst-case lower bound.

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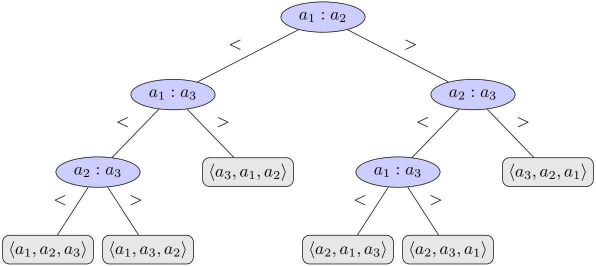

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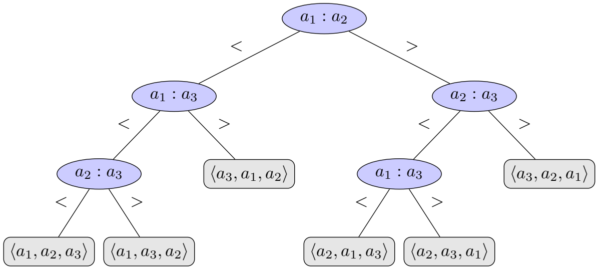

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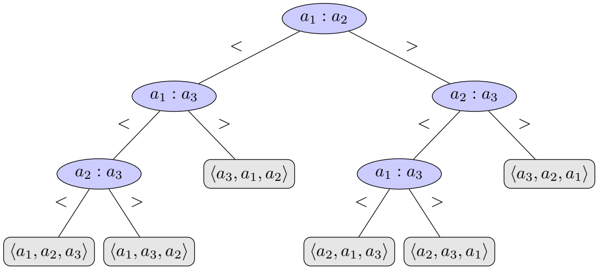

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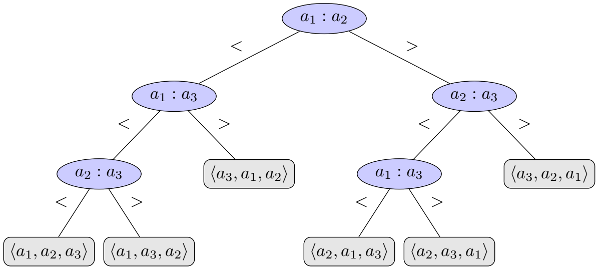

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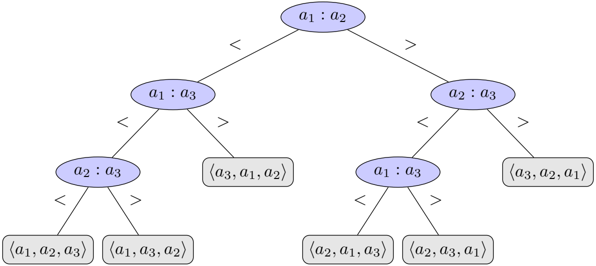

$W(3) = 3 \qquad B(3) = 2$

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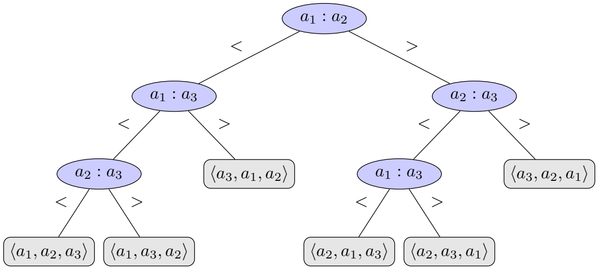

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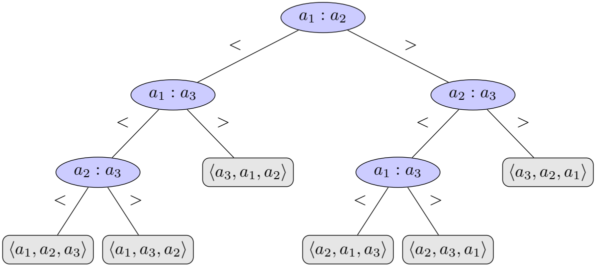

