A Guided Tour of Chapter 5: Dynamic Programming

Ashwin Rao

ICME, Stanford University

Dynamic Programming for Prediction and Control

- Prediction: Compute the Value Function of an MRP
- Control: Compute the Optimal Value Function of an MDP
- (Optimal Policy can be extracted from Optimal Value Function)
- Planning versus Learning: access to the $\mathcal{P}_R$ function ("model")
- Original use of *DP* term: MDP Theory *and* solution methods
- Bellman refered to DP as the *Principle of Optimality*
- Later, the usage of the term DP diffused out to other algorithms
- In CS, it means "recursive algorithms with overlapping subproblems"
- We restrict the term DP to: "Algorithms for Prediction and Control"
- Specifically applied to the setting of FiniteMarkovDecisionProcess
- Later we cover extensions such as Asynchronous DP, Approximate DP

Solving the Value Function as a *Fixed-Point*

- We will be covering 3 Dynamic Programming algorithms
- Each of the 3 algorithms is founded on the Bellman Equations
- Each is an iterative algorithm converging to the true Value Function
- Each algorithm is based on the concept of *Fixed-Point*

Definition

The Fixed-Point of a function $f : \mathcal{X} \rightarrow \mathcal{X}$ (for some arbitrary domain $\mathcal{X}$) is a value $x \in \mathcal{X}$ that satisfies the equation: $x = f(x)$.

- Some functions have multiple fixed-points, some have none
- DP algorithms are based on functions with a unique fixed-point
- Simple example: $f(x) = \cos(x)$, Fixed-Point: $x^* = \cos(x^*)$
- For any x_0, $\cos(\cos(\ldots \cos(x_0) \ldots))$ converges to fixed-point x^*
- Why does this work? How fast does it converge?

Banach Fixed-Point Theorem

Theorem (Banach Fixed-Point Theorem)

Let $\mathcal{X}$ be a non-empty set equipped with a complete metric $d : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$. Let $f : \mathcal{X} \rightarrow \mathcal{X}$ be such that there exists a $L \in [0, 1)$ such that $d(f(x_1), f(x_2)) \leq L \cdot d(x_1, x_2)$ for all $x_1, x_2 \in \mathcal{X}$. Then,

- *There exists a unique Fixed-Point $x^* \in \mathcal{X}$, i.e.,*

$$x^* = f(x^*)$$

- *For any $x_0 \in \mathcal{X}$, and sequence $[x_i | i = 0, 1, 2, \ldots]$ defined as $x_{i+1} = f(x_i)$ for all $i = 0, 1, 2, \ldots$,*

$$\lim_{i \rightarrow \infty} x_i = x^*$$

If you have a complete metric space $\langle \mathcal{X}, d \rangle$ and a contraction f (with respect to d), then you have an algorithm to solve for the fixed-point of f.

Policy Evaluation (for *Prediction*)

- MDP with $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}, \mathcal{N} = \{s_1, s_2, \ldots, s_m\}$
- Given a policy π, compute the Value Function of π-implied MRP
- $\mathcal{P}_R^\pi : \mathcal{N} \times \mathcal{D} \times \mathcal{S} \rightarrow [0, 1]$ is given as a data structure
- Extract (from $\mathcal{P}_R^\pi$) $\mathcal{P}^\pi : \mathcal{N} \times \mathcal{S} \rightarrow [0, 1]$ and $\mathcal{R}^\pi : \mathcal{N} \rightarrow \mathbb{R}$
- For non-large spaces, we can compute (in vector notation):

$$\boldsymbol{V}^\pi = (\boldsymbol{I_m} - \gamma \boldsymbol{\mathcal{P}}^\pi)^{-1} \cdot \boldsymbol{\mathcal{R}}^\pi$$

- Note: $\boldsymbol{V}^\pi, \boldsymbol{\mathcal{R}}^\pi$ are m-column vectors ($\in \mathbb{R}^m$) and $\boldsymbol{\mathcal{P}}^\pi$ is $m \times m$ matrix
- So we look for an iterative algorithm to solve MRP Bellman Equation:

$$\boldsymbol{V}^\pi = \boldsymbol{\mathcal{R}}^\pi + \gamma \boldsymbol{\mathcal{P}}^\pi \cdot \boldsymbol{V}^\pi$$

Bellman Policy Operator and it's Fixed-Point

- Define the *Bellman Policy Operator* $\boldsymbol{B}^\pi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ as:

$$\boldsymbol{B}^\pi(\boldsymbol{V}) = \boldsymbol{\mathcal{R}}^\pi + \gamma \boldsymbol{\mathcal{P}}^\pi \cdot \boldsymbol{V} \text{ for any Value Function vector } \boldsymbol{V} \in \mathbb{R}^m$$

- $\boldsymbol{B}^\pi$ is an affine transformation on vectors in $\mathbb{R}^m$
- So, the MRP Bellman Equation can be expressed as:

$$\boldsymbol{V}^\pi = \boldsymbol{B}^\pi(\boldsymbol{V}^\pi)$$

- This means $\boldsymbol{V}^\pi \in \mathbb{R}^m$ is a Fixed-Point of $\boldsymbol{B}^\pi : \mathbb{R}^m \rightarrow \mathbb{R}^m$
- Metric $d : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ defined as L^∞ norm:

$$d(\boldsymbol{X}, \boldsymbol{Y}) = \|\boldsymbol{X} - \boldsymbol{Y}\|_\infty = \max_{s \in \mathcal{N}} |(\boldsymbol{X} - \boldsymbol{Y})(s)|$$

