

# Time Response for State Space Models

In this section we shall determine the time response of a system represented by a state-space model. We shall take as a starting point the transformed state space model developed in the [last section \(../2/tf4ss\)](#).

It is possible to derive the time response directly in the time domain from the state-equations. I believe however, that the following development will be easier to understand. I advise you to consult any text book for the time domain development.

In the [last section \(../2/tf4ss\)](#), we showed that the state space model was

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

and that the transformed equation was

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

In this section we will show how the time response of the state space model may be derived from the transformed model.

## Zero Input Response

We first consider the response of the system to its initial conditions and zero input. This is the *zero input response* or homogeneous response of the system.

With the input transform  $U(s) = 0$  the state equation becomes

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{x}(0).$$

We define

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1}$$

so that

$$\mathbf{X}(s) = \Phi(s)\mathbf{x}(0)$$

is the transform of the zero input state response.

In the time domain, the corresponding state response is given by the inverse Laplace transform of equation (1)

$$\mathbf{x}(t) = \mathcal{L}^{-1} \{ \Phi(s)\mathbf{x}(0) \} ,$$

which, since the initial condition vector  $\mathbf{x}(0)$  is a vector of constants becomes:

$$\begin{aligned}\mathbf{x}(t) &= \mathcal{L}^{-1} \{ \Phi(s) \} \mathbf{x}(0) \\ &= \phi(t) \mathbf{x}(0).\end{aligned}$$

## State Transition Matrix

The vector of time functions  $\phi(t)$  is known as the *state transition matrix*. It defines the motion (or *trajectory*) of the state variables through the  $n$ -dimensional *state space* from a given set of initial states.

## Resolvent matrix

The *resolvent matrix* is defined as

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\text{adj}[s\mathbf{I} - \mathbf{A}]}{|s\mathbf{I} - \mathbf{A}|}.$$

The resolvent matrix is a matrix of rational polynomials in the Laplace transform variable  $s$ .

The state transition matrix is

## $\phi(t)$

$\mathcal{L}^{-1} \left\{ \frac{\text{adj}[s\mathbf{I} - \mathbf{A}]}{|s\mathbf{I} - \mathbf{A}|} \right\}$  What is the form of this function?

## Characteristic Polynomial

The adjoint matrix  $\text{adj}[s\mathbf{I} - \mathbf{A}]$  is a matrix of polynomials each of which is of order  $n - 1$  or less.

The determinant

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s - a_{nn} \end{vmatrix}$$

is called the *characteristic polynomial*. It is always a polynomial of degree  $n$ .

Therefore each element of the resolvent matrix  $\Phi(s)$  is of the form

$$\frac{b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}.$$

## System Characteristic (or Eigen) Values

The coefficients  $b_i$  of each element of the resolvent matrix  $\Phi(s)$  depend on the form of the system. The characteristic polynomial  $|s\mathbf{I} - \mathbf{A}|$  is the common denominator of all elements of  $\Phi(s)$ .

It may be factorized:

$$\begin{aligned} |s\mathbf{I} - \mathbf{A}| &= s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0 \\ &= (s - p_1)(s - p_2) \dots (s - p_n) \end{aligned}$$

The values  $p_i$  are known as the *characteristic* (or *eigen*) values of the state matrix  $\mathbf{A}$ . The characteristic values are the *poles* of the system!

## Zero-input response

If we make a partial fraction expansion of the characteristic equation, then for the  $i$ - $j$ th element of the resolvent matrix we have

$$\Phi_{ij}(s) = \frac{r_1}{s - p_1} + \frac{r_2}{s - p_2} + \dots + \frac{r_n}{s - p_n}$$

(where the coefficient  $r_i$  is the so called *residue* of the partial fraction expansion determined at the value  $s = p_i$ ).

Inverse Laplace transforming this equation we finally obtain the *state transition* function:

$$\phi_{ij}(t) = r_1 e^{p_1 t} + r_2 e^{p_2 t} + \dots + r_n e^{p_n t}.$$

## System 'modes'

The state transition matrix  $\phi(t)$  is an  $n \times n$  matrix, each element of which is a linear combination of the  $n$  "mode functions" (or simply *modes*)

$$e^{p_1 t}, e^{p_2 t}, \dots, e^{p_n t}.$$

Each mode is constructed from the eigenvalues ( $p_1, p_2, \dots, p_n$ ) of the state matrix  $\mathbf{A}$ .

The eigenvalues are also known as the *poles* of the system.

## Example

If

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 \\ 6 & s + 5 \end{bmatrix}$$

so

$$|s\mathbf{I} - \mathbf{A}| = s^2 + 5s + 6 = (s + 2)(s + 3).$$

The characteristic values of the system are  $p_1 = -2$  and  $p_2 = -3$  so the modes are  $e^{-2t}$  and  $e^{-3t}$ .

