Digital System Models and System Response

Digital System Models

The equivalent of the differential equation model for continuous systems is the difference equation for digital systems.

Replacing the differential operator $\frac{d}{dt}$ by the advance operator $\triangle$ gives the general form of the difference equation as

$$\triangle^n y + a_1 \triangle^{n-1} y + \ldots + a_n y = b_0 \triangle^n u + b_1 \triangle^{n-1} u + \ldots + b_n u.$$

The equivalent of the differential equation model for continuous systems is the difference equation for digital systems.

Difference equation

Now, given the definition of the advance operator derived in a previous lecture

$$\triangle^n v_k = v_{k+n}$$

we can re-write equation (1) as the *difference equation*

$$y_{k+n} + a_1 y_{k+n-1} + \ldots + a_n y_k = b_0 u_{k+n} + b_1 u_{k+n-1} + \ldots + b_n u_k.$$

Difference equation in terms of the delay operator

Unlike the differential equation, however, which is hardly ever expressed in an integral form, the difference equation is more usually expressed in terms of the delay operator ∇.

Applying the operator ∇^n to equation (1) gives

$$\begin{aligned} y + a_1 \nabla y + \ldots + a_n \nabla^n y &= b_0 u + b_1 \nabla u + \ldots + b_n \nabla^n u \\ y_k + a_1 y_{k-1} + \ldots + a_n y_{k-n} &= b_0 u_k + b_1 u + \ldots + b_n u_{k-n}. \end{aligned}$$

z-transform of difference equation

Applying the z transform directly to the difference equation with the delay operator (3) gives

$$\begin{aligned} Y + a_1 z^{-1} Y + \ldots + a_n z^{-n} Y &= b_0 U + b_1 z^{-1} U + \ldots + b_n z^{-n} U \\ (1 + a_1 z^{-1} + \ldots + a_n z^{-n}) Y &= (b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}) U \end{aligned}$$

z Transfer Function

Given that

$$(1 + a_1 z^{-1} + \ldots + a_n z^{-n}) Y(z) = (b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}) U(z)$$

The z transfer function is

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

A digital system in this general form is known as a "*pole-zero, infinite impulse response, recursive auto-regressive moving average digital filter(!)*"

z Transfer Function (2)

If $b_1 = b_2 = \cdots = b_n = 0$ then the transfer function is

$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}}$$

A digital system in this form is known as an "*all pole, infinite impulse response, recursive auto-regressive digital filter.*"

z Transfer Function (2)

When $a_1 = a_2 = \cdots = a_n = 0$ then the transfer function is

$$H(z) = b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}$$

A digital system in this form is known as an "*all zero, finite impulse response, non recursive moving average digital filter.*"

Other forms of digital transfer function

The transfer function can also be expressed in the zero-pole-gain form

$$H(z) = \frac{k(z - z_1)(z - z_2) \cdots (z - z_n)}{(z - p_1)(z - p_2) \cdots (z - p_n)}$$

and the partial fraction form

$$H(z) = \frac{r_1}{z - p_1} + \frac{r_2}{z - p_2} \cdots \frac{r_n}{z - p_n}$$

Canonical Forms

With the transfer function written as

$$H(z) = \frac{b_0 z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_n}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + b_n}$$

there is a direct analogy with the general form of the continuous system transfer function with s instead of z.

This was implemented with the physically realistic integral operator $\int dt$ for which the digital equivalent is the delay operator ∇.

End of Pre-Class Presentation

This concludes the pre-class presentation.

In the class we will look at system response and compute the impulse and step responses of and example system.

Digital System Response

As in the case of a continuous system, the response of a digital signal comprises the sum of a free response and a forced response. The free response is dependent on the initial conditions of a digital system states, and as these are taken as zero here the free response is also zero and will not be considered further.

Digital System Response

The response of a digital system with transfer function $H(z)$ to a digital input signal u is the digital output signal y given in transform form as

$$Y(z) = H(z)U(z)$$

The inverse transform needed to determine the digital system response is obtained using the inverse z transform methods, e.g. polynomial division and partial fraction expansion, discussed in a previous lecture.

Taking the inverse transform gives the digital system response as

$$y_k = \mathcal{Z}^{-1} Y(z) = \mathcal{Z}^{-1} \left\{ H(z)U(z) \right\}$$

Response to Singularity Signals

The elemental singularity signals in a digital system response include the digital impulse signal and the digital step input.

Impulse response

Impulse signal

The digital impulse signal is given by

$$v = \delta = \{\delta_k\}$$

where $\delta_0 = 1$ when $k = 0$, and $\delta_k = 0$ otherwise.

Therefore the sequence for the impulse is simply

$$\delta_k = 1,\ 0,\ 0,\ 0,\ \ldots .$$

The transform of the digital impulse signal is

$$V = \Delta = \sum_{k=0}^{\infty} \delta_k z^{-k} = 1$$

Example 1: Impulse Response Calculate the impulse

response of the digital system with transfer function

$$H(z) = \frac{4z^2 - 16}{z^2 - 0.25}$$

Consider the system

$$H(z) = \frac{4z^2 - 16}{z^2 - 0.25}$$

The impulse response will be

$$Y(z) = H(z) \times 1 = \frac{4z^2 - 16}{z^2 - 0.25}$$

We shall determine this response using the partial fraction expansion.

$$\begin{aligned} Y(z) &= \frac{4 - 16z^{-2}}{1 - 0.25z^{-2}} \\ &= \frac{4(4 - 16z^{-2})}{4 - z^{-2}} \\ &= \frac{4(2 - 4z^{-1})(2 + 4z^{-1})}{(2 - z^{-1})(2 + z^{-1})} \end{aligned}$$

Assuming a partial fraction expansion of the form

$$Y(z) = \frac{A}{2 - z^{-1}} + \frac{B}{2 + z^{-1}} + C$$

we have

$$\begin{aligned} \frac{4(2 - 4z^{-1})(2 + 4z^{-1})}{(2 - z^{-1})(2 + z^{-1})} &= \frac{A(2 + z^{-1}) + B(2 - z^{-1}) + C(2 - z^{-1})(2 + z^{-1})}{(2 - z^{-1})(2 + z^{-1})} \\ 16 - 64z^{-2} &= 2A + Az^{-1} + 2B - Bz^{-1} + 4C - Cz^{-2} \end{aligned}$$

Gathering terms and equating coefficients

$$\begin{aligned} 16 &= 2A + 2B + 4C \\ 0 &= A - B \\ -64 &= -C \end{aligned}$$

Hence

$$\begin{aligned} C &= 64 \\ A &= B \\ 16 &= 4A + 256 \\ A &= B = -60 \end{aligned}$$

Thus

$$\begin{aligned} Y(z) &= 64 - \frac{60}{2 - z^{-1}} - \frac{60}{2 + z^{-1}} \\ &= 64 - \frac{30}{1 - 1/2z^{-1}} - \frac{30}{1 + 1/2z^{-1}} \\ y_k &= \left\{ 64\delta_k - 30\left(\frac{1}{2}\right)^k - 30\left(-\frac{1}{2}\right)^k \right\} \\ &= \{4,\ 0,\ -15,\ 0,\ -3.75,\ 0,\ -0.9375,\ \ldots\} \end{aligned}$$

Step response

Step signal

The digital step signal is

$$v = \epsilon = \{\epsilon_k\}$$

where $\epsilon_k = 1$ when $k \geq 0$, and $\epsilon_k = 0$ otherwise.

Therefore the sequence for the step is simply

$$\epsilon_k = 1,\ 1,\ 1,\ 1,\ \ldots .$$

z-transform of step signal

The transform of the digital step signal is

$$\begin{aligned} V = E &= \sum_{k=0}^{\infty} \epsilon_k z^{-k} \\ &= \sum_{k=0}^{\infty} z^{-k} \\ &= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots z^{-n} + \cdots \\ &= \frac{1}{1 - z^{-1}} = \frac{z}{z-1}. \end{aligned}$$

Example 2: Step Response

Calculate the step response of the digital system with transfer function

$$H(z) = \frac{4z^2 - 16}{z^2 - 0.25}$$

The step response of the example system is

$$Y(z) = H(z) \times \frac{z}{z-1} = \frac{z(4z^2 - 16)}{(z-1)(z^2 - 0.25)}$$

We shall determine this response using the partial fraction expansion.

$$\begin{aligned} Y(z) &= \frac{4z^3 - 16z}{z^3 - z^2 - 0.25z + 0.25} \\ &= \frac{4 - 16z^{-2}}{1 - z^{-1} - 0.25z^{-2} + 0.25z^{-3}} \end{aligned}$$

Earlier we showed that the result of the partial fraction expansion was

$$\frac{30}{1 - 1/2z^{-1}} - \frac{10}{1 + 1/2z^{-1}} - \frac{16}{1 - z^{-1}}$$

and the corresponding sequence is

$$y_k = \left\{30(1/2)^k - 10(-1/2)^k - 16\epsilon_k\right\}.$$