Computer-aided building ventilation system design — a system-theoretic approach

Fariborz Haghighat and **Jiwu Rao**
Centre for Building Studies, Concordia University, Montreal, Que., H3G 1M8 (Canada)

(Received March 11, 1990; accepted September 2, 1990; revised paper received October 24, 1990)

Abstract

An understanding of the pattern of airflow is an important element in the design of ventilation systems. The airflows are caused by wind effect, stack effect and mechanical ventilation systems. Several models have been developed to analyse the airflow in buildings. So far, the focus has been on the modelling process. In this paper, the modelling and analysis of building ventilation systems using a system-theoretic procedure are presented. The emphasis is on presenting a theoretic derivation of nodal-governing equations for building airflow systems and obtaining efficient procedures for automatic formulation of the system equations. This approach is based on matrix algebra, therefore is easy to implement on a computer.

Introduction

The trend in modern construction towards more airtight buildings and the increased use of synthetic materials in construction, furnishings, and maintenance operations have increased the indoor airborne contaminant concentration levels. This, in addition to concerns about moisture transport and accumulation indoors, has raised concerns about the indoor air quality in modern buildings from both health and durability perspectives. The transport of airborne contaminants throughout a building is caused by air movements driven by pressure differences between individual zones. The concern for better indoor environment and energy conservation requires the implementation of the best ventilation system and its optimal operation. This goal calls for an improved understanding of airflow circulation patterns within and through buildings, and for procedures for simulations of buildings and ventilation systems with respect to airflow behaviour, and for the evaluation of their performance under different operating conditions.

Computer simulation of airflow in buildings is the primary research tool with which ventilation performance can be predicted. The modelling process involves the breaking of the total system (building and HVAC systems) into components (rooms, ducts and fans), and formulating flow equations for each component. The set of governing equations is derived from air mass continuity in any space in a building. Typical input data for these simulators are: meteorological data, building and HVAC characteristics. Typical output data from these simulators are airflow rates either throughout the building or in selected zones.

Several models have been developed for predicting airflows, dispersion of contaminants and moisture, and fire-induced smoke migration in buildings. Reviews of these models are given in refs. 1–4. The majority of these models use the network approach to obtain a minimum set of equations which are then solved to obtain the unknown pressures. Walton [5], for example, has developed a multi-zonal airflow network model which accounts for a variety of airflow elements such as power law resistance, ducts, fans, and doorways. The set of governing equations is derived from mass balance of each room/node. The solution algorithm is a modified Newton–Raphson's method, which simultaneously updates the pressure of

0378-7788/91/$3.50

all the nodes and uses a convergence acceleration routine to reduce the number of iterations.

In this paper, the system-theoretic technique is used to model the air infiltration and ventilation mechanism. The building airflow system is decomposed into its elementary components. The system theory is used to formulate a set of governing equations. A detailed analysis of the process and results is then given to show an efficient procedure for automatic formulation of system equations.

The order of this paper is as follows: the fundamentals of airflow through the building are discussed briefly and related equations are given. This is followed by a brief presentation of the system-theoretic approach. An example is given.

Driving forces

The airflow in a building is caused by pressure differences which are generated by wind, thermal force and mechanical systems. These driving forces will be discussed briefly.

(a) Wind pressure effect

When wind blows onto a building, it creates a positive pressure on the windward face of the building, and negative pressures on the other facades. The wind-induced pressure at one location of the building envelope is generally expressed in terms of a wind pressure coefficient and the wind dynamic pressure [6, 7], i.e.,

$$P^{w}=C_{p}\left(\frac{1}{2}\rho V_{h}^{2}\right) \tag{1}$$

where C_p is the pressure coefficient, ρ is the ambient air density, and V_h is the wind velocity on site at the roof height. Wind velocities are generally recorded in an open area, at an elevation of 10 m above the ground. The wind velocity at the building site may be determined from

$$\frac{V_h}{V_o}=ah^{b} \tag{2}$$

where V_h is the wind velocity at height h above ground and V_o is the wind speed at a height of 10 meters. Coefficients a and b have been related to four types of terrain conditions defined as open, country, urban and city in order of increasing terrain roughness [7].

