Lecturer

Set up MATLAB

```
cd matlab
pwd
clear all
imatlab_export_fig('print-svg')  % Static svg figures.
format compact
```

```
ans =
    '/Users/eechris/code/src/github.com/cpjobling/eglm03-textbook/03/5/matlab'
```

# 3.5. Analytical Root-Locus Design of Phase-Lead Compensators

This MATLAB Live Script presents an analytical procedure for phase-lead design. It is based on Section 7.8 of Phillips and Harbor *Feedback Control Systems*, Prentice Hall, 1988[1]. For the procedure it is convenient to write the compensator transfer function as

$$D(s) = \frac{a_1 s + a_0}{b_1 s + 1}$$

In this procedure we choose $a_1$, $a_0$, and $b_1$ such that given $s_1$, the equation

$$KD(s)G(s)H(s)|_{s=s_1} = -1$$

is satisfied; that is we are designing a compensator that places a root of the closed-loop characteristic equation at $s = s_1$.

In equation (2) we have four unknowns, including $K$, and only two relationships (magnitude and phase) that must be satisfied. Hence, we can arbitrarily assign values to two of the unknowns. $K$ is easily eliminated since

$$KD(s) = \frac{Ka_1 s + Ka_0}{b_1 + 1}$$

so if we assume that $K = 1$ for the design procedure we eliminate one of the unknowns. The other unknown that can be eliminated is $a_0$ which can be seen to be the DC gain of the compensator. Its value can therefore be chosen to satisfy the steady-state error requirements of the design and we need only to determine values for $a_1$ and $b_1$.

The design proceeds as follows. First, we express the desired closed loop pole position

$$s_1 = |s_1|e^{j\beta}$$

and

$$G(s_1)H(s_1) = |G(s_1)H(s_1)|\, e^{j\psi}$$

Then the design equations (derived in Appendix B of Phillips and Harbor, 1988) are

$$\begin{aligned} a_1 &= \frac{\sin\beta + a_0\,|G(s_1)H(s_1)|\,\sin(\beta-\psi)}{|s_1|\,|G(s_1)H(s_1)|\,\sin\psi} \\ b_1 &= \frac{\sin(\beta-\psi) + a_0\,|G(s_1)H(s_1)|\,\sin\psi}{-\,|s_1|\,\sin\psi} \end{aligned}$$

Given $a_0$, $G(s)H(s)$, and the desired closed-loop pole location $s_1$, (5) and (6) give the remaining compensator coefficients. This procedure places a closed-loop pole at $s = s_1$; however, the locations of the remaining poles are unknown and may be unsatisfactory. In fact, some may be unstable!

For the case that $\psi$ is either $0^\circ$ or $180^\circ$, equations (5) must be modified to give the single equation

$$a_1|s_1|\cos\beta \pm \frac{b_1|s_1|}{|G(s_1)H(s_1)|} \pm \frac{1}{|G(s_1)H(s_1)|} + a_0 = 0$$

where the plus sign applies to the case $\psi = 0^\circ$ and the minus sign applies to $\psi = 180^\circ$. For this case, the value of either $a_1$ or $b_1$ can also be assigned. An example is now given to illustrate the procedure.

## 3.5.1. Example

An executable version of this document is available as a MATLAB Live Script analrloc.mlx. You can use it to design a Lead Compensator for other systems by downoading that script and changing the set-up parameters.

### 3.5.1.1. Definitions (change these to change design)

The plant transfer function is :

```
G = tf(1,[1 0 0]);
```

The feedback transfer function is $H(s) = 1$:

```
H = tf(1,1);
```

So $G(s)H(s)$ is:

```
GH=series(G,H)
```

```
GH =
```

```

    1

   ---

   s^2


 Continuous-time transfer function.
```

The desired closed-loop poles are:

```
s1 = -2 + 2j;
```

Now the DC gain of this type 2 system will be:

$$
\begin{aligned}
K_a &= s^2 D(s) G(s) H(s)\Big|_{s=0} \\
&= s^2 \frac{a_1 s + a_0}{b_1 + 1} \times \frac{1}{s^2}\Big|_{s=0} \\
&= a_0.
\end{aligned}
$$

For the purpose of illustration let us arbitrarily take a value of $a_0 = 8/3$:

```
a0 = 8/3;
```

## 3.5.1.2. Calculations

**(You shouldn't need to change these commands)**

Polar form of $s_1$

```
m_s1=abs(s1),  p_s1 = (angle(s1)*180/pi + 90) % degrees
```

```
m_s1 =
    2.8284
```

```
p_s1 =
   225
```

Transfer function evaluated at $s_1 = G(s_1)H(s_1)$in polar form:

```
[numGH,denGH] = tfdata(GH,'v');GHs1=polyval(numGH,s1)/polyval(denGH,s1)
```

```
GHs1 =
   0.0000 + 0.1250i
```

