Computational Graphics: Lecture 4

Alberto Paoluzzi

Mon, Feb 7, 2016

Outline: larlib

1. Simplicial complexes
2. Introduction to `larlib`

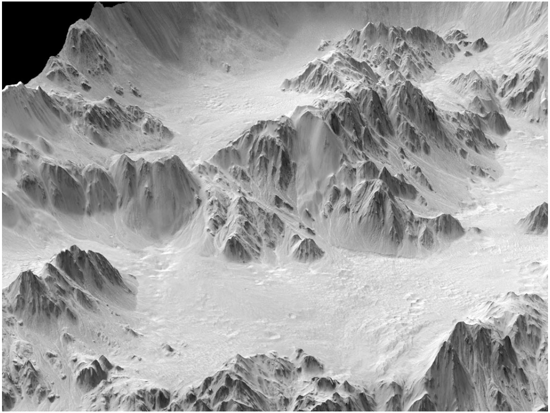
Simplices and simplicial complexes

- A. Paoluzzi, F. Bernardini, C. Cattani and V. Ferrucci: Dimension-Independent Modeling with Simplicial Complexes. ACM Transactions on Graphics. 12(1): 56-102 (1993)

Simplicial complexes

Examples

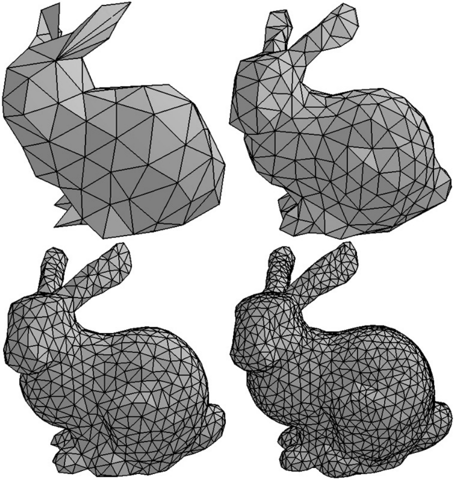
Digital Terrain model



Enhanced 3D Model of Mars Crater Edge Shows Ups and Downs

Examples

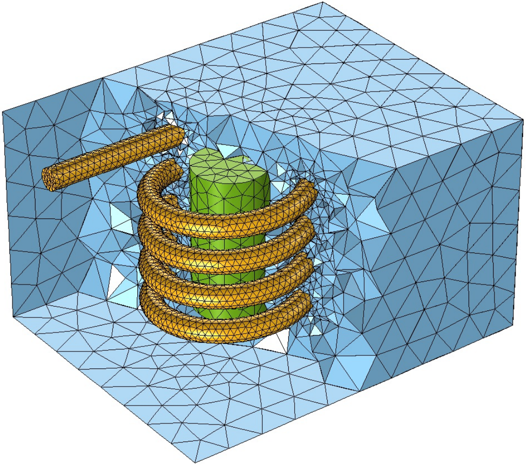
Typical graphical model



Stanford bunny, by Marc Levoy

Examples

Typical engineering model mesh



COMSOL Multiphysics Software Product Suite

First definitions

The join of two sets P, $Q \subset \Re^n$ is the set $PQ = \{\gamma p + \lambda q,\ \ p \in P, q \in Q\}$, where γ, $\lambda \in \Re$, γ, $\lambda \geq 0$, and $\gamma + \lambda = 1$. The join operation is associative and commutative. A simplex $\sigma \subset \Re^n$ of order d, or d-simplex, is the join of $d + 1$ affinely independent points, called vertices. The $n + 1$ points $p_0, \ldots, p_n$ are affinely independent when the n vectors $\boldsymbol{p}_1 - \boldsymbol{p}_0, \ldots, \boldsymbol{p}_n - \boldsymbol{p}_0$ are linearly independent. A d-simplex can be seen as a d-dimensional triangle: 0-simplex is a point, 1-simplex is a segment, 2-simplex is a triangle, 3-simplex is a tetrahedron, and so on. Any subset of $s + 1$ vertices $(0 \leq s \leq d)$ of a d-simplex σ defines an s-simplex, which is called s-face of σ.