(b) Stack effect

Temperature in a node is assumed to be uniform, and the pressure on the same elevation is considered to be constant. However, stack stratification exists along the vertical direction. The pressure decrease due to the stack effect depends on the air density and the vertical distance, as:

$$P^{s}=\rho gh \tag{3}$$

where $\rho=P_a/(R_a T)$ is the air density, h is the vertical displacement, and P_a and T are the atmospheric pressure and zonal temperature (in Kelvin), respectively. Consequently, the differences in air temperature and hence densities between zones, and between the inside air and outside air affect airflow through the building. The pressure difference, ΔP^s, due to the stack effect can be obtained by:

$$\Delta P^{s}=g(\rho_j h_j-\rho_i h_i) \tag{4}$$

where ρ_i and ρ_j are the air densities in zones i and j, respectively, h_i and h_j are the opening heights with respect to nodal reference heights of i and j. Using the Ideal Gas Law, eqn. (4) can be rewritten as:

$$\Delta P^{s}=\frac{P_a}{R_a}g\left(\frac{h_j}{T_j}-\frac{h_i}{T_i}\right) \tag{5}$$

where T_i and T_j are the air temperatures in zones i and j.

(c) Mechanical system

Another mechanism responsible for air movement in buildings is any mechanical ventilation system, which can play an important role in controlling contaminant movement. Mechanical systems cause pressure differences between two zones.

Flow equation

In general, pressure drops may occur due to flow resistances along flow paths. These flow paths may be either:

(1) purpose-provided openings (ducts),

(2) component openings (cracks around door), and

(3) background leakage (permeable materials).

Two forms of flow equations are usually used to describe the flow through component openings and background cracks: the quadratic form and the power law form. The quadratic equation forms the basis of simple models and is defined by [8]:

$$\Delta P = a_1 Q + a_2 Q^2 \tag{6}$$

where Q is the volume flow rate, and a_1 and a_2 are flow coefficients for a given geometry, i.e., the shape and size of the opening. Another versatile way of describing the relationship between airflow and pressure difference is by the power law form [6] as in:

$$Q = K \Delta P^n \tag{7}$$

where the component permeability, K, is a function of Reynold's number and the ratio of the opening size to the entire surface. The value of exponent n is in the range of 0.5 to 1.0 depending on the type of the flow; $n = 0.5$ for fully developed turbulent flow, and $n = 1$ for fully developed laminar flow. The exponent n is strongly related with the pressure difference. Values of K and n can be measured by the fan pressurization technique (DC) or the AC pressurization technique [9].

Liddament [10] compared the performance of the quadratic and power law equations against experimental data, and found that the power law gave a better agreement with the experimental data.

Solution algorithm

The non-linear dependence of the flow rate and pressure difference results in solving a set of non-linear mass balance equations. Due to the characteristic of the set of equations, the internal node pressure must be calculated by iterative techniques.

Most of the existing programs use the well-known Newton–Raphson technique, or its modified versions. The solution of this set of equations is obtained by an iterative technique, which progressively adjusts the node pressures until the convergence criterion is satisfied. The convergence criterion could be, say, when the percentage of flow rates from the current iteration to the previous one is less then a specified tolerance,

$$P^{k+1} = P^k + \text{correction} \tag{8}$$

Different types of algorithms for calculating the adjustment or correction term have been used in airflow models. As an example, the standard Newton–Raphson algorithm is used in the Oscar Faber model [11], while the Relaxed Newton–Raphson is used in the NBS airflow model [12].

System-theoretic approach and formulation technique

The system-theoretic approach has been successfully applied, in both time and frequency domains, to many types of physical systems, including mechanical, hydraulic and energy systems. This technique is fully described in refs. 13 and 14. A four-room building with eight openings is chosen as an example to demonstrate the modelling and formulation procedure. The building and room dimensions, opening characteristics, temperature distribution and wind-induced pressures are given in Fig. 1.

A building and building ventilation system can be considered as a collection of several types of components which are interconnected in a predetermined manner, called a *terminal graph*. The graph consists of a set of points and lines, where a line connecting a pair of points is called an *edge*. The edges in a graph are connected to their end points called *nodes*

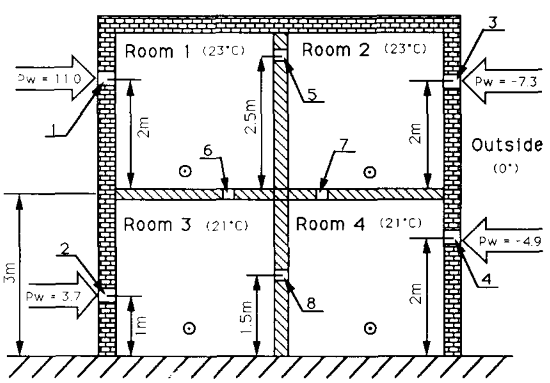
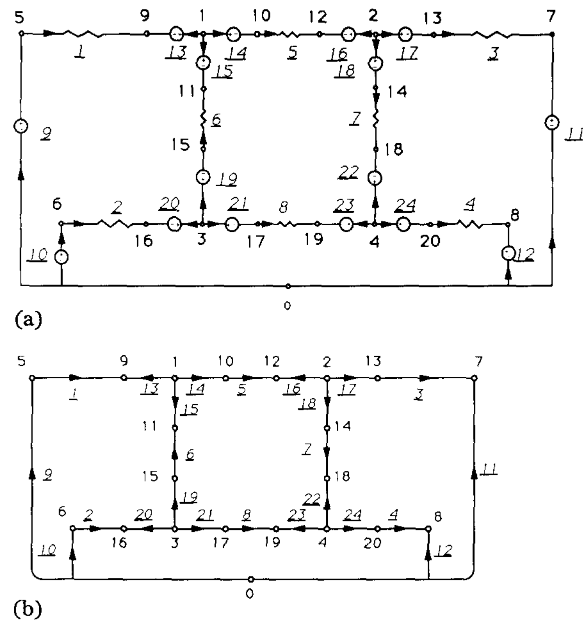


