Polyhedral geometry 5

Computational Visual Design (CVD-Lab), DIA, "Roma Tre" University, Rome, Italy

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Homogenization

A map

$$\mathrm{homog}\ : \mathbb{E}^n \to \mathbb{E}^{n+1} : \mathbf{x} \mapsto \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}$$

is said to homogenize the points of a set $A \subset \mathbb{E}^n$.

A set of points $A = \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_d\}$ is affinely independent if and only if the vector images of points in $\mathrm{homog}\,(A)$ are linearly independent.

Affine coordinates

To compute the Affine coordinates $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_d)$ of a point $\mathbf{x} \in \mathbb{E}^d$, either internal or external to a simplex $\mathrm{conv}\,\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_d\}$, it is sufficient to solve the simultaneous set of $d+1$ linear equations in $d+1$ unknowns, whose first equation codifies the constraint that λ must be a partition of the unity:

$$\begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{x}_0 & \mathbf{x}_1 & \cdots & \mathbf{x}_d \end{pmatrix} \lambda,$$

so that

$$\lambda = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathbf{x}_0 & \mathbf{x}_1 & \cdots & \mathbf{x}_d \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}.$$

Implementation

TODO

Section 1

Polyhedral sets

Polyhedra and cones are collectively called polyhedral sets.

They are often referred to as $\mathcal{H}$-polyhedra, which means that they can be presented as the intersection of closed halfspaces.

Polyhedron

A set $P \in \mathbb{E}^n$ is a polyhedron if and only if it is the intersection of a finite number of closed halfspaces.

So, we define:

$$\mathcal{H}(P) := P(\mathbf{A}, \mathbf{b}) := \{\mathbf{x} \in \mathbb{E}^n | \mathbf{A}\mathbf{x} \leq \mathbf{b}\},$$

for some $\mathbf{A} \in \mathbb{R}^m_n$ and some $\mathbf{b} \in \mathbb{R}^m$.

Polyhedral cone

A set $C \in \mathbb{E}^n$ is said a polyhedral cone if and only if

$$\mathcal{H}(C) := C(\mathbf{A}, \mathbf{0}) := \{\mathbf{x} \in \mathbb{E}^n | \mathbf{A}\mathbf{x} \leq \mathbf{0}\},$$

for some $\mathbf{A} \in \mathbb{R}^m_n$.

Such a kind of set is both a cone and a polyhedron.

Section 2

Boundary structure

Let us suppose in this section that $C \subset \mathbb{E}^n$ is a non-empty closed convex set.

Relative interior

The relative interior of a set is a refinement of the concept of the interior, useful when dealing with low-dimensional sets placed in higher-dimensional spaces.

The relative interior of a set contains all points which are not on the "edge" of the set, relative to the smallest subspace in which this set lies.

Formally, the relative interior of a set C (denoted $\operatorname{relint} C$) is defined as its interior within the affine hull of C:

$$\operatorname{relint} C := \{x \in C : \exists \epsilon > 0 : (N_\epsilon(x) \cap \operatorname{aff} C) \subseteq C\}$$

where $N_\epsilon(x)$ is a ball of radius ϵ centered on x.

Faces

- A face of C can be defined as a convex subset $F \subseteq C$ such that $\mathbf{x}, \mathbf{y} \in C$ and $(\mathbf{x} + \mathbf{y})/2 \in F$ implies $\mathbf{x}, \mathbf{y} \in F$.
- Notice that both C and $\emptyset$ are faces of C.
- A non-empty face F is said to be proper when F is a proper subset of C.
- The set of faces of C will be denoted as $\mathcal{F}(C)$.
- A face of dimension d will be called a d-face, and $\mathcal{F}_d(C) \subset \mathcal{F}(C)$ will denote the set of d-faces.

Extremal points

- An extremal point of a convex set C is a point $z \in C$ that cannot be written in the form

$$z = (1 - \lambda)x + \lambda y,$$

for $x, y \in C$ and $\lambda \in (0, 1)$.

- If $\{z\} \subset A$ is a face, then z is an extremal point.
- It is very important to write a closed convex set as the convex hull of a much smaller finite set.
- The Minkowski's theorem states that each convex body is the convex hull of its extremal points

Face partition

The relative interiors of any two faces in $\mathcal{F}(C)$ are disjointed.

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Thus, the family

$$\{\mathrm{relint}\, F | F \in \mathcal{F}(C)\} \cup \mathcal{F}_0(C)$$

of relative interiors of faces of C, united with the set of 0-dimensional faces $\mathcal{F}_0(C)$, i.e.$\sim$of faces $\{\mathbf{x}\}$, where $\mathbf{x}$ is an extreme point, gives a partition of C into disjoint subsets.

Section 3

Polytopes

Polytopes

We already know that a polyhedron is the solution set of a system of linear inequalities $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, and that a polytope is a bounded polyhedron.

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We already know that a polyhedron is the solution set of a system of linear inequalities $\mathbf{Ax} \leq \mathbf{b}$, and that a polytope is a bounded polyhedron.

In particular we know that a polyhedron is a polytope when it does not contain a cone, i.e.$\sim$when the associated homogeneous system $\mathbf{Ax} \leq \mathbf{0}$ has no solutions different from the zero vector.

Polytope properties

1. Each polytope is the intersection of finitely many closed halfspaces.
2. Each bounded intersection of finitely many closed halfspaces is a polytope.
3. The join, intersection, sum, product, projection of polytopes is a polytope.
4. The projection of a simplex is a polytope.
5. The polar body of a polytope is a polytope.
6. Each face of a polytope is a polytope.
7. Each proper face of a polytope P is contained in some facet of P.