- $\boldsymbol{B}^\pi$ is a contraction function under L^∞ norm: For all $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^m$,

$$\max_{s \in \mathcal{N}} |(\boldsymbol{B}^\pi(\boldsymbol{X}) - \boldsymbol{B}^\pi(\boldsymbol{Y}))(s)| = \gamma \cdot \max_{s \in \mathcal{N}} |(\boldsymbol{\mathcal{P}}^\pi \cdot (\boldsymbol{X} - \boldsymbol{Y}))(s)|$$
$$\leq \gamma \cdot \max_{s \in \mathcal{N}} |(\boldsymbol{X} - \boldsymbol{Y})(s)|$$

Policy Evaluation Convergence Theorem

Invoking the Banach Fixed-Point Theorem for $\gamma < 1$ gives:

Theorem (Policy Evaluation Convergence Theorem)

For a Finite MDP with $|\mathcal{N}| = m$ and $\gamma < 1$, if $\boldsymbol{V}^\pi \in \mathbb{R}^m$ is the Value Function of the MDP when evaluated with a fixed policy $\pi : \mathcal{N} \times \mathcal{A} \rightarrow [0, 1]$, then $\boldsymbol{V}^\pi$ is the unique Fixed-Point of the Bellman Policy Operator $\boldsymbol{B}^\pi : \mathbb{R}^m \rightarrow \mathbb{R}^m$, and

$$\lim_{i \to \infty} (\boldsymbol{B}^\pi)^i(\boldsymbol{V_0}) \rightarrow \boldsymbol{V}^\pi \text{ for all starting Value Functions } \boldsymbol{V_0} \in \mathbb{R}^m$$

Policy Evaluation algorithm

- Start with any Value Function $\boldsymbol{V_0} \in \mathbb{R}^m$
- Iterating over $i = 0, 1, 2, \ldots$, calculate in each iteration:

$$\boldsymbol{V_{i+1}} = \boldsymbol{B}^\pi(\boldsymbol{V_i}) = \boldsymbol{\mathcal{R}}^\pi + \gamma \boldsymbol{\mathcal{P}}^\pi \cdot \boldsymbol{V_i}$$

- Stop when $d(\boldsymbol{V_i}, \boldsymbol{V_{i+1}}) = \max_{s \in \mathcal{N}} |(\boldsymbol{V_i} - \boldsymbol{V_{i+1}})(s)|$ is small enough

Banach Fixed-Point Theorem also assures speed of convergence (dependent on choice of starting Value Function $\boldsymbol{V_0}$ and on choice of γ).

Running time of each iteration is $O(m^2)$. Constructing the MRP from the MDP and the policy takes $O(m^2 k)$ operations, where $m = |\mathcal{N}|, k = |\mathcal{A}|$.

Greedy Policy

- Now we move on to solving the MDP *Control* problem
- We want to iterate *Policy Improvements* to drive to an *Optimal Policy*
- *Policy Improvement* is based on a "greedy" technique
- The *Greedy Policy Function* $G : \mathbb{R}^m \rightarrow (\mathcal{N} \rightarrow \mathcal{A})$ (interpreted as a function mapping a Value Function vector $\boldsymbol{V}$ to a deterministic policy $\pi'_D : \mathcal{N} \rightarrow \mathcal{A}$) is defined as:

$$G(\boldsymbol{V})(s) = \pi'_D(s) = \arg\max_{a \in \mathcal{A}} \{\mathcal{R}(s, a) + \gamma \sum_{s' \in \mathcal{N}} \mathcal{P}(s, a, s') \cdot \boldsymbol{V}(s')\}$$

Definition (Value Function Comparison)

We say $X \geq Y$ for Value Functions $X, Y : \mathcal{N} \rightarrow \mathbb{R}$ of an MDP iff:

$$X(s) \geq Y(s) \text{ for all } s \in \mathcal{N}$$

We say π_1 better ("improvement") than π_2 if $\boldsymbol{V}^{\pi_1} \geq \boldsymbol{V}^{\pi_2}$

Policy Improvement Theorem

Theorem (Policy Improvement Theorem)

For a finite MDP, for any policy π,

$$\boldsymbol{V}^{\pi'_D} = \boldsymbol{V}^{G(\boldsymbol{V}^\pi)} \geq \boldsymbol{V}^\pi$$

- Note that applying $\boldsymbol{B}^{\pi'_D} = \boldsymbol{B}^{G(\boldsymbol{V}^\pi)}$ repeatedly, starting with $\boldsymbol{V}^\pi$, will converge to $\boldsymbol{V}^{\pi'_D}$ (Policy Evaluation with policy $\pi'_D = G(\boldsymbol{V}^\pi)$):

$$\lim_{i \to \infty} (\boldsymbol{B}^{\pi'_D})^i(\boldsymbol{V}^\pi) = \boldsymbol{V}^{\pi'_D}$$

- So the proof is complete if we prove that:

$$(\boldsymbol{B}^{\pi'_D})^{i+1}(\boldsymbol{V}^\pi) \geq (\boldsymbol{B}^{\pi'_D})^i(\boldsymbol{V}^\pi) \text{ for all } i = 0, 1, 2, \ldots$$

