Computational topology: Lecture 4

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1. Simplicial Complexes
2. Cellular Complexes
3. Chain Complexes

Simplicial Complexes

Join operation

The join of two sets $P, Q \subset \mathbb{E}^n$ is the set of convex combination of their points:

$$PQ = \{\alpha \mathbf{x} + \beta \mathbf{y} |\ \mathbf{x} \in P, \mathbf{y} \in Q\},$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

The join operation is associative and commutative.

Implementation

How ?

Simplex

A d-simplex $\sigma_d \subset \mathbb{E}^n$ $(0 \leq d \leq n)$ may be defined as the repeated join of $d+1$ affinely independent points, called vertices.

A d-simplex can be seen as a d-dimensional triangle: a 0-simplex is a point, a 1-simplex is a segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and so on.

The set $\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d\}$ of vertices of σ_d is called the 0-skeleton of σ_d.

The s-simplex generated from any subset of $s+1$ vertices $(0 \leq s \leq n)$ of σ_d is called an s-face of σ_d.

Let us notice, from the definition, that a simplex may be considered both as a purely combinatorial object and as a geometric object, i.e. as the compact point set defined by the convex hull of a discrete set of points.

Implementation

How ?

Simplicial complex

Definition

A set Σ of simplices is called a triangulation.

A simplicial complex, often simply denoted as complex, is a triangulation Σ that verifies the following conditions:

1. if $\sigma \in \Sigma$, then any face of σ belongs to Σ;
2. if $\sigma, \tau \in \Sigma$, then either $\sigma \cap \tau = \emptyset$, or $\sigma \cap \tau$ is a face of both σ and τ.

Simplicial complex

Definitions

A simplicial complex can be considered a “well-formed" triangulation.

Such kind of triangulations are widely used in engineering analysis, e.g., in topography or in finite element methods.

The order of a complex is the maximum order of its simplices. A complex Σ_d of order d is also called a d-complex.

A d-complex is said to be regular or pure if each simplex is a face of a d-simplex.

A regular d-complex is homogeneously d-dimensional.

Implementation

How ?

Combinatorial boundary

The combinatorial boundary $\Sigma_{d-1} = \partial\sigma_d$ of a simplex σ_d is a simplicial complex consisting of all proper s-faces ($s < d$) of σ_d.

Two simplices σ and τ in a complex Σ are called s-adjacent if they have a common s-face.

With some abuse of language, we call (combinatorial) s-skeletons the sets $\mathcal{K}_s$ ($s \leq d$). Geometric carrier $|\Sigma|$, also called the support space, is the point-set union of simplices in Σ.

Implementation

How ?

```
help?> Lar.simplexFacets
```

Orientation

The ordering of the 0-skeleton of a simplex implies an orientation of it. The simplex can be oriented according to the even or odd permutation class of its 0-skeleton.

The two opposite orientations of a simplex will be denoted as $+\sigma$ and $-\sigma$.

Two simplices are coherently oriented when their common faces have opposite orientation. A complex is orientable when all its simplices can be coherently oriented.

It is assumed that:

1. The two orientations of a simplex may represent its relative interior and exterior;
2. the two orientations of an orientable simplicial complex analogously represent the relative interior and exterior of the complex, respectively;
3. the boundary of a complex maintains the same orientation of the complex.

Orientation and volume

The volume associated with an orientation of a simplex (or complex) is positive, while the one associated with the opposite orientation has the same absolute value and opposite sign.

It is assumed that the bounded object has positive volume.

It is also assumed that either a minus sign or a multiplying factor -1 denote a complementation, i.e. an opposite orientation of the simplex, which can be explicitly obtained by swapping two vertices in its ordered 0-skeleton:

$$\begin{aligned} +\sigma_3 &= \langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle \\ -\sigma_3 &= \langle \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3 \rangle \end{aligned}$$

Implementation

How ?

```
help?> Lar.surface
```

Face extraction

The oriented facets $\sigma_{d-1,(i)}$ $(0 \leq i \leq d)$ of the oriented d-simplex $\sigma_d = +\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d \rangle$ are obtained by removing the i-th vertex $\mathbf{v}_i$ from the 0-skeleton of σ_d:

$$\sigma_{d-1,(i)} = (-1)^i (\sigma_d - \langle \mathbf{v}_i \rangle), \qquad 0 \leq i \leq d. \tag{1}$$

where the minus sign denotes a subtraction between 0-skeletons.

