Analytical Design of a PID Compensator

This section presents an analytical procedure for PID compensator design. It is based on Section 7.11 of Phillips and Harbor *Feedback_ Control Systems*, Prentice Hall, 1988[1].

The compensator transfer function is assumed to be

$$D(s) = \frac{K_D s^2 + K_{\mathrm{prop}}\ s + K_I}{s}$$

where K_{prop} is the proportional gain, K_D is the derivative gain and K_I is the integral gain. In this procedure we choose the PID gain parameters such that, given a desired location for one of the closed-loop poles s_1, the equation

$$D(s)G(s)H(s)|_{s=s_1} = -1$$

is satisfied; that is we are designing a compensator that places a root of the closed-loop characteristic equation at $s = s_1$.

The design proceeds as follows. First we express the desired closed loop pole position

$$s_1 = |s_1|e^{j\psi}$$

and

$$G(s_1)H(s_1) = |G(s_1)H(s_1)|\ e^{j\psi}$$

Then the design equations (derived in Appendix B of Phillips and Harbor, 1988) are

$$K_{\mathrm{prop}} = -\frac{\sin(\beta - \psi)}{|G(s_1)H(s_1)|\sin\beta} - \frac{2K_I\cos\beta}{|s_1|}$$

$$K_{\mathrm{prop}} = -\frac{\sin\psi}{|s_1|\ |G(s_1)H(s_1)|\sin\beta} - \frac{K_I}{|s_1|^2}$$

Since there are three unknowns and only two relationships that must be satisfied, one of the gains may be chosen to satisfy a different design specification, such as choosing K_I to achieve a certain steady-state response. These equations can also be used for PI and P+D controllers by setting the appropriate gain to zero. We now illustrate the design procedure with an example.

Example

Definitions (change these to change design)

The plant transfer function is

$$G(s) = \frac{1}{(s+1)(5s+1)}$$

In [1]:

```
G = tf(1,conv([1 1],[5 1]));
```

The feedback transfer function is $H(s) = 1$:

In [2]:

```
H=tf(1,1);
```

So $G(s)H(s)$ is:

In [3]:

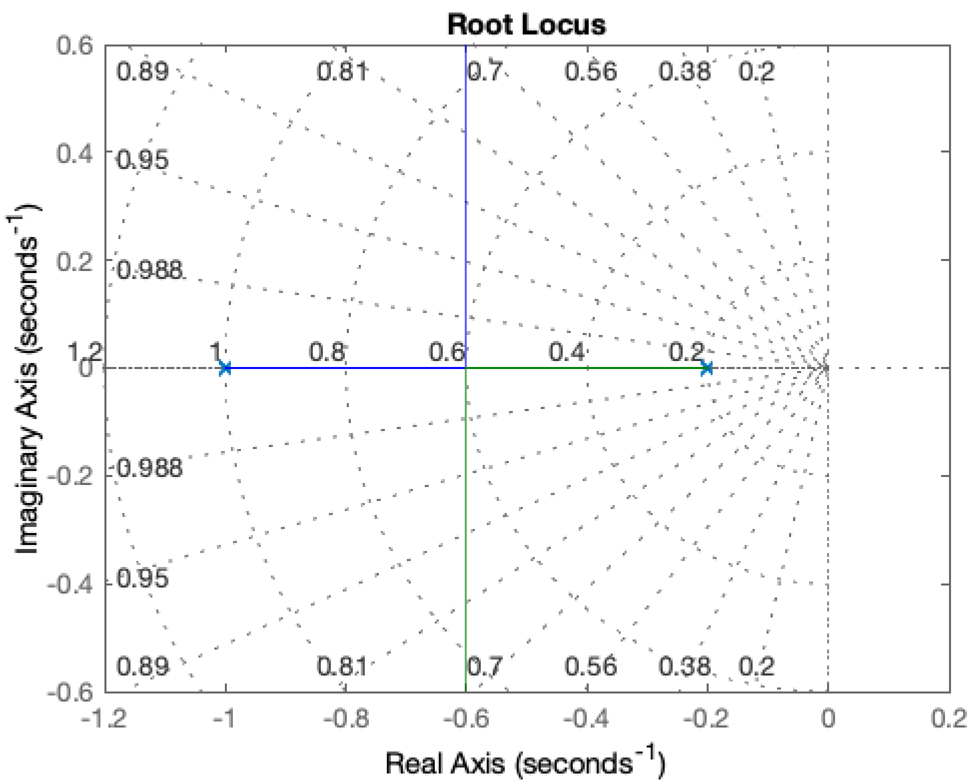
```
GH=G*H
```

```
GH =

         1
  ---------------
  5 s^2 + 6 s + 1

Continuous-time transfer function.
```

The root locus of the uncompensated system is:

In [4]:

```
clf, sgrid(1/sqrt(2),0.25:0.25:2), hold on, rlocus(GH),hold off
```



From the root locus diagram, it is clear that for ideal damping the natural frequency of the closed-loop poles would be about 0.9 rad/s with a settling time of:

$T_s = \frac{4.6}{\zeta\omega_n} = \frac{4.6}{5/8} = 7.36$ s

Suppose we wish to half the settling time then we need to double the natural frequency to $\omega_n = 2$rad/s.

