Computational Algebraic Topology: Lecture 10

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LAR – Topological operators

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Unsigned Boundary-Coboundary

Unsigned boundary operator

Let us consider the incidence operator $\mathcal{I}_{p-1,p} : C_p \to C_{p-1}$ and its matrix $[\mathcal{I}_{p-1,p}] := M_{p-1,p} = M_{p-1}\, M_p^t$. The entry $M_{p-1,p}(i,j)$ stores the value of the application of the $(p-1)$-cochain generator μ^i_{p-1} on the p-chain generator λ^j_p:

$$M_{p-1,p}(i,j) = \langle \mu^i_{p-1}, \lambda^j_p \rangle = \mu^i_{p-1}(\lambda^j_p)$$

Unsigned boundary operator

By standard multiplication in $\mathbb{Z}$ (not $\mathbb{Z}_2$), we get

$$
\begin{aligned}
M_{p-1,p}(i,j) &= \sum_{h=0}^{k_0-1} (M_{p-1}(i,h))\,(M_p(j,h)) \\
&= \sharp(\mu_{p-1}^i \cap \lambda_p^j),
\end{aligned}
$$

thus computing the number of vertices of the intersection (i.e., of the common face) between $\mu_{p-1}^i \subset \Lambda_0$ and $\lambda_p^j \subset \Lambda_0$. This face coincides with μ_{p-1}^i if and only if

$$\sharp(\mu_{p-1}^i \cap \lambda_p^j) = \sharp\mu_{p-1}^i.$$

In such a case, we have

$$\mu_{p-1}^i(\lambda_p^j) = \mu_{p-1}^i \in \partial\lambda_p^j.$$

Algorithm

Therefore, as a computational procedure to calculate the unoriented boundary operator, we have the following algorithm:

1. Compute $M_{p-1,p} := M_{p-1} M_p^t$ by standard matrix product of sparse matrices.
2. For each $0 \leq i \leq k_{p-1} - 1$, set $k = \sharp \mu_{p-1}^i$
3. For each $0 \leq j \leq k_p - 1$:

$$[\,\partial_p\,](i,j) = \begin{array}{ll} 1 & \text{if } M_{p-1,p}(i,j) = k \\ 0 & \text{otherwise} \end{array}$$

Signed Boundary-Coboundary

Cellular complex and Chain spaces

Let X be a topological space, and $\Lambda(X) = \bigcup_k \Lambda_k$ ($k \in 0, 1, \ldots, d$) be a partition of X, with Λ_k a set of open k-cells. CW-structure on the space X is a filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X = \bigcup_d X_d$, such that, for each k, the skeleton X_k is homeomorphic to a space obtained from X_{k-1} by attachment of k-cells in $\Lambda_k = \Lambda_k(X)$.

Cellular complex and Chain spaces

CW-complex is a space X endowed with a CW-structure, and is also called a cellular complex. A cellular complex is finite when it contains a finite number of cells.

A regularized d-complex is a complex where very k-cell $(k < d)$ is contained in the boundary of a d-cell.

Cellular complex and Chain spaces

Let $(G, +)$ be a nontrivial abelian (i.e., commutative) group. A p-chain of X with coefficients in G is a mapping $c_p : X \to G$ such that, for each $\sigma \in X_p$, reversing a cell orientation changes the sign of the chain value:

$$c_p(-\sigma) = -c_p(\sigma).$$

Chain addition is defined by addition of chain values: if c_{p_1}, c_{p_2} are p-chains, then $(c_{p_1} + c_{p_2})(\sigma) = c_{p_1}(\sigma) + c_{p_2}(\sigma)$, for each $\sigma \in X_p$. The resulting group is denoted $C_p(X; G)$. When clear from the context, the group G is often left implied, writing $C_p(X)$.

Cellular complex and Chain spaces

Let σ be an oriented cell in X and $g \in G$. The elementary chain whose value is g on σ, $-g$ on $-\sigma$ and 0 on any other cell in X is denoted $g\sigma$. Each chain can then be written in a unique way as a (finite) sum of elementary chains.

Chains are often thought of as attaching orientation and multiplicity to cells: if coefficients are taken from the group $G = \{-1, 0, 1\}$, then cells can only be discarded or selected, possibly inverting their orientation.

Conventional orientation

Recall some notions about p-chains as linear combinations of oriented $(p-1)$-chains, for $0 \leq p \leq d$

It is useful to select a conventional choice to orient the singleton chains (single cells) automatically.