The 0-skeleton of $\sigma_{d-1,(i)}$ is therefore obtained by removing the i-th vertex from the 0-skeleton of σ_d and either by swapping a pair of vertices or, better, by inverting the simplex sign, when i is odd.

Implementation

How ?

```
help?> Lar.simplexFacets
```

Facet extraction

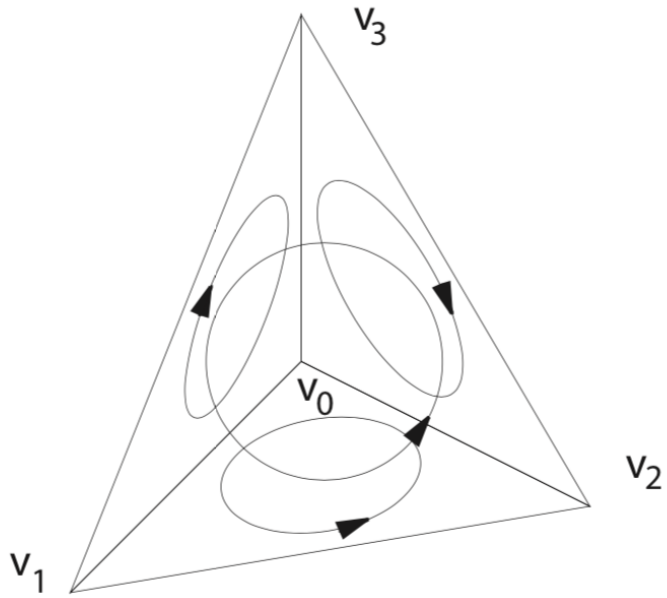

Figure 1: Coherent orientation of the facets of a 3-simplex

Implementation? See:

function Lar.simplexFacets(simplices::Cells)::Cells

Simplicial prism

The prism over a simplex $\sigma_d = \langle \mathbf{v}_0, \ldots, \mathbf{v}_d \rangle$, defined as the set $P_{d+1} := \sigma_d \times [a, b]$, with $[a, b] \subset \mathbb{E}$, will be called simplicial $(d+1)$-prism.

An oriented complex which triangulates P_{d+1} can be defined combinatorially, by using a closed form formula for its $\mathcal{K}_{d+1}$ skeleton:

$$\mathcal{K}_{d+1} = \{\sigma_{d+1,(i)} = (-1)^{id} \langle \mathbf{v}_i^a, \mathbf{v}_{i+1}^a, \ldots, \mathbf{v}_d^a, \mathbf{v}_0^b, \mathbf{v}_1^b, \ldots, \mathbf{v}_i^b \rangle | 0 \leq i \leq d\}$$

where $\mathbf{v}_i^a = (\mathbf{v}_i, a)$ and $\mathbf{v}_i^b = (\mathbf{v}_i, b)$.

Exercise: extract the boundary of implicial prism

extract the boundary of the simplicial cube

- write dimension-independent code
- apply to a grid of such decomposed cubes

Hint

See and modify the code of `Lar.simplexGrid`

Cellular Complexes

Preliminary definitions

Let X be a topological space, and $\Lambda(X) = \bigcup \Lambda_p$ $(p \in \{0, 1, \ldots, d\})$ a partition of X, with Λ_p a set of $\{$(relatively) open$\}$, connected, and manifold p-cells.

A CW-structure on the space X is a filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_{d-1} \subset X = \bigcup_p X_p$, such that, for each p, the skeleton X_p is homeomorphic to a space obtained from X_{p-1} by attachment of p-cells in $\Lambda_p = \Lambda_p(X) \sim$[**?**].

