Chapter 9
Homology Groups

Given a topological space, its homology is a formal, algebraic way to talk about its connectivity. Better known than the homology groups are their ranks, which are the Betti numbers of the space. In contrast to most other topological formalisms that capture connectivity, homology groups have fast algorithms.

9.1 Little Creatures

We motivate the sequence definitions needed to construct homology groups with a simple example: the 2-dimensional torus. Embedding it in $\mathbb{R}^3$, it looks like a tire filled with air; see Fig. 9.1. Imagine the torus finely triangulated, too fine to show the vertices and edges. There is a large population of little creatures that explore the surfacethrough traveling. After some time, they all deliver a report describing in great detail the traveled path. Some never get back to where they started, and we find their reports not very useful in determining anything about the global connectivity of the surface. Many return to their original position, and from each we get a closed path as the record of the trip. There is a large number of records, some similar and some different. To bring order into the collection, we first select the closed paths that surround a piece of real estate. A closed path that does not fall into this category has the property that the real estate on its left is mysteriously connected to the real estate on its right. To find out, we consult the open path records we have prematurely labeled as useless for our exploration. But there are too many closed paths in the second category, so we further simplify by grouping closed paths that together surround a piece of real estate.

After this elaborate classification effort, we get the closed paths separated into four categories. In the first, all paths surround real estate, and in each of the other three, every pair of closed paths surrounds real estate. Strangely, there is also a relationship between the latter three categories: picking one closed path from each, the triplet again surrounds real estate, as shown in Fig. 9.1.

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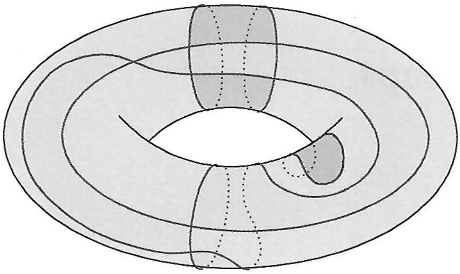

Fig. 9.1 The torus embedded in $\mathbb{R}^3$. Of the six paths, one surrounds real estate by itself, two form a pair with a ring-shaped piece of real estate between them, and three collaborate to surround a complicated piece of real estate that reaches all the way around the torus

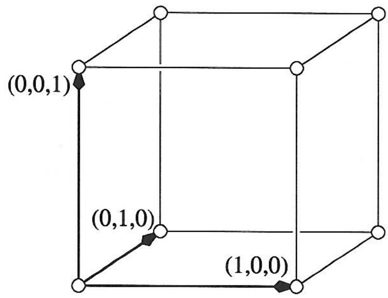

Fig. 9.2 Vector space interpretation of the chain group

9.2 Chain Groups

To formalize the exploration effort, we now sort the records into an algebraic structure. First, we specify what exactly these records are. Assuming a simplicial complex, K, a *p-chain* is a subset of the p-simplices in K. To relate this concept to our motivating example, we set $p = 1$, but we no longer insist that a chain is connected. Indeed, we can think of a chain as the travel record of a team of little creatures. This is useful because we may want to combine teams and their records to form new records. Formally, the *sum* of two p-chains is the symmetric difference between the two sets of p-simplices, which is again a p-chain. We write $(\mathsf{C}_p, +)$ for the thus constructed group of p-chains, or usually just C_p.

Before we continue, we point out that C_p may be large but it has the simple structure of a vector space. This vector space is generated by the s_p p-simplices in K, which we draw as the unit coordinate vectors of this s_p-dimensional space. Each coordinate takes only two values: 0 or 1, so the vector space consists of the vertices of the s_p-dimensional unit cube; see Fig. 9.2.

9.3 Chain Complex

To prepare the next step, we define the *boundary* of a p-simplex as the set of $(p-1)$-faces. There are $p+1$ of them. The boundary of a p-chain is the sum of the boundaries of its p-simplices: $\partial_p c = \sum_{\sigma \in c} \partial_p \sigma$. Note that the boundary is a $(p-1)$-chain. Hence, we can formalize this operation as a map between vector spaces:

$$\partial_p : \mathsf{C}_p \to \mathsf{C}_{p-1}, \tag{9.1}$$

called the *p-th boundary map* or *boundary homomorphism*. Indeed, it has the property of a homomorphism, namely that the map commutes with the group operation: $\partial_p(c+c') = \partial_p c + \partial_p c'$. If we now line up the groups and the maps, we get the *chain complex* of K:

$$\ldots \xrightarrow{\partial_{p+2}} \mathsf{C}_{p+1} \xrightarrow{\partial_{p+1}} \mathsf{C}_p \xrightarrow{\partial_p} \mathsf{C}_{p-1} \xrightarrow{\partial_{p-1}} \ldots. \tag{9.2}$$

Only the groups of dimension $0 \le p \le \dim K$ are non-trivial, but it is convenient to add trivial groups at both ends so we do not have to worry about where the line ends. Doing so requires that the entire group of 0-chains maps to 0, but this makes sense because the boundary of a vertex is empty in any case. Similarly, we will usually drop the index and not distinguish between maps in different dimensions.