Fig. 1. A four-room building.

Opening characteristics

No.	n	K	No.	n	K
1	0.65	0.005	5	0.5	0.015
2	0.65	0.008	6	0.5	0.020
3	0.65	0.007	7	0.5	0.020
4	0.65	0.009	8	0.5	0.015

or *terminals*. In a building airflow system, individual driving forces and airflow resistances are modelled as edges. As shows in Table 1, the edges can represent the flow resistance, wind pressure, stack effect and mechanical system. A physical component may be represented by one or a combination of several of these edges. The airflow system of the example building (Fig. 1) can be represented by a graph of 20 nodes and 24 edges (Fig. 2). Each opening is considered as a flow resistance and is represented by an edge (edges 1 to 8). In the inside of the building, the pressure differences due to stack effect, from the nodal reference heights to the heights of the openings, are known and are represented by separate edges (edges 13 to 24). At the outside, the pressure difference between the end of an opening and the atmosphere node, which is the sum of stack effect and wind-induced pressure, is represented as one edge

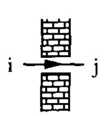
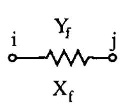
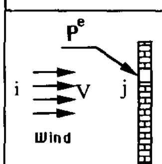
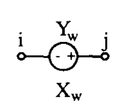
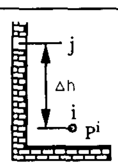
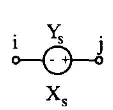
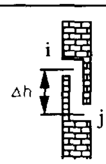
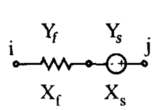
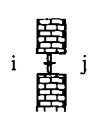
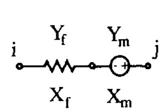


(a)

(b)

Fig. 2. (a) Network and (b) graph representation of the four-room building.

TABLE 1. Modelling of building components

Components	Electrical equivalent	Graph representation flow equation
		i j $Y_f = F(X_f)$
		i j $X_w = C_p \frac{1}{2} \rho V^2$
		i j $X_s = \rho g \Delta h$
		i j
		i j

(edges 9 to 12). Four nodes (1 to 4) represent the four rooms and 16 other nodes represent the intersections between the flow resistance edges and the known pressure edges.

The state of each edge is described by a pair of measurements called *across variable* and *through variable*. In most commonly accepted system-theoretic models of fluid dynamics, pressure differences are considered as the across variables and the airflow rates as the through variables. Any mathematical model of an opening requires the pertinent information about the behaviour of the edge with reference to the through and across variables. The behaviour of an opening in the graph is independently measured either experimentally (i.e., fan pressurization techniques) or empirically. The mathematical relations between the across and through variables associated with an edge are called *terminal equations*. For flow resistance edges, this relationship can be expressed either in terms of through variables, $\Delta P = g(Q)$, or in terms of across variables, $Q = f(\Delta P)$. The edges that represent driving forces have specified across variables, and are called '*drivers*', or are referred to as the known pressure edges.

In addition to the terminal equations, two topological relationships exist which are necessary to fully describe the system. These equations are based on the oriented linear graph and, in system-theoretic terms, are called vertex postulates (conservation of mass) and circuit postulates (compatibility law). The vertex equation in matrix form is:

$$\mathbf{I}^*\boldsymbol{Y}=0 \tag{9}$$

where $\boldsymbol{Y}$ is the vector of through variables and $\mathbf{I}^*$ is the *incidence matrix* derived from the node-edge topology of the graph; it has v rows and e columns corresponding to v nodes and e edges in the graph. Elements of the incidence matrix are defined as follows:

$$I^*_{ij}=\begin{cases} 1 & \text{if } j\text{th edge is connected to node } i \text{ and oriented away from it} \\ -1 & \text{if } j\text{th edge is connected to node } i \text{ and oriented towards it, and} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Each column of the incidence matrix has one $+1$ and one -1. Thus, the rows of the incidence matrix are linearly dependent. To obtain a linearly independent set of rows, one of the rows is deleted. The resulting matrix with $(v-1)$ rows is called the *reduced incidence matrix* $\mathbf{I}$, and:

$$\mathbf{I}\boldsymbol{Y}=\mathbf{0} \tag{11}$$

The vertex which corresponds to the row of $\mathbf{I}^*$ and which is not in $\mathbf{I}$ is called the *reference vertex* or *node*, and is usually chosen to be the one that represents the atmosphere (the outside of the building).

