4.2.4. Analytical Design of a PID Compensator

This section presents an analytical procedure for PID compensator design. It is based on Section 7.11 of Phillips and Harbor *Feedback_ Control Systems*, Prentice Hall, 1988[1].

The compensator transfer function is assumed to be

$$D(s) = \frac{K_D s^2 + K_{\mathrm{prop}}\ s + K_I}{s}$$

where K_{prop} is the proportional gain, K_D is the derivative gain and K_I is the integral gain. In this procedure we choose the PID gain parameters such that, given a desired location for one of the closed-loop poles s_1, the equation

$$D(s)G(s)H(s)|_{\ s=s_1} = -1$$

is satisfied; that is we are designing a compensator that places a root of the closed-loop characteristic equation at $s = s_1$.

The design proceeds as follows. First we express the desired closed loop pole position

$$s_1 = |s_1|e^{j\psi}$$

and

$$G(s_1)H(s_1) = |G(s_1)H(s_1)|\ e^{j\psi}$$

Then the design equations (derived in Appendix B of Phillips and Harbor, 1988) are

$$K_{\mathrm{prop}} = -\frac{\sin(\beta - \psi)}{|G(s_1)H(s_1)|\ \sin\beta} - \frac{2K_I\ \cos\beta}{|s_1|}$$

$$K_{\mathrm{prop}} = -\frac{\sin\psi}{|s_1|\ |G(s_1)H(s_1)|\ \sin\beta} - \frac{K_I}{|s_1|^2}$$

Since there are three unknowns and only two relationships that must be satisfied, one of the gains may be chosen to satisfy a different design specification, such as choosing K_I to achieve a certain steady-state response. These equations can also be used for PI and P+D controllers by setting the appropriate gain to zero. We now illustrate the design procedure with an example.

4.2.4.1. Example

Definitions (change these to change design)

The plant transfer function is

$$G(s) = \frac{1}{(s+1)(5s+1)}$$

```
imatlab_export_fig('print-svg')  % Static svg figures.
```

```
G = tf(1,conv([1 1],[5 1]));
```

The feedback transfer function is $H(s) = 1$:

```
H=tf(1,1);
```

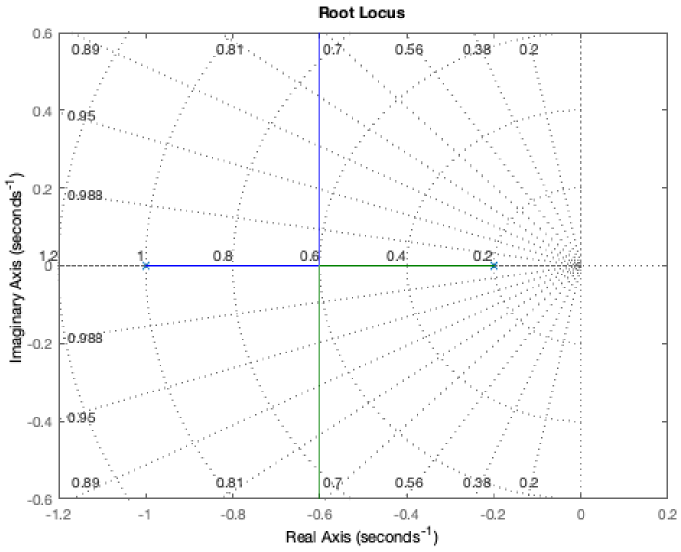
So $G(s)H(s)$ is:

```
GH=G*H
```

```
GH =


          1

  ---------------

  5 s^2 + 6 s + 1


Continuous-time transfer function.
```

The root locus of the uncompensated system is:

```
clf, sgrid(1/sqrt(2),0.25:0.25:2), hold on, rlocus(GH),hold off
```



From the root locus diagram, it is clear that for ideal damping the natural frequency of the closed-loop poles would be about 0.9 rad/s with a settling time of:

$T_s = \frac{4.6}{\zeta\omega_n} = \frac{4.6}{5/8} = 7.36$ s

Suppose we wish to half the settling time then we need to double the natural frequency to $\omega_n = 2$rad/s.

