Lemma 1. *There is no proof of $A \to A$ in the Łukasiewicz system that is less than 5 steps long.*

Proof. In order to prove that no such proof exists we will attempt to construct such a proof. We will despite our best efforts fail to do so demonstrating the impossibility of the task.

We will start our proof at the end. We know that the statement $A \to A$ must appear in the proof, and that any steps after it are extraneous and can be removed. Thus $A \to A$ must be the last step of the proof. We will label this step α

$$A \to A \tag{α}$$

We also know that $A \to A$ does not fit the form of any of our axioms. This means that we must have arrived at it from modus ponens.

$$A \to A \qquad \text{Modus Ponens } (\alpha)$$

Since we arrived at this from modus ponens we know that there must be earlier statements of the form ϕ and $\phi \to (A \to A)$. Since all statements are finite we also know that there is no ϕ such that

$$\phi = \phi \to (A \to A)$$

Thus the two statements must be separate.

$$\begin{aligned} &\phi && (\gamma) \\ &\phi \to (A \to A) && (\beta) \\ &A \to A && \text{Modus Ponens } (\alpha) \end{aligned}$$

We can express $\phi \to (A \to A)$ as a statement of L.S.1 if $\phi = A$, however doing so would mean that our proof would need to include a proof of A. Since A clearly is independent of our axioms we know that the statement cannot be a reference to L.S.1.

If we try to write the same statement as L.S.2 we will find it impossible.

$$\begin{gathered} \phi \to (A \to A) = (\psi \to (\chi \to \omega)) \to ((\psi \to \chi) \to (\psi \to \omega)) \\ \phi = (\psi \to (\chi \to \omega)), A \to A = (\psi \to \chi) \to (\psi \to \omega) \\ \phi = (\psi \to (\chi \to \omega)), A = \psi \to \chi, A = \psi \to \omega \end{gathered}$$

Since A cannot be of the form $\psi \to \chi$ (nor the form $\psi \to \omega$) there is no instantiation of L.S.2 of the form $\phi \to (A \to A)$.

If we set ϕ equal to $\neg A \to \neg A$ we will find that L.S.3 allows us to conclude our statement.

$$\begin{aligned} &\neg A \to \neg A && (\gamma) \\ &(\neg A \to \neg A) \to (A \to A) && \text{L.S.3 } (\beta) \\ &A \to A && \text{Modus Ponens } (\alpha) \end{aligned}$$

Now using the same thought process as $A \to A$ we can show that $\neg A \to \neg A$ must be derived via modus ponens. This gives us a proof of the form:

$$
\begin{array}{cr}
\psi & (\eta) \\
\psi \to (\neg A \to \neg A) & (\delta) \\
(\neg A \to \neg A) \to (A \to A) & \text{L.S.3 } (\beta) \\
\neg A \to \neg A & \text{Modus Ponens } (\gamma) \\
A \to A & \text{Modus Ponens } (\alpha)
\end{array}
$$

Now since we are looking for a proof with 4 steps we know that it must be the case that ψ is equal to $(\neg A \to \neg A) \to (A \to A)$, otherwise we would have 5 steps.

$$
\begin{array}{cr}
((\neg A \to \neg A) \to (A \to A)) \to (\neg A \to \neg A) & \\
(\neg A \to \neg A) \to (A \to A) & \text{L.S.3} \\
\neg A \to \neg A & \text{Modus Ponens} \\
A \to A & \text{Modus Ponens}
\end{array}
$$

Now we need to check whether the first statement can be a instantiation of one of our axioms. If we check we find that there are no values of ϕ, ψ and χ for which the our statement is an instantiation of any of the axioms.

Now we backtrace to the last descision we made. We chose to represent $\phi \to (A \to A)$ as L.S.3. Since that arrived us at an incorrect conclusion we know that it cannot be the case that in a four step proof that step is introduced by L.S.3. Since we have removed all of the axioms as possibilites to introduce $\phi \to (A \to A)$ we know that it is introduced by modus ponens.

$$
\begin{array}{cr}
\phi & \\
\psi & \\
\psi \to (\phi \to (A \to A)) & \\
\phi \to (A \to A) & \text{Modus Ponenes} \\
A \to A & \text{Modus Ponens}
\end{array}
$$

Since we have 5 claims here we know that in order to reduce our proof to 4 steps we must have two of them that are equal. It is clear that no statement containing ϕ can be equal to ϕ and the same goes for ψ. We also know that if a statement χ that relies on a statement ω, it must be the case that $\chi \neq \omega$ otherwise our proof would be circular.

Of the remaining statments that could be equal there is only one pair that has the same form. This leaves us to say that $\phi = \psi$.

$$
\begin{array}{cr}
\phi & \\
\phi \to (\phi \to (A \to A)) & \\
\phi \to (A \to A) & \text{Modus Ponenes} \\
A \to A & \text{Modus Ponens}
\end{array}
$$

Our remaining statements must be instantiations of our axioms because any use of modus ponens would add new steps to the proof. If we start with the sentence $\phi \to (\phi \to (A \to A))$ we will find it can only be instantiated by L.S.1.

$$\begin{array}{cr} A \to A & \\ (A \to A) \to ((A \to A) \to (A \to A)) & \text{L.S.1} \\ (A \to A) \to (A \to A) & \text{Modus Ponenes} \\ A \to A & \text{Modus Ponens} \end{array}$$

Now in order to make this proof valid we must proof $A \to A$ in 1 step. This would require the instantiation of an axiom and we already know that this is impossible.

Thus there is no proof of $A \to A$ that is 4 steps or shorter.

$\square$