- Non-decreasing tower of Value Functions $[(\boldsymbol{B}^{\pi'_D})^i(\boldsymbol{V}^\pi) | i = 0, 1, 2, \ldots]$ with repeated applications of $\boldsymbol{B}^{\pi'_D}$

Proof by Induction

- To prove the base case (of proof by induction), note that:

$$\boldsymbol{B}^{\pi'_D}(\boldsymbol{V}^{\pi})(s) = \max_{a \in \mathcal{A}} \{\mathcal{R}(s,a) + \gamma \sum_{s' \in \mathcal{N}} \mathcal{P}(s,a,s') \cdot \boldsymbol{V}^{\pi}(s')\} = \max_{a \in \mathcal{A}} Q^{\pi}(s,a)$$

- $\boldsymbol{V}^{\pi}(s)$ is weighted average of $Q^{\pi}(s, \cdot)$ while $\boldsymbol{B}^{\pi'_D}(\boldsymbol{V}^{\pi})(s)$ is maximum

$$\boldsymbol{B}^{\pi'_D}(\boldsymbol{V}^{\pi}) \geq \boldsymbol{V}^{\pi}$$

- Induction step is proved by monotonicity of $\boldsymbol{B}^{\pi}$ operator (for any π):

$$\text{Monotonicity Property of } \boldsymbol{B}^{\pi} : \boldsymbol{X} \geq \boldsymbol{Y} \Rightarrow \boldsymbol{B}^{\pi}(\boldsymbol{X}) \geq \boldsymbol{B}^{\pi}(\boldsymbol{Y})$$

$$\text{So } (\boldsymbol{B}^{\pi'_D})^{i+1}(\boldsymbol{V}^{\pi}) \geq (\boldsymbol{B}^{\pi'_D})^{i}(\boldsymbol{V}^{\pi}) \Rightarrow (\boldsymbol{B}^{\pi'_D})^{i+2}(\boldsymbol{V}^{\pi}) \geq (\boldsymbol{B}^{\pi'_D})^{i+1}(\boldsymbol{V}^{\pi})$$

Intuitive Understanding of Policy Improvement Theorem

- Non-decreasing tower of Value Functions $[(\boldsymbol{B}^{\pi'_D})^i(\boldsymbol{V}^{\pi}) | i = 0, 1, 2, \ldots]$
- Each stage of further application of $\boldsymbol{B}^{\pi'_D}$ improves the Value Function
- Stage 0: Value Function $\boldsymbol{V}^{\pi}$ means execute policy π throughout
- Stage 1: VF $\boldsymbol{B}^{\pi'_D}(\boldsymbol{V}^{\pi})$ means execute improved policy π'_D for the 1st time step, then execute policy π for all further time steps
- Improves the VF from Stage 0: $\boldsymbol{V}^{\pi}$ to Stage 1: $\boldsymbol{B}^{\pi'_D}(\boldsymbol{V}^{\pi})$
- Stage 2: VF $(\boldsymbol{B}^{\pi'_D})^2(\boldsymbol{V}^{\pi})$ means execute improved policy π'_D for first 2 time steps, then execute policy π for all further time steps
- Improves the VF from Stage 1: $\boldsymbol{B}^{\pi'_D}(\boldsymbol{V}^{\pi})$ to Stage 2: $(\boldsymbol{B}^{\pi'_D})^2(\boldsymbol{V}^{\pi})$
- Each stage applies policy π'_D instead of π for an extra time step
- These stages are the iterations of *Policy Evaluation* (using policy π'_D)
- Builds non-decreasing tower of VFs that converge to VF $\boldsymbol{V}^{\pi'_D}$ ($\geq \boldsymbol{V}^{\pi}$)

Repeating Policy Improvement and Policy Evaluation

- Policy Improvement Theorem says:
  - Start with Value Function $\boldsymbol{V}^{\pi}$ (for policy π)
  - Perform a "greedy policy improvement" to create policy $\pi'_D = G(\boldsymbol{V}^{\pi})$
  - Perform Policy Evaluation (for policy π'_D) with starting VF $\boldsymbol{V}^{\pi}$
  - This results in VF $\boldsymbol{V}^{\pi'_D} \geq$ starting VF $\boldsymbol{V}^{\pi}$
- We can repeat this process starting with $\boldsymbol{V}^{\pi'_D}$
- Creating an improved policy π''_D and improved VF $\boldsymbol{V}^{\pi''_D}$.
- ... and we can keep going to create further improved policies/VFs
- ... until there is no further improvement
- This in fact is the *Policy Iteration* algorithm

Policy Iteration algorithm

- Start with any Value Function $\boldsymbol{V_0} \in \mathbb{R}^m$
- Iterating over $j = 0, 1, 2, \ldots$, calculate in each iteration:

$$\text{Deterministic Policy } \pi_{j+1} = G(\boldsymbol{V_j})$$

$$\text{Value Function } \boldsymbol{V_{j+1}} = \lim_{i \to \infty} (\boldsymbol{B}^{\pi_{j+1}})^i (\boldsymbol{V_j})$$

- Stop when $d(\boldsymbol{V_j}, \boldsymbol{V_{j+1}}) = \max_{s \in \mathcal{N}} |(\boldsymbol{V_j} - \boldsymbol{V_{j+1}})(s)|$ is small enough

