```
and that the transformed equation was
                                                                        s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)
                                                                                     \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)
In this section we will show how the time response of the state space model may be derived from the transformed model.
Zero Input Response
We first consider the response of the system to its initial conditions and zero input. This is the zero input response or
 homogeneous response of the system.
With the input transform U(s) = 0 the state equation becomes
                                                                               \mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0).
We define
                                                                                   \Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1}
so that
                                                                                     \mathbf{X}(s) = \Phi(s)\mathbf{x}(0)
is the transform of the zero input state response.
In the time domain, the corresponding state response is given by the inverse Laplace transform of equation (1)
                                                                               \mathbf{x}(t) = \mathcal{L}^{-1} \left\{ \Phi(s) \mathbf{x}(0) \right\},\,
which, since the initial condition vector \mathbf{x}(0) is a vector of constants becomes:
                                                                               \mathbf{x}(t) = \mathcal{L}^{-1} \left\{ \Phi(s) \right\} \mathbf{x}(0)
                                                                                       = \phi(t)\mathbf{x}(0).
Resolvent matrix
The resolvent matrix is defined as
                                                                    \Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\text{adj } [s\mathbf{I} - \mathbf{A}]}{|s\mathbf{I} - \mathbf{A}|}.
The resolvent matrix is a matrix of rational polynomials in the Laplace transform variable s.
The state transition matrix is
                                                                        \phi(t) = \mathcal{L}^{-1} \left\{ \frac{\text{adj} [s\mathbf{I} - \mathbf{A}]}{|s\mathbf{I} - \mathbf{A}|} \right\}.
What is the form of this function?
State Transition Matrix
The vector of time functions \phi(t) is known as the state transition matrix. It defines the motion (or trajectory) of the state
variables through the n-dimensional state space from a given set of initial states.
Characteristic Polynomial
The adjoint matrix \operatorname{adj}[s\mathbf{I} - \mathbf{A}] is a matrix of polynomials each of which is of order n-1 or less.
The determinant
                                                           |s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \end{vmatrix}
is called the characteristic polynomial. It is always a polynomial of degree n.
Therefore each element of the resolvent matrix \Phi(s) is of the form
System Characteristic (or Eigen) Values
The coefficients b_i of each element of the resolvent matrix \Phi(s) depend on the form of the system. The characteristic
polynomial |s\mathbf{I} - \mathbf{A}| is the common denominator of all elements of \Phi(s).
\Phi(s) may be factorized:
                                                               |s\mathbf{I} - \mathbf{A}| = s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0
                                                                              = (s - p_1)(s - p_2) \cdots (s - p_n)
The values p_i are known as the characteristic (or eigen) values of the state matrix A. The characteristic values are the poles of
the system!
Zero-input response
If we make a partial fraction expansion of the characteristic equation, then for the i-jth element of the resolvent matrix we have
                                                               \Phi_{ij}(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \dots + \frac{r_n}{s - p_n}
(where the coefficient r_i is the so called residue of the partial fraction expansion determined at the value s = p_i).
Inverse Laplace transforming this equation we finally obtain the state transition function:
                                                                    \phi_{i,i}(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \dots + r_n e^{p_n t}.
System 'modes'
The state transition matrix \phi(t) is an n \times n matrix, each element of which is a linear combination of the n "mode functions" (or
simply modes)
                                                                                  e^{p_1t}, e^{p_2t}, \ldots, e^{p_nt}.
Each mode is constructed from the eigenvalues (p_1, p_2, \ldots, p_n) of the state matrix A.
The eigenvalues are also known as the poles of the system.
Example
lf
                                                                               s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}
SO
                                                                |s\mathbf{I} - \mathbf{A}| = s^2 + 5s + 6 = (s+2)(s+3).
The characteristic values of the system are p_1 = -2 and p_2 = -3 so the modes are e^{-2t} and e^{-3t}.
Now
                                                                          \operatorname{adj}\left[s\mathbf{I} - \mathbf{A}\right] = \begin{bmatrix} s+5 & 1\\ -6 & s \end{bmatrix}
so the resolvent matrix is
                                                                      \Phi(s) = \begin{bmatrix} \frac{s+5}{(s+2)(s+3)} & \frac{1}{(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} & \frac{s}{(s+2)(s+3)} \end{bmatrix}.
 Expanding each element of \Phi(s) we get
