

## Lecturer

### Set up MATLAB

In [2]:

```
cd matlab  
pwd  
clear all  
format compact
```

ans =

```
'/Users/eechris/dev/eglm03-textbook/content/03/4/matlab'
```

# Cascade Lead compensation

## Introduction

The proportional plus derivative compensator has the unfortunate property that its high frequency gain is infinite. This means that high frequency effects, such as sensor noise and un-modelled high-frequency dynamics, e.g. resonance terms, will be amplified with potentially disastrous effects. Of course, a real physical derivative operator cannot be implemented and any implementation will actually have poles that will limit the high-frequency gain.

Recognizing this, an alternative to the pure P+D

$$D_{P+D} = K_D s + K_{\text{prop}}$$

is the so-called "lead compensator"

$$D_{\text{lead}}(s) = K_c \left( \frac{s - z_0}{s - p_0} \right)$$

where  $|p_0| > |z_0|$ .

Considering the frequency response of  $D_{\text{lead}}$

$$D_{\text{lead}}(j\omega) = K_c \left( \frac{j\omega - z_0}{j\omega - p_0} \right)$$

The low and high-frequency gains are:

$$D_{\text{lead}}(j\omega)|_{\omega \rightarrow 0} = K_c \left( \frac{z_0}{p_0} \right)$$

$$D_{\text{lead}}(j\omega)|_{\omega \rightarrow \infty} = K_c$$

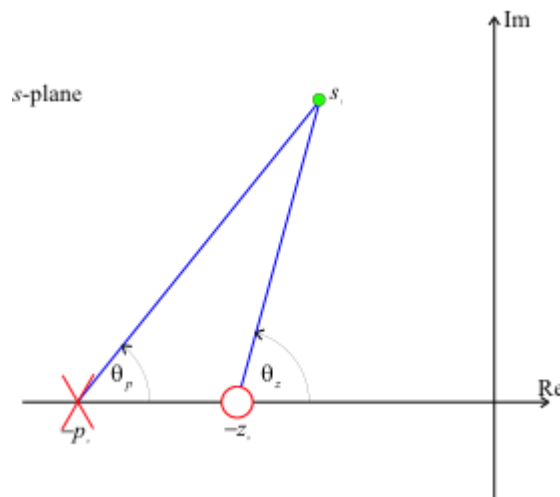
so that the ratio of high-to-low frequency gain is

$$\frac{D_{\text{lead}}(j\infty)}{D_{\text{lead}}(j0)} = \frac{p_0}{z_0} > 0$$

The lead compensator is still a high-pass filter but the pole at  $s = p_0$  limits the high frequency gain. Typically, the ratio of  $p_0$  to  $z_0$  is kept to below 10.

## Properties of the Cascade Lead Compensator

As  $|p_0| > |z_0|$ , the angle contributed by the compensator to some arbitrary point  $s_1$  at on the s-plane is illustrated in Figure 1.



**Figure 1** Angle contribution of a lead compensator

The net contribution is

$$\phi_c = \theta_z - \theta_p > 0$$

so that the lead compensator always makes a positive contribution to the angle criterion.

This has the effect of allowing the closed-loop poles to move to the left in the s-plane.

The problem is then how to choose the relative location of the pole and the zero.

We reproduce the advice of D'Azzo and Houpis (1975).

## Method 1

Use the zero to cancel a low frequency real pole. This can simplify the root locus and reduce the complexity of the problem. The compensator pole is then placed such that  $s_1$  becomes a point on the desired root-locus.

For a Type 1 system, the real pole (excluding the pole at zero) that is closest to the origin should be cancelled.

For a Type 0 system, the second closest pole to the origin should be cancelled.

## Example 1

The following Matlab code illustrates these principles for the system with

Type 1 open-loop transfer function

$$G_1(s) = \frac{1}{s(s+1)}$$

Define the plant

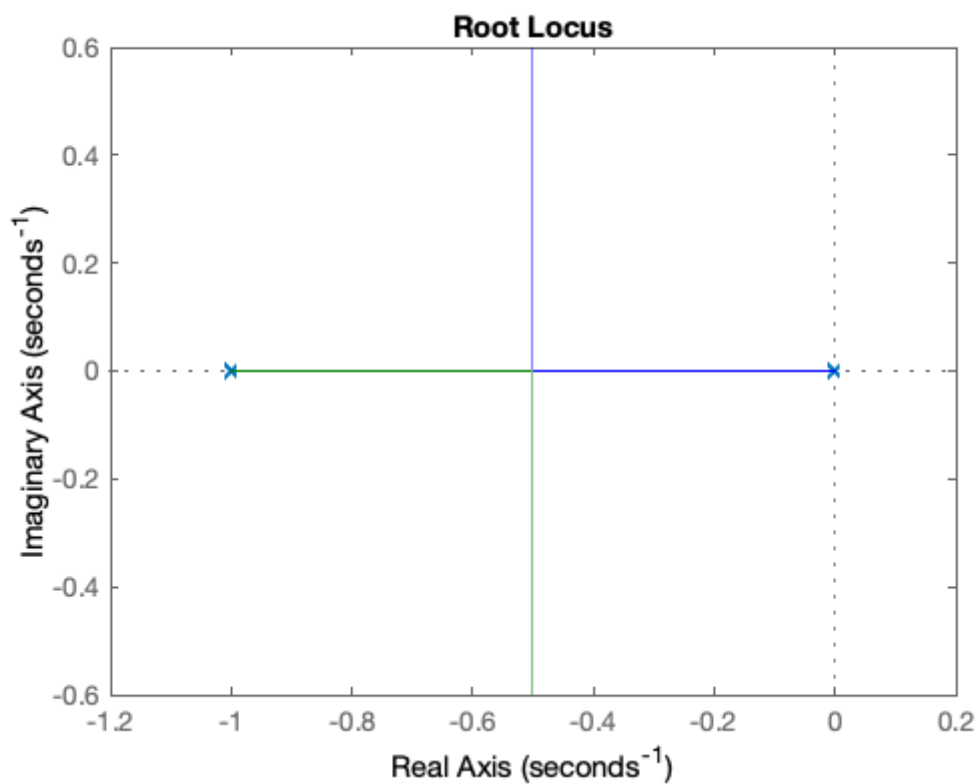
In [3]:

```
G1 = tf(1,conv([1, 0],[1, 1])); H=1;
```

Plot root-locus

In [4]:

```
rlocus(G1*H)
```



Clearly, we cannot achieve a closed-loop pole at  $s_1 = -2 + j2$  without some dynamic compensation.

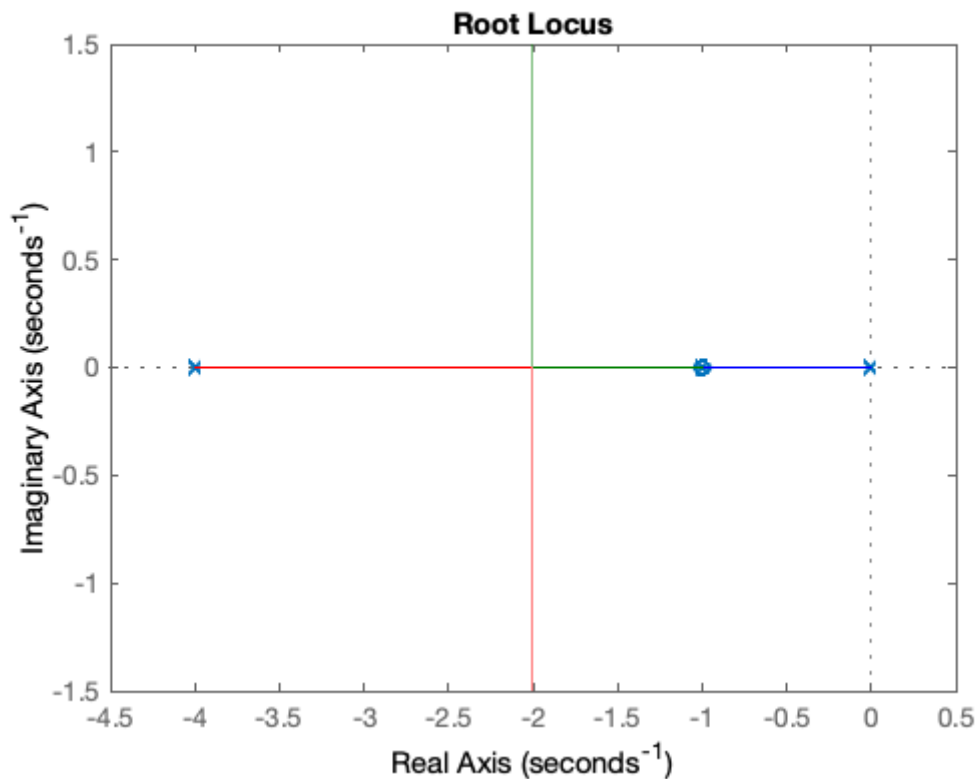
However, if we use the zero of a cascade lead compensator to cancel the pole at  $s = -1$  and place the pole at  $s = -4$  we get:

In [5]:

```
D1 = zpk([-1],[ -4],1);  
Go1 = D1*G1*H;
```

In [12]:

```
rlocus(Go1)
```



which will have a closed-loop pole at the desired location when the gain is

In [6]:

```
Kc = rlocfind(Go1, -2+2j)
```

```
Kc =  
8
```

## Example 2

For a Type 0 system

$$G_2(s) = \frac{1}{(s+1)(s+2)}$$

the zero should be used to cancel the pole at  $s = -2$ . We leave it as an exercise to prove that the compensator

$$D_2(s) = 2.5 \left( \frac{s+2}{s+3} \right)$$

gives the desired closed-loop poles.

### Note

You should be aware that the lead compensator zero will still appear in the closed-loop transfer function, and you should verify that the closed-loop step response is acceptable.

## Method 2

The following graphical method maximizes the ratio between pole and zero for any given angle contribution. This minimizes the additional compensator gain needed to satisfy the gain criterion.

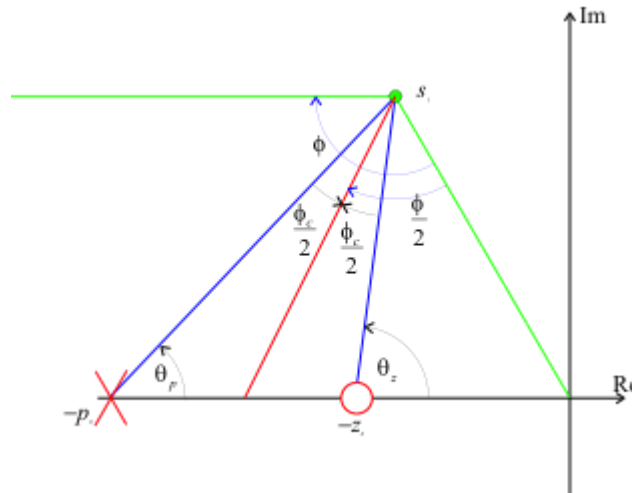


Figure 2 Graphical construction for locating the pole and zero of a lead compensator.

The steps in the location of the lead-compensator pole and zero are as follows (refer to Figure 2).

- Locate the desired closed-loop pole  $s_1$ . Draw a line from the origin to  $s_1$  and a horizontal line through  $s_1$  to the left.
- Bisect the angle between the two lines drawn in step 1.
- Measure the angle  $\phi_c$  either side of the line drawn in step 2.
- The intersections of these lines with the real axis locate the compensator pole  $p_0$  and zero  $z_0$ .