  At termination: $\boldsymbol{V_j} = (\boldsymbol{B}^{G(\boldsymbol{V_j})})^i (\boldsymbol{V_j}) = \boldsymbol{V_{j+1}}$ for all $i = 0, 1, 2, \ldots$

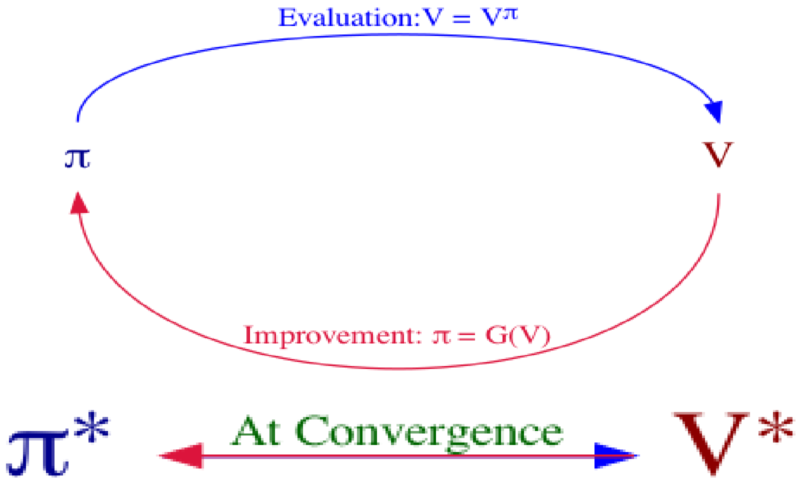
Specializing this to $i = 1$, we have for all $s \in \mathcal{N}$:

$$\boldsymbol{V_j}(s) = \boldsymbol{B}^{G(\boldsymbol{V_j})}(\boldsymbol{V_j})(s) = \max_{a \in \mathcal{A}} \{\mathcal{R}(s, a) + \gamma \sum_{s' \in \mathcal{N}} \mathcal{P}(s, a, s') \cdot \boldsymbol{V_j}(s')\}$$

This means $\boldsymbol{V_j}$ satisfies the MDP Bellman Optimality Equation and so,

$$\boldsymbol{V_j} = \boldsymbol{V}^{\pi_j} = \boldsymbol{V}^*$$

Policy Iteration algorithm



Policy Iteration Convergence Theorem

Theorem (Policy Iteration Convergence Theorem)

For a Finite MDP with $|\mathcal{N}| = m$ and $\gamma < 1$, Policy Iteration algorithm converges to the Optimal Value Function $\boldsymbol{V}^* \in \mathbb{R}^m$ along with a Deterministic Optimal Policy $\pi_D^* : \mathcal{N} \rightarrow \mathcal{A}$, no matter which Value Function $\boldsymbol{V_0} \in \mathbb{R}^m$ we start the algorithm with.

Running time of Policy Improvement is $O(m^2 k)$ where $|\mathcal{N}| = m, |\mathcal{A}| = k$

Running time of each iteration of Policy Evaluation is $O(m^2 k)$

Bellman Optimality Operator

- Tweak the definition of Greedy Policy Function ($\arg\max$ to $\max$)
- *Bellman Optimality Operator* $\boldsymbol{B}^* : \mathbb{R}^m \rightarrow \mathbb{R}^m$ defined as:

$$\boldsymbol{B}^*(\boldsymbol{V})(s) = \max_{a \in \mathcal{A}} \{\mathcal{R}(s, a) + \gamma \sum_{s' \in \mathcal{N}} \mathcal{P}(s, a, s') \cdot \boldsymbol{V}(s')\}$$

- Think of this as a non-linear transformation of a VF vector $\boldsymbol{V} \in \mathbb{R}^m$
- The action a producing the max is the action prescribed by $G(\boldsymbol{V})$. So,

$$\boldsymbol{B}^{G(\boldsymbol{V})}(\boldsymbol{V}) = \boldsymbol{B}^*(\boldsymbol{V}) \text{ for all } \boldsymbol{V} \in \mathbb{R}^m$$

- Specializing $\boldsymbol{V}$ to be the Value Function $\boldsymbol{V}^\pi$ for a policy π, we get:

$$\boldsymbol{B}^{G(\boldsymbol{V}^\pi)}(\boldsymbol{V}^\pi) = \boldsymbol{B}^*(\boldsymbol{V}^\pi)$$

- This is the 1st stage of Policy Evaluation with improved policy $G(\boldsymbol{V}^\pi)$

Fixed-Point of Bellman Optimality Operator

- $\boldsymbol{B}^{\pi}$ was motivated by the MDP Bellman Policy Equation
- Similarly, $\boldsymbol{B}^*$ is motivated by the MDP Bellman Optimality Equation:

$$\boldsymbol{V}^*(s) = \max_{a \in \mathcal{A}} \{\mathcal{R}(s, a) + \gamma \sum_{s' \in \mathcal{N}} \mathcal{P}(s, a, s') \cdot \boldsymbol{V}^*(s')\} \text{ for all } s \in \mathcal{N}$$

