Transforms and Time Responses for State Space Models

- Laplace Transform of State Space Models
- Time Responses for State Space Models
- Detailed example (in class)
- Problems (homework)

Laplace Transforms of State Space Models

The Laplace transform can be used to convert a differential equation into a transfer function. It can also be used to convert a state space model into a transfer function. In this lecture we demonstrate how this is done and we give an example.

Laplace transform of a vector of functions

The Laplace transform of a vector $\mathbf{v}(t)$ is a vector $\mathbf{V}(s)$. The elements of $\mathbf{V}(s)$ are the Laplace transforms of the corresponding elements of the vector $\mathbf{v}(t)$.

For array

$$\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_n(t) \end{bmatrix}$$

The transformed variables are

$$\mathcal{L}\mathbf{v}(t) = \begin{bmatrix} \mathcal{L}v_1(t) \\ \mathcal{L}v_2(t) \\ \vdots \\ \mathcal{L}v_n(t) \end{bmatrix} = \begin{bmatrix} V_1(s) \\ V_2(s) \\ \vdots \\ V_n(s) \end{bmatrix} = \mathbf{V}(s)$$

For example, if[1]

$$\mathbf{v}(t) = \begin{bmatrix} \epsilon(t) \\ e^{-at} \\ \sin bt \end{bmatrix}$$

then

$$\mathbf{V}(s) = \begin{bmatrix} 1/s \\ 1/(s+a) \\ b/(s^2+b^2) \end{bmatrix}$$

Transform of State Equations

Let us now transform the generalized form of the state equations obtained in the last lecture.

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Applying the Laplace transform to both sides of this matrix equation gives the transform equations

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$
$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

where $\mathbf{x}(0)$ is the vector of initial conditions vector of the states; $\mathbf{X}(s)$ is the state transform vector; $\mathbf{U}(s)$ input transform vector; $\mathbf{Y}(s)$ is output transform vector.

Transformed State-Equations for Example 1 from Section

For the system in the example the state vector is defined as $\mathbf{x} = [v_{31}, i_1]^T$, the input current is u, and the output variables are all the currents and voltages in the circuit $\mathbf{y} = [v_{31}, i_1, v_{32}, v_{21}, i_2]^T$.

The transformed state space model is therefore:

$$s\begin{bmatrix} V_{31} \\ I_1 \end{bmatrix} - \begin{bmatrix} v_{31}(0) \\ i_1(0) \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix} \begin{bmatrix} V_{31} \\ I_1 \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} [U]$$

$$\begin{bmatrix} V_{31} \\ I_1 \\ V_{32} \\ V_{21} \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -R \\ 0 & R \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_{31} \\ I_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} [U].$$

Transfer function from State-Space Models

The transform equations may be solved as follows (the Laplace transform operator s is omitted for brevity).

Substituting $\mathbf{X}$ from (1) into (2) gives

$$\mathbf{Y} = \left[\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{BU} + \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) \right] + \mathbf{DU}$$

which after gathering terms and simplifying gives

$$\mathbf{Y} = \left[\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D} \right] \mathbf{U} + \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0)$$

When the initial conditions of the state-variables are all zero, this reduces to the transfer matrix model

$$\mathbf{Y} = \left[\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D} \right] \mathbf{U}$$

The matrix $\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$ is the *system transfer matrix*.

The element of the i-th row and j-th column is the transfer function that relates the i-th output transform Y_i to the j-th input transform U_j.

For a single-input, single-output (SISO) system, the system transfer matrix reduces to a single element transfer function.

The matrix $[s\mathbf{I} - \mathbf{A}]^{-1}$ is very important.

It is known as the *resolvent matrix* of the system.

