Appendix D: Extension for partially observed B encounters

Our LR model assumes B encounters are never observed, while our LRB model assumes all B encounters are observed. However, in some sampling scenarios, perhaps B encounters are partially observed. For example, this may occur when multiple photographs are collected from individuals detected at least once within a sampling occasion. In this case, each individual photograph may be of the left side only, the right side only, or both sides simultaneously. This situation can be accommodated by extending the model to an additional encounter state (for an independent development of a similar model, see Bonner and Holmberg 2013), where there are now five encounter types based on the photographs(s) for each individual: no encounter (0), photographed on the left side only (L), photographed on the right side only (R), photographed at least once on the left side and at least once on the right side (A), or photographed at least once on both sides simultaneously (B). Allowing both A and B encounters requires an additional parameter, the probability of recording a B given both sides were photographed (α). The LR and LRB models are then special cases of this LRAB model with $\alpha = 0$ and $\alpha = 1$, respectively.

The 5^T latent histories for the LRAB model are indexed by

$$i = 1 + \sum_{t=1}^{T} \epsilon_t 5^{T-t},$$

where $\epsilon_t = 0$ for no encounters, $\epsilon_t = 1$ for L encounters, $\epsilon_t = 2$ for R encounters, $\epsilon_t = 3$ for A encounters, and $\epsilon_t = 4$ for B encounters at time t. Similar to the other models, the $5^T - 4^T + 2^{T+1} - 2$ recorded histories are indexed using the same ordering but with the unobservable recorded histories (e.g., 000, 00A, 0LR) removed. For example, with $T = 3$, the latent encounter history $\boldsymbol{\omega}_{10} = $ 0LB only produces the recorded history $\tilde{\boldsymbol{\omega}}_6 = $ 0LB because the true encounter history for an individual with at least one B detection is observed without error. For model M_t, this latent encounter history has probability

$$\pi_{10} = (1 - p_1) p_2 \delta^L p_3 \delta^B \alpha.$$

Similarly, the latent history $\boldsymbol{\omega}_{20} = 0\text{AB}$ only produces the recorded history $\tilde{\boldsymbol{\omega}}_{10} = 0\text{AB}$ with probability

$$\pi_{20} = (1 - p_1)p_2\delta^B(1 - \alpha)p_3\delta^B\alpha.$$

The latent encounter history $\boldsymbol{\omega}_9 = 0\text{LA}$ produces the recorded histories $\tilde{\boldsymbol{\omega}}_2 = 00\text{R}$ and $\tilde{\boldsymbol{\omega}}_5 = 0\text{LL}$ with probability

$$\pi_9 = (1 - p_1)p_2\delta^L p_3\delta^B(1 - \alpha).$$

Note that A encounters are only observed for individuals with at least one B detection. As with the other models, the $5^T \times (5^T - 4^T + 2^{T+1} - 2)$ $\mathbf{A}$ matrix can be constructed from the contributed records column by replacing each dot (.) entry with a zero and any other entry with a one (see Table D1 for a complete example with $T = 2$). Code to construct the $\mathbf{A}$ matrix and automate selection of a set of $r = 4^T - 2^{T+1} + 2$ basis vectors for bilateral photograph analyses for the LRAB model is provided in *Supplement*.

We repeated the simulation experiments described in *Simulation Study* (and Appendix B) to compare estimator performance with the LR and the LRAB data types when $\alpha = 0.5$ for the closed population abundance model M_t. These results are summarized in Table D2. In terms of bias and precision of the abundance estimators, we found relatively little was gained over the LR model when incorporating A and B detections using the LRAB model. This is consistent with our findings from our original simulation study comparing the LR model and the LRB model when $\alpha = 1$.

LITERATURE CITED

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Table D1: Latent histories ($\boldsymbol{\omega}$) and recorded histories ($\tilde{\boldsymbol{\omega}}$) for the LRAB model with $T = 2$. Contributed records columns show the recorded histories arising from specific latent histories. For example, latent history 16, $\boldsymbol{\omega}_{16} =$ A0 , gives rise to recorded histories $j = 4$ and $j = 7$, $\tilde{\boldsymbol{\omega}}_4 =$ L0 and $\tilde{\boldsymbol{\omega}}_7 =$ R0. For illustration, we include the probability of each latent history, $\Pr(\boldsymbol{\omega} \mid \boldsymbol{\theta})$, for the closed population abundance model M_t allowing time variation in detection probability p_t ($t = 1, \ldots, T$), where $q_t = 1 - p_t$. This presentation mirrors that of Link et al. (2010).