Now

$$\text{adj}[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

so the resolvent matrix is

$$\Phi(s) = \begin{bmatrix} \frac{s+5}{(s+2)(s+3)} & \frac{1}{(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} & \frac{s}{(s+2)(s+3)} \end{bmatrix}.$$

Expanding each element of  $\Phi(s)$  we get

$$\Phi(s) = \begin{bmatrix} \frac{3}{s+2} - \frac{2}{s+3} & \frac{1}{s+2} - \frac{1}{s+3} \\ \frac{-6}{s+2} + \frac{6}{s+3} & -\frac{2}{s+2} + \frac{3}{s+3} \end{bmatrix}.$$

Inverse Laplace transforming this we get the final state transition matrix, representing the zero-input response of the system:

$$\phi(t) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}.$$

Note that, by an alternative derivation<sup>1</sup>, we can show that

$$\phi(t) = e^{\mathbf{A}t}$$

where  $e^{\mathbf{A}t}$  is called the matrix exponential.

## Unforced system response

To determine the response of the system to the initial state we recall that

$$\mathbf{x} = \mathbf{B}\Phi(s)\mathbf{x}(0)$$

and putting this together with the definition of  $\mathbf{Y}(s)$  we can determine the unforced system response of a system.

## Zero Input System Response

For non-zero initial conditions and zero input we have

$$\begin{aligned} \mathbf{X}(s) &= [s\mathbf{I} - \mathbf{A}]^{-1} \mathbf{x}(0) \\ &= \Phi(s)\mathbf{x}(0) \end{aligned}$$

The zero-input system output transform is therefore given by

$$\mathbf{Y}_{zi}(s) = \mathbf{C}\Phi(s)\mathbf{x}(0).$$

Because we have already shown that  $\phi(t)$  will be a linear combination of modes, the unforced system response  $y(t)$  will be another linear combination of the same modes.

We now consider the case where the initial condition vector  $\mathbf{x}(0) = \mathbf{0}$  and we will obtain the response of the system to an input. This is called the *zero state response* of the system.

## The Zero State Response

For zero initial conditions we have

$$\begin{aligned}\mathbf{X}(s) &= [s\mathbf{I} - \mathbf{A}]^{-1} \mathbf{B}\mathbf{U}(s) \\ &= \Phi(s)\mathbf{B}\mathbf{U}(s)\end{aligned}$$

The system output transform (forced response) is given by

$$\mathbf{Y}_{zs}(s) = [\mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}] \mathbf{U}(s)$$

(where  $\mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}$  is the system transfer matrix).

## Full System Response

The full system response for the state-space model is simply the sum of the zero-state and zero-input responses:

$$\begin{aligned}\mathbf{Y}_{\text{full}}(s) &= \mathbf{Y}_{zs}(s) + \mathbf{Y}_{zi}(s) \\ &= \mathbf{C}\Phi(s) [\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)] + \mathbf{D}\mathbf{U}(s).\end{aligned}$$

The full system response for the state-space model is simply the sum of the zero-state and zero-input responses:

$$\begin{aligned}\mathbf{Y}_{\text{full}}(s) &= \mathbf{Y}_{zs}(s) + \mathbf{Y}_{zi}(s) \\ &= \mathbf{C}\Phi(s) [\mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)] + \mathbf{D}\mathbf{U}(s).\end{aligned}$$

In the class we will reinforce these concepts by working through an example in detail.

## Example

Let

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{u} \\ \mathbf{y} &= \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \mathbf{x} \\ \mathbf{u} &= \begin{bmatrix} e^{-t} \\ \epsilon(t) \end{bmatrix}.\end{aligned}$$

Calculate the full system response for this system given that the initial conditions are  $\mathbf{x}(0) = [-1, 1]^T$ .

## Solution to Example

The transfer function is

$$\begin{aligned}\mathbf{G}(s) &= \mathbf{C}\Phi(s)\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+5}{(s+2)(s+3)} & \frac{1}{(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} & \frac{s}{(s+2)(s+3)} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2s+9}{(s+2)(s+3)} & \frac{s+5}{(s+2)(s+3)} \\ \frac{-s-12}{(s+2)(s+3)} & \frac{-6}{(s+2)(s+3)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{5s+30}{(s+2)(s+3)} & \frac{2s+16}{(s+2)(s+3)} \\ \frac{-s-12}{(s+2)(s+3)} & \frac{-6}{(s+2)(s+3)} \end{bmatrix}\end{aligned}$$

The transform of the input vector is

$$\mathbf{U}(s) = \mathcal{L} \begin{bmatrix} e^{-t} \\ \epsilon(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s} \end{bmatrix}.$$

Hence

$$\begin{aligned}\mathbf{Y}(s) &= \begin{bmatrix} \frac{5s+30}{(s+2)(s+3)} & \frac{2s+16}{(s+2)(s+3)} \\ \frac{-s-12}{(s+2)(s+3)} & \frac{-6}{(s+2)(s+3)} \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s} \end{bmatrix} \\ &= \begin{bmatrix} \frac{7s^2+48s+16}{s(s+1)(s+2)(s+3)} \\ \frac{-(s^2+18s+6)}{s(s+1)(s+2)(s+3)} \end{bmatrix} = \begin{bmatrix} \frac{8/3}{s} + \frac{25/2}{s+1} - \frac{26}{s+2} + \frac{65/6}{s+3} \\ -\frac{1}{s} - \frac{11/2}{s+1} + \frac{13}{s+2} - \frac{13/2}{s+3} \end{bmatrix}\end{aligned}$$

Inverse Laplace transforming the previous result gives the zero-state output response of the system:

$$\begin{bmatrix} \frac{8}{3}\epsilon(t) + \frac{25}{2}e^{-t} - 26e^{-2t} + \frac{65}{6}e^{-3t} \\ -\epsilon(t) - \frac{11}{2}e^{-t} + 13e^{-2t} - \frac{13}{2}e^{-3t} \end{bmatrix}.$$

Note that response is now a linear combination of the system modes ( $e^{-2t}$  and  $e^{-3t}$ ) and the input modes ( $\epsilon(t)$  [the unit step function] and  $e^{-t}$ ).

If the initial condition vector  $\mathbf{x}(0) = [-1, 1]^T$  what is the total response of the system?

$$\begin{aligned}\mathbf{Y}(s) &= \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+5}{(s+2)(s+3)} & \frac{1}{(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} & \frac{s}{(s+2)(s+3)} \end{bmatrix} \\ &\quad \times \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{s} \end{bmatrix} \right\} \\ &= \begin{bmatrix} \frac{2s+16}{(s+2)(s+3)} & \frac{-s+2}{(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} & \frac{s}{(s+2)(s+3)} \end{bmatrix} \begin{bmatrix} \frac{-s^2+2s+1}{s(s+1)} \\ \frac{s^2}{s(s+1)} \end{bmatrix}\end{aligned}$$

$$\mathbf{Y}(s) = \begin{bmatrix} \frac{-3s^2 - 10s^2 + 34s + 16}{s(s+1)(s+2)(s+3)} \\ \frac{s^3 + 6s^2 - 12s - 6}{s(s+1)(s+2)(s+3)} \end{bmatrix}$$

$$\left[ \frac{8}{s} e(t) + \frac{25}{s} e^{-t} - 34e^{-2t} + \frac{59}{s} e^{-3t} \right]$$

## Solving this problem in MATLAB

We can use the symbolic toolbox!

Set up the problem ... we can use ordinary matrices for this

In [55]:

```
format compact
A = [0, 1; -6, -5]; B = [2, 1; -1, 0];
C = [2, -1; 0, 1]; D = [0, 0; 0, 0];
x0 = [-1, 1]';
```

Now define  $s$  and  $t$  as symbolic variables

In [56]:

```
syms t s
```

Define input vector  $\mathbf{U}(s)$

In [57]:

```
u = [exp(-t); heaviside(t)];
Us = laplace(u)
```

```
Us =
 1/(s + 1)
 1/s
```

Compute  $\Phi(s)$

In [58]:

```
PhiS = inv(s*eye(2) - A)

PhiS =
[ (s + 5)/(s^2 + 5*s + 6), 1/(s^2 + 5*s + 6) ]
[ -6/(s^2 + 5*s + 6), s/(s^2 + 5*s + 6) ]
```

Now compute zero state response:

$$\mathbf{Y}_{zs}(s) = (\mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D})$$



In [59]:

```
Yzs = (C*PhiS*B + D)*Us
```

```
Yzs =
(s/(s^2 + 5*s + 6) + 10/(s^2 + 5*s + 6) + (4*(s + 5))/(s^2 + 5*s +
6))/(s + 1) + (6/(s^2 + 5*s + 6) + (2*(s + 5))/(s^2 + 5*s + 6))/s
- 6/(s*(s^2 + 5*s + 6)) - (s/(s^2 + 5*s + 6) + 12/(s^2 + 5*s + 6))/(s + 1)
```

Unforced (zero-initial condition) response:

$$\mathbf{Y}_{zi}(s) = (\mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D})$$

In [60]:

```
Yzi = C*PhiS*x0
```

```
Yzi =
- s/(s^2 + 5*s + 6) - 4/(s^2 + 5*s + 6) - (2*(s + 5))/(s^2 + 5*s +
6)
s/(s^2 + 5*s + 6) + 6/(s^2 + 5*s +
6)
```

Total response:

$$\mathbf{Y}(s) = \mathbf{Y}_{zs}(s) + \mathbf{Y}_{zi}(s)$$

In [61]:

```
Ytotal = Yzs + Yzi
```

```
Ytotal =
(s/(s^2 + 5*s + 6) + 10/(s^2 + 5*s + 6) + (4*(s + 5))/(s^2 + 5*s +
6))/(s + 1) - 4/(s^2 + 5*s + 6) - s/(s^2 + 5*s + 6) - (2*(s + 5))/(s
^2 + 5*s + 6) + (6/(s^2 + 5*s + 6) + (2*(s + 5))/(s^2 + 5*s + 6))/s
s/(s^2 + 5*s + 6) + 6/(s^2 + 5*s + 6) - 6/(s*(s^2 + 5*s + 6)) - (s/
(s^2 + 5*s + 6) + 12/(s^2 + 5*s + 6))/(s + 1)
```

Finally compute the time response  $y(t)$ 

In [62]:

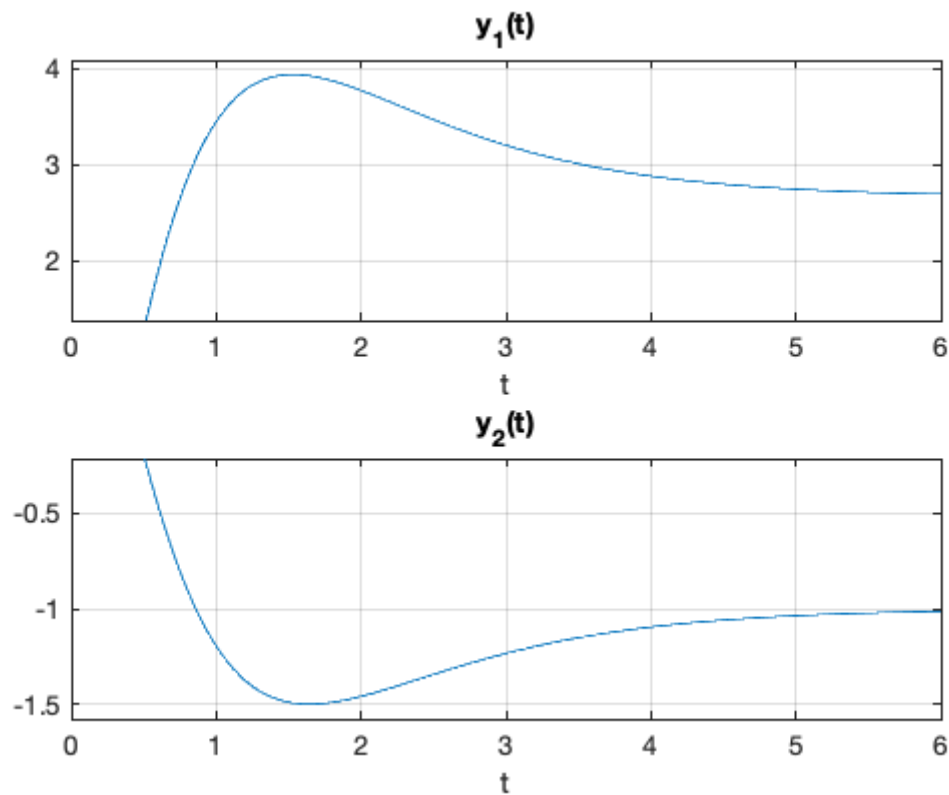
```
y = ilaplace(Ytotal)
```

```
y =
(25*exp(-t))/2 - 34*exp(-2*t) + (95*exp(-3*t))/6 + 8/3
17*exp(-2*t) - (11*exp(-t))/2 - (19*exp(-3*t))/2 - 1
```

Plot response

In [69]:

```
subplot(211)
ezplot(y(1),[0,6]),grid,title('y_1(t)')
subplot(212)
ezplot(y(2),[0,6]),grid,title('y_2(t)')
```



## Simulation of State Space Models in Matlab

- The functions `step` and `impz` will produce step impulse responses of a system with zero initial conditions.
- The function `lsim` can be used to determine the state space response to arbitrary inputs.
- If you need to simulate a system with arbitrary inputs and predefined initial conditions, you should use the *Linear Simulation Tool* which is opened when you execute `lsim(model)`.

Visit the MATLAB Control Systems Toolbox help pages to find out more.

## Footnote

1. For a first order differential equation

$$\frac{dx}{dt} = ax$$

so the Laplace transform is

$$sX(s) - x(0) = aX(s)$$

$$X(s)(s - a) = x(0)$$

$$X(s) = \frac{x(0)}{s - a}.$$

The inverse laplace transform of the final equation gives

$$x(t) = e^{at}.$$

For the vector case

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$$

so by comparison

$$\mathbf{x}(t) = e^{\mathbf{A}t}.$$

The proper proof is of course a little more involved and will be explored in [Section 7.6](#) ([../5/gensolution](#))!