It may be written as

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\text{adj}\,[s\mathbf{I} - \mathbf{A}]}{\det\,[s\mathbf{I} - \mathbf{A}]}.$$

Resolvent matrix for the example

For the system in the example, the resolvent matrix is developed as

$$\mathbf{A} = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix} = \begin{bmatrix} s & +1/C \\ -1/L & s + R/L \end{bmatrix}$$

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\begin{bmatrix} s + R/L & -1/C \\ +1/L & s \end{bmatrix}}{s(s + R/L) + 1/(LC)} = \frac{\begin{bmatrix} s + R/L & -1/C \\ +1/L & s \end{bmatrix}}{s^2 + (R/L)s + 1/(LC)}$$

When $[s\mathbf{I} - \mathbf{A}]^{-1}$ has been obtained, then the system transfer function is easily obtained through $\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$. For the system in the example, when all outputs are measured, the system transfer matrix is:

Transfer matrix for example

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -R \\ 0 & R \\ 0 & -1 \end{bmatrix} \left\{ \frac{\begin{bmatrix} s + R/L & -1/C \\ +1/L & s \end{bmatrix}}{s^2 + (R/L)s + 1/(LC)} \right\} \begin{bmatrix} 1/C \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -R \\ 0 & R \\ 0 & -1 \end{bmatrix} \left\{ \frac{\left[\begin{array}{c} (1/C)s + R/(LC) \\ +1/(LC) \end{array} \right]}{s^2 + (R/L)s + 1/(LC)} \right\} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2 + (R/L)s + 1/(LC)} \begin{bmatrix} (1/C)s + R/(LC) \\ 1/(LC) \\ (1/C)s \\ R/(LC) \\ -1/(LC) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{(1/C)s + R/(LC)}{s^2 + (R/L)s + 1/(LC)} \\[2ex] \dfrac{1/(LC)}{s^2 + (R/L)s + 1/(LC)} \\[2ex] \dfrac{(1/C)s}{s^2 + (R/L)s + 1/(LC)} \\[2ex] \dfrac{R/(LC)}{s^2 + (R/L)s + 1/(LC)} \\[2ex] -\dfrac{1/(LC)}{s^2 + (R/L)s + 1/(LC)} + 1 \end{bmatrix}$$

In matrix form, when combined with the input and output transforms we have the situation illustrated below. Each transfer function relates the corresponding output transform to the input transform.

For example

$$V_{31} = \frac{(1/C)s + R/(LC)}{s^2 + (R/L)s + 1/(LC)} U.$$

Transform Equations for Example
$$\mathbf{Y}(s) = \left[\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D} \right] \mathbf{U}(s)$$
$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$$

$$\begin{bmatrix} V_{31}(s) \\ I_1(s) \\ V_{32}(s) \\ V_{21}(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{(1/C)s + R/(LC)}{s^2 + (R/L)s + 1/(LC)} \\ \dfrac{1/(LC)}{s^2 + (R/L)s + 1/(LC)} \\ \dfrac{(1/C)s}{s^2 + (R/L)s + 1/(LC)} \\ \dfrac{R/(LC)}{s^2 + (R/L)s + 1/(LC)} \\ -\dfrac{1/(LC)}{s^2 + (R/L)s + 1/(LC)} + 1 \end{bmatrix} \mathbf{U}(s).$$

Note that the denominator is the same for each transfer function, and that the order of the numerator is less than the denominator except for one case, for which

$$\begin{aligned} I_2 &= \left(-\frac{1/(LC)}{s^2 + (R/L)s + 1/(LC)} + 1 \right) U \\ &= \frac{-1/LC + s^2 + (R/L)s + 1/(LC)}{s^2 + (R/L)s + 1/(LC)} U \\ &= \frac{s^2 + (R/L)s}{s^2 + (R/L)s + 1/(LC)} U \end{aligned}$$

Replacing s by $\frac{d}{dt}$ gives the corresponding differential equations relating the dependant variable to the input.

Converting SS to TF in Matlab

We will do this in class

Continuing example from [Section 7.1 (../1/intro2ss)](../1/intro2ss):

In [11]:

```
clear all
format compact
```

Define some values for capacitance, inductance and resitance

In [12]:

```
Cap = 1; L = 1; R = 1;
```

Define state space model and label states inputs and outputs

```
A = [0 -1/Cap; 1/L -R/L];
B = [1/Cap; 0];
C = [1 0; 0 1; 1 -R; 0 R; 0 -1];
D = [0; 0; 0; 0; 1];
circ_ss = ss(A, B, C, D, ...
'statename',{'v31' 'i1'}, ...
'inputname', 'u', ...
'outputname', {'v31' 'i1' 'v32' 'v21' 'i2'});
```
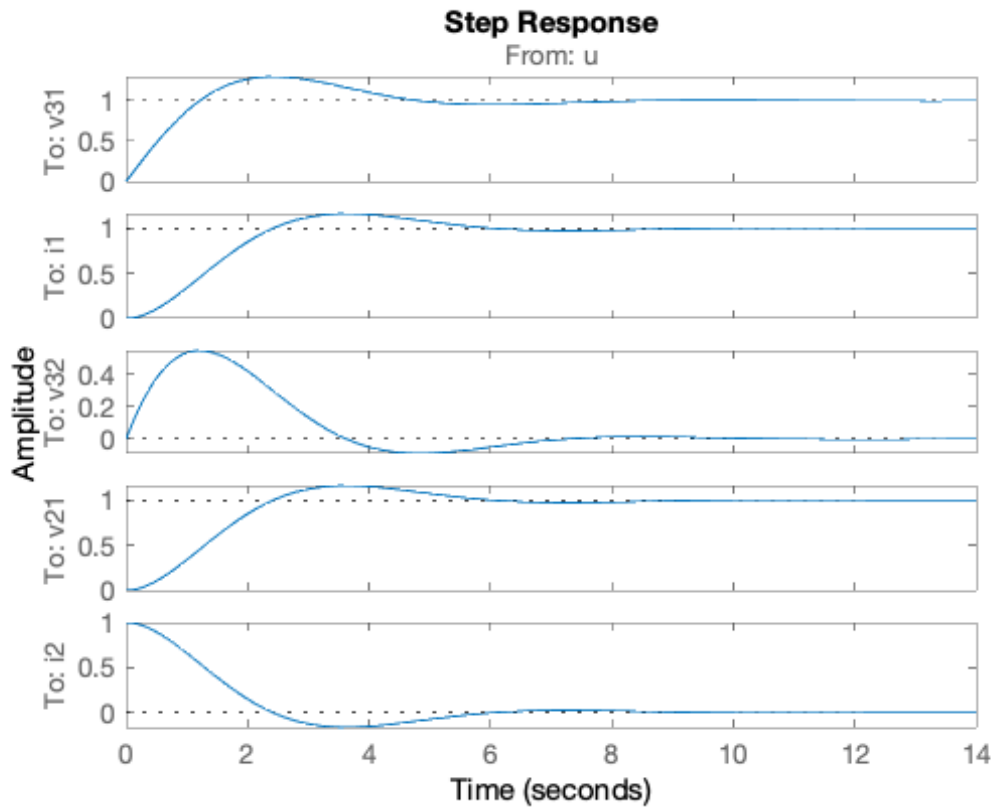
Show model

```
circ_ss
```

```
circ_ss =

  A =
          v31    i1
   v31     0    -1
   i1      1    -1

  B =
          u
   v31    1
   i1     0

  C =
          v31    i1
   v31     1     0
   i1      0     1
   v32     1    -1
   v21     0     1
   i2      0    -1

  D =
          u
   v31    0
   i1     0
   v32    0
   v21    0
   i2     1
```

Continuous-time state-space model.

Plot a step response

```
step(circ_ss)
```


Convert to transfer functiom matrix

The function `tf(ss_model)` returns a vector of transfer functions.

```
circ_tf = tf(circ_ss)
```

circ_tf =

```
  From input "u" to output...
            s + 1
   v31:  -----------
         s^2 + s + 1


             1
   i1:   -----------
         s^2 + s + 1


             s
   v32:  -----------
         s^2 + s + 1


             1
   v21:  -----------
         s^2 + s + 1


           s^2 + s
   i2:   -----------
         s^2 + s + 1
```

Continuous-time transfer function.

Determine poles and zeros

```
circ_zpk=zpk(circ_ss)
```

circ_zpk =

  From input "u" to output...
              (s+1)
    v31:   -------------
           (s^2 + s + 1)

               1
    i1:   -------------
           (s^2 + s + 1)

               s
    v32:   -------------
           (s^2 + s + 1)

               1
    v21:   -------------
           (s^2 + s + 1)

            s (s+1)
    i2:   -------------
           (s^2 + s + 1)

Continuous-time zero/pole/gain model.

The state transition matrix

Calculated using the symbolic math tools provided by MATLAB See `help`
`symbolic`

```
syms phi t s
phi = inv(s*eye(2) - A)
```

phi =
[(s + 1)/(s^2 + s + 1),   -1/(s^2 + s + 1)]
[1/(s^2 + s + 1),   s/(s^2 + s + 1)]

The state transfer matrix

```
G = C*phi*B + D
```

G =

$$
(s + 1)/(s^2 + s + 1)
$$
$$
1/(s^2 + s + 1)
$$
$$
(s + 1)/(s^2 + s + 1) - 1/(s^2 + s + 1)
$$
$$
1/(s^2 + s + 1)
$$
$$
1 - 1/(s^2 + s + 1)
$$

```
G = simplify(G)
```

G =

$$
(s + 1)/(s^2 + s + 1)
$$
$$
1/(s^2 + s + 1)
$$
$$
s/(s^2 + s + 1)
$$
$$
1/(s^2 + s + 1)
$$
$$
1 - 1/(s^2 + s + 1)
$$

```
pretty(G)
```

```
/          s + 1          \
|        ----------        |
|          2               |
|        s   + s + 1       |
|                          |
|            1             |
|        ----------        |
|          2               |
|        s   + s + 1       |
|                          |
|            s             |
|        ----------        |
|          2               |
|        s   + s + 1       |
|                          |
|            1             |
|        ----------        |
|          2               |
|        s   + s + 1       |
|                          |
|                1         |
|  1  -  ----------        |
|            2             |
\          s   + s + 1    /
```

A executable script version of this example is available as ssmodels.mlx (matlab/ssmodels.mlx).

Some Important Properties

System poles

Clearly the denominator of the transfer function is generated by the matrix inverse which produces the term:

$$\det[s\mathbf{I} - \mathbf{A}]$$

This evaluates to the denominator polynomial and the poles of the system are the roots of the system's characteristic equation:

$$\det[s\mathbf{I} - \mathbf{A}] = 0.$$

The system poles are solutions to the system's characteristic equation

$$\det[s\mathbf{I} - \mathbf{A}] = 0.$$

System zeros

What is the corresponding numerator polynomial of the transfer function, whose roots give the zeros of the system?

The zeros are those values of s for which the output is zero when the input and states are not zero.

Thus:

$$(s\mathbf{I} - \mathbf{A})\mathbf{X} - \mathbf{B}U = 0$$
$$\mathbf{C}\mathbf{X} + dU = Y = 0$$

In matrix form:

$$\begin{bmatrix} s\mathbf{I} - \mathbf{A} & \vdots & \mathbf{B} \\ \cdots & \cdots & \cdots \\ \mathbf{C} & \vdots & d \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \cdots \\ U \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \cdots \\ 0 \end{bmatrix}$$

The only way this can have non-zero solutions in $\mathbf{X}$ and U is if:

$$\det \begin{bmatrix} s\mathbf{I} - \mathbf{A} & \vdots & \mathbf{B} \\ \cdots & \cdots & \cdots \\ \mathbf{C} & \vdots & d \end{bmatrix} = 0$$

This is another polynomial in s whose roots give the system zeros and therefore corresponds to the numerator polynomial of the TF.

Given this result, an alternative expression for the TF is:

$$\frac{Y(s)}{U(s)} = \frac{\det \begin{bmatrix} s\mathbf{I} - \mathbf{A} & \vdots & \mathbf{B} \\ \cdots & \cdots & \cdots \\ \mathbf{C} & \vdots & d \end{bmatrix}}{\det[s\mathbf{I} - \mathbf{A}]}$$

Time Responses from Transfer Function Matrices

In the [next section (../3/tr4ss)](../3/tr4ss) we will consider how we can use the transfer function model to compute time responses from state-space models.

Footnote

1. $\epsilon(t)$ is the unit step function $\epsilon(t) = 0$ for $t < 0$; $\epsilon(t) = 1$ for $t \geq 0$.