

# Nonparaxial TM and TE beams in free space

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Expressions for the fields of TM and TE laser beams in free space that are rigorous solutions to Maxwell's equations are given in a closed form. The electric and the magnetic fields are both expressed in terms of nonparaxial elegant Laguerre–Gaussian beams that are exact solutions of the Helmholtz equation. These solutions involve well-known functions, such as spherical Bessel and associated Legendre functions. Radially and azimuthally polarized beams of arbitrary order are considered, and the lowest-order radially polarized beam (TM<sub>01</sub> beam) is investigated in detail. © 2008 Optical Society of America  
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Many authors have studied TM electromagnetic beams and TE beams in the paraxial approximation [1]. However, in the case of tightly focused beams nonparaxial beams have to be considered. Moreover, the vectorial nature of light cannot be ignored to correctly describe nonparaxial beams. Consequently, the expressions for the electric and the magnetic fields of a nonparaxial beam must be exact solutions to Maxwell's equations. Some authors have proposed expressions for TM and TE beams that are solutions to Maxwell's equations. Quabis *et al.* [2], Youngworth and Brown [3], and Kozawa and Sato [4] have used the integral representation for the electric field, as introduced by Richards and Wolf [5], but they have to solve integrals numerically. Salamin [6] has employed a method that consists in adding nonparaxial correction terms to the paraxial solution and that has its origins in the work of Lax and co-workers [7]; however, the series is analytically truncated to some degree of accuracy. Deng *et al.* [8] have made use of the vectorial Rayleigh–Sommerfeld formulas to evaluate the electric field distribution of nonparaxial beams, but the expressions obtained include inherent approximations. Hence, in all the cases previously mentioned, no closed-form expressions for the electromagnetic fields that are rigorous solutions to Maxwell's equations valid for all space are provided. The aim of this Letter is to present such analytical expressions for the fields of TM and TE optical beams. The field components are expressed in terms of functions that reduce to elegant Laguerre–Gaussian modes in the paraxial limit. These functions are in turn written as a simple linear combination of functions that involve spherical Bessel and associated Legendre functions of complex arguments and that can ultimately be expressed only in terms of trigonometric functions.

In this Letter, it is assumed that the propagation axis is the  $z$  axis and that the fields have a time dependence of the form  $\exp(j\omega t)$ , where  $\omega$  is the angular frequency. The wavenumber of the monochromatic fields is  $k = \omega/c$ , where  $c$  is the speed of light in vacuum. The complex source-point (CSP) model is a theory especially appropriate to describe nonparaxial beams. According to the CSP method, a source is assumed to be located at an imaginary distance along the propagation axis, i.e., the coordinate  $z$  is replaced

by the complex quantity  $z + ja$ , where  $a$  is a real constant. For instance, the use of the CSP method is a convenient way to convert a spherical wave into a nonparaxial Gaussian beam (for details regarding the CSP model see [9] and references therein). We introduce complex spherical coordinates  $(\tilde{R}, \tilde{\theta}, \phi)$ :  $\tilde{R} = [x^2 + y^2 + (z + ja)^2]^{1/2}$ ,  $\cos \tilde{\theta} = (z + ja)/\tilde{R}$ , and  $\phi$  is the standard azimuthal angle. For a known wavenumber  $k$  the confocal parameter  $a$  that appears naturally in the oblate spheroidal coordinates can be related to physical parameters such as the waist spot size  $w_o$ , the Rayleigh range  $z_R = \frac{1}{2}kw_o^2$ , or the angle of divergence  $\delta = \arctan(w_o/z_R)$  of the beam by  $a = w_o[1 + (\frac{1}{2}kw_o)^2]^{1/2} = z_R[1 + 2/(kz_R)]^{1/2} = 2/(k \sin \delta \tan \delta)$ . In a previous paper [10] we have presented a function  $\tilde{U}_{p,m}^\sigma$  that is an exact solution to the Helmholtz equation  $\nabla^2 \tilde{U}_{p,m}^\sigma + k^2 \tilde{U}_{p,m}^\sigma = 0$ . The superscript indices  $\sigma = \{e, o\}$  refer to even or odd modes, respectively. This function is given by

$$\tilde{U}_{p,m}^\sigma = 2^{p+2} K_{p,m} \left( \frac{kw_o}{2} \right)^{2p+m+2} \sum_{s=0}^p \binom{p+m}{s+m} \times \frac{(4s+2m+1)(2s-1)!!}{(2p+2s+2m+1)!!} \tilde{\psi}_{2s+m,m}^\sigma, \quad (1)$$

