Laplace Transforms of State Space Models

The Laplace transform can be used to convert a differential equation into a transfer function. It can also be used to convert a state space model into a transfer function. In this lecture we demonstrate how this is done and we give an example.

The Laplace transform of a vector $\mathbf{v}(t)$ is a vector $\mathbf{V}(s)$. The elements of $\mathbf{V}(s)$ are the Laplace transforms of the corresponding elements of the vector $\mathbf{v}(t)$.

For array

$$\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_n(t) \end{bmatrix}$$

The transformed variables are

$$\mathcal{L}\mathbf{v}(t) = \begin{bmatrix} \mathcal{L}v_1(t) \\ \mathcal{L}v_2(t) \\ \vdots \\ \mathcal{L}v_n(t) \end{bmatrix} = \begin{bmatrix} V_1(s) \\ V_2(s) \\ \vdots \\ V_n(s) \end{bmatrix} = \mathbf{V}(s)$$

For example, if[1]

$$\mathbf{v}(t) = \begin{bmatrix} \epsilon(t) \\ e^{-at} \\ \sin bt \end{bmatrix}$$

then

$$\mathbf{V}(s) = \begin{bmatrix} 1/s \\ 1/s + a \\ b/(s^2 + b^2) \end{bmatrix}$$

Let us now transform the generalized form of the state equations obtained in the last lecture.

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Applying the Laplace transform to both sides of this matrix equation gives the transform equations

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$
$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

where $\mathbf{x}(0)$ is the vector of initial conditions vector of the states; $\mathbf{X}(s)$ is the state transform vector; $\mathbf{U}(s)$ input transform vector; $\mathbf{Y}(s)$ is output transform vector.

Transformed State-Equations for Example 1 from Section

For the system in the example the state vector is defined as $\mathbf{x} = [v_{31}, i_1]^T$, the input current is u, and the output variables are all the currents and voltages in the circuit $\mathbf{y} = [v_{31}, i_1, v_{32}, v_{21}, i_2]^T$.

The transformed state space model is therefore:

$$s \begin{bmatrix} V_{31} \\ I_1 \end{bmatrix} - \begin{bmatrix} v_{31}(0) \\ i_1(0) \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix} \begin{bmatrix} V_{31} \\ I_1 \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} [U]$$

$$\begin{bmatrix} V_{31} \\ I_1 \\ V_{32} \\ V_{21} \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -R \\ 0 & R \\ 0 & -1 \end{bmatrix} \begin{bmatrix} V_{31} \\ I_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} [U].$$

Laplace Transform from State-Space Models

Substituting $\mathbf{X}$ from (1) into (2) gives

$$\mathbf{Y} = \left[\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U} + \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) \right] + \mathbf{D}\mathbf{U}$$

which after gathering terms and simplifying gives

$$\mathbf{Y} = \left[\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D} \right] \mathbf{U} + \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0)$$

When the initial conditions of the state-variables are all zero, this reduces to the transfer matrix model

$$\mathbf{Y} = \left[\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D} \right] \mathbf{U}$$

The matrix $\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$ is the *system transfer matrix*.

The element of the i-th row and j-th column is the transfer function that relates the i-th output transform Y_i to the j-th input transform U_j.

For a single-input, single-output (SISO) system, the system transfer matrix reduces to a single element transfer function.

The matrix $[s\mathbf{I} - \mathbf{A}]^{-1}$ is very important.

It is known as the *resolvent matrix* of the system.

It may be written as

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\text{adj}\,[s\mathbf{I} - \mathbf{A}]}{\det\,[s\mathbf{I} - \mathbf{A}]}.$$

Resolvent matrix for the example

For the system in the example, the resolvent matrix is developed as

$$\mathbf{A} = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix} = \begin{bmatrix} s & +1/C \\ -1/L & s + R/L \end{bmatrix}$$

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\begin{bmatrix} s + R/L & -1/C \\ +1/L & s \end{bmatrix}}{s(s + R/L) + 1/(LC)} = \frac{\begin{bmatrix} s + R/L & -1/C \\ +1/L & s \end{bmatrix}}{s^2 + (R/L)s + 1/(LC)}$$

When $[s\mathbf{I} - \mathbf{A}]^{-1}$ has been obtained, then the system transfer function is easily obtained through $\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$. For the system in the example, when all outputs are measured, the system transfer matrix is:

