A Guided Tour of Chapter 5: Dynamic Programming

Ashwin Rao

ICME, Stanford University

Dynamic Programming for Prediction and Control

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- Specifically applied to the setting of FiniteMarkovDecisionProcess
- Later we cover extensions such as Asynchronous DP, Approximate DP

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- Why does this work? How fast does it converge?

Banach Fixed-Point Theorem

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Theorem (Banach Fixed-Point Theorem)

Let $\mathcal{X}$ be a non-empty set equipped with a complete metric $d : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$. Let $f : \mathcal{X} \rightarrow \mathcal{X}$ be such that there exists a $L \in [0, 1)$ such that $d(f(x_1), f(x_2)) \leq L \cdot d(x_1, x_2)$ for all $x_1, x_2 \in \mathcal{X}$. Then,

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If you have a complete metric space $\langle \mathcal{X}, d \rangle$ and a contraction f (with respect to d), then you have an algorithm to solve for the fixed-point of f.

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- So we look for an iterative algorithm to solve MRP Bellman Equation:

$$\boldsymbol{V}^\pi = \boldsymbol{\mathcal{R}}^\pi + \gamma \boldsymbol{\mathcal{P}}^\pi \cdot \boldsymbol{V}^\pi$$

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- $\boldsymbol{B}^\pi$ is a contraction function under L^∞ norm: For all $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^m$,

$$\max_{s \in \mathcal{N}} |(\boldsymbol{B}^\pi(\boldsymbol{X}) - \boldsymbol{B}^\pi(\boldsymbol{Y}))(s)| = \gamma \cdot \max_{s \in \mathcal{N}} |(\boldsymbol{\mathcal{P}}^\pi \cdot (\boldsymbol{X} - \boldsymbol{Y}))(s)|$$

$$\leq \gamma \cdot \max_{s \in \mathcal{N}} |(\boldsymbol{X} - \boldsymbol{Y})(s)|$$

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Theorem (Policy Evaluation Convergence Theorem)

For a Finite MDP with $|\mathcal{N}| = m$ *and* $\gamma < 1$*, if* $\boldsymbol{V}^\pi \in \mathbb{R}^m$ *is the Value Function of the MDP when evaluated with a fixed policy* $\pi : \mathcal{N} \times \mathcal{A} \rightarrow [0, 1]$*, then* $\boldsymbol{V}^\pi$ *is the unique Fixed-Point of the Bellman Policy Operator* $\boldsymbol{B}^\pi : \mathbb{R}^m \rightarrow \mathbb{R}^m$*, and*

$$\lim_{i \to \infty} (\boldsymbol{B}^\pi)^i(\boldsymbol{V_0}) \rightarrow \boldsymbol{V}^\pi \text{ for all starting Value Functions } \boldsymbol{V_0} \in \mathbb{R}^m$$

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Running time of each iteration is $O(m^2)$. Constructing the MRP from the MDP and the policy takes $O(m^2 k)$ operations, where $m = |\mathcal{N}|, k = |\mathcal{A}|$.

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$$G(\boldsymbol{V})(s) = \pi'_D(s) = \arg\max_{a \in \mathcal{A}} \{\mathcal{R}(s,a) + \gamma \sum_{s' \in \mathcal{N}} \mathcal{P}(s,a,s') \cdot \boldsymbol{V}(s')\}$$

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- Induction step is proved by monotonicity of $\boldsymbol{B}^{\pi}$ operator (for any π):

$$\text{Monotonicity Property of } \boldsymbol{B}^{\pi} : \boldsymbol{X} \geq \boldsymbol{Y} \Rightarrow \boldsymbol{B}^{\pi}(\boldsymbol{X}) \geq \boldsymbol{B}^{\pi}(\boldsymbol{Y})$$

$$\text{So } (\boldsymbol{B}^{\pi'_D})^{i+1}(\boldsymbol{V}^{\pi}) \geq (\boldsymbol{B}^{\pi'_D})^{i}(\boldsymbol{V}^{\pi}) \Rightarrow (\boldsymbol{B}^{\pi'_D})^{i+2}(\boldsymbol{V}^{\pi}) \geq (\boldsymbol{B}^{\pi'_D})^{i+1}(\boldsymbol{V}^{\pi})$$

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- This in fact is the *Policy Iteration* algorithm