where  $\tilde{\psi}_{n,m}^e = \exp(-ka) j_n(k\tilde{R}) P_n^m(\cos \tilde{\theta}) \cos(m\phi)$ ,  $p = 0, 1, 2, \dots$  is the radial mode number,  $m = 0, 1, 2, \dots$  is the angular mode number,  $K_{p,m}$  is a normalization constant (that is arbitrary and generally depends on the mode numbers  $p, m$ ),  $j_n(k\tilde{R})$  is the spherical Bessel function of the first kind of order  $n$ , and  $P_n^m(\cos \tilde{\theta})$  is the associated Legendre function. The definition used in this Letter for  $P_n^m(\cos \tilde{\theta})$  is taken from Arfken [11]. Superscript index  $e$  in function  $\tilde{\psi}_{n,m}^e$  stands for even modes, with the even function  $\cos(m\phi)$  for the azimuthal dependence; odd modes, denoted by  $\tilde{\psi}_{n,m}^o$ , are simply obtained by replacing  $\cos(m\phi)$  by the odd function  $\sin(m\phi)$ . This particular linear combination of functions  $\tilde{\psi}_{n,m}^\sigma$ , as given by Eq. (1), has the property to reduce to the well-known elegant Laguerre–Gaussian beams in the

paraxial limit ( $ka \gg 1$ ) [10]. Actually, another independent function denoted by  $\tilde{V}_{p,m}^\sigma$  that is required to express analytic solutions for nonparaxial TM beams has the same property. Whereas  $\tilde{U}_{p,m}^\sigma$  is written as a linear combination of  $\tilde{\psi}_{n,m}^\sigma$  for which  $n-m$  is even,  $\tilde{V}_{p,m}^\sigma$  is a linear combination of  $\tilde{\psi}_{n,m}^\sigma$  for which  $n-m$  is odd:

$$\tilde{V}_{p,m}^\sigma = -j2^{p+2}K_{p,m} \left( \frac{kw_o}{2} \right)^{2p+m+2} \sum_{s=0}^p \binom{p+m}{s+m} \times \frac{(4s+2m+3)(2s+1)!!}{(2p+2s+2m+3)!!} \tilde{\psi}_{2s+m+1,m}^\sigma. \quad (2)$$

It is possible to encounter functions  $\tilde{U}_{p,m}^\sigma$  and  $\tilde{V}_{p,m}^\sigma$  with a negative angular number, and they can be converted to functions with the corresponding positive angular number. With Eq. (1) one can derive the following useful relationships:  $p!K_{p-m,m}\tilde{U}_{p,-m}^\sigma = (-1)^m(p-m)!K_{p,-m}\tilde{U}_{p-m,m}^\sigma$  and  $p!K_{p-m,m}\tilde{U}_{p,-m}^\sigma = (-1)^{m+1}(p-m)!K_{p,-m}\tilde{U}_{p-m,m}^\sigma$ . Similar relations can be derived for the function  $\tilde{V}_{p,-m}^\sigma$ , and they may be obtained directly from those for  $\tilde{U}_{p,-m}^\sigma$  by making the substitution  $\tilde{U} \rightarrow \tilde{V}$ . It can also be shown that the functions  $\tilde{U}_{p,m}^\sigma$  and  $\tilde{V}_{p,m}^\sigma$  have the following first derivatives:

$$\frac{\partial \tilde{U}_{p,m}^e}{\partial x} = \frac{1}{w_o} \left[ \frac{(p+1)K_{p,m}}{K_{p+1,m-1}} \tilde{U}_{p+1,m-1}^e - \frac{K_{p,m}}{K_{p,m+1}} \tilde{U}_{p,m+1}^e \right], \quad (3a)$$

$$\frac{\partial \tilde{U}_{p,m}^\sigma}{\partial y} = -\frac{1}{w_o} \left[ \frac{(p+1)K_{p,m}}{K_{p+1,m-1}} \tilde{U}_{p+1,m-1}^\sigma + \frac{K_{p,m}}{K_{p,m+1}} \tilde{U}_{p,m+1}^\sigma \right], \quad (3b)$$

$$\frac{\partial \tilde{U}_{p,m}^\sigma}{\partial z} = -jk\tilde{V}_{p,m}^\sigma, \quad (3c)$$

$$\frac{\partial \tilde{V}_{p,m}^\sigma}{\partial z} = -jk \left[ \tilde{U}_{p,m}^\sigma - \frac{4(p+1)}{k^2 w_o^2} \frac{K_{p,m}}{K_{p+1,m}} \tilde{U}_{p+1,m}^\sigma \right]. \quad (3d)$$

