Computational topology: Lecture 4

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March 14, 2019

Simplicial Complexes

Join operation

The join of two sets $P, Q \subset \mathbb{E}^n$ is the set of convex combination of their points:

$$PQ = \{\alpha \mathbf{x} + \beta \mathbf{y} |\ \mathbf{x} \in P, \mathbf{y} \in Q\},$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

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Implementation

How ?

Simplex

A *d*-simplex $\sigma_d \subset \mathbb{E}^n$ $(0 \leq d \leq n)$ may be defined as the repeated join of $d+1$ affinely independent points, called vertices.

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A simplicial complex, often simply denoted as complex, is a triangulation Σ that verifies the following conditions:

1. if $\sigma \in \Sigma$, then any face of σ belongs to Σ;
2. if $\sigma, \tau \in \Sigma$, then either $\sigma \cap \tau = \emptyset$, or $\sigma \cap \tau$ is a face of both σ and τ.

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With some abuse of language, we call (combinatorial) s-skeletons the sets $\mathcal{K}_s$ ($s \leq d$). Geometric carrier $|\Sigma|$, also called the support space, is the point-set union of simplices in Σ.

Implementation

How ?

```
help?> Lar.simplexFacets
```

Orientation

The ordering of the 0-skeleton of a simplex implies an orientation of it. The simplex can be oriented according to the even or odd permutation class of its 0-skeleton.

The two opposite orientations of a simplex will be denoted as $+\sigma$ and $-\sigma$.

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It is assumed that:

1. The two orientations of a simplex may represent its relative interior and exterior;
2. the two orientations of an orientable simplicial complex analogously represent the relative interior and exterior of the complex, respectively;
3. the boundary of a complex maintains the same orientation of the complex.

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$$
\begin{aligned}
+\sigma_3 &= \langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle \\
-\sigma_3 &= \langle \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3 \rangle
\end{aligned}
$$

Implementation

How ?

```
help?> Lar.surface
```

Face extraction

The oriented facets $\sigma_{d-1,(i)}$ $(0 \leq i \leq d)$ of the oriented d-simplex $\sigma_d = +\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d \rangle$ are obtained by removing the i-th vertex $\mathbf{v}_i$ from the 0-skeleton of σ_d:

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Implementation

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help?> Lar.simplexFacets
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Facet extraction

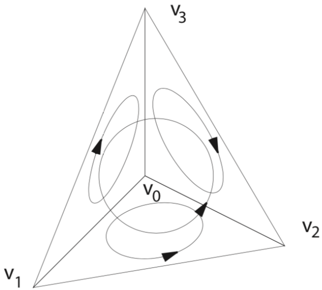

Figure 1: Coherent orientation of the facets of a 3-simplex

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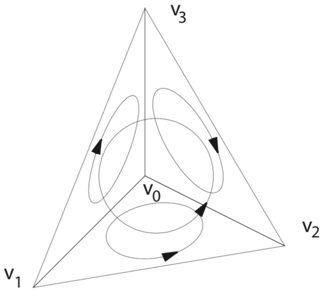

Figure 1: Coherent orientation of the facets of a 3-simplex

Implementation? See:

function Lar.simplexFacets(simplices::Cells)::Cells

Simplicial prism

The prism over a simplex $\sigma_d = \langle \mathbf{v}_0, \dots, \mathbf{v}_d \rangle$, defined as the set $P_{d+1} := \sigma_d \times [a, b]$, with $[a, b] \subset \mathbb{E}$, will be called simplicial $(d+1)$-prism.

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$$\mathcal{K}_{d+1} = \{\sigma_{d+1,(i)} = (-1)^{id} \langle \mathbf{v}_i^a, \mathbf{v}_{i+1}^a, \dots, \mathbf{v}_d^a, \mathbf{v}_0^b, \mathbf{v}_1^b, \dots, \mathbf{v}_i^b \rangle | 0 \le i \le d\}$$

where $\mathbf{v}_i^a = (\mathbf{v}_i, a)$ and $\mathbf{v}_i^b = (\mathbf{v}_i, b)$.

