Chapter 6 – Maneuvering Models

An alternative to the seakeeping formalism is to use maneