# Continuous System Equivalence

## Introduction

In many cases, e.g. signal processing, control systems, etc., we want to design a digital system so that it behaves (dynamically and in steady-state) the same as a continuous system. A digital system that has the same input-behaviour as a (sampled) continuous system is called a *continuous equivalent*.

## Agenda

In the pre-lecture presentation we will start by discussing the relationship of $s$ to $z$. We will then present four ways to convert a transfer function $H(s)$ into its digital equivalent $H(z)$. These are:

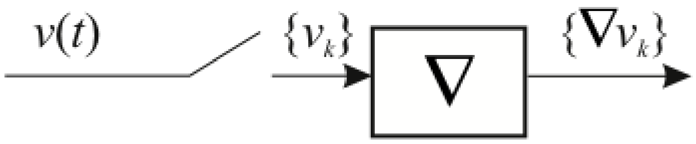
- The zero-order hold equivalent
- The Tustin bilinear transform equivalent
- Matched pole zero equivalent
- Modified matched pole-zero equivalent

Before we can describe what we might mean by a continuous equivalent system, it is necessary to establish the relationship between digital operations, such as the shift, and continuous operations.

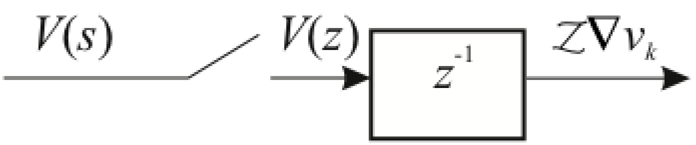
## Equivalence of s and z

### Sampling a Delayed Signal

Consider a simple operation of sampling with a delay

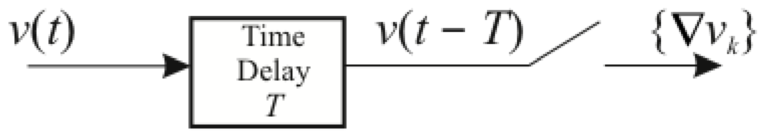



This can be represented in transform as

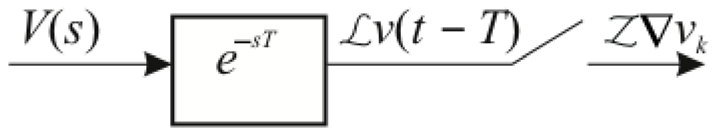


### Delaying a Sampled Signal

The same result could be obtained by delaying the continuous signal and then sampling.




Which can be represented in transform as




### Relationship of z to s

From the preceding arguments

$$z^{-1} = e^{-sT}$$

That is

$$z = e^{sT}$$

or

$$s = \frac{1}{T} \ln z$$

This is the fundamental relationship of equivalence. Before using it, we must see how a continuous signal is reconstructed from a digital signal. This is accomplished by means of a "*Digital-to-Analogue Converter*"

## Digital-to-Analogue Converter

### Modelling a DAC with a Zero-Order Hold

The simplest converter is a "*Zero-Order Hold*" (see Fig. 1). This acts the opposite way to a sampler.

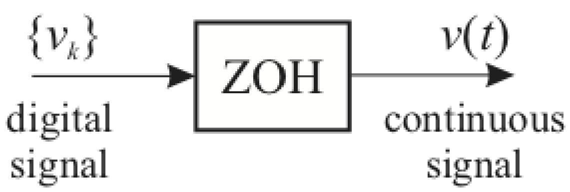



**Figure 1: zero-order Hold**

## Operation of the Zero-order Hold

During each sample period, the device holds the output $v(t)$ constant at the current value of the digital signal $v_k$.

That is

$$v(t) = v_k \text{ for } kT \leq t < (k+1)T$$

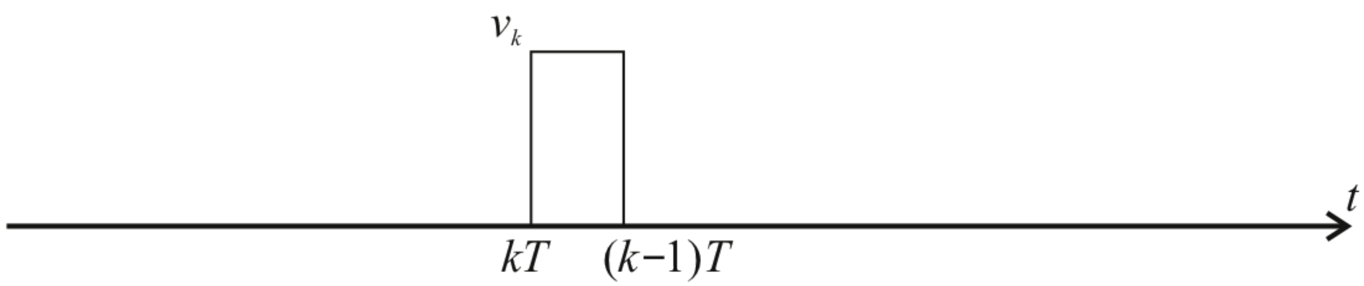
This generates a stepwise continuous signal $v(t)$ which at the sampling instants is equal to the continuous signal from which the digital signal $v_k$ was generated.