The derivatives of  $\tilde{V}_{p,m}^e$  with respect to  $x$  and  $y$  have the same form as those of  $\tilde{U}_{p,m}^e$ . Also, the derivatives with respect to  $x$  and  $y$  of odd modes can be obtained from those of even modes by making the following substitutions in Eqs. (3):  $\tilde{U}^e \rightarrow \tilde{U}^\sigma$  and  $\tilde{U}^\sigma \rightarrow -\tilde{U}^e$  as well as  $\tilde{V}^e \rightarrow \tilde{V}^\sigma$  and  $\tilde{V}^\sigma \rightarrow -\tilde{V}^e$ . The second derivatives of the functions  $\tilde{U}_{p,m}^\sigma$  and  $\tilde{V}_{p,m}^\sigma$  can be found from Eqs. (3). After straightforward calculations, it can be

verified that  $\tilde{U}_{p,m}^\sigma$  and  $\tilde{V}_{p,m}^\sigma$  are both exact solutions of the Helmholtz equation, that is,  $\nabla^2 \tilde{U}_{p,m}^\sigma + k^2 \tilde{U}_{p,m}^\sigma = 0$  and  $\nabla^2 \tilde{V}_{p,m}^\sigma + k^2 \tilde{V}_{p,m}^\sigma = 0$ .

Nonparaxial TM beams may be obtained from the vector magnetic potential  $\mathbf{A}$  oriented along the propagation axis (i.e.,  $z$  axis) [1]. With the Lorenz gauge the vector magnetic potential  $\mathbf{A}$  satisfies the vectorial Helmholtz equation  $\nabla^2 \mathbf{A} + k^2 \mathbf{A} = \mathbf{0}$ . By definition the magnetic flux density  $\mathbf{B}$  is related to  $\mathbf{A}$  by  $\mathbf{B} = \nabla \times \mathbf{A}$ . With the constitutive relation  $\mathbf{B} = \mu_o \mathbf{H}$  ( $\mu_o$  is the permeability of free space), one can deduce the magnetic field  $\mathbf{H}$  from vector  $\mathbf{A}$  with  $\mathbf{H} = \nabla \times \mathbf{A} / \mu_o$ . Then the electric field  $\mathbf{E}$  can be found from Maxwell's equation  $\mathbf{E} = \nabla \times \mathbf{H} / (j\omega\epsilon_o)$  ( $\epsilon_o$  is the permittivity of free space) in the spectral domain. To obtain expressions for TM modes we consider solutions in which  $\mathbf{A}$  is oriented in the axial direction, i.e.,  $\mathbf{A} = A_o \tilde{U}_{p,m}^e \hat{\mathbf{a}}_z$ , where  $A_o$  is a constant amplitude,  $\tilde{U}_{p,m}^e$  is defined in Eq. (1), and  $\hat{\mathbf{a}}_z$  is a unit vector oriented along the  $z$  axis. Because the nonzero component of  $\mathbf{A}$  is proportional to  $\tilde{U}_{p,m}^e$  it follows that  $\mathbf{A}$  automatically satisfies the vectorial Helmholtz equation. With Eqs. (3) one can show by direct calculations that Maxwell's equations give  $H_z = 0$ , which is expected for TM beams, and the following nonzero fields components:

$$H_x = -\frac{H_o}{2} \left[ \frac{(p+1)K_{p,m}}{K_{p+1,m-1}} \tilde{U}_{p+1,m-1}^\sigma + \frac{K_{p,m}}{K_{p,m+1}} \tilde{U}_{p,m+1}^\sigma \right], \quad (4a)$$

$$H_y = -\frac{H_o}{2} \left[ \frac{(p+1)K_{p,m}}{K_{p+1,m-1}} \tilde{U}_{p+1,m-1}^e - \frac{K_{p,m}}{K_{p,m+1}} \tilde{U}_{p,m+1}^e \right], \quad (4b)$$

$$E_x = -\frac{E_o}{2} \left[ \frac{(p+1)K_{p,m}}{K_{p+1,m-1}} \tilde{V}_{p+1,m-1}^e - \frac{K_{p,m}}{K_{p,m+1}} \tilde{V}_{p,m+1}^e \right], \quad (4c)$$