Simplicial and simple polytopes

- A polytope $P \subset \mathbb{E}^d$ is said to be simplicial when all its facets are simplices.
- It is said to be simple when each vertex is generated as intersection of the minimal number d of facets.
- Examples of simplicial polytopes are the octahedron and the icosahedron, where each face is a triangle.
- Examples of simple polytopes are the cube and the dodecahedron, where each vertex is the intersection of three facets.
- Simplices are the only polytopes which are both simple and simplicial.

Platonic solids

only 5 regular polytopes

the faces of a Platonic solid are congruent regular polygons, with the same number of faces meeting at each vertex; all edges and angles are congruent

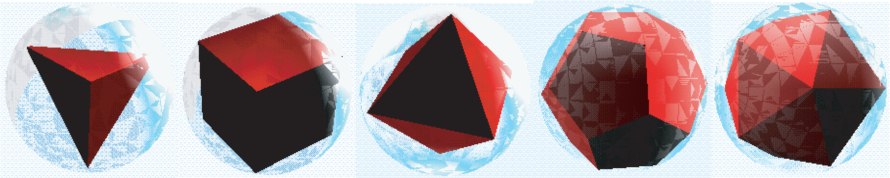

Figure: Tetrahedron, hexahedron, octahedron, dodecahedron and icosahedron inscribed in a unit sphere

Platonic solids (Exercise)

Construct in Javascript one of the Platonic solids.

Make it as inscribed in the unit sphere centered at the origin

Section 4

Simplicial complexes

Join operation

- The join of two sets $P, Q \subset \mathbb{E}^n$ is the set

$$PQ = \{\alpha \mathbf{x} + \beta \mathbf{y} | \; \mathbf{x} \in P, \mathbf{y} \in Q\},$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

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The join operation is associative and commutative.

Simplex

- A d-simplex $\sigma_d \subset \mathbb{E}^n$ $(0 \leq d \leq n)$ may be defined as the repeated join of $d+1$ affinely independent points, called vertices.
- A d-simplex can be seen as a d-dimensional triangle: a 0-simplex is a point, a 1-simplex is a segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and so on.
- The set $\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d\}$ of vertices of σ_d is called the 0-skeleton of σ_d.

Faces of a simplex

The s-simplex generated from any subset of $s + 1$ vertices $(0 \leq s \leq d)$ of σ_d is called an s-face of σ_d.

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Let us notice, from the definition, that a simplex may be considered both as a purely combinatorial object and as a geometric object, i.e. as the compact point set defined by the convex hull of a discrete set of points.

Simplicial complex

Definition

A set Σ of simplices is called a triangulation.

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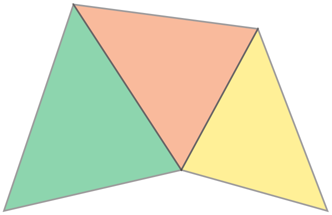
A simplicial complex, often simply denoted as complex, is a triangulation Σ that verifies the following conditions:

1. if $\sigma \in \Sigma$, then any face of σ belongs to Σ;
2. if $\sigma, \tau \in \Sigma$, then either $\sigma \cap \tau = \emptyset$, or $\sigma \cap \tau$ is a face of both σ and τ.

Simplicial complex

A simplicial complex can be considered a "well-formed" triangulation.



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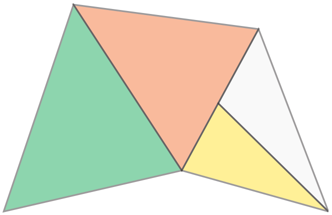

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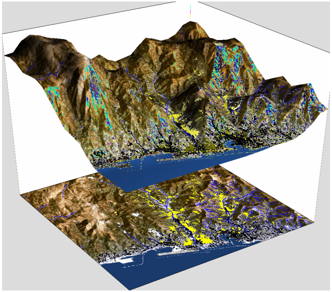
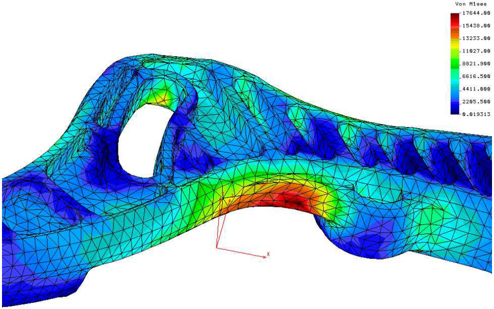
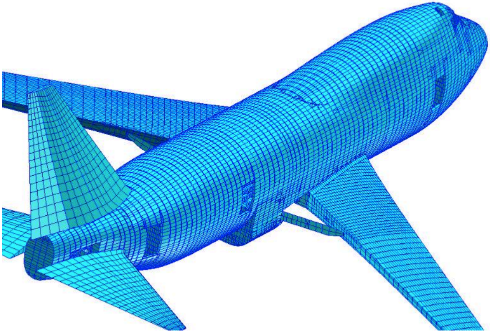
Simplicial complex

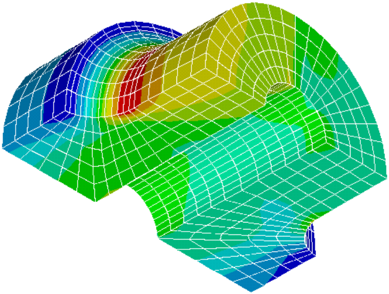
Such kind of triangulations are widely used in engineering analysis, e.g., in topography or in finite element methods.





Cuboidal complex





Simplicial complex

The order of a complex is the maximum order of its simplices.

A complex Σ_d of order d is also called a d-complex.

A d-complex is said to be regular or pure if each simplex is a face of a d-simplex.

A regular d-complex is homogeneously d-dimensional.

EXERCISE: Building model

Figure: 126 Anniversary of Mies van der Rohe birthday