The nodal variables refer to measurements made at each node with respect to the datum or reference node. In the airflow graph, the nodal variables are the pressures at the nodes with respect to atmospheric pressure. System theory shows that any across variable can be given as a linear combination of the nodal variables. This relation is called the *nodal transformation equation*, and takes the following form:

$$\boldsymbol{X}=\mathbf{I}^{\mathrm{T}}\boldsymbol{\Phi} \tag{12}$$

where $\boldsymbol{X}$ is the vector of across variables, $\boldsymbol{\Phi}$ is the vector of nodal variables, and the superscript $^{\mathrm{T}}$ denotes the matrix transpose operation. The nodes can be divided into those which represent rooms, n-nodes, and the remaining k-nodes. This partitions the nodal variable vector into:

$$\boldsymbol{\Phi}=[\boldsymbol{\Phi}_n\ \ \boldsymbol{\Phi}_k]^{\mathrm{T}} \tag{13}$$

Derivation of governing equations

In the graph of a building airflow system, the across variable of an edge that represents a specified driver, such as the stack effect, or a combination of stack and wind pressure, is given and known. The across variables of flow resistance edges, on the other hand, are unknown. Accordingly, edges are partitioned into flow resistance edges, f-edges, and driver (known across variable) edges, k-edges. The edges in the graph of Fig. 2, for example, can be divided into two sets of 1 to 8 and 9 to 24. The corresponding partitions of the across and through variable vectors are expressed as:

$$\boldsymbol{X}=\begin{bmatrix}\boldsymbol{X}_f\\ \boldsymbol{X}_k\end{bmatrix},\qquad \boldsymbol{Y}=\begin{bmatrix}\boldsymbol{Y}_f\\ \boldsymbol{Y}_k\end{bmatrix}$$

The vertex eqn. (11) can then be written in partitioned form as:

$$\begin{bmatrix}\boldsymbol{I}_{nf}\ \boldsymbol{I}_{nk}\\ \boldsymbol{I}_{kf}\ \boldsymbol{I}_{kk}\end{bmatrix}\begin{bmatrix}\boldsymbol{Y}_f\\ \boldsymbol{Y}_k\end{bmatrix}=\mathbf{0} \tag{14}$$

The row of the reduced incidence matrix is partitioned according to eqn. (13).

The partitioned incidence matrix in the above equation can be simplified if the following conventions are used in assigning orientations and numbering to nodes and edges. The orientations of the edges that represent pressure 'drivers' at the outside of the building are chosen so that they are all oriented away from the reference node (e.g., edges 9–12 are all oriented away from node 0 in Fig. 2). The edges that represent inside stack effects are assigned orientations away from n-nodes (e.g., edges 13, 14 and 15 have orientations away from node 1 in Fig. 2). The flow resistance edges of openings can have arbitrary orientations. The numbering of nodes and edges is related. The n-nodes and flow resistance edges (f-edges) are numbered first, in consecutive order, respectively (nodes 1 to 4, edges 1 to 8 in Fig. 2). The k-nodes can then be numbered in any consecutive order, but preferably with the outside before the inside of the building. The numbering of the k-edges must be in concordance with the numbering of the k-

nodes, so that the order of k-edges and the order of k-nodes are the same.

With the above conventions concerning orientations and numbering of nodes and edges, the sub-matrix $\mathbf{I}_{nf}$ will be equal to a zero matrix (since any f-edge is not directly incident on any n-node), and the sub-matrix $\mathbf{I}_{kk}$ will be a negative identity matrix (diagonal elements are equal to -1 and off-diagonal elements zeros). Let $\mathbf{I}_1 = \mathbf{I}_{nk}$ and $\mathbf{I}_2 = \mathbf{I}_{kf}$ then the vertex eqn. (14) can be simplified to:

$$\begin{bmatrix} \mathbf{0} & \mathbf{I}_1 \\ \mathbf{I}_2 & -\mathbf{U} \end{bmatrix} \begin{bmatrix} \boldsymbol{Y}_f \\ \boldsymbol{Y}_k \end{bmatrix} = \mathbf{0} \tag{15}$$

where $\mathbf{U}$ is the identity matrix. For the example case, the two matrices $\mathbf{I}_1$ and $\mathbf{I}_2$ can be formed from Fig. 2 as:

$$\mathbf{I}_1 = \begin{bmatrix} 0&0&0&0&1&1&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1 \end{bmatrix} \tag{16}$$

$$\mathbf{I}_2^{\mathrm{T}} = \begin{bmatrix} 1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&-1&0&0&0&0 \\ 0&0&-1&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&-1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&-1&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&-1&0 \end{bmatrix} \tag{17}$$

The lower part of eqn. (15) is

$$\mathbf{I}_2 \boldsymbol{Y}_f - \boldsymbol{Y}_k = \mathbf{0} \tag{18}$$

and it can be rewritten as

$$\boldsymbol{Y}_k = \mathbf{I}_2 \boldsymbol{Y}_f \tag{19}$$

Substituting $\boldsymbol{Y}_k$ into the upper part of the vertex eqn. (15) yields:

$$(\mathbf{I}_1 \mathbf{I}_2) \boldsymbol{Y}_f = \mathbf{0} \tag{20}$$

Let

$$\mathbf{II} = \mathbf{I}_1 \mathbf{I}_2 \tag{21}$$

then the set of vertex equations is obtained as:

$$\mathbf{II}\, \boldsymbol{Y}_f = \mathbf{0} \tag{22}$$

The number of rows in this equation is equal to the number of room nodes. This equation provides the minimum set of equations, and will be referred to as the reduced vertex equation.

For the case study, matrix $\mathbf{II}$ can be calculated as:

$$\mathbf{II} = \begin{bmatrix} -1&0&0&0&1&-1&0&0 \\ 0&0&1&0&-1&0&1&0 \\ 0&-1&0&0&0&1&0&1 \\ 0&0&0&1&0&0&-1&-1 \end{bmatrix}$$

A general observation about matrix $\mathbf{II}$ is that this matrix is the reduced incidence matrix of a graph of the building's physical connection (Fig. 3), with the atmospheric node as the reference vertex. This physical connection graph consists of rooms (plus atmosphere) as nodes and openings as edges. The orientations of the edges are the same as those of the flow resistance edges of Fig. 2.

Using the same division of edges and nodes, the nodal transformation eqn. (12) can be written as:

$$\begin{bmatrix} \boldsymbol{X}_f \\ \boldsymbol{X}_k \end{bmatrix} = \mathbf{I}^{\mathrm{T}} \boldsymbol{\Phi} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_2^{\mathrm{T}} \\ \mathbf{I}_1^{\mathrm{T}} & -\mathbf{U} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_n \\ \boldsymbol{\Phi}_k \end{bmatrix} \tag{23}$$

From the lower part, the relation between the known across variables and the nodal variables can be written as:

$$\boldsymbol{X}_k = \mathbf{I}_1^{\mathrm{T}} \boldsymbol{\Phi}_n - \boldsymbol{\Phi}_k \tag{24}$$

From this equation the nodal variable $\boldsymbol{\Phi}_k$ of k-nodes is expressed as:

$$\boldsymbol{\Phi}_k = \mathbf{I}_1^{\mathrm{T}} \boldsymbol{\Phi}_n - \boldsymbol{X}_k \tag{25}$$

that is, $\boldsymbol{\Phi}_k$ can be expressed as linear combinations of $\boldsymbol{X}_k$ and $\boldsymbol{\Phi}_n$. Substituting this relation into the upper part of eqn. (23), a reduced nodal transformation equation can be obtained as:

$$\begin{aligned} \boldsymbol{X}_f &= \mathbf{I}_2^{\mathrm{T}} \boldsymbol{\Phi}_k \\ &= \mathbf{I}_2^{\mathrm{T}} (\mathbf{I}_1^{\mathrm{T}} \boldsymbol{\Phi}_n - \boldsymbol{X}_k) = (\mathbf{I}_1 \mathbf{I}_2)^{\mathrm{T}} \boldsymbol{\Phi}_n - \mathbf{I}_2^{\mathrm{T}} \boldsymbol{X}_k \end{aligned} \tag{26}$$

or:

$$\boldsymbol{X}_f = \mathbf{II}^{\mathrm{T}} \boldsymbol{\Phi}_n - \mathbf{I}_2^{\mathrm{T}} \boldsymbol{X}_k \tag{27}$$

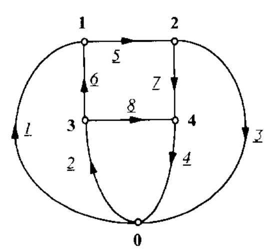


Fig. 3. A simple graph for the building.

The second term on the right side of eqn. (27) can be calculated as:

$$-\mathbf{I}_2^{\mathrm{T}}\boldsymbol{X}_k=\begin{bmatrix} X_{13}-X_9 \\ X_{20}-X_{10} \\ X_{11}-X_{17} \\ X_{12}-X_{24} \\ X_{16}-X_{14} \\ X_{15}-X_{19} \\ X_{22}-X_{18} \\ X_{23}-X_{20} \end{bmatrix} \tag{28}$$

Comparing it to the graph in Fig. 2, it can be concluded that each element of the vector on the right side of the above equation represents the combination effect of wind and stack driving forces exerted on an opening. Let the vector of those combinatory driving forces be:

$$\boldsymbol{P}_f=-\mathbf{I}_2^{\mathrm{T}}\boldsymbol{X}_k \tag{29}$$

then, the nodal transformation eqn. (27) can be replaced by:

$$\boldsymbol{X}_f=\mathbf{II}^{\mathrm{T}}\boldsymbol{\Phi}_n+\boldsymbol{P}_f \tag{30}$$

For an opening (l) which connects rooms l_i and l_j the stack effect acting on the opening is calculated according to eqn. (5) as:

$$[\boldsymbol{P}_f]_l=\frac{P_{\mathrm{a}}}{R_{\mathrm{a}}}g\left(\frac{h_{l_j}}{T_{l_j}}-\frac{h_{l_i}}{T_{l_i}}\right) \tag{31}$$

while for an opening (l) which connects the inside and the outside of the building, the combination of wind and stack forces is:

$$[\boldsymbol{P}_f]_l=P_l^{\mathrm{w}}+\frac{P_{\mathrm{a}}}{R_{\mathrm{a}}}g\left(\frac{h_{l_j}}{T_{l_j}}-\frac{h_{l_i}}{T_{l_i}}\right) \tag{32}$$

where P_l^{w} is the wind-induced pressure acting on the opening.

Flow equations (or terminal equations) of the flow resistance edges are represented by:

$$\boldsymbol{Y}_f=\mathbf{F}(\boldsymbol{X}_f) \tag{33}$$

where $\mathbf{F}()$ is a diagonal matrix and each diagonal element represents a function of the corresponding flow equation. Combining the minimum set of vertex eqn. (22), the reduced nodal transformation eqn. (30) and the flow eqn. (33), the system equation in matrix form can finally be obtained as:

$$\mathbf{II}\{\boldsymbol{F}(\mathbf{II}^{\mathrm{T}}\boldsymbol{\Phi}_n+\boldsymbol{P}_f)\}=\mathbf{0} \tag{34}$$

This is the matrix form of the governing equations for the given building airflow system. The only unknowns in the equation are the room nodal pressures (with respect to atmospheric pressure).

Equation (34) is a set of non-linear equations and is usually solved by iterative algorithms. The most popular solution approach is the Newton–Raphson method. Basically, this method calculates new values of the unknowns, at each iteration, by linearizing the flow function $\mathbf{F}()$ and solving for the resultant set of linear equations. Each iteration brings the set of unknown variables closer to the actual values. Convergence and the speed of this method prove to be a problem [15]. Modifications, such as the relaxation technique, have been used to avoid the convergence problem and to accelerate the rate of convergence [5, 16].

According to the Newton–Raphson method, the iteration algorithm is given by:

$$\boldsymbol{\Phi}_n^{k+1}=\boldsymbol{\Phi}_n^k-\left[\frac{\partial}{\partial\boldsymbol{\Phi}_n}\{(\mathbf{F}(\mathbf{II}^{\mathrm{T}}\boldsymbol{\Phi}_n+\boldsymbol{P}_f)\}\right]^{-1} \times[\mathbf{IIF}(\mathbf{II}^{\mathrm{T}}\boldsymbol{\Phi}_n+\boldsymbol{P}_f)] \tag{35}$$

