Analytical Root-Locus Design of Phase-Lead Compensators

This MATLAB Live Script presents an analytical procedure for phase-lead design. It is based on Section 7.8 of Phillips and Harbor "*Feedback Control Systems,*" Prentice Hall, 1988[1]. For the procedure it is convenient to write the compensator transfer function as

$$D(s) = \frac{a_1 s + a_0}{b_1 s + 1} \qquad (1)$$

In this procedure we choose a_1, a_0, and b_1 such that given s_1, the equation

$$KD(s)G(s)H(s)|_{s=s_1} = -1 \qquad (2)$$

is satisfied; that is we are designing a compensator that places a root of the closed-loop characteristic equation at $s = s_1$.

In equation (2) we have four unknowns, including *K*, and only two relationships (magnitude and phase) that must be satisfied. Hence, we can arbitrarily assign values to two of the unknowns. *K* is easily eliminated since

$$KD(s) = \frac{Ka_1 s + Ka_0}{b_1 + 1}$$

so if we assume that $K = 1$ for the design procedure we eliminate one of the unknowns. The other unknown that can be eliminated is a_0 which can be seen to be the DC gain of the compensator. Its value can therefore be chosen to satisfy the steady-state error requirements of the design and we need only to determine values for a_1 and b_1.

The design proceeds as follows. First, we express the desired closed loop pole position

$$s_1 = |s_1|e^{j\beta} \qquad (3)$$

and

$$G(s_1)H(s_1) = |G(s_1)H(s_1)|e^{j\psi} \qquad (4)$$

Then the design equations (derived in Appendix B of Phillips and Harbor, 1988) are

$$a_1 = \frac{\sin\beta + a_0|G(s_1)H(s_1)|\sin(\beta - \psi)}{|s_1||G(s_1)H(s_1)|\sin\psi}$$

$$b_1 = \frac{\sin(\beta - \psi) + a_0|G(s_1)H(s_1)|\sin\psi}{-|s_1|\sin\psi} \qquad (5)$$

Given a_0, $G(s)H(s)$, and the desired closed-loop pole location s_1, (5) gives the remaining compensator coefficients. This procedure places a closed-loop pole at $s = s_1$; however, the locations of the remaining poles are unknown and may be unsatisfactory. In fact, some may be unstable!

For the case that ψ is either 0° or 180°, equations (5) must be modified to give the single equation

$$a_1|s_1|\cos\beta \pm \frac{b_1|s_1|}{|G(s_1)H(s_1)|} \pm \frac{1}{|G(s_1)H(s_1)|} + a_0 = 0 \qquad (6)$$

where the plus sign applies to the case $\psi = 0^\circ$ and the minus sign applies to $\psi = 180^\circ$. For this case, the value of either a_1 or b_1 can also be assigned. An example is now given to illustrate the procedure.

Example

[This example is presented using actual MATLAB commands. The example is the satellite attitude control problem given in lectures, but this script can be used to solve similar problems if use is made of the MATLAB Live Script interface. All that is necessary is to change the definitions for the plant transfer function , the feedback transfer function , the desired closed-loop pole position and the DC gain . The command Run to End from the command ribbon menu should then be executed to recalculate the values]

Definitions (change these to change design)

The plant transfer function is :

```
G = tf(1,[1 0 0]);
```

The feedback transfer function is $H(s) = 1$:

```
H = tf(1,1);
```

So $G(s)H(s)$ is:

```
GH=series(G,H)
```

```
GH =

    1
   ---
   s^2

Continuous-time transfer function.
```

The desired closed-loop poles are:

```
s1 = -2 + 2j;
```

Now the DC gain of this type 2 system will be:

For the purpose of illustration let us arbitrarily take a value of $a_0 = 8/3$:

```
a0 = 8/3;
```

Calculations (shouldn't need to change these commands)

Polar form of s_1

```
m_s1=abs(s1),  p_s1 = (angle(s1)*180/pi + 90) % degrees
```

```
m_s1 = 2.8284
p_s1 = 225
```

Transfer function evaluated at $s_1 = G(s_1)H(s_1)$ in polar form:

```
[numGH,denGH] = tfdata(GH,'v');GHs1=polyval(numGH,s1)/polyval(denGH,s1)
```

```
GHs1 = 0.0000 + 0.1250i
```

Magnitude:

```
mGHs1=abs(GHs1)
```

```
mGHs1 = 0.1250
```

Phase:

```
pGHs1=angle(GHs1)*180/pi - 180 % degrees
```

```
pGHs1 = -90
```

Hence angles are:

```
beta = p_s1*pi/180
```

```
beta = 3.9270
```

```
psi = pGHs1*pi/180  % radians
```

```
psi = -1.5708
```

From (5)

```
a1 = (sin(beta) + a0*mGHs1*sin(beta - psi))/(m_s1*mGHs1*sin(psi))
```

```
a1 = 2.6667
```

```
b1 = (sin(beta + psi) + a0*mGHs1*sin(beta))/(-(m_s1)*sin(psi))
```

```
b1 = 0.1667
```

Compensator is therefore given by

```
numD = [a1, a0], denD = [b1, 1]
```

```
numD = 1×2
    2.6667    2.6667
denD = 1×2
    0.1667    1.0000
```

which in normal form:

$$D(s) = K_c\left(\frac{s+z_1}{s+p_1}\right)$$

has

```
Kc = a1/b1, z0 = a0/a1, p0 = 1/b1
```

```
Kc = 16.0000
z0 = 1
p0 = 6.0000
```

Now make a transfer function

```
D = tf(Kc*[1, z0],[1, p0])
```

```
D =

  16 s + 16
  ---------
    s + 6

Continuous-time transfer function.
```

Evaluation of Design

Open loop transfer function:

```
Go = series(D,GH)
```

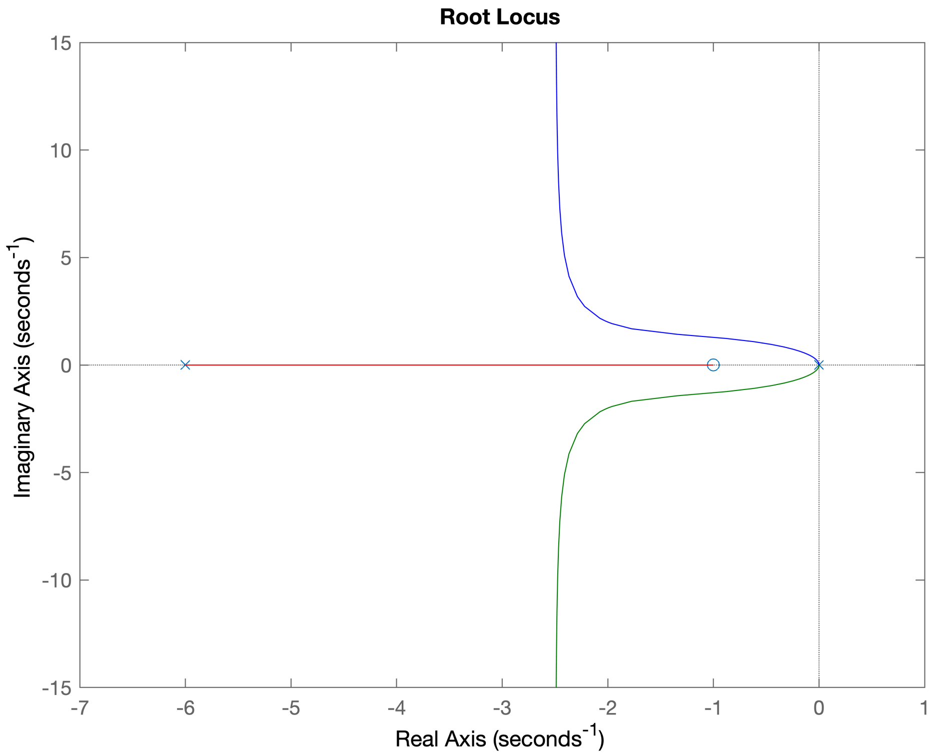
```
Go =

   16 s + 16
  -----------
  s^3 + 6 s^2

Continuous-time transfer function.
```

Root locus:

```
rlocus(Go)
```



Closed-loop transfer function:

```
DG=series(D,G)
```

```
DG =

   16 s + 16
  -----------
  s^3 + 6 s^2

Continuous-time transfer function.
```

```
Gc=feedback(DG,H)
```

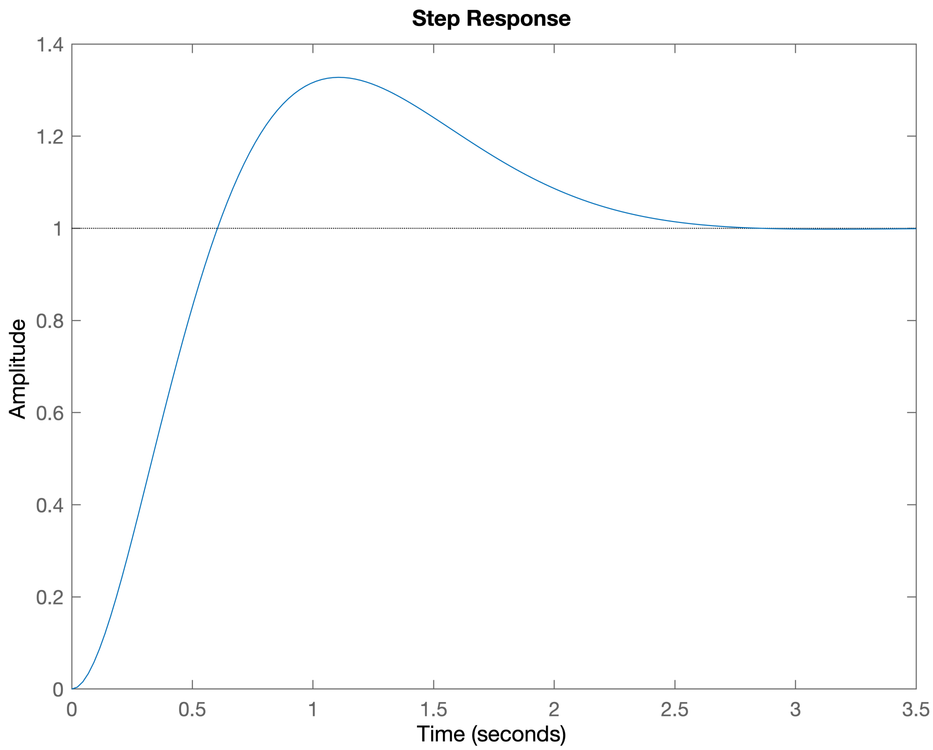
```
Gc =

         16 s + 16
  -----------------------
  s^3 + 6 s^2 + 16 s + 16

Continuous-time transfer function.
```

Step response:

```
step(Gc)
```



As an exercise, you should examine the effect of designing for a range of DC gains in the range $0 \le K_a \le 10$.

Footnotes

[1] The proofs of the formulae given are derived in Appendix B of that text.