That is:

```
zeta = 1/sqrt(2); wn=2;
s1 = -zeta*wn+j*wn*sqrt(1-zeta^2)
```

```
s1 =
  -1.4142 + 1.4142i
```

The steady state error of the uncompensated type 0 system is:

$$\frac{1}{1+G(s)H(s)|_{s=0}} = \frac{1}{1+\frac{1}{(5s+1)(s+1)}\Big|_{s=0}} = \frac{1}{2}$$

For the compensated system, which is type 1:

$$K_v = sD(s)G(s)H(s)|_{s=0} = \frac{s\left(K_D\, s^s + K_{\text{prod}}\, s + K_I\right)}{s}\Bigg|_{s=0} = K_I$$

So if we want a steady-state _velocity _error of 20% we need

```
Ki=20;
```

4.2.4.2. Calculations

Having set up your problem, you shouldn?t need to change these commands

Polar form of s_1

```
m_s1=abs(s1),  p_s1 = angle(s1)*180/pi % degrees
```

```
m_s1 =
     2
```

```
p_s1 =
   135
```

Transfer function evaluated at s_1 is $G(s_1)H(s_1)$in polar form:

```
[numGH,denGH] = tfdata(GH,'v');
GHs1=polyval(numGH,s1)/polyval(denGH,s1)
```

```
GHs1 =
  -0.0397 + 0.0610i
```

Magnitude:

```
mGHs1=abs(GHs1)
```

```
mGHs1 =
    0.0728
```

Phase[2]:

```
pGHs1=-angle(GHs1)*180/pi - 90 % degrees
```

```
pGHs1 =
 -213.0264
```

Hence:

```
beta = p_s1*pi/180; psi = pGHs1*pi/180;  % radians
```

From (5) and (6)

```
Kprop = (-sin(beta+psi))/(mGHs1*sin(beta)) - (2*Ki*cos(beta)/m_s1)
```

```
Kprop =
   33.1421
```

```
Kd = (sin(psi)/(m_s1*mGHs1*sin(beta))) + Ki/(m_s1^2)
```

```
Kd =
   10.2929
```

Compensator is therefore given by

```
D = tf([Kd, Kprop, Ki],[1, 0])
```

```
D =


  10.29 s^2 + 33.14 s + 20

  ------------------------

             s


Continuous-time transfer function.
```

4.2.4.3. Evaluation of Design

Open loop transfer function:

```
Go=D*GH
```

```
Go =


  10.29 s^2 + 33.14 s + 20

  ------------------------

     5 s^3 + 6 s^2 + s


Continuous-time transfer function.
```

4.2.4.3.1. Root locus:

rlocus(Go)

4.2.4.3.2. Closed-loop transfer function:

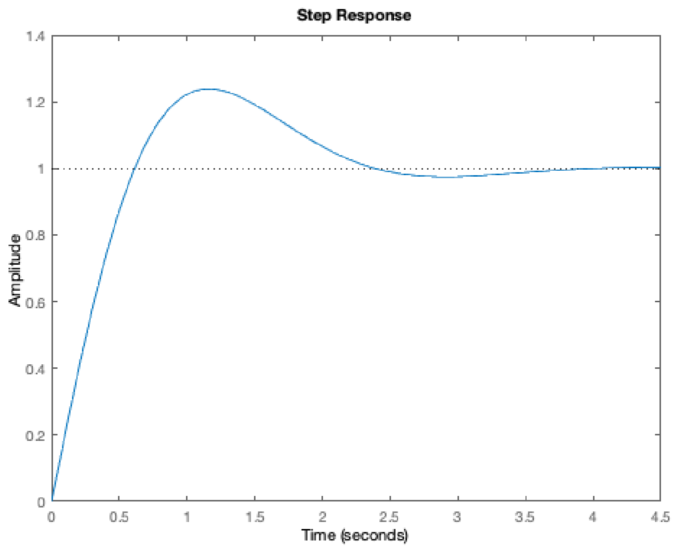
```
DG = D*G
Gc = feedback(DG,H)
```

```
DG =


  10.29 s^2 + 33.14 s + 20

  ------------------------

     5 s^3 + 6 s^2 + s


Continuous-time transfer function.

Gc =


      10.29 s^2 + 33.14 s + 20
```

```
  -------------------------------

  5 s^3 + 16.29 s^2 + 34.14 s + 20


Continuous-time transfer function.
```

4.2.4.3.3. Step response:

```
step(Gc)
```



4.2.4.4. Footnotes

[1] The proofs of the formulae given are derived in Appendix B of this text.

[2] You must be careful with angles when using packages like MATLAB, and indeed pocket calculators. It is nearly always beneficial to have a sketch so that you can correct the results. In this case a correction of -90° was needed.

4.2.4.5. Resources

An executable version of this document is available to download as the MATLAB Live Script file analrloc.mlx.

By Dr Chris P. Jobling

This page was created by Dr Chris P. Jobling for Swansea University.