6.2. Digital System Models and System Response

6.2.1. Digital System Models

The equivalent of the differential equation model for continuous systems is the difference equation for digital systems.

Replacing the differential operator $\frac{d}{dt}$ by the advance operator $\triangle$ gives the general form of the difference equation as

$$\triangle^n y + a_1 \triangle^{n-1} y + \ldots + a_n y = b_0 \triangle^n u + b_1 \triangle^{n-1} u + \ldots + b_n u.$$

The equivalent of the differential equation model for continuous systems is the difference equation for digital systems.

6.2.1.1. Difference equation

Now, given the definition of the advance operator derived in a previous lecture

$$\triangle^n v_k = v_{k+n}$$

we can re-write equation (1) as the *difference equation*

$$y_{k+n} + a_1 y_{k+n-1} + \ldots + a_n y_k = b_0 u_{k+n} + b_1 u_{k+n-1} + \ldots + b_n u_k.$$

6.2.1.2. Difference equation in terms of the delay operator

Unlike the differential equation, however, which is hardly ever expressed in an integral form, the difference equation is more usually expressed in terms of the delay operator ∇.

Applying the operator ∇^n to equation (1) gives

$$\begin{aligned} y + a_1 \nabla y + \ldots + a_n \nabla^n y = &\; b_0 u + b_1 \nabla u + \ldots + b_n \nabla^n u \\ y_k + a_1 y_{k-1} + \ldots + a_n y_{k-n} = &\; b_0 u_k + b_1 u_{k-1} + \ldots + b_n u_{k-n}. \end{aligned}$$

z-transform of difference equation

Applying the z transform directly to the difference equation with the delay operator (3) gives

$$\begin{aligned} Y + a_1 z^{-1} Y + \ldots + a_n z^{-n} Y = &\; b_0 U + b_1 z^{-1} U + \ldots + b_n z^{-n} U \\ (1 + a_1 z^{-1} + \ldots + a_n z^{-n}) Y = &\; (b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}) U \end{aligned}$$

6.2.1.3. z Transfer Function

Given that

$$(1 + a_1 z^{-1} + \ldots + a_n z^{-n}) Y(z) = (b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}) U(z)$$

The z transfer function is

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

A digital system in this general form is known as a "*pole-zero, infinite impulse response, recursive auto-regressive moving average digital filter(!)*"

6.2.1.4. z Transfer Function (2)

If $b_1 = b_2 = \cdots = b_n = 0$ then the transfer function is

$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

A digital system in this form is known as an "*all pole, infinite impulse response, recursive auto-regressive digital filter.*"

6.2.1.5. z Transfer Function (3)

When $a_1 = a_2 = \cdots = a_n = 0$ then the transfer function is

$$H(z) = b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}$$

A digital system in this form is known as an "*all zero, finite impulse response, non recursive moving average digital filter.*"

6.2.1.6. Other forms of digital transfer function

The transfer function can also be expressed in the zero-pole-gain form

$$H(z) = \frac{k(z - z_1)(z - z_2) \cdots (z - z_n)}{(z - p_1)(z - p_2) \cdots (z - p_n)}$$

and the partial fraction form

$$H(z) = \frac{r_1}{z - p_1} + \frac{r_2}{z - p_2} \cdots \frac{r_n}{z - p_n}$$

6.2.1.7. Canonical Forms

With the transfer function written as

$$H(z) = \frac{b_0 z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_n}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + b_n}$$

there is a direct analogy with the general form of the continuous system transfer function with s instead of z.

This was implemented with the physically realistic integral operator $\int dt$ for which the digital equivalent is the delay operator ∇.

6.2.2. End of Pre-Class Presentation

This concludes the pre-class presentation.

In the class we will look at system response and compute the impulse and step responses of an example system.

6.2.3. Digital System Response

As in the case of a continuous system, the response of a digital signal comprises the sum of a free response and a forced response. The free response is dependent on the initial conditions of a digital system states, and as these are taken as zero here the free response is also zero and will not be considered further.

6.2.3.1. Digital System Response

The response of a digital system with transfer function $H(z)$ to a digital input signal u is the digital output signal y given in transform form as

$$Y(z) = H(z)U(z)$$

The inverse transform needed to determine the digital system response is obtained using the inverse z transform methods, e.g. polynomial division and partial fraction expansion, discussed in a previous lecture.