Simplicial complex

A simplicial complex is a set of simplices Σ, verifying the following conditions: (a) if $\sigma \in \Sigma$, then any s-face of σ belongs to Σ; (b) if σ, $\tau \in \Sigma$, then either $\sigma \cap \tau = \emptyset$, or $\sigma \cap \tau$ is an s-face of σ and τ. Geometric carrier $[\Sigma]$ is the pointset union of simplices in Σ.

Order and skeleton

The order of a complex is the maximum order of its simplices. A complex Σ^d of order d is also called a d-complex. A d-complex is regular if each simplex is an s-face of a d-simplex. Two simplices σ_1 and σ_2 in a complex Σ are s-adjacent if they have a common s-face; they are s-connected if a sequence of simplices in Σ exists, beginning with σ_1 and ending with σ_2, such that any two consecutive terms of the sequence are s-adjacent. In the following, face and adjacency (without prefix) of a d-simplex stand for $(d-1)$-face and $(d-1)$-adjacency. $K^s(\Sigma^d)$ $(0 \leq s \leq d)$ denotes the set of s-simplices belonging to Σ^d, and $|K^s|$ denotes their number. With some abuse of language, we call K^s the s-skeleton. The set of vertices of Σ^d is therefore $K^0(\Sigma^d)$, and the set of d-simplices is $K^d(\Sigma^d)$.

Polyhedra

The set of all linear d-polyhedra embedded in $\Re^n$ will be denoted as $\mathscr{P}^{d,n}$. A polyhedron $P \in \mathscr{P}^{d,n}$ coincides with the geometric carrier of a simplicial d-complex, and we write $P = [\Sigma^d]$. As an extreme example, $o \in \mathscr{P}^{0,0}$ is the 0-polyhedron consisting of a single point—the set $\mathscr{P}^{0,0}$ is a singleton and contains only o. A polyhedron is regular if any associated complex is regular.

Boundary of a polyhedron

The boundary ∂P of a regular polyhedron $P = [\Sigma^d]$ is the geometric carrier of a $(d-1)$-complex whose $(d-1)$-simplices are faces of exactly one d-simplex in Σ^d. Notice that if P is regular then $\partial\partial P = \varnothing$. The set of vertices of a polyhedron $P = [\Sigma^d]$ is defined to be $K^0(\Sigma^d)$ and is concisely indicated by

$K^0(P)$. Notice that such a set of vertices may be redundant (e.g., lying in the interior), according to the use of "nodes" in FEM decompositions. For sake of brevity, we will occasionally soften the distinction between the polyhedron $P \in \mathscr{P}^{d,n}$ and an associated complex: the meaning of $K^s(P)$ is "the s-skeleton of Σ^d such that $[\Sigma^d] = P$".

Coherent orientation of faces

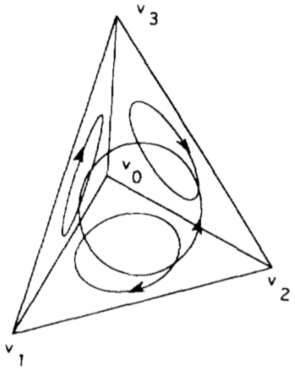


Fig. 1. Coherent orientation of the 2-faces of a 3-simplex.

Extraction of facets

The choice of an ordering for the vertices of a simplex implies its orientation, according to the even or odd permutation of the ordering. The two opposite orientations will be denoted as $+\sigma$ and $-\sigma$. A complex is orientable when all its simplices can be coherently oriented. The oriented $(d-1)$-faces of the d-simplex $\sigma_i = \langle v_{i,0}, \dots, v_{i,d} \rangle$ are given by the formula:

$$\sigma_{i,j} = (-1)^j \langle v_{i,0}, \dots, v_{i,j-1}, v_{i,j+1}, \dots, v_{i,d} \rangle, \quad 0 \le j \le d, \qquad (1)$$

where $\sigma_{i,j}$ and $v_{i,j}$ denote the jth face and the jth vertex of σ_i, respectively. A similar notation for the oriented $(d-1)$-faces of a d-simplex is attributed by Dieudonné [20] to Eilenberg and Mac Lane. Two adjacent simplices are coherently oriented when their common face has opposite orientations (see Figure 1).

