On the Truncation Error of Fourier Series

July 14, 2020

1 Notation

1.1 Fourier Series

We will consider the notion of a generalized Fourier series with wave period L. We will write the Fourier expansion of a function f in its exponential form,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi}{L} inx} \tag{1}$$

where $c_n \in \mathbb{C}$ are determined by the series expansion of a function using the following expression.

$$c_n = \frac{1}{L} \int_0^L f(x) e^{-\frac{2\pi}{L} inx} dx \tag{2}$$

In practical settings, we cannot compute the entirety of a Fourier series. Instead, we compute the truncated Fourier series which does not contain all infinitely many terms. The truncated Fourier series of degree N is denoted as

$$\hat{f}_N(x) = \sum_{n=-N}^{N} \hat{c}_n e^{\frac{2\pi}{L} inx} \tag{3}$$

where $\hat{c}_n$ are the same coefficients as c_n if included in the truncation. That is

$$\hat{c}_n = \begin{cases} c_n, & |n| \leq N, \\ 0, & |n| > N \end{cases} \tag{4}$$

1.2 Function Norms

We use function norms to measure the magnitude of a function over a compact support. In general, the L_p norm of an integrable function f is defined by:

$$\|f\|_p = \left[\int_0^L |f(x)|^p dx \right]^{\frac{1}{p}} \tag{5}$$

Specifically, we will be using the following function norms throughout this text:

$$\|f\|_1 = \int_0^L |f(x)| \ dx \tag{6}$$

$$\|f\|_2 = \sqrt{\int_0^L |f(x)|^2 \ dx} \tag{7}$$

2 Error Equality via Integration

We would like to determine the accuracy of a truncated Fourier series function compared to its exact function. To do this, we find the L^2 norm of the difference between the two functions.

Lemma 1 (Parseval's Identity)**.** *Suppose that f is a square integrable function ($\int_0^L |f(x)|^2\, dx$ exists) whose Fourier series converges uniformly to f. Then,*

$$\|f\|_2^2 = \int_0^L |f(x)|^2\, dx = L \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{8}$$

Corollary 2. *Suppose that f is a square integrable function whose Fourier series converges uniformly to f. Then,*

$$\|f - \hat{f}_N\|_2^2 = \|f\|_2^2 - \|\hat{f}_N\|_2^2$$

Proof. Using Parseval's Identity, we can rewrite the LHS of our equation.

$$\begin{aligned}
\|f - \hat{f}_N\|_2^2 &= L \sum_{n=-\infty}^{\infty} |c_n - \hat{c}_n|^2 &(9)\\
&= L \sum_{n=-\infty}^{\infty} (c_n^2 - 2c_n\hat{c}_n + \hat{c}_n^2) &(10)\\
&= L \sum_{n=-\infty}^{\infty} c_n^2 + L \sum_{n=-\infty}^{\infty} \hat{c}_n^2 - 2L \sum_{n=-\infty}^{\infty} c_n\hat{c}_n &(11)\\
&= \|f\|_2^2 + \|\hat{f}_N\|_2^2 - 2L \sum_{n=-\infty}^{\infty} c_n\hat{c}_n &(12)
\end{aligned}$$

We now analyze this final summation term. By substituting the exact expression for $\hat{c}_n$, we find

$$\begin{aligned}
L \sum_{n=-\infty}^{\infty} c_n\hat{c}_n &= L \sum_{n=-\infty}^{\infty} \begin{cases} c_n^2, & |n| \le N, \\ 0, & |n| > N \end{cases} &(13)\\
&= L \sum_{n=-N}^{N} c_n^2 = L \sum_{n=-N}^{N} \hat{c}_n^2 = L \sum_{n=-\infty}^{\infty} \hat{c}_n^2 &(14)\\
&= \|\hat{f}_N\|_2^2. &(15)
\end{aligned}$$

Therefore, plugging this result back in, we obtain the RHS of the equation.

$$\|f - \hat{f}_N\|_2^2 = \|f\|_2^2 + \|\hat{f}_N\|_2^2 - 2\|\hat{f}_N\|_2^2 = \|f\|_2^2 - \|\hat{f}_N\|_2^2. \tag{16}$$

This completes the proof. □

Thus, if we know both f and f_N, we can compute the error:

$$\int_0^L |(f - \hat{f}_N)(x)|^2\, dx = \int_0^L |f(x)|^2\, dx - \int_0^L |\hat{f}_N(x)|^2\, dx$$

3 Error Bound via Continuity

Often, we wish to know the degree of terms N we need to calculate of the truncated Fourier series $\hat{f}_N$ before computing it. For this purpose, we form an inequality on N to achieve some level of accuracy ε.

Lemma 3. *Let f be a function that has p continuous derivatives ($f \in \mathcal{C}^p$). Then, in the Fourier series of f,*

$$|c_n| \le \frac{\|f^{(p)}\|_1}{|n|^p} \tag{17}$$

Corollary 4. *Let f be a function that has p continuous derivatives. Suppose that we wish to bound the truncation error $\|f - \hat{f}_N\|_2^2 \leq \varepsilon$. Then, the following condition suffices:*

$$N \geq \sqrt[2p-1]{\frac{2\|f^{(p)}\|_1^2}{(2p-1)\varepsilon}} \tag{18}$$

The following proof was based on the Giardina and Chirlian proof of a similar result for functions of bounded variation.

Proof. We start by rewriting the error calculation we found previously:

$$\begin{aligned} \|f - \hat{f}_N\|_2^2 &= \|f\|_2^2 - \|\hat{f}_N\|_2^2 &(19)\\ &= L \sum_{n=-\infty}^{\infty} (|c_n|^2 - |\hat{c}_n|^2) &(20)\\ &= L \sum_{n=N+1}^{\infty} (|c_n|^2 + |c_{-n}|^2) &(21) \end{aligned}$$

Then, we apply the lemma in order to find a bound.

$$\|f - \hat{f}_N\|_2^2 \leq 2L \sum_{n=N+1}^{\infty} \frac{\|f^{(p)}\|_1^2}{n^{2p}} \tag{22}$$

Since n^{-2p} is monotonically decreasing for $p \geq 0$, we can convert this into an integral inequality.

$$2L \sum_{n=N+1}^{\infty} \frac{\|f^{(p)}\|_1^2}{n^{2p}} \leq 2L \int_N^{\infty} \frac{\|f^{(p)}\|_1^2}{x^{2p}}\, dx \tag{23}$$

Simplifying, we obtain:

$$\|f - \hat{f}_N\|_2^2 \leq \frac{2L\|f^{(p)}\|_1^2}{(2p-1)N^{2p-1}} \tag{24}$$

Finally, we solve for N by upper bounding our entire expression by ε.

$$\sqrt[2p-1]{\frac{2L\|f^{(p)}\|_1^2}{(2p-1)\varepsilon}} \leq N \tag{25}$$

$\square$

Corollary 5. *Giardina and Chilian proved that:*
Let f be a continuous, bounded function such that $\sup_x |f(x)| = B < \infty$. Suppose that we wish to bound the truncation error $\|f - \hat{f}_N\|_2^2 \leq \varepsilon$. Then, the following condition suffices:

$$N \geq \frac{2LB^2}{\varepsilon} \tag{26}$$

4 References

1. C. Giardina and P. Chirlian, "Bounds on the truncation error of periodic signals," in IEEE Transactions on Circuit Theory, vol. 19, no. 2, pp. 206-207, March 1972, doi: 10.1109/TCT.1972.1083433.