A Nonlinear Schrödinger Wave Equation With Linear Quantum Behavior

Chris D. Richardson, Peter Schlagheck, John Martin, Nicolas Vandewalle, and Thierry Bastin

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arXiv:1403.2177 [quant-ph]

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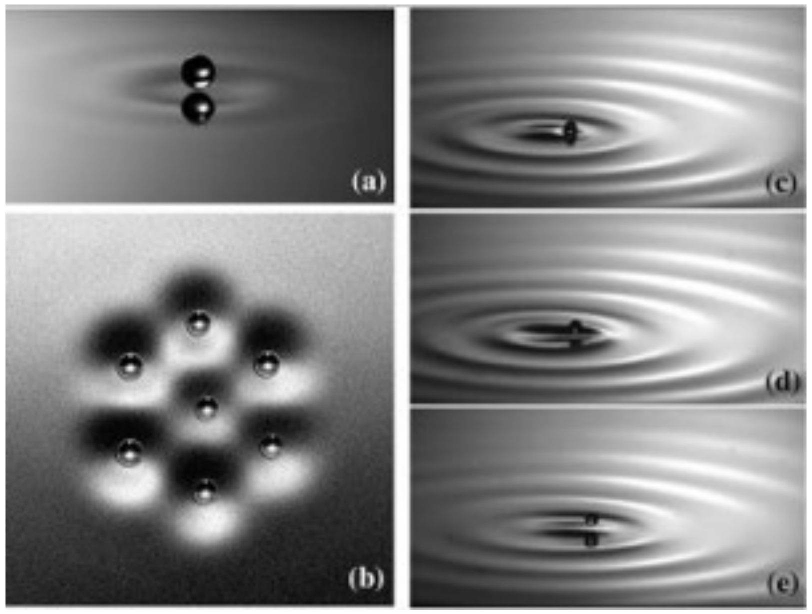
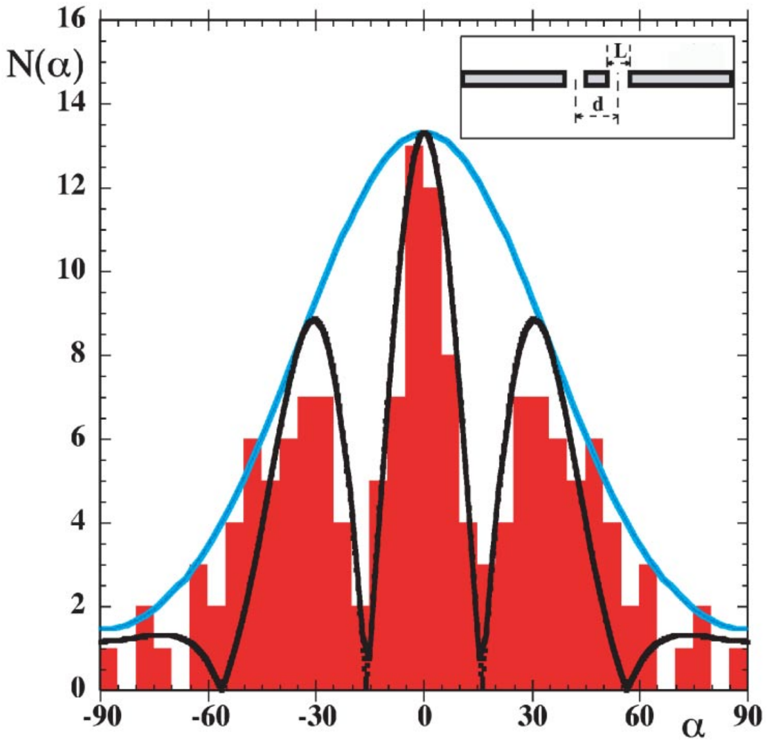
Motivation from exploring the quantum classical transition.





- Completely classical system.
- Exhibits behavior similar that is usually only seen in quantum mechanics.
- How does one describe a transition?

S. Protière, A. Boudaoud, and Y. Couder, *Particle-wave association on a fluid interface*, J. Fluid. Mech. **554**, 85 (2006).
Y. Couder and E. Fort, Single-Particle Diffraction and Interference at a Macroscopic Scale, Phys. Rev. Lett. 97, 154101 (2006).

What's this about?

Linear quantum behavior can be encountered in a non-linear wave equation.

- To show this we first express linear quantum mechanics and non-linear classical mechanics in a unified language.
- We define an equation that transitions between the two regimes.
- We show, both analytically and numerically, that this equation is equivalent to a linear scaled Schrödinger equation.

Quantum Mechanics in Classical Language

Quantum Language	Classical Language
Polar Form of the Wavefunction: $\psi(\mathbf{r},t) = A(\mathbf{r},t)e^{iS(\mathbf{r},t)/\hbar}$	

E. Madelung, Z Phys **40**, 322326 (1926).
D. Bohm, Phys. Rev. **85**, 166 (1952).

Quantum Mechanics in Classical Language

Quantum Language	Classical Language
Polar Form of the Wavefunction: $\psi(\mathbf{r},t) = A(\mathbf{r},t)e^{iS(\mathbf{r},t)/\hbar}$ $\downarrow$ Schrödinger Equation: $i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r},t)\psi(\mathbf{r},t)$	

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Quantum Mechanics in Classical Language

Quantum Language	Classical Language

Polar Form of the Wavefunction:

$$\psi(\mathbf{r},t) = A(\mathbf{r},t)e^{iS(\mathbf{r},t)/\hbar}$$

Schrödinger Equation:

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r},t)\psi(\mathbf{r},t)$$

$$\frac{\partial S(\mathbf{r},t)}{\partial t} = -\frac{1}{2m}\left[\nabla S(\mathbf{r},t)\right]^2 - \left[V(\mathbf{r},t) + U(\mathbf{r},t)\right]$$