Taking the inverse transform gives the digital system response as

$$y_k = \square^{-1}\, Y(z) = \square^{-1}\, \{H(z)U(z)\}$$

6.2.4. Response to Singularity Signals

The elemental singularity signals in a digital system response include the digital impulse signal and the digital step input.

6.2.4.1. Impulse response

6.2.4.1.1. Impulse signal

The digital impulse signal is given by

$$v = \delta = \{\delta_k\}$$

where $\delta_0 = 1$ when $k = 0$, and $\delta_k = 0$ otherwise.

Therefore the sequence for the impulse is simply

$$\delta_k = 1,\ 0,\ 0,\ 0,\ \ldots.$$

The transform of the digital impulse signal is

$$V = \Delta = \sum_{k=0}^{\infty} \delta_k z^{-k} = 1$$

6.2.4.1.2. Example 1: Impulse Response

Calculate the impulse response of the digital system with transfer function

$$H(z) = \frac{4z^2 - 16}{z^2 - 0.25}$$

Consider the system

$$H(z) = \frac{4z^2 - 16}{z^2 - 0.25}$$

The impulse response will be

$$Y(z) = H(z) \times 1 = \frac{4z^2 - 16}{z^2 - 0.25}$$

We shall determine this response using the partial fraction expansion.

$$
\begin{aligned}
Y(z) &= \frac{4 - 16z^{-2}}{1 - 0.25z^{-2}} \\
&= \frac{4(4 - 16z^{-2})}{4 - z^{-2}} \\
&= \frac{4(2 - 4z^{-1})(2 + 4z^{-1})}{(2 - z^{-1})(2 + z^{-1})}
\end{aligned}
$$

Assuming a partial fraction expansion of the form

$$
Y(z) = \frac{A}{2 - z^{-1}} + \frac{B}{2 + z^{-1}} + C
$$

we have

$$
\begin{aligned}
\frac{4(2 - 4z^{-1})(2 + 4z^{-1})}{(2 - z^{-1})(2 + z^{-1})} &= \frac{A(2 + z^{-1}) + B(2 - z^{-1}) + C(2 - z^{-1})(2 + z^{-1})}{(2 - z^{-1})(2 + z^{-1})} \\
16 - 64z^{-2} &= 2A + Az^{-1} + 2B - Bz^{-1} + 4C - Cz^{-2}
\end{aligned}
$$

Gathering terms and equating coefficients

$$
\begin{aligned}
16 &= 2A + 2B + 4C \\
0 &= A - B \\
-64 &= -C
\end{aligned}
$$

Hence

$$
\begin{aligned}
C &= 64 \\
A &= B \\
16 &= 4A + 256 \\
A &= B = -60
\end{aligned}
$$

Thus

$$
\begin{aligned}
Y(z) &= 64 - \frac{60}{2 - z^{-1}} - \frac{60}{2 + z^{-1}} \\
&= 64 - \frac{30}{1 - 1/2z^{-1}} - \frac{30}{1 + 1/2z^{-1}} \\
y_k &= \left\{ 64\delta_k - 30\left(\frac{1}{2}\right)^k - 30\left(-\frac{1}{2}\right)^k \right\} \\
&= \{4,\ 0,\ -15,\ 0,\ -3.75,\ 0,\ -0.9375,\ \ldots\}
\end{aligned}
$$

6.2.4.2. Step response

6.2.4.2.1. Step signal

The digital step signal is

$$
v = \epsilon = \{\epsilon_k\}
$$

where $\epsilon_k = 1$ when $k \geq 0$, and $\epsilon_k = 0$ otherwise.