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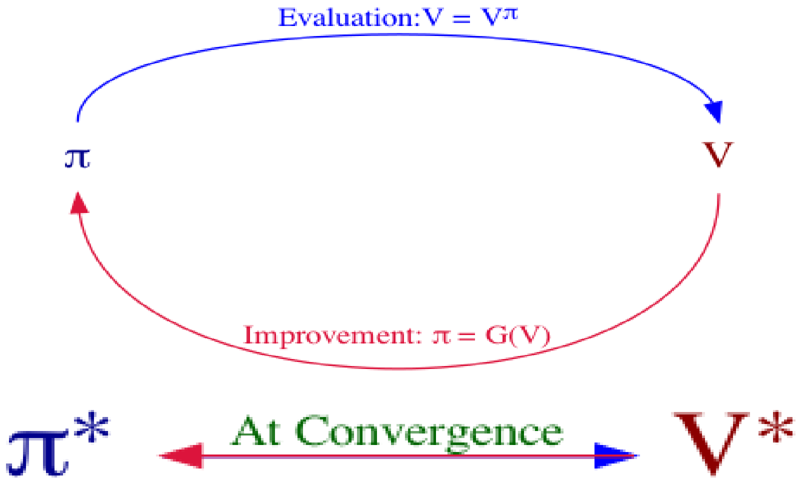
This means $\boldsymbol{V_j}$ satisfies the MDP Bellman Optimality Equation and so,

$$\boldsymbol{V_j} = \boldsymbol{V}^{\pi_j} = \boldsymbol{V}^*$$

Policy Iteration algorithm



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For a Finite MDP with $|\mathcal{N}| = m$ and $\gamma < 1$, Policy Iteration algorithm converges to the Optimal Value Function $\boldsymbol{V}^* \in \mathbb{R}^m$ along with a Deterministic Optimal Policy $\pi_D^* : \mathcal{N} \rightarrow \mathcal{A}$, no matter which Value Function $\boldsymbol{V_0} \in \mathbb{R}^m$ we start the algorithm with.

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- The action a producing the max is the action prescribed by $G(\boldsymbol{V})$. So,

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- This is the 1st stage of Policy Evaluation with improved policy $G(\boldsymbol{V}^\pi)$

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Theorem (Value Iteration Convergence Theorem)

For a Finite MDP with $|\mathcal{N}| = m$ and $\gamma < 1$, if $\boldsymbol{V}^* \in \mathbb{R}^m$ is the Optimal Value Function, then $\boldsymbol{V}^*$ is the unique Fixed-Point of the Bellman Optimality Operator $\boldsymbol{B}^* : \mathbb{R}^m \rightarrow \mathbb{R}^m$, and

$$\lim_{i \rightarrow \infty} (\boldsymbol{B}^*)^i(\boldsymbol{V_0}) \rightarrow \boldsymbol{V}^* \textit{ for all starting Value Functions } \boldsymbol{V_0} \in \mathbb{R}^m$$

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Running time of each iteration of Value Iteration is $O(m^2 k)$ where $|\mathcal{N}| = m$ and $|\mathcal{A}| = k$

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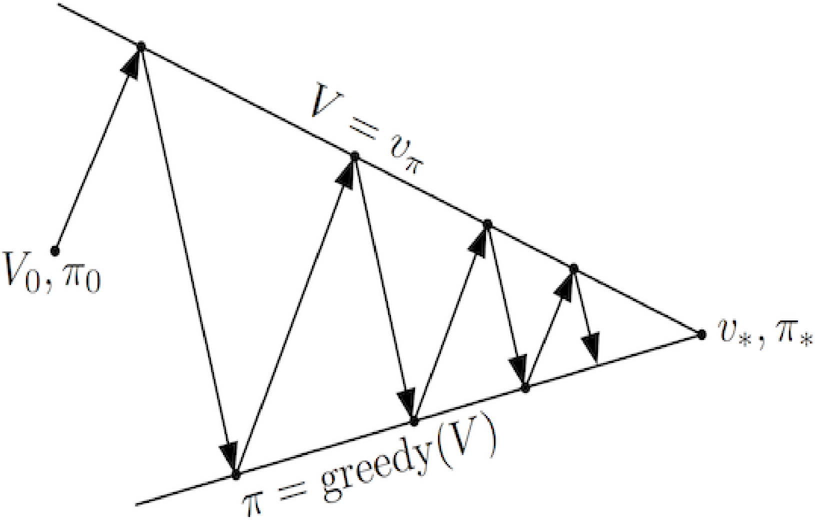
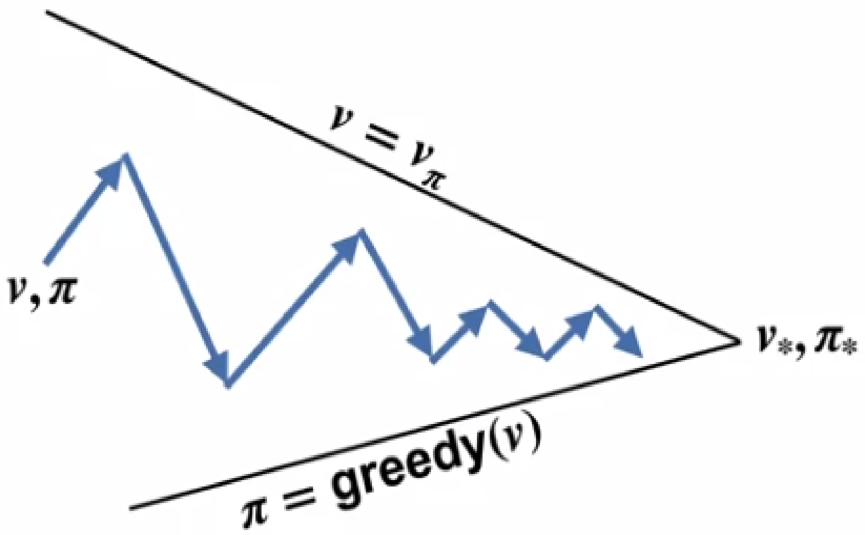
  - π being consistent with/close to Greedy Policy $G(\boldsymbol{V})$ of VF $\boldsymbol{V}$.

