17-1-5-091 - CSE - B_2
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Q1. For the following data, calculate the differences and obtain the forward and backward difference polynomials. Interpolate at $x = 0.25$ and $x = 0.35$.

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.40	1.56	1.76	2.00	2.28

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	1.40			
		0.16		
0.2	1.56		0.04	
		+0.2		0 ~~0.04~~
0.3	1.76		+0.04	
		+0.24		
0.4	2.00			
		+0.28		
0.5	2.28			

$x = 0.25$

$x_0 = 0.2$

$h = 0.1$

$\therefore x = x_0 + ph$

$0.25 = 0.2 + h(0.1)$

$\frac{0.25 - 0.2}{0.1} = p$

$\therefore p = 0.5$

$\therefore y_1 = 1.56, \Delta y_1 = -0.2, \Delta^2 y_1 = -0.04$

$y_{0.25} = y_0 + p\Delta y_1 + \frac{p(p-1)}{2!}\Delta^2 y_1 \ldots$ by newton's forward interpolation formula

$y_{0.25} = 1.56 + (0.5)(+0.2) + \frac{(0.5)(0.5-1)}{2}(+0.04) \ldots$

$= 1.56 + 0.1 + (-0.125)(+0.04)$

$= 1.56 + 0.1 \mp 0.005$

$= 1.655$

$x = 0.35$

$x_0 = 0.4$

$h = 0.1$

$p = -0.5$

$x = x_0 + ph$

$0.35 = 0.4 + (0.1) p$

$\frac{0.35 - 0.4}{0.1} = p$

$p = -0.5$

$y_{0.4} = 2.00, \ \nabla y_{0.4} = 0.24, \ \nabla^2 y_{0.4} = 0.04$

$$y_{0.35} = y_{0.4} + p \nabla y_{0.4} + \frac{p(p+1)}{2} \nabla^2 y_{0.4}$$

... by newton's backward interpolation.

$$y_{0.35} = 2.00 + (-0.5)(0.24) + \frac{(-0.5)(0.5)}{2} \, 0.04$$

$$= 2 - 0.12 - 0.005$$

$$y_{0.35} = 1.875$$

Ex:2. Prove the following relations

(i) $\nabla - \Delta = -\Delta\nabla$

$\Delta = (1 - E^{-1})$

$\nabla = E - 1$

$\therefore \ -\Delta\nabla = (1 - E^{-1})(E - 1)$

$= E - 1 - 1 + E^{-1}$

$= E - 1 - (1 - E^{-1})$

$= \nabla - \Delta$

(ii) $\Delta + \nabla = \Delta/\nabla - \nabla/\Delta$

$$\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{\Delta^2 - \nabla^2}{\nabla\Delta} = \frac{(\Delta - \nabla)(\Delta + \nabla)}{\nabla\Delta}$$

$$= \frac{(\Delta - \nabla)(\Delta + \nabla)}{(\Delta - \nabla)} \quad \because \nabla\Delta = \Delta - \nabla$$

$$= \Delta + \nabla$$

Ex:3 The following table of values represents a polynomial of degree $n \leq 3$. Locate any error in the table of values.

x	$f(x) = y$	Δy	$\Delta^2 y$
0	2.00	0.11	0.06
0.1	2.11	0.17	-0.06
0.2	2.28	0.11	0.06
0.3	2.39	0.17	
0.4	2.56		

2 difference must be constant

$\therefore (1-t)^2 = {}^2C_0 \quad {}^2C_1 \quad {}^2C_2$

$\quad\quad\quad\quad 1 \quad -2 \quad 1$

$0.06 = a + t$

$-0.06 = a - 2t$

$0.06 = a + t$

$2a + 2t = 0.12$

$a - 2t = -0.06$

$3a = 0.06$

$\boxed{a = 0.02}$

$\boxed{t = 0.04}$

$\therefore y + t = 2.28$

$\boxed{y = 2.24}$

Ex-4 If $u_{-1} = 10,\ u_1 = 8,\ u_2 = 10,\ u_4 = 50$. find u_0 and u_3

x	-1	1	2	4
y	10	8	10	50

$$y = \frac{(x-1)(x-2)(x-4)}{(-1-1)(-1-2)(-1-4)}\,10 + \frac{(x+1)(x-2)(x-4)}{(1+1)(1-2)(1-4)}\,8 +$$