- So we can express the MDP Bellman Optimality Equation neatly as:

$$\boldsymbol{V}^* = \boldsymbol{B}^*(\boldsymbol{V}^*)$$

- Therefore, $\boldsymbol{V}^* \in \mathbb{R}^m$ is a Fixed-Point of $\boldsymbol{B}^* : \mathbb{R}^m \rightarrow \mathbb{R}^m$
- We want to prove that $\boldsymbol{B}^*$ is a contraction function (under L^{∞} norm)
- So we can take advantage of Banach Fixed-Point Theorem
- And solve the Control problem by iterative applications of $\boldsymbol{B}^*$

Proof that B^* is a contraction

- We need to utilize two key properties of $\boldsymbol{B}^*$

$$\text{Monotonicity Property: } \boldsymbol{X} \geq \boldsymbol{Y} \Rightarrow \boldsymbol{B}^*(\boldsymbol{X}) \geq \boldsymbol{B}^*(\boldsymbol{Y})$$

$$\text{Constant Shift Property: } \boldsymbol{B}^*(\boldsymbol{X} + c) = \boldsymbol{B}^*(\boldsymbol{X}) + \gamma c$$

- With these two properties, we can prove that:

$$\max_{s \in \mathcal{N}} |(\boldsymbol{B}^*(\boldsymbol{X}) - \boldsymbol{B}^*(\boldsymbol{Y}))(s)| \leq \gamma \cdot \max_{s \in \mathcal{N}} |(\boldsymbol{X} - \boldsymbol{Y})(s)|$$

Theorem (Value Iteration Convergence Theorem)

For a Finite MDP with $|\mathcal{N}| = m$ and $\gamma < 1$, if $\boldsymbol{V}^* \in \mathbb{R}^m$ is the Optimal Value Function, then $\boldsymbol{V}^*$ is the unique Fixed-Point of the Bellman Optimality Operator $\boldsymbol{B}^* : \mathbb{R}^m \rightarrow \mathbb{R}^m$, and

$$\lim_{i \rightarrow \infty} (\boldsymbol{B}^*)^i(\boldsymbol{V_0}) \rightarrow \boldsymbol{V}^* \text{ for all starting Value Functions } \boldsymbol{V_0} \in \mathbb{R}^m$$

Value Iteration algorithm

- Start with any Value Function $\boldsymbol{V_0} \in \mathbb{R}^m$
- Iterating over $i = 0, 1, 2, \ldots$, calculate in each iteration:

$$\boldsymbol{V_{i+1}}(s) = \boldsymbol{B}^*(\boldsymbol{V_i})(s) \text{ for all } s \in \mathcal{N}$$

- Stop when $d(\boldsymbol{V_i}, \boldsymbol{V_{i+1}}) = \max_{s \in \mathcal{N}} |(\boldsymbol{V_i} - \boldsymbol{V_{i+1}})(s)|$ is small enough

Running time of each iteration of Value Iteration is $O(m^2 k)$ where $|\mathcal{N}| = m$ and $|\mathcal{A}| = k$

Optimal Policy from Optimal Value Function

- Note that Value Iteration does not deal with any policy (only VFs)
- Extract Optimal Policy π^* from Optimal VF V^* such that $\boldsymbol{V}^{\pi^*} = \boldsymbol{V}^*$
- Use Greedy Policy function G. We know:

$$\boldsymbol{B}^{G(\boldsymbol{V})}(\boldsymbol{V}) = \boldsymbol{B}^*(\boldsymbol{V}) \text{ for all } \boldsymbol{V} \in \mathbb{R}^m$$

- Specializing $\boldsymbol{V}$ to $\boldsymbol{V}^*$, we get:

$$\boldsymbol{B}^{G(\boldsymbol{V}^*)}(\boldsymbol{V}^*) = \boldsymbol{B}^*(\boldsymbol{V}^*)$$

- But we know $\boldsymbol{V}^*$ is the Fixed-Point of $\boldsymbol{B}^*$, i.e., $\boldsymbol{B}^*(\boldsymbol{V}^*) = \boldsymbol{V}^*$. So,

$$\boldsymbol{B}^{G(\boldsymbol{V}^*)}(\boldsymbol{V}^*) = \boldsymbol{V}^*$$

- So $\boldsymbol{V}^*$ is the Fixed-Point of the Bellman Policy Operator $\boldsymbol{B}^{G(\boldsymbol{V}^*)}$
- But we know $\boldsymbol{B}^{G(\boldsymbol{V}^*)}$ has a unique Fixed-Point ($= \boldsymbol{V}^{G(\boldsymbol{V}^*)}$). So,

$$\boldsymbol{V}^{G(\boldsymbol{V}^*)} = \boldsymbol{V}^*$$

- Evaluating MDP with greedy policy extracted from $\boldsymbol{V}^*$ achieves $\boldsymbol{V}^*$
- So, $G(\boldsymbol{V}^*)$ is a (Deterministic) Optimal Policy

Value Function Progression in Policy Iteration

$$\pi_1 = G(\boldsymbol{V_0}) : \boldsymbol{V_0} \rightarrow \boldsymbol{B}^{\pi_1}(\boldsymbol{V_0}) \rightarrow \dots (\boldsymbol{B}^{\pi_1})^i(\boldsymbol{V_0}) \rightarrow \dots \boldsymbol{V}^{\pi_1} = \boldsymbol{V_1}$$

$$\pi_2 = G(\boldsymbol{V_1}) : \boldsymbol{V_1} \rightarrow \boldsymbol{B}^{\pi_2}(\boldsymbol{V_1}) \rightarrow \dots (\boldsymbol{B}^{\pi_2})^i(\boldsymbol{V_1}) \rightarrow \dots \boldsymbol{V}^{\pi_2} = \boldsymbol{V_2}$$

