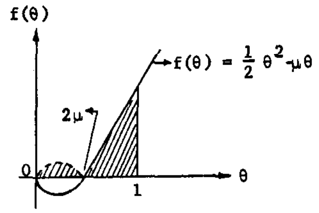

Fig. 1.

as follows (see Fig. 1):

$$\int_0^1 \left|\left(\frac{1}{2}\theta^2 - \mu\theta\right)\right| d\theta = -\int_0^{2\mu}\left(\frac{1}{2}\theta^2 - \mu\theta\right) d\theta + \int_{2\mu}^{1}\left(\frac{1}{2}\theta^2 - \mu\theta\right) d\theta = 4/3\mu^3 - \mu/2 + 1/6.$$

Hence (8) becomes

$$|e(\mu)| \le h^2\left(\frac{1}{2} - \mu\right)|\ddot{X}(t+h)| + h^3 \underset{0\le\theta\le1}{M}\left(4/3\mu^3 - \frac{1}{2}\mu + \frac{1}{6}\right)$$

which is exactly (5′). Equations (6′) and (7′) follow by substituting $\mu=0$ and $\frac{1}{2}$, respectively, in (5′).

Despite Russo's claim, (7′), and not (7), is a well-known result. (See, for example, [2, p. 31] or [3, p. 165].)

2) The relation $\mu = 1/q - [1/(e^q - 1)]$, and the statement that follows, i.e., "μ is a monotonically decreasing function of $q \,\forall\, q$[2]," are given in [1], and therefore reference to [2] is unnecessary.

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Bounds on the Truncation Error of Periodic Signals

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Abstract—Bounds on the truncation error as a function of the number of terms used in a Fourier series are obtained. Two different error criteria, based on the Hilbert norm and the Chebyshev norm, are used.

Introduction

When Fourier series are used for computation, only a finite number of terms can be worked with. Truncation of the series results in an error, and it is often tedious to obtain this error. In this correspondence, bounds on the truncation error as a function of the number of terms used in the Fourier series are derived. These bounds are simple to use.

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Two different error criteria are used. One is based on the Hilbert norm, or mean-squared-error criteria. The other is based on the Chebyshev norm, and it restricts the maximum pointwise difference between the actual signal and the truncated one. The derivation of these bounds is based on derivations of the effective bandwidth of a signal [1], [2]. Related material is discussed in [3], [4].

Bounds on the Truncation Error

We shall work with functions of time that are real and periodic with period T. These functions are those which can be expressed by the Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \tag{1}$$

where

$$\omega_0 = 2\pi/T \tag{2}$$

and

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t}\, dt. \tag{3}$$

We shall approximate $f(t)$ by $f_m(t)$ where

$$f_m(t) = \sum_{n=-m}^{m} C_n e^{jn\omega_0 t}. \tag{4}$$

We wish to determine a value of m which guarantees that the error between $f(t)$ and $f_m(t)$ is less than some specified value.

Two different error criteria will be used. The first is the mean square or Hilbert norm given by

$$\epsilon_{MS}^2 = \|f - f_m\|_2^2 = \int_{-T/2}^{T/2} [f(t) - f_m(t)]^2\, dt. \tag{5}$$

The second error criterion is the maximum error or the Chebyshev norm

$$\epsilon_{\max} = \|f - f_m\| = \sup |f(t) - f_m(t)|, \qquad -\frac{T}{2} \le t \le \frac{T}{2}. \tag{6}$$

This is the maximum pointwise difference between $f(t)$ and $f_m(t)$.

We shall start with the Hilbert norm.

Theorem 1

If a periodic $f(t)$ is of bounded variation whose total variation over one period is bounded by N, that is

$$\underset{-T/2}{\overset{T/2}{V}} [f(t)] \le N$$

then the mean square error is bounded by

$$\epsilon_{MS}^2 \le \frac{N^2}{\pi\omega_0 m}.$$

Proof:

$$\epsilon_{MS}^2 = \int_{-T/2}^{T/2} [f(t) - f_m(t)]^2\, dt. \tag{7}$$

Using Parseval's theorem, we have

$$\epsilon_{MS}^2 = 2T \sum_{n=m+1}^{\infty} |C_n|^2. \tag{8}$$

From (3), we have

$$|C_n| = \frac{1}{T}\left|\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt\right|. \tag{9}$$

Putting this into the form of a Stieltjes integral, we obtain

$$|C_n| = \frac{1}{\omega_0 T n}\left|\int_{-T/2}^{T/2} f(t)\, d e^{-jn\omega_0 t}\right|. \tag{10}$$

Integrating by parts and noting that $f(t)e^{-jn\omega_0 t}$ is periodic yields

$$|C_n| = \frac{1}{\omega_0 T n}\left|\int_{-T/2}^{T/2} e^{-jn\omega_0 t}\, df(t)\right|. \tag{11}$$

Bounding the Stieltjes integral and substituting (2), we obtain

$$|C_n| \leq \frac{1}{2\pi n} \overset{T/2}{\underset{-T/2}{V}} [f(t)] \leq \frac{N}{2\pi n}. \tag{12}$$

Substituting this result into (8), we have

$$\epsilon_{MS}^2 \leq \frac{TN^2}{2\pi^2} \sum_{n=m+1}^{\infty} \frac{1}{n^2}. \tag{13}$$

We can bound the summation using the integral relation

$$\sum_{n=m+1}^{\infty} \frac{1}{n^2} \leq \int_m^{\infty} \frac{1}{x^2}\, dx = \frac{1}{m}. \tag{14}$$

Substituting (14) and (2) into (13), we obtain

$$\epsilon_{MS}^2 \leq \frac{N^2}{\pi\omega_0 m}. \tag{15}$$

This completes the proof.

Let us assume that a maximum mean square error ϵ is specified. We wish to obtain a lower bound on m which guarantees that the actual mean square error ϵ_{MS} is less than ϵ. To accomplish this we set the right-hand side of (15) equal to ϵ^2. We call the value of m which produces this equality M. Then,

$$M = \frac{N^2}{\pi\omega_0\epsilon^2}. \tag{16}$$

Then any m which satisfies

$$m > M \tag{17}$$

will result in an error less than the specified ϵ.

As a corollary to the theorem, we can state the following.

Corollary 1

Consider a periodic $f(t)$ whose first $q-1$ derivatives are continuous, $q=1, 2, 3, \cdots$, and whose qth derivative is piecewise defined and is of bounded variation with total variation

$$\overset{T/2}{\underset{-T/2}{V}} [d^q f(t)/dt^q] \leq N_q.$$

It is assumed that at the jumps, the qth derivative is defined to be normalized to one-half of the jump value. If $f(t)$ satisfies these conditions then the mean square error is bounded by

$$\epsilon_{MS}^2 \leq \frac{N_q^2}{\pi(2q+1)(\omega_0 m)^{2q+1}}.$$

Proof: The proof follows that of Theorem 1 except that the integral of (9) is integrated by parts q times. The proof then proceeds as before.

Now let us consider the Chebyshev norm.

Theorem 2

Consider a periodic $f(t)$ whose first $q-1$ derivatives are continuous $q=1, 2, 3, \cdots$, and whose qth derivative is piecewise defined and of bounded variation with total variation

$$\overset{T/2}{\underset{-T/2}{V}} [d^q f(t)/dt^q] \leq N_q.$$

It is assumed that at the jumps, the q derivative is defined to be normalized to one-half the jump value. If $f(t)$ satisfies these conditions, then the Chebyshev norm is bounded by

$$\epsilon_{\max} \leq \frac{N_q}{\pi q(\omega_0 m)^q}.$$

Proof:

$$\epsilon_{\max} = \sup\left|\sum_{n=m+1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=-(m+1)}^{-\infty} C_n e^{jn\omega_0 t}\right|. \tag{18}$$

This utilizes the pointwise convergence of the Fourier series. Then

$$\epsilon_{\max} \leq 2 \sum_{n=m+1}^{\infty} |C_n|. \tag{19}$$

Differentiating (9) q times and proceeding as in Theorem 1, we obtain

$$|C_n| \leq \frac{N_q}{2\pi\omega_0^q n^{q+1}}, \qquad q \geq 1. \tag{20}$$

Substitution in (19) yields

$$\epsilon_{\max} \leq \frac{N_q}{\pi\omega_0^q} \sum_{n=m+1}^{\infty} \frac{1}{n^{q+1}}. \tag{21}$$

Then, proceeding as in Theorem 1, we have

$$\epsilon_{\max} \leq \frac{N_q}{\pi q(\omega_0 m)^q}. \tag{22}$$

This completes the proof.

Again let us assume that a maximum error ϵ is specified and that we wish to obtain a bound on m such that $\epsilon_{\max} < \epsilon$. Then, proceeding as before, we have

$$M = \frac{1}{\omega_0}\left[\frac{N_q}{\pi q\epsilon}\right]^{1/q}. \tag{23}$$

If $m > M$, the actual error will be less than the specified value.

Let us consider an example which illustrates the use of these bounds. Suppose that

$$\begin{aligned} f(t) &= \pi + t, \qquad -\pi \leq t \leq 0 \\ &= \pi - t, \qquad 0 < t \leq \pi \end{aligned}$$

and

$$f(t + 2\pi) = f(t).$$

Then $T = 2\pi$; hence $\omega_0 = 1$. The first derivative is given by

$$\begin{aligned} f'(t) &= 1, \qquad -\pi < t < 0 \\ &= -1, \qquad 0 < t < \pi. \end{aligned}$$

The total variation of the derivative then is

$$\overset{T/2}{\underset{-T/2}{V}} f'(t) = 4.$$

Let us choose the maximum error as $\epsilon = 0.1$. Then substituting in (23) we have

$$M = \frac{4}{\pi(0.1)} = 12.75.$$

Thus we should use 13 terms and achieve an error which is less than 0.1.

Conclusion

Easily evaluated bounds on the number of terms required to make the error in a Fourier series less than a specified value have been obtained. These bounds are based on both mean-square- and maximum-error criteria and provide a very simple way of deciding where a Fourier series should be truncated.

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