Policy Iteration



Generalized Policy Iteration (GPI)



Value Iteration and Reinforcement Learning as GPI

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- RL updates either a subset of states or just one state at a time

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- RL updates either a subset of states or just one state at a time
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Value Iteration and Reinforcement Learning as GPI

- Value Iteration takes only one step of Policy Evaluation

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- These can be thought of as *partial* Policy Evaluation/Improvement

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$$\text{Gaps } g(s) = |V(s) - \max_{a \in \mathcal{A}} \{\mathcal{R}(s,a) + \gamma \cdot \sum_{s' \in \mathcal{N}} \mathcal{P}(s,a,s') \cdot V(s')\}|$$

But this requires us to know the reverse transitions to resort queue

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- Next we cover a special case of DAG MDPs: finite-horizon MDPs

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  gamma: float
) -> Iterator[Tuple[
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    FiniteDeterministicPolicy[S, A]
]]

def unwrap_finite_horizon_MDP(
  process: FiniteMarkovDecisionProcess[WithTime[S], A]
) -> Sequence[StateActionMapping[S, A]]

def finite_horizon_MDP(
  process: FiniteMarkovDecisionProcess[S, A],
  limit: int
) -> FiniteMarkovDecisionProcess[WithTime[S], A]
```

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- Dynamic Pricing: Core to many businesses, flexing to supply/demand
- We consider special case of products being sold at end of life/season
- Assume we are T days from season-end and our inventory is M units
- Assume no more incoming inventory during these final T days
- Set prices daily to max *Expected Total Sales Revenue* over T days
- Price for a given day picked from prices $P_1, P_2, \ldots, P_N \in \mathbb{R}$
- Customer daily demand is $Poisson(\lambda_i)$ if Price P_i is picked for the day
- Note that demand can exceed inventory on any day, Sales $\leq$ Inventory

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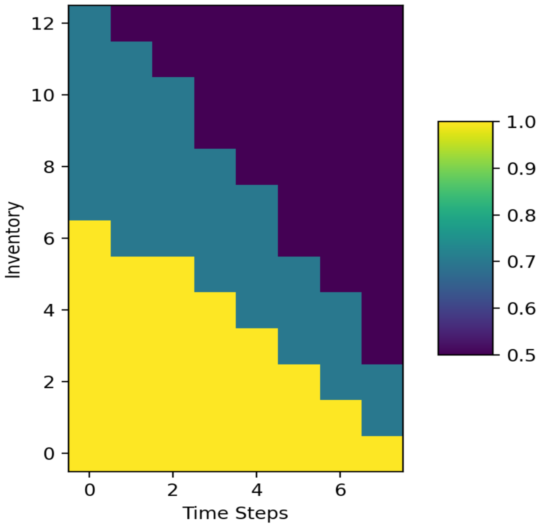
$$(\mathcal{P}_R)_t(I_t, i, r_{t+1}, I_t - k) = \begin{cases} \frac{e^{-\lambda_i} \lambda_i^k}{k!} & \text{if } k < I_t \text{ and } r_{t+1} = k \cdot P_i \\ \sum_{j=I_t}^{\infty} \frac{e^{-\lambda_i} \lambda_i^j}{j!} & \text{if } k = I_t \text{ and } r_{t+1} = k \cdot P_i \\ 0 & \text{otherwise} \end{cases}$$

Optimal Dynamic Pricing



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- These generalizations known as *Approximate Dynamic Programming*

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- Backward Induction: A straightforward method to solve finite-horizon MDPs by simply walking backwards and *setting* the Value Function from the horizon-end to the start.