The partial derivatives of functions $\mathbf{F}()$ with respect to nodal pressure at $\boldsymbol{\Phi}_n=\boldsymbol{\Phi}_n^k$ is:

$$\begin{aligned}\mathbf{J}&=\frac{\partial}{\partial\boldsymbol{\Phi}_n}\{\mathbf{IIF}(\mathbf{II}^{\mathrm{T}}\boldsymbol{\Phi}_n^k+\boldsymbol{P}_f)\}\\&=\mathbf{II}\frac{\partial}{\partial\boldsymbol{\Phi}_n}\{\mathbf{F}(\mathbf{II}^{\mathrm{T}}\boldsymbol{\Phi}_n^k+\boldsymbol{P}_f)\}\\&=\mathbf{II}\left\{\frac{\mathrm{d}\mathbf{F}(\boldsymbol{X}_f)}{\mathrm{d}\boldsymbol{X}_f}\times\frac{\partial}{\partial\boldsymbol{\Phi}_n}(\mathbf{II}^{\mathrm{T}}\boldsymbol{\Phi}_n^k+\boldsymbol{P}_f)\right\}\\&=\mathbf{II}\times\left\{\frac{\mathrm{d}\mathbf{F}(\boldsymbol{X})_f}{\mathrm{d}\boldsymbol{X}_f}\right\}\times\mathbf{II}^{\mathrm{T}}\end{aligned} \tag{36}$$

or:

$$\mathbf{J}=\mathbf{IIDII}^{\mathrm{T}} \tag{37}$$

where $\mathbf{D}=\mathrm{d}\mathbf{F}(\boldsymbol{X}_f)/\mathrm{d}\boldsymbol{X}_f$ is a diagonal matrix. Each diagonal element $D_{ii}=\mathrm{d}F_i(\boldsymbol{X}_f)/\mathrm{d}\boldsymbol{X}_i$ is called the dynamic admittance of the flow opening i.

Let $\mathbf{J}$ of eqn. (37) be called the dynamic nodal-admittance matrix. For the four-room building case, this matrix can be calculated as:

$$\mathbf{J}=\begin{bmatrix} D_1+D_5+D_6 & -D_5 & -D_6 & 0 \\ -D_5 & D_3+D_5+D_7 & 0 & -D_7 \\ -D_6 & 0 & D_2+D_6+D_8 & -D_8 \\ 0 & -D_7 & -D_8 & D_4+D_7+D_8 \end{bmatrix}$$

where

$$D_i = \frac{\mathrm{d}}{\mathrm{d}X_i}[K_i X_i^{n_i}] = n_i K_i (X_i^k)^{(n_i - 1)}$$

The relation between the nodal-admittance matrix and the topology of the physical connection graph for the building is as follows:

(a) the diagonal elements J_{ii} are composed of the sum of the admittances of edges that are connected to the ith (room) node, the sign is positive, and

(b) the off-diagonal elements J_{ij} are composed of the sum of the dynamic admittances of edges that connect (room) nodes i and j, the sign is negative. The symmetric and non-singular properties of the nodal-admittance matrix $\mathbf{J}$ ensure the solution to the update eqn. (35).

The iterative formula (eqn. (35)) can be expressed in update form:

$$\boldsymbol{\Phi}_n^{k+1} = \boldsymbol{\Phi}_n^k + \boldsymbol{C}^k \tag{38}$$

where the update (or correction) vector $\boldsymbol{C}^k$ is computed by:

$$\mathbf{\Pi D \Pi}^{\mathrm{T}} \boldsymbol{C}^k = -\mathbf{\Pi} \boldsymbol{Y}_f^k \tag{39}$$

The initial value for the nodal across variable, $\boldsymbol{\Phi}_n^0$, is obtained by approximating each airflow component as a linear relation. This is equivalent to letting the vertex equation be:

$$\boldsymbol{Y}_f = \mathbf{F}(\boldsymbol{X}_f) = \boldsymbol{A} + \mathbf{B}\boldsymbol{X}_f \tag{40}$$

The vertex equation is thus a set of linear equations of the form:

$$\mathbf{\Pi}[\boldsymbol{A} + \mathbf{B}(\mathbf{\Pi}^{\mathrm{T}} \boldsymbol{\Phi}_n^0 + \boldsymbol{P}_f)] = \mathbf{0} \tag{41}$$

which leads to:

$$\boldsymbol{\Phi}_n^0 = -[\mathbf{\Pi B \Pi}^{\mathrm{T}}]^{-1} \mathbf{\Pi}[\boldsymbol{A} + \mathbf{B}\boldsymbol{P}_f] \tag{42}$$

The procedure given herein is called the *Nodal Tableau Formulation Method*; it considers the fundamental driving forces and resistance forces. The influence of each component is clearly indicated in the procedure and the final set of equations. Therefore, it leads to a better understanding of the mechanism and modelling process of the airflow in buildings.

The calculated nodal pressures and airflow rates of the case study are shown in Fig. 4. Comparing values to the results of Walton's model [5], the difference in nodal pressures is within 0.02%.