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Quantum Mechanics in Classical Language

Quantum Language

Polar Form of the Wavefunction:

$$\psi(\mathbf{r},t) = A(\mathbf{r},t)e^{iS(\mathbf{r},t)/\hbar}$$

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$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r},t)\psi(\mathbf{r},t)$$

Classical Language

Hamilton-Jacobi Equation:

$$\frac{\partial S(\mathbf{r},t)}{\partial t} = -\frac{1}{2m}\left[\nabla S(\mathbf{r},t)\right]^2 - V(\mathbf{r},t)$$

$$\frac{\partial S(\mathbf{r},t)}{\partial t} = -\frac{1}{2m}\left[\nabla S(\mathbf{r},t)\right]^2 - \left[V(\mathbf{r},t) + U(\mathbf{r},t)\right]$$

Bohm's Quantum Potential:

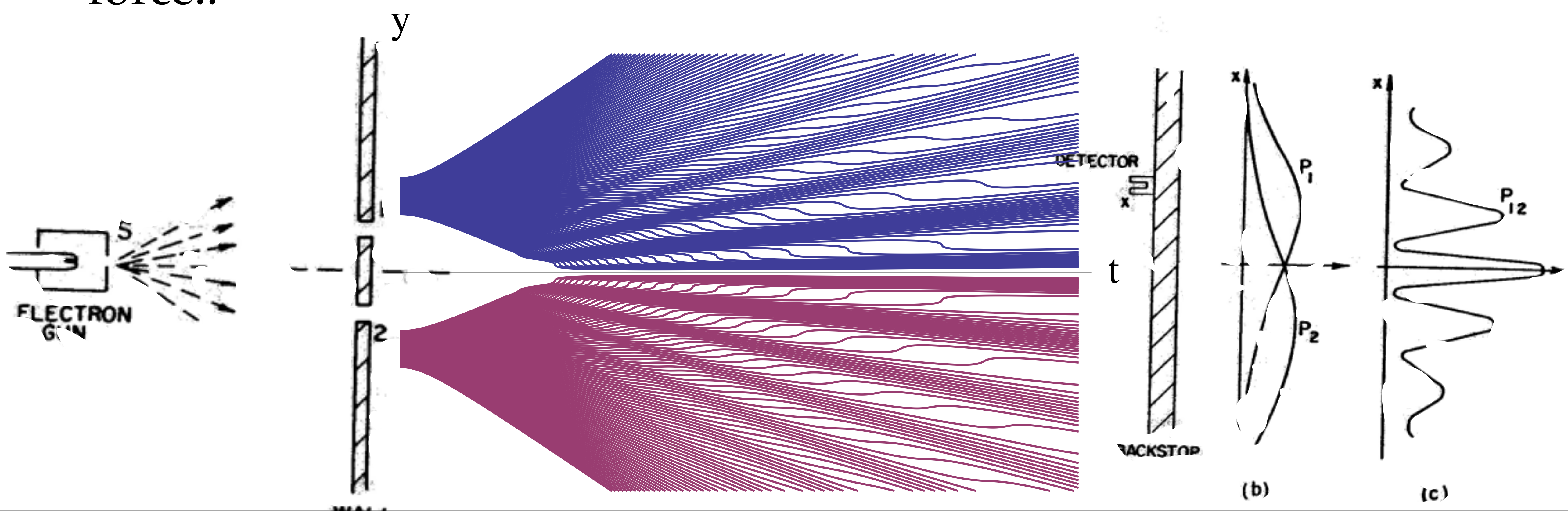
$$\boxed{U(\mathbf{r},t) = -\frac{\hbar^2}{2m}\frac{\nabla^2 A(\mathbf{r},t)}{A(\mathbf{r},t)} = -\frac{\hbar^2}{2m}\frac{\nabla^2\left|\psi(\mathbf{r},t)\right|}{\left|\psi(\mathbf{r},t)\right|}}$$

E. Madelung, Z Phys **40**, 322326 (1926).
D. Bohm, Phys. Rev. **85**, 166 (1952).

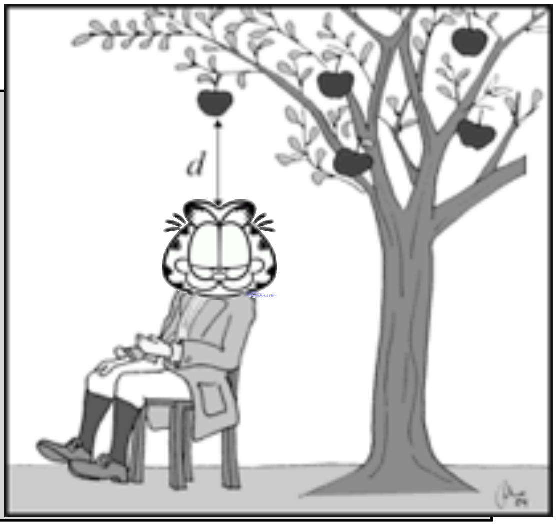
Bohmian Interpretation

$$m\ddot{\mathbf{r}}(t) = -\nabla\left[V(\mathbf{r},t) + U(\mathbf{r},t)\right] = -\nabla\left[V(\mathbf{r},t) - \frac{\hbar^2}{2m}\frac{\nabla^2|\psi(\mathbf{r},t)|}{\psi(\mathbf{r},t)}\right]$$

- Quantum dynamics from a classical force.
- Interference between two wave packets can be simulated using this force..



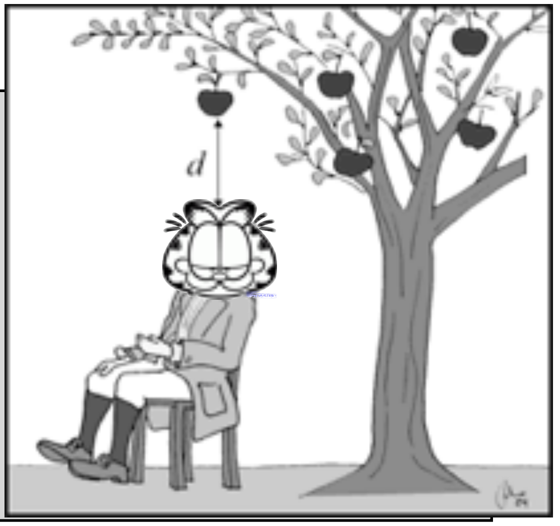
Classical Mechanics in Quantum Language



Quantum Language	Classical Language
	Initial Uncertain Positions of an Ensemble of Particles: $A_c^2(\mathbf{r},0)\,\mathrm{d}^3r$ Initial Classical Action: $S_c(\mathbf{r},0)$

X. Oriols and J. Mompart, *Applied Bohmian Mechanics: From Nanoscale Systems to Cosmology* (Editorial Pan Stanford Publishing Pte. Ltd, 2012) Chap. 1, pp. 15–147.

Classical Mechanics in Quantum Language



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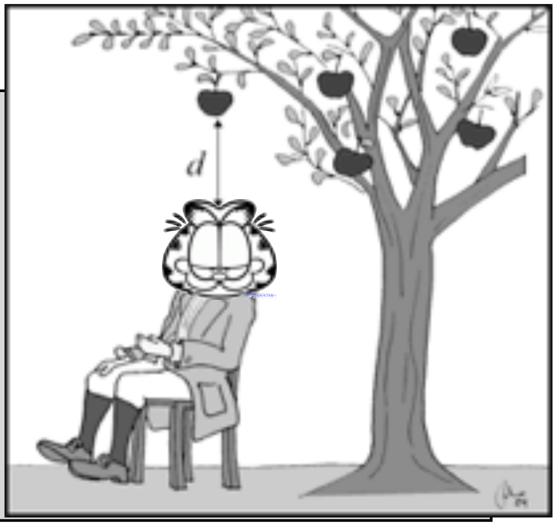
Classical Mechanics in Quantum Language



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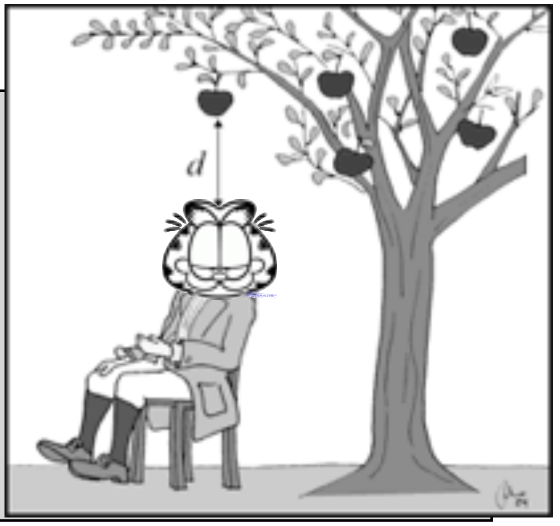
Classical Mechanics in Quantum Language



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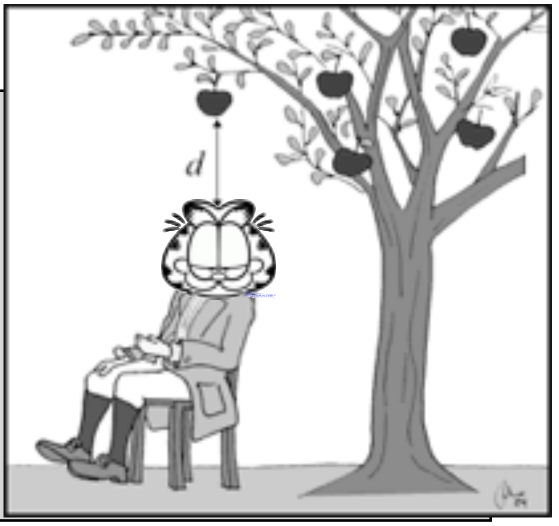
Classical Mechanics in Quantum Language



Quantum Language	Classical Language
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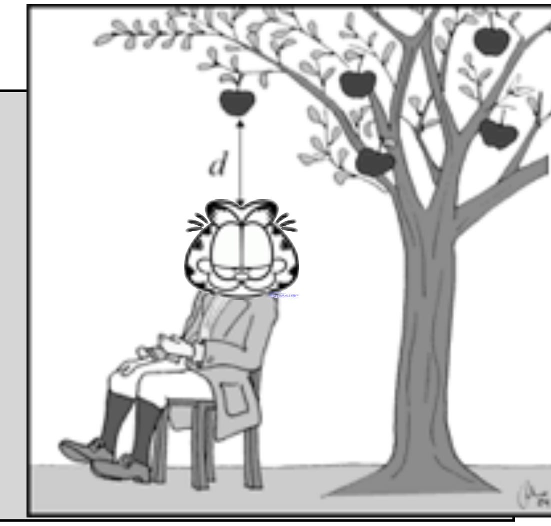
Classical Mechanics in Quantum Language



Quantum Language

Classical

Polar Form of the Wavefunction:

$$\psi_c(\mathbf{r},t) = A_c(\mathbf{r},t)e^{iS_c(\mathbf{r},t)/\hbar}$$

Classical Language

Positions of an Ensemble of Particles:

$$A_c^2(\mathbf{r},t)\,\mathrm{d}^3r$$

Classical Action:

$$S_c(\mathbf{r},t)$$

Local Conservation Law

$$\frac{\partial A_c^2(\mathbf{r},t)}{\partial t} + \nabla\left(\nabla S_c(\mathbf{r},t)A_c^2(\mathbf{r},t)\right) = 0$$

Hamilton-Jacobi Equation:

$$\frac{\partial S_c(\mathbf{r},t)}{\partial t} = -\frac{1}{2m}\left[\nabla S_c(\mathbf{r},t)\right]^2 - V(\mathbf{r},t)$$

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Classical Mechanics in Quantum Language

Quantum Language
Classical Polar Form of the Wavefunction: $\psi_c(\mathbf{r},t) = A_c(\mathbf{r},t) e^{iS_c(\mathbf{r},t)/\hbar}$ Classical Wave Equation: $i\hbar \frac{\partial \psi_c(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi_c(\mathbf{r},t) + V(\mathbf{r},t)\psi_c(\mathbf{r},t) + \frac{\hbar^2}{2m}\frac{\nabla^2 \lvert\psi_c(\mathbf{r},t)\rvert}{\lvert\psi_c(\mathbf{r},t)\rvert}\psi_c(\mathbf{r},t)\ ,$

Classical Language
Positions of an Ensemble of Particles: $A_c^2(\mathbf{r},t)\,\mathrm{d}^3 r$ Classical Action: $S_c(\mathbf{r},t)$ Local Conservation Law $\frac{\partial A_c^2(\mathbf{r},t)}{\partial t} + \nabla\left(\nabla S_c(\mathbf{r},t) A_c^2(\mathbf{r},t)\right) = 0$ Hamilton-Jacobi Equation: $\frac{\partial S_c(\mathbf{r},t)}{\partial t} = -\frac{1}{2m}\left[\nabla S_c(\mathbf{r},t)\right]^2 - V(\mathbf{r},t)$

X. Oriols and J. Mompart, *Applied Bohmian Mechanics: From Nanoscale Systems to Cosmology* (Editorial Pan Stanford Publishing Pte. Ltd, 2012) Chap. 1, pp. 15–147.

Classical Wave Equation Behavior

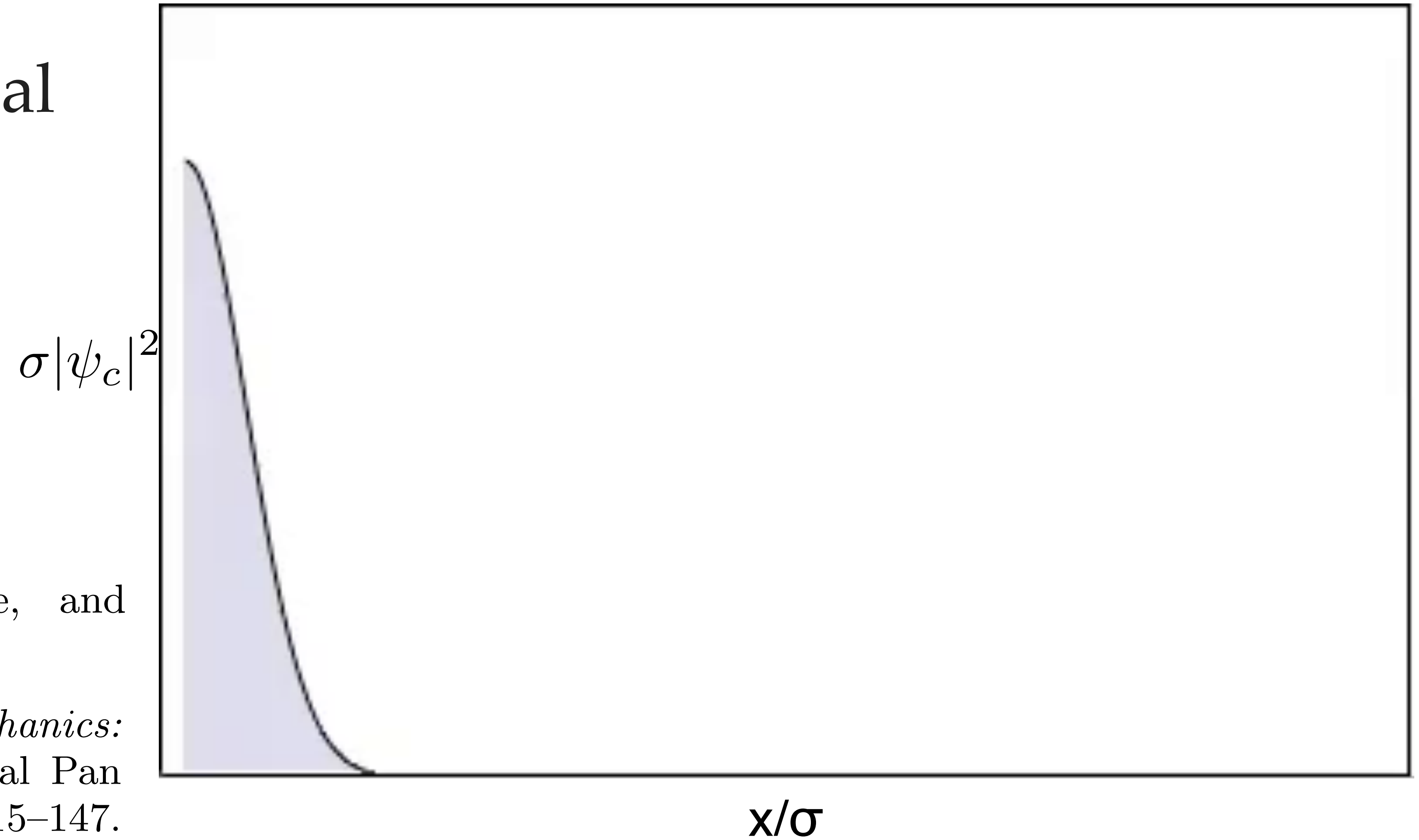
$$i\hbar \frac{\partial \psi_c(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_c(\mathbf{r},t) + V(\mathbf{r},t)\psi_c(\mathbf{r},t) + \frac{\hbar^2}{2m}\frac{\nabla^2|\psi_c(\mathbf{r},t)|}{|\psi_c(\mathbf{r},t)|}\psi_c(\mathbf{r},t) \; ,$$

Bohm's Quantum Potential

$$-\frac{\hbar^2}{2m}\frac{\nabla^2|\psi_c(\mathbf{r},t)|}{|\psi_c(\mathbf{r},t)|}$$

Classicality Enforcing Potential

$$\frac{\hbar^2}{2m}\frac{\nabla^2|\psi_c(\mathbf{r},t)|}{|\psi_c(\mathbf{r},t)|}$$



W. P. Schleich, D. M. Greenberger, D. H. Kobe, and M. O. Scully, PNAS **110**, 5374 (2013).

X. Oriols and J. Mompart, *Applied Bohmian Mechanics: From Nanoscale Systems to Cosmology* (Editorial Pan Stanford Publishing Pte. Ltd, 2012) Chap. 1, pp. 15–147.

Linearity == Quantum Mechanics

$$i\hbar\frac{\partial\psi_c(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_c(\mathbf{r},t) + V(\mathbf{r},t)\psi_c(\mathbf{r},t)$$
$$\cancel{+\frac{\hbar^2}{2m}\frac{\nabla^2|\psi_c(\mathbf{r},t)|}{|\psi_c(\mathbf{r},t)|}}\psi_c(\mathbf{r},t)\ ,$$

- Schleich et al. argue that eliminating the non-linearity is the way to return to quantum mechanics.

- Quantum mechanics is, after all, absolutely linear.

- We explore if this requirement can be relaxed.

W. P. Schleich, D. M. Greenberger, D. H. Kobe, and M. O. Scully, PNAS **110**, 5374 (2013).

Transition Equation

Classical Wave Equation

$$i\hbar\frac{\partial\psi_c(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_c(\mathbf{r},t) + V(\mathbf{r},t)\psi_c(\mathbf{r},t) + \frac{\hbar^2}{2m}\frac{\nabla^2|\psi_c(\mathbf{r},t)|}{|\psi_c(\mathbf{r},t)|}\psi_c(\mathbf{r},t)\ ,$$

Transition Equation

$$i\hbar\frac{\partial\psi_\epsilon(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_\epsilon(\mathbf{r},t) + V(\mathbf{r},t)\psi_\epsilon(\mathbf{r},t) + (1-\epsilon)\frac{\hbar^2}{2m}\frac{\nabla^2|\psi_\epsilon(\mathbf{r},t)|}{|\psi_\epsilon(\mathbf{r},t)|}\psi_\epsilon(\mathbf{r},t)$$

Scaling Factor

ε = Degree of Quantumness

- Reduces to the Schrödinger equation when the degree of quantumness goes to one, $\varepsilon \to 1$.
- Equal to the classical wave equation for $\varepsilon \to 0$.

Equivalent to a Scaled Schrödinger Equation

$$\psi_\epsilon(\mathbf{r}, t) = A_\epsilon(\mathbf{r}, t) e^{iS_\epsilon(\mathbf{r}, t)/\hbar}$$

Equivalent to a Scaled Schrödinger Equation

$$\psi_\epsilon(\mathbf{r},t) = A_\epsilon(\mathbf{r},t)e^{iS_\epsilon(\mathbf{r},t)/\hbar}$$

$$\downarrow$$

$$i\hbar\frac{\partial\psi_\epsilon(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_\epsilon(\mathbf{r},t) + V(\mathbf{r},t)\psi_\epsilon(\mathbf{r},t)$$
$$+(1-\epsilon)\frac{\hbar^2}{2m}\frac{\nabla^2\left|\psi_\epsilon(\mathbf{r},t)\right|}{\left|\psi_\epsilon(\mathbf{r},t)\right|}\psi_\epsilon(\mathbf{r},t)$$

Equivalent to a Scaled Schrödinger Equation

$$\psi_\epsilon(\mathbf{r},t) = A_\epsilon(\mathbf{r},t)e^{iS_\epsilon(\mathbf{r},t)/\hbar}$$

$$\downarrow$$

$$i\hbar\frac{\partial\psi_\epsilon(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_\epsilon(\mathbf{r},t) + V(\mathbf{r},t)\psi_\epsilon(\mathbf{r},t) + (1-\epsilon)\frac{\hbar^2}{2m}\frac{\nabla^2|\psi_\epsilon(\mathbf{r},t)|}{|\psi_\epsilon(\mathbf{r},t)|}\psi_\epsilon(\mathbf{r},t)$$

$$\longrightarrow \quad \frac{\partial\rho(\vec{r},t)}{\partial t} + \vec{\nabla}\vec{j}(\vec{r},t) = 0$$

Equivalent to a Scaled Schrödinger Equation

$$\psi_\epsilon(\mathbf{r},t) = A_\epsilon(\mathbf{r},t)e^{iS_\epsilon(\mathbf{r},t)/\hbar}$$

$$\downarrow$$

$$i\hbar\frac{\partial\psi_\epsilon(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_\epsilon(\mathbf{r},t) + V(\mathbf{r},t)\psi_\epsilon(\mathbf{r},t) + (1-\epsilon)\frac{\hbar^2}{2m}\frac{\nabla^2|\psi_\epsilon(\mathbf{r},t)|}{|\psi_\epsilon(\mathbf{r},t)|}\psi_\epsilon(\mathbf{r},t)$$

$$\longrightarrow \quad \frac{\partial A_\epsilon^2(\mathbf{r},t)}{\partial t} + \nabla\left(\nabla S_\epsilon(\mathbf{r},t)A_\epsilon^2(\mathbf{r},t)\right) = 0$$

Equivalent to a Scaled Schrödinger Equation

$$\psi_\epsilon(\mathbf{r},t) = A_\epsilon(\mathbf{r},t)e^{iS_\epsilon(\mathbf{r},t)/\hbar}$$

$$i\hbar\frac{\partial\psi_\epsilon(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_\epsilon(\mathbf{r},t) + V(\mathbf{r},t)\psi_\epsilon(\mathbf{r},t) + (1-\epsilon)\frac{\hbar^2}{2m}\frac{\nabla^2|\psi_\epsilon(\mathbf{r},t)|}{|\psi_\epsilon(\mathbf{r},t)|}\psi_\epsilon(\mathbf{r},t)$$

$$\frac{\partial A_\epsilon^2(\mathbf{r},t)}{\partial t} + \nabla\left(\nabla S_\epsilon(\mathbf{r},t)A_\epsilon^2(\mathbf{r},t)\right) = 0$$

$$\frac{\partial S_\epsilon(\mathbf{r},t)}{\partial t} = -\frac{1}{2m}[\nabla S_\epsilon(\mathbf{r},t)]^2 - \left[V(\mathbf{r},t) - \epsilon\frac{\hbar^2}{2m}\frac{\nabla^2 A_\epsilon(\mathbf{r},t)}{A_\epsilon(\mathbf{r},t)}\right]$$

Equivalent to a Scaled Schrödinger Equation

$$\psi_\epsilon(\mathbf{r},t) = A_\epsilon(\mathbf{r},t)e^{iS_\epsilon(\mathbf{r},t)/\hbar}$$

$$i\hbar\frac{\partial\psi_\epsilon(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_\epsilon(\mathbf{r},t) + V(\mathbf{r},t)\psi_\epsilon(\mathbf{r},t) + (1-\epsilon)\frac{\hbar^2}{2m}\frac{\nabla^2|\psi_\epsilon(\mathbf{r},t)|}{|\psi_\epsilon(\mathbf{r},t)|}\psi_\epsilon(\mathbf{r},t)$$

$$\frac{\partial A_\epsilon^2(\mathbf{r},t)}{\partial t} + \nabla\left(\nabla S_\epsilon(\mathbf{r},t)A_\epsilon^2(\mathbf{r},t)\right) = 0$$

$$\frac{\partial S_\epsilon(\mathbf{r},t)}{\partial t} = -\frac{1}{2m}[\nabla S_\epsilon(\mathbf{r},t)]^2 - \left[V(\mathbf{r},t) - \epsilon\frac{\hbar^2}{2m}\frac{\nabla^2 A_\epsilon(\mathbf{r},t)}{A_\epsilon(\mathbf{r},t)}\right]$$

$$\boxed{\tilde{\hbar} = \hbar\sqrt{\epsilon}}$$

$$\frac{\partial S_\epsilon(\mathbf{r},t)}{\partial t} = -\frac{1}{2m}[\nabla S_\epsilon(\mathbf{r},t)]^2 - \left[V(\mathbf{r},t) - \frac{\tilde{\hbar}^2}{2m}\frac{\nabla^2 A_\epsilon(\mathbf{r},t)}{A_\epsilon(\mathbf{r},t)}\right]$$

Equivalence to a Scaled Schrödinger Equation

$$\psi_\epsilon(\mathbf{r},t) = A_\epsilon(\mathbf{r},t)e^{iS_\epsilon(\mathbf{r},t)/\hbar}$$

$$i\hbar\frac{\partial\psi_\epsilon(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_\epsilon(\mathbf{r},t) + V(\mathbf{r},t)\psi_\epsilon(\mathbf{r},t) + (1-\epsilon)\frac{\hbar^2}{2m}\frac{\nabla^2|\psi_\epsilon(\mathbf{r},t)|}{|\psi_\epsilon(\mathbf{r},t)|}\psi_\epsilon(\mathbf{r},t)$$

$$\frac{\partial A_\epsilon^2(\mathbf{r},t)}{\partial t} + \nabla\left(\nabla S_\epsilon(\mathbf{r},t)A_\epsilon^2(\mathbf{r},t)\right) = 0$$

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$$\boxed{\tilde{\hbar} = \hbar\sqrt{\epsilon}}$$

$$\frac{\partial S_\epsilon(\mathbf{r},t)}{\partial t} = -\frac{1}{2m}[\nabla S_\epsilon(\mathbf{r},t)]^2 - \left[V(\mathbf{r},t) - \frac{\tilde{\hbar}^2}{2m}\frac{\nabla^2 A_\epsilon(\mathbf{r},t)}{A_\epsilon(\mathbf{r},t)}\right]$$

$$\tilde{\psi}(\mathbf{r},t) = A_\epsilon(\mathbf{r},t)e^{iS_\epsilon(\mathbf{r},t)/\tilde{\hbar}}$$

$$\boxed{i\tilde{\hbar}\frac{\partial\tilde{\psi}(\mathbf{r},t)}{\partial t} = -\frac{\tilde{\hbar}^2}{2m}\nabla^2\tilde{\psi}(\mathbf{r},t) + V(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)}$$

Single Particle Interference

1-D Scaled Schrödinger Equation

$$\boxed{i\tilde{\hbar}\frac{\partial\tilde{\psi}}{\partial t} = -\frac{\tilde{\hbar}^2}{2m}\frac{\partial^2\tilde{\psi}}{\partial x^2}}$$

$$\tilde{\hbar} = \hbar\sqrt{\epsilon}$$

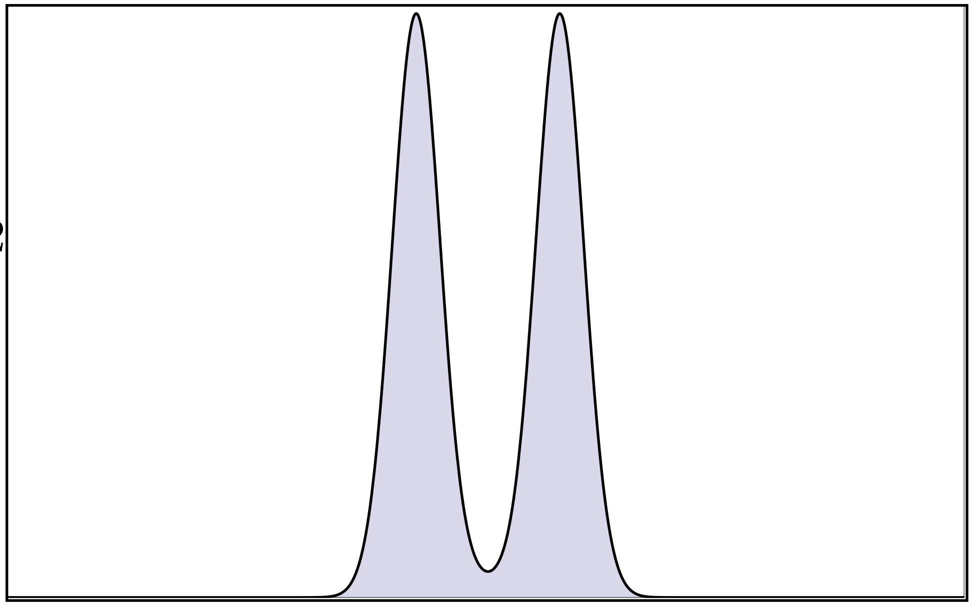
$$\overset{?}{=}$$

1-D Transition Equation

$$\boxed{\begin{aligned} i\hbar\frac{\partial\psi_\epsilon(x,t)}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\psi_\epsilon(x,t)}{\partial x^2} \\ &\quad + \frac{\hbar^2}{2m}\frac{1-\epsilon}{|\psi_\epsilon(x,t)|}\frac{\partial^2|\psi_\epsilon(x,t)|}{\partial x^2}\psi_\epsilon(x,t) \end{aligned}}$$

Initial Condition:

$$\tilde{\psi}(x,0) = \sqrt{N_0}\left[e^{-(x-d)^2/4\sigma^2} + e^{-(x+d)^2/4\sigma^2}\right]$$

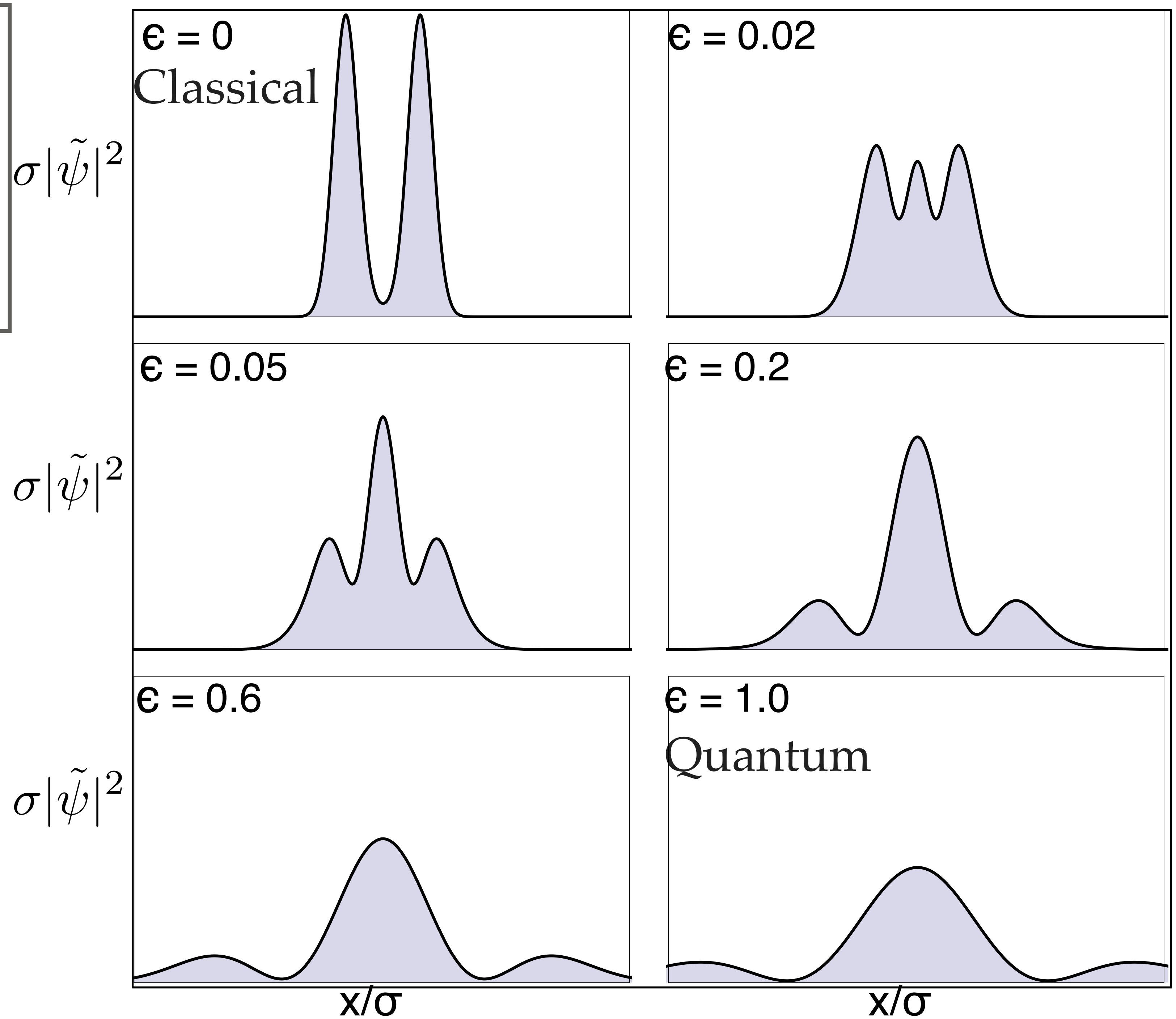


Analytic Solution for $i\tilde{\hbar}\frac{\partial\tilde{\psi}}{\partial t} = -\frac{\tilde{\hbar}^2}{2m}\frac{\partial^2\tilde{\psi}}{\partial x^2}$

$$\left|\tilde{\psi}(x,t)\right|^2 = \frac{N_0}{\tilde{\sigma}_t}\Bigg[\left(e^{-(x-d)^2/4\tilde{\sigma}_t^2} + e^{-(x+d)^2/4\tilde{\sigma}_t^2}\right)^2 - 4e^{-\left(x^2+d^2\right)/2\tilde{\sigma}_t^2}\sin^2\left(\frac{\tilde{\hbar}txd}{4m\sigma^2\tilde{\sigma}_t^2}\right)\Bigg],$$

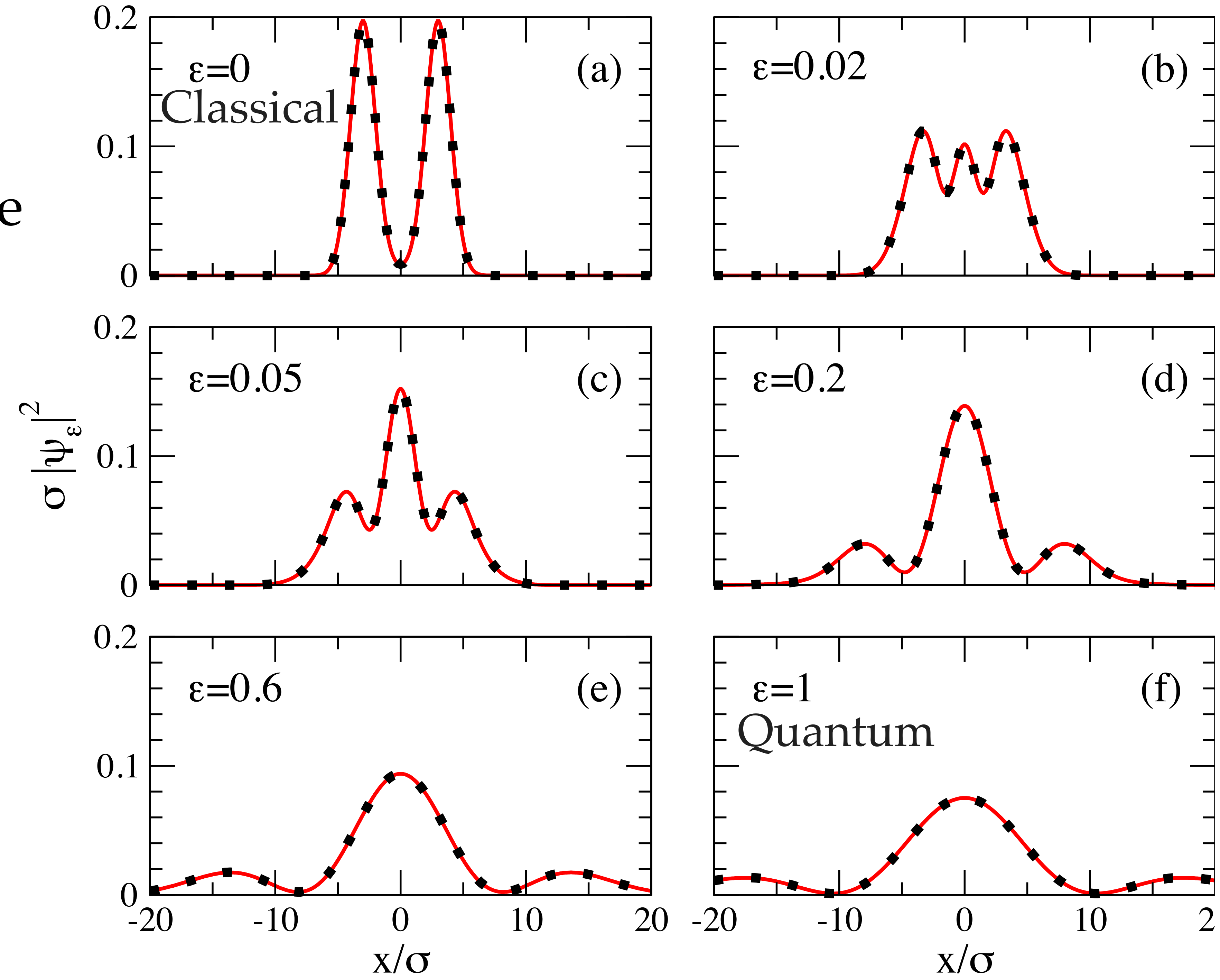
$$\tilde{\sigma}_t^2 = \frac{\tilde{\hbar}^2 t^2}{4m^2\sigma^2} + \sigma^2$$

- Visibility will go to one eventually for all values of the degree of quantumness.



Numerical Simulation

- All at the same time step. Epsilon is changing.
- Visibility will go to one eventually.



Conclusion

$$i\hbar\frac{\partial\psi_\epsilon(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi_\epsilon(\mathbf{r},t) + V(\mathbf{r},t)\psi_\epsilon(\mathbf{r},t) + (1-\epsilon)\frac{\hbar^2}{2m}\frac{\nabla^2|\psi_\epsilon(\mathbf{r},t)|}{|\psi_\epsilon(\mathbf{r},t)|}\psi_\epsilon(\mathbf{r},t)$$

- The linear Schrödinger equation with a scaled hbar is obtained from a non-linear wave equation.
- Classical mechanics is special singular case when the degree of quantumness is zero.
- A nonlinear wave equation does not necessarily lead to nonlinear dynamics.
- Thanks for your time and attention.

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arXiv:1403.2177 [quant-ph]