                                                                 \Phi(s) = \begin{bmatrix} \frac{3}{s+2} - \frac{2}{s+3} & \frac{1}{s+2} - \frac{1}{s+3} \\ \frac{-6}{s+2} + \frac{6}{s+3} & -\frac{2}{s+2} + \frac{3}{s+3} \end{bmatrix}.
Inverse Laplace transforming this we get the final state transition matrix, representing the zero-input response of the system:
                                                            \phi(t) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}.
Note that, by an alternative derivation<sup>1</sup>, we can show that
                                                                                          \phi(t) = e^{\mathbf{A}t}
where e^{\mathbf{A}t} is called the matrix exponential.
Zero Input System Response
 For non-zero initial conditions and zero input we have
                                                                               \mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0)
                                                                                         =\Phi(s)\mathbf{x}(0)
The zero-input system output transform is therefore given by
                                                                                 \mathbf{Y}_{\mathrm{zi}}(s) = \mathbf{C}\Phi(s)\mathbf{x}(0).
 Because we have already shown that \phi(t) will be a linear combination of modes, the unforced system response v(t) will be
another linear combination of the same modes.
We now consider the case where the initial condition vector \mathbf{x}(0) = \mathbf{0} and we will obtain the response of the system to an
input. This is called the zero state response of the system.
The Zero State Response
For zero initial conditions we have
                                                                             \mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)
                                                                                       = \Phi(s)\mathbf{B}\mathbf{U}(s)
The system output transform (forced response) is given by
                                                                          \mathbf{Y}_{zs}(s) = [\mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}]\mathbf{U}(s)
(where \mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D} is the system transfer matrix).
Full System Response
The full system response for the state-space model is simply the sum of the zero-state and zero-input responses:
                                                               \mathbf{Y}_{\text{full}}(s) = \mathbf{Y}_{zs}(s) + \mathbf{Y}_{zi}(s)
                                                                            = \mathbf{C}\Phi(s) \left[ \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s) \right] + \mathbf{D}\mathbf{U}(s).
End of Pre-Class Presentation
In the class we will reinforce these concepts by working through an example in detail.
Example
Let
                                                                     \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{u}
                                                                      \mathbf{y} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x}
                                                                     \mathbf{u} = \begin{bmatrix} e^{-t} \\ \epsilon(t) \end{bmatrix}.
Calculate the full system response for this system given that the initial conditions are \mathbf{x}(0) = [-1, 1]^T.
Solution to Example
The transfer function is
                                           \mathbf{G}(s) = \mathbf{C}\Phi(s)\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+5}{(s+2)(s+3)} & \frac{1}{(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} & \frac{s}{(s+2)(s+3)} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}
                                                    = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2s+9}{(s+2)(s+3)} & \frac{s+5}{(s+2)(s+3)} \\ \frac{-s-12}{(s+2)(s+3)} & \frac{-6}{(s+2)(s+3)} \end{bmatrix}
= \begin{bmatrix} \frac{5s+30}{(s+2)(s+3)} & \frac{2s+16}{(s+2)(s+3)} \\ \frac{-s-12}{(s+2)(s+3)} & \frac{-6}{(s+2)(s+3)} \end{bmatrix}
The transform of the input vector is
                                                                         \mathbf{U}(s) = \mathcal{L} \begin{bmatrix} e^{-t} \\ \epsilon(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s+1} \end{bmatrix}.
Hence
                                                  \mathbf{Y}(s) = \begin{bmatrix} \frac{5s+30}{(s+2)(s+3)} & \frac{2s+16}{(s+2)(s+3)} \\ \frac{-s-12}{(s+2)(s+3)} & \frac{-6}{(s+2)(s+3)} \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s} \end{bmatrix}
= \begin{bmatrix} \frac{7s^2+48s+16}{s(s+1)(s+2)(s+3)} \\ \frac{-(s^2+18s+6)}{s(s+1)(s+2)(s+3)} \end{bmatrix} = \begin{bmatrix} \frac{8/3}{s} + \frac{25/2}{s+1} - \frac{26}{s+2} + \frac{65/6}{s+3} \\ -\frac{1}{s} - \frac{11/2}{s+1} + \frac{13}{s+2} - \frac{13/2}{s+3} \end{bmatrix}
Inverse Laplace transforming the previous result gives the zero-state output response of the system:
                                                                   \begin{bmatrix} \frac{8}{3}\epsilon(t) + \frac{25}{2}e^{-t} - 26e^{-2t} + \frac{65}{6}e^{-3t} \\ -\epsilon(t) - \frac{11}{2}e^{-t} + 13e^{-2t} - \frac{13}{2}e^{-3t} \end{bmatrix}.
Note that response is now a linear combination of the system modes (e^{-2t} and e^{-3t}) and the input modes (e^{-t}) [the unit step
function] and e^{-t} ).
If the initial condition vector \mathbf{x}(0) = [-1, 1]^T what is the total response of the system? \mathbf{Y}(s) = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+5}{(s+2)(s+3)} & \frac{1}{(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} & \frac{s}{(s+2)(s+3)} \end{bmatrix}
                                                                 \times \left\{ \begin{bmatrix} -1\\1 \end{bmatrix} + \begin{bmatrix} 2 & 1\\-1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1}\\\frac{1}{s} \end{bmatrix} \right\}
```

Time Response for State Space Models

starting point the transformed state space model developed in the last section.

In the last section, we showed that the state space model was

In this section we shall determine the time response of a system represented by a state-space model. We shall take as a

It is possible to derive the time response directly in the time domain from the state-equations. I believe however, that the

following development will be easier to understand. I advise you to consult any text book for the time domain development.

 $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$

 $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$

Set up the problem ... we can use ordinary matrices for this In [1]: format compact imatlab_export_fig('print-svg') % Static svg figures.

x0 = [-1, 1]';

Define input vector $\mathbf{U}(s)$

In [4]: u = [exp(-t); heaviside(t)];

Us = laplace(u)

1/s

In [5]: PhiS = inv(s*eye(2) - A)

In [6]: Yzs = (C*PhiS*B + D)*Us

Now compute zero state response:

In [3]: syms t s

Us =

1/(s + 1)

Compute $\Phi(s)$

Phis =

Yzs =

In [7]: Yzi = C*PhiS*x0

Total response:

In [8]: Ytotal = Yzs + Yzi

2 + 5*s + 6))/s

*s + 6))/(s + 1)

In [9]: y = ilaplace(Ytotal)

3

2.5

1.5

-0.6

-0.8

-1

-1.2

-1.4

Footnote

so the Laplace transform is

Finally compute the time response $\mathbf{y}(t)$

Ytotal =

Yzi =

In [2]: A = [0, 1; -6, -5]; B = [2, 1; -1, 0];

Now define *s* and *t* as symbolic variables

C = [2, -1; 0, 1]; D = [0, 0; 0, 0];

 $[(s + 5)/(s^2 + 5*s + 6), 1/(s^2 + 5*s + 6)]$

s + 6) + $(2*(s + 5))/(s^2 + 5*s + 6))/s$

 $-6/(s^2 + 5*s + 6)$, $s/(s^2 + 5*s + 6)$]

We can use the symbolic toolbox!