Therefore the sequence for the step is simply

$$
\epsilon_k = 1,\ 1,\ 1,\ 1,\ \ldots .
$$

6.2.4.2.2. z-transform of step signal

The transform of the digital step signal is

$$
\begin{aligned}
V = E &= \sum_{k=0}^{\infty} \epsilon_k z^{-k} \\
&= \sum_{k=0}^{\infty} z^{-k} \\
&= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots z^{-n} + \cdots \\
&= \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}.
\end{aligned}
$$

6.2.4.2.3. Example 2: Step Response

Calculate the step response of the digital system with transfer function

$$
H(z) = \frac{4z^2 - 16}{z^2 - 0.25}
$$

The step response of the example system is

$$
Y(z) = H(z) \times \frac{z}{z - 1} = \frac{z(4z^2 - 16)}{(z - 1)(z^2 - 0.25)}
$$

We shall determine this response using the partial fraction expansion.

$$
\begin{aligned}
Y(z) &= \frac{4z^3 - 16z}{z^3 - z^2 - 0.25z + 0.25} \\
&= \frac{4 - 16z^{-2}}{1 - z^{-1} - 0.25z^{-2} + 0.25z^{-3}}
\end{aligned}
$$

Earlier we showed that the result of the partial fraction expansion was

$$\frac{30}{1 - 1/2z^{-1}} - \frac{10}{1 + 1/2z^{-1}} - \frac{16}{1 - z^{-1}}$$

and the corresponding sequence is

$$y_k = \left\{30(1/2)^k - 10(-1/2)^k - 16\epsilon_k\right\}.$$

6.2.5. Computing Digital System Responses with MATLAB

We will run this part of the presentation in class. There is an executable version as a MATLAB Live Script available as digiresp.mlx.

6.2.5.1. Control System Toolbox Help

What functions do we have in the controls system toolbox?

```
imatlab_export_fig('print-svg')  % Static svg figures.
```

```
doc control
```

6.2.5.1.1. Modelling

Modeling looks likely

```
doc ctrlmodels
```

6.2.5.1.2. Transfer function

From this, the transfer function looks likely

```
help tf
```

```
TF  Construct transfer function or convert to transfer function.

  Construction:
    SYS = TF(NUM,DEN) creates a continuous-time transfer function SYS with
    numerator NUM and denominator DEN. SYS is an object of type TF when
    NUM,DEN are numeric arrays, of type GENSS when NUM,DEN depend on tunable
    parameters (see REALP and GENMAT), and of type USS when NUM,DEN are
    uncertain (requires Robust Control Toolbox).

    SYS = TF(NUM,DEN,TS) creates a discrete-time transfer function with
    sample time TS (set TS=-1 if the sample time is undetermined).

    S = TF('s') specifies the transfer function H(s) = s (Laplace variable).
    Z = TF('z',TS) specifies H(z) = z with sample time TS.
    You can then specify transfer functions directly as expressions in S
    or Z, for example,
       s = tf('s');  H = exp(-s)*(s+1)/(s^2+3*s+1)

    SYS = TF creates an empty TF object.
    SYS = TF(M) specifies a static gain matrix M.

    You can set additional model properties by using name/value pairs.
    For example,
       sys = tf(1,[1 2 5],0.1,'Variable','q','IODelay',3)
    also sets the variable and transport delay. Type "properties(tf)"
    for a complete list of model properties, and type
       help tf.<PropertyName>
    for help on a particular property. For example, "help tf.Variable"
    provides information about the "Variable" property.

    By default, transfer functions are displayed as functions of 's' or 'z'.
    Alternatively, you can use the variable 'p' in continuous time and the
    variables 'z^-1', 'q', or 'q^-1' in discrete time by modifying the
    "Variable" property.

  Data format:
    For SISO models, NUM and DEN are row vectors listing the numerator
    and denominator coefficients in descending powers of s,p,z,q or in
    ascending powers of z^-1 (DSP convention). For example,
       sys = tf([1 2],[1 0 10])
    specifies the transfer function (s+2)/(s^2+10) while
       sys = tf([1 2],[1 5 10],0.1,'Variable','z^-1')
    specifies (1 + 2 z^-1)/(1 + 5 z^-1 + 10 z^-2).

    For MIMO models with NY outputs and NU inputs, NUM and DEN are
    NY-by-NU cell arrays of row vectors where NUM{i,j} and DEN{i,j}
    specify the transfer function from input j to output i. For example,
       H = tf( {-5 ; [1 -5 6]} , {[1 -1] ; [1 1 0]})
    specifies the two-output, one-input transfer function
       [     -5 /(s-1)      ]
       [ (s^2-5s+6)/(s^2+s) ]

  Arrays of transfer functions:
    You can create arrays of transfer functions by using ND cell arrays
    for NUM and DEN above. For example, if NUM and DEN are cell arrays
    of size [NY NU 3 4], then
       SYS = TF(NUM,DEN)
    creates the 3-by-4 array of transfer functions
       SYS(:,:,k,m) = TF(NUM(:,:,k,m),DEN(:,:,k,m)),  k=1:3,  m=1:4.
    Each of these transfer functions has NY outputs and NU inputs.

    To pre-allocate an array of zero transfer functions with NY outputs
    and NU inputs, use the syntax
       SYS = TF(ZEROS([NY NU k1 k2...])) .

  Conversion:
    SYS = TF(SYS) converts any dynamic system SYS to the transfer function
    representation. The resulting SYS is always of class TF.

  See also TF/EXP, FILT, TFDATA, ZPK, SS, FRD, GENSS, USS, DYNAMICSYSTEM.

   Documentation for tf
      doc tf

   Other functions named tf

      DynamicSystem/tf    idParametric/tf    mpc/tf    StaticModel/tf
```