## Example 3

We return to the satellite attitude control problem with

$$G(s) = \frac{1}{s^2}$$

Requiring a closed-loop pole  $S_1 = -2 + j2$ , the geometry of the problem is illustrated in Figure 3.

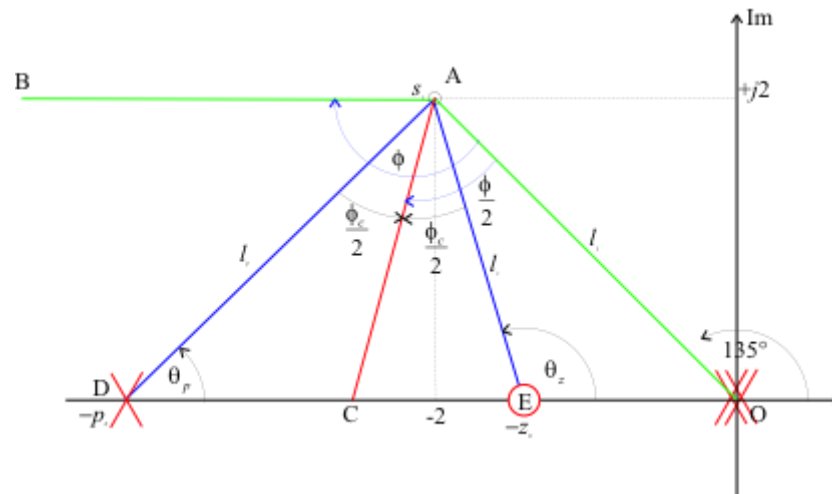


Figure 3 Lead compensator design for the satellite attitude control problem.

Note that the line drawn from the origin to the point  $s_1$  subtends an angle of  $135^\circ$  to the positive real axis.

We can use MATLAB to help to work through the trigonometry. The angle contribution of the plant and feedback at  $s_1$  is obtained as follows.

In [7]:

```
G = tf(1,[1,0,0]);
H = 1;
GH = G*H;
s1 = -2+2j;
```

The total contribution of the plant poles and zeros can be calculated directly using the Matlab equivalent of the angle criterion

In [8]:

```
[zeros,poles, gain]=zpkdata(GH,'v');
```

contribution in degrees

In [9]:

```
contrib = (180/pi)*(sum(angle(s1 - zeros)) - sum(angle(s1 - poles)))
contrib =
    -270
```

The root locus angle criterion gives lead contribution

$$\angle G(s_1)H(s_1) + \phi_c = -180^\circ$$

$$\phi_c = -180^\circ - \angle G(s_1)H(s_1)$$

In [10]:

```
phi_c = -180 - contrib
```

```
phi_c =  
90
```

In [11]:

```
half_phi_c = phi_c/2
```

```
half_phi_c =  
45
```

Because the line BA and OD are parallel, the angle subtended by the line OAB is also  $135^\circ$ . Thus

In [12]:

```
angle_OAB = 135;  
angle_BAD = angle_OAB/2 - half_phi_c;  
angle_BEO = angle_OAB/2 + half_phi_c;
```

and by parallel line theory

In [13]:

```
theta_p = angle_BAD
```

```
theta_p =  
22.5000
```

In [14]:

```
theta_z = angle_BEO
```

```
theta_z =  
112.5000
```

The pole and zero locations are given by

In [15]:

```
p0 = -2-2/tan(theta_p*pi/180)
```

```
p0 =  
-6.8284
```

In [16]:

```
z0 = -2-2/tan(theta_z*pi/180)
```

```
z0 =  
-1.1716
```

The compensator gain is obtained using the gain criterion. With MATLAB, this can be calculated directly from the gain formula:

$$K_0 = \frac{\left| \frac{(s_1 - p_0)}{(s - z_0)} \right|}{\left| \prod_{i=1}^{n-1} (s_1 - z_i) \right|}$$

$$\cdot \left| \prod_{j=1}^{n-1} (s_1 - z_j) \right|$$

In [17]:

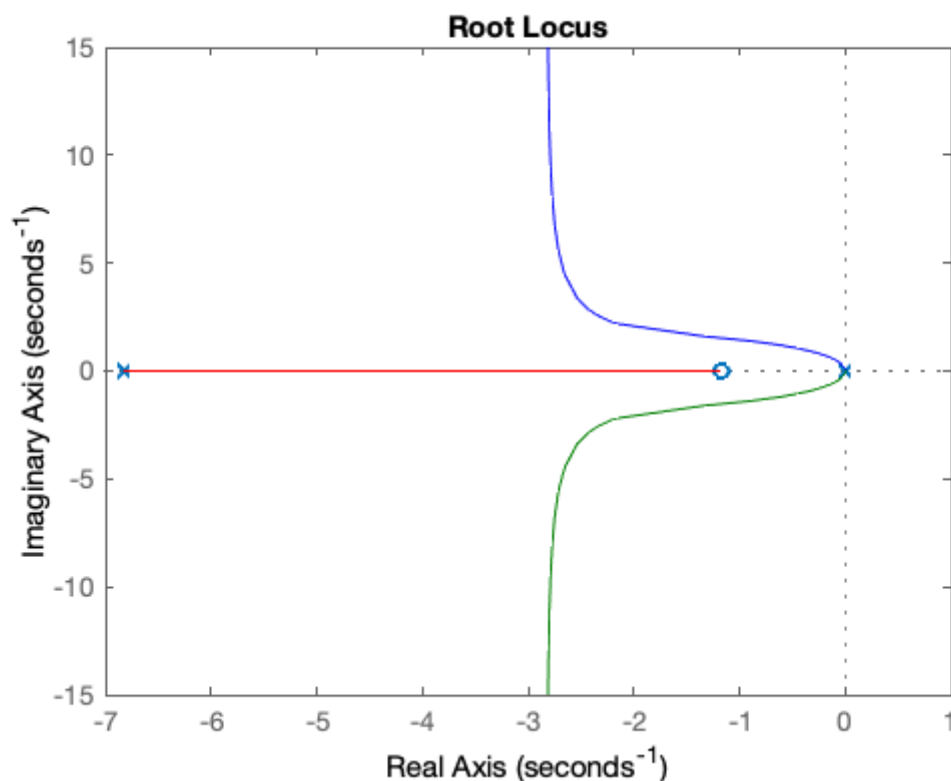
```
Ko = (abs(s1-p0)*prod(abs(s1-poles)))/(abs(s1-z0)*prod(abs(s1-zeros)))
```

```
Ko =  
19.3137
```

Let us also check this result using the root locus.

In [18]:

```
D = zpk(z0,p0,1);  
Go=D*GH;  
rlocus(Go)
```





In [19]:

```
Kc = rlocfind(Go,s1)
```

```
Kc =  
19.3137
```

Finally, let us calculate the step response and compare it with the result achieved with velocity feedback

$$G_1(s) = \frac{8}{s^2 + 4s + 8}$$

and proportional + derivative compensation

$$G_2(s) = \frac{4(s + 2)}{s^2 + 4s + 8}$$

In [20]:

```
G1 = tf(8,[1, 4, 8]);  
G2 = tf(4*[1, 2],[1, 4, 8]);  
G3 = feedback(Kc*D*G,H)
```

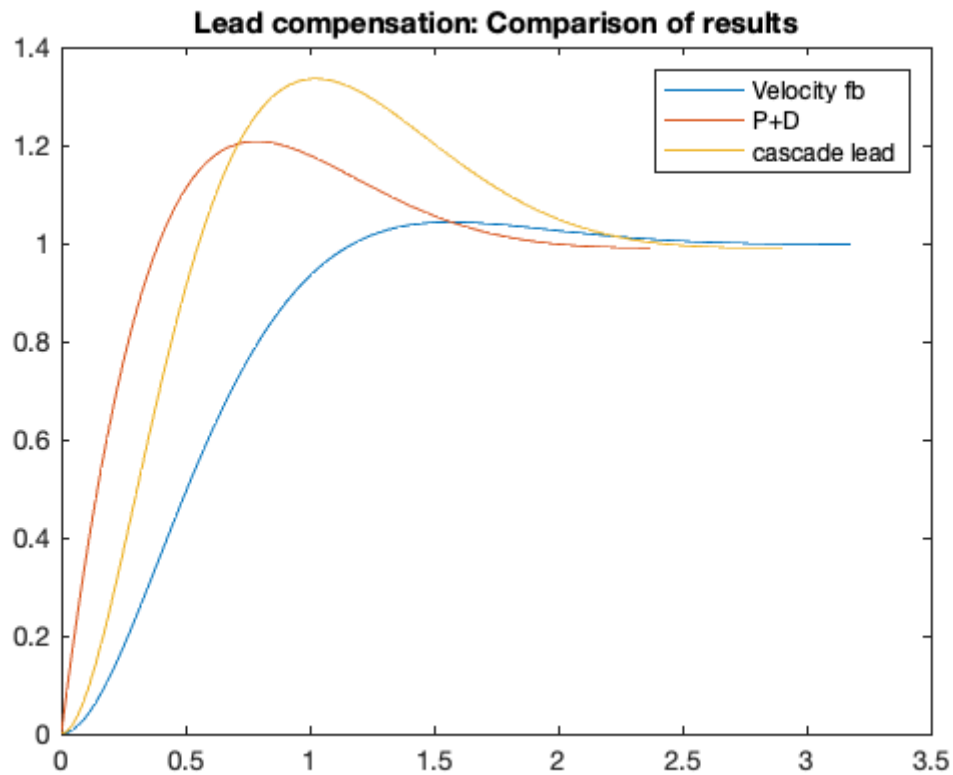
```
G3 =
```

$$\frac{19.314 (s+1.172)}{(s+2.828) (s^2 + 4s + 8)}$$

Continuous-time zero/pole/gain model.

In [36]:

```
[y1,t1]=step(G1);  
[y2,t2]=step(G2);  
[y3,t3]=step(G3);  
plot(t1,y1,t2,y2,t3,y3),legend('Velocity fb','P+D','cascade lead'),title('Lead c  
ompensation: Comparison of results')
```



When evaluating the third design you should take into account the location of the compensator zero and the third closed-loop pole (at  $s = -2.828$ ) relative to the desired closed-loop pole at  $s_1$ .

## Method 3

The third method referenced in D'Azzo and Houpis addresses a problem with lead compensator design that has so far not been addressed. That is that only the desired transient performance, and hence the desired location of the dominant closed-loop poles, is considered. The desired system gain is not specified. A method of achieving both gain and desired pole location has been proposed by Phillips and Harbour (1988) and is considered in the [Analytic Root Locus Design \(../5/analrloc\)](#) section (**not assessed**).

## References

John J. D'Azzo and Constantne Houpis, (1975) *Linear Control System Analysis and Design (Conventional and Modern)*, McGraw & Hill, 1975 and later editions.

Phillips and Harbor (1988), *Feedback Control Systems*, Prentice Hall.

## Resources

An executable version of this document is available to download as a MATLAB Live Script file [cclead.mlx](#) ([matlab/cclead.mlx](#)).

The Simulink model which compares the results of the satellite attitude control system compensated with velocity feedback, P+D compensation *and* lead compensation is [lead\\_compensation.slx](#) ([matlab/lead\\_compensation.slx](#)).