The signal may be considered as an infinite number of pulses of which the $k$-th is that shown in the next slide.

## Modelling the ZOH Mathematically

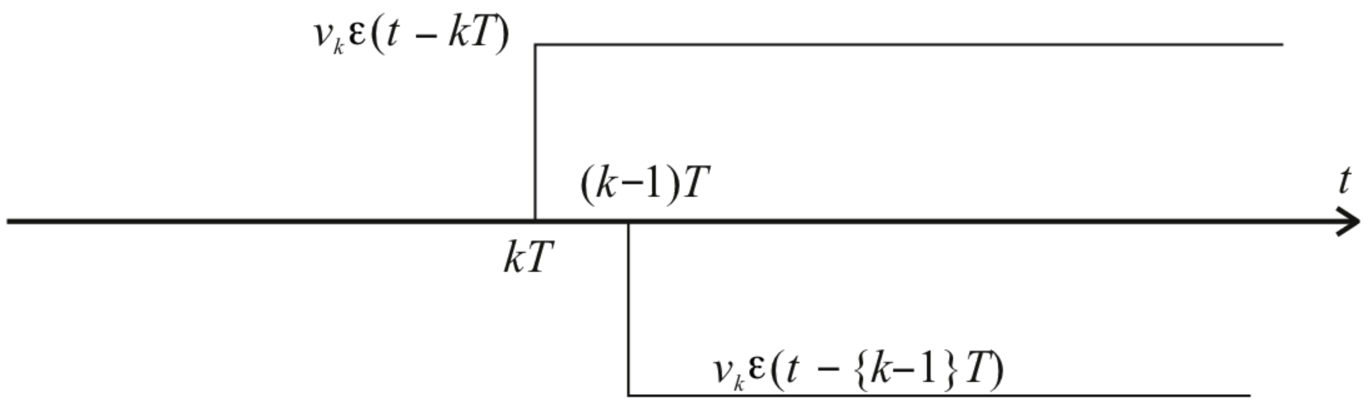
### Step-wise continuous signal

This represents the output of the zero-order hold $v(t) = v_k$ for $kT \leq t < (k+1)T$.




To model such a signal we use the so-called "*gating*" property of the time-delayed unit-step function $\epsilon(t)$ illustrated in the next slide.

### The Gating Function



The opening "gate" is given by $v_k\epsilon(t - kT)$, a step of height $v_k$, which is activated at $t = kT$ seconds. The gate is "closed" by a negative going unit step, also of height $v_k$, which is activated at $t = \{k+1\}T$ seconds.

The sum of these two signals is

$$\begin{aligned} p(t) &= v_k\epsilon(t - kT) - v_k\epsilon(t - \{k+1\}T) \\ &= v_k\left[\epsilon(t - kT) - \epsilon(t - \{k+1\}T)\right] \end{aligned}$$

So for the sequence:

$$v(t) = \sum_{k=0}^{\infty} v_k\left[\epsilon(t - kT) - \epsilon(t - \{k+1\}T)\right]$$

In transform form this is

$$\begin{aligned} V(s) = & \sum_{k=0}^{\infty} v_k\left(\frac{1}{s}e^{-kTs} - \frac{1}{s}e^{-\{k+1\}Ts}\right) \\ &= \frac{1}{s}\left(1 - e^{-Ts}\right)\sum_{k=0}^{\infty} v_k e^{-kTs} \\ &= \frac{1}{s}\left(1 - z^{-1}\right)\sum_{k=0}^{\infty} v_k z^{-k} \\ &= \frac{1}{s}\,\frac{z-1}{z}\,V(z) \end{aligned}$$

So the zero-order hold is represented by the mixed transfer function

$$G_{\mathrm{zoh}} = \frac{1}{s}\,\frac{z-1}{z}$$

as shown in Fig. 2.

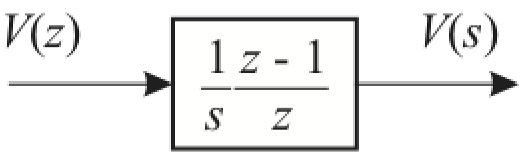
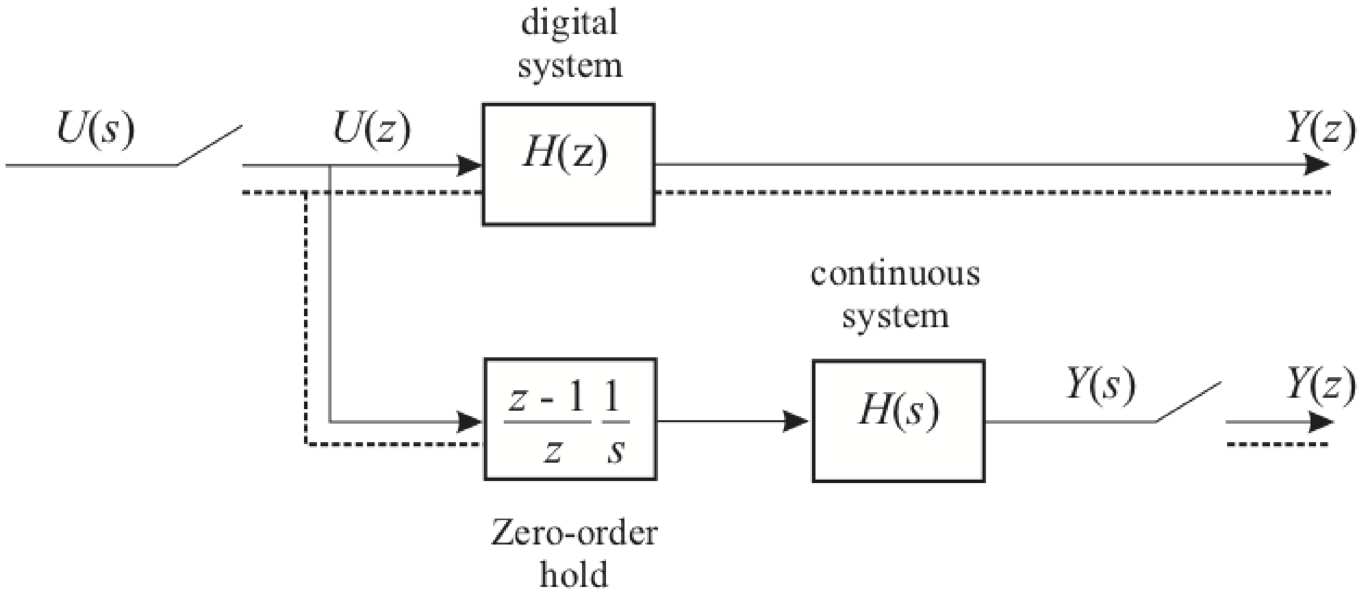



**Figure 2: Transfer Function of the Zero-Order Hold**

We can now design a 'hold-equivalent' digital system.

**Hold-Equivalent Digital System**




From the diagram in the previous slide.

$$
\begin{aligned}
Y(z) &= H(z)U(z) \\
Y(z) &= \mathcal{Z}Y(s) \\
&= \mathcal{Z}\frac{1}{s}H(s)\frac{z-1}{z}U(z)
\end{aligned}
$$

so

$$H(z) = \frac{z-1}{z}\mathcal{Z}\frac{H(s)}{s}$$

**Example 1**

If

$$H(s) = \frac{a}{s+a}$$

then find the Zero-Order Hold Equivalent $H(z)$

***Solution***

$$
\begin{aligned}
H(z) &= \frac{z-1}{z}\mathcal{Z}\frac{a}{s(s+a)} \\
&= \frac{z-1}{z}\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})} \\
&= \frac{1-e^{-aT}}{z-e^{-aT}}.
\end{aligned}
$$

**Malab note**: the zero-order-hold equivalent is the default system used for continuous system equivalence in *MATLAB*. To convert a continous system in any *lti* format[1] use:

```
lti_d = c2d(lti, Ts); % You must provide a sampling time or use -1 if unde
fined
```

# Other Continuous System Equivalences

## Approximation based on numerical integration

An alternative approach is to use the relationship

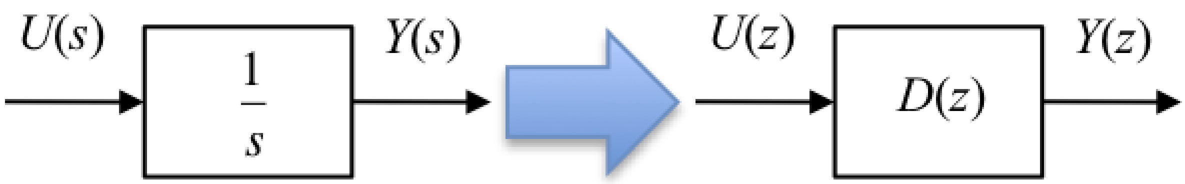
$$s = \frac{1}{T}\ln z.$$

This cannot be substituted into a transfer function directly as the result is not rational, but an approximation may be used.

