ASSIGNMENT 1

Q1. For the following data, calculate the differences and obtain the forward and backward difference polynomials. Interpolate at $x = 0.25$ and $x = 0.35$

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.40	1.56	1.76	2.00	2.28

Soln -

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	1.40			
		0.16		
0.2	1.56		0.04	
		0.2		0
0.3	1.76		0.04	
		0.24		0
0.4	2.00		0.04	
		0.28		
0.5	2.28			

Thus, the polynomial is of degree 2.

For forward interpolation,

$x = x_0 + ph = 0.1 + 0.1p \Rightarrow p = \frac{x-0.1}{0.1} \longrightarrow ①$

$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0$

$= 1.4 + p(0.16) + \frac{p(p-1)}{2}(0.04)$

$y_x = 1.4 + \left(\frac{x-0.1}{0.1}\right)(0.16) + \frac{\left(\frac{x-0.1}{0.1}\right)\left(\frac{x-0.1}{0.1}-1\right)}{2}(0.04)$

$= 1.4 + 1.6x - 0.16 + (x-0.1)(x-0.2)\times 2$

$= 1.4 + 1.6x - 0.16 + (x^2 - 0.3x + 0.02)\times 2$

$\boxed{y_x = 2x^2 + x + 1.28}$

$y(0.25) = 2(0.25)^2 + 0.25 + 1.28 = 1.655$

$y(0.35) = 2(0.35)^2 + 0.35 + 1.28 = 1.875.$

For backward interpolation,

$x = x_n + ph = 0.5 + 0.1p \Rightarrow p = \frac{x-0.5}{0.1} \longrightarrow ②$

$y_p = y_n + p\Delta y_n + \frac{p(p+1)}{2!}\Delta^2 y_n$

$= 2.28 + p(0.28) + \frac{p(p+1)}{2}(0.04)$

$y_x = 2.28 + \left(\frac{x-0.5}{0.1}\right)(0.28) + \frac{\left(\frac{x-0.5}{0.1}\right)\left(\frac{x-0.5}{0.1}+1\right)}{2}(0.04)$

$= 2.28 + 2.8x - 1.4 + (x-0.5)(x-0.4)2$

$$\boxed{y_x = 2x^2 + x + 1.28}$$

$y(0.25) = 2(0.25)^2 + 0.25 + 1.28 = 1.655$

$y(0.35) = 2(0.35)^2 + 0.35 + 1.28 = 1.875$

Q2. Prove the following relations.

i> $\nabla - \Delta = -\Delta\nabla$

we know that,

$$\begin{Bmatrix} \Delta = E-1, & E = \Delta+1 \\ \nabla = 1-E^{-1} & \end{Bmatrix}$$

$$\text{LHS} = \nabla - \Delta = 1 - 1/E - E + 1 = \frac{-E^2 + 2E - 1}{E}$$

$$= \frac{-(E^2 - 2E + 1)}{E}$$

$$= -\frac{(E-1)^2}{E}$$

$$\text{RHS} = -\Delta\nabla = -(E-1)(1-E^{-1})$$

$$= -(E-1)\frac{(E-1)^0}{E} = \frac{-(E-1)^2}{E}$$

Hence LHS = RHS.

ii> $\Delta + \nabla = \Delta/\nabla - \nabla/\Delta$

$$\text{RHS} = \Delta/\nabla - \nabla/\Delta = \frac{E-1}{1-1/E} - \frac{1-1/E}{E-1} = \frac{E-1}{E-1} \times E - \frac{E-1}{(E-1)E}$$

$$= E - \frac{1}{E}$$

$$= E - 1 + 1 - \frac{1}{E}$$

$$= \Delta + \nabla = \text{LHS}$$

Hence LHS = RHS.

Q4. If $u_{-1} = 10$, $u_1 = 8$, $u_2 = 10$, $u_4 = 50$, find u_0 and u_3

Sol^n:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-1	10				
		y_0-10			
0	y_0		$18-2y_0$		
		$8-y_0$		$3y_0-24$	
1	8		y_0-6		y_3-4y_0+18
		2		y_3-y_0-6	
2	10		y_3-12		y_0-4y_3+78
		y_3-10		$72-3y_3$	
3	y_3		$60-2y_3$		
		$50-y_3$			
4	50				

Now,

$3y_0-24=72-3y_3$ and $y_3-4y_0+18=0$

$\Rightarrow 3y_0+3y_3=96$

$\Rightarrow y_0+y_3=32$

$\Rightarrow y_0=32-y_3$

$\Rightarrow y_3-4(32-y_3)+18=0$

$\Rightarrow y_3-128+4y_3+18=0$

$\Rightarrow 5y_3=110$

$\Rightarrow y_3=22$

$\therefore y_0=32-22$

$=10$

$\therefore u_0=10,\ u_3=22$

Q5. Prove that Lagrange's interpolation formula can be put in the following form;

$$P_n(x)=\sum_{r=0}^{n}\frac{\phi(x)f(x_r)}{(x-x_r)\phi'(x_r)} \quad \text{where } \phi(x)=\prod_{r=0}^{n}(x-x_r)$$

Sol^n: According to Newton's interpolation formula,

$$y(x)=\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_n$$

$$+\cdots$$

$$+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n$$

$$=\frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_n)f(x_0)}{(x-x_0)\{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)\}}+\frac{(x-x_0)(x-x_2)\cdots(x-x_n)f(x_1)}{(x-x_1)\{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)\}}$$

$$+\cdots+\frac{(x-x_0)(x-x_1)\cdots(x-x_{r-1})\cancel{(x-x_r)}\cdots(x-x_{n-1})(x-x_n)}{(x-x_n)\{(x_r-x_0)(x_r-x_1)\cdots(x_r-x_{r-1})(x_r-x_{r+1})\cdots(x_r-x_{n-1})}y_n$$

