1. If A is upper triangular,

let $Z = A^{-1}$, then

$I = AZ$

$$d[j][c] = \sum_{i=1}^{m} \underbrace{a[j][i]}_{\substack{0 \text{ for } i<j \\ \neq 0 \text{ for } i \geq j}} z[i][j] = \begin{cases} 1 & c=j \\ 0 & c \neq j \end{cases}$$

so for $j=0$

$$d[0][c] \quad [1\ 0\ 0\ 0\ \ldots] = \underbrace{[a_1\ a_2 \ldots a_m]}_{\text{all nonzero}}^{A \text{ row } 0} \begin{array}{c} Z \text{ col } 0 \\ \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix} \end{array}$$

so $a_1 z_1 = 1 \quad \Leftarrow \quad z_1 \neq 0$

$\left.\begin{array}{l} a_2 z_2 = 0 \\ \vdots \\ a_m z_m = 0 \end{array}\right\}$ for these to be 0, $z_2 \ldots z_m = 0$

more generally, since $a[j][i] = \begin{cases} 0 & i<j \\ \neq 0 & i \geq j \end{cases}$ $\quad Z[i][j] = \begin{cases} \text{nonzero} & i \leq j \\ 0 & i > j \end{cases}$

so the columns of Z are nonzero for $i \leq j$

thus, if A is upper triangular $\rightarrow Z = A^{-1}$ is upper triangular.

Pt 1. ∎

Since A is unitary, $A^* = A^{-1}$, so A^* is upper triangular by Pt 1.

A and A^* are both upper triangular iff A is diagonal

// If A is lower triangular, then for Pt 1 above,

$$d[0][c] = [1\ 0 \ldots 0] = [\underset{\text{nonzero}}{a_1}\ \underbrace{a_2\ a_3 \ldots a_m}_{0}] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}$$

all nonzero,

$a[j][i] = \begin{cases} \neq 0 & i \leq j \\ 0 & i > j \end{cases}$

$Z[i][j] = \begin{cases} 0 & i<j \\ \neq 0 & i \geq j \end{cases}$

So if A lower tri., $Z = A^{-1}$ is lower tri., and if A is unitary,

A^* is also lower triangular, so A^* and A are lower triangular iff A is diagonal

2. a. If λ is eigenvalue of A,

by def $Av = \lambda v$ $\longrightarrow$ $A^{-1}Av = A^{-1}\lambda v$

$= \lambda A^{-1} v$

if A is invertible, $A^{-1}Av = v$

$\lambda A^{-1} v = v$

iff $A^{-1} v = \frac{1}{\lambda} v$

aka, $\frac{1}{\lambda}$ is eigenvalue of A^{-1}

b. If λ is eigenvalue of AB,

so by def $\lambda v = ABv$

let $y = Bv$, then $BAy = BABv$

$= B\lambda v$

$= \lambda Bv$

$BAy = \lambda y$

So λ is eigenvalue of BA

2c. Eigenvalues are solutions to

$$\det(A-\lambda I)=0$$

$$(A-\lambda I)^t = A^t - \lambda I^t = A^t - \lambda I$$

and since $\det(M) = \det(M^t)$

$$\det(A-\lambda I) = \det(A^t - \lambda I)$$

$$\downarrow \qquad\qquad \downarrow$$

solutions to this $=0$ = solutions to this $=0$

↳ $\underline{A, A^t \text{ have same eigenvalues}}$

3. a.

$$Ax = \lambda x$$

$$\bar{x}^T(Ax) = \bar{x}^T \lambda x$$

$$= \lambda \bar{x}^T x$$

$$\bar{x}^T(Ax) = \underline{\lambda} \|x\|$$

take complex conjugate both sides

$$\underbrace{\bar{x}^T \bar{A}^T x}_{\downarrow} = \underline{\bar{\lambda} \|x\|}$$

$\bar{A}^T = A$, so $\bar{x}^T \bar{A}^T x = \bar{x}^T A x$

$$= \bar{x}^T \lambda x$$

$$= \underline{\lambda \|x\|}$$

$$\lambda \|x\| = \bar{\lambda} \|x\|$$

$$\lambda = \bar{\lambda}$$

so $\underline{\lambda \text{ is real}}$

b. $\lambda_1 x = Ax$ $\quad \lambda_1 \neq \lambda_2$

$\lambda_2 y = Ay$

$$Ax \cdot y = x \cdot A^T y$$
$$= x \cdot Ay$$
$$\lambda_1 x \cdot y = x \cdot \lambda_2 y$$
$$\lambda_1 \, x \cdot y = \lambda_2 x \cdot y$$
$$\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0}(x \cdot y) = 0$$

$\neq 0$ so $\underline{(x \cdot y) = 0}$ $\Rightarrow$ they are orthogonal.