Magnitude:

```
mGHs1=abs(GHs1)
```

```
mGHs1 =
    0.1250
```

Phase:

```
pGHs1=angle(GHs1)*180/pi - 180 % degrees
```

```
pGHs1 =
   -90
```

Hence angles are:

```
beta = p_s1*pi/180
psi = pGHs1*pi/180  % radians
```

```
beta =
    3.9270
```

```
psi =
   -1.5708
```

From (5)

```
a1 = (sin(beta) + a0*mGHs1*sin(beta - psi))/(m_s1*mGHs1*sin(psi))
b1 = (sin(beta + psi) + a0*mGHs1*sin(beta))/(-(m_s1)*sin(psi))
```

```
a1 =
    2.6667
```

```
b1 =
    0.1667
```

Compensator is therefore given by

```
numD = [a1, a0], denD = [b1, 1]
```

```
numD =
    2.6667    2.6667
```

```
denD =
    0.1667    1.0000
```

which in normal form:

$$D(s) = K_c\left(\frac{s+z_1}{s+p_1}\right)$$

has

```
Kc = a1/b1, z0 = a0/a1, p0 = 1/b1
```

```
Kc =
   16.0000
```

```
z0 =
     1
```

```
p0 =
    6.0000
```

Now make a transfer function

```
D = tf(Kc*[1, z0],[1, p0])
```

```
D =
 

  16 s + 16

  ---------

    s + 6

 
Continuous-time transfer function.
```

### 3.5.1.3. Evaluation of Design

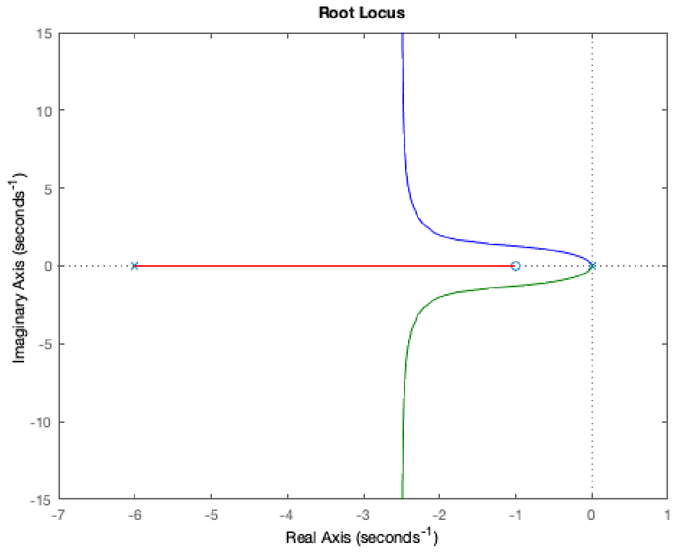
Open loop transfer function:

```
Go = series(D,GH)
```

```
Go =
 

   16 s + 16

  -----------

  s^3 + 6 s^2

 
Continuous-time transfer function.
```

*Root locus:*

```
rlocus(Go)
```



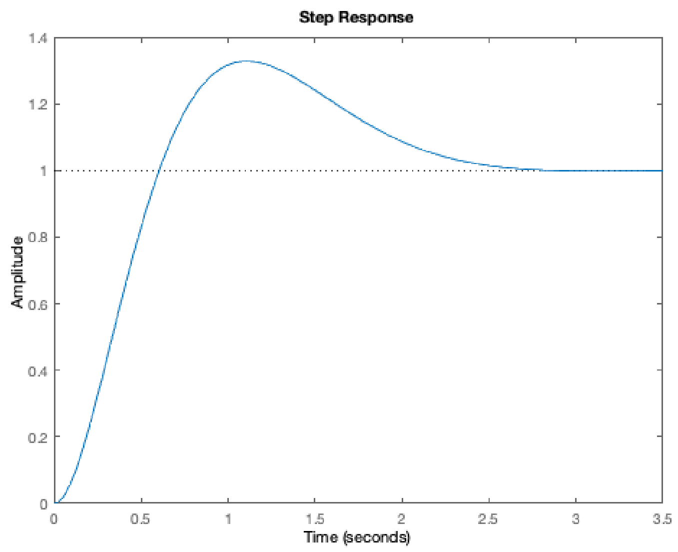

*Closed-loop transfer function:*

```
DG=series(D,G)
Gc=feedback(DG,H)
```

```
DG =


   16 s + 16

  -----------

  s^3 + 6 s^2


Continuous-time transfer function.

Gc =


         16 s + 16

  ----------------------

  s^3 + 6 s^2 + 16 s + 16


Continuous-time transfer function.
```

*Step response:*

```
step(Gc)
```




As an exercise, you should examine the effect of designing for a range of DC gains in the range $0 \leq K_a \leq 10$.

## 3.5.2. Footnotes

[1] The proofs of the formulae given are derived in Appendix B of that text.