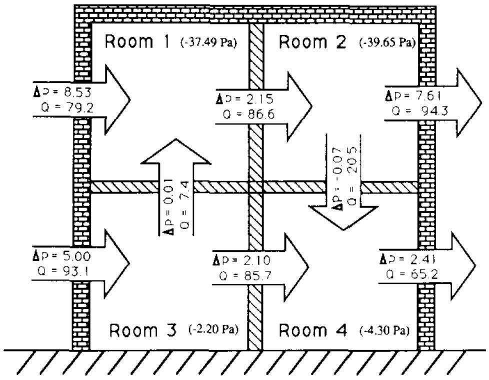


Fig. 4. Calculated airflow rates and pressures (units: pressure – Pascal, airflow rate – kg/h).

Automatic formulation

The above procedure shows the system-theoretic formulation and solution method for the governing equations of the airflow system of a building. It utilizes the graph which is composed of the fundamental driving forces and resistance forces. The influence of each individual component is clearly indicated in the formulation.

Analysis of the system eqn. (35) indicates that it can be formulated from the physical connecting graph of the building airflow system. Therefore, in computational implementation, inputs only require the physical connection data and opening characteristics (which include the flow equations and locations of openings).

The formulation can be further 'automated' by taking advantage of the regular composition of the $\mathbf{J}$ matrix in the iterative update calculation. Since the elements of the $\mathbf{J}$ matrix can be assembled by simple rules, this matrix can be automatically generated from flow equations and the physical connection data.

Conclusions

It was demonstrated that the building airflow system can be modelled as a graph which has the fundamental driving forces and flow resistances as edges. The Nodal Tableau Method is used to formulate a minimum set of non-linear governing equations with room nodal pressures as unknown variables. The analysis of the resultant equations concludes that, due to inherent properties of the airflow systems, the graph has certain patterns which enable direct formulation of the governing equations without consulting the complete graph. Governing equations can be derived from the physical connection data and opening characteristics.

The introduction of the complete graph helps understanding of the mechanism of the airflow caused by the driving forces. The presented analysis on nodal equations clearly exhibits the formulation procedure of the system equations for building airflow caused by the driving forces and flow resistance of the airflow.

Acknowledgements

Financial support through research grants from the Department of Energy, Mines and Resources Canada and National Science and Engineering Research of Canada is acknowledged with special thanks.

References

1 F. Haghighat, Air infiltration and indoor air quality models — a review, *Int. J. Ambient Energy, 10* (3) (1989) 115–122.
2 N. A. Said, A review of smoke control models, *ASHRAE J., 30* (4) (1988) 36–40.
3 H. E. Feustel and V. M. Kendon, Infiltration models for multicellular structures — a literature review, *Energy Build., 8* (1985) 123–136.
4 M. Liddament and C. Thompson, Mathematical models of air infiltration — a brief review and bibliography, *Tech. Note AIC 9*, Air Infiltration Centre, Bracknell, U.K., 1982.
5 G. N. Walton, Airflow network models for element-based building airflow modelling, *ASHRAE Trans., 95* (2) (1989) 611–104.
6 *ASHRAE Handbook — 1985 Fundamentals*, American Society of Heating, Refrigeration, and Air Conditioning Engineers, Inc., Atlanta.
7 M. N. Liddament, *Air Infiltration Calculation Techniques — An Applications Guide*, Air Infiltration and Ventilation Centre, Bracknell, U.K., 1986.
8 D. W. Etheridge, Crack flow equations and scale effect, *Building Environ., 12* (1977) 181–189.
9 P. S. Charlesworth, *Air Exchange Rate and Airtightness Measurement Techniques — An Application Guide*, Air Infiltration and Ventilation Centre, Bracknell, U.K., 1988.
10 M. N. Liddament, Power law rules — OK?, *Air Infiltration Rev., 8* (2) (1987) 4–6.
11 S. J. Irving, The computer simulation of smoke movement during building fires, *Fire Prev. Sci. Technol., 22* (1979) 3–8.
12 G. N. Walton, *Thermal Analysis Research Program Reference Manual, NBSIR 83-2655*, U.S. Department of Commerce, 1983.
13 F. Haghighat and M. Chandrashekar, System-theoretic models for building thermal analysis, *J. Solar Energy Eng., 109* (1987) 79–88.
14 H. K. Kesavan and M. Chandrashekar, Graph-theoretic models for pipe network analysis, *ASCE J. Hydraulics Div., 98* (1972) 345–364.
15 D. J. Wood and A. G. Rayes, Reliability of algorithms for pipe network analysis, *ASCE J. Hydraulics Div., 107* (1981) 1145–1163.
16 H. Feustel and A. Raynor-Hoosen (eds.), *Fundamentals of the multizone air flow model — COMIS, Document AIC-TN-29-90*, Air Infiltration and Ventilation Centre, Bracknell, U.K., 1990.