Exercise: extract the boundary of implicial prism

extract the boundary of the simplicial cube

- write dimension-independent code

Hint

See and modify the code of `Lar.simplexGrid`

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- write dimension-independent code
- apply to a grid of such decomposed cubes

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Cellular Complexes

Preliminary definitions

Let X be a topological space, and $\Lambda(X) = \bigcup \Lambda_p$ ($p \in \{0, 1, \ldots, d\}$) a partition of X, with Λ_p a set of {(relatively) open}, connected, and manifold p-cells.

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Let X be a topological space, and $\Lambda(X) = \bigcup \Lambda_p$ $(p \in \{0, 1, \ldots, d\})$ a partition of X, with Λ_p a set of {(relatively) open}, connected, and manifold p-cells.

A CW-structure on the space X is a filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_{d-1} \subset X = \bigcup_p X_p$, such that, for each p, the skeleton X_p is homeomorphic to a space obtained from X_{p-1} by attachment of p-cells in $\Lambda_p = \Lambda_p(X)$~[**?**].

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- The support space $|\sigma|$ of a cell σ is its compact point-set.

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Chain Complexes

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A p-chain of X with coefficients in G is a mapping $c_p : X \to G$ such that, for each $\sigma \in X_p$, reversing a cell orientation changes the sign of the chain value:

$$c_p(-\sigma) = -c_p(\sigma).$$

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Chain addition is defined by addition of chain values: if c_p^1, c_p^2 are p-chains, then $(c_p^1 + c_p^2)(\sigma) = c_p^1(\sigma) + c_p^2(\sigma)$, for each $\sigma \in X_p$.

The resulting group is denoted $C_p(X; G)$. When clear from the context, the group G is often left implied, writing $C_p(X)$.

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Let σ be an oriented cell in X and $g \in G$. The elementary chain whose value is g on σ, $-g$ on $-\sigma$ and 0 on any other cell in X is denoted $g\sigma$.

Each chain can be written in a unique way as a sum of elementary chains.

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A p-cycle is a closed p-chain, i.e., a p-chain without boundary.

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- Finally, a d-cell may be oriented as the sign of its oriented volume.

Chain spaces

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To allow not only for chain addition, but also for linear combination of chains, coefficients should be taken from a set endowed with the structure of a field, such as $(\mathbb{F}, +, \times, 0, 1)$,

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Each chain can be written in a unique way as a linear combination of unit chains $u \in U$, if the outer cell is not taken into account.

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Canonical representation and standard basis

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In particular, $\#U_d = \#\Lambda_d - 1$.

Often, with some abuse of notation, one does not distinguish between a p-cell and the corresponding unit p-chain.

Characteristic matrices

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A matrix M_p, whose rows are indexed by unit p-chains and columns are indexed by unit 0-chains, provides a useful representation of a basis for the linear space C_p.

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While chains are mostly presented as formal sums of cells, in the actual implementation their signed coordinate vectors are used as sparse arrays, and in particular as CSC (Compressed Sparse Column) maps : $\mathbb{N} \to \{-1, 0, 1\}$.

Characteristic matrices

Implementation

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How ?

help?> Lar.characteristicMatrix

WHY Characteristic matrices are useful

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There are 9 incidence relations between pairs of cells in such a V, E, F decomposition, and traditional graph-based representations are chosen by optimizing the trade-off between space requirements and efficiency of queries.

With LAR, all such queries are sparse matrix–vector multiplications without any additional space requirements.

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With LAR, all such queries are sparse matrix–vector multiplications without any additional space requirements.

