Time Response for State Space Models

In this section we shall determine the time response of a system represented by a state-space model. We shall take as a starting point the transformed state space model developed in the last section (../2/tf4ss).

It is possible to derive the time response directly in the time domain from the state-equations. I believe however, that the following development will be easier to understand. I advise you to consult any text book for the time domain development.

In the last section (../2/tf4ss), we showed that the state space model was

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

and that the transformed equation was

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

In this section we will show how the time response of the state space model may be derived from the transformed model.

Zero Input Response

We first consider the response of the system to its initial conditions and zero input. This is the *zero input response* or homogeneous response of the system.

With the input transform $U(s) = 0$ the state equation becomes

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0).$$

We define

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1}$$

so that

$$\mathbf{X}(s) = \Phi(s)\mathbf{x}(0)$$

is the transform of the zero input state response.

In the time domain, the corresponding state response is given by the inverse Laplace transform of equation (1)

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left\{\Phi(s)\mathbf{x}(0)\right\},$$

which, since the initial condition vector $\mathbf{x}(0)$ is a vector of constants becomes:

$$\begin{aligned}\mathbf{x}(t) &= \mathcal{L}^{-1}\left\{\Phi(s)\right\}\mathbf{x}(0)\\ &= \phi(t)\mathbf{x}(0).\end{aligned}$$

Resolvent matrix

The *resolvent matrix* is defined as

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\mathrm{adj}\,[s\mathbf{I} - \mathbf{A}]}{|s\mathbf{I} - \mathbf{A}|}.$$

The resolvent matrix is a matrix of rational polynomials in the Laplace transform variable s.

The state transition matrix is

$$\phi(t) = \mathcal{L}^{-1}\left\{\frac{\mathrm{adj}\,[s\mathbf{I} - \mathbf{A}]}{|s\mathbf{I} - \mathbf{A}|}\right\}.$$

What is the form of this function?

State Transition Matrix

The vector of time functions $\phi(t)$ is known as the *state transition matrix*. It defines the motion (or *trajectory*) of the state variables through the n-dimensional *state space* from a given set of initial states.

Characteristic Polynomial

The adjoint matrix $\mathrm{adj}\,[s\mathbf{I} - \mathbf{A}]$ is a matrix of polynomials each of which is of order $n-1$ or less.

The determinant

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1n}\\ -a_{21} & s - a_{22} & \cdots & -a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ -a_{n1} & -a_{2n} & \cdots & s - a_{nn}\end{vmatrix}$$

is called the *characteristic polynomial*. It is always a polynomial of degree n.

Therefore each element of the resolvent matrix $\Phi(s)$ is of the form

$$\frac{b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}.$$

System Characteristic (or Eigen) Values

The coefficients b_i of each element of the resolvent matrix $\Phi(s)$ depend on the form of the system. The characteristic polynomial $|s\mathbf{I} - \mathbf{A}|$ is the common denominator of all elements of $\Phi(s)$.

It may be factorized:

$$\begin{aligned}|s\mathbf{I} - \mathbf{A}| &= s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \\ &= (s - p_1)(s - p_2)\cdots(s - p_n)\end{aligned}$$

The values p_i are known as the *characteristic* (or *eigen*) values of the state matrix $\mathbf{A}$. The characteristic values are the *poles* of the system!

Zero-input response

If we make a partial fraction expansion of the characteristic equation, then for the i-jth element of the resolvent matrix we have

$$\Phi_{ij}(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \cdots + \frac{r_n}{s - p_n}$$

(where the coefficient r_i is the so called *residue* of the partial fraction expansion determined at the value $s = p_i$).

Inverse Laplace transforming this equation we finally obtain the *state transition* function:

$$\phi_{ij}(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \cdots + r_n e^{p_n t}.$$

System 'modes'

The state transition matrix $\phi(t)$ is an $n \times n$ matrix, each element of which is a linear combination of the n "mode functions" (or simply *modes*)

$$e^{p_1 t},\ e^{p_2 t},\ \ldots,\ e^{p_n t}.$$

Each mode is constructed from the eigenvalues ($p_1,\ p_2,\ \ldots,\ p_n$) of the state matrix $\mathbf{A}$.

The eigenvalues are also known as the *poles* of the system.

Example

If

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}$$

so

$$|s\mathbf{I} - \mathbf{A}| = s^2 + 5s + 6 = (s+2)(s+3).$$

The characteristic values of the system are $p_1 = -2$ and $p_2 = -3$ so the modes are e^{-2t} and e^{-3t}.

Now

$$\mathrm{adj}\,[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

so the resolvent matrix is

$$\Phi(s) = \begin{bmatrix} \frac{s+5}{(s+2)(s+3)} & \frac{1}{(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} & \frac{s}{(s+2)(s+3)} \end{bmatrix}.$$

Expanding each element of $\Phi(s)$ we get

$$\Phi(s) = \begin{bmatrix} \frac{3}{s+2} - \frac{2}{s+3} & \frac{1}{s+2} - \frac{1}{s+3} \\ \frac{-6}{s+2} + \frac{6}{s+3} & -\frac{2}{s+2} + \frac{3}{s+3} \end{bmatrix}.$$

Inverse Laplace transforming this we get the final state transition matrix, representing the zero-input response of the system:

$$\phi(t) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}.$$

Note that, by an alternative derivation[1], we can show that

$$\phi(t) = e^{\mathbf{A}t}$$

where $e^{\mathbf{A}t}$ is called the matrix exponential.

Unforced system response

To determine the response of the system to the initial state we recall that

$$\mathbf{x} = \mathbf{B\Phi(s)x}(0)$$

and putting this together with the definition of $\mathbf{Y(s)}$ we can determine the unforced system response of a system.

Zero Input System Response

For non-zero initial conditions and zero input we have

$$\begin{aligned} \mathbf{X}(s) &= [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0) \\ &= \Phi(s)\mathbf{x}(0) \end{aligned}$$

The zero-input system output transform is therefore given by

$$\mathbf{Y}_{\mathrm{zi}}(s) = \mathbf{C}\Phi(s)\mathbf{x}(0).$$

Because we have already shown that $\phi(t)$ will be a linear combination of modes, the unforced system response $y(t)$ will be another linear combination of the same modes.

We now consider the case where the initial condition vector $\mathbf{x}(0) = \mathbf{0}$ and we will obtain the response of the system to an input. This is called the *zero state response* of the system.

The Zero State Response

For zero initial conditions we have

$$\begin{aligned}\mathbf{X}(s) &= [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U}(s)\\ &= \Phi(s)\mathbf{B}\mathbf{U}(s)\end{aligned}$$

The system output transform (forced response) is given by

$$\mathbf{Y}_{\mathrm{zs}}(s) = [\mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}]\,\mathbf{U}(s)$$

(where $\mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}$ is the system transfer matrix).

Full System Response

The full system response for the state-space model is simply the sum of the zero-state and zero-input responses:

$$\begin{aligned}\mathbf{Y}_{\mathrm{full}}(s) &= \mathbf{Y}_{\mathrm{zs}}(s) + \mathbf{Y}_{\mathrm{zi}}(s)\\ &= \mathbf{C}\Phi(s)\,[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)] + \mathbf{D}\mathbf{U}(s).\end{aligned}$$

The full system response for the state-space model is simply the sum of the zero-state and zero-input responses:

$$\begin{aligned}\mathbf{Y}_{\mathrm{full}}(s) &= \mathbf{Y}_{\mathrm{zs}}(s) + \mathbf{Y}_{\mathrm{zi}}(s)\\ &= \mathbf{C}\Phi(s)\,[\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)] + \mathbf{D}\mathbf{U}(s).\end{aligned}$$

End of Pre-Class Presentation

In the class we will reinforce these concepts by working through an example in detail.

Example

Let

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{u} \\
\mathbf{y} &= \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x} \\
\mathbf{u} &= \begin{bmatrix} e^{-t} \\ \epsilon(t) \end{bmatrix}.
\end{aligned}
$$

Calculate the full system response for this system given that the initial conditions are $\mathbf{x}(0) = [-1, 1]^T$.

Solution to Example

The transfer function is

$$\mathbf{G}(s) = \mathbf{C}\Phi(s)\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+5}{(s+2)(s+3)} & \frac{1}{(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} & \frac{s}{(s+2)(s+3)} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2s+9}{(s+2)(s+3)} & \frac{s+5}{(s+2)(s+3)} \\ \frac{-s-12}{(s+2)(s+3)} & \frac{-6}{(s+2)(s+3)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5s+30}{(s+2)(s+3)} & \frac{2s+16}{(s+2)(s+3)} \\ \frac{-s-12}{(s+2)(s+3)} & \frac{-6}{(s+2)(s+3)} \end{bmatrix}$$

The transform of the input vector is

$$\mathbf{U}(s) = \mathcal{L}\begin{bmatrix} e^{-t} \\ \epsilon(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s} \end{bmatrix}.$$

Hence

$$\mathbf{Y}(s) = \begin{bmatrix} \frac{5s+30}{(s+2)(s+3)} & \frac{2s+16}{(s+2)(s+3)} \\ \frac{-s-12}{(s+2)(s+3)} & \frac{-6}{(s+2)(s+3)} \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7s^2+48s+16}{s(s+1)(s+2)(s+3)} \\ \frac{-(s^2+18s+6)}{s(s+1)(s+2)(s+3)} \end{bmatrix} = \begin{bmatrix} \frac{8/3}{s} + \frac{25/2}{s+1} - \frac{26}{s+2} + \frac{65/6}{s+3} \\ -\frac{1}{s} - \frac{11/2}{s+1} + \frac{13}{s+2} - \frac{13/2}{s+3} \end{bmatrix}$$

Inverse Laplace transforming the previous result gives the zero-state output response of the system:

$$\begin{bmatrix} \frac{8}{3}\epsilon(t) + \frac{25}{2}e^{-t} - 26e^{-2t} + \frac{65}{6}e^{-3t} \\ -\epsilon(t) - \frac{11}{2}e^{-t} + 13e^{-2t} - \frac{13}{2}e^{-3t} \end{bmatrix}.$$

Note that response is now a linear combination of the system modes (e^{-2t} and e^{-3t}) and the input modes ($\epsilon(t)$ [the unit step function] and e^{-t}).

If the initial condition vector $\mathbf{x}(0) = [-1, 1]^T$ what is the total response of the system?

$$\mathbf{Y}(s) = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+5}{(s+2)(s+3)} & \frac{1}{(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} & \frac{s}{(s+2)(s+3)} \end{bmatrix}$$

$$\times \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s} \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \frac{2s+16}{(s+2)(s+3)} & \frac{-s+2}{(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} & \frac{s}{(s+2)(s+3)} \end{bmatrix} \begin{bmatrix} \frac{-s^2+2s+1}{s(s+1)} \\ \frac{s^2}{s(s+1)} \end{bmatrix}$$

$$\mathbf{Y}(s) = \begin{bmatrix} \frac{-3s^2-10s^2+34s+16}{s(s+1)(s+2)(s+3)} \\ s^3+6s^2-12s-6 \end{bmatrix}$$

Solving this problem in MATLAB

We can use the symbolic toolbox!

Set up the problem ... we can use ordinary matrices for this

In [55]:

```
format compact
A = [0, 1; -6, -5]; B = [2, 1; -1, 0];
C = [2, -1; 0, 1]; D = [0, 0; 0, 0];
x0 = [-1, 1]';
```

Now define s and t as symbolic variables

In [56]:

```
syms t s
```

Define input vector $\mathbf{U}(s)$

In [57]:

```
u = [exp(-t);heaviside(t)];
Us = laplace(u)
```

```
Us =
 1/(s + 1)
       1/s
```

Compute $\Phi(s)$

In [58]:

```
PhiS = inv(s*eye(2) - A)
```

```
PhiS =
[ (s + 5)/(s^2 + 5*s + 6), 1/(s^2 + 5*s + 6)]
[       -6/(s^2 + 5*s + 6), s/(s^2 + 5*s + 6)]
```

Now compute zero state response:

$$\mathbf{Y}_{\mathrm{zs}}(s) = (\mathbf{C}\mathbf{\Phi}(s)\mathbf{B} + \mathbf{D})$$

In [59]:

```
Yzs = (C*PhiS*B + D)*Us
```

```
Yzs =
 (s/(s^2 + 5*s + 6) + 10/(s^2 + 5*s + 6) + (4*(s + 5))/(s^2 + 5*s +
6))/(s + 1) + (6/(s^2 + 5*s + 6) + (2*(s + 5))/(s^2 + 5*s + 6))/s
                                                                   - 6/(s*(s
^2 + 5*s + 6)) - (s/(s^2 + 5*s + 6) + 12/(s^2 + 5*s + 6))/(s + 1)
```

Unforced (zero-initial condition) response:

$$\mathbf{Y}_{\mathrm{zi}}(s) = (\mathbf{C}\mathbf{\Phi}(s)\mathbf{B} + \mathbf{D})$$

In [60]:

```
Yzi = C*PhiS*x0
```

```
Yzi =
 - s/(s^2 + 5*s + 6) - 4/(s^2 + 5*s + 6) - (2*(s + 5))/(s^2 + 5*s +
6)
                                 s/(s^2 + 5*s + 6) + 6/(s^2 + 5*s +
6)
```

Total response:

$$\mathbf{Y}(s) = \mathbf{Y}_{\mathrm{zs}}(s) + \mathbf{Y}_{\mathrm{zi}}(s)$$

In [61]:

```
Ytotal = Yzs + Yzi
```

```
Ytotal =
 (s/(s^2 + 5*s + 6) + 10/(s^2 + 5*s + 6) + (4*(s + 5))/(s^2 + 5*s +
6))/(s + 1) - 4/(s^2 + 5*s + 6) - s/(s^2 + 5*s + 6) - (2*(s + 5))/(s
^2 + 5*s + 6) + (6/(s^2 + 5*s + 6) + (2*(s + 5))/(s^2 + 5*s + 6))/s

s/(s^2 + 5*s + 6) + 6/(s^2 + 5*s + 6) - 6/(s*(s^2 + 5*s + 6)) - (s/
(s^2 + 5*s + 6) + 12/(s^2 + 5*s + 6))/(s + 1)
```

Finally compute the time response $\mathbf{y}(t)$

In [62]:

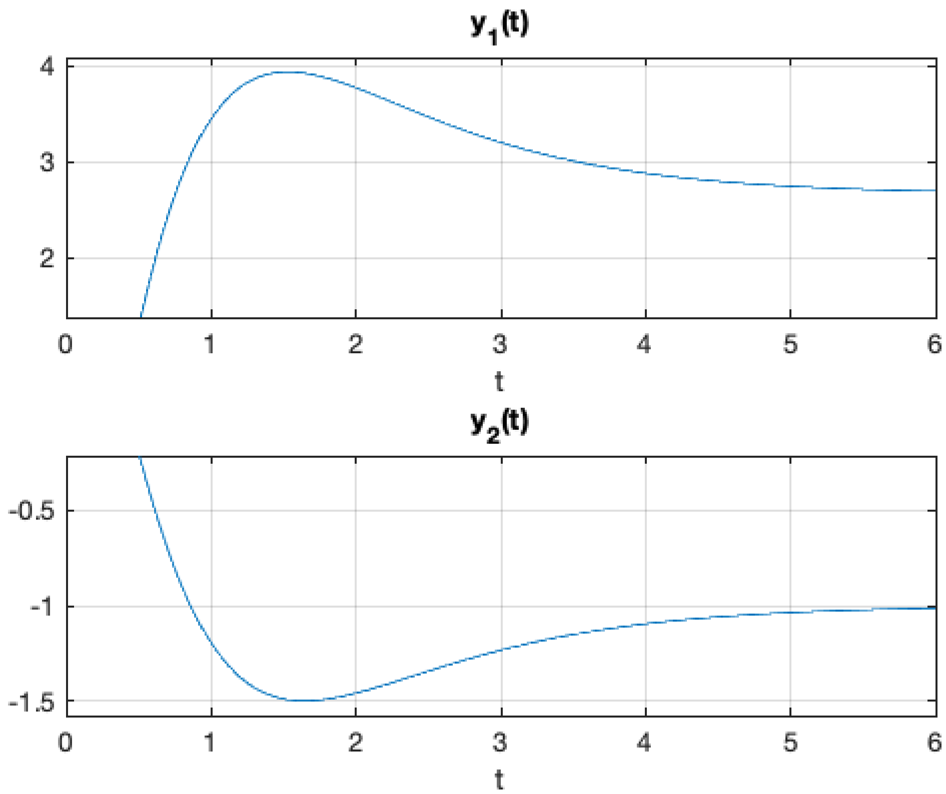
```
y = ilaplace(Ytotal)
```

```
y =
 (25*exp(-t))/2 - 34*exp(-2*t) + (95*exp(-3*t))/6 + 8/3
   17*exp(-2*t) - (11*exp(-t))/2 - (19*exp(-3*t))/2 - 1
```

Plot response

In [69]:

```
subplot(211)
ezplot(y(1),[0,6]),grid,title('y_1(t)')
subplot(212)
ezplot(y(2),[0,6]),grid,title('y_2(t)')
```



Simulation of State Space Models in Matlab

- The functions `step` and `impulse` will produce step impulse responses of a system with zero initial conditions.
- The function `lsim` can be used to determine the state space response to arbitrary inputs.
- If you need to simulate a system with arbitrary inputs and predefined initial conditions, you should use the *Linear Simulation Tool* which is opened when you execute `lsim(model)`.

Visit the MATLAB Control Systems Toolbox help pages to find out more.

Footnote

1. For a first order differential equation

$$\frac{dx}{dt} = ax$$

so the Laplace transform is

$$\begin{aligned} sX(s) - x(0) &= aX(s) \\ X(s)(s-a) &= x(0) \\ X(s) &= \frac{x(0)}{s-a}. \end{aligned}$$

The inverse laplace transform of the final equation gives

$$x(t) = e^{at}.$$

For the vector case

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{AX(s)}$$

so by comparison

$$\mathbf{x}(t) = e^{\mathbf{A}t}.$$

The proper proof is of course a little more involved and will be explored in Section 7.6 (../5/gensolution)!