Laplace Transforms of State Space Models

The Laplace transform can be used to convert a differential equation into a transfer function. It can also be used to convert a state space model into a transfer function. In this lecture we demonstrate how this is done and we give an example.

The Laplace transform of a vector $\mathbf{v}(t)$ is a vector $\mathbf{V}(s)$. The elements of $\mathbf{V}(s)$ are the Laplace transforms of the corresponding elements of the vector $\mathbf{v}(t)$.

For array

$$\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_n(t) \end{bmatrix}$$

The transformed variables are

$$\mathcal{L}\mathbf{v}(t) = \begin{bmatrix} \mathcal{L}v_1(t) \\ \mathcal{L}v_2(t) \\ \vdots \\ \mathcal{L}v_n(t) \end{bmatrix} = \begin{bmatrix} V_1(s) \\ V_2(s) \\ \vdots \\ V_n(s) \end{bmatrix} = \mathbf{V}(s)$$

For example, if[1]

$$\mathbf{v}(t) = \begin{bmatrix} \epsilon(t) \\ e^{-at} \\ \sin bt \end{bmatrix}$$

then

$$\mathbf{V}(s) = \begin{bmatrix} 1/s \\ 1/s+a \\ b/(s^2+b^2) \end{bmatrix}$$

Let us now transform the generalized form of the state equations obtained in the last lecture.

$$\begin{aligned} \frac{d\mathbf{x}(t)}{dt} &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned}$$

Applying the Laplace transform to both sides of this matrix equation gives the transform equations

$$\begin{aligned} s\mathbf{X}(s) - \mathbf{x}(0) &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) &= \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{aligned}$$

where $\mathbf{x}(0)$ is the vector of initial conditions vector of the states; $\mathbf{X}(s)$ is the state transform vector; $\mathbf{U}(s)$ input transform vector; $\mathbf{Y}(s)$ is output transform vector.

Transformed State-Equations for Example 1 from Section

For the system in the example the state vector is defined as $\mathbf{x} = [v_{31}, i_1]^T$, the input current is u, and the output variables are all the currents and voltages in the circuit $\mathbf{y} = [v_{31}, i_1, v_{32}, v_{21}, i_2]^T$.

The transformed state space model is therefore:

$$s\begin{bmatrix} V_{31} \\ I_1 \end{bmatrix} - \begin{bmatrix} v_{31}(0) \\ i_1(0) \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix}\begin{bmatrix} V_{31} \\ I_1 \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix}[U]$$

$$\begin{bmatrix} V_{31} \\ I_1 \\ V_{32} \\ V_{21} \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -R \\ 0 & R \\ 0 & -1 \end{bmatrix}\begin{bmatrix} V_{31} \\ I_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}[U].$$

Laplace Transform from State-Space Models

Substituting $\mathbf{X}$ from (1) into (2) gives

$$\mathbf{Y} = \left[\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{U} + \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0)\right] + \mathbf{D}\mathbf{U}$$

which after gathering terms and simplifying gives

$$\mathbf{Y} = \left[\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U} + \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0)$$

When the initial conditions of the state-variables are all zero, this reduces to the transfer matrix model

$$\mathbf{Y} = \left[\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}$$

The matrix $\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$ is the *system transfer matrix*.

The element of the i-th row and j-th column is the transfer function that relates the i-th output transform Y_i to the j-th input transform U_j.

For a single-input, single-output (SISO) system, the system transfer matrix reduces to a single element transfer function.

The matrix $[s\mathbf{I} - \mathbf{A}]^{-1}$ is very important.

It is known as the *resolvent matrix* of the system.

It may be written as

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\mathrm{adj}\,[s\mathbf{I} - \mathbf{A}]}{\det\,[s\mathbf{I} - \mathbf{A}]}.$$

Resolvent matrix for the example

For the system in the example, the resolvent matrix is developed as

$$\mathbf{A} = \left[\begin{array}{cc} 0 & -1/C \\ 1/L & -R/L \end{array}\right]$$

$$s\mathbf{I} - \mathbf{A} = \left[\begin{array}{cc} s & 0 \\ 0 & s \end{array}\right] - \left[\begin{array}{cc} 0 & -1/C \\ 1/L & -R/L \end{array}\right] = \left[\begin{array}{cc} s & +1/C \\ -1/L & s + R/L \end{array}\right]$$

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\left[\begin{array}{cc} s + R/L & -1/C \\ +1/L & s \end{array}\right]}{s(s + R/L) + 1/(LC)} = \frac{\left[\begin{array}{cc} s + R/L & -1/C \\ +1/L & s \end{array}\right]}{s^2 + (R/L)s + 1/(LC)}$$

When $[s\mathbf{I} - \mathbf{A}]^{-1}$ has been obtained, then the system transfer function is easily obtained through $\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$. For the system in the example, when all outputs are measured, the system transfer matrix is:

Transfer matrix for example

$$\begin{equation*} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & -R \\ 0 & R \\ 0 & -1 \end{array}\right] \left{\frac{\left[\begin{array}{cc} s+R/L & -1/C \\ +1/L & s \end{array}\right]}{s^2 + (R/L)s +1/(LC)}\right} \left[\begin{array}{c} 1/C \\ 0 \end{array}\right]

- \left[

$$\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array}$$

\right] \end{equation*}$$

```
$$\begin{equation*} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & -R \\ 0 & R \\ 0 & -1 \end{array}\right]
\left{\frac{\left[ \begin{array}{c} (1/C)s+R/(LC) \\ +1/(LC) \end{array}\right]}{s^2 + (R/L)s +1/(LC)}\right}
```

- \left[

$$\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array}$$

\right]\end{equation*}$$

$$= \frac{1}{s^2 + (R/L)s + 1/(LC)} \left[\begin{array}{c} (1/C)s + R/(LC) \\ 1/(LC) \\ (1/C)s \\ R/(LC) \\ -1/(LC) \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array}\right]$$

$$= \left[\begin{array}{c} \frac{(1/C)s+R/(LC)}{s^2+(R/L)s+1/(LC)} \\ \frac{1/(LC)}{s^2+(R/L)s+1/(LC)} \\ \frac{(1/C)s}{s^2+(R/L)s+1/(LC)} \\ \frac{R/(LC)}{s^2+(R/L)s+1/(LC)} \\ -\frac{1/(LC)}{s^2+(R/L)s+1/(LC)} + 1 \end{array}\right]$$

$$= \frac{1}{s^2 + (R/L)s + 1/(LC)} \left[\begin{array}{c} (1/C)s + R/(LC) \\ 1/(LC) \\ (1/C)s \\ R/(LC) \\ -1/(LC) \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array}\right]$$

$$= \left[\begin{array}{c} \frac{(1/C)s+R/(LC)}{s^2+(R/L)s+1/(LC)} \\ \frac{1/(LC)}{s^2+(R/L)s+1/(LC)} \\ \frac{(1/C)s}{s^2+(R/L)s+1/(LC)} \\ \frac{R/(LC)}{s^2+(R/L)s+1/(LC)} \\ -\frac{1/(LC)}{s^2+(R/L)s+1/(LC)} + 1 \end{array}\right]$$

In matrix form, when combined with the input and output transforms we have the situation illustrated below. Each transfer function relates the corresponding output transform to the input transform. For example

$$V_{31} = \frac{(1/C)s + R/(LC)}{s^2 + (R/L)s + 1/(LC)}\ U.$$

Transform Equations for Example

$$\mathbf{Y}(s) = \left[\mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}\right] \mathbf{U}(s)$$
$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$$

$$\begin{bmatrix} V_{31}(s) \\ I_1(s) \\ V_{32}(s) \\ V_{21}(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \frac{(1/C)s+R/(LC)}{s^2+(R/L)s+1/(LC)} \\ \frac{1/(LC)}{s^2+(R/L)s+1/(LC)} \\ \frac{(1/C)s}{s^2+(R/L)s+1/(LC)} \\ \frac{R/(LC)}{s^2+(R/L)s+1/(LC)} \\ -\frac{1/(LC)}{s^2+(R/L)s+1/(LC)} + 1 \end{bmatrix} \mathbf{U}(s).$$

Note that the denominator is the same for each transfer function, and that the order of the numerator is less than the denominator except for one case, for which

$$\begin{aligned} I_2 &= \left(-\frac{1/(LC)}{s^2 + (R/L)s + 1/(LC)} + 1\right) U \\ &= \frac{-1/LC + s^2 + (R/L)s + 1/(LC)}{s^2 + (R/L)s + 1/(LC)}\ U \\ &= \frac{s^2 + (R/L)s}{s^2 + (R/L)s + 1/(LC)}\ U \end{aligned}$$

Replacing s by $\frac{d}{dt}$ gives the corresponding differential equations relating the dependant variable to the input.

Converting SS to TF in Matlab

Continuing example from Section 7.1 (../1/intro2ss):

In [43]:

```
clear all
format compact
```

Define some values for capacitance, inductance and resitance

In [44]:

```
Cap = 1; L = 1; R = 1;
```

Define state space model and label states inputs and outputs

In [45]:

```
A = [0 -1/Cap; 1/L -R/L];
B = [1/Cap; 0];
C = [1 0; 0 1; 1 -R; 0 R; 0 -1];
D = [0; 0; 0; 0; 1];
circ_ss = ss(A, B, C, D, ...
'statename',{'v31' 'i1'}, ...
'inputname', 'u', ...
'outputname', {'v31' 'i1' 'v32' 'v21' 'i2'});
```

Show model

In [46]:

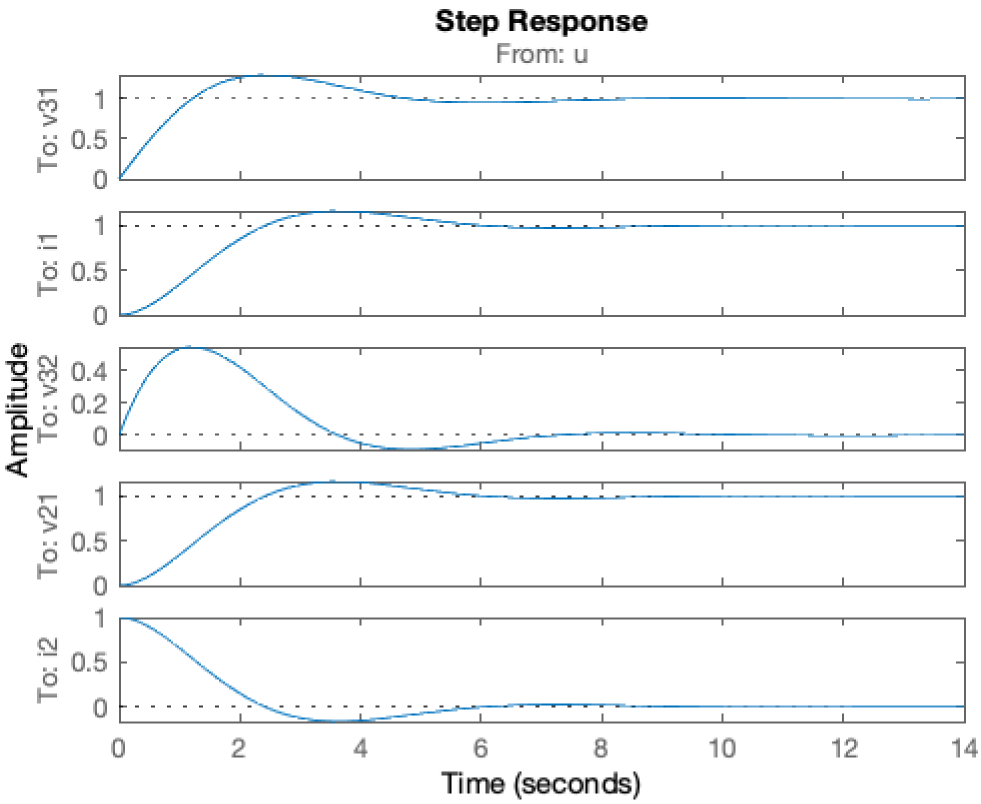
```
circ_ss
```

```
circ_ss =

  A =
        v31   i1
   v31    0   -1
   i1     1   -1

  B =
        u
   v31  1
   i1   0

  C =
        v31   i1
   v31    1    0
   i1     0    1
   v32    1   -1
   v21    0    1
   i2     0   -1

  D =
        u
   v31  0
   i1   0
   v32  0
   v21  0
   i2   1

Continuous-time state-space model.
```

Plot a step response

In [47]:

```
step(circ_ss)
```



Convert to transfer functiom matrix

The function `tf(ss_model)` returns a vector of transfer functions.

In [48]:

```
circ_tf = tf(circ_ss)
```

```
circ_tf =

  From input "u" to output...
            s + 1
   v31:  -----------
         s^2 + s + 1

             1
   i1:  -----------
        s^2 + s + 1

              s
   v32:  -----------
         s^2 + s + 1

              1
   v21:  -----------
         s^2 + s + 1

          s^2 + s
   i2:  -----------
        s^2 + s + 1

Continuous-time transfer function.
```

Determine poles and zeros

In [49]:

```
circ_zpk=zpk(circ_ss)
```

```
circ_zpk =

  From input "u" to output...
               (s+1)
   v31:  -------------
         (s^2 + s + 1)

               1
   i1:  -------------
        (s^2 + s + 1)

               s
   v32:  -------------
         (s^2 + s + 1)

               1
   v21:  -------------
         (s^2 + s + 1)

            s (s+1)
   i2:  -------------
        (s^2 + s + 1)

Continuous-time zero/pole/gain model.
```

The state transition matrix

Calculated using the symbolic math tools provided by MATLAB See `help symbolic`

In [50]:

```
syms phi t s
phi = inv(s*eye(2) - A)
```

```
phi =
[ (s + 1)/(s^2 + s + 1), -1/(s^2 + s + 1)]
[       1/(s^2 + s + 1),  s/(s^2 + s + 1)]
```

The state transfer matrix

In [54]:

```
G = C*phi*B + D
```

```
G =
                   (s + 1)/(s^2 + s + 1)
                         1/(s^2 + s + 1)
 (s + 1)/(s^2 + s + 1) - 1/(s^2 + s + 1)
                         1/(s^2 + s + 1)
                     1 - 1/(s^2 + s + 1)
```

In [55]:

```
G = simplify(G)
```

```
G =
 (s + 1)/(s^2 + s + 1)
       1/(s^2 + s + 1)
       s/(s^2 + s + 1)
       1/(s^2 + s + 1)
   1 - 1/(s^2 + s + 1)
```

In [52]:

```
pretty(G)
```

```
/       s + 1      \
|   ----------     |
|    2             |
|   s  + s + 1     |
|                  |
|         1        |
|   ----------     |
|    2             |
|   s  + s + 1     |
|                  |
|         s        |
|   ----------     |
|    2             |
|   s  + s + 1     |
|                  |
|         1        |
|   ----------     |
|    2             |
|   s  + s + 1     |
|                  |
|           1      |
| 1 - ----------   |
|      2           |
\     s  + s + 1   /
```

A executable script version of this example is available as ssmodels.mlx (matlab/ssmodels.mlx).

Some Important Properties

System poles

Clearly the denominator of the transfer function is generated by the matrix inverse which produces the term:

$$\det[s\mathbf{I} - \mathbf{A}]$$

This evaluates to the denominator polynomial and the poles of the system are the roots of the system's characteristic equation:

$$\det[s\mathbf{I} - \mathbf{A}] = 0.$$

The system poles are solutions to the system's characteristic equation

$$\det[s\mathbf{I} - \mathbf{A}] = 0.$$

System zeros

What is the corresponding numerator polynomial of the transfer function, whose roots give the zeros of the system?

The zeros are those values of s for which the output is zero when the input and states are not zero.

Thus:

$$\begin{aligned}(s\mathbf{I} - \mathbf{A})\mathbf{X} - \mathbf{B}U &= \mathbf{0}\\ \mathbf{C}\mathbf{X} + dU = Y &= 0\end{aligned}$$

In matrix form:

$$\begin{bmatrix} s\mathbf{I} - \mathbf{A} & \vdots & \mathbf{B} \\ \cdots & \cdots & \cdots \\ \mathbf{C} & \vdots & d \end{bmatrix}\begin{bmatrix} \mathbf{X} \\ \cdots \\ U \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \cdots \\ 0 \end{bmatrix}$$

Zeros are those values of s for which the system output is zero when the input and states are not zero

$$\begin{aligned}(s\mathbf{I} - \mathbf{A})\mathbf{X} - \mathbf{B}U = &\quad \mathbf{0}\\ \mathbf{C}\mathbf{X} + dU = Y = &\quad 0\end{aligned}$$

In matrix form:

$$\begin{bmatrix} s\mathbf{I} - \mathbf{A} & \vdots & \mathbf{B} \\ \cdots & \cdots & \cdots \\ \mathbf{C} & \vdots & d \end{bmatrix}\begin{bmatrix} \mathbf{X} \\ \cdots \\ U \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \cdots \\ 0 \end{bmatrix}$$

The only way this can have non-zero solutions in $\mathbf{X}$ and U is if:

$$\det\begin{bmatrix} s\mathbf{I} - \mathbf{A} & \vdots & \mathbf{B} \\ \cdots & \cdots & \cdots \\ \mathbf{C} & \vdots & d \end{bmatrix} = 0$$

This is another polynomial in s whose roots give the system zeros and therefore corresponds to the numerator polynomial of the TF.

Given this result, an alternative expression for the TF is:

$$\frac{Y(s)}{U(s)} = \frac{\det \begin{bmatrix} s\mathbf{I} - \mathbf{A} & \vdots & \mathbf{B} \\ \cdots & \cdots & \cdots \\ \mathbf{C} & \vdots & d \end{bmatrix}}{\det[s\mathbf{I} - \mathbf{A}]}$$

Time Responses from Transfer Function Matrices

In the next section (../3/sstr) we will consider how we can use the transfer function model to compute time responses from state-space models.

Footnote

1. $\epsilon(t)$ is the unit step function $\epsilon(t) = 0$ for $t < 0$; $\epsilon(t) = 1$ for $t \geq 0$.