- The 0-cells are considered all positive.
- The p-cells, for $1 \leq p \leq d-1$, can be given a coherent (internal) orientation according to the orientation of the first $(p-1)$-cell in their canonical (sorted on facet indices) representation.
- Finally, a d-cell may be oriented as the sign of its oriented volume.

Extraction of a 3-cell

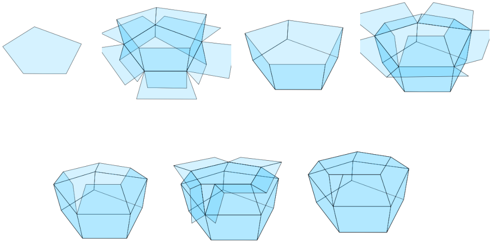
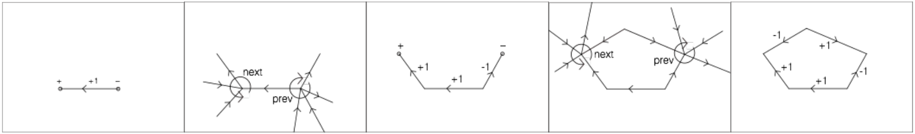


Figure 1: Extraction of a minimal 2-cycle from $\mathcal{A}(X_2)$: (a) the initial value for $c \in C_2$; (b) cyclic subgroups on $\delta\partial c$; (c) new value of c; (d) cyclic subgroups on $\delta\partial c$; (e) new value of c; (f) cyclic subgroups on $\delta\partial c$; (g) new value of c, such that $\partial c = 0$, hence stop.

Extraction of a 2-cell



(a) (b) (c) (d) (e)

Figure 2: Extraction of a minimal 1-cycle from $\mathcal{A}(X_1)$: (a) the initial value for $c \in C_1$ and the signs of its boundary elements; (b) cyclic subgroups on $\delta\partial c$; (c) new (coherently oriented) value of c and signs of ∂c; (d) cyclic subgroups on $\delta\partial c$; (e) final value of c, with $\partial c = \emptyset$.

Extraction of a 2-cell

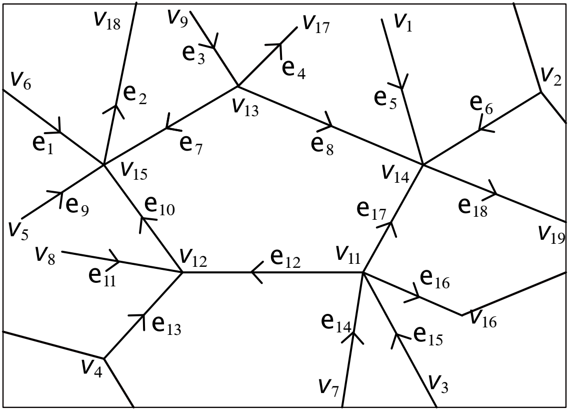


Figure 3: The portion of a 1-complex used by Example **??**.

Extraction of a 2-cell

(a) Set $c = e_{12}$. Then $\partial c = v_{12} - v_{11}$;

(b) then $\delta\partial c = \delta v_{12} - \delta v_{11}$ by linearity. Hence,
$\delta\partial c = (e_{10} + e_{11} + e_{12} + e_{13}) - (+e_{12} + e_{14} + e_{15} + e_{16} + e_{17})$.

(c) Actually,by computing stripe(c) we get

$$\begin{aligned} \text{stripe}(c) &= c + \text{next}(c \cap \delta\partial c) \\ &= c + \text{next}(e_{12})(\delta v_{12}) - \text{next}(e_{12})(\delta v_{11}) \\ &= e_{12} + \text{next}(e_{12})(\delta v_{12}) + \text{prev}(e_{12})(\delta v_{11}) \\ &= e_{12} + e_{10} + e_{17} \end{aligned}$$

If we orient coherently c, we get $c = e_{10} + e_{12} - e_{17}$, and
$\partial c = v_{15} - v_{12} + v_{12} - v_{11} + v_{11} - v_{14} = v_{15} - v_{14}$.

(d) As before, we repeat and reorient coherently the computed 1-chain:

$$\begin{aligned} \text{stripe}(c) &= c + \text{next}(c \cap \delta\partial c) \\ &= c + \text{next}(e_{10})(\delta v_{15}) - \text{next}(e_{17})(\delta v_{14}) \\ &= e_{10} + e_{12} - e_{17} + \text{next}(e_{10})(\delta v_{15}) + \text{prev}(e_{17})(\delta v_{14}) \\ &= e_{10} + e_{12} - e_{17} - e_7 + e_8 \end{aligned}$$

(e) Finally, $\partial\,$stripe$(c) = \emptyset$ and the extraction algorithm terminates, giving $e_{10} + e_{12} - e_{17} - e_7 + e_8$ as a basis element for $C_2(X)$, with $X = \mathcal{A}(X_1)$, and hence as a column for the oriented matrix of the unknown $\partial_2 : C_2 \to C_1$.