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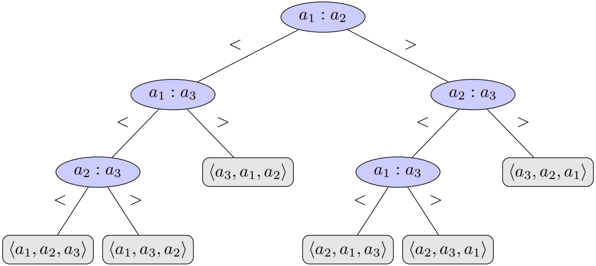

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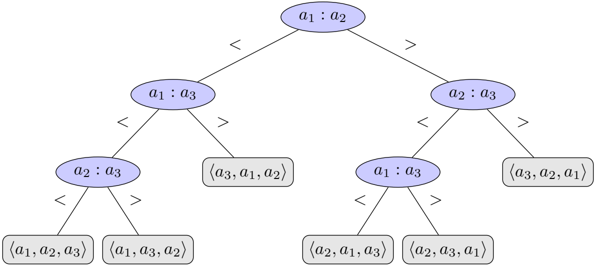

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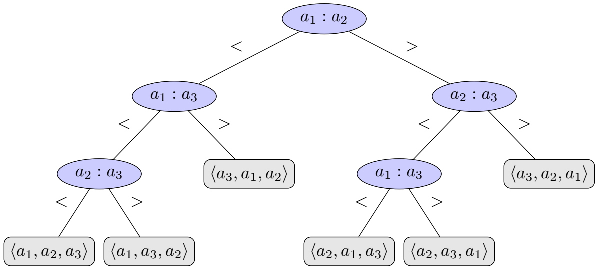

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3-element Median Seletion (Problem 1.2)

(1) Design an algorithm for selecting the median of 3 distinct elements.

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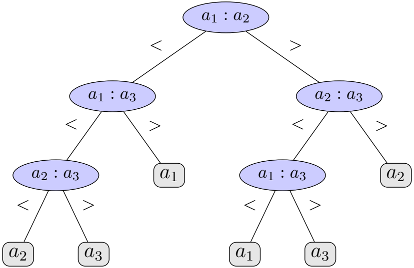

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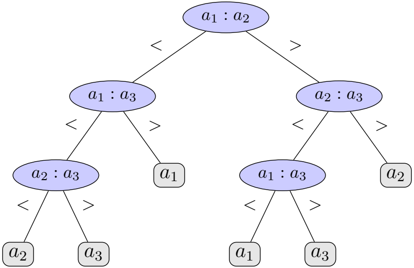

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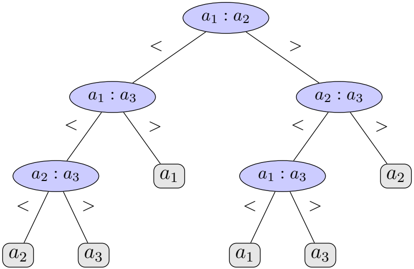

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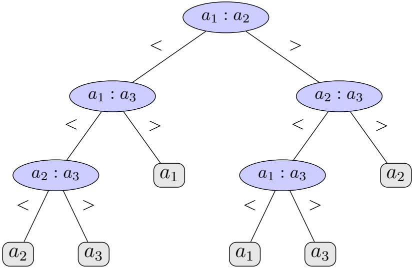

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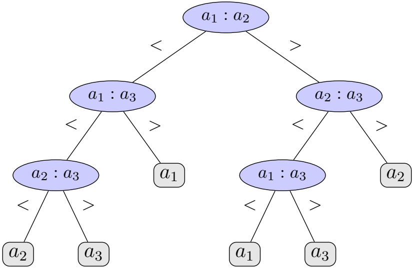

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$$\mathsf{LB}(3) = 3 \qquad (\mathsf{LB}(3) \geq \frac{3n}{2} - \frac{3}{2})$$