Cellular complex

A CW-complex is a space X endowed with a CW-structure, and is also called a cellular complex.

A cellular complex is finite when it contains a finite number of cells.

A $\{$regular$\}$ d-complex is a complex where each closed d-cell equates the closure of its interior, and where p-cells ($p < d$) are contained in the boundary of a d-cell.

- Two d-cells are coherently oriented when their common $(d-1)$-cells have opposite orientations.
- A cellular d-complex X is orientable when its d-cells can be coherently oriented.
- The support space $|\sigma|$ of a cell σ is its compact point-set.

Implementation

How ?

Chain Complexes

Chain groups

aaaaa

Chains are defined by attaching coefficients to cells.

Since one wishes to add chains, one has to pick coefficients from a set endowed with the structure of a commutative group, or stronger.

Let $(G, +, 0)$ be a nontrivial commutative group, whose identity element is denoted 0.

A p-chain of X with coefficients in G is a mapping $c_p : X \to G$ such that, for each $\sigma \in X_p$, reversing a cell orientation changes the sign of the chain value:

$$c_p(-\sigma) = -c_p(\sigma).$$

Chain groups

aaaaa

Chain addition is defined by addition of chain values: if c_p^1, c_p^2 are p-chains, then $(c_p^1 + c_p^2)(\sigma) = c_p^1(\sigma) + c_p^2(\sigma)$, for each $\sigma \in X_p$.

The resulting group is denoted $C_p(X; G)$. When clear from the context, the group G is often left implied, writing $C_p(X)$.

Let σ be an oriented cell in X and $g \in G$. The elementary chain whose value is g on σ, $-g$ on $-\sigma$ and 0 on any other cell in X is denoted $g\sigma$.

Each chain can be written in a unique way as a sum of elementary chains.

Chain groups

aaaaa

Chains are often thought of as attaching orientation and/or multiplicity to cells

If coefficients are taken from the group $G = (\{-1, 0, 1\}, +, 0)$, then cells can only be discarded or selected, possibly inverting their orientation

A p-cycle is a closed p-chain, i.e., a p-chain without boundary.

Chain orientation

It is useful to select a conventional orientation to orient cells automatically.

- 0-cells are considered all positive.
- Closed p-cells can be given a coherent (internal) orientation according with the orientation of the first $(p-1)$-cell in their canonical representation (sorted on indices) of their $(p-1)$-cycles.
- Finally, a d-cell may be oriented as the sign of its oriented volume.

Chain spaces

aaaaa

To allow not only for chain addition, but also for linear combination of chains, coefficients should be taken from a set endowed with the structure of a field, such as $(\mathbb{F}, +, \times, 0, 1)$,

where 0 and $1 \neq 0$ denote, respectively, the additive and multiplicative identities.

Unit chains are elementary chains whose value is $u = 1\sigma$ for some cell σ.

Each chain can be written in a unique way as a linear combination of unit chains $u \in U$, if the outer cell is not taken into account.

Chain spaces

Canonical representation and standard basis

Hence, the space of p-chains C_p is endowed with a standard (or natural) basis, comprised of all the independent unit p-chains.

The standard basis may be the one associated with the canonical representation ...

In particular, $\#U_d = \#\Lambda_d - 1$.

Often, with some abuse of notation, one does not distinguish between a p-cell and the corresponding unit p-chain.

Characteristic matrices

Given a set $S = \{s_j\}$, the characteristic function $\chi_A : S \to \{0, 1\}$ takes value 1 for all elements of $A \subseteq S$ and 0 at all elements of S not in A.

We call characteristic matrix M of a collection of subsets $A_i \subseteq S$ $(i = 1, \ldots, n)$ the binary matrix $M = (m_{ij})$, with $m_{ij} = \chi_{A_i}(s_j)$.

A matrix M_p, whose rows are indexed by unit p-chains and columns are indexed by unit 0-chains, provides a useful representation of a basis for the linear space C_p.