#### Approximation by Numerical Integration

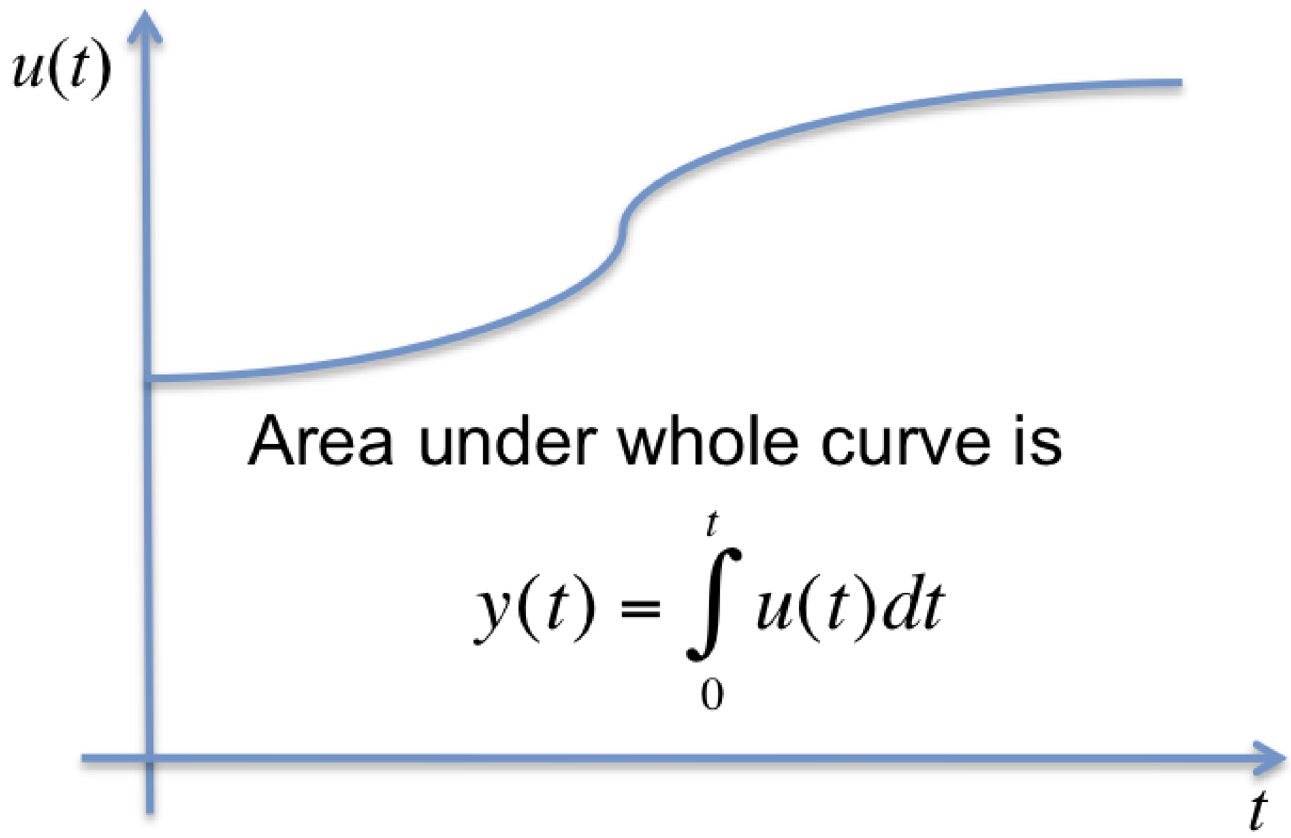
We wish to find a transfer function $T(z)$ that is equivalent to $T(s) = s$.

Let us instead seek a transfer function $D(z)$ that is equivalent to $D(s) = 1/s$.




Thus, the transfer function we are seeking will in fact be an approximation of the integral $y(t) = \int u(t)dt$. We can illustrate this as shown in the next slide.

**Model of Integration**




If we sample the curve $u(t)$ and consider the situation at the $n$-th sampling instant, we will have