i	Latent history ($\boldsymbol{\omega}_i$)	$\Pr(\boldsymbol{\omega}_i \mid \boldsymbol{\theta})$	Contributed records (j from i)	j	Recorded history ($\tilde{\boldsymbol{\omega}}_j$)
1	00	$q_1 q_2$	....	1	0L
2	0L	$q_1 p_2 \delta^L$	1.............	2	0R
3	0R	$q_1 p_2 \delta^R$	.2............	3	0B
4	0A	$q_1 p_2 \delta^B (1-\alpha)$	12............	4	L0
5	0B	$q_1 \delta^B \alpha$	..3...........	5	LL
6	L0	$p_1 \delta^L q_2$	...4..........	6	LB
7	LL	$p_1 \delta^L p_2 \delta^L$	....5.........	7	R0
8	LR	$p_1 \delta^L p_2 \delta^R$	.2.4..........	8	RR
9	LA	$p_1 \delta^L p_2 \delta^B (1-\alpha)$	.2..5.........	9	RB
10	LB	$p_1 \delta^L p_2 \delta^B \alpha$	.....6.........	10	AB
11	R0	$p_1 \delta^R q_2$	......7........	11	B0
12	RL	$p_1 \delta^R p_2 \delta^L$	1.....7........	12	BL
13	RR	$p_1 \delta^R p_2 \delta^R$	.......8.......	13	BR
14	RA	$p_1 \delta^R p_2 \delta^B (1-\alpha)$	1......8.......	14	BA
15	RB	$p_1 \delta^R p_2 \delta^B \alpha$	........9......	15	BB
16	A0	$p_1 \delta^B (1-\alpha) q_2$	...4..7........		
17	AL	$p_1 \delta^B (1-\alpha) p_2 \delta^L$	....5.7........		
18	AR	$p_1 \delta^B (1-\alpha) p_2 \delta^R$	...4...8.......		
19	AA	$p_1 \delta^B (1-\alpha) p_2 \delta^B (1-\alpha)$	....5..8.......		
20	AB	$p_1 \delta^B (1-\alpha) p_2 \delta^B \alpha$	.........10.....		
21	B0	$p_1 \delta^B \alpha q_2$	..........11....		
22	BL	$p_1 \delta^B \alpha p_2 \delta^L$	...........12...		
23	BR	$p_1 \delta^B \alpha p_2 \delta^R$	............13..		
24	BA	$p_1 \delta^B \alpha p_2 \delta^B (1-\alpha)$	.............14.		
25	BB	$p_1 \delta^B \alpha p_2 \delta^B \alpha$	15		

Table D2: Simulation results for abundance estimation assuming different combinations of underlying abundance (N), detection probability (p), and bilateral observation probabilities (δ^L and δ^R). For all simulations, the probability of recording a B given both sides were photographed (α) is 0.5. Estimator performance is summarized by proportional relative bias ('bias'), coefficient of variation (CV), and 80% credible interval coverage (Cov). Subscripts on performance measures indicate the different estimators used, which included the methods developed in this paper for the LRAB data type (1) and the LR data type (2), and traditional mark-recapture analysis using one side of the animal (3). One hundred simulations were performed at each design point.

N	p	δ^L	δ^R	bias_1	bias_2	bias_3	Cov_1	Cov_2	Cov_3	CV_1	CV_2	CV_3
100	0.2	0.2	0.2	-0.02	-0.02	-0.03	0.80	0.78	0.76	0.12	0.12	0.14
100	0.2	0.4	0.4	-0.04	-0.04	-0.06	0.81	0.80	0.68	0.15	0.15	0.19
100	0.2	0.5	0.3	-0.02	-0.02	-0.03	0.75	0.74	0.73	0.15	0.16	0.18
100	0.4	0.2	0.2	0.00	0.00	-0.01	0.92	0.92	0.90	0.04	0.04	0.05
100	0.4	0.4	0.4	0.00	0.00	0.00	0.88	0.89	0.89	0.06	0.06	0.08
100	0.4	0.5	0.3	-0.01	-0.01	-0.01	0.85	0.86	0.80	0.06	0.06	0.07
500	0.2	0.2	0.2	0.00	0.00	0.00	0.81	0.77	0.81	0.06	0.05	0.07
500	0.2	0.4	0.4	-0.01	-0.01	-0.02	0.77	0.76	0.78	0.07	0.07	0.09
500	0.2	0.5	0.3	0.00	0.00	0.01	0.81	0.84	0.78	0.07	0.07	0.09
500	0.4	0.2	0.2	0.00	0.00	0.00	0.80	0.80	0.82	0.02	0.02	0.02
500	0.4	0.4	0.4	0.00	0.00	-0.01	0.83	0.84	0.93	0.03	0.03	0.04
500	0.4	0.5	0.3	0.00	0.00	0.00	0.84	0.87	0.93	0.03	0.03	0.03