6.2.5.1.3. Digital transfer function block

It seems that that the third argument is sampling period. Set this to -1 for “unspecified”.

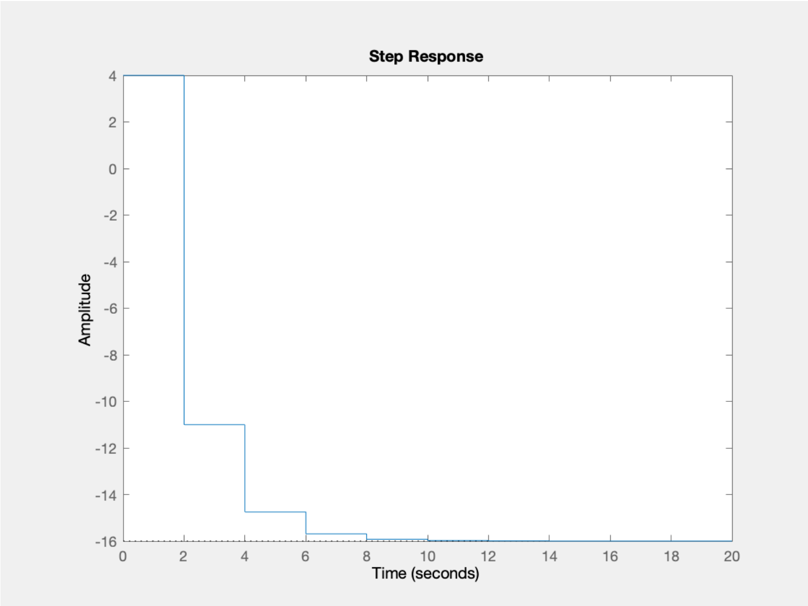
```
H = tf([4, 0, -16],[1, 0, -0.25],-1)
```

```
H =


  4 z^2 - 16

  ----------

  z^2 - 0.25


Sample time: unspecified

Discrete-time transfer function.
```

6.2.5.2. Step Response

Now do a step response

```
step(H)
```



What about the sequence values?

```
ys = step(H)
```

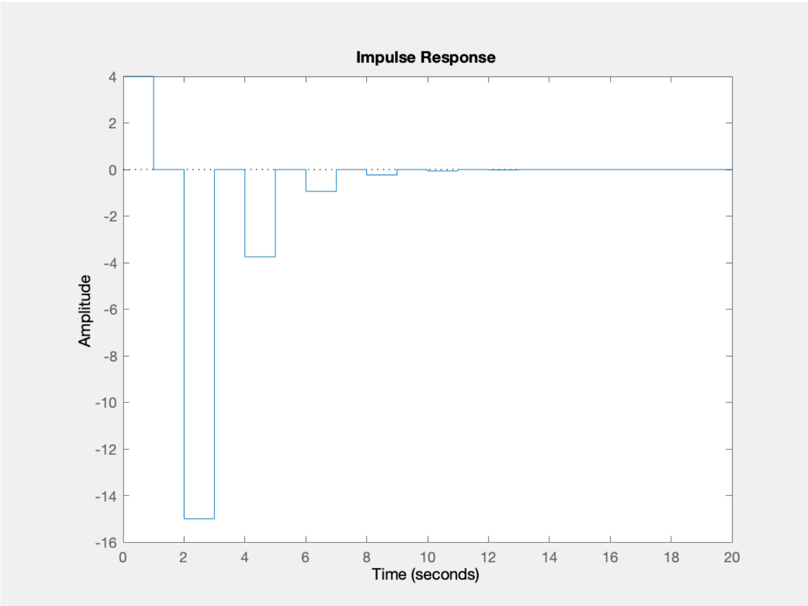
```
ys =
    4.0000
    4.0000
  -11.0000
  -11.0000
  -14.7500
  -14.7500
  -15.6875
  -15.6875
  -15.9219
  -15.9219
  -15.9805
  -15.9805
  -15.9951
  -15.9951
  -15.9988
  -15.9988
  -15.9997
  -15.9997
  -15.9999
  -15.9999
  -16.0000
```