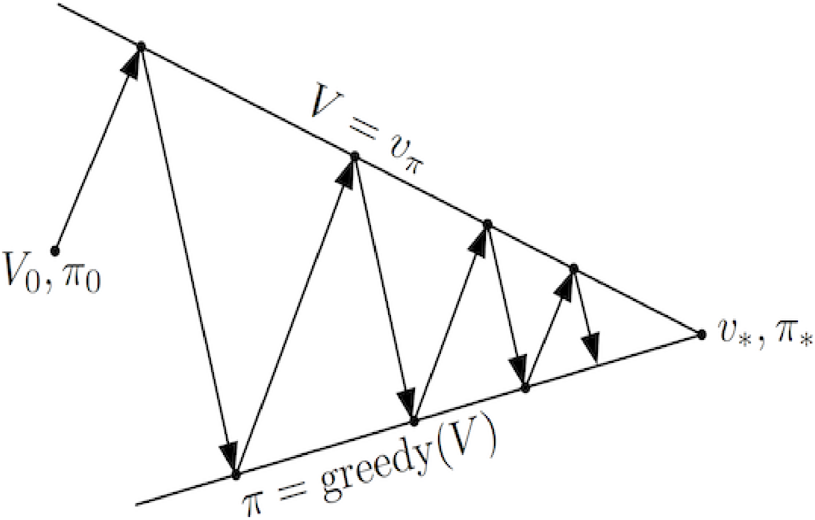
$$\dots$$

$$\dots$$

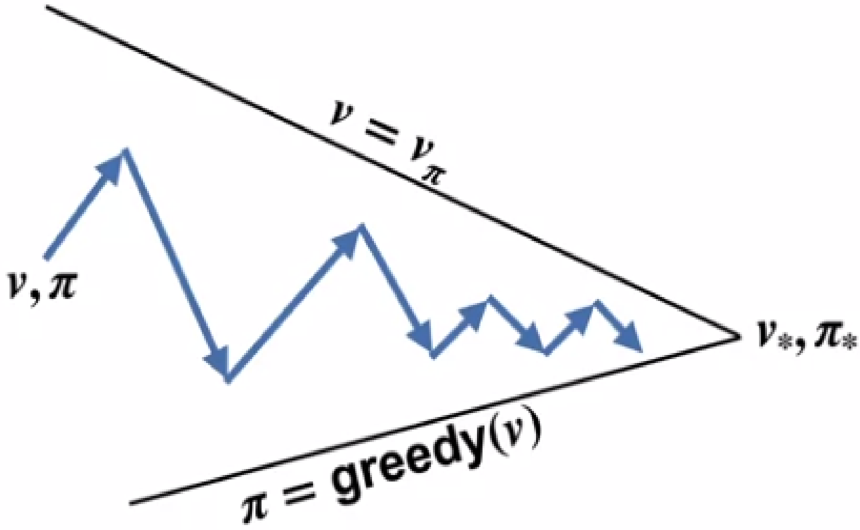
$$\pi_{j+1} = G(\boldsymbol{V_j}) : \boldsymbol{V_j} \rightarrow \boldsymbol{B}^{\pi_{j+1}}(\boldsymbol{V_j}) \rightarrow \dots (\boldsymbol{B}^{\pi_{j+1}})^i(\boldsymbol{V_j}) \rightarrow \dots \boldsymbol{V}^{\pi_{j+1}} = \boldsymbol{V}^*$$

- Policy Evaluation and Policy Improvement alternate until convergence
- In the process, they simultaneously compete and try to be consistent
- There are actually two notions of consistency:
  - VF $\boldsymbol{V}$ being consistent with/close to VF $\boldsymbol{V}^\pi$ of the policy π.
  - π being consistent with/close to Greedy Policy $G(\boldsymbol{V})$ of VF $\boldsymbol{V}$.

Policy Iteration



Generalized Policy Iteration (GPI)



Value Iteration and Reinforcement Learning as GPI

- Value Iteration takes only one step of Policy Evaluation

$$\pi_1 = G(\boldsymbol{V_0}) : \boldsymbol{V_0} \rightarrow \boldsymbol{B}^{\pi_1}(\boldsymbol{V_0}) = \boldsymbol{V_1}$$

$$\pi_2 = G(\boldsymbol{V_1}) : \boldsymbol{V_1} \rightarrow \boldsymbol{B}^{\pi_2}(\boldsymbol{V_1}) = \boldsymbol{V_2}$$

$$\dots$$

$$\dots$$

$$\pi_{j+1} = G(\boldsymbol{V_j}) : \boldsymbol{V_j} \rightarrow \boldsymbol{B}^{\pi_{j+1}}(\boldsymbol{V_j}) = \boldsymbol{V}^*$$

- RL updates either a subset of states or just one state at a time
- Large-scale RL updates function approximations of a VF
- These can be thought of as *partial* Policy Evaluation/Improvement

Asynchronous Dynamic Programming

The DP algorithms we've covered are qualified as *Synchronous DP*:

