Computational topology: Lecture 6

Alberto Paoluzzi

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Topology computing with chains

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$[1, 1, 1, 0, 1, 0]^t$ is the coordinate vector of c with respect to the (ordered) basis $(u_1, u_2, \ldots, u_6) = (1\nu_1, 1\nu_2, \ldots, 1\nu_6)$.
Analogously for the 1-chain $d \in C_1$ and the 2-chain $e \in C_2$, written as $d = \eta_2 + \eta_3 + \eta_5$ and $e = \gamma_1 + \gamma_3$, with coordinate vectors $[0, 1, 1, 0, 1, 0, 0, 0]^t$ and $[1, 0, 1]^t$, respectively.

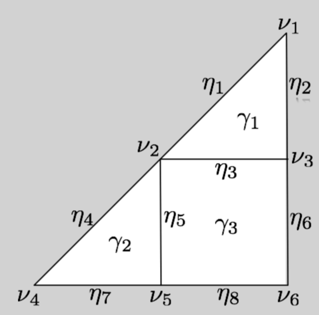

Figure 1: Cellular 2-complex

Example A.3 (Orientation).

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The oriented 1-chain w first vertex ν_1 and last vertex ν_5 is given as $d' = \eta_2 - \eta_3 + \eta_5$, with coordinate vector $[0, 1, -1, 0, 1, 0, 0, 0]^t$

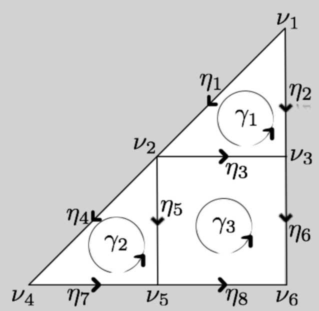

Figure 2: Oriented 2-complex

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In discrete geometric calculus, cochains are functions from chains to reals.
Colored numbers on 1- and 2-cells are exactly the evaluation $\phi^k(u_k) = \langle \phi^k, u_k \rangle$ of the dual elementary cochain on each elementary chain

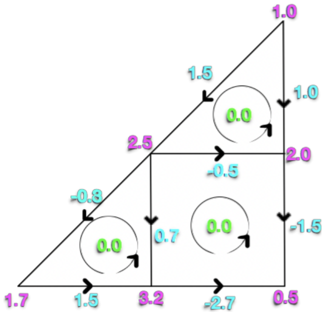

Figure 3: Elementary cochains

Definition (Boundary)

Boundary operators are maps $C_p \to C_{p-1}$, with $1 \leq p \leq d$.

Hence for a 2-complex we have two operators, denoted as

$$\partial_2 : C_2 \to C_1 \quad \text{and} \quad \partial_1 : C_1 \to C_0$$

As linear maps between linear spaces, may be represented by matrices of coefficients $[\partial_2]$ and $[\partial_1]$ from the underlying field $\mathbb{F}$.

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For the unsigned and the signed case (see previous slides) we have:

$$[\partial_2] = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \quad [\partial_2'] = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

Example (Boundary)

Analogously, for the unsigned ∂_1 and the signed ∂_1' operators we have:

$$[\partial_1] = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad [\partial_1'] = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Example A.5 (Boundary)

As a check, let us compute:

1. the 0-boundary of the coordinate representations of the unsigned 1-chain $[d] = [0, 1, 1, 0, 1, 0, 0, 0]^t$ and

$$\partial_1 d = [\partial_1][d] \ mod \ 2 = [1, 0, 0, 0, 1, 0]^t = \nu_1 + \nu_5 \in C_0,$$

where the matrix product is computed *mod* 2, and

$$\partial_1' d' = [\partial_1'][d'] = [-1, 0, 0, 0, 1, 0]^t = \nu_5 - \nu_1 \in C_0'.$$

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Example A.5 (Boundary)

```
julia> B_1 = [
        1  1  0  0  0  0  0  0;
        1  0  1  1  1  0  0  0;
        0  1  1  0  0  1  0  0;
        0  0  0  1  0  0  1  0;
        0  0  0  0  1  0  1  1;
        0  0  0  0  0  1  0  1]
6×8 Array{Int64,2}:
...

julia> d = [0,1,1,0,1,0,0,0];

julia> B_1 * d
6-element Array{Int64,1}:
1 2 2 0 1 0

julia> B_1 * d .% 2
6-element Array{Int64,1}:
1 0 0 0 1 0
```