Does the sequence match the theory?

6.2.5.3. Impulse Response

How about impulse response?

```
impulse(H)
```



The plot isn't quite right – it's a "hold-equivalent" result. We don't know values between sampling instants.

Sequence?

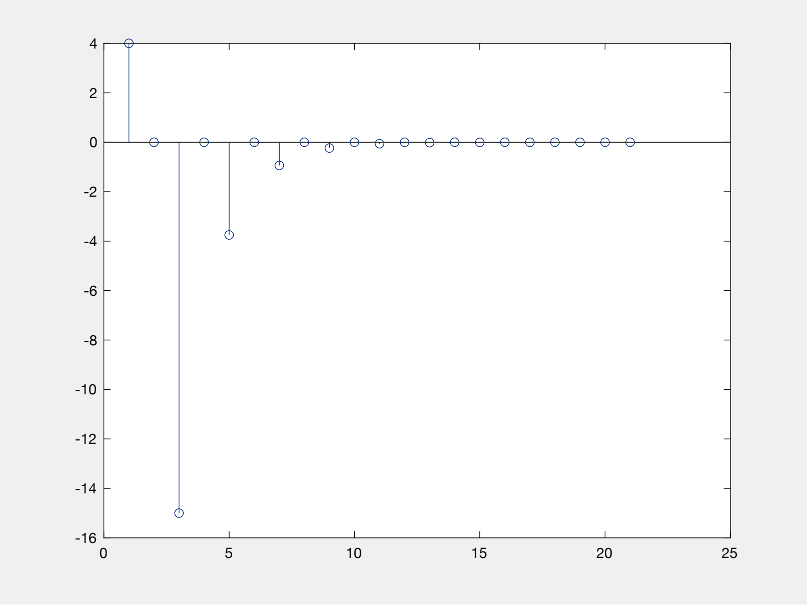
```
yi = impulse(H)
```

```
yi =
    4.0000
         0
  -15.0000
         0
   -3.7500
         0
   -0.9375
         0
   -0.2344
         0
   -0.0586
         0
   -0.0146
         0
   -0.0037
         0
   -0.0009
         0
   -0.0002
         0
   -0.0001
```