Note that for $p \ge 2$, every $(p-2)$-face of a p-simplex, σ, belongs to exactly two $(p-1)$-faces. Hence, $\partial\partial\sigma = 0$, and this also holds for $p < 2$. It follows that applying the boundary map twice to a chain gives 0. This may be obvious but it is important so we state it explicitly.

Fundamental Lemma $\partial\partial c = 0$.

9.4 Cycles and Boundaries

Next, we focus on chains that close up, like the closed paths traveled by our little creatures. A *p-cycle* is a p-chain whose boundary is empty. Since the boundary map commutes with the group operation, the sum of two cycles is again a cycle: $\partial(z+z') = \partial z + \partial z' = 0$. We use this as a reminder that a cycle does not need to be connected, and its components may cross one another. The p-cycles thus form a subgroup of the p-chains, denoted as Z_p, for every dimension p. Recall that the *kernel* of a map is everything in the domain that maps to 0, so $\mathsf{Z}_p = \ker \partial_p$; see Fig. 9.3. Recall that for $p = 0$, the entire chain group maps to zero. This implies $\mathsf{Z}_0 = \ker \partial_0 = \mathsf{C}_0$.

A special type of cycle is obtained by taking the boundary of a higher-dimensional chain. A *p-boundary* is the boundary of a $(p+1)$-chain: $b = \partial_{p+1} c$, and since $\partial b = \partial\partial c = 0$, the p-boundary is indeed a p-cycle. Similarly, if $b' = \partial_{p+1} c'$ then

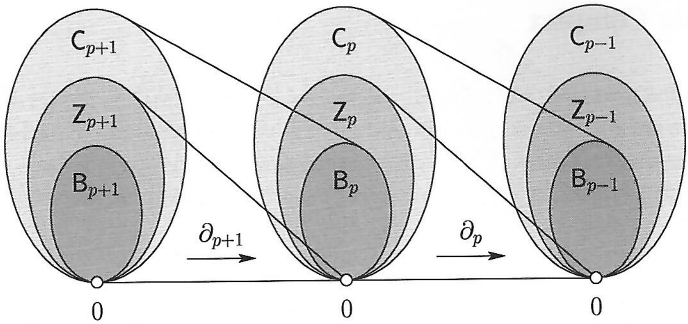

Fig. 9.3 Three consecutive groups in the chain complex. The cycle and boundary subgroups are shown as kernels and images of the boundary maps

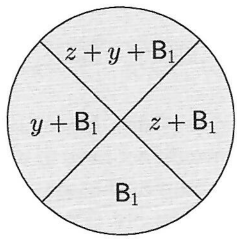

Fig. 9.4 Partition of the first cycle group of the torus into homology classes

$b + b' = \partial_{p+1}(c + c')$, so the sum of two p-boundaries is again a p-boundary. In other words, the p-boundaries form a subgroup of the p-cycles, as indicated in Fig. 9.3. We denote this subgroup by B_p. Recall that the *image* of a map consists of all elements in the range reached by elements in the domain. Hence, $B_p = \operatorname{im} \partial_{p+1}$.

9.5 Homology

The boundary group collects all records that—in the language of our motivational example—surround real estate. We still have to categorize the cycles that do not bound. The idea is to not distinguish between two such cycles if they differ only by a boundary. In other words, we say $z, z' \in Z_p$ are equivalent if there exists $b \in B_p$ such that $z = z' + b$. Since this implies the existence of $c \in C_{p+1}$ with $\partial c = b$, we conclude that $z - z' = z + z'$ is the boundary of a $(p + 1)$-chain. In this case, we write $z \sim z'$ and say that z and z' are *homologous*.

We can now categorize the cycles into equivalence classes, called *homology classes*; see Fig. 9.4. One such class is B_p, another is $z + B_p$, where z is a p-cycle not in B_p. Note that $z' \in z + B_p$ iff z and z' are homologous. Hence, $z \in z' + B_p$, so it really does not matter whether we use z or z' to represent the class. The homology classes that partition Z_p are the elements of the *p-th homology group*, and we write $\mathsf{H}_p = \mathsf{Z}_p/\mathsf{B}_p$, where taking the quotient is the algebraic way of saying that we partition into equivalence classes. Homology classes can also be added, simply by choosing representatives and adding them: $(z + \mathsf{B}_p) + (y + \mathsf{B}_p) = (z + y) + \mathsf{B}_p$. Indeed, if z' and y' are different representatives of the same classes, then $z' = z + b_1$ and $y' = y + b_2$, and therefore

$$(z' + y') + \mathsf{B}_p = (z + y + b_1 + b_2) + \mathsf{B}_p \tag{9.3}$$
$$= (z + y) + \mathsf{B}_p. \tag{9.4}$$