Solving this problem in MATLAB

 $5*s + 6) + 12/(s^2 + 5*s + 6))/(s + 1)$ Unforced (zero-initial condition) response: $\mathbf{Y}_{zi}(s) = (\mathbf{C}\mathbf{\Phi}(s)\mathbf{B} + \mathbf{D})$

 $-s/(s^2 + 5*s + 6) - 4/(s^2 + 5*s + 6) - (2*(s + 5))/(s^2 + 5*s + 6)$

 $\mathbf{Y}(s) = \begin{bmatrix} \frac{-3s - 10s + 34s + 10}{s(s+1)(s+2)(s+3)} \\ \frac{s^3 + 6s^2 - 12s - 6}{s(s+1)(s+2)(s+3)} \end{bmatrix}$ $\mathbf{y}(t) = \begin{bmatrix} \frac{8}{3}\epsilon(t) + \frac{25}{2}e^{-t} - 34e^{-2t} + \frac{59}{6}e^{-3t} \\ -\epsilon(t) - \frac{11}{2}e^{-t} + 17e^{-2t} - \frac{19}{2}e^{-3t} \end{bmatrix}.$

 $\mathbf{Y}_{zs}(s) = (\mathbf{C}\mathbf{\Phi}(s)\mathbf{B} + \mathbf{D})$

 $(s/(s^2 + 5*s + 6) + 10/(s^2 + 5*s + 6) + (4*(s + 5))/(s^2 + 5*s + 6))/(s + 1) + (6/(s^2 + 5*s + 6))$

 $s/(s^2 + 5*s + 6) + 6/(s^2 + 5*s + 6)$

 $\mathbf{Y}(s) = \mathbf{Y}_{zs}(s) + \mathbf{Y}_{zi}(s)$

 $(s/(s^2 + 5*s + 6) + 10/(s^2 + 5*s + 6) + (4*(s + 5))/(s^2 + 5*s + 6))/(s + 1) - 4/(s^2 + 5*s$ $+ 6) - s/(s^2 + 5*s + 6) - (2*(s + 5))/(s^2 + 5*s + 6) + (6/(s^2 + 5*s + 6) + (2*(s + 5))/(s^2 + 5*s + 6)$

 $s^2 + 5*s + 6 + 6/(s^2 + 5*s + 6) - 6/(s*(s^2 + 5*s + 6)) - (s/(s^2 + 5*s + 6) + 12/(s^2 + 5*s + 6))$

 $-6/(s*(s^2 + 5*s + 6)) - (s/(s^2 +$

s/(

 $(25*\exp(-t))/2 - 34*\exp(-2*t) + (95*\exp(-3*t))/6 + 8/3$ $17*\exp(-2*t) - (11*\exp(-t))/2 - (19*\exp(-3*t))/2 - 1$ Plot response In [10]: subplot(211) ezplot(y(1),[0,6]),grid,title('y 1(t)') ezplot(y(2),[0,6]),grid,title('y_2(t)') y₁(t) 3.5

y₂(t)

2

Simulation of State Space Models in Matlab • The functions step and impulse will produce step impulse responses of a system with zero initial conditions. • The function lsim can be used to determine the state space response to arbitrary inputs. • If you need to simulate a system with arbitrary inputs and predefined initial conditions, you should use the *Linear* Simulation Tool which is opened when you execute <code>lsim(model)</code>. Visit the MATLAB Control Systems Toolbox help pages to find out more. 1. For a first order differential equation $\frac{dx}{dt} = ax$ sX(s) - x(0) = aX(s)X(s)(s-a) = x(0) $X(s) = \frac{x(0)}{s-a}$.

5

5

 $x(t) = x(0)e^{at}.$

 $\mathbf{x}(t) = \mathbf{x}(0)e^{\mathbf{A}t}.$ The proper proof is of course a little more involved and will be explored in <u>Section 7.6!</u>

For the vector case $s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$ so by comparison

The inverse laplace transform of the final equation gives