6.2.5.3.1. Stem plot

To plot this properly, we use a stem plot

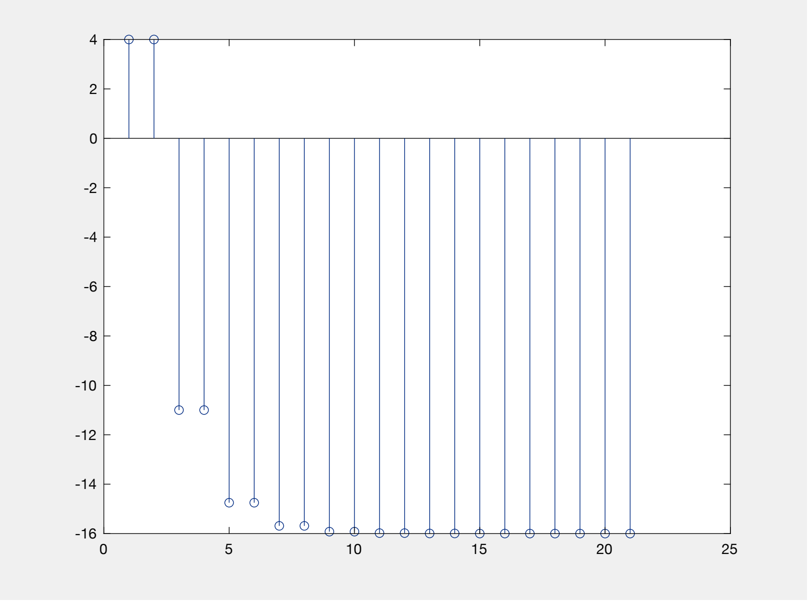
```
stem(yi)
```



6.2.5.3.2. Step response as stem plot

Should do same for step response too

```
stem(ys)
```



6.2.5.4. Partial fractions

Can we use partial fractions for inverse z transforms?

```
help residue
```

```
RESIDUE Partial-fraction expansion (residues).
   [R,P,K] = RESIDUE(B,A) finds the residues, poles and direct term of
   a partial fraction expansion of the ratio of two polynomials B(s)/A(s).
   If there are no multiple roots,
      B(s)       R(1)       R(2)             R(n)
      ----  =  -------- + -------- + ... + -------- + K(s)
      A(s)     s - P(1)   s - P(2)         s - P(n)
   Vectors B and A specify the coefficients of the numerator and
   denominator polynomials in descending powers of s.  The residues
   are returned in the column vector R, the pole locations in column
   vector P, and the direct terms in row vector K.  The number of
   poles is n = length(A)-1 = length(R) = length(P). The direct term
   coefficient vector is empty if length(B) < length(A), otherwise
   length(K) = length(B)-length(A)+1.

   If P(j) = ... = P(j+m-1) is a pole of multplicity m, then the
   expansion includes terms of the form
                R(j)        R(j+1)                R(j+m-1)
              -------- + ------------   + ... + ------------
              s - P(j)   (s - P(j))^2           (s - P(j))^m

   [B,A] = RESIDUE(R,P,K), with 3 input arguments and 2 output arguments,
   converts the partial fraction expansion back to the polynomials with
   coefficients in B and A.

   Warning: Numerically, the partial fraction expansion of a ratio of
   polynomials represents an ill-posed problem.  If the denominator
   polynomial, A(s), is near a polynomial with multiple roots, then
   small changes in the data, including roundoff errors, can make
   arbitrarily large changes in the resulting poles and residues.
   Problem formulations making use of state-space or zero-pole
   representations are preferable.

   Class support for inputs B,A,R:
      float: double, single

   See also POLY, ROOTS, DECONV.

   Documentation for residue
      doc residue
```

6.2.5.4.1. PFE for for inverse z-transform of impulse response

```
[r,p,k] = residue([4,0,-16],[1,0,-0.25])
```

```
r =
   -15
    15
```

```
p =
    0.5000
   -0.5000
```

```
k =
     4
```

6.2.5.4.2. PFE for inverse z-transform for step response

```
[r,p,k] = residue(conv([1, 0],[4,0,-16]),conv([1, -1],[1,0,-0.25]))
```

```
r =
  -16.0000
   15.0000
    5.0000
```

```
p =
    1.0000
    0.5000
   -0.5000
```

```
k =
     4
```

Do these results match the theory?

6.2.5.4.3. Another approach

Use LTI block to define the TF then extract num/den for PFE

```
U = tf([1, 0], [1, -1], -1)
Y = series(U, H)
```

```
U =
```

```
    z
```

```
  -----
```

```
  z - 1
```

```
Sample time: unspecified
```

```
Discrete-time transfer function.
```

```
Y =
```

```

```

```
        4 z^3 - 16 z
```

```
  -------------------------
```

```
  z^3 - z^2 - 0.25 z + 0.25
```

```

```

```
Sample time: unspecified
```

```
Discrete-time transfer function.
```

extract numerator & denominator

```
[num,den] = tfdata(Y,'v')
```

```
num =
     4     0   -16     0
```

```
den =
    1.0000   -1.0000   -0.2500    0.2500
```

get pfe

```
[r,p,k]=residue(num,den)
```

```
r =
  -16.0000
   15.0000
    5.0000
```

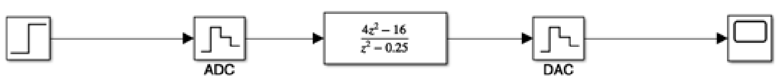
```
p =
    1.0000
    0.5000
   -0.5000
```

```
k =
     4
```

