Random Harmonic Series, Byron Schmuland

This paper by Schmuland [1] investigates the distribution and density of the a.s. limit X of the harmonic random series. Schmuland computes a lower bound on the probability that the series lands within any arbitrary interval of the real line, guaranteeing that the series takes on any real value. To study the density function of X, the author uses the fact that $\sum_{j=1}^{\infty} \frac{\epsilon_j}{2^j}$ has uniform distribution on $[-1,1]$. This is because for the sum where the ϵ_j are replaced with fair Bernoulli, the values of the ϵ_j are essentially choices of the jth base-2 digit (after the decimal point) of a number in $[0,1]$.

Using this fact, the author uses an essential trick to decompose the infinite sum of $\frac{\epsilon_j}{j}$ into sums

$$U_j = \sum_{i=0}^{\infty} \frac{\epsilon_{(2j+1)2^i}}{(2j+1)2^i} = \frac{2}{2j+1} \sum_{j=1}^{\infty} \frac{\epsilon_{(2j+1)2^{i-1}}}{2^i}$$

which again gives us the uniform random variable infinite sum of the right, so that each U_j is a scaled uniform random variable on a smaller interval. It is important that the U_j are independent. Moreover, X can now be approximated by finitely many of the U_j, $X \approx U_0 + \ldots + U_n$, and this approximation converges in L^2 to X. The partial sums of the U_j are a convergent martingale, so the limit of the sum of the U_j is in fact X almost surely, since it must agree with the L^2 convergence.

This decomposition allows the author to study properties of the density of X. He finds that the density is smooth since all of the moments of X exist (as bounded by partial products of the characteristic functions of U_j). Also, by using convolutions of the densities of the partial sums of U_j, he shows that the these densities converge uniformly to the density of X. Then he is able to prove strict bounds on the density at the value 0 and 2—taking limits of the partial sum densities.

Below I explore distributions of some random variables that are limits of related random series and test some numerical computations of densities. I have found this paper very interesting; the various tools that the author uses to prove multiple results about the distribution and density of X are fascinating. In this case, the random variable is not defined from a distribution or density, so that creative techniques have to be used to deduce properties about it.

Investigation of $\sum_{j=1}^{\infty} \frac{\xi_j}{3^j}$

We investigate the distribution of $\sum_{j=1}^{\infty} \frac{\xi_j}{3^j}$. First, we know that the sequence of partial sums converges almost surely to some random variable X, by the martingale convergence theorem. We see that the magnitude of the first 4 values of the sum are

$$\frac{1}{3} = .3333\ldots \quad \frac{1}{9} = .1111\ldots \quad \frac{1}{27} = .0370\ldots \quad \frac{1}{81} = .0123\ldots$$

the decay is quick and thus the first few terms for the most part determine the value of X. Unlike the random harmonic series, this series very clearly does not have a distribution with full support on the real line. $|X|$ is bounded above by $1/2$ since the sum of the magnitudes is geometric

$$|X| \leq \sum_{j=1}^{\infty} \frac{1}{3^j} = \frac{1}{2}$$

and it is bounded below by $\frac{1}{6}$, since the first summand has magnitude $\frac{1}{3}$, but the remaining terms have magnitude $\sum_{j=2}^{\infty} \frac{1}{3^j} = \frac{1}{6}$. In fact, these bounds continue down with j, after the nth step, the sum will never return back to $(S_{n-1} - \frac{1}{2 \cdot 3^n}, S_{n-1} + \frac{1}{2 \cdot 3^n})$, where S_k is the kth partial sum and $S_0 = 0$. To see this, we suppose that $\epsilon_n = 1$, and note that for $k \geq n$, we have

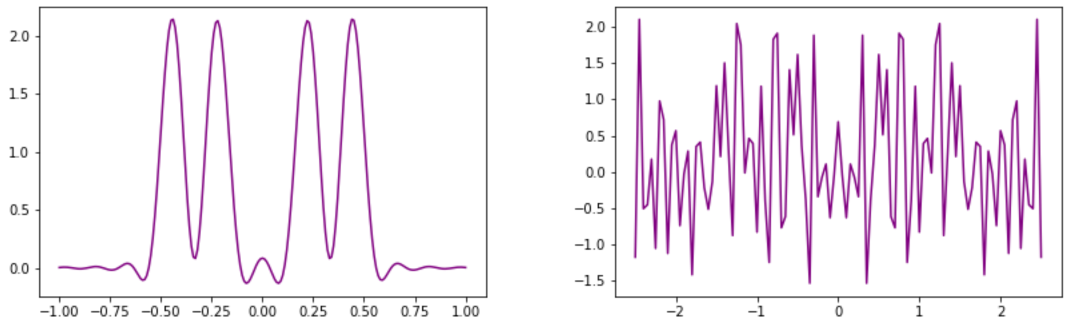
$$\begin{aligned} S_k &= S_{n-1} + \frac{1}{3^n} + \sum_{j=n+1}^{k} \frac{\epsilon}{3^j} \\ &> S_{n-1} + \frac{1}{3^n} + \sum_{j+n+1}^{\infty} \frac{-1}{3^j} \\ &= S_{n-1} + \frac{1}{3^n} - \frac{1/3^{n+1}}{1 - 1/3} \\ &= S_{n-1} + \frac{1}{2 \cdot 3^n} \end{aligned}$$

Thus, the support of the distribution of X is on a Cantor set of some sort. This distribution is not absolutely continuous with respect to the Lebesgue measure and it does not have anything close to continuous density. Anyways, for curiosity's sake we numerically integrate the inversion formula (which clearly does not apply here) using the exact characteristic function of the limit, following Morrison's *Random Walks with Decreasing Steps*: [3]

$$\frac{1}{\pi} \int_0^{\infty} \cos(xt) \prod_{j=1}^{\infty} \cos\left(\frac{t}{3^j}\right) dt$$

Using numerical integration, we truncate the number of terms in the product to about 50 (the terms in the product converge to 1 quite quickly for the $t \in [0, 45]$ that we integrate over).

Note that there are 4 peaks in the left plot, which correspond to each of the 4 possible combinations of the first two summands. These terms indeed dominate the value of X. We use normal double-precision floating point arithmetic on the left. When integrating instead in high-precision arithmetic with high-precision integration techniques, the result is erratic. Compare the approximation with that of the numerically computed value of $\sum \frac{\epsilon_j}{j^2}$, which has a smooth density. The numerical integration of the latter has proper, reasonable support, with low approximation error.



(a) Standard precision

(b) High precision

Figure 1: Approximate "density" of $\sum \frac{\epsilon_j}{3^j}$. The plot on the left is computed in standard precision and numerical integration along a grid in numpy. The plot on the right uses high precision and precise integration techniques in mpmath.

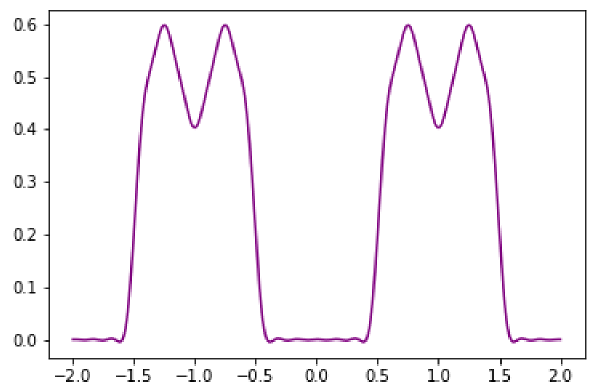


Figure 2: Density of $\sum \frac{\epsilon_j}{j^2}$ as numerically computed by the inversion formula with standard precision.

References

[1] Byron Schmuland. *Random Harmonic Series*. `http://www.stat.ualberta.ca/people/schmu/preprints/rhs.pdf`

[2] Kent Morrison. *Cosine Products, Fourier Transforms, and Random Sums*. `https://web.calpoly.edu/~kmorriso/Research/cosine.pdf`

[3] Kent Morrison. *Random Walks with Decreasing Steps*. `https://web.calpoly.edu/~kmorriso/Research/RandomWalks.pdf`