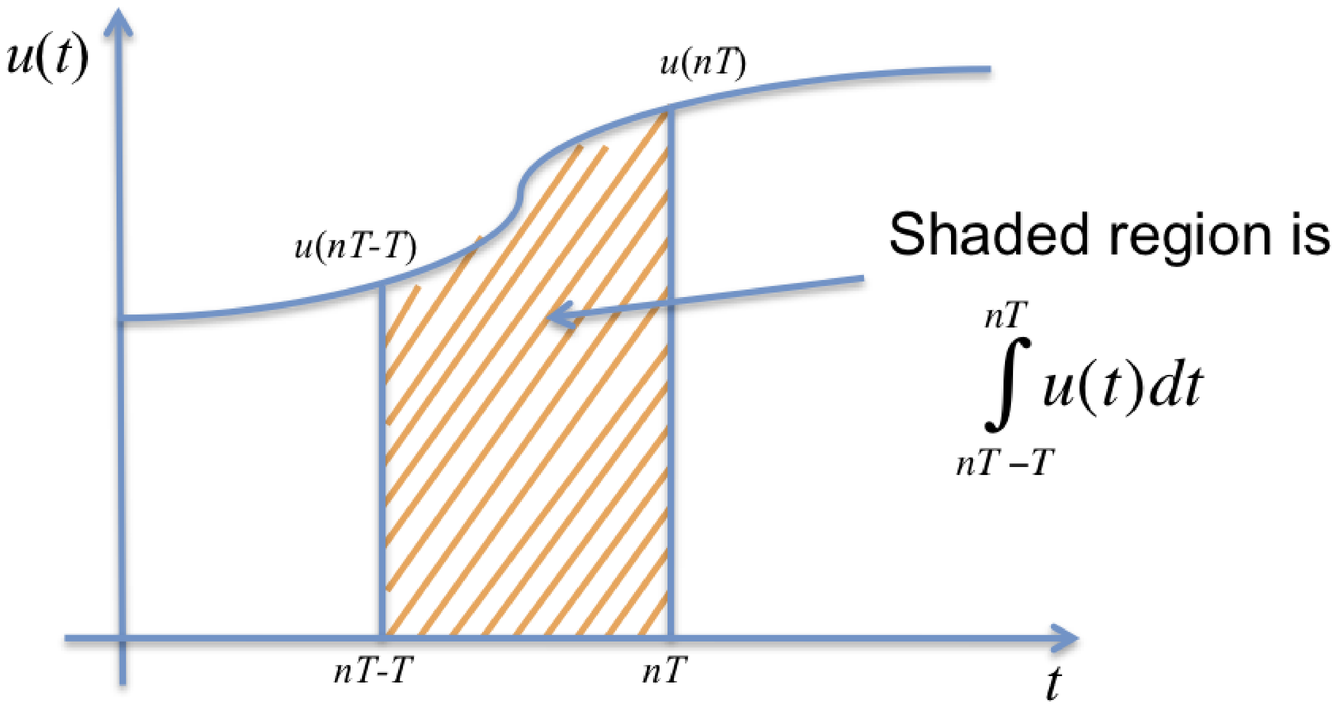
$$y(nT) = \int_0^{nT} u(t)dt$$

We can rewrite this as

$$y(nT) = \int_0^{nT-T} u(t)dt + \int_{nT-T}^{nT} u(t)dt$$

where the second integral term is the shaded area shown in the next slide,

**Sampled Model of Integration**



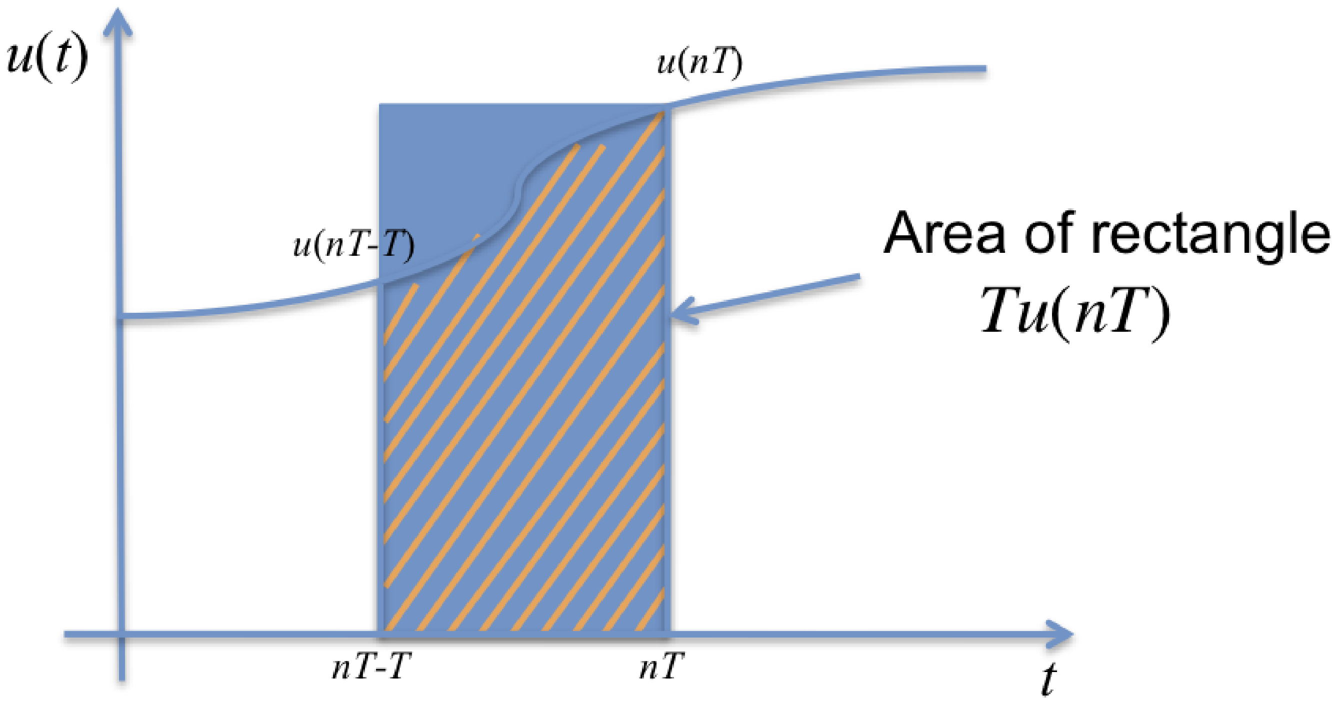

Now if we assume that the first integral term was approximated by the digital integrator in the previous sampling instant, then $y(nT-T) = \int_0^{nT-T} u(t)dt$ is known, and consequently we have