In other words, H_p together with the addition operation forms again a group, and in the particular case in which we add chains by taking the symmetric difference, it is a vector space. The rank of the p-th homology group is called the *p-th Betti number* of K: $\beta_p = \text{rank}\,\mathsf{H}_p$. Recall that the elements in H_p are classes of size 2^r, where $r = \text{rank}\,\mathsf{B}_p$. Since the classes partition Z_p, we have

$$2^{\text{rank}\,\mathsf{Z}_p} = 2^{\text{rank}\,\mathsf{B}_p} \cdot 2^{\text{rank}\,\mathsf{H}_p}, \tag{9.5}$$

and therefore $\beta_p = \text{rank}\,\mathsf{Z}_p - \text{rank}\,\mathsf{B}_p$. This rank is a measure of the difference between the p-th cycle group and the p-th boundary group.

9.6 Euler-Poincaré Formula

Recall that the Euler characteristic of the simplicial complex is $\chi = s_0 - s_1 + s_2 - \cdots \pm s_k$, where $k = \dim K$. Recall also that $s_p = \text{rank}\,\mathsf{C}_p$. We may partition the chain group by calling two chains equivalent if they differ by a cycle. The resulting quotient is known as the *coimage* of the map: $\text{coi}\,\partial_p = \mathsf{C}_p/\mathsf{Z}_p$. All chains in an equivalence class map to the same $(p-1)$-boundary, which implies $\text{rank coi}\,\partial_p = \text{rank}\,\mathsf{B}_{p-1}$ and therefore $\text{rank}\,\mathsf{C}_p = \text{rank}\,\mathsf{Z}_p + \text{rank}\,\mathsf{B}_{p-1}$. We use this equation to rewrite the Euler characteristic:

$$\chi = \sum_{p=0}^{k} (-1)^p \left(\text{rank}\,\mathsf{Z}_p + \text{rank}\,\mathsf{B}_{p-1}\right) \tag{9.6}$$
$$= \sum_{p=0}^{k} (-1)^p \left(\text{rank}\,\mathsf{Z}_p - \text{rank}\,\mathsf{B}_p\right) \tag{9.7}$$
$$= \sum_{p=0}^{k} (-1)^p \beta_p. \tag{9.8}$$

In words, the Euler characteristic is equal to the alternating sum of Betti numbers. This result is known as the Euler-Poincaré Formula. It was established by Henri Poincaré, who thus achieved a vast generalization of Euler's relation, which was originally perceived for convex polytopes in $\mathbb{R}^3$. See [1] for a discussion of the mathematical effort that culminated in Poincaré's proof.

It is important to note that the Betti numbers do not depend on the triangulation but only on the triangulated space. While this is not obvious, it is true and justifies the practice to speak of the Betti numbers of the space and the Euler characteristic of the space, rather than of the triangulation. Similarly, we can talk about the homology groups of the space. However, here we need to mention that the groups depend on the triangulation; after all, the simplices are the elements of the cycles, which are the elements of the classes. Nonetheless, the homology groups of two different triangulations, K and L, of the same space are *isomorphic* to each other: $\mathsf{H}_p(K) \simeq \mathsf{H}_p(L)$. By this we mean that there is a bijection between the classes that commutes with the group operations.

9.7 Some Computations

To comprehend the meaning of the Betti numbers, it is helpful to know what they are for a few spaces we understand. We begin with a general statement about the meaning of the zeroth Betti number.

Claim A *β_0 is the number of connected components.*

Proof Recall that $\mathsf{H}_0 = \mathsf{Z}_0/\mathsf{B}_0$ and assume first that there is only one component. Every set of vertices is a cycle, hence the size of Z_0 is 2^{s_0}. Every even number of vertices is a boundary, and every odd set is not, which implies that the size of B_0 is 2^{s_0-1}. Finally, $\beta_0 = s_0 - (s_0 - 1) = 1$, as claimed. The argument for $j > 1$ the cycle group. The size of the cycle group is still 2^{s_0}. For a set of vertices to be a boundary, it needs to have an even number of vertices in each component. Writing n_i for the number of vertices in the i-th component, the number of boundaries is therefore

$$2^{n_1-1} \cdot 2^{n_2-1} \cdot \ldots \cdot 2^{n_j-1} = 2^{s_0-j}. \tag{9.9}$$

The rank of the zeroth homology group is therefore $\beta_0 = s_0 - (s_0 - j) = j$, as claimed. $\square$

Next, we consider two special spaces.

Claim B *The non-zero Betti numbers of the torus and the Klein bottle are $\beta_0 = 1$, $\beta_1 = 2$, and $\beta_2 = 1$.*

The proof is not very difficult, in particular for $p = 0, 2$. For $p = 1$, we use the help of our little creatures, and we put them to work on the Klein bottle for which they generate the same result. Claim B illustrates the limitations of the discriminative power of homology groups. In particular, using modulo 2 arithmetic, homology

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Reference

1. Lakatos I (1976) Proofs and refutations: the logic of mathematical discovery. Cambridge University Press, Cambridge, England