Transfer matrix for example

$$\begin{equation*} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & -R \\ 0 & R \\ 0 & -1 \end{array}\right] \left[\frac{\left[\begin{array}{cc} s+R/L & -1/C \\ +1/L & s \end{array}\right]}{s^2 + (R/L)s +1/(LC)}\right] \left[\begin{array}{c} 1/C \\ 0 \end{array}\right]$$

- \left[

$$0$$

$$0$$

$$0$$

$$0$$

$$1$$

\right] \end{equation*}$$

$$\begin{equation*} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & -R \\ 0 & R \\ 0 & -1 \end{array}\right]$
$\left[\frac{\left[\begin{array}{c} (1/C)s+R/(LC) \\ +1/(LC) \end{array}\right]}{s^2 + (R/L)s +1/(LC)}\right]$$

- \left[

$$0$$
$$0$$
$$0$$
$$0$$
$$1$$

\right]\end{equation*}$$

$$= \frac{1}{s^2 + (R/L)s + 1/(LC)}\begin{bmatrix} (1/C)s + R/(LC) \\ 1/(LC) \\ (1/C)s \\ R/(LC) \\ -1/(LC) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(1/C)s+R/(LC)}{s^2+(R/L)s+1/(LC)} \\ \frac{1/(LC)}{s^2+(R/L)s+1/(LC)} \\ \frac{(1/C)s}{s^2+(R/L)s+1/(LC)} \\ \frac{R/(LC)}{s^2+(R/L)s+1/(LC)} \\ -\frac{1/(LC)}{s^2+(R/L)s+1/(LC)} + 1 \end{bmatrix}$$

$$= \frac{1}{s^2 + (R/L)s + 1/(LC)}\begin{bmatrix} (1/C)s + R/(LC) \\ 1/(LC) \\ (1/C)s \\ R/(LC) \\ -1/(LC) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(1/C)s+R/(LC)}{s^2+(R/L)s+1/(LC)} \\ \frac{1/(LC)}{s^2+(R/L)s+1/(LC)} \\ \frac{(1/C)s}{s^2+(R/L)s+1/(LC)} \\ \frac{R/(LC)}{s^2+(R/L)s+1/(LC)} \\ -\frac{1/(LC)}{s^2+(R/L)s+1/(LC)} + 1 \end{bmatrix}$$

In matrix form, when combined with the input and output transforms we have the situation illustrated below. Each transfer function relates the corresponding output transform to the input transform. For example

$$V_{31} = \frac{(1/C)s + R/(LC)}{s^2 + (R/L)s + 1/(LC)}\, U.$$

Transform Equations for Example

$$\mathbf{Y}(s) = \left[\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D} \right] \mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$$

$$\begin{bmatrix} V_{31}(s) \\ I_1(s) \\ V_{32}(s) \\ V_{21}(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{(1/C)s + R/(LC)}{s^2 + (R/L)s + 1/(LC)} \\ \frac{1/(LC)}{s^2 + (R/L)s + 1/(LC)} \\ \frac{(1/C)s}{s^2 + (R/L)s + 1/(LC)} \\ \frac{R/(LC)}{s^2 + (R/L)s + 1/(LC)} \\ -\frac{1/(LC)}{s^2 + (R/L)s + 1/(LC)} + 1 \end{bmatrix} \mathbf{U}(s).$$

Note that the denominator is the same for each transfer function, and that the order of the numerator is less than the denominator except for one case, for which

$$\begin{aligned} I_2 &= \left(-\frac{1/(LC)}{s^2 + (R/L)s + 1/(LC)} + 1 \right) U \\ &= \frac{-1/LC + s^2 + (R/L)s + 1/(LC)}{s^2 + (R/L)s + 1/(LC)} U \\ &= \frac{s^2 + (R/L)s}{s^2 + (R/L)s + 1/(LC)} U \end{aligned}$$

Replacing s by $\frac{d}{dt}$ gives the corresponding differential equations relating the dependant variable to the input.