That is:

In [5]:

```
zeta = 1/sqrt(2); wn=2;
s1 = -zeta*wn+j*wn*sqrt(1-zeta^2)
```

```
s1 =

  -1.4142 + 1.4142i
```

The steady state error of the uncompensated type 0 system is:

$$\frac{1}{1 + G(s)H(s)|_{s=0}} = \frac{1}{1 + \left.\frac{1}{(5s+1)(s+1)}\right|_{s=0}} = \frac{1}{2}$$

For the compensated system, which is type 1:

$$K_v = sD(s)G(s)H(s)|_{s=0} = \left.\frac{s\left(K_D\, s^s + K_{\text{prod}}\, s + K_I\right)}{s}\right|_{s=0} = K_I$$

So if we want a steady-state _velocity _error of 20% we need

In [6]:

```
Ki=20;
```

Calculations

Having set up your problem, you shouldn?t need to change these commands

Polar form of s_1

In [11]:

```
m_s1=abs(s1),  p_s1 = angle(s1)*180/pi % degrees
```

```
m_s1 =

     2


p_s1 =

   135
```

Transfer function evaluated at s_1 is $G(s_1)H(s_1)$ in polar form:

In [7]:

```
[numGH,denGH] = tfdata(GH,'v');
GHs1=polyval(numGH,s1)/polyval(denGH,s1)
```

```
GHs1 =

  -0.0397 + 0.0610i
```

Magnitude:

In [8]:

```
mGHs1=abs(GHs1)
```

```
mGHs1 =

    0.0728
```

Phase[2]:

In [9]:

```
pGHs1=-angle(GHs1)*180/pi - 90 % degrees
```

```
pGHs1 =

 -213.0264
```

Hence:

In [12]:

```
beta = p_s1*pi/180; psi = pGHs1*pi/180;  % radians
```

From (5) and (6)

In [13]:

```
Kprop = (-sin(beta+psi))/(mGHs1*sin(beta)) - (2*Ki*cos(beta)/m_s1)
```

```
Kprop =

   33.1421
```

In [14]:

```
Kd = (sin(psi)/(m_s1*mGHs1*sin(beta))) + Ki/(m_s1^2)
```

```
Kd =

   10.2929
```

Compensator is therefore given by

In [15]:

```
D = tf([Kd, Kprop, Ki],[1, 0])
```

```
D =

  10.29 s^2 + 33.14 s + 20
  ------------------------
             s

Continuous-time transfer function.
```

Evaluation of Design

Open loop transfer function:

In [16]:

```
Go=D*GH
```

```
Go =

  10.29 s^2 + 33.14 s + 20
  ------------------------
     5 s^3 + 6 s^2 + s

Continuous-time transfer function.
```

Root locus:

rlocus(Go)

Closed-loop transfer function:

In [17]:

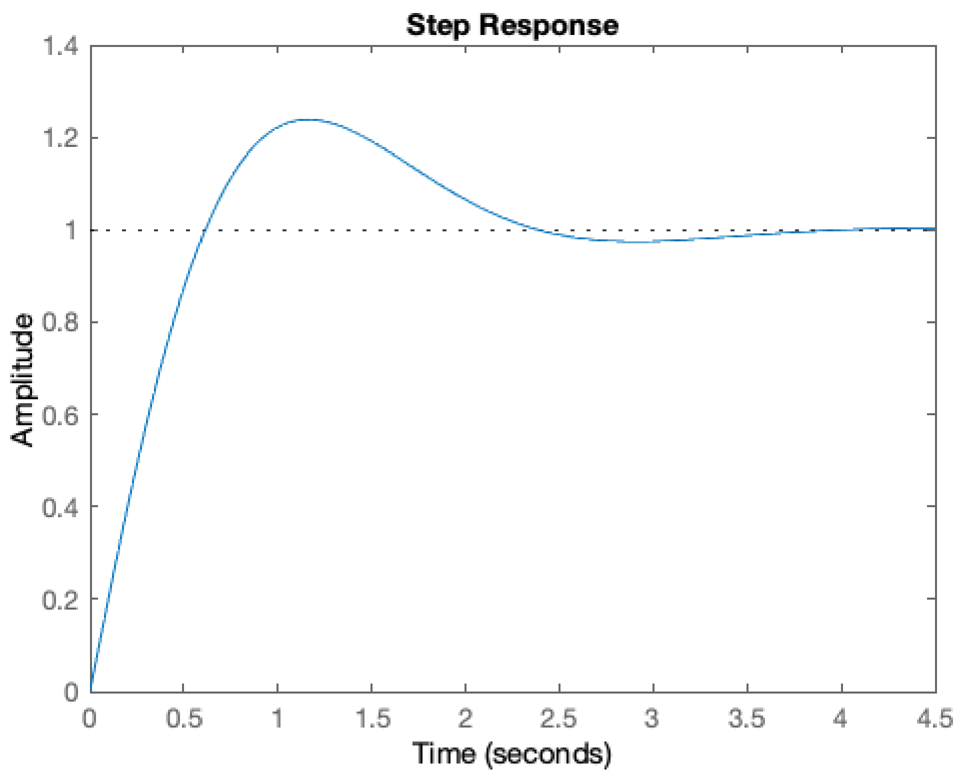
```
DG = D*G
Gc = feedback(DG,H)
```

```
DG =

  10.29 s^2 + 33.14 s + 20
  ------------------------
     5 s^3 + 6 s^2 + s

Continuous-time transfer function.


Gc =

      10.29 s^2 + 33.14 s + 20
  --------------------------------
  5 s^3 + 16.29 s^2 + 34.14 s + 20

Continuous-time transfer function.
```

Step response:

In [18]:

```
step(Gc)
```



Footnotes

[1] The proofs of the formulae given are derived in Appendix B of this text.

[2] You must be careful with angles when using packages like MATLAB, and indeed pocket calculators. It is nearly always beneficial to have a sketch so that you can correct the results. In this case a correction of -90° was needed.

Resources

An executable version of this document is available to download as the MATLAB Live Script file analrloc.mlx (analrloc.mlx).