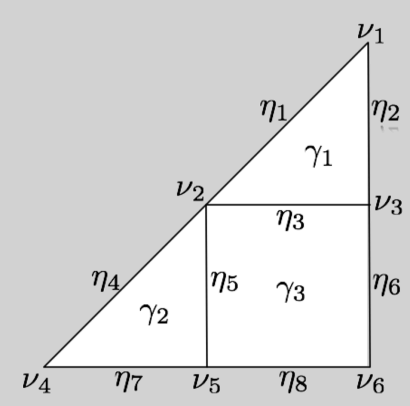

Figure 4: Cellular 2-complex

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```
julia> B_1 = [
       -1 -1  0  0  0  0  0  0;
        1  0 -1 -1 -1  0  0  0;
        0  1  1  0  0 -1  0  0;
        0  0  0  1  0  0 -1  0;
        0  0  0  0  1  0  1 -1;
        0  0  0  0  0  1  0  1]
6×8 Array{Int64,2}:
...

julia> d = [0,1,-1,0,1,0,0,0]
8-element Array{Int64,1}:
0 1 -1 0 1 0 0 0

julia> B_1 * d
6-element Array{Int64,1}:
-1 0 0 0 1 0
```

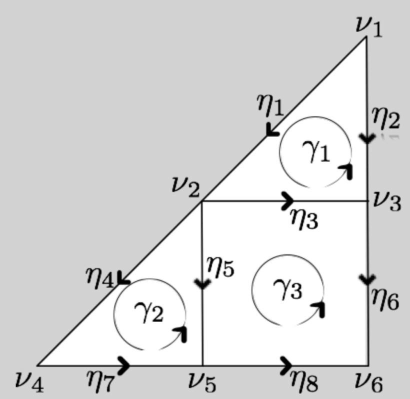

Figure 5: Cellular 2-complex

Example A.6 (Cell with a hole).

```
V = [[0.,0.],[3.,3.],[1.,2.],[2.,1.],[3.,0.],[1.,1.],[0.,3.],[2.,2.]]
FV = [[1,2,3,4,5,6,7,8],[3,4,6,8]]
EV = [[1,5],[1,7],[2,5],[2,7],[3,6],[3,8],[4,6],[4,8]]
```

```
M_1 = [
1   0   0   0   1   0   0   0;
1   0   0   0   0   0   1   0;
0   1   0   0   1   0   0   0;
0   1   0   0   0   0   1   0;
0   0   1   0   0   1   0   0;
0   0   1   0   0   0   0   1;
0   0   0   1   0   1   0   0;
0   0   0   1   0   0   0   1]

M_2 = [
1   1   1   1   1   1   1   1;
0   0   1   1   0   1   0   1]
```

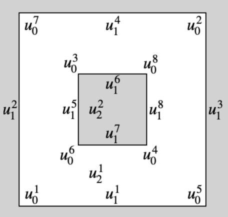

Figure 6: Cellular 2-complex

```
convert(
Array{Int64,2},(M_1 * M_2')
8×2 Array{Int64,2}:
 1  0
 1  0
 1  0
 1  0
 1  1
 1  1
 1  1
 1  1