6.2.5.5. Simulink Model

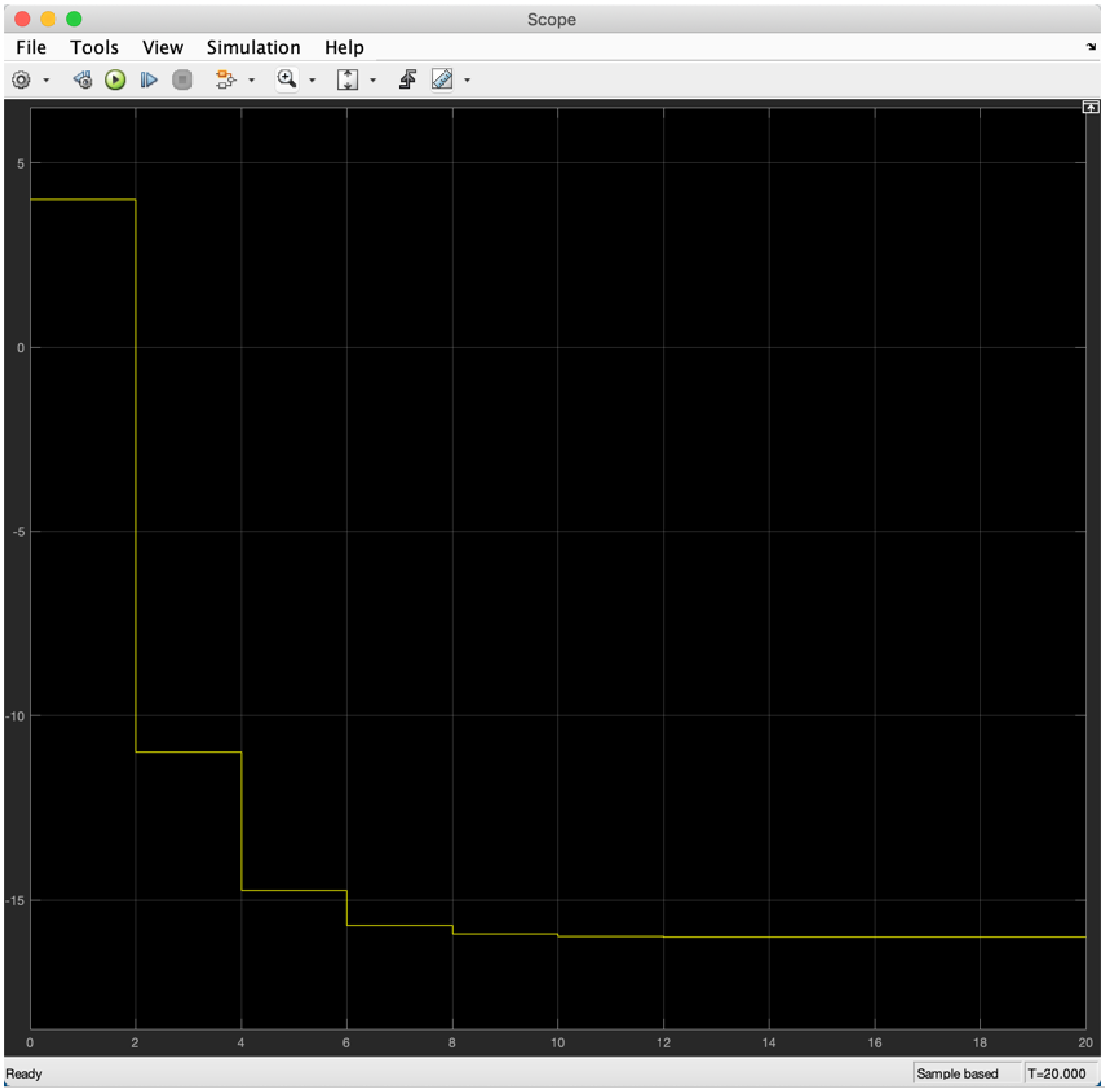
A simulink model of the digital implementation is here digiresp.slx.

```
cd matlab
digiresp_model
```



Note that in Simulink, you use the ZOH both as an ADC and DAC operator. The z-transfer function block is used for the digital transfer function.

6.2.5.5.1. Response



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