The chain operators corresponding to the incidence relations are

$$
\begin{array}{lll}
\mathcal{V}\mathcal{V} : C_0 \to C_0, & \mathcal{E}\mathcal{V} : C_0 \to C_1, & \mathcal{F}\mathcal{V} : C_0 \to C_2; \\
\mathcal{V}\mathcal{E} : C_1 \to C_0, & \mathcal{E}\mathcal{E} : C_1 \to C_1, & \mathcal{F}\mathcal{E} : C_1 \to C_2; \\
\mathcal{V}\mathcal{F} : C_2 \to C_0, & \mathcal{E}\mathcal{F} : C_2 \to C_1, & \mathcal{F}\mathcal{F} : C_2 \to C_2.
\end{array}
$$

WHY Characteristic matrices are useful

Consider the incidence queries that arise in a cellular decomposition $\Lambda(X)$ of a 2D space, such as 2D triangulation or the boundary of a manifold 3-solid.

There are 9 incidence relations between pairs of cells in such a V, E, F decomposition, and traditional graph-based representations are chosen by optimizing the trade-off between space requirements and efficiency of queries.

With LAR, all such queries are sparse matrix–vector multiplications without any additional space requirements.

The chain operators corresponding to the incidence relations are

$$\begin{array}{lll}
\mathcal{V}\mathcal{V} : C_0 \to C_0, & \mathcal{E}\mathcal{V} : C_0 \to C_1, & \mathcal{F}\mathcal{V} : C_0 \to C_2; \\
\mathcal{V}\mathcal{E} : C_1 \to C_0, & \mathcal{E}\mathcal{E} : C_1 \to C_1, & \mathcal{F}\mathcal{E} : C_1 \to C_2; \\
\mathcal{V}\mathcal{F} : C_2 \to C_0, & \mathcal{E}\mathcal{F} : C_2 \to C_1, & \mathcal{F}\mathcal{F} : C_2 \to C_2.
\end{array}$$

Characteristic matrices: Examples

aaaaa

The corresponding CSR matrices are readily computed:

$$
\begin{aligned}
&\mathcal{V}\mathcal{V} = \mathcal{V}\mathcal{E} \circ \mathcal{E}\mathcal{V} = \mathcal{E}\mathcal{V}^{\top} \circ \mathcal{E}\mathcal{V} \Rightarrow [\mathcal{V}\mathcal{V}] = M_1^t M_1 \\
&\mathcal{V}\mathcal{E} = \mathcal{E}\mathcal{V}^{\top} \Rightarrow [\mathcal{V}\mathcal{E}] = M_1^t \\
&\mathcal{V}\mathcal{F} = \mathcal{F}\mathcal{V}^{\top} \Rightarrow [\mathcal{V}\mathcal{F}] = M_2^t \\
&\mathcal{E}\mathcal{V} \quad [\mathcal{E}\mathcal{V}] = M_1 \\
&\mathcal{E}\mathcal{E} = \mathcal{E}\mathcal{V} \circ \mathcal{V}\mathcal{E} = \mathcal{E}\mathcal{V} \circ \mathcal{E}\mathcal{V}^{\top} \Rightarrow [\mathcal{E}\mathcal{E}] = M_1 M_1^t \\
&\mathcal{E}\mathcal{F} = \mathcal{E}\mathcal{V} \circ \mathcal{V}\mathcal{F} = \mathcal{E}\mathcal{V} \circ \mathcal{F}\mathcal{V}^{\top} \Rightarrow [\mathcal{E}\mathcal{F}] = M_1 M_2^t \\
&\mathcal{F}\mathcal{V} \quad [\mathcal{F}\mathcal{V}] = M_2 \\
&\mathcal{F}\mathcal{E} = \mathcal{F}\mathcal{V} \circ \mathcal{V}\mathcal{E} = \mathcal{F}\mathcal{V} \circ \mathcal{E}\mathcal{V}^{\top} \Rightarrow [\mathcal{F}\mathcal{E}] = M_2 M_1^t \\
&\mathcal{F}\mathcal{F} = \mathcal{F}\mathcal{V} \circ \mathcal{V}\mathcal{F} = \mathcal{F}\mathcal{V} \circ \mathcal{F}\mathcal{V}^{\top} \Rightarrow [\mathcal{F}\mathcal{F}] = M_2 M_2^t.
\end{aligned}
$$