julia> (B_2 * [1,1]) .% 2
8-element Array{Int64,1}:
1 1 1 1 0 0 0 0
```

Boundary and coboundary

Coboundary operator

The coboundary operator $\delta^p : C^p \to C^{p+1}$ acts on p-cochains as the dual of the boundary operator ∂_{p+1} on $(p+1)$-chains. For all $\phi^p \in C^p$ and $c_{p+1} \in C_{p+1}$:

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Since we use dual bases, matrices representing dual operators are the transpose of each other: for all $p = 0, \dots, d-1$:

$$[\delta^p]^t = [\partial_{p+1}]$$

Interval trees

From Wikipedia

In computer science, an interval tree is a tree data structure to hold intervals.

It allows to efficiently find all intervals that overlap with any given interval or point.

It is often used for windowing queries, for instance, to find all roads on a computerized map inside a rectangular viewport, or to find all visible elements inside a three-dimensional scene.

A similar data structure is the segment tree.

The data structure

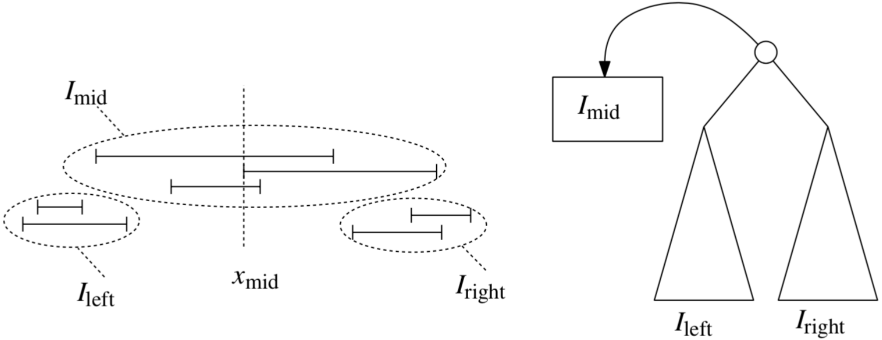
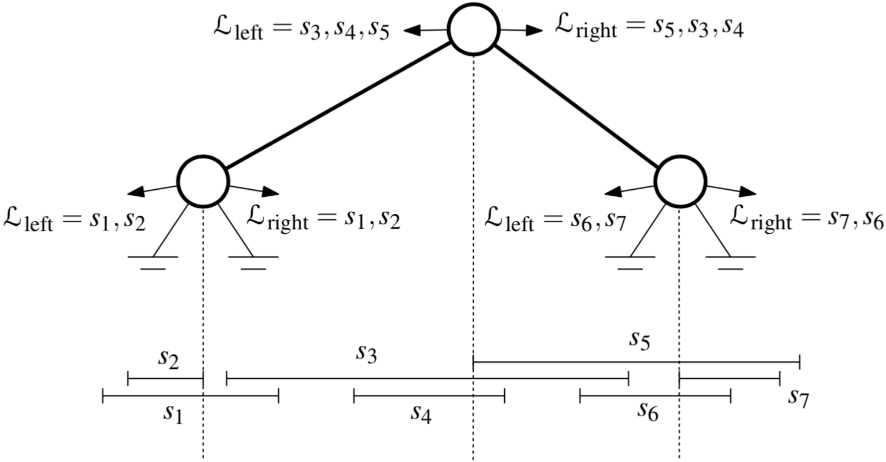


Figure 7: Classification of segments with respect to x_{mid}

Example

Dotted vertical segments indicate the values x_{mid} for each node



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  - $I_{right} = \{[x_j : x_j'] \in I : x_{mid} < x_j\}$.

Algorithm

Algorithm CONSTRUCTINTERVALTREE(I)

Input. A set I of intervals on the real line.

Output. The root of an interval tree for I.

1. **if** $I = \emptyset$
2. **then return** an empty leaf
3. **else** Create a node ν. Compute x_{mid}, the median of the set of interval endpoints, and store x_{mid} with ν.
4. Compute I_{mid} and construct two sorted lists for I_{mid}: a list $\mathcal{L}_{\text{left}}(\nu)$ sorted on left endpoint and a list $\mathcal{L}_{\text{right}}(\nu)$ sorted on right endpoint. Store these two lists at ν.
5. $lc(\nu) \leftarrow$ CONSTRUCTINTERVALTREE(I_{left})
6. $rc(\nu) \leftarrow$ CONSTRUCTINTERVALTREE(I_{right})
7. **return** ν

Figure 9: from: de Berg, Otfried Cheong, van Kreveld, Overmars: *Computational Geometry, Algorithms and Applications*, Third Edition, Springer.

Julia Package

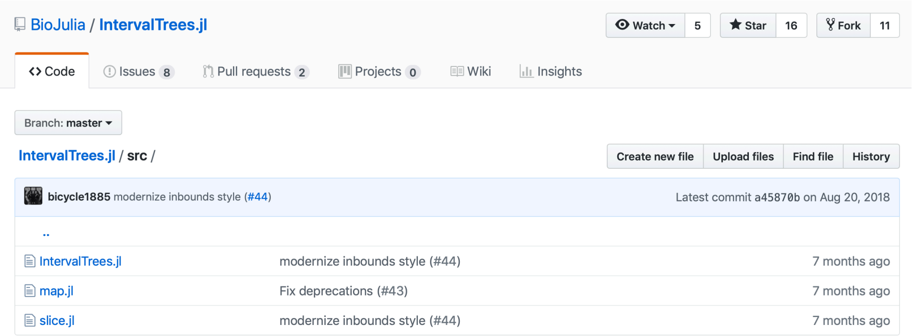

Figure 10: BioJulia/IntervalTrees.jls