Converting SS to TF in Matlab

Continuing example from Section 7.1 (../1/intro2ss):

In [43]:

```
clear all
format compact
```

Define some values for capacitance, inductance and resitance

In [44]:

```
Cap = 1; L = 1; R = 1;
```

Define state space model and label states inputs and outputs

In [45]:

```
A = [0 -1/Cap; 1/L -R/L];
B = [1/Cap; 0];
C = [1 0; 0 1; 1 -R; 0 R; 0 -1];
D = [0; 0; 0; 0; 1];
circ_ss = ss(A, B, C, D, ...
'statename',{'v31' 'i1'}, ...
'inputname', 'u', ...
'outputname', {'v31' 'i1' 'v32' 'v21' 'i2'});
```

Show model

In [46]:

```
circ_ss
```

```
circ_ss =

  A =
         v31    i1
    v31    0    -1
    i1     1    -1

  B =
         u
    v31  1
    i1   0

  C =
         v31    i1
    v31    1     0
    i1     0     1
    v32    1    -1
    v21    0     1
    i2     0    -1

  D =
         u
    v31  0
    i1   0
    v32  0
    v21  0
    i2   1

  Continuous-time state-space model.
```
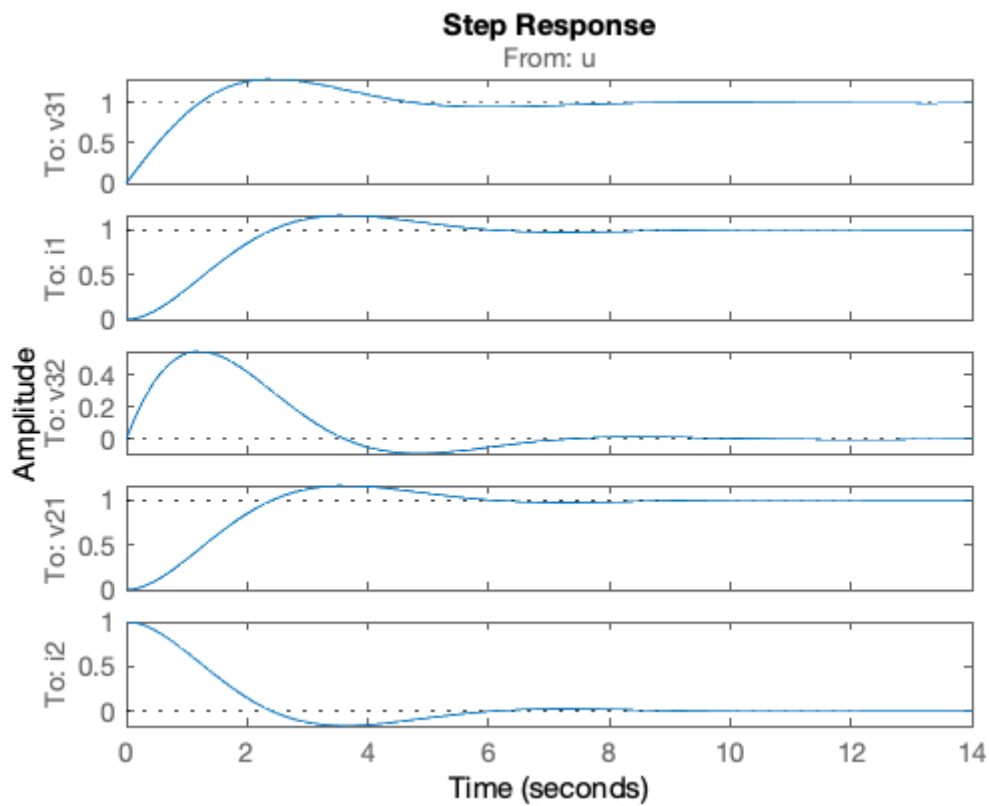
Plot a step response

In [47]:

```
step(circ_ss)
```


Convert to transfer functiom matrix

The function `tf(ss_model)` returns a vector of transfer functions.

```
In [48]:
```

```
circ_tf = tf(circ_ss)
```

```
circ_tf =

  From input "u" to output...
            s + 1
  v31:  -----------
        s^2 + s + 1


             1
  i1:   -----------
        s^2 + s + 1


             s
  v32:  -----------
        s^2 + s + 1


             1
  v21:  -----------
        s^2 + s + 1


          s^2 + s
  i2:   -----------
        s^2 + s + 1

Continuous-time transfer function.
```

Determine poles and zeros

```
In [49]:
```

```
circ_zpk=zpk(circ_ss)
```

```
circ_zpk =

  From input "u" to output...
             (s+1)
    v31:  -------------
          (s^2 + s + 1)


              1
    i1:   -------------
          (s^2 + s + 1)


               s
    v32:  -------------
          (s^2 + s + 1)


              1
    v21:  -------------
          (s^2 + s + 1)


           s (s+1)
    i2:   -------------
          (s^2 + s + 1)


Continuous-time zero/pole/gain model.
```

