On the Composition of Fourier and Power Series

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1 Power Variable under Transformation of a Fourier Function

1.1 Notation

Let $x \in \mathbb{R}$ be a scalar. Let $y \in \mathbb{R}$ be a scalar. Let $\sigma \in \mathcal{C}(\mathbb{R})$ be an Fourier function.

Then, σ can be represented as

$$\sigma(x) = \sum_{n \in \mathbb{Z}} s_n e^{2\pi i n x} \tag{1}$$

Suppose that x and y share the relation

$$y = \sigma(x). \tag{2}$$

Suppose that x can be represented as a Fourier series of z, another scalar.

$$x = \sum_{k=0}^{\infty} a_k z^k \tag{3}$$

Suppose that wish to find a similar Fourier representation for y in terms of z.

$$y = \sum_{k=0}^{\infty} b_k z^k \tag{4}$$

1.2 Expansion

We expand the relation given in Equation (2) by using Equations (1) and (3).

$$\begin{aligned}
y &= \sigma(x) \\
&= \sum_{n \in \mathbb{Z}} s_n e^{2\pi i n x} \\
&= \sum_{n \in \mathbb{Z}} s_n e^{2\pi i n \sum_{k=0}^{\infty} a_k z^k} \\
&= \sum_{n \in \mathbb{Z}} s_n \sum_{q=0}^{\infty} \frac{\left(2\pi i n \sum_{k=0}^{\infty} a_k z^k\right)^q}{q!} \\
&= \sum_{n \in \mathbb{Z}} s_n \sum_{q=0}^{\infty} \frac{(2\pi i n)^q}{q!} \left(\sum_{k=0}^{\infty} a_k z^k\right)^q \\
&= \sum_{n \in \mathbb{Z}} s_n \sum_{q=0}^{\infty} \frac{(2\pi i n)^q}{q!} \left(\sum_{q_0 + q_1 + \cdots = q} \binom{q}{q_0, q_1, \ldots} \prod_{k=0}^{\infty} \left(a_k z^k\right)^{q_k}\right) \\
&= \sum_{n \in \mathbb{Z}} s_n \sum_{q=0}^{\infty} \frac{(2\pi i n)^q}{q!} \left(\sum_{q_0 + q_1 + \cdots = q} \binom{q}{q_0, q_1, \ldots} \prod_{k=0}^{\infty} a_k^{q_k} z^{q_k k}\right)
\end{aligned} \tag{5}$$

1.3 Coefficient Extraction

In order to find the coefficients b_i for Equation (4), we must extract the index and power constraints from Equation (5).

The index constraint is

$$q = \sum_{m=0}^{\infty} q_m. \tag{6}$$

The power constraint is

$$i = \sum_{m=0}^{\infty} m q_m. \tag{7}$$

Therefore, the expression for the coefficient g_i is

$$b_i = \sum_{n=0}^{\infty} s_n \sum_{q}^{\infty} \frac{(2\pi i n)^q}{q!} \sum_{\substack{q_0+q_1+\cdots q_i = q \\ 0q_0+1q_1+\cdots i q_i = i}} \binom{q}{q_0, q_1, \ldots, q_i} \prod_{k=0}^{\infty} a_k^{q_k}. \tag{8}$$

1.4 Analysis of Sine Transformation

We can let the transformation $\sigma(x) = \sin(x)$. We then have

$$\sigma(x) = \sin(x) = \frac{i}{2} e^{-ix} - \frac{i}{2} e^{ix} \tag{9}$$

so that $s_{-1} = i/2$, $s_1 = -i/2$, and $s_k = 0$ for $k \neq \pm 1$. Inserting these coefficients into Equation (8), we get

$$\begin{aligned}
b_j &= \sum_{n \in \{-1,1\}}^{\infty} s_n \sum_{q=0}^{\infty} \frac{(2\pi i n)^q}{q!} \sum_{\substack{q_0+q_1+\cdots q_j = q \\ 0q_0+1q_1+\cdots j q_j = j}} \binom{q}{q_0, q_1, \ldots, q_j} \prod_{k=0}^{\infty} a_k^{q_k} \\
&= \frac{i}{2} \left[\sum_{q=0}^{\infty} \frac{(-1)^q (2\pi i)^q}{q!} \sum_{\substack{q_0+q_1+\cdots q_i = q \\ 0q_0+1q_1+\cdots j q_j = j}} \binom{q}{q_0, q_1, \ldots, q_j} \prod_{k=0}^{\infty} a_k^{q_k} \right] - \frac{i}{2} \left[\sum_{q=0}^{\infty} \frac{(2\pi i)^q}{q!} \sum_{\substack{q_0+q_1+\cdots q_j = q \\ 0q_0+1q_1+\cdots j q_j = j}} \binom{q}{q_0, q_1, \ldots, q_j} \prod_{k=0}^{\infty} a_k^{q_k} \right] \\
&= \frac{i}{2} \sum_{q=0}^{\infty} \left[\frac{(2\pi i)^q}{q!} \left((-1)^q - 1\right) \sum_{\substack{q_0+q_1+\cdots q_j = q \\ 0q_0+1q_1+\cdots j q_j = j}} \binom{q}{q_0, q_1, \ldots, q_j} \prod_{k=0}^{\infty} a_k^{q_k} \right] \\
&= -i \sum_{q=0}^{\infty} \left[\frac{(2\pi i)^{2q+1}}{(2q+1)!} \sum_{\substack{q_0+q_1+\cdots q_j = 2q+1 \\ 0q_0+1q_1+\cdots j q_j = j}} \binom{2q+1}{q_0, q_1, \ldots, q_j} \prod_{k=0}^{\infty} a_k^{q_k} \right] \\
&= \sum_{q=0}^{\infty} \left[\frac{(-1)^q (2\pi)^{2q+1}}{(2q+1)!} \sum_{\substack{q_0+q_1+\cdots q_j = 2q+1 \\ 0q_0+1q_1+\cdots j q_j = j}} \binom{2q+1}{q_0, q_1, \ldots, q_j} \prod_{k=0}^{\infty} a_k^{q_k} \right]
\end{aligned} \tag{10}$$

This can also be found using a similar expansion as in Equation (**??**) with $y = sin(x)$. We will use this expansion to prove the following theorem about the instability of the Taylor expansion of the iterated sine function.

Theorem 1. *Let* $\sin^{(k)}(x)$ *be the iterated sine function. That is* $\sin^{(1)}(x) = \sin(x)$*,* $\sin^{(2)}(x) = \sin(\sin(x))$*,* $\sin^{(3)}(x) = \sin(\sin(\sin(x)))$*, etc. Then, each iterated sine function* $\sin^{(k)}(x)$ *has a Maclaurin Series representation*

$$\sin^{(k)}(x) = s_0^{(k)} + s_1^{(k)}x + s_2^{(k)}x^2 + \cdots \tag{11}$$

and $\left|s_i^{(m)}\right| \geq \left|s_i^{(n)}\right|$ *if* $m \geq n$ *for all* $i \in \mathbb{N}$*.*

Proof. We will prove this theorem by strong induction on m.

First, we will start with $m = 1$. Then, either $n = 0$ or $n = 1$.

1. When $n = 0$, $\sin^{(n)}(x) = id(x)$ is the identity of x which has $s_1^{(0)} = 1$ and $s_i^{(0)} = 0$ for $i \neq 1$. The Maclaurin Series expansion of $\sin(x)$ has $s_{2n}^{(1)} = 0$ and $s_{2n+1}^{(1)} = \frac{(-1)^n}{(2n+1)!}$. Since, $s_1^{(1)} = \frac{(-1)^0}{1!} = 1$, then $\left|s_i^{(1)}\right| \geq \left|s_i^{(0)}\right|$, $\forall i \in \mathbb{N}$.

2. When $n = 1 = m$, we trivially obtain $\left|s_i^{(m)}\right| \geq \left|s_i^{(n)}\right|$, $\forall i \in \mathbb{N}$.

Second, we will prove that the theorem holds for $m = k$ under the assumption that the theorem holds for $m < k$ for $k > 1$. In this case, either $n < m$ or $n = m$.

1. When $n < m$, then, by assumption, we have $\left|s_i^{(m-1)}\right| \geq \left|s_i^{(n)}\right|$, $\forall i \in \mathbb{N}$ and we need only show that $\left|s_i^{(m)}\right| \geq \left|s_i^{(m-1)}\right|$, $\forall i \in \mathbb{N}$. We start by supposing that

$$x(z) = sin^{(m-1)}(z) = s_0^{(m-1)} + s_1^{(m-1)}z + s_2^{(m-1)}z^2 + \cdots . \tag{12}$$

We also suppose

$$\sigma(z) = sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}. \tag{13}$$

Finally, we wish to find $s_0^{(m)}, s_1^{(m)} \ldots$ to satisfy

$$y(z) = sin^{(m)}(z) = s_0^{(m)} + s_1^{(m)}z + s_2^{(m)}z^2 + \cdots . \tag{14}$$

From our previous results, we know that

$$s_i^{(m)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left[\sum_{\substack{k_0+k_1+\cdots+k_i=k \\ 0k_0+1k_1+\cdots ik_i=k}} \binom{k}{k_0, k_1, \cdots, k_i} \prod_{j=0}^{i} \left(s_j^{(m-1)}\right)^{k_j} \right]. \tag{15}$$

2. When $n = m$, we trivially obtain $\left|s_i^{(m)}\right| \geq \left|s_i^{(n)}\right|$, $\forall i \in \mathbb{N}$.

Thus, we have $\left|s_i^{(m)}\right| \geq \left|s_i^{(n)}\right|$ if $m \geq n$ for all $i \in \mathbb{N}$ so the theorem is proven. □

For instance, $\sin^{(k)}$ has the following first 10 coefficients for $k = 0, 1, \ldots, 7$.

Theorem 2. *The sequence of values* $s_{2n+1}^{(k)}$ *as a function of* k *is a polynomial of degree* n*.*

PROOF NOT COMPLETE HERE. □

k	$c_0^{(k)}$	$c_1^{(k)}$	$c_2^{(k)}$	$c_3^{(k)}$	$c_4^{(k)}$	$c_5^{(k)}$	$c_6^{(k)}$	$c_7^{(k)}$	$c_8^{(k)}$	$c_9^{(k)}$
0	0	1	0	0	0	0	0	0	0	0
1	0	1	0	$-1/6$	0	$1/120$	0	$-1/5040$	0	$1/362880$
2	0	1	0	$-2/6$	0	$12/120$	0	$-128/5040$	0	$1872/362880$
3	0	1	0	$-3/6$	0	$33/120$	0	$-731/5040$	0	$25857/362880$
4	0	1	0	$-4/6$	0	$64/120$	0	$-2160/5040$	0	$121600/362880$
5	0	1	0	$-5/6$	0	$105/120$	0	$-4765/5040$	0	$368145/362880$
6	0	1	0	$-6/6$	0	$156/120$	0	$-8896/5040$	0	$873936/362880$
7	0	1	0	$-7/6$	0	$217/120$	0	$-14903/5040$	0	$1776817/362880$

Therefore, not only are the coefficients increasing, but higher-order coefficients grow more quickly than lower-order coefficients.

Corollary 3. *Conjecture: For $k > 1$, the radius of convergence of the power series of $\sin^{(k)}(x)$ is $\rho = \pi^{-{}^{k-1}2}$ about $x = 0$.*

PROOF NOT COMPLETE HERE. ☐

Thus, for $k > 1$, the power series of $\sin^{(k)}(x)$ has a finite radius of convergence contrary to the infinite radius of convergece of the power series of $\sin(x)$.

2 Fourier Variable under Transformation of an Power Function

2.1 Notation

Let $x \in \mathbb{R}$ be a scalar. Let $y \in \mathbb{R}$ be a scalar. Let $\sigma \in \mathcal{C}(\mathbb{R})$ be a function with a power representation.

Then, σ can be represented as

$$\sigma(x) = \sum_{n=0}^{\infty} s_n x^n. \tag{16}$$

Suppose that x and y share the relation

$$y = \sigma(x). \tag{17}$$

Suppose that x can be represented as a Fourier series of z, another scalar.

$$x = \sum_{k \in \mathbb{Z}} f_k e^{2\pi i k z} \tag{18}$$

Suppose that wish to find a similar Fourier representation for y in terms of z.

$$y = \sum_{k \in \mathbb{Z}} g_k e^{2\pi i k z} \tag{19}$$

2.2 Expansion

We expand the relation given in Equation (17) by using Equations (16) and (18).

$$
\begin{aligned}
y &= \sigma(x) \\
&= \sum_{n=0}^{\infty} s_n x^n \\
&= \sum_{n=0}^{\infty} s_n \left(\sum_{k\in\mathbb{Z}} f_k e^{2\pi i k z} \right)^n \\
&= \sum_{n=0}^{\infty} s_n \left(\sum_{\cdots k_{-1}+k_0+k_1+\cdots=n} \binom{n}{\ldots, k_{-1}, k_0, k_1, \ldots} \prod_{t\in\mathbb{Z}} \left(f_t e^{2\pi i t z} \right)^{k_t} \right) \\
&= \sum_{n=0}^{\infty} s_n \left(\sum_{\cdots k_{-1}+k_0+k_1+\cdots=n} \binom{n}{\ldots, k_{-1}, k_0, k_1, \ldots} \prod_{t\in\mathbb{Z}} f_t^{k_t} e^{2\pi i t k_t z} \right)
\end{aligned} \tag{20}
$$

2.3 Coefficient Extraction

In order to find the coefficients g_i for Equation (19), we must recognize an index constraint and a power constraint on the term $e^{2\pi i t k_t z}$ from Equation (20).

The index constraint is

$$n = \sum_{m\in\mathbb{Z}} k_m. \tag{21}$$

The power constraint is

$$i = \sum_{t\in\mathbb{Z}} t k_t \tag{22}$$

Therefore, the expression for the coefficient g_i is

$$g_i = \sum_{n=0}^{\infty} s_n \left(\sum_{\substack{n=\sum_{m\in\mathbb{Z}} k_m \\ i=\sum_{t\in\mathbb{Z}} t k_t}} \binom{n}{\ldots, k_{-1}, k_0, k_1, \ldots} \prod_{t\in\mathbb{Z}} f_t^{k_t} \right). \tag{23}$$

This cannot be simplified as in the variable power series case since $i = \sum_{t\in\mathbb{Z}} t k_t$ cannot be reduced. All modes must be considered.

3 Fourier Variable under Transformation of an Fourier Function

3.1 Notation

Let $x \in \mathbb{R}$ be a scalar. Let $y \in \mathbb{R}$ be a scalar. Let $\sigma \in \mathcal{C}(\mathbb{R})$ be a function with a Fourier representation.

Then, σ can be represented as

$$\sigma(x) = \sum_{n\in\mathbb{Z}} s_n e^{2\pi i n x}. \tag{24}$$

Suppose that x and y share the relation

$$y = \sigma(x). \tag{25}$$

Suppose that x can be represented as a Fourier series of z, another scalar.

$$x = \sum_{k\in\mathbb{Z}} f_k e^{2\pi i k z} \tag{26}$$

Suppose that wish to find a similar Fourier representation for y in terms of z.

$$y = \sum_{k\in\mathbb{Z}} g_k e^{2\pi ikz} \tag{27}$$

3.2 Expansion

We expand the relation given in Equation (25) by using Equations (24) and (26).

$$\begin{aligned}
y &= \sigma(x) \\
&= \sum_{n\in\mathbb{Z}} s_n e^{2\pi inx} \\
&= \sum_{n\in\mathbb{Z}} s_n e^{2\pi in\left(\sum_{k\in\mathbb{Z}} f_k e^{2\pi ikz}\right)} \\
&= \sum_{n\in\mathbb{Z}} s_n \sum_{q=0}^{\infty} \frac{\left(2\pi in\left[\sum_{k\in\mathbb{Z}} f_k e^{2\pi ikz}\right]\right)^q}{q!} \\
&= \sum_{n\in\mathbb{Z}} s_n \sum_{q=0}^{\infty} \frac{(2\pi in)^q}{q!} \left(\sum_{k\in\mathbb{Z}} f_k e^{2\pi ikz}\right)^q \\
&= \sum_{n\in\mathbb{Z}} s_n \sum_{q=0}^{\infty} \frac{(2\pi in)^q}{q!} \sum_{\cdots+q_{-1}+q_0+q_1+\cdots=q} \binom{q}{\ldots,q_{-1},q_0,q_1,\ldots} \prod_{m\in\mathbb{Z}} (f_m e^{2\pi imz})^{q_m} \\
&= \sum_{n\in\mathbb{Z}} s_n \sum_{q=0}^{\infty} \frac{(2\pi in)^q}{q!} \sum_{\cdots+q_{-1}+q_0+q_1+\cdots=q} \binom{q}{\ldots,q_{-1},q_0,q_1,\ldots} \prod_{m\in\mathbb{Z}} f_m^{q_m} e^{2\pi q_m imz}
\end{aligned} \tag{28}$$

3.3 Coefficient Extraction

In order to find the coefficients g_i for Equation (27), we must recognize an index constraint and a power constraint on the term $e^{2\pi q_m imz}$ from Equation (28).

The index constraint is

$$q = \sum_{m\in\mathbb{Z}} q_m. \tag{29}$$

The power constraint is

$$i = \sum_{t\in\mathbb{Z}} mq_m \tag{30}$$

Therefore, the expression for the coefficient g_i is

$$g_i = \sum_{n=0}^{\infty} s_n \sum_{q=0}^{\infty} \frac{(2\pi in)^q}{q!} \left(\sum_{\substack{q=\sum_{m\in\mathbb{Z}} q_m \\ i=\sum_{m\in\mathbb{Z}} mq_m}} \binom{n}{\ldots,q_{-1},q_0,q_1,\ldots} \prod_{m\in\mathbb{Z}} f_t^{q_m} \right). \tag{31}$$

This cannot be simplified as in the variable power series case since $i = \sum_{m\in\mathbb{Z}} mq_m$ cannot be reduced. All modes must be considered.