5.5 Filtration

Next we vary α and consider the complete range of possible values, which is from 0 to ∞. For $\alpha < \alpha'$, we have $D_s(\alpha) \subseteq D_s(\alpha')$ and therefore $R_s(\alpha) \subseteq R_s(\alpha')$. Recall that st is an edge in the α-complex iff $R_s(\alpha) \cap R_t(\alpha) \neq \emptyset$. Since the regions grow with the radius, this implies $R_s(\alpha') \cap R_t(\alpha') \neq \emptyset$, and therefore st is also an edge in the α'-complex. Similarly, every triangle in the α-complex belongs to the α'-complex. In summary, $A(\alpha) \subseteq A(\alpha')$ whenever $\alpha \leq \alpha'$. It thus makes sense to ask—for each vertex, edge, and triangle σ in the Delaunay triangulation—what the smallest value of α is for which σ belongs to $A(\alpha)$. Denoting this value by α_σ, we can construct the α-complex simply by collecting all vertices, edges, and triangles that have a value not larger than α:

$$A(\alpha) = \{\sigma \in K \mid \alpha_\sigma \leq \alpha\}, \tag{5.2}$$

where K is the Delaunay triangulation of S. Computing this value is easiest for the vertices, since we have $\alpha_s = 0$ for every $s \in S$. It is also easy for triangles, for which α_{stu} is the radius of the circumcircle; see Fig. 5.5. The computation of the smallest α-value is slightly more difficult for edges. Here, we distinguish between two cases. First, the edge st may intersect the dual Voronoi edge in its interior; as in Fig. 5.5. In this case, st belongs to the α-complex as soon as the two disks meet, which happens when the radius reaches half the distance between the sites: $\alpha_{st} = \frac{1}{2}\|s - t\|$. If st is shared by the triangles stu and stv in K, then the condition of st intersecting the dual Voronoi edge in its interior is equivalent to having acute angles at u and v. This leads us to the second case in which one of these two angles is obtuse, say the angle at u. Then it is not enough that the two disks meet; they also need to reach the Voronoi edge, which happens when the triangle stu enters the α-complex. Hence, $\alpha_{st} = \alpha_{stu}$. Now that we have the threshold value for every vertex, edge, and triangle in the Delaunay triangulation, we can sort them such that

$$\alpha_{\sigma_1} \leq \alpha_{\sigma_2} \leq \ldots \leq \alpha_{\sigma_n}. \tag{5.3}$$

The corresponding sequence of simplices is called a *filter*. Here, we make sure that every simplex is preceded by its faces. We get this from the formulas already, since the value of every edge is smaller than or equal to the values of the triangles it belongs to. However, in case the value of the edge is equal to that of an incident triangle, we make sure we order the edge before the triangle. With this, every prefix of the filter is a complex. Writing K_j for the collection of simplices σ_i with $i \leq j$, we get an increasing sequence of complexes,

$$\emptyset = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n = K. \tag{5.4}$$

We call such an increasing sequence as a *filtration*. The not necessarily contiguous subsequence of alpha complexes is sometimes referred to as the *alpha complex*