- *All* states' values are updated in each iteration
- "Simultaneous" state updates implemented by updating a copy of VF

Asynchronous DP can update subset of states, or update in any order

1. *In-place* updates enable updated values to be used immediately
2. *Prioritized Sweeping* keeps states sorted by their Value Function *gaps*

$$\text{Gaps } g(s) = |V(s) - \max_{a \in \mathcal{A}} \{\mathcal{R}(s,a) + \gamma \cdot \sum_{s' \in \mathcal{N}} \mathcal{P}(s,a,s') \cdot V(s')\}|$$

   But this requires us to know the reverse transitions to resort queue
3. *Real-Time Dynamic Programming (RTDP)* runs DP *while* the agent is experiencing real-time interaction with the environment
   - A state is updated when it is visited during the real-time interaction
   - The choice of action is governed by real-time VF-extracted policy

Episodic MDPs with unique state visits

- A fairly common specialization of MDPs enables great tractability:
  - All random sequences terminate within fixed time steps (episodic MDP)
  - A state is encountered at most once in an episode
- This can be conceptualized as a Directed Acyclic Graph (DAG)
- Each node in the DAG is a (state, action) pair
- Prediction/Control solved by “backwards walk” from terminal nodes
- Bellman Equation enables simply *setting the VF* of a visited node
- Avoids the expensive “iterate to convergence” method of classical DP
- States visited (and VFs set) in order of reverse Topological Sort
- Next we cover a special case of DAG MDPs: finite-horizon MDPs

Finite-Horizon MDPs

- Finite-Horizon Markov Decision Processes are characterized by:
  - Each sequence terminates within a finite number of time steps T
  - Each time step has it's own state space
- Denote states at time t as $\mathcal{S}_t$, terminal states as $\mathcal{T}_t$, non-terminal states as $\mathcal{N}_t = \mathcal{S}_t - \mathcal{T}_t$ (note: $\mathcal{N}_T = \emptyset$), actions as $\mathcal{A}_t$, rewards as $\mathcal{D}_t$
- Augment each state to include time-index: augmented state is (t, s_t)

$$\text{Entire MDP's States } \mathcal{S} = \{(t, s_t) | t = 0, 1, \ldots, T, s_t \in \mathcal{S}_t\}$$

- Each t gets its own state-reward transition probability function

$$(\mathcal{P}_R)_t : \mathcal{N}_t \times \mathcal{A}_t \times \mathcal{D}_{t+1} \times \mathcal{S}_{t+1} \rightarrow [0, 1]$$

- Likewise, each t gets its own policy $\pi_t : \mathcal{N}_t \times \mathcal{A}_t \rightarrow [0, 1]$
- An overall policy $\pi : \mathcal{N} \times \mathcal{A} \rightarrow [0, 1]$ composed of $(\pi_0, \pi_1, \ldots, \pi_{T-1})$

Backward Induction for Finite MRP with Finite-Horizon

- VF for a given policy π can be represented by time-sequenced VFs

$$V_t^\pi : \mathcal{N}_t \rightarrow \mathbb{R}$$

- So Bellman Equation for π-implied Finite-Horizon MRP becomes:

$$V_t^\pi(s_t) = \sum_{s_{t+1} \in \mathcal{S}_{t+1}} \sum_{r_{t+1} \in \mathbb{D}_{t+1}} (\mathcal{P}_R^{\pi_t})_t(s_t, r_{t+1}, s_{t+1}) \cdot (r_{t+1} + \gamma \cdot V_{t+1}^\pi(s_{t+1}))$$

$$(\mathcal{P}_R^{\pi_t})_t(s_t, r_{t+1}, s_{t+1}) = \sum_{a_t \in \mathcal{A}_t} \pi_t(s_t, a_t) \cdot (\mathcal{P}_R)_t(s_t, a_t, r_{t+1}, s_{t+1})$$

- "Backward Induction" algorithm for *finite MRP* with Finite-Horizon
- Decrementing t from $T - 1$ to 0, and calculating V_t^π from V_{t+1}^π
- Running time is $O(m^2 T)$ where $|\mathcal{N}_t|$ is $O(m)$
- $O(m^2 kT)$ to convert MDP to π-implied MRP ($|\mathcal{A}_t|$ is $O(k)$)

Backward Induction for Finite MDP with Finite-Horizon

- Optimal VF V^* can be represented by time-sequenced Optimal VFs

$$V_t^* : \mathcal{N}_t \to \mathbb{R}$$

- So MDP Bellman Optimality Equation becomes:

$$V_t^*(s_t) = \max_{a_t \in \mathcal{A}_t} \sum_{s_{t+1}} \sum_{r_{t+1}} (\mathcal{P}_R)_t(s_t, a_t, r_{t+1}, s_{t+1}) \cdot (r_{t+1} + \gamma \cdot V_{t+1}^*(s_{t+1}))$$

- "Backward Induction" (Control) for *finite MDP* with Finite-Horizon
- Decrementing t from $T-1$ to 0, and calculating V_t^* from V_{t+1}^*
- (Associated) Optimal (Deterministic) Policy $(\pi_D^*)_t : \mathcal{N}_t \to \mathcal{A}_t$ is

$$(\pi_D^*)_t(s_t) = \arg\max_{a_t \in \mathcal{A}_t} \sum_{s_{t+1}} \sum_{r_{t+1}} (\mathcal{P}_R)_t(s_t, a_t, r_{t+1}, s_{t+1}) \cdot (r_{t+1} + \gamma \cdot V_{t+1}^*(s_{t+1}))$$