Permuting (reindexing) either rows or columns provides a different basis.

While chains are mostly presented as formal sums of cells, in the actual implementation their signed coordinate vectors are used as sparse arrays, and in particular as CSC (Compressed Sparse Column) maps : $\mathbb{N} \to \{-1, 0, 1\}$.

Characteristic matrices

Implementation

Implementation

How ?

help?> Lar.characteristicMatrix

WHY Characteristic matrices are useful

Consider the incidence queries that arise in a cellular decomposition $\Lambda(X)$ of a 2D space, such as 2D triangulation or the boundary of a manifold 3-solid.

There are 9 incidence relations between pairs of cells in such a V, E, F decomposition, and traditional graph-based representations are chosen by optimizing the trade-off between space requirements and efficiency of queries.

With LAR, all such queries are sparse matrix–vector multiplications without any additional space requirements.

The chain operators corresponding to the incidence relations are

$$\begin{array}{lll} \mathcal{V}\mathcal{V} : C_0 \to C_0, & \mathcal{E}\mathcal{V} : C_0 \to C_1, & \mathcal{F}\mathcal{V} : C_0 \to C_2; \\ \mathcal{V}\mathcal{E} : C_1 \to C_0, & \mathcal{E}\mathcal{E} : C_1 \to C_1, & \mathcal{F}\mathcal{E} : C_1 \to C_2; \\ \mathcal{V}\mathcal{F} : C_2 \to C_0, & \mathcal{E}\mathcal{F} : C_2 \to C_1, & \mathcal{F}\mathcal{F} : C_2 \to C_2. \end{array}$$

Characteristic matrices: Examples

From incidence/adjacency operators to their matrices

The corresponding CSR matrices are readily computed:

$$\mathcal{V}\mathcal{V} = \mathcal{V}\mathcal{E} \circ \mathcal{E}\mathcal{V} = \mathcal{E}\mathcal{V}^\top \circ \mathcal{E}\mathcal{V} \Rightarrow [\mathcal{V}\mathcal{V}] = M_1^t M_1$$

$$\mathcal{V}\mathcal{E} = \mathcal{E}\mathcal{V}^\top \Rightarrow [\mathcal{V}\mathcal{E}] = M_1^t$$

$$\mathcal{V}\mathcal{F} = \mathcal{F}\mathcal{V}^\top \Rightarrow [\mathcal{V}\mathcal{F}] = M_2^t$$

$$\mathcal{E}\mathcal{V} \quad [\mathcal{E}\mathcal{V}] = M_1$$

$$\mathcal{E}\mathcal{E} = \mathcal{E}\mathcal{V} \circ \mathcal{V}\mathcal{E} = \mathcal{E}\mathcal{V} \circ \mathcal{E}\mathcal{V}^\top \Rightarrow [\mathcal{E}\mathcal{E}] = M_1 M_1^t$$

$$\mathcal{E}\mathcal{F} = \mathcal{E}\mathcal{V} \circ \mathcal{V}\mathcal{F} = \mathcal{E}\mathcal{V} \circ \mathcal{F}\mathcal{V}^\top \Rightarrow [\mathcal{E}\mathcal{F}] = M_1 M_2^t$$

$$\mathcal{F}\mathcal{V} \quad [\mathcal{F}\mathcal{V}] = M_2$$

$$\mathcal{F}\mathcal{E} = \mathcal{F}\mathcal{V} \circ \mathcal{V}\mathcal{E} = \mathcal{F}\mathcal{V} \circ \mathcal{E}\mathcal{V}^\top \Rightarrow [\mathcal{F}\mathcal{E}] = M_2 M_1^t$$

$$\mathcal{F}\mathcal{F} = \mathcal{F}\mathcal{V} \circ \mathcal{V}\mathcal{F} = \mathcal{F}\mathcal{V} \circ \mathcal{F}\mathcal{V}^\top \Rightarrow [\mathcal{F}\mathcal{F}] = M_2 M_2^t.$$