Incidence-Adjacency operators

Incidence and Adjacency between boundary elements

Consider the incidence queries that typically arise in a cellular decomposition $\Lambda(X)$ of a 2D space, such as 2D triangulation or the boundary of a solid 3-shape.

There are 9 incidence relations between pairs of cells in such a V, E, F decomposition, and traditional graph-based representations are chosen by optimizing the trade-off between the space requirements and efficiency of queries

Table 1: Incidence and adjacency relations

	V	E	F
V	VV	VE	VF
E	EV	EE	EF
F	FV	FE	FF

Table 2: Incidence and adjacency cardinalities

	V	E	F
V	$2\|E\|$	$2\|E\|$	$2\|E\|$
E	$2\|E\|$	$4\|E\|$	$2\|E\|$
F	$2\|E\|$	$2\|E\|$	$2\|E\|$

Chain operators

With LAR, all such queries are sparse matrix–vector multiplications without any additional space requirements. The chain operators corresponding to the incidence relations $VV \subset V \times V$, $VEsubsetV \times E$, and $VFsubsetV \times F$ are given below:

$$
\begin{array}{lll}
\mathcal{V}\mathcal{V} : C_0 \to C_0, & \mathcal{E}\mathcal{V} : C_0 \to C_1, & \mathcal{F}\mathcal{V} : C_0 \to C_2; \\
\mathcal{V}\mathcal{E} : C_1 \to C_0, & \mathcal{E}\mathcal{E} : C_1 \to C_1, & \mathcal{F}\mathcal{E} : C_1 \to C_2; \\
\mathcal{V}\mathcal{F} : C_2 \to C_0, & \mathcal{E}\mathcal{F} : C_2 \to C_1, & \mathcal{F}\mathcal{F} : C_2 \to C_2.
\end{array}
$$

Figure 4: operators

Matrix operators

$$
\begin{aligned}
&\mathcal{V}\mathcal{V} = \mathcal{V}\mathcal{E} \circ \mathcal{E}\mathcal{V} = \mathcal{E}\mathcal{V}^{\top} \circ \mathcal{E}\mathcal{V} \Rightarrow [\mathcal{V}\mathcal{V}] = M_1^t M_1 \\
&\mathcal{V}\mathcal{E} = \mathcal{E}\mathcal{V}^{\top} \Rightarrow [\mathcal{V}\mathcal{E}] = M_1^t \\
&\mathcal{V}\mathcal{F} = \mathcal{F}\mathcal{V}^{\top} \Rightarrow [\mathcal{V}\mathcal{F}] = M_2^t \\
&\mathcal{E}\mathcal{V} \quad [\mathcal{E}\mathcal{V}] = M_1 \\
&\mathcal{E}\mathcal{E} = \mathcal{E}\mathcal{V} \circ \mathcal{V}\mathcal{E} = \mathcal{E}\mathcal{V} \circ \mathcal{E}\mathcal{V}^{\top} \Rightarrow [\mathcal{E}\mathcal{E}] = M_1 M_1^t \\
&\mathcal{E}\mathcal{F} = \mathcal{E}\mathcal{V} \circ \mathcal{V}\mathcal{F} = \mathcal{E}\mathcal{V} \circ \mathcal{F}\mathcal{V}^{\top} \Rightarrow [\mathcal{E}\mathcal{F}] = M_1 M_2^t \\
&\mathcal{F}\mathcal{V} \quad [\mathcal{F}\mathcal{V}] = M_2 \\
&\mathcal{F}\mathcal{E} = \mathcal{F}\mathcal{V} \circ \mathcal{V}\mathcal{E} = \mathcal{F}\mathcal{V} \circ \mathcal{E}\mathcal{V}^{\top} \Rightarrow [\mathcal{F}\mathcal{E}] = M_2 M_1^t \\
&\mathcal{F}\mathcal{F} = \mathcal{F}\mathcal{V} \circ \mathcal{V}\mathcal{F} = \mathcal{F}\mathcal{V} \circ \mathcal{F}\mathcal{V}^{\top} \Rightarrow [\mathcal{F}\mathcal{F}] = M_2 M_2^t.
\end{aligned}
$$

Figure 5: operators

References

Papers

- Linear Algebraic Representation for Topological Structures
- Arrangements of cellular complexes