$\text{LB} = 2$

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procedure $\textsc{Median}(a, b, c)$
    if $(a - b)(a - c) < 0$ **then**
        return a
    if $(b - a)(b - c) < 0$ **then**
        return b
    return c

$\mathsf{LB} = 2$

```
procedure MEDIAN(a, b, c)
    if (a − b)(a − c) < 0 then
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    if (b − a)(b − c) < 0 then
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```

Not comparison-based!

Exercise

$$n = 5$$

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Reference

"The Art of Computer Programming, Vol 3: Sorting and Searching (Section 5.3.1)" by Donald E. Knuth

$$S(21) = 66$$

Analysis of Bubblesort (Problem 3.2)

(a) Correctness

(b) $W(n)$ & $A(n)$

(c) Improved version $A'(n)$:

$$l \triangleq \max_k \left\{ \textsc{Swap}(A[k], A[k+1]) \right\}$$

```
procedure BUBBLESORT(A[1 ··· n])
    for i ← n downto 2 do
        for j ← 1 to i − 1 do
            if A[j] > A[j + 1] then
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$$A'(n) = \frac{1}{2}\Big(n^2 - n \ln n - (\gamma + \ln 2 - 1)n\Big) + O(\sqrt{n}) =$$

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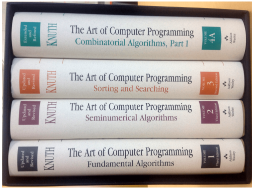
People who analyze algorithms have double happiness.

— Donald E. Knuth (1995)



People who analyze algorithms have double happiness.

Then they receive a practical payoff when their theories make it possible to get other jobs done more quickly and more economically.

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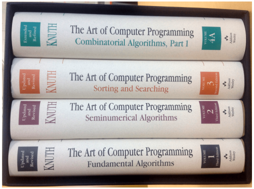

People who analyze algorithms have double happiness.

First of all they experience the sheer beauty of elegant mathematical patterns that surround elegant computational procedures.

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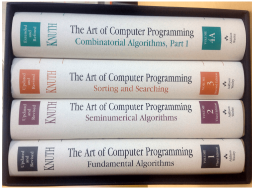

Mathematical Induction

Horner's rule (Problem 1.5)

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

```
procedure HORNER(A[0...n], x)          ▷ A : {a_0 ... a_n}
    p ← A[n]
    for i ← n − 1 downto 0 do
        p ← px + A[i]
    return p
```

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  $p \leftarrow A[n]$
  for $i \leftarrow n - 1$ **downto** 0 **do**
    $p \leftarrow px + A[i]$
  return p

Loop invariant (after the k-th loop):

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When you are in an exam:

20% : Finding $\mathcal{I}$

80% : Proving $\mathcal{I}$ by PMI

Prove by mathematical induction on the number k of loops.

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Base Case: $k = 0$.

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Inductive Hypothesis: $\mathcal{I}$ is valid after the k-th $(k \geq 0)$ loop.

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$$\boxed{\left(\sum_{i=n}^{i=n-k} a_i x^{k-(n-i)} \right) \cdot x + A[n-k-1] = \sum_{i=n}^{i=n-(k+1)} a_i x^{(k+1)-(n-i)}}$$

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Termination

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Termination

(a) $i \leftarrow n-1$ **downto** 0

(b) $k = n \implies p = \sum_{i=0}^{i=n} a_i x^i$

Integer Multiplication (Problem 1.6)

---

procedure $\textsc{Int-Mult}(y, z)$ $\triangleright\ y, z \geq 0;\ y, z \in \mathbb{Z}$

  if $z = 0$ **then**

    return 0

  return $\textsc{Int-Mult}(cy, \lfloor \frac{z}{c} \rfloor) + y(z \bmod c)$ $\triangleright\ c \geq 2$

Integer Multiplication (Problem 1.6)

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procedure INT-MULT(y, z)                          ▷ y, z ≥ 0; y, z ∈ ℤ
    if z = 0 then
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Prove by mathematical induction on

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Prove by mathematical induction on the non-negative integer z.