Simplicial meshes

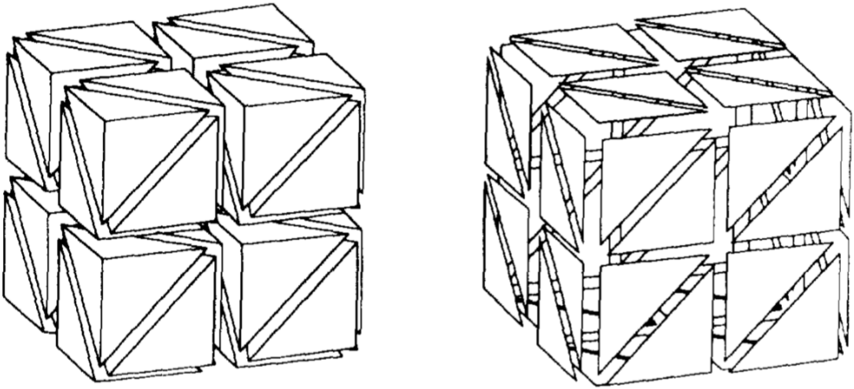
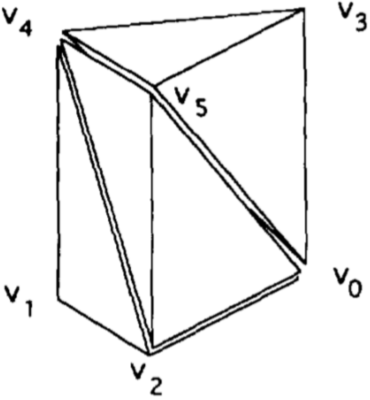
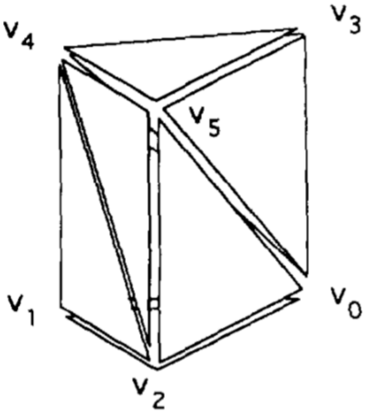


Fig. 3. Decompositive versus boundary representation.

Extrusion of a triangle





Winged representation

$$\mathscr{W}(P) = \begin{cases} \sigma_0 = +\langle v_1, v_2, v_0, v_4 \rangle & \mathscr{A}(\sigma_0) = \langle \sigma_1, \perp, \perp, \perp \rangle \\ \sigma_1 = +\langle v_2, v_0, v_4, v_5 \rangle & \mathscr{A}(\sigma_1) = \langle \sigma_2, \perp, \perp, \sigma_0 \rangle \\ \sigma_2 = +\langle v_0, v_4, v_5, v_3 \rangle & \mathscr{A}(\sigma_2) = \langle \perp, \perp, \perp, \sigma_1 \rangle \end{cases}$$