$(x-x_r)$

$$+\cdots+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})\cancel{(x-x)}(x-x_n)}{(x-x_r)\{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})\}}f(x_n)$$

$$= \cancel{\sum_{r=0}^{n} \frac{(x-x_r) f(x_r)}{}}$$

$$= \sum_{r=0}^{n} \frac{\phi(x)\, f(x_r)}{(x-x_r)\,\phi(x_r)} \qquad \text{where } \phi(x) = \prod_{r=0}^{n} (x-x_r)$$

Q6. Given the table of values

x	50	52	54	56
y	3.684	3.732	3.779	3.825

Use Lagrange's intp formula to find x when $\sqrt[3]{x} = 3.756$

Sol

$\sqrt[3]{x} = 3.756$

$x = (3.756)^3 = 52.988$

$$y = f_x = \frac{(x-52)(x-54)(x-56)}{(-2)(-4)(-6)} 3.684 + \frac{(x-50)(x-54)(x-56)}{(2)(-2)(-4)} 3.732$$

$$+ \frac{(x-50)(x-52)(x-56)}{(4)(2)(-2)} 3.779 + \frac{(x-50)(x-52)(x-54)}{(6)(4)(2)} 3.825$$

$$= \cancel{\frac{(x^2-106x+2808)(x-56)}{-48} \times 3.684)}$$

$$y(52.988) = \frac{(0.988)(-1.012)(-3.012)}{-48} \times 3.684 + \frac{(2.988)(-1.012)(-3.012)}{16} 3.732$$

$$+ \frac{(2.988)(0.988)(-3.012)}{-16} 3.779 + \frac{(2.988)(0.988)(-1.012)}{48} 3.825$$

$= -0.231 \cancel{-0.238} + 2.124 + 2.1 - 0.238$

$= 3.755$

$$x = \frac{(3.756-3.732)(3.756-3.779)(3.756-3.825)}{(3.684-3.732)(3.684-3.779)(3.684-3.825)} \times 50 +$$

$$\frac{(3.756-3.684)(3.756-3.779)(3.756-3.825)}{(3.732-3.684)(3.732-3.779)(3.732-3.825)} \times 52 +$$

$$\frac{(3.756-3.684)(3.756-3.732)(3.756-3.825)}{(3.779-3.684)(3.779-3.732)(3.779-3.825)} \times 54 +$$

$$\frac{(3.785-3.684)(3.785-3.732)(3.785-3.779)}{(3.825-3.684)(3.825-3.732)(3.825-3.779)} \times 56$$

$= -2.962 + 28.319 + 31.347 - 3.689$

$= 53.015$

$x = 53.015$ //

Q7. Using Lagrange's interpolation formula, express the function

$$\frac{x^2+6x-1}{(x^2-1)(x-4)(x-6)}$$ as a sum of partial fractions.

Sol^n. For

$y = x^2+6x-1$

and $x=1$, $x=-1$, $x=4$, and $x=6$

x	-1	1	4	6
y	8	6	39	71

$$y = \frac{(x-1)(x-4)(x-6)}{(-2)(-5)(-7)}(-8) + \frac{(x+1)(x-4)(x-6)}{2.(-3)(-5)}.6 + \frac{(x+1)(x-1)(x-6)}{5.3.(-2)}.39$$

$$+ \frac{(x+1)(x-1)(x-4)}{7.5.2}.71$$

$$= \frac{4}{35}(x-1)(x-4)(x-6) + \frac{1}{5}(x+1)(x-4)(x-6) - \frac{13}{10}(x+1)(x-1)(x-6)$$

$$+ \frac{71}{70}(x+1)(x-1)(x-4)$$

Now,

$$\frac{x^2+6x-1}{(x^2-1)(x-4)(x-6)} = \frac{4}{35}.\frac{1}{(x+1)} + \frac{1}{5}.\frac{1}{(x-1)} - \frac{13}{10}.\frac{1}{(x-4)} + \frac{71}{70}.\frac{1}{(x-6)}$$

Q8. Show that nth order divided differences of a polynomial of nth degree are constant.

Proof:- Considering the divided difference of x^n

$$\Delta x^n = \frac{(x+h)^n - x^n}{x+h-x} = \frac{nhx^{n-1} + \cdots}{h}$$

which is a polynomial of degree $(n-1)$

Also, since divided difference operator is a linear operator.

Δ of any Nth degree polynomial is a (N-1)th degree polynomial

Similarly, Δ^2 of any Nth degree polynomial is a (N-2)th degree polynomial and so on.

Hence, Δ^n of an N^{th} degree polynomial is a constant.

Q9. Write the merits and demerits of Lagrange's interpolation method.

Soln- Merits:-

1. Used in simultaneous optimization of norms of derivatives of Lagrange polynomials.
2. The answers for higher order polynomials will be more accurate.
3. For higher order polynomials the approximate result converges to the exact solution very quickly.
4. For higher order derivatives where n refers to the order of the lagrange polynomial, the error decreases by 2^{n+1} if we decrease the distance between the interpolation points by 2.

Demerits:-

1. It becomes a tedious job to do when the polynomial order ~~because~~ increases because the number of points increases and we need to evaluate approximate solutions for each point

Q10. Given that $f(1)+f(2)+f(3) = 25$, $f(4) = 29$ and $f(5)+f(6) = 113$, estimate the value of $f(7)$.

Soln- $F(3) = f(1)+f(2)+f(3) = 25$

$F(4) = F(3) + f(4) = 54$

$F(6) = F(4) + f(5) + f(6) = 167$

x	3	4	6
$F(x)$	25	54	167

$$F(x) = \frac{(x-4)(x-6)}{(-1)(-3)}(25) + \frac{(x-3)(x-6)}{1.(-2)}(54) + \frac{(x-3)(x-4)}{3.2}(167)$$

$$F(7) = \frac{3.1}{3}.25 + \frac{4.1}{-2}.54 + \frac{4.3}{3.2}.167$$

$$= 25 - 108 + 334 = 251$$

$f(7) = F(7) - F(6) = 251 - 167 = 84$//

Q12. Given that.

$\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 7 = 0.8451$, $\log 13 = 1.1139$,

$\log 19 = 1.2788$, $\log 37 = 1.5682$

Use appropriate formula to find $\log 37.2$

Solⁿ. $\log 5 = 0.6989 = \log 2 - \log 3 = \log(2+3)$

$\log 35 = \log(5.7) = \log 5 + \log 7 = 1.544$

$\log 6 = \log(3.2) = \log 3 + \log 2 = 0.7781$

$\log 36 = \log 6^2 = 2\log 6 = 1.5562$

$\log 37 = 1.5682$

$\log 38 = \log(19.2) = \log 19 + \log 2 = 1.5798$

$\log 39 = \log(13.3) = \log 13 + \log 3 = 1.591$

$\log 40 = \log(2^3.5) = 3\log 2 + \log 5 = 1.6019$

x	y	Δy	$\Delta^2 y$
35	1.544		
		0.0122	
36	1.5562		-0.0002
		0.012	
37	1.5682		-0.0004
		0.0116	
38	1.5798		-0.0004
		0.0112	
39	1.591		-0.0003
		0.0109	
40	1.6019		

$x = x_0 + ph \Rightarrow 37.2 = 37 + p$

$\Rightarrow p = 0.2$

$$y_p = y_0 + p\Delta y + \frac{p(p-1)}{2!}\Delta^2 y$$

$$= 1.5682 + (0.2)(0.0116)$$

$$= 1.5705$$

Q11. Prove the following identity

$$u_1 x + u_2 x^2 + u_3 x^3 + \cdots = \frac{x}{1-x}u_1 + \frac{x^2}{(1-x)^2}u_1 + \frac{x^3}{(1-x)^3}u_1 + \cdots$$

Soln- $u_1 x + u_2 x^2 + u_3 x^3 + \cdots = (x + x^2 E + x^3 E^3 + \cdots \infty) u_1$

$$= (1 + xE + x^2E^2 + \cdots \infty) x u_1$$

$$= \frac{x u_1}{1 - xE}$$

Using $E = \Delta + 1$

$$\Rightarrow \frac{x u_1}{1 - x(\Delta + 1)}$$

$$= \frac{x u_1}{(1-x) - x\Delta} = \frac{1}{(1-x)}\left(1 - \frac{x\Delta}{1-x}\right)^{-1} x u_1$$

$$= \frac{1}{(1-x)}\left\{1 + \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x)^2} + \cdots\right\} x u_1$$

$$= \frac{x u_1}{1-x} + \frac{x^2 \Delta u_1}{(1-x)^2} + \frac{x^3 \Delta^2 u_1}{(1-x)^3} + \cdots$$

Proved.

Q3. The following values represents a polynomial of degree $n \leq 3$. Locate any error in the table of values.

Soln-

x	y	Δy	$\Delta^2 y$
0	2.00	0.11	0.06
0.1	2.11	0.17	-0.06
0.2	2.28	0.11	0.06
0.3	2.39	0.17	
0.4	2.56		

So, we get

$$-2e = -0.06$$

$$e = 0.03$$

So corrected value = given value − error

$$= \cancel{2.}\,2.28 - 0.03$$

$$= 2.25 \;//$$

HOMEWORK

Q. 1.1 Find the missing value in the following table

x	0	5	10	15	20	25
y	6	10	-	17	-	31

Find $y(10)$ and $y(20)$

Soln.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	6				
		4			
5	10		$y_{10}-14$		
		$y_{10}-10$		$41-3y_{10}$	
10	y_{10}		$27-2y_{10}$		$y_{20}+6y_{10}-102$
		$17-y_{10}$		$y_{20}+3y_{10}-61$	
15	17		$y_{20}+y_{10}-34$		$143-4y_{20}-4y_{10}$
		$y_{20}-17$		$82-3y_{20}-y_{10}$	
20	y_{20}		$48-2y_{20}$		
		$31-y_{20}$			
25	31				

We have 4 available values, hence the polynomial needs to be of degree 3. The 3rd order diff are constant and higher order diff are 0.

$41-3y_{10} = y_{20}+3y_{10}-61$ and $143-4y_{20}-4y_{10}=0$

$\Rightarrow y_{20}+6y_{10} = 102$

$\Rightarrow y_{20} = 102-6y_{10} \longrightarrow (1)$

$\Rightarrow \frac{143}{4} = y_{20}+y_{10}$

$\Rightarrow \frac{143}{4} = 102-6y_{10}+y_{10}$

$\Rightarrow 5y_{10} = 102-\frac{143}{4}$

$\Rightarrow y_{10} = \frac{265}{20} = \frac{53}{4}$

and $y_{20} = 102 - 6\times\frac{53}{4}$

$= \frac{45}{2}$

Q. 1.2. Construct a difference table from the following data

x	2	4	6	8	10	12	14	16
y	31	52	61	79	110	150	162	199

Find the value of $\Delta^3 y(6)$, $\Delta^4 y_8$, ∇y_{12}, $\nabla^2 y_{13}$, $\nabla^5 y_{16}$

Soln-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
2	31						
		21					
4	52		-12				
		9		21			
6	61		9		-17		
		18		4		9	
8	79		-13		-8		-34
		31		-4		-25	
10	110		9		-33		148
		40		-37		123	
12	150		-28		90		
		12		53			
14	162		25				
		37					
16	199						

$\Delta^3 y_6 = -4$ $\quad$ $\nabla y_{12} = 40$ $\quad$ $\nabla^5 y_{16} = 123.$

$\Delta^4 y_8 = 90$ $\quad$ $\nabla^2 y_{14} = -28$

Q 2.1. $\Delta \tan^{-1}\left(\frac{n-1}{n}\right) = \tan^{-1}\left(\frac{1}{2n^2}\right)$ Prove.

Soln- $LHS = \Delta \tan^{-1}\left(\frac{n-1}{n}\right) = \tan^{-1}\left(\frac{n}{n+1}\right) - \tan^{-1}\left(\frac{n-1}{n}\right)$

$$= \tan^{-1}\left(\frac{\frac{n}{n+1} - \frac{n-1}{n}}{1 + \frac{n-1}{n+1}}\right) = \tan^{-1}\left(\frac{1}{2n^2}\right) = RHS$$

Hence proved.

Q.2.2. Solve $\Delta(1/x)$

Solve: $\Delta(1/x) = \frac{1}{x+1} - \frac{1}{x} = \frac{x - x - 1}{x(x+1)} = \frac{-1}{x(x+1)}$

Q2.3. Solve $\Delta^n \sin(ax+b)$

Soln- $\Delta^n \sin(ax+b)$

$= \Delta^{n-1}(\Delta \sin(ax+b)) = \Delta^{n-1}(\sin(ax+a+b) - \sin(ax+b))$

$= \Delta^{n-1}(2\cos(ax+b+a/2)\sin(a/2))$

$= 2\sin(a/2)\, \Delta^{n-2}(\cos(ax+b+3a/2) - \cos(ax+b+a/2))$

$= 2\sin(a/2)\, \Delta^{n-2}(-2\sin(a/2)\sin(ax+b+a))$

$= (2\sin a/2)^n (-1)^{n+1} \sin\left(ax+b + \frac{na}{2} + \frac{n\pi}{2}\right)$

Q.3.1. Find the value of $f(5.8)$ given that $f(4) = 270$, $f(5) = 648$, $\Delta f(5) = 682$, $\Delta^3 f(4) = 132$.

It is given that $f(x)$ is a polynomial of degree 4.

Soln. $\Delta f(5) = f(6) - f(5)$

$\Rightarrow 682 = f(6) - 648$

$\Rightarrow f(6) = 1330$

$\Delta^3 f(4) = f(7) - 3f(6) + 3f(5) - f(4)$

$\Rightarrow 132 = f(7) - 3990 + 1944 - 270$

$\Rightarrow f(7) = 2448.$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	270			
		378		
			304	
5	648			132.
		682		
			436	
6	1330			
		1118		
7	2448			

$5.8 = 5 + p \cdot 1$

$\Rightarrow p = 0.8$

Applying Gauss forward interpolation.

$$y_{5.8} = 648 + p(682) + \frac{p(p-1)}{2}(304) + \frac{(p+1)p(p-1)}{6}(132)$$

$$= 648 + 545.6 - 24.32 - 6.336$$

$$= 1162.944$$

Q.3.2. $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 7 = 0.8451$, $\log 13 = 1.1139$, $\log 19 = 1.2788$, $\log 37 = 1.5682$

... interpolation formula to find $\log 37.2$

x	y	Δy	$\Delta^2 y$
35	1.544		
		0.0122	
36	1.5562		-0.0002
		0.012	
37	1.5682		-0.0004
		0.116	
38	1.5798		-0.0004
		0.0112	
39	1.591		-0.0003
		0.0109	
40	1.6099		

$x = x_0 + ph$

$37.2 = 37 + p$

$p = 0.2$

$$y_p = y_0 + p\Delta y + \frac{p(p-1)}{2}\Delta^2 y$$

$$= 1.5682 + (0.2)(0.116) + \frac{(0.2)(-0.8)}{2}(-0.0004)$$

$$= 1.5882 + 0.0232 + 0.000032$$

$$= 1.59143$$

Q. 4.1. Applying Lagrange's intp. formula prove that

i) $y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$

Soln.-

x	-3	-1	1	3
y	y_{-3}	y_{-1}	y_1	y_3

$$y_x = \frac{(x+1)(x-1)(x-3)}{(-2)(-4)(-6)}(y_{-3}) + \frac{(x+3)(x-1)(x-3)}{(2)(-2)(-4)}y_{-1} + \frac{(x+3)(x+1)(x-3)}{(4)(2)(-2)}y_1 + \frac{(x+3)(x+1)(x-1)}{(6)(4)(2)}y_3$$

Now

$$y_0 = \frac{-3}{48}y_{-3} + \frac{9}{16}y_{-1} + \frac{9}{16}y_1 - \frac{3}{48}y_3$$

$$= \frac{1}{2}(y_1 + y_{-1}) + \frac{1}{16}y_{-1} + \frac{1}{16}y_1 - \frac{1}{16}y_{-3} - \frac{1}{16}y_3$$

$$= \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{16}\left[-y_{-1} - y_1 + y_{-3} + y_3\right]$$

$$= \frac{1}{2}(y_1 - y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$$

Proved.

iv> $y_3 = 0.05(y_0+y_6) - 0.3(y_1+y_5) + 0.75(y_2+y_4)$

Sol^n-

x	0	1	2	4	5	6
y	y_0	y_1	y_2	y_4	y_5	y_6

$$y_x = \frac{(x-1)(x-2)(x-4)(x-5)(x-6)}{(-1)(-2)(-4)(-5)(-6)}y_0 + \frac{(x-0)(x-2)(x-4)(x-5)(x-6)}{(-1)(-3)(-4)(-5)}y_1$$

$$+ \frac{(x)(x-1)(x-4)(x-5)(x-6)}{(2)(-2)(-3)(-4)}y_2 + \frac{(x)(x-1)(x-2)(x-5)(x-6)}{(4)(3)(2)(-1)(-2)}y_4$$

$$+ \frac{(x)(x-1)(x-2)(x-4)(x-6)}{(5)(4)(3)(1)(-1)}y_5 + \frac{(x)(x-1)(x-2)(x-4)(x-5)}{(6)(5)(4)(2)(1)}y_6$$

$$y_3 = 0.05y_0 - 0.3y_1 + 0.75y_2 + 0.75y_4 - 0.3y_5 + 0.05y_6$$

$$= 0.05(y_0+y_6) - 0.3(y_1+y_5) + 0.75(y_2+y_4)$$

Proved.

Q4.2. Apply Lagrange's interpolation formula to find the value of $f(x)$ for $x=0$, from the table.

Sol^n-

x	-1	-2	2	4
y	-1	-9	11	69

~~y(0) = (0+2)(0-2)(0-~~

$$y(x) = \frac{(x+2)(x-2)(x-4)}{(1)(-3)(-5)}(-1) + \frac{(x+1)(x-2)(x-4)}{(-1)(-4)(-6)}(-9) +$$

$$\frac{(x+1)(x+2)(x-4)}{(3)(4)(-2)}(11) + \frac{(x+1)(x+2)(x-2)}{(5)(6)(2)}(69)$$

$$y(0) = \frac{(2)(-2)(-4)}{(-3)(-5)}(-1) + \frac{(-2)(-4)(-9)}{(-1)(-4)(-6)} + \frac{(2)(-4)(11)}{(3)(4)(-2)} + \frac{(2)(-2)(69)}{(5)(6)(2)}$$

$$= -1.0667 + 3 + 3.6667 - 4.6$$

$$= 1$$

Q.4.3. Determine the function U_x in power of $(x-1)$ given that $u_0 = 8$, $u_1 = 11$, $u_4 = 68$ and $u_5 = 123$.

Sol^n-

x	y	Δ	Δ^2	Δ^3
0	8			
		3		
1	11		4	1
		19		
4	68		9	
		55		
5	123			

$y = 8 + (x)[3] + x(x-1)(4) + x(x-1)(x-4)(1)$

$= 8 + 3x + (x^2 - x)4 + (x^2 - 5x + 4)(x)$

$= 8 + 3x + 4x^2 - 4x + x^3 - 5x^2 + 4x$

$= x^3 - x^2 + 3x + 8$

$= (x-1)^3 + 2x^2 + 9$

$= (x-1)^3 + 2(x-1)^2 + 4x + 7$

$= (x-1)^3 + 2(x-1)^2 + 4(x-1) + 11$ //

Q.5.1 Use bisection method to find the roots.

i> $f(x) = x^3 - x - 1$

Sol^n- $f(x) = x^3 - x - 1$

$f(0) = -1 < 0$ $\quad f(1) = -1 < 0$

$f(1.5) = 0.875 > 0$

$f(0)\, f(1.5) < 0$

the real root is in the interval $[0, 1.5]$

app.	a	b	$c = \frac{a+b}{2}$	f(c)
1	0	1.5	0.75	-1.3281
2	0.75	1.5	1.125	-0.7011
3	1.125	1.5	1.3125	-0.0515
4	1.3125	1.5	1.40625	0.37466
5	1.3125	1.40625	1.359375	0.1526
6	1.3125	1.359375	1.3359375	0.0483
7	1.3125	1.336	1.32425	-0.00199
8	1.32425	1.336	1.330125	

$x = 1.33$ (approx).

ii) $x \log_{10} x = 1.2$ $[2, 3]$ (Bisection

sol^n. $f(x) = x \log x - 1.2$

$f(2) f(3) < 0$

app	a	b	$c = \frac{a+b}{2}$	$f(c)$
1	2	3	2.5	-0.2051
2	2.5	3	2.75	0.00816
3	2.5	2.75	2.625	-0.0997
4	2.625	2.75	2.6875	-0.0461
5	2.6875	2.75	2.71875	-0.019
6	2.71875	2.75	2.7344	-0.0054
7	2.7344	2.75	2.7422	0.00135
8	2.7344	2.7422	2.7383	-0.00204
9	2.7383	2.7422	2.74025	-0.0003
10	2.7402	2.7422		

$x = 2.74$ (approx).

iii) $x^3 - 5x + 3 = 0$ (Bisection method)

sol $f(x) = x^3 - 5x + 3$

$f(0) = 3 > 0$ $f(1) = -1 < 0$

$f(0.5) = 0.625 > 0$

$f(0.5) f(1) < 0$

real root is in the interval $[0.5, 1]$

app	a	b	$c = \frac{a+b}{2}$	$f(c)$
1	0.5	1	0.75	-0.328
2	0.5	0.75	0.625	0.1191
3	0.625	0.75	0.6875	-0.112
4	0.625	0.6875	0.65625	0.0013
5	0.65625	0.6875	0.671875	-0.056
6	0.65625	0.671875	0.664	-0.027
7	0.664	0.6719	0.6679	-0.041
8	0.6679	0.6719	0.67	-0.049
9	0.67	0.6719		

$x = 0.67$ (approx)

iv) $x^3 - 12.2x^2 + 7.45x + 42 = 0$ (Bisection method)

Sol^n. $f(x) = x^3 - 12.2x^2 + 7.45x + 42$

$f(2.4) = 3.432 > 0 \quad f(2.6) = -3.526 < 0$

$f(2.4)\,f(2.6) < 0$

real root lies between $[2.4, 2.6]$

app	a	b	c	f(c)
1	2.4	2.6	2.45	1.728
2	2.45	2.6	2.525	-0.8229
3	2.45	2.525	2.4875	0.4342
4	2.4875	2.525	2.506	-0.208
5	2.4875	2.506	2.49675	0.113
6	2.49675	2.506	2.5013	-0.047
7	2.49675	2.5013	2.4990	0.0334
8	2.499	2.5013	2.5	

$x = 2.5$ (approx).

Q.5.2. Find the real root of $\cos x = 3x - 1$ correct upto 3 decimal place.

Sol^n. $f(x) = \cos x - 3x + 1$

$f(0) = 2 > 0 \quad f(\pi/2) = < 0 \quad$ root lies in $(0, \pi/2)$

$x = \dfrac{\cos x + 1}{3}$

$\phi(x) = \dfrac{\cos x + 1}{3}$

$|\phi'(x)| = \left|\dfrac{\sin x}{3}\right| \quad |\phi'(\pi/4)| = \left|\dfrac{\sin(\pi/4)}{3}\right| = 0.2357 < 1$

$\phi(x) = \dfrac{\cos x + 1}{3}$ is a proper choice $\quad \phi_0(x) = \pi/4 = 0.785$

Itr no	x_n	x_{n+1}
0	0.785	0.5691
1	0.5691	0.6141
2	0.6141	0.6057
3	0.6057	0.6073
4	0.6073	0.6070

$x = 0.607$ (approx)

which is the required root.

Q.5.3. $x = e^{-x}$ (Iterative method)

<u>Solⁿ</u> $f(x) = e^{x}x - 1$

$f(0) < 0 \quad f(1) > 0$

real root lies in the interval (0,1)

now,

$\phi(x) = e^{-x}$

$|\phi'(0.5)| = |e^{-0.5}| = 0.6065 < 1$

thus $\phi(x) = e^{-x}$ is a proper choice

now $x_0 = 0.5$

$x_{n+1} = e^{-x_n}$

Itr no.	x_n	x_{n+1}
0	0.5	0.6065
1	0.6065	0.5452
2	0.5452	0.5796
3	0.5796	0.56
4	0.56	0.5711
5	0.5711	0.5648
6	0.5648	0.5684
7	0.5684	0.5664
8	0.5664	0.5675
9	0.5675	0.5669
10	0.5669	0.5672
11	0.5672	0.567

$x = 0.567$ (approx)

is the real root

Q.5.4. Find the real root of the eqⁿ $x^3 - 5x + 3 = 0$ (Iterative method)

<u>Solⁿ</u> $f(x) = x^3 - 5x + 3 = 0$

$f(1) = -1 < 0 \quad f(2) = 1 > 0$

real root lies in the interval (1,2)

now, $\phi(x) = \dfrac{x^3+3}{5}$

$\phi'(x) = \dfrac{3x^2}{5} \quad |\phi'(1.5)| = 1.35 > 1$

which is not a proper choice

Now, $\phi(x) = \sqrt[3]{5x-3}$

$$\phi'(x) = \frac{5}{3(5x-3)^{2/3}}$$

$$|\phi'(1.5)| = \left|\frac{5}{3(5(1.5)-3)^{2/3}}\right| = 0.611 < 1$$

which is a proper choice.

$x_0 = 0.5$

$x_{n+1} = \sqrt[3]{5x_n - 3}$

Itr no.	x_n	x_{n+1}
0	0.5	-0.793
1	-0.793	-1.910
2	-1.91	-2.323
3	-2.323	-2.445
4	-2.445	-2.478
5	-2.478	-2.487
6	-2.487	-2.489
~~7~~	~~-2.489~~	

$x = -2.48$ (approx) is a real root of the eqn.

Q.55. $2x - \log_{10} x = 7$ (Iterative method)

Soln.- $f(x) = 2x - \log_{10} x - 7$

$f(3) = -1.477 < 0$ $\quad f(4) = 0.397 > 0$

real root lies in the interval (3,4)

Now, $x = \dfrac{7+\log_{10} x}{2}$

$$\phi_1(x) = \frac{7+\log_{10} x}{2}$$

$$|\phi_1'(x)| = \frac{1}{2x \ln 10} = \frac{\log_{10} e}{2x} = \frac{\ln e}{2x} = \left|\frac{0.4342}{2x}\right|$$

$|\phi_1'(3.5)| = 0.062 < 1$

which is a proper choice.

$x_0 = 3.5$

$$x_{n+1} = \frac{7+\log_{10} x_n}{2}$$

Itr no.	x_n	x_{n+1}
0	3.5	3.772
1	3.772	3.788
2	3.788	3.789
3	3.789	3.7892

$\therefore x = 3.7892$ (approx)

is the real root of the given equation

Q.5.6. $e^x \tan x = 1$ (Iterative method)

Soln. $f(x) = e^x \tan x - 1$

$f(0) = -1 < 0$ $\quad f(1) = 3.233 > 0$

$f(0)f(1) < 0$

The real root lies in the interval $[0,1]$.

Now,

$$x = -\ln(\tan x)$$

$$\phi(x) = -\ln(\tan x)$$

$$|\phi'(x)| = \left|\frac{\sec^2 x}{\tan x}\right| = \left|\frac{\cos x}{\cos^2 x \sin x}\right| = \left|\frac{1}{\sin x \cos x}\right|$$

$$|\phi'(0.5)| = 2.376 > 1$$

which is not a proper choice.

Now,

$$x = \tan^{-1}(e^{-x})$$

$$\phi(x) = \tan^{-1}(e^{-x})$$

$$|\phi'(x)| = \left|\frac{e^{-x}}{1+e^{-2x}}\right|$$

$$|\phi'(0.5)| = 0.443 < 1$$

which is a proper choice

Itr no.	x_n	x_{n+1}
0	0.5	0.5452
1	0.5452	0.5253
2	0.5253	0.5340
3	0.5340	0.5302
4	0.5302	0.5318
5	0.5318	0.5311

$x = 0.531$ (approx)

is the required real root.

Q.5.7. $1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \cdots = 0$

Solⁿ- Writing the equⁿ as:

$$x = 1 + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \cdots\cdots = \phi(x)$$

Omitting x^2 & higher powers of x, we get $\phi(x) = 1$ approximately

Taking $x_0 = 1$, we obtain

$$x_1 = \phi(x_0) = 1 + \frac{1}{(2!)^2} - \frac{1}{(3!)^2} + \frac{1}{(4!)^2} - \cdots$$

$$= 1.22395$$

~~$x_2 = \phi(x_1) =$~~

Itr no.	x_n	x_{n+1}
0	1	1.22395
1	1.22395	1.32748
2	1.32748	1.38096
3	1.38096	1.40992
4	1.40992	1.42597
5	1.42597	1.43498
6	1.43498	1.44
7	1.44	1.442

$x = 1.44$ (approx)

is the required real root.

Q5.8. ~~Evaluate $\sqrt{30}$ by iterative method and correct upto 4 decimal places.~~

~~solⁿ: $x^2 = 30$~~

~~$\Rightarrow x^2 - 30 = 0$~~

~~$f(x) = x^2 - 30$~~

~~$f(6) = 6 > 0 \quad f(5) = -5 < 0$~~

~~real root lies in the interval [6,5]~~

~~$\cancel{\phi(x) = \sqrt{30}}$ $\phi(x) = \frac{30}{x}$~~

~~$|\phi'(x)| = \left|\frac{30}{x^2}\right| = 0.991 < 1$~~

~~which is a proper choice.~~

~~$x_{n+1} = \frac{30}{x_n}$~~

It no	x_n	x_{n+1}
0	5.5	5.454
1	5.454	5.5

$x = 5.45$ (approx)

is the result of $\sqrt{30}$.

Q 6.0. $x \log_{10} x - 1.2 = 0$ (Regular Falsi method)

Soln. $f(x) = x \log_{10} x - 1.2$

$f(1) = -ve \quad f(2) = -ve \quad f(3) = +ve$

$\therefore$ root lies in the interval $(2, 3)$

$x_0 = 2, \; x_1 = 3 \quad f(x_0) = -0.597 \quad f(x_1) = 0.231$

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

It no	x_0	x_1	x_2	$f(x_2)$
0	2	3	2.721	-0.017
1	2.721	3	2.7401	-0.0004
2	2.7401	3	2.7405	-0.00001

$\therefore x = 2.74$ (approx)

is the reqd root.

Q.6.1. $x^6 - x^4 - x^3 - 1 = 0$ (Regular Falsi method)

Soln. $f(x) = x^6 - x^4 - x^3 - 1$

$f(0.5) = -1.17 < 0 \quad f(1.5) = 1.953 > 0$

$f(0.5) f(1.5) < 0$

root lies in the interval $(0.5, 1.5)$

It no	x_0	x_1	x_2	$f(x_2)$
0	0.5	1.5	0.874	-1.805
1	0.874	1.5	1.174	-1.899
2	1.174	1.5	1.334	-0.905
3	1.334	1.5	1.386	-0.263
4	1.386	1.5	1.399	-0.071
5	1.399	1.5	1.402	-0.025
6	1.402	1.5	1.403	-0.009
7	1.403	1.5	1.403	-0.009

$x = 1.403$ (approx) is the reqd root.

.00944 / 1000

Q.6.2. $2x = 6 + \log x$ (Regular-Falsi method)

Sol - $f(x) = \log x + 6 - 2x$

$f(3) = 0.477 > 0$ $f(4) = -1.397 < 0$

$f(3) f(4) < 0$

the real root lies in the interval $(3, 4)$

Here, $x_0 = 3$ & $x_1 = 4$

Itn no	x_0	x_1	x_2	$f(x_2)$
0	3	4	3.254	0.004
1	3.254	4	3.256	0.0068
2	3.256	4	3.259	-0.0049
3	3.256	3.259	3.273	

$\therefore x = 3.25$ (approx)

is the real root of the equation

Q6.3. $x^3 - 3x - 5 = 0$ (Regular Falsi method)

Soln- $f(x) = x^3 - 3x - 5$

$f(2.2) = -0.952 < 0$ $f(2.4) = 1.624 > 0$

$f(2.2) f(2.4) < 0$

the real root lies in the interval $(2.2, 2.4)$

Here $x_0 = 2.2$ & $x_1 = 2.4$

Itn no	x_0	x_1	x_2	$f(x_2)$
0	2.2	2.4	2.273	-0.075
1	2.273	2.4	2.278	-0.012
2	2.278	2.4	2.228	-0.012

$\therefore x = 2.278$ (approx)

is the real root of the equation

Q6.4. Evaluate using N-R method.

$e^{-x} = \sin x$

Soln- $f(x) = e^x \sin x - 1$

$f'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$

$f(0) = -1 < 0$ $f(1) = 1.287 > 0$

$f(0) f(1) < 0$ root lies in the interval $(0, 1)$

Itr no.	x_n	$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$
0	0.5	0.5936
1	0.5936	0.5885
2	0.5885	0.5885

$x = 0.5885$ (approx)

is the required root.

Q 6.5. $x\log_{10} x - 1.2 = 0$ (Newton Raphson method)

Soln- $f(x) = x\log_{10} x - 1.2$

$f'(x) = \log_{10} x + \log_{10} e$

$f(2) = -0.597$ $\quad f(3) = 0.231$

$f(2)\,f(3) < 0$

the root lies in the interval (2,3)

Itr no.	x_n	$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$
0	2.1	2.7917
1	2.7917	2.7408
2	2.7408	2.7406
3	2.7406	2.7406

$x = 2.7406$ (approx)

is the required root.

Q.6.6- $\sin x = \frac{x+1}{x-1}$ (Newton-Raphson method.

Soln- $f(x) = \cancel{(x-1)\sin x}\ (x+1) + (1-x)\sin x$

$f'(x) = 1 - \sin x + (1-x)\cos x$

$f(-1) = -1.682 < 0$ $\quad f(0) = 1 > 0$

$f(-1)f(0) < 0$

$x \in (-1, 1)$

Itr no.	x_n	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	-1.2	-0.3754
1	-0.3754	-0.4208
2	-0.4208	-0.4203
3	-0.4203	-0.4203

$\therefore x = -0.4203$ is the reqd root of the eqn. (approx)

Q. 6.7. $x^x + 2x - 6 = 0$ (Newton-Raphson method)

Soln- $f(x) = x^x + 2x - 6$

$f'(x) = x^x(\ln x + 1) + 2$

$f(1.8) = 0.48 > 0$ $\quad f(1.3) = -1.993 < 0$

$f(1.3)\, f(1.8) < 0$

$\therefore x \in (1.3, 1.8)$

Itr no	x_n	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1.5	1.753
1	1.753	1.723
2	1.723	1.723

$x = 1.723$ (approx) is the reqd. root of the eqn

Q 6.8. $x\sin x + \cos x = 0$ (Newton- Raphson method)

Soln- $f(x) = x\sin x + \cos x$

$f'(x) = \sin x + x\cos x - \sin x = x\cos x$

$f(0.2) = 1.019 > 0$ $\quad f(3.5) = -2.164 < 0$

$f(0.2)\, f(3.5) < 0$

$x \in (0.2, 3.5)$

Itr no.	x_n	$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$
0	~~0.3~~ 3.4	2.841
1	2.841	2.799
2	2.799	2.798
3	2.798	2.798

$x = 2.798$ (approx) is the reqd. root

Q.7.1 Find the value of $1/\sqrt{\sin x}$ where $x=0.5$ is the starting value of the root.

sol^n - $x = \frac{1}{\sqrt{\sin x}}$ $\Rightarrow x^2 \sin x - 1 = 0$

$f(x) = x^2 \sin x - 1$

$f'(x) = 2x \sin x + x^2 \cos x$

~~f(0) =~~ given, $x_0 = 0.5$

Itr no.	x_n	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	0.5	1.759
1	1.759	1.049
2	1.049	1.068
3	1.068	1.068

$x = 1.068$ (approx)

$\therefore 1/\sqrt{\sin x} = 1.068$ (approx)

Q.7.2. The bacteria count in a reservoir varies as $C = 4e^{-2t} + e^{-0.1t}$. Using N-R method, calculate the time reqd for the bacteria conc. to be 0.5.

sol^n - $4e^{-2t} + e^{-0.1t} = 0.5$

$f(x) = 4e^{-2t} + e^{-0.1t} - 0.5$

$f'(x) = -8e^{-2t} - 0.1e^{-0.1t}$

$f(0) = 4.5 > 0$ $f(1) = 0.9416 > 0$

$f(10) = -0.13 < 0$

$f(1) f(10) < 0$

$x \in (1, 10)$

Itr no	x_n	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	9	6.701
1	6.701	6.928
2	6.928	6.931
3	6.931	6.931

$\therefore x = 6.931$ (approx)

$t = 6.931$ is reqd for the bacteria conc to be 0.5

...ing value of

$e^{-2t} + e^{-0.1t}$.

backwincare.

Q.7.3. Current i in an electric ~~current~~ circuit is given by $i = 10e^{-t}\sin 2\pi t$ where t is in seconds. Using N-R method, find the value of t correct to 3 decimal places for $i = 2A$.

Sol^n. $f(x) = 10e^{-t}\sin(2\pi t) - 2$

$f'(x) = -10e^{-t}\sin(2\pi t) + 10e^{-t}\cos(2\pi t)2\pi$

$f(2) = \cancel{-1.705} < 0 \qquad f(1$

-2