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BC: $z = 0: \quad \textsc{Int-Mult}(y, 0) = 0 = y \cdot 0.$

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I.H.: $< z \ (z > 0): \quad \textsc{Int-Mult}(y, z) = yz.$

Integer Multiplication (Problem 1.6)

---

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  if $z = 0$ **then**
    return 0
  return $\textsc{Int-Mult}(cy, \lfloor \frac{z}{c} \rfloor) + y(z \bmod c)$ $\quad\triangleright\ c \geq 2$

Prove by mathematical induction on the non-negative integer z.

BC: $z = 0: \quad \textsc{Int-Mult}(y, 0) = 0 = y \cdot 0.$

I.H.: $< z \ (z > 0): \quad \textsc{Int-Mult}(y, z) = yz.$

I.S.: $= z.$

Integer Multiplication (Problem 1.6)

```
procedure INT-MULT(y, z)                      ▷ y, z ≥ 0; y, z ∈ ℤ
    if z = 0 then
        return 0
    return INT-MULT(cy, ⌊z/c⌋) + y(z mod c)    ▷ c ≥ 2
```

Prove by mathematical induction on the non-negative integer z.

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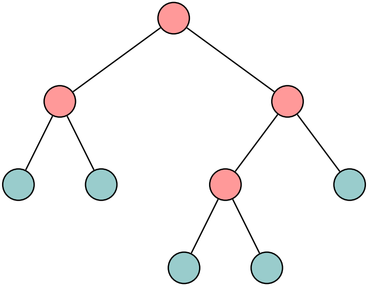
I.S.: $= z.$

$$\begin{aligned}\textsc{Int-Mult}(y, z) &= \textsc{Int-Mult}(cy, \lfloor \frac{z}{c} \rfloor) + y(z \bmod c) \\ &= cy \cdot \lfloor \frac{z}{c} \rfloor + y(z \bmod c) = yz\end{aligned}$$

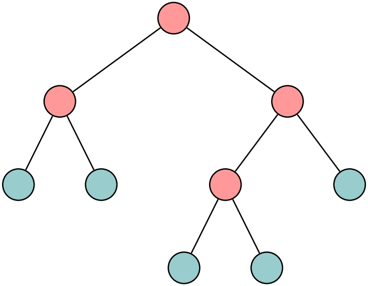
2-tree; Full Binary Tree (Problem 2.5)



$$n_0 = n_2 + 1$$

Prove by mathematical induction on

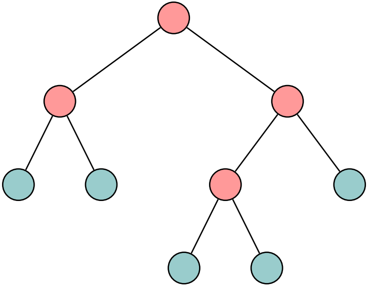
2-tree; Full Binary Tree (Problem 2.5)



$$n_0 = n_2 + 1$$

Prove by mathematical induction on *the size of binary tree.*

2-tree; Full Binary Tree (Problem 2.5)



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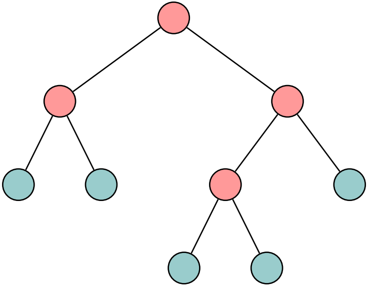
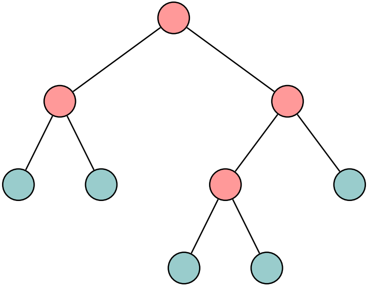


$$n_{0L} = n_{2L} + 1$$

$$n_{0R} = n_{2R} + 1$$

2-tree; Full Binary Tree (Problem 2.5)