Winged representation of the boundary

$$\mathscr{W}_\partial(P) = \begin{cases} \sigma_{0,1} = -\langle v_1, v_0, v_4 \rangle & \mathscr{A}(\sigma_{0,1}) = \langle \sigma_{2,2}, \sigma_{0,2}, \sigma_{0,3} \rangle \\ \sigma_{0,2} = +\langle v_1, v_2, v_4 \rangle & \mathscr{A}(\sigma_{0,2}) = \langle \sigma_{1,1}, \sigma_{0,1}, \sigma_{0,3} \rangle \\ \sigma_{0,3} = -\langle v_1, v_2, v_0 \rangle & \mathscr{A}(\sigma_{0,3}) = \langle \sigma_{1,2}, \sigma_{0,1}, \sigma_{0,2} \rangle \\ \sigma_{1,1} = -\langle v_2, v_4, v_5 \rangle & \mathscr{A}(\sigma_{1,1}) = \langle \sigma_{2,0}, \sigma_{1,2}, \sigma_{0,2} \rangle \\ \sigma_{1,2} = +\langle v_2, v_0, v_5 \rangle & \mathscr{A}(\sigma_{1,2}) = \langle \sigma_{2,1}, \sigma_{1,1}, \sigma_{0,3} \rangle \\ \sigma_{2,0} = +\langle v_4, v_5, v_3 \rangle & \mathscr{A}(\sigma_{2,0}) = \langle \sigma_{2,1}, \sigma_{2,2}, \sigma_{1,1} \rangle \\ \sigma_{2,1} = -\langle v_0, v_5, v_3 \rangle & \mathscr{A}(\sigma_{2,1}) = \langle \sigma_{2,0}, \sigma_{2,2}, \sigma_{1,2} \rangle \\ \sigma_{2,2} = +\langle v_0, v_4, v_3 \rangle & \mathscr{A}(\sigma_{2,2}) = \langle \sigma_{2,0}, \sigma_{2,1}, \sigma_{0,1} \rangle \end{cases}$$

Sweeping and extrusion

Gaspar Monge used the sweeping operation in the 18th Century, as a method for generating curves and surfaces by moving a point or a curve, respectively. When sweeping is applied to space curves or surfaces, it produces space surfaces or solids, respectively. Sweeping, extrude, and revolve operations are largely used in CAD systems (see, e.g., Pegna [50] and Weld [64]). In this section we analyze the translational extrusion of a polyhedron, which can be considered a basic operation to generate higher-dimensional polyhedra.

Sweeping vs extrusion

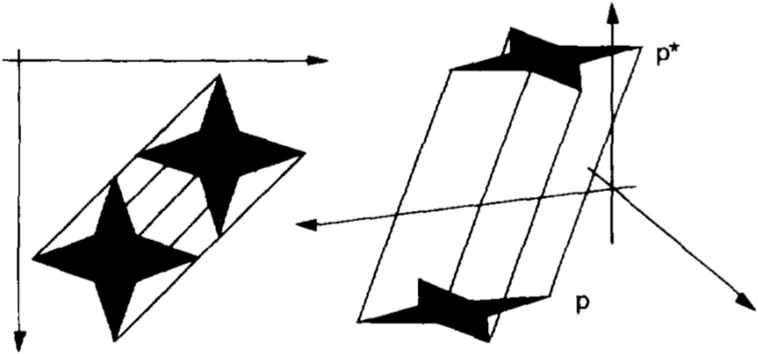


Fig. 5. Sweeping versus extrusion.

Extrusion of simplices

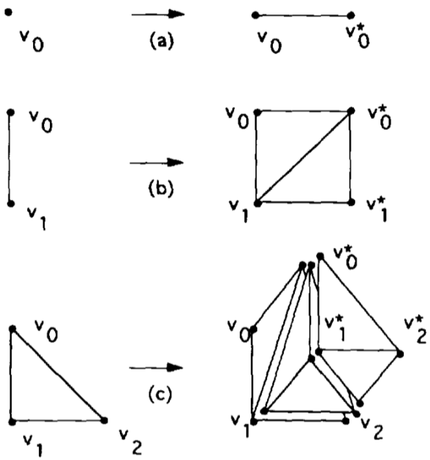


Fig. 6. Extrusion of (a) a point; (b) a straight line segment; (c) a triangle.

Extrusion of a simplex

$$K^{d+1}(\sigma \times I) = \left\{\tau_i : \tau_i = (-1)^{id} \langle v_i, \ldots, v_d, v_0^*, \ldots, v_i^* \rangle, 0 \leq i \leq d\right\}$$

Introduction to `larlib`

LAR Basics

- A. DiCarlo, V. Shapiro, and A. Paoluzzi, Linear Algebraic Representation for Topological Structures, Computer-Aided Design, Volume 46, Issue 1 , January 2014, Pages 269-274

References

Python Scientific Lecture Notes