$$E_y = \frac{E_o}{2} \left[ \frac{(p+1)K_{p,m}}{K_{p+1,m-1}} \tilde{V}_{p+1,m-1}^\sigma + \frac{K_{p,m}}{K_{p,m+1}} \tilde{V}_{p,m+1}^\sigma \right], \quad (4d)$$

$$E_z = E_o \frac{2(p+1)}{jk w_o} \frac{K_{p,m}}{K_{p+1,m}} \tilde{U}_{p+1,m}^e, \quad (4e)$$

where  $H_o = 2A_o / (\mu_o w_o)$ ,  $E_o = \eta_o H_o$ , and  $\eta_o = (\mu_o / \epsilon_o)^{1/2}$  is the intrinsic impedance of free space. Such a TM beam, as expressed by Eqs. (4), will be specifically designated as a  $\text{TM}_{p,m+1}$  beam. The expressions for the electromagnetic fields of TE modes can be easily found from those of TM modes (and vice versa) by means of the duality transformation  $\mathbf{E} \rightarrow \eta_o \mathbf{H}$  and  $\mathbf{H} \rightarrow -\mathbf{E} / \eta_o$ ; such TE beams may then be denoted  $\text{TE}_{p,m+1}$  beams.

One can obtain radially polarized beams by setting  $m=0$  in Eqs. (4). In fact,  $E_\phi = -E_x \sin \phi + E_y \cos \phi \neq 0$  if  $m=0$ . Moreover,  $H_r = H_x \cos \phi + H_y \sin \phi \neq 0$  if  $m=0$  so that the magnetic field is purely azimuthal. Consequently, radially polarized beams (of order  $p$ ) are denoted  $\text{TM}_{p,1}$  beams. Explicitly, the nonzero electro-

magnetic components of a radially polarized beam (TM<sub>p1</sub> beam) are in general given by

$$H_\phi = H_o \frac{K_{p,0}}{K_{p,1}} (\tilde{U}_{p,1}^o \sin \phi + \tilde{U}_{p,1}^e \cos \phi), \quad (5a)$$

$$E_r = E_o \frac{K_{p,0}}{K_{p,1}} (\tilde{V}_{p,1}^e \cos \phi + \tilde{V}_{p,1}^o \sin \phi), \quad (5b)$$

$$E_z = E_o \frac{2(p+1)}{jk\omega_o} \frac{K_{p,0}}{K_{p+1,0}} \tilde{U}_{p+1,0}^e. \quad (5c)$$

It can be verified that Eqs. (5) are  $\phi$  independent. It should be pointed out that the expressions for the electromagnetic components of an azimuthally polarized beam (of order  $p$ ), denoted the TE<sub>p1</sub> beam, may be easily found from those of a radially polarized beam by means of the previously mentioned duality transformation. The well-known TM<sub>01</sub> beam is the lowest-order member of the family of the TM<sub>p1</sub> beams and is obtained by setting  $p=0$  in Eqs. (5). Using Eqs. (1) and (2) one finds

$$H_\phi = H'_o \exp(-ka) j_1(k\tilde{R}) \sin \tilde{\theta}, \quad (6a)$$

$$E_r = -\frac{1}{2} j E'_o \exp(-ka) j_2(k\tilde{R}) \sin(2\tilde{\theta}), \quad (6b)$$

$$E_z = -\frac{2}{3} j E'_o \exp(-ka) [j_0(k\tilde{R}) + j_2(k\tilde{R}) P_2(\cos \tilde{\theta})], \quad (6c)$$

where  $H'_o = \frac{1}{2} H_o k^3 \omega_o^3 K_{0,0}$ ,  $E'_o = \eta_o H'_o$ ,  $j_n(k\tilde{R})$  is the spherical Bessel function of order  $n$ , and  $P_2(\cos \tilde{\theta}) = \frac{1}{4} [1 + 3 \cos(2\tilde{\theta})]$  is the Legendre polynomial of degree 2. If we set  $E'_o = j_2^{\frac{3}{2}} \exp(ka)$ , Eqs. (6b) and (6c) are identical to the expressions for the TM<sub>01</sub> electric field found by Sheppard and Saghaei [12]. Figure 1 shows the variation in the electric energy density (defined