$$y(nT) = y(nT-T) + \int_{nT-T}^{nT} u(t)dt$$

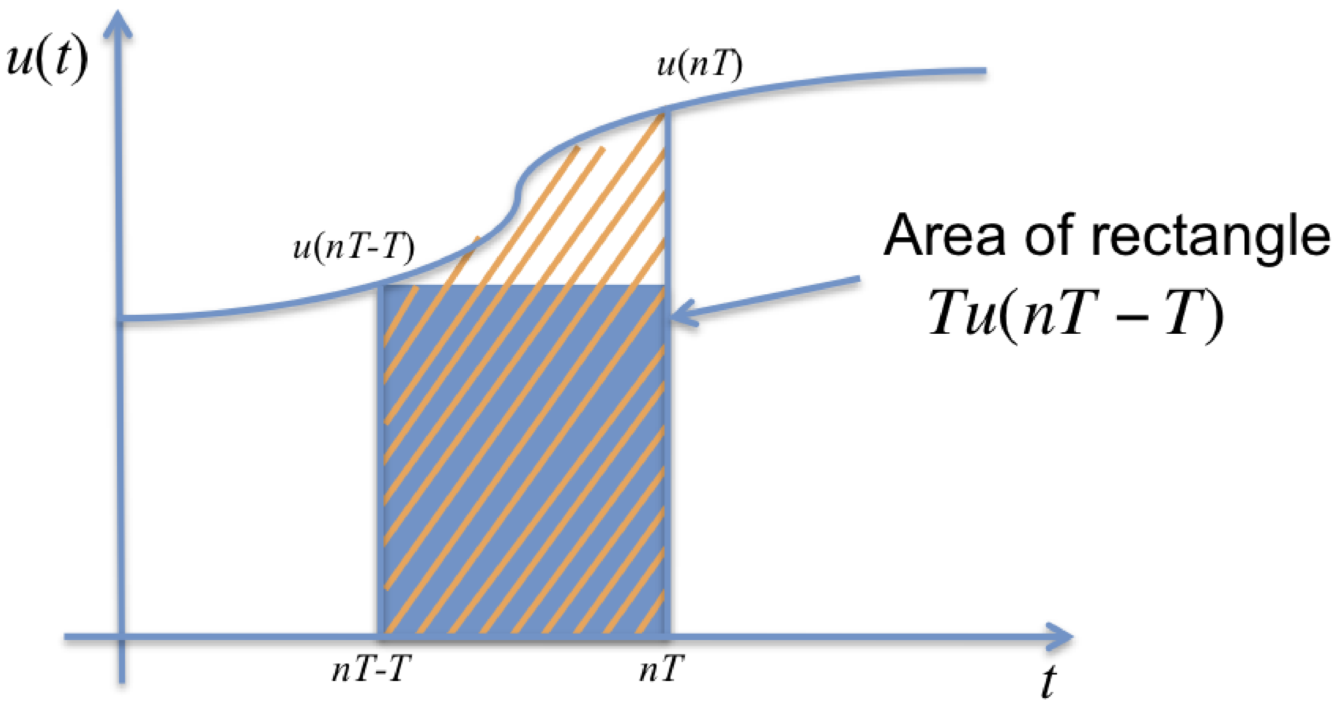
and we only need to determine the area of the shaded region to approximate $y(nT)$.

The most obvious approximations are illustrated in the next three slides.