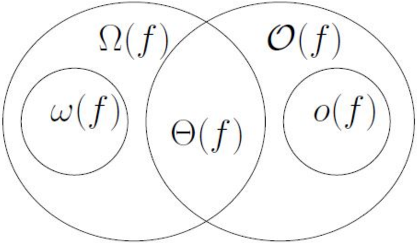


$$n_{0L} = n_{2L} + 1$$

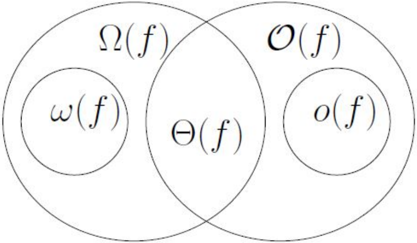
$$n_{0R} = n_{2R} + 1$$

$$n_{0L} + n_{0R} \textit{ vs. } n_{2L} + n_{2R} + 1$$

Asymptotics



Asymptotics



$$Q: \theta(f)?$$

$$O(g(n)) = \{f(n) \mid \exists c > 0, \exists n_0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n)\}$$

$$\Omega(g(n)) = \{f(n) \mid \exists c > 0, \exists n_0, \forall n \geq n_0 : 0 \leq cg(n) \leq f(n)\}$$

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$$\Theta(g(n)) = \{f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0, \forall n \geq n_0 : \\ 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$$

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$$\omega(g(n)) = \{f(n) \mid \forall c > 0, \exists n_0, \forall n \ge n_0 : 0 \le cg(n) \le f(n)\}$$

$$f(n) \sim g(n) \iff \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$$

Asymptotics (Problem 2.6 (4))

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \land f(n) = \Omega(g(n))$$

Asymptotics (Problem 2.6 (4))

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$$\Theta(g(n)) \cap o(g(n)) = \emptyset$$

$Q : f(n) = O(g(n)) \lor g(n) = \Omega(f(n))?$

Asymptotics (Problem 2.6 (4))

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$$\Theta(g(n)) \cap o(g(n)) = \emptyset$$

$Q : f(n) = O(g(n)) \lor g(n) = \Omega(f(n))?$

$$f(n) = n, \quad g(n) = n^{1+\sin n}$$

Asymptotics (Problem 2.6 (4))

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Asymptotics (Problem 2.6 (6))

$$\Theta(g(n)) \cap o(g(n)) = \emptyset$$

$Q : f(n) = O(g(n)) \lor g(n) = \Omega(f(n))?$

$$f(n) = n, \quad g(n) = n^{1+\sin n}$$

Reference:

"Big Omicron and Big Omega and Big Theta" by Donald E. Knuth, 1976.

Asymptotics (Problem 2.7 (2))

$$(\log n)^2 \quad \textit{vs.} \quad \sqrt{n}$$

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$$(\log n)^2 \quad \textit{vs.} \quad \sqrt{n}$$

$$(\log n)^{c_1} = O(n^{c_2}) \quad c_1, c_2 > 0$$

Asymptotics (Problem 2.10)

$$\log(n!) = \Theta(n \log n)$$

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Stirling Formula (by *James Stirling*):

$$n! = \Theta\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)$$

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$$\log(n!) \leq n \log n \qquad \log(n!) \geq \frac{n}{2} \log \frac{n}{2}$$

Summation (Problem 2.20)

```
procedure CONUNDRUM(n)
    r ← 0
    for i ← 1 to n do
        for j ← i + 1 to n do
            for k ← i + j − 1 to n do
                r ← r + 1
    return r
```

Summation (Problem 2.20)