The state transition matrix

Calculated using the symbolic math tools provided by MATLAB See `help symbolic`

```
In [50]:
```

```
syms phi t s
phi = inv(s*eye(2) - A)
```

```
phi =
[ (s + 1)/(s^2 + s + 1), -1/(s^2 + s + 1)]
[       1/(s^2 + s + 1),  s/(s^2 + s + 1)]
```

The state transfer matrix

```
In [54]:
```

```
G = C*phi*B + D
```

```
G =
                      (s + 1)/(s^2 + s + 1)
                            1/(s^2 + s + 1)
 (s + 1)/(s^2 + s + 1) - 1/(s^2 + s + 1)
                            1/(s^2 + s + 1)
                      1 - 1/(s^2 + s + 1)
```

In [55]:

```
G = simplify(G)
```

```
G =
  (s + 1)/(s^2 + s + 1)
        1/(s^2 + s + 1)
        s/(s^2 + s + 1)
        1/(s^2 + s + 1)
    1 - 1/(s^2 + s + 1)
```

In [52]:

```
pretty(G)
```

```
/          s + 1        \
|       ----------      |
|         2             |
|       s   + s + 1     |
|                       |
|            1          |
|       ----------      |
|         2             |
|       s   + s + 1     |
|                       |
|            s          |
|       ----------      |
|         2             |
|       s   + s + 1     |
|                       |
|            1          |
|       ----------      |
|         2             |
|       s   + s + 1     |
|                       |
|               1       |
|   1 - ----------      |
|           2           |
\          s   + s + 1 /
```

A executable script version of this example is available as ssmodels.mlx (matlab/ssmodels.mlx).

Some Important Properties

System poles

Clearly the denominator of the transfer function is generated by the matrix inverse which produces the term:
$$\det[s\mathbf{I} - \mathbf{A}]$$

This evaluates to the denominator polynomial and the poles of the system are the roots of the system's characteristic equation:
$$\det[s\mathbf{I} - \mathbf{A}] = 0.$$

The system poles are solutions to the system's characteristic equation

$$\det[s\mathbf{I} - \mathbf{A}] = 0.$$

System zeros

What is the corresponding numerator polynomial of the transfer function, whose roots give the zeros of the system?

The zeros are those values of s for which the output is zero when the input and states are not zero.

Thus:

$$(s\mathbf{I} - \mathbf{A})\mathbf{X} - \mathbf{B}U = \mathbf{0}$$
$$\mathbf{C}\mathbf{X} + dU = Y = 0$$

In matrix form:

$$\begin{bmatrix} s\mathbf{I} - \mathbf{A} & \vdots & \mathbf{B} \\ \cdots & \cdots & \cdots \\ \mathbf{C} & \vdots & d \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \cdots \\ U \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \cdots \\ 0 \end{bmatrix}$$

Zeros are those values of s for which the system output is zero when the input and states are not zero

$$(s\mathbf{I} - \mathbf{A})\mathbf{X} - \mathbf{B}U = \quad \mathbf{0}$$
$$\mathbf{C}\mathbf{X} + dU = Y = \quad 0$$

In matrix form:

$$\begin{bmatrix} s\mathbf{I} - \mathbf{A} & \vdots & \mathbf{B} \\ \cdots & \cdots & \cdots \\ \mathbf{C} & \vdots & d \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \cdots \\ U \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \cdots \\ 0 \end{bmatrix}$$

The only way this can have non-zero solutions in $\mathbf{X}$ and U is if:

$$\det \begin{bmatrix} s\mathbf{I} - \mathbf{A} & \vdots & \mathbf{B} \\ \cdots & \cdots & \cdots \\ \mathbf{C} & \vdots & d \end{bmatrix} = 0$$

This is another polynomial in s whose roots give the system zeros and therefore corresponds to the numerator polynomial of the TF.

Given this result, an alternative expression for the TF is:

$$\frac{Y(s)}{U(s)} = \frac{\det \begin{bmatrix} s\mathbf{I} - \mathbf{A} & \vdots & \mathbf{B} \\ \cdots & \cdots & \cdots \\ \mathbf{C} & \vdots & d \end{bmatrix}}{\det[s\mathbf{I} - \mathbf{A}]}$$

Time Responses from Transfer Function Matrices

In the <u>next section (../3/tr4ss)</u> we will consider how we can use the transfer function model to compute time responses from state-space models.

Footnote

1. $\epsilon(t)$ is the unit step function $\epsilon(t) = 0$ for $t < 0$; $\epsilon(t) = 1$ for $t \geq 0$.