Lemma 1. *There is no proof of $A \to A$ in the Lukasiewicz system that is less than 5 steps long.*

Proof. In order to prove that no such proof exists we will attempt to construct such a proof. We will despite our best efforts fail to do so demonstrating the impossibility of the task.

We will start our proof at the end. We know that the statement $A \to A$ must appear in the proof, and that any steps after it are extraneous and can be removed. Thus $A \to A$ must be the last step of the proof.

$$A \to A$$

We also know that $A \to A$ does not fit the form of any of our axioms. This means that we must have arrived at it from modus ponens.

$$A \to A \qquad \text{Modus Ponens}$$

Since we arrived at this from modus ponens we know that there must be earlier statements of the form ϕ and $\phi \to (A \to A)$. Since all statements are finite we also know that there is no ϕ such that

$$\phi = \phi \to (A \to A)$$

Thus the two statements must be separate.

$$\begin{array}{cr} \phi & \\ \phi \to (A \to A) & \\ A \to A & \text{Modus Ponens} \end{array}$$

We can express $\phi \to (A \to A)$ as a statement of L.S.1 if $\phi = A$, however doing so would mean that our proof would need to include a proof of A. Since A clearly is independent of our axioms we know that the statement cannot be a reference to L.S.1.

If we try to write the same statement as L.S.2 we will find it impossible.

$$\phi \to (A \to A) = (\psi \to (\chi \to \omega)) \to ((\psi \to \chi) \to (\psi \to \omega))$$
$$\phi = (\psi \to (\chi \to \omega)), A \to A = (\psi \to \chi) \to (\psi \to \omega)$$
$$\phi = (\psi \to (\chi \to \omega)), A = \psi \to \chi, A = \psi \to \omega$$

Since A cannot be of the form $\psi \to \chi$ (nor the form $\psi \to \omega$) there is no instantiation of L.S.2 of the form $\phi \to (A \to A)$.

If we set ϕ equal to $\neg A \to \neg A$ we will find that L.S.3 allows us to conclude our statement.

$$\begin{array}{cr}
\neg A \to \neg A & \\
(\neg A \to \neg A) \to (A \to A) & \text{L.S.3} \\
A \to A & \text{Modus Ponens}
\end{array}$$

Now using the same thought process as $A \to A$ we can show that $\neg A \to \neg A$ must be derived via modus ponens. This gives us a proof of the form:

$$\begin{array}{cr}
\psi & \\
\psi \to (\neg A \to \neg A) & \\
(\neg A \to \neg A) \to (A \to A) & \text{L.S.3} \\
\neg A \to \neg A & \text{Modus Ponens} \\
A \to A & \text{Modus Ponens}
\end{array}$$

Now since we are looking for a proof with 4 steps we know that it must be the case that ψ is equal to $(\neg A \to \neg A) \to (A \to A)$, otherwise we would have 5 steps.

$$\begin{array}{cr}
((\neg A \to \neg A) \to (A \to A)) \to (\neg A \to \neg A) & \\
(\neg A \to \neg A) \to (A \to A) & \text{L.S.3} \\
\neg A \to \neg A & \text{Modus Ponens} \\
A \to A & \text{Modus Ponens}
\end{array}$$

Now we need to check whether the first statement can be a instantiation of one of our axioms. If we check we find that there are no values of ϕ, ψ and χ for which the our statement is an instantiation of any of the axioms.

Now we backtrace to the last descision we made. We chose to represent $\phi \to (A \to A)$ as L.S.3. Since that arrived us at an incorrect conclusion we know that it cannot be the case that in a four step proof that step is introduced by L.S.3. Since we have removed all of the axioms as possibilites to introduce $\phi \to (A \to A)$ we know that it is introduced by modus ponens.

$$
\begin{aligned}
&\phi \\
&\psi \\
&\psi \to (\phi \to (A \to A)) \\
&\phi \to (A \to A) && \text{MP} \\
&A \to A && \text{Modus Ponens}
\end{aligned}
$$

Since we have 5 claims here we know that in order to reduce our proof to 4 steps we must have two of them that are equal. It is clear that no statement containing ϕ can be equal to ϕ and the same goes for ψ. We also know that if a statement χ that relies on a statement ω, it must be the case that $\chi \neq \omega$ otherwise our proof would be circular.

Of the remaining statments that could be equal there is only one pair that has the same form. This leaves us to say that $\phi = \psi$.

$$
\begin{aligned}
&\phi \\
&\phi \to (\phi \to (A \to A)) \\
&\phi \to (A \to A) && \text{MP} \\
&A \to A && \text{Modus Ponens}
\end{aligned}
$$

□