$$\frac{(x+1)(x-1)(x-4)}{(2+1)(2-1)(2-4)}\,10 + \frac{(x+1)(x-1)(x-2)}{(4+1)(4-1)(4-2)}\,50$$

$$y = \frac{(x-1)(x-2)(x-4)}{-30}10 + \frac{(x+1)(x-2)(x-4)}{6}8 +$$

$$\frac{(x+1)(x-1)(x-4)}{-6}(10) + \frac{(x+1)(x-1)(x-2)}{-30}(50)$$

$$y_0 = \frac{(+1)(+2)(+4)}{+3\cancel{0}}(1\cancel{0}) + \frac{(1)(+2)(-4)(8)}{\cancel{6}3} + \frac{(1)(+1)(\overset{2}{\cancel{+4}})(10)}{-\cancel{6}3}$$

$$+ \frac{(1)(-1)(+2)}{+3\cancel{0}}(5\cancel{0})$$

$$y_0 = \frac{8}{3} + \frac{32}{3} - \frac{20}{3} - \frac{10}{3} = \frac{8+32-30}{3} = 10/3$$

$$y_3 = \frac{(2)(1)(+1)(1\cancel{0})}{+3\cancel{0}} + \frac{(\overset{2}{\cancel{4}})(1)(-1)(8)}{3\cancel{6}} + \frac{(-4)(\cancel{2})(+1)(10)}{+\cancel{6}3} + \frac{(4)(2)(1)(5}{-3\cancel{0}}$$

$$= \frac{2}{3} + \frac{16}{3} + \frac{40}{3} - \frac{40}{3} = \frac{18}{3} = 6$$

Ex-5 Prove that Lagrange's interpolation formula can be put in the following form.

$$P_n(x) = \sum_{r=0}^{n} \frac{\phi(x)\, f(x_r)}{(x-x_r)\,\phi'(x_r)}$$

where $\phi(x) = \prod_{r=0}^{n} (x-x_r)$

Ex-6 Given the table of values.

x	50	52	54	56
$y=\sqrt[3]{x}$	3.684	3.732	3.779	3.825

$\sqrt[3]{x} = 3.756$, $x = ?$

$$x = \frac{(y-3.732)(y-3.779)(y-3.825)}{(3.684-3.732)(3.684-3.779)(3.684-3.779)}(50)$$

$$+ \frac{(y-3.684)(y-3.779)(y-3.825)}{(3.732-3.684)(3.732-3.779)(3.732-3.825)}(52)$$

$$+ \frac{(y-3.684)(y-3.732)(y-3.825)}{(3.779-3.684)(3.779-3.732)(3.779-3.825)}(54)$$

$$+ \frac{(y-3.684)(y-3.732)(y-3.779)}{(3.825-3.684)(3.825-3.732)(3.825-3.779)}(56$$

$y = 3.756$

$x = \cancel{(3.756)(3.75}$

$$x = \frac{(3.756-3.732)(3.756-3.779)(3.756-3.825)}{-4.332\times10^{-4}} \times 50$$

$\cancel{(3.56}$

$$+ \frac{(3.756-3.684)(3.756-3.779)(3.756-3.825)}{2.09808\times10^{-4}} \times 52$$

$$+ \frac{(3.\cancel{779}^{756}-3.684)(3.\cancel{756}^{756}-3.732)(3.\cancel{756}^{756}-3.825)}{-2.0539\times10^{-4}}(54)$$