by  $w_e(r, \phi, z) \equiv \frac{1}{2} \epsilon_o |\mathbf{E}(r, \phi, z)|^2$ ) for a TM<sub>01</sub> beam, normalized to the maximum electric energy density, in the beam waist for different values of  $ka$ . In the paraxial limit ( $ka \gg 1$ ), the energy density on the propagation axis is small compared to its maximum value, providing to the beam a dark center, as it is well known. As the beam is focused (as the value of  $ka$  decreases) the energy density associated with the axial electric component is enhanced and the dark center gradually disappears. For values of  $ka$  smaller than 10, the longitudinal component of the electric field dominates the radial component, so that the energy density is maximum at the center of the beam. For values of  $ka$  as small as 1, there are oscillations in the energy density profile in the beam waist. It is instructive to compute the FWHM of the energy density profile when the TM<sub>01</sub> beam is ultimately focused ( $ka=0$ ); this gives  $0.402\lambda$ , a value close to the beam size calculated by Quabis *et al.* [2], despite the fact that these values have been calculated with two different approaches. Also, the theoretical energy density profile of the TM<sub>01</sub> beam with  $ka=1$  (numerical aperture approximately equal to 0.9) calculated with Eqs. (6) and illustrated in Fig. 1 is in agreement with the experimental results presented by Dorn *et al.* [13].

In conclusion, we have obtained closed-form expressions for the electromagnetic fields of non-paraxial TM and TE beams of any order, termed TM<sub>p,m+1</sub> and TE<sub>p,m+1</sub> beams, respectively, that are rigorous solutions to Maxwell's equations. These analytical expressions may be employed to investigate efficiently, for instance, the focusing properties of the TM<sub>01</sub> laser beam and of radially polarized laser beams of higher order ( $p \geq 1$ ).

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## References

1. L. W. Davis and G. Patsakos, Opt. Lett. **6**, 22 (1981).
2. S. Quabis, R. Dorn, M. Eberler, O. Glöckl, and G. Leuchs, Opt. Commun. **179**, 1 (2000).
3. K. S. Youngworth and T. G. Brown, Opt. Express **7**, 77 (2000).
4. Y. Kozawa and S. Sato, J. Opt. Soc. Am. A **24**, 1793 (2007).
5. B. Richards and E. Wolf, Proc. R. Soc. London, Ser. A **253**, 358 (1959).
6. Y. I. Salamin, Opt. Lett. **31**, 2619 (2006).
7. M. Lax, W. H. Louisell, and W. B. McKnight, Phys. Rev. A **11**, 1365 (1975).
8. D. Deng, J. Opt. Soc. Am. B **23**, 1228 (2006).
9. C. J. R. Sheppard and S. Saghaei, J. Opt. Soc. Am. A **16**, 1381 (1999).
10. A. April, Opt. Lett. **33**, 1392 (2008).
11. G. Arfken, *Mathematical Methods for Physicists*, 3rd ed. (Academic, 1985).
12. C. J. R. Sheppard and S. Saghaei, Opt. Lett. **24**, 1543 (1999).
13. R. Dorn, S. Quabis, and G. Leuchs, Phys. Rev. Lett. **91**, 233901 (2003).

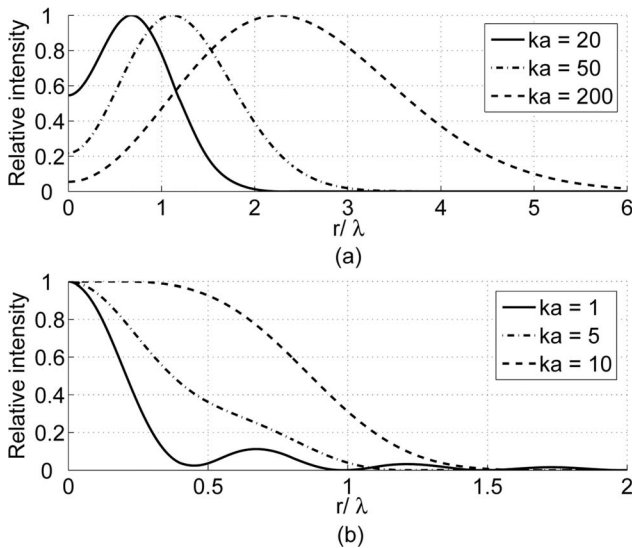


Fig. 1. Transverse variation of the electric energy density (in arbitrary units) in the beam waist ( $z=0$ ) for a TM<sub>01</sub> beam, (a)  $ka=20, 50, 200$  and (b)  $ka=1, 5, 10$ .