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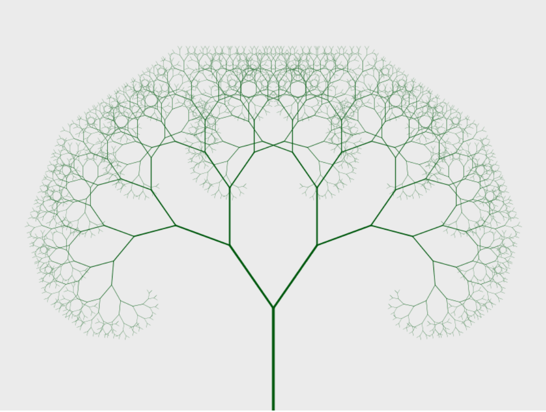
$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 =$$

Summation (Problem 2.20)

```
1: procedure CONUNDRUM(n)
2:     r ← 0
3:     for i ← 1 to n do
4:         for j ← i + 1 to n do
5:             for k ← i + j − 1 to n do
6:                 r ← r + 1
7:     return r
```

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \frac{n^2 - n}{2} = \Theta(n^2)$$

Recurrences



$$T(n) = aT(n/b) + f(n) \quad (a > 0, b > 1)$$

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Assume that $T(n)$ is constant for sufficiently small n.

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$$\left.\begin{array}{r} f(n) \\ af(\frac{n}{b}) \\ a^2 f(\frac{n}{b^2}) \\ \vdots \\ a^{\log_b n} f(c) = \Theta(n^{\log_b a}) \end{array}\right\}$$

$$T(n) = aT(n/b) + f(n) \quad (a > 0, b > 1)$$

Assume that $T(n)$ is constant for sufficiently small n.

$$\left.\begin{array}{r} f(n) \\ af(\frac{n}{b}) \\ a^2 f(\frac{n}{b^2}) \\ \vdots \\ a^{\log_b n} f(c) = \Theta(n^{\log_b a}) \end{array}\right\} \sum \quad f(n) \underline{\text{vs.}} \; n^E \begin{cases} n^{\log_b a} & f(n) = O(n^{E-\epsilon}) \\ n^{\log_b a} \log n & f(n) = \Theta(n^E) \\ f(n) & f(n) = \Omega(n^{E+\epsilon}) \end{cases}$$

Solving Recurrences (Problem 2.15)

(1) $\Theta(n^{\log_3 2})$

(2) $\Theta(\log^2 n)$

(3) $\Theta(n)$

(4) $\Theta(n \log n)$

(5) $\Theta(n \log^2 n)$

(6) $\Theta(n^2)$

(7) $\Theta(n^{\frac{3}{2}} \log n)$

(8) $\Theta(n)$

(9) $\Theta(n^{c+1})$

(10) $\Theta(c^{n+1})$

(11) $\cdots$

$$T(n) = T(n/2) + \log n$$

$$T(n) = 2T(n/2) + n \log n$$

Solving Recurrences (Problem 2.15)

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(11) $\cdots$

$$T(n) = T(n/2) + \log n$$

$$T(n) = 2T(n/2) + n \log n$$

Reference:

$$f(n) = \Theta(n^{\log_b a} \log^k n) \implies \Theta(n^{\log_b a} \log^{k+1} n)$$

Solving Recurrences (Problem 2.15)

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Gaps in Master Theorem (Problem 2.18)

Solving Recurrences (Problem 2.15)

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(2) $\Theta(\log^2 n)$

(3) $\Theta(n)$

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Gaps in Master Theorem (Problem 2.18)

$$T(n) = 2T(n/2) + \frac{n}{\log n}$$

Solving Recurrences (Problem 2.15)

(1) $\Theta(n^{\log_3 2})$

(2) $\Theta(\log^2 n)$

(3) $\Theta(n)$

(4) $\Theta(n \log n)$

(5) $\Theta(n \log^2 n)$

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Reference:

$$f(n) = \Theta(n^{\log_b a} \log^k n) \implies \Theta(n^{\log_b a} \log^{k+1} n)$$