$\cancel{(37}$

$$+ \frac{(3.756-3.684)(3.756-3.732)(3.756-3.779)}{6.03198\times10^{-4}} \times 56$$

$$x = \frac{0.38088}{-4.332}\times 50 + \frac{(\cancel{-1.19232})^{1.14264}}{(2.09808)}\times 52 + \frac{(-1.19232)}{(-2.0539)}\times 54$$

$$\times \frac{-0.39744}{6.03198}\times 56 = -4.3961 + 28.3198 + 31.3478 - 3.6897$$

$= 51.5818$

$$x = \frac{(y-3.732)(y-3.779)(y-3.825)(50)}{(3.684-3.732)(3.684-3.779)(3.684-3.825)}$$

$$+ \frac{(y-3.684)(y-3.779)(y-3.825)(52)}{(3.732-3.684)(3.732-3.779)(3.732-3.825)}$$

$$+ \frac{(y-3.684)(y-3.732)(y-3.825)(54)}{(3.779-3.684)(3.779-3.732)(3.779-3.825)}.$$

$$+ \frac{(y-3.684)(y-3.732)(y-3.779)\,56}{(3.825-3.684)(3.825-3.732)(3.825-3.779)}$$

$y = 3.756$.

$\therefore x = -2.9619 + 28.3198 + 31.3478 - 3.6897$

$x = 53.016$

Ex-7 Using lagrange's intp formula, express the function

$$\frac{x^2+6x-1}{(x^2-1)(x-4)(x-6)}$$ as a sum of partial fractions

$\therefore x$ — 1 — 4 — 6

f(x) Let us evaluate x^2+6x-1 for values $-1, +1, +4, +6$

$\therefore x: \quad -1 \quad +1 \quad +4 \quad +6$

$y: \quad -6 \quad 6 \quad 39 \quad 71$

By lagrange's formula.

$$y = \frac{(x-1)(x-4)(x-6)}{(-1-1)(-1-4)(-1-6)}(-6) + \frac{(x+1)(x-4)(x-6)}{(1+1)(1-4)(1-6)}(+6)$$

$$+ \frac{(x+1)(x-1)(x-6)}{(4+1)(4-1)(4-6)}(71) + \frac{(x+1)(x-1)(x-4)}{(6+1)(6-1)(6-4)}71$$

$$\therefore y = \frac{(x-1)(x-4)(x-6)(-6)}{-70} + \frac{(x+1)(x-4)(x-6)}{30}(6)$$

$$\frac{(x+1)(x-1)(x-6)(71)}{-30} + \frac{(x+1)(x-1)(x-4)\,71}{70}$$

Thus $$\frac{x^2+6x-1}{\cancel{(x^2+1)6}\,(x+1)(x-1)(x-4)(x-6)}$$

$$= \frac{6}{70(x+1)} + \frac{6}{30(x-1)} \mp \frac{71}{30(x-4)} + \frac{71}{70(x-6)}$$

Ex 8 Show that n^{th} order divided differences of a polynomial of n^{th} degree are constant

Sol:

Consider the divided difference of x^n,

$$Dx^n = \frac{(x+h)^n - x^n}{x+h-x} = \frac{nhx^{n-1} + \cdots}{h}$$

= a polynomial of degree $(n-1)$

Also since divided difference operator is a linear operator, D of any N^{th} degree polynomial is an $(N-1)^{th}$ degree polynomial and second D is an $(N-2)$ degree polynomial, so on the N^{th} divided difference of an N^{th} degree polynomial is a constant.

Ex-9 Write the merits and demerits of a lagrange's method

Merits of lagrange's method:

① Used in simultaneous optimization of norms of derivatives of lagrange polynomials.

② The answers for higher order polynomials will be more accurate.

③ For higher order polynomials the approximate result converges to the exact solution very quickly.

④ For higher order derivatives where n refers to the order of lagrange polynomial, the error decreases by 2^{n+1} if we decrease the distance b/w the interpolating points by 2.

Demerits

①. It becomes a tedious job to do when the polynomial order increases because the number of points increases and we need to evaluate the approximate sol's for each point, Ex-10

Given that $f(1)+f(2)+f(3)=25$, $f(4)=29$ and $f(5)+f(6)=113$, estimate the value of $f(7)$.

Ex-11 Prove the following identity

$$u_1x + u_2x^2 + u_3x^3 + \cdots$$
$$= \frac{x}{1-x}u_1 + \frac{x^2}{(1-x)^2}\Delta u_1 + \frac{x^3}{(1-x)^3}\Delta^2 u_1 + \cdots$$

$u_{x+h} = E^h u_x$

$\therefore u_1 = E u_0 , u_2 = E^2 u_0$

$\therefore u_1x + u_2x^2 + u_3x^3 + \cdots$

$= \cancel{(x E + x^2 E^2 + x^3 E^3 + \cdots \infty) u_1}$

$= (x + x^2E + x^3E^2 + \cdots \infty) u_1$

$= \cancel{xE(1 + xE + x^2 + \cdots \infty) u_0}$

$= \frac{xE}{1-xE} u_0 = \frac{x(1+\Delta)}{1-x(1+\Delta)} u_0$

$= \frac{(1+\Delta)}{(1+\Delta)} \frac{x}{\frac{1}{1+\Delta} - x} u_0$

$= u_0 x \left[\frac{1}{1+\Delta} - x\right]^{-1}$

$u_1x + u_2x^2 + u_3x^3$

$= (x + x^2E + x^3E^2 + \cdots \infty)u_1$

$= x\left[(1 + xE + x^2E^2 + x^3E^3 + \cdots \infty)u_1\right]$

$= x\left[\frac{1}{1-xE}\cdot u_1\right] = \frac{x}{1-x(1+\Delta)}u_1$

$= \frac{x}{(1-x)-x\Delta}u_1 = \frac{x}{(1-x)}\left\{1 - \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x^2)} + \cdots\right\}u_1$

$= \frac{xu_1}{1-x} + \frac{x^2}{(1-x)^2}\Delta u_1 + \frac{x^3}{(1-x)^3}\Delta^2 u_1 + \cdots$

Ex-12 Given that

$\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 7 = 0.8451$
$\log 13 = 1.1139$, $\log 19 = 1.2788$, $\log 37 = 1.5682$
Use appropriate formula to find $\log 37.2$.

x	y	y	Δy	$\Delta^3 y$	$\Delta^4 y$
35	log(35)	1.5441			
			0.0121		
36	log(36)	1.5562		-0.0001	
			0.012		-3×10^{-4}
37	log(37)	1.5682		-0.0004	
			0.0116		0
38	log(38)	1.5798		-0.0004	
			0.0112		2×10^{-4}
39	log(39)	1.5910		-0.0002	
			0.011		
40	log(40)	1.6020			

$\log(35) = \log(5) + \log(7) = \log(10/2) + \log 7$
$= \log(10) - \log(2) + \log(7)$
$= 1 - 0.3010 + 0.8451$
$\log(35) = 1.5441$

$\log(36) = \log(9\times4) = \log(9) + \log(4) = 2\log 3 + 2\log 2$
$= 2(0.4771) + 2(0.3010)$
$= 1.5562$

$\log(37) = 1.5682$

$\log(38) = \cancel{2}\log 2 + \log 19 = 0.3010 + 1.2788 = 1.5798$

$\log(39) = \log 3 + \log 13 = 1.1139 + 0.4771 = 1.5910$

$\log(40) = 2\log(2) + 1 = 1 + 0.6020 = 1.6020$

$x = x_0 + ph$
$h = 1$
$x_0 = 37$
$x = 37.2$

$\therefore 37.2 = 37 + p(1)$

$\frac{0.2}{1} = \boxed{p = 0.2}$

according to newton's central backward diff. int. formula

$y_p = 1.5682 + (0.2)\,\overset{0.0116}{\cancel{1.5682}} + \frac{(0.2+1)(0.2)}{2}\,\overset{-0.004}{(\cancel{0.0116})} + 0$

$= 1.5682 + \overset{0.00232}{\cancel{0.31364}} + \cancel{0.00139}$

$= 1.5705$