4. If A had $\lambda = 0$, then there will be $Ax = 0x = 0$,

$x^T Ax = 0$ for some eigenvector x. contradiction.

If A had $\lambda < 0$, then $\exists$ eigenvector x s.t. $Ax = \lambda x$

$x^T Ax = x^T \lambda x = \lambda x^T x = \lambda \|x\| < 0$, since $\|x\| > 0$.

<u>contradiction</u>

So A must have strictly <u>positive eigenvalues</u>

5. a. by def, if A is unitary, $AA^* = I$

$$Ax = \lambda x$$
$$A^*Ax = A^* \lambda x = Ix$$

since eigenvalues of transpose are the same (Question 2c),

$$A^* \lambda x = \lambda \lambda x = Ix$$
$$\|\lambda\|^2 = 1$$
$$|\lambda| = 1$$

b.

$$\|A\|_F = \sqrt{trace(A^*A)} = \sqrt{trace(I_n)}$$
$$= \sqrt{n} \neq 1$$

6. a. Following the proof in 3a.

$$\bar{x}^T \bar{A}^T x = \bar{\lambda} \|x\|$$
$$\downarrow$$
$$\bar{A}^T = -A, \text{ so}$$
$$\bar{x}^T - Ax = \bar{\lambda} \|x\|$$
$$\bar{x}^T(-\lambda)x = \bar{\lambda} \|x\|$$
$$-\lambda\|x\| = \bar{\lambda} \|x\|$$
$$-\lambda = \bar{\lambda} \Rightarrow \lambda \text{ is pure imaginary}$$

b. If $I - A$ is singular,

$\exists x$ s.t. $(I-A)x = 0$

$\Rightarrow x = Ax$

$\Rightarrow A$ has eigenvalue of 1

from a.), this is not possible, contradiction.

So $I-A$ is <u>not</u> singular.

7.) let λ be the eigenvalue, x be nonzero eigenvector.

$Ax = \lambda x$ then let $X = \{ x_1 | x_2 | x_3 \ldots \}$

all eigenvectors.

so $AX = \lambda X$

so $\underline{|\lambda| \, \|X\|} = \|\lambda X\| = \|AX\| \leq \underline{\|A\| \, \|X\|}$

so $|\lambda| \leq \|A\|$

so $\rho(A) \leq |\lambda| \leq \|A\|$

$\rho(A) \leq \|A\|$

8. a. $\|A\|_2 = \sqrt{\rho(A^*A)}$

$= \sqrt{\rho(uv^*vu^*)}$

$= \sqrt{\rho(uu^*v^*v)}$

$= \sqrt{\rho(uu^*)\rho(v^*v)}$

$= \sqrt{\rho(uu^*)}\sqrt{\rho(v^*v)}$

$= \|u\|_2 \, \|v\|_2$

b. $\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=n}^{n}(u_i v_j)^2}$

$= \sqrt{\sum_{i=1}^{m}(u_i)^2 \sum_{j=1}^{n}(v_j)^2}$

$= \sqrt{\sum_{i=1}^{m}(u_i)^2}\sqrt{\sum_{j=1}^{n}(v_j)^2}$

$= \|u\|_2 \, \|v\|_2$

$= \|u\|_F \, \|v\|_F$

a) $\|AQ\|_2 = \sup\limits_{\substack{x \neq 0 \\ x \in \mathbb{C}^m}} \dfrac{\|AQx\|_2}{\|x\|_2}$