Gaps in Master Theorem (Problem 2.18)

$$T(n) = 2T(n/2) + \frac{n}{\log n} = \Theta(n \log \log n)$$

Solving Recurrences (Problem 2.15)

(1) $\Theta(n^{\log_3 2})$

(2) $\Theta(\log^2 n)$

(3) $\Theta(n)$

(4) $\Theta(n \log n)$

(5) $\Theta(n \log^2 n)$

(6) $\Theta(n^2)$

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(9) $\Theta(n^{c+1})$

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(11) $\cdots$

$$T(n) = T(n-1) + n^c \quad c \geq 1$$

$$T(n) = T(n-1) + c^n \quad c > 1$$

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8)$$

Solving Recurrences (Problem 2.15 (11))

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Solving Recurrences (Problem 2.15 (11))

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$$\boxed{T(n) = \Theta(n^{\alpha})}$$

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8)$$

$$\boxed{T(n) = \Theta(n^{0.879146})}$$

$$\boxed{T(n) = \Theta(n^{\alpha})}$$

$$2^{-\alpha} + 4^{-\alpha} + 8^{-\alpha} = 1$$

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8)$$

$$\boxed{T(n) = \Theta(n^{0.879146})}$$

$$\boxed{T(n) = \Theta(n^{\alpha})}$$

$$2^{-\alpha} + 4^{-\alpha} + 8^{-\alpha} = 1$$

```
Solve[2^{-x} + 4^{-x} + 8^{-x} == 1, x] // N
```

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8) + n$$

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8) + n$$

By recursion-tree.

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$$T(n) = T(n/2) + T(n/4) + T(n/8) + n$$

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Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8) + n$$

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$$\boxed{T(n) = \Theta(n)}$$

Exercise: Prove it by mathematical induction.

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8) + n$$

By recursion-tree.

$$\boxed{T(n) = \Theta(n)}$$

Exercise: Prove it by mathematical induction.

Reference:

"On the Solution of Linear Recurrence Equations" by Akra & Bazzi, 1996.

$$T(n) = \sum_{i=1}^{k} a_i T(n/b_i) + f(n)$$

Solving Recurrences (Problem 2.17)

$$
\begin{aligned}
\mathsf{T}(n) &= \sqrt{n}\ \mathsf{T}(\sqrt{n}) + n \\
&= n^{\frac{1}{2}}\ \mathsf{T}\left(n^{\frac{1}{2}}\right) + n \\
&= n^{\frac{1}{2}}\left(n^{\frac{1}{2^2}}\ \mathsf{T}\left(n^{\frac{1}{2^2}}\right) + n^{\frac{1}{2}}\right) + n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}}\ \mathsf{T}\left(n^{\frac{1}{2^2}}\right) + 2n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}}\left(n^{\frac{1}{2^3}}\ \mathsf{T}\left(n^{\frac{1}{2^3}}\right) + n^{\frac{1}{2^2}}\right) + 2n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}}\ \mathsf{T}\left(n^{\frac{1}{2^3}}\right) + 3n \\
&= \cdots \\
&= n^{\sum_{i=1}^{k}\frac{1}{2^i}}\ \mathsf{T}\left(n^{\frac{1}{2^k}}\right) + kn
\end{aligned}
$$

$$n^{\frac{1}{2^k}} = 2$$

$$n^{\frac{1}{2^k}} = 2 \implies k = \log\log n$$

$$n^{\frac{1}{2^k}} = {\color{red}2} \implies k = \log\log n$$

$$\begin{aligned}
\mathsf{T}(n) &= n^{\sum_{i=1}^{k} \frac{1}{2^i}}\ \mathsf{T}\left(n^{\frac{1}{2^k}}\right) + kn \\
&= n^{\sum_{i=1}^{\log\log n} \frac{1}{2^i}}\ \mathsf{T}(2) + n\log\log n
\end{aligned}$$

