Homework 5 Math/ISyE 6767 Fall 2013

The assignment is based on the option spread, described as the "redemption amount" in the document at

http://secfilings.nyse.com/filing.php?doc=1&attach=ON&ipage=9203111&rid=23

There are 2 underlyings: the S and P 500 index and the Russell 2000 index. We will denote the value of the S and P 500 by S and the value of the Russell 2000 by R. Each index is equally weighted. So we will evaluate the payoff at the values $\frac{R(T)+S(T)}{2}$ over many simulations and take the discounted average.

We will assume that the volatility of each is stochastic and that its square (the "variance") is given by Heston's mean reverting Stochastic Differential Equation which, in the case of a single underlying, has the form:

$$dv_t = -\lambda(v_t - \theta)dt + \eta\sqrt{v_t}dW_t$$

The parameter θ is the mean variance; this is the number that the variance is drawn to.

λ is the rate of reversion to the mean.

η is the volatility of the mean reversion process.

We will have such an SDE associated with each of the underlyings. The underlyings will satisfy:

$$dX = \mu X dt + \sqrt{v_t} X dW_t$$

where the drift is taken to be the risk-free rate minus the dividend yield.

Moreover, we will eventually assume that the random drivers of all our stochastic processes are correlated.

In summary we assume that that the dynamics of S and R are given by:

$$dS_t = \mu_S S_t dt + \sqrt{v_t^{(S)}} S_t dW_t^{(1)} \tag{1}$$

$$dR_t = \mu_R R_t dt + \sqrt{v_t^{(R)}} R_t dW_t^{(2)} \tag{2}$$

where the volatilities $\sqrt{v_t^{(S)}}$ and $\sqrt{v_t^{(R)}}$ satisfy:

$$dv_t^{(S)} = -\lambda(v_t^{(S)} - \theta_S)dt + \eta\sqrt{v_t^{(S)}}dW_t^{(3)} \tag{3}$$

$$dv_t^{(R)} = -\lambda(v_t^{(R)} - \theta_R)dt + \eta\sqrt{v_t^{(R)}}dW_t^{(4)} \tag{4}$$

and the drift parameters have the form (risk-free rate) - (dividend yield).

Parameter Values. Use these parameter values. Many are related to the market place.

risk free rate (r): 1%
Time to Expiration: 1 year
dividend yield of S and P 500**:** 1.93%
dividend yield of Russell 2000**:** 1.56%
λ**:** 10

η**:** 1
θ_S**:** 0.034
θ_R**:** 0.0529
S_0**:** 16000
R_0**:** 1120
$v_0^{(S)}$**:** 0.0121
$v_0^{(R)}$**:** 0.0256
Correlations among the $dW^{(i)}$ terms: See the matrix below.

About the Brownian Increments. We should do two types of pricing.

- We can assume that the Brownian increments are independent.
- We can assume that the $dW^{(i)}$ for $i = 1, 2, 3, 4$ are not independent and that their correlation matrix is $\begin{bmatrix} 1 & 0.7 & -0.7 & -0.8 \\ 0.7 & 1 & -0.6 & -0.7 \\ -0.7 & -0.6 & 1 & 0.8 \\ -0.8 & -0.7 & 0.8 & 1 \end{bmatrix}$

Numerical Simulation. You have to time step. The Euler-Maruyama discretization of the Heston SDE is as follows, given v_0, compute v_i from v_{i-1} by:

$$v_i = v_{i-1} - \lambda(v_{i-1} - \theta)\Delta t + \eta\sqrt{v_{i-1}}\sqrt{\Delta t}Z \tag{5}$$

where Z is a Gaussian sample. The Euler-Maruyama method is ok, but it does have the possibility of giving negative volatilties. So you have to check for this. If it turns out that $v_i < 0$, replace it by $-v_i$. (We have supressed the superscript on v here.) That is, change the sign of the volatility if it becomes negative. An alternative is to use Milstein's method for time stepping. We will give the formula later.

We would then use v_i to step from S_i to S_{i+1}:

$$S_{i+1} = S_i + \mu S_i\Delta t + \sqrt{v_i}S_i Z' \tag{6}$$

where Z' is a sample from a Gaussian whose correlation with the sample used in (5) is correct.

What to do. Price the option with both independent random processes and correlated processes. Use the parameter information given above. Discuss your results from points of view of math, finance, and programming.

You will need to implement the Cholesky factorization to do the correlated case. The TNT library should work fine for this.

APPENDIX: Milstein's Method. To get v_{i+1} from v_i, first get a Gaussian Z and then calculate:

$$v_{i+1} = v_i - \lambda(v_i - \theta)\Delta t + \eta\sqrt{v_i}\sqrt{\Delta t}Z + \frac{\eta^2}{4}\Delta t(Z^2 - 1) \tag{7}$$

$$= (\sqrt{v_i} + \frac{\eta}{2}\sqrt{\Delta t}Z)^2 - \lambda(v_i - \theta)\Delta t - \frac{\eta^2}{4}\Delta t \tag{8}$$