Probability Distributions

Wednesday, 2 December 2020 14:17

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Functions:

- Gamma function
  - factorials of complex numbers (real part > 0)
  - $\Gamma(n) = (n-1)!$
- Digamma function
  - $\Psi(x) = \frac{d \log \Gamma(x)}{dx}$
  - $\Psi(x+1) = \Psi(x) + \frac{1}{x}$
- Beta function
  - $\mathrm{B(a, b)} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(\mathrm{a+b})}$
- Multivariate Beta function
  - $\mathrm{B}(\boldsymbol{\alpha}) = \frac{\prod_{i=1}^{k} \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^{k} \alpha_i)}$

Permutations and Combinations

Number of permutations (order matters) of *n* things taken *r* at a time:

$$P(n,r) = \frac{n!}{(n-r)!}$$

Number of combinations (order does not matter) of *n* things taken *r* at a time:

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

Number of different permutations of *n* objects where there are n_1 repeated items, n_2 repeated items, ... n_k repeated items

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

Distributions:

- Bernoulli distribution
  - throw a coin
  - $\mathrm{P(X{=}1)} = p$
  - $\mathrm{P(X{=}0)} = 1 - p$
  - $\mathrm{E[X]} = p$
  - $\mathrm{Var[X]} = p(1-p)$
- Binomial distribution B(n, p)
  - n Bernoulli trials, got k successes
  - n=1: Bernoulli
  - PMF: $\mathrm{P(X{=}k)} = C(n,k)\, p^k\, (1-p)^{n-k}$
  - $\mathrm{E[X]} = np$
  - $\mathrm{Var[X]} = np(1-p)$
- Geometric distribution
  - k Bernoulli trials until 1 success (including)
  - PMF: $\mathrm{P(X{=}k)} = (1-p)^{k-1}p$
  - $\mathrm{E[X]} = \frac{1}{p}$
  - $\mathrm{Var[X]} = \frac{1-p}{p^2}$
- Categorical distribution
  - roll a k-sided die
  - PMF:

    $f(x = i \mid p_1, \dots, p_k) = p_i \qquad \sum_{i=1}^{k} p_i = 1$

    or

    $f(\mathbf{x} \mid p_1, \dots, p_k) = \prod_{i=1}^{k} p_i^{\mathbf{x}_i}$

where $\mathbf{x}$ is a k-dimensional one-hot encoding vector

- Multinomial distribution
  - n trials of rolling a k-sided die
  - k=2, n=1: Bernoulli
  - k=2, n>1: Binomial
  - PMF:

    x is a k-dimensional vector with counts for each possible outcome

    $$f(x_1, \dots, x_k \mid n, p_1, \dots, p_k) = P(X_1 = x_1, \dots, X_k = x_k)$$
    $$= \frac{n!}{x_1! \dots x_k!} p_1^{x_1} \dots p_k^{x_k} \qquad \sum_{i=1}^{\mathrm{k}} x_i = n$$

    or

    $$f(x_1, \dots, x_k \mid p_1, \dots, p_k) = \frac{\Gamma(\sum_i x_i + 1)}{\prod_i \Gamma(x_i + 1)} \prod_{i=1}^{k} p_i^{\mathbf{x}_i}$$
  - $\mathrm{E}[X_i] = np_i$
  - $\mathrm{Var}[X_i] = np_i(1 - p_i)$

- Beta distribution
  - defined on interval [0, 1]
  - $\mathrm{E}[\mathrm{X}] = \frac{a}{a+b}$
  - $\mathrm{Var}[\mathrm{X}] = \frac{ab}{(a+b)^2(a+b+1)}$
- Poisson distribution $\quad$ Poisson(λ)
  - $\mathrm{B(n, p)}$ when $\mathrm{np} = \lambda$, $\mathrm{n} \to \infty$
- Gamma distribution
  - continuous version of Poisson distribution on positive real numbers
  - a: shape parameter; b: rate parameter ($\frac{1}{\mathrm{b}}$: scale parameter)
  - PDF: $f(x \mid a, b) = \frac{b^a x^{a-1} e^{-bx}}{\Gamma(\mathrm{a})} \qquad x, a, b > 0$
  - $\mathrm{E}[\mathrm{X}] = \frac{\mathrm{a}}{\mathrm{b}}$
  - $\mathrm{Var}[\mathrm{X}] = \frac{\mathrm{a}}{\mathrm{b}^2}$
- Dirichlet distribution $\quad$ $\mathrm{Dir}(\alpha_1, \dots, \alpha_k)$
  - Beta distribution in high dimensionalities
  - PDF:

    $$f(x_1, \dots, x_k \mid \alpha_1, \dots, \alpha_k) = \frac{1}{\mathrm{B}(\boldsymbol{\alpha})} \prod_{i=1}^{k} x_i^{\alpha_i - 1} \qquad \sum_{i=1}^{\mathrm{k}} x_i = 1 \quad x_i \geq 0$$

    $$\mathrm{B}(\boldsymbol{\alpha}) = \frac{\prod_{i=1}^{\mathrm{k}} \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^{k} \alpha_i)}$$
  - $\mathrm{E}[X_i] = \frac{\alpha_i}{\sum_{i=1}^{k} \alpha_i}$

More about conjugate distributions:

Recall the Bayes' Rule:

1. Choose prior that is conjugate to likelihood

$$p(\theta|D) = \frac{p(D|\theta)p(\theta)}{\int p(D|\theta)p(\theta)d\theta}$$

2. The posterior will have same form as conjugate prior distribution, i.e. closed-form.

(E.g., Beta-Binomial distribution & Dirichlet-Multinomial distribution)