let $AQ = B$ then

$$= \sup \frac{\|AQx\|_2}{\|x\|_2} = \sup \frac{\|Bx\|_2}{\|x\|_2} = \|B\|_2$$

and since $\|AQ\|_2 = \|A\|_2 = \|B\|_2$,

$\underline{\|AQ\|_2 = \|A\|_2}$

b $\|AQ\|_F = \sqrt{Tr[(AQ)^* AQ]} = \sqrt{Tr[Q^*A^*AQ]}$

$= \sqrt{Tr(AQ^*QA)}$

$= \sqrt{Tr(A^*A)}$

$= \underline{\|A\|_F}$

$*$

$= \sqrt{Tr(A^*Q^*QA)}$

$= \sqrt{Tr[(QA)^* QA]}$

$= \underline{\|QA\|_F}$

10 a. $\|A\|_F = \|QBQ^*\|_F$

then by proofs in 9), RHS = $\|BQ^*\|_F = \|B\|_F$

so $\|A\|_F = \|B\|_F$

and since Frobenius norm is L2 norm of singular values,

A and B have the same L2 of singular values

$$\sqrt{\sum_i \sigma_i^2} \text{ of } A = \sqrt{\sum_i \sigma_i^2} \text{ of } B.$$

~~So A's $\sum_i \sigma_i^2$ = B's $\sum_i \sigma_i^2$~~

and since every matrix has unique singular values

this is only possible if A's σ values = B's σ values

//

$$A = QBQ^*$$

$$= Q(U\Sigma_B V^*)Q^*$$

$$= \underbrace{QU}_{}\Sigma_B \underbrace{V^*Q^*}_{}$$

unitary matrix product stays unitary

still a singular value form,

$$= U'\Sigma_A V^*$$

thus A's Σ = B's Σ

10b. Let $\Sigma_A = \Sigma_B$, so singular values are <u>equal</u>

Let $A = (QU + E \text{ for nonzero } E)\, \Sigma_A\, (QV + F \text{ for nonzero } F)^*$
for $B = U \Sigma_B V^*$
then there exists A, B with same Σ s.t.

$$A - QBQ^* \neq 0$$ aka not ~~unilaterally~~ Unitarily equivalent.

11. a. $f(x_1, x_2) = x_1 + x_2$

$$J = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right) = (1, 1)$$

$\|J\|_\infty = 2$

$$k(x_1, x_2) = \frac{\|J\|_\infty \|x_0\|_\infty}{\|f(x_1, x_2)\|_\infty} = \frac{2 \max\{|x_1|, |x_2|\}}{|x_1 + x_2|}$$

If $|x_1 + x_2| \approx 0$, ill-conditioned.
$x_1 \approx -x_2$

b. $f(x_1, x_2) = x_1 x_2$

$$J = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right) = (x_2, x_1)$$

$\|J\|_\infty = |x_1| + |x_2|$

$$k(x_1, x_2) = \frac{(|x_1| + |x_2|) \cdot \max\{|x_1|, |x_2|\}}{|x_1 x_2|}$$

if $|x_1| > |x_2|$

$$k = \frac{(|x_1| + |x_2|)\, x_1}{|x_1 x_2|} = \left|\frac{x_1}{x_2}\right| + 1$$

ill-conditioned if $|x_1| >> |x_2|$

if $|x_2| > |x_1|$

$$k = \left|\frac{x_2}{x_1}\right| + 1$$

ill conditioned if $|x_2| >> |x_1|$

11c. $f(x) = (x-2)^9$

$J = \frac{df}{dx} = 9(x-2)^8$

$\boxed{J_{x=2} = 0, \text{ so well-conditioned at } x=2}$

From the Ex 12.5 in the book the $g(x)$ Jacobian is infinite and it is ill-conditioned.

or by def $k = \left|\frac{dx_j}{x_j}\right| / \left|\frac{da_i}{a_i}\right| = \left|\frac{a_i x_j^{i-1}}{p'(x_j)}\right|$

So for the $g(x)$ expression / expansion of $f(x)$,

k of $x_{j=0}$ wrt $a_{i=1}$ $= \left|\frac{a_1 x_{j=0}^0}{p'(x_j)}\right| = \left|\frac{512}{\frac{g'(x)}{x_j'}}\right|$

etc.

these k values can be very large, i.e. ill-conditioned.

k of x_j wrt a_6 $= \left|\frac{a_6 x_j^5}{\frac{g'(x_j)}{x_j'}}\right| = \left|\frac{2016 x_j^5}{\frac{g'(x_j)}{x_j^?}}\right|$ $x_j = 2$, so near $x=2$ $\left|\frac{2016 \times 2}{9(x-2)^8}\right|$ can be very large.