$$n^{\frac{1}{2^k}} = 2 \implies k = \log\log n$$

$$\begin{aligned} \mathsf{T}(n) &= n^{\sum_{i=1}^{k} \frac{1}{2^i}}\ \mathsf{T}\left(n^{\frac{1}{2^k}}\right) + kn \\ &= n^{\sum_{i=1}^{\log\log n} \frac{1}{2^i}}\ \mathsf{T}(2) + n \log\log n \end{aligned}$$

$$\sum_{i=1}^{\log_2 \log_2(n)} \frac{1}{2^i} < 1 \implies T(n) = \Theta(n \log\log n)$$

$$n^{\frac{1}{2^k}} = 2 \implies k = \log\log n$$

$$\begin{aligned}\mathsf{T}(n) &= n^{\sum_{i=1}^{k} \frac{1}{2^i}}\ \mathsf{T}\left(n^{\frac{1}{2^k}}\right) + kn \\ &= n^{\sum_{i=1}^{\log\log n} \frac{1}{2^i}}\ \mathsf{T}(2) + n\log\log n\end{aligned}$$

$$\sum_{i=1}^{\log_2\log_2(n)} \frac{1}{2^i} < 1 \implies T(n) = \Theta(n\log\log n)$$

Exercise: Prove it by mathematical induction.

$$\boxed{T(n) = \sqrt{n}T(\sqrt{n}) + n}$$

$$\boxed{T(n) = \sqrt{n}T(\sqrt{n}) + n}$$

$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

$$\boxed{T(n) = \sqrt{n}T(\sqrt{n}) + n}$$

$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

$$n \leftrightarrow 2^m$$

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$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

$$n \leftrightarrow 2^m$$

$$\frac{T(2^m)}{2^m} = \frac{T(2^{m/2})}{2^{m/2}} + 1$$

$$\boxed{T(n) = \sqrt{n}T(\sqrt{n}) + n}$$

$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

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$$\frac{T(2^m)}{2^m} = \frac{T(2^{m/2})}{2^{m/2}} + 1$$

$$S(m) \leftrightarrow \frac{T(2^m)}{2^m}$$

$$\boxed{T(n) = \sqrt{n}T(\sqrt{n}) + n}$$

$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

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$$\frac{T(2^m)}{2^m} = \frac{T(2^{m/2})}{2^{m/2}} + 1$$

$$S(m) \leftrightarrow \frac{T(2^m)}{2^m}$$

$$S(m) = S(m/2) + 1 =$$

$$\boxed{T(n) = \sqrt{n}T(\sqrt{n}) + n}$$

$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

$$n \leftrightarrow 2^m$$

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Maximal Sum Subarray (Problem 1.3.5)

- array $A[1 \cdots n], a_i >=< 0$
- to find (the sum of) an MS in A

$$A[-2, 1, -3, \boxed{4, -1, 2, 1}, -5, 4]$$

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Trial and error.

- try subproblem MSS$[i]$: the sum of the MS (MS[i]) in $A[1 \cdots i]$
- goal: mss = MSS$[n]$
- question: Is $a_i \in$ MS$[i]$?
- recurrence:

$$\text{MSS}[i] = \max\{\text{MSS}[i-1], ???\}$$

Solution.

- subproblem MSS$[i]$: the sum of the MS *ending with* a_i or 0
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- initialization: MSS$[0] = 0$

```
procedure MSS(A[1 ··· n])
    MSS[0] ← 0
    for i ← 1 to n do
        MSS[i] ← max {MSS[i − 1] + A[i], 0}
    return max_{1≤i≤n} MSS[i]
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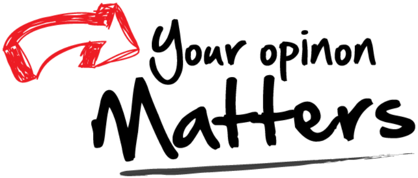
```
procedure MSS(A[1 ··· n])
    mss ← 0
    MSS ← 0

    for i ← 1 to n do
        MSS ← max {MSS + A[i], 0}
        mss ← max {mss, MSS}

    return mss
```


Thank
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn