Math: Matrices

What is a matrix?

A **matrix** is a rectangular arrangement of numbers into rows and columns. Like a **vector,** the order of the elements matters.

$$\begin{array}{cc} & \text{3 columns} \\ \text{2 rows} & \begin{array}{ccc}\downarrow & \downarrow & \downarrow\end{array} \\ \begin{array}{c}\rightarrow \\ \rightarrow\end{array} & \begin{bmatrix} -2 & 5 & 6 \\ 5 & 2 & 7 \end{bmatrix} \end{array}$$

Matrices also have dimensions, but now the dimension is represented by two numbers.

In the example above, the matrix has **two rows** and **three columns**, and so we say it is a **2 x 3 matrix.**

What can we do with matrices?

Like vectors, we can **add** them (*if they are the same size*):

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 4 & 8 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 4+1 & 8+0 \\ 3+5 & 7+2 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 8 & 9 \end{bmatrix}$$

We can **scale** them:

$$3 \cdot \begin{bmatrix} 4 & 8 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 4 & 3 \cdot 8 \\ 3 \cdot 2 & 3 \cdot 1 \end{bmatrix}$$

What can we do with matrices?

If we can scale them and add them, we can also subtract them:

$$\begin{bmatrix} 2 & 8 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 11 & 3 \end{bmatrix} = \begin{bmatrix} 2-5 & 8-6 \\ 0-11 & 9-3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -11 & 6 \end{bmatrix}$$

We can multiply them (if a certain condition is met):

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 10 & 11 \\ 20 & 21 \\ 30 & 31 \end{bmatrix} = \begin{bmatrix} 1\times10 + 2\times20 + 3\times30 & 1\times11 + 2\times21 + 3\times31 \\ 4\times10 + 5\times20 + 6\times30 & 4\times11 + 5\times21 + 6\times31 \end{bmatrix} = \begin{bmatrix} 140 & 146 \\ 320 & 335 \end{bmatrix}$$

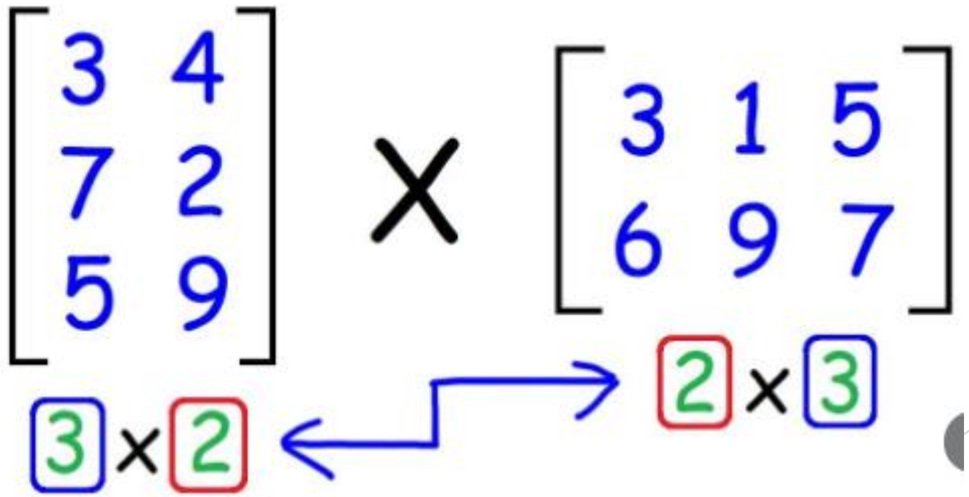
What can we do to Matrices?

A little bit more about matrix multiplication…

$$\underset{\text{Matrix } \mathbf{A} \text{ is } 3x4}{\begin{bmatrix} 8 & 3 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}} \underset{\text{Matrix } \mathbf{B} \text{ is } 4x4}{\begin{bmatrix} 5 & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}} = \underset{\text{Matrix } \mathbf{C} \text{ is } 3x4}{\begin{bmatrix} 53 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}$$

$$\text{because } c_{11} = \sum_{k=1}^{4} a_{1k} b_{k1} = 8 \cdot 5 + 3 \cdot 4 + 0 \cdot 3 + 1 \cdot 1 = 53$$

Pro Tip! Look at the inner dimensions:



Let's try some examples:

$$\begin{bmatrix} 1 & 2 & -1 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} 9 & 8 & -11 \\ 6 & 10 & 2 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix} = \begin{bmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 2\pi & 4 \\ 14 & -3 & 8 \end{bmatrix} + \begin{bmatrix} 19 & 1.4 \\ -6 & 2 \\ 2 & 2 \end{bmatrix} = \, ? = \mathit{You\ can't\ add\ these!}\ \text{Why not?}$$

$$4 * \begin{bmatrix} 19 & 1.4 \\ -6 & \pi \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} 76 & 5.6 \\ -24 & 4\pi \\ 8 & 8 \end{bmatrix}$$

Let's try some examples:

I won’t make you do matrix multiplication, but tell me the dimensions of the product matrices:

$$A_{4x2} \text{ x } B_{2x4} = C_? = C_{4x4}$$

$$A_{100\,x\,9} \text{ x } B_{52x9} = C_? = \text{Cannot be done!}$$

$$A_{n\,x\,p} \text{ x } B_{p\,x\,m} = C_? = C_{n\,x\,m}$$

$$A_{85\,x\,1}\ x\ B_{1\,x\,85} = C_? = C_{85\,x\,85}$$

$$A_{1\,x\,85} \text{ x } B_{85\,x\,1} = C_? = C_{1\,x\,1}$$

Matrix: a Collection of Vectors

Remember our Reptile vectors?

Features **Label**

Name	Egg-laying	Scales	Poisonous	Cold-blooded	# legs	Reptile
Cobra	True	True	True	True	0	Yes
Rattlesnake	True	True	True	True	0	Yes
Boa constrictor	False	True	False	True	0	Yes
Chicken	True	True	False	False	2	No
Alligator	True	True	False	True	4	Yes
Dart frog	True	False	True	False	4	No
Salmon	True	True	False	True	0	No
Python	True	True	False	True	0	Yes

Matrix: a Collection of Vectors

We can represent this entire table as a **matrix**:

$$\begin{bmatrix} Cobra \\ Rattlesnake \\ Boa\ Constrictor \\ Chicken \\ Alligator \\ Dart\ Frog \\ Salmon \\ Python \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 & 4 & 1 \\ 1 & 0 & 1 & 0 & 4 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

If we represented our animals like this data matrix in Python:

Animal_matrix[: , 2] would return all the data for "Poisonous"

Animal_matrix[1 , :] would return all the data for "Rattlesnake"

Animal_matrix[1 , 2] would return the "Poisonous" datapoint for "Rattlesna
(Animal_matrix[1, 2] = 1)

Matrix: a System of Equations (SOE)

Example: Solve

- $x + y + z = 6$
- $2y + 5z = -4$
- $2x + 5y - z = 27$

You've probably been asked to solve a system of equations like this before.

SOE are extremely common in solving real-world engineering and finance problems.

Matrix: a System of Equations (SOE)

Example: Solve

- $x + y + z = 6$
- $2y + 5z = -4$
- $2x + 5y - z = 27$

$$\begin{array}{ccccccc} x & + & y & + & z & = & 6 \\ & & 2y & + & 5z & = & -4 \\ 2x & + & 5y & - & z & = & 27 \end{array}$$

Matrix: a System of Equations (SOE)

They could be turned into a table of numbers like this:

$$\begin{array}{cccccc} 1 & 1 & 1 & = & 6 \\ 0 & 2 & 5 & = & -4 \\ 2 & 5 & -1 & = & 27 \end{array}$$

We could even separate the numbers before and after the "=" into:

$$\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{array} \quad \text{and} \quad \begin{array}{c} 6 \\ -4 \\ 27 \end{array}$$

Now it looks like we have 2 Matrices.

Matrix: a System of Equations (SOE)

Now it looks like we have 2 Matrices.

In fact we have a third one, which is [x y z]:

$$\underset{\boldsymbol{A}}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix}} \underset{\vec{\boldsymbol{v}}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{\vec{\boldsymbol{b}}}{\begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}}$$

$$A\vec{v} = \vec{b}$$

We want $\vec{v}$!

Matrix: a System of Equations (SOE)

The inverse of a matrix is its reciprocal: $\boldsymbol{A^{-1} * A = A * A^{-1} = "1"}$

$$A\vec{v} = \vec{b}$$

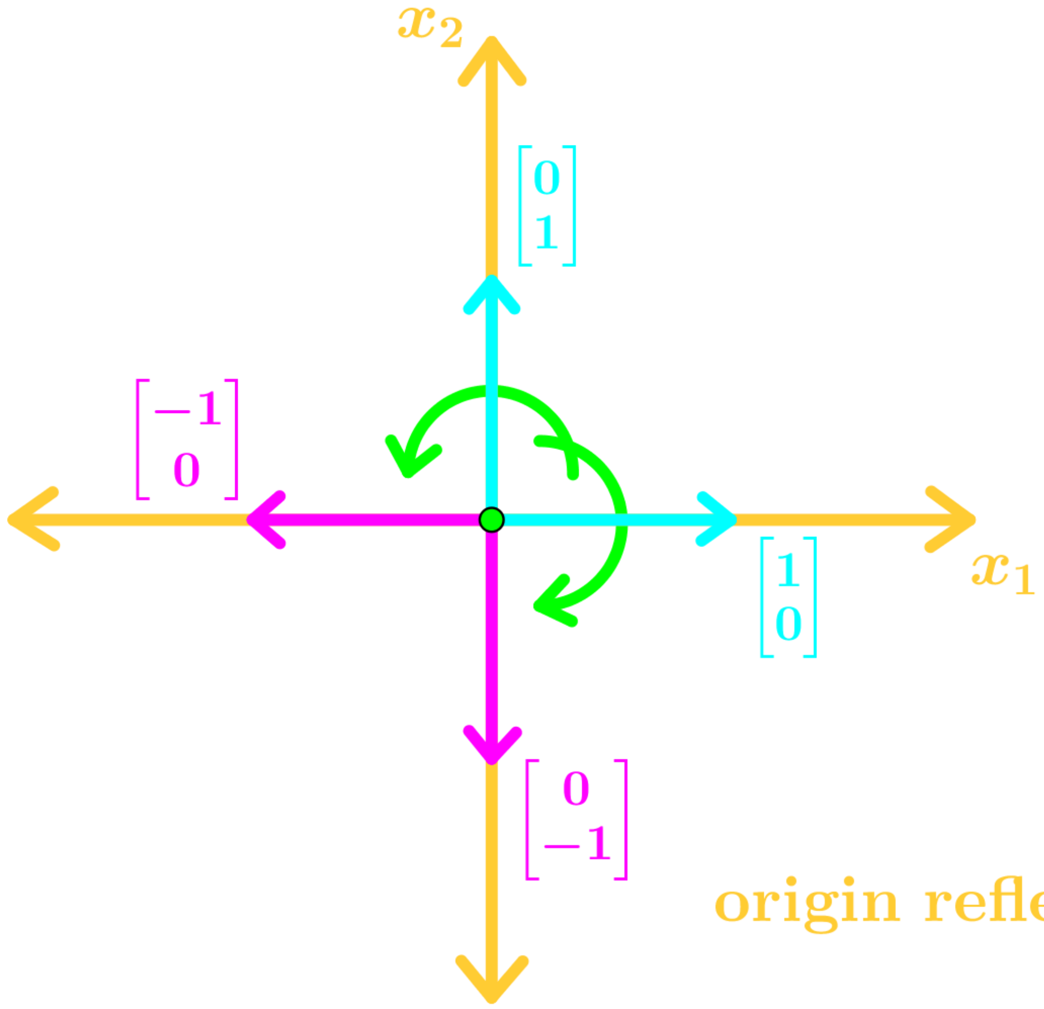
$$(A^{-1})A\vec{v} = (A^{-1})\vec{b}$$

$$\vec{v} = A^{-1}\vec{b}$$

Using the inverse of our coefficient matrix (A^{-1}), a computer can quickly solve for ***x***, ***y*** and ***z***

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-21}\begin{bmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix} = \frac{1}{-21}\begin{bmatrix} -105 \\ -63 \\ 42 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

Matrix: a Linear Transformation



Standard Matrix

$$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$$

origin reflection

Matrix: a Linear Transformation

[Linear transformations and matrices | Chapter 3, Essence of linear algebra]