- Running time is $O(m^2 k T)$ where $|\mathcal{N}_t|$ is $O(m)$ and $|\mathcal{A}_t|$ is $O(k)$

@abstractclass MarkovDecisionProcess

```python
def optimal_vf_and_policy(
  steps: Sequence[StateActionMapping[S, A]],
  gamma: float
) -> Iterator[Tuple[
    Mapping[NonTerminal[S], float],
    FiniteDeterministicPolicy[S, A]
]]

def unwrap_finite_horizon_MDP(
  process: FiniteMarkovDecisionProcess[WithTime[S], A]
) -> Sequence[StateActionMapping[S, A]]

def finite_horizon_MDP(
  process: FiniteMarkovDecisionProcess[S, A],
  limit: int
) -> FiniteMarkovDecisionProcess[WithTime[S], A]
```

Dynamic Pricing for End-of-Life/End-of-Season

- Dynamic Pricing: Core to many businesses, flexing to supply/demand
- We consider special case of products being sold at end of life/season
- Assume we are T days from season-end and our inventory is M units
- Assume no more incoming inventory during these final T days
- Set prices daily to max *Expected Total Sales Revenue* over T days
- Price for a given day picked from prices $P_1, P_2, \ldots, P_N \in \mathbb{R}$
- Customer daily demand is $Poisson(\lambda_i)$ if Price P_i is picked for the day
- Note that demand can exceed inventory on any day, Sales $\leq$ Inventory

Dynamic Pricing Model for End-of-Life/End-of-Season

$$\mathcal{S}_t = \{(t, I_t) | I_t \in \mathbb{Z}, 0 \leq I_t \leq M\} \text{ for all } 0 \leq t \leq T$$

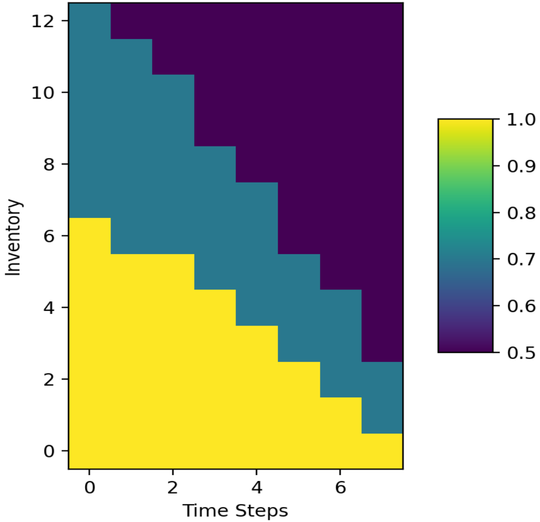
$$\mathcal{N}_t = \mathcal{S}_t \text{ and } \mathcal{A}_t = \{1, 2, \ldots, N\} \text{ for all } 0 \leq t < T, \text{ and } \mathcal{N}_T = \emptyset$$

$$I_0 = M \text{ and } I_{t+1} = \max(0, I_t - d_t) \text{ where } d_t \sim Poisson(\lambda_i) \text{ if } a_t = i$$

$$\text{Sales Revenue on day } t \text{ is equal to } \min(I_t, d_t) \cdot P_i$$

$$(\mathcal{P}_R)_t(I_t, i, r_{t+1}, I_t - k) = \begin{cases} \frac{e^{-\lambda_i} \lambda_i^k}{k!} & \text{if } k < I_t \text{ and } r_{t+1} = k \cdot P_i \\ \sum_{j=I_t}^{\infty} \frac{e^{-\lambda_i} \lambda_i^j}{j!} & \text{if } k = I_t \text{ and } r_{t+1} = k \cdot P_i \\ 0 & \text{otherwise} \end{cases}$$

Optimal Dynamic Pricing



Generalizations to Non-Tabular Algorithms

- Finite MDP algorithms we covered known as "tabular" algorithms
- "Tabular" means MDP is specified as a finite data structure
- More importantly, Value Function represented as a "table"
- These algorithms typically sweep through all states in each iteration
- Cannot do this for large finite spaces or for infinite spaces
- Requires us to generalize to function approximation of Value Function
  - Sample an appropriate subset of states
  - Calculate the Value Function for those states (Bellman calculation)
  - Create/Update a func approx with the sampled states' calculated values
- The fundamental structure of the algorithms is still the same
- Fundamental principles (Fixed-Point/Bellman Operators) still same
- These generalizations known as *Approximate Dynamic Programming*

Key Takeaways from this Chapter

- Fixed-Point of Functions and Fixed-Point Theorem: Enables iterative algorithms to solve a variety of problems cast as Fixed-Point.
- Generalized Policy Iteration: Powerful idea of alternating between *any* method for Policy Evaluation and *any* method for Policy Improvement, including methods that are partial applications of Policy Evaluation or Policy Improvement. This generalized perspective unifies almost all of the algorithms that solve MDP Control problems.
- Backward Induction: A straightforward method to solve finite-horizon MDPs by simply walking backwards and *setting* the Value Function from the horizon-end to the start.