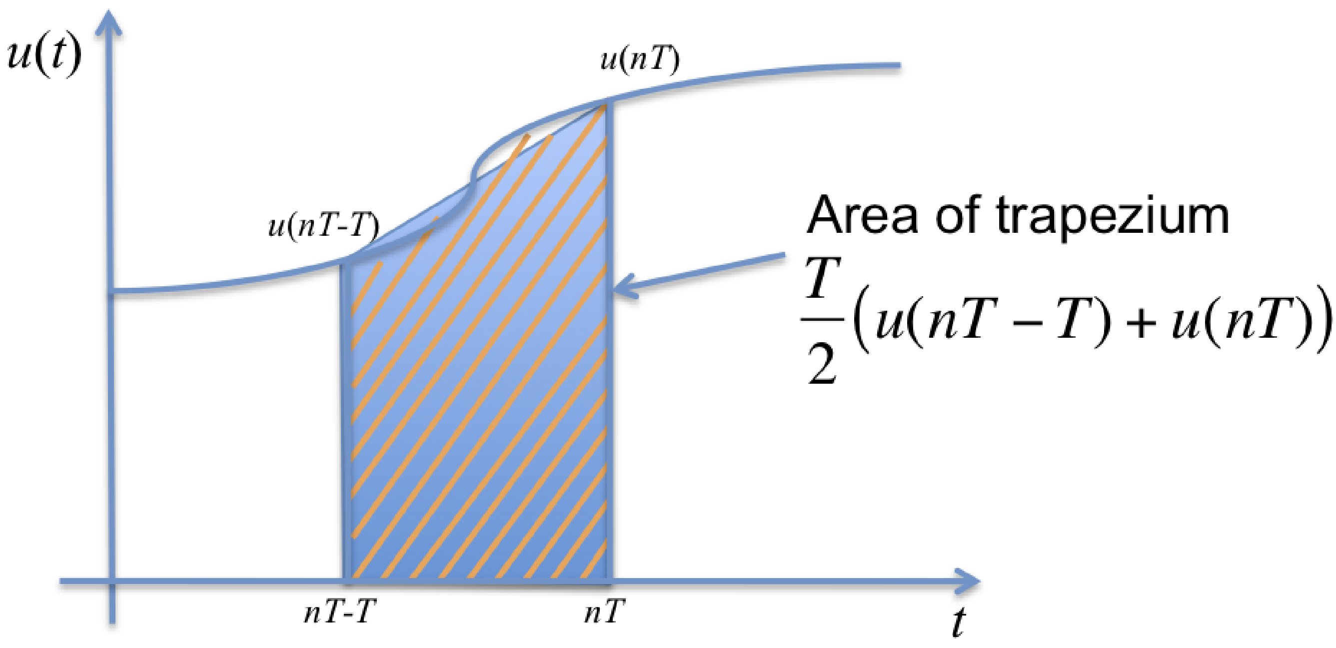
**Forward Rectangular Approximation**




**Backward Rectangular Approximation**



**Trapezoidal Approximation**




It is clear that if we are to disallow any further subdivision of the shaded area, the *trapezoidal approximation* will provide the most accurate result.

**z-transform of trapezoidal approximation**

Completing the analysis, we can show that

$$y(nT) \approx y(nT-T) + \frac{T}{2}\{u(nT-T) + u(T)\}$$

which on taking z-transforms gives

$$Y(z) = z^{-1}Y(z) + \frac{T}{2}\{z^{-1}U(z) + U(z)\}$$

thus

$$\frac{Y(z)}{U(z)} = D(z) = \frac{T}{2}\left\{\frac{1+z^{-1}}{1-z^{-1}}\right\}.$$

**Approximation of s by numerical integration**

By comparison with continuous integration $Y(s)/U(s) = 1/s$, this result, obtained by *numerical integration* allows us to say that

$$s \approx \frac{2}{T}\{\frac{1-z^{-1}}{1+z^{-1}}\}.$$

These results are summarised in the next few slides.

## Summary of Numerical Integration Methods

#### Trapezoidal approximation

$$\begin{aligned} s = & \quad \frac{2}{T}\frac{z-1}{z+1} \\ z = & \quad \frac{1+1/2sT}{1-1/2sT} \end{aligned}$$

compare expansion

$$z=(1+ (1/2) sT)(1- (1/2) sT)^{-1} = 1 + sT + (1/2) s^2T^2 + (1/4) s^3T^3

- \cdots$$ with

$$z = e^{sT} = 1 + sT + (1/2)s^2T^2 + (1/6)s^3T^3 + \cdots$$

This approximation is known as "*Tustin*'s *Bilinear Transformation*".

#### Forward rectangular approximation

$$\begin{aligned} s = & \quad \frac{z-1}{T} \\ z = & \quad 1 + sT \end{aligned}$$

compare with

$$z = e^{sT} = 1 + sT + (1/2)s^2T^2 + (1/6)s^3T^3 + \cdots$$

#### Backward rectangular approximation

$$\begin{aligned} s = & \quad \frac{z-1}{Tz} \\ z = & \quad \frac{1}{1+sT} \end{aligned}$$

compare expansion

$$z=(1-sT)^{-1} = 1 + sT + s^2T^2 + s^3T^3

- \cdots

  $$with$$

  z = e^{sT} = 1 + sT + (1/2) s^2T^2 + (1/6) s^3T^3 + \cdots$$

## Example 2

If

$$H(s) = \frac{a}{s+a},$$

determine the equivalent $H(z)$ using Tustin's bilinear transformation

### Solution

Tustin's bilinear transformation gives a digital system with transfer function

$$\begin{aligned} H(z) &= \frac{a}{\frac{2}{T}\frac{z-1}{z+1} + a} \\ &= \frac{\left(\frac{(aT)/2}{1+(aT)/2}\right)(z+1)}{z - \left(\frac{1-(aT)/2}{1+(aT)/2}\right)}. \end{aligned}$$

**Malab note**: the bilinear transform equivalent is obtained by passing the argument `'tustin'` to the *c2d* method:

```
lti_d = c2d(lti, Ts, 'tustin');
```

## Matched pole-zero approximations

Another way to obtain a digital approximation of a continuous transfer function is to use the relationship $z = e^{sT}$ to map poles and zeros of the continuous transfer function into poles and zeros in the z-transfer function.

Since continuous transfer functions often have more poles than zeros, that is $n - m$ zeros at infinity, zeros at infinity are replaced in the z-transfer function by zeros at $z = -1$ which is equivalent to half the sampling frequency $\omega_s/2$ (i.e. the highest frequency possible in the z-domain). Thus if $n - m = 1$ we add $(1 + z^{-1})$, if $n - m = 2$, $(1 + z^{-1})^2$, etc.

### Matched-Pole Zero method

Idea is that all poles and zeros of continuous transfer function $D(s)$ can become poles and zeros of digital transfer function $D(z)$ if the mapping $z = e^{sT}$ is used.

Method:

1. Map finite poles and zeros of $D(s)$ to poles and zeros of $D(z)$ according to $z = e^{sT}$.

1. Add zeros at $z = -1$ for each infinite zero in $D(s)$.

1. Match DC or low-frequency gain.

### Example 3

Use the Matched-Pole-Zero (MPZ) approximation to give the z-transfer function equivalent to

$$D(s) = \frac{s+a}{s+b}.$$

### *Solution*

Order of numerator and denominator are equal so there are no infinite zeros and by matching poles and zeros

$$D(z) = k\left(\frac{1-e^{-aT}z^{-1}}{1-e^{-bT}z^{-1}}\right).$$

For $D(s)$ the DC gain is $D(s)|_{s=0} = a/b$.

The DC gain for $D(z)$ is $D(z)|_{z=1}$ (from final value theorem) that is

$$k\left(\frac{1-e^{-aT}}{1-e^{-bT}}\right) = \frac{a}{b}.$$

We choose $k$ so that the DC gains match, i.e.

$$k = \frac{a}{b}\left(\frac{1-e^{-bT}}{1-e^{-aT}}\right).$$

### Example 4

Use the Matched-Pole-Zero (MPZ) approximation to give the z-transfer function equivalent to

$$D(s) = \frac{s+a}{s(s+b)}.$$

***Solution***

Here, the order of the denominator is one greater than the numerator so there is $n - m = 2 - 1 = 1$ infinite zero. Placing this zero at $z = -1$ makes the MPZ transfer function

$$D(z) = k\left(\frac{(1+z^{-1})(1-e^{-aT}z^{-1})}{(1-z^{-1})(1-e^{-bT}z^{-1})}\right).$$

As $D(s)$ is type 1, we can't use the value of $D(0)$ to compute the DC gain. Instead, let us compute the gain at $s = -1$:

$$\begin{aligned} D(-1) = & \left.\frac{s+a}{s(s+b)}\right|_{s=-1} \\ = & \frac{-1+a}{(-1)(-1+b)} = \frac{a-1}{(1-b)} \end{aligned}$$

The equivalent value $s = -1$ in the $z$-plane is $z = e^{-T}$.

$$\begin{aligned} k\left.\frac{(1+z^{-1})(1-e^{-aT}z^{-1})}{(1-e^{-bT}z^{-1})}\right|_{z=k} = & \frac{(1+e^{T})(1-e^{-aT}e^{T})}{(1-e^{-bT}e^{T})} \\ = & k\frac{\left(1+e^{T}\right)\left(1-e^{-(1-a)T}\right)}{\left(1-e^{-(1-b)T}\right)} \end{aligned}$$

Again, we choose $k$ so that the equivalent gains match, i.e.

$$k = \left(\frac{1-b}{a-1}\right)\frac{\left(1-e^{-(1-b)T}\right)}{\left(1+e^{T}\right)\left(1-e^{-(1-a)T}\right)}.$$

We would implement $D(z)$ as a *difference equation* defined in terms of the sampled inputs and outputs as shown for Example 3.

#### Implementation of an MPZ Approximation

If

$$\frac{Y(z)}{U(z)} = D(z) = k\frac{1-\alpha z^{-1}}{1-\beta z^{-1}}$$

then the digital implementation will be

$$y(n) = \beta y(n-1) + k(u(n) - \alpha u(n-1))$$

$y(n)$ is the current calculated value, $y(n-1)$ is previous calculated value, $u(n)$ is current sample, and $u(n-1)$ is previous sample.

Implementation only works if computation rate $<<$ significant dynamics of the sampled system and a small fraction of sampling time $T$.

In the implementation, you should notice that it is actually physically impossible to sample $u(t)$, compute $y(n)$ and output $y(n)$ all at the same instant of time. Hence the equation is actually impossible to implement. However, if the computation is sufficiently fast, the delay between sampling $u(t)$ and outputting $y(t)$ will be small and can often be neglected. A rule of thumb has computation delay $< t_r/20$ of the dominant pole. It should certainly be a small fraction of the sampling period $T$.

**Modified MPZ**

Used if constraint on computation time cannot be met[2].

Allow a zero at infinity in $D(z)$ so that order of numerator is one less than denominator.

This ensures that only past values of $u$ and $y$ appear in the implementation equation and a whole sample period is available for computation

**Example 5**

Re-implement Example 3 using the Modified MPZ method.

***Solution***

If we allow an infinite zero

$$D(s) = \frac{s+a}{s(s+b)}$$

becomes

$$D(z) = k\frac{(1-e^{-aT}z^{-1})}{(1-z^{-1})(1-e^{-bT}z^{-1})}$$

and

$$k = \left(\frac{1-b}{a-1}\right)\frac{\left(1-e^{-(1-b)T}\right)}{\left(1-e^{-(1-a)T}\right)}.$$

The implementation is now

$$y(n) = (1-e^{-bT})y(n-1) - e^{-bT}y(n-2) + k(u(n-1) - e^{-aT}u(n-2))$$

and now the computation for $y(n)$ is performed only on past values of $u(n)$ and $y(n)$ and a whole sample period is available for computation.

**Matlab note**: the matched-pole zero equivalent of an LTI system is obtained by passing `'matched'` as the third argument to *c2d*:

```
lti_d = c2d(lti, Ts, 'matched')
```

There isn't a built-in method that returns the *modified*-matched-pole zero equivalent.

# Footnotes

1: The LTI functions are `tf`, `ss`, and `zpk`. For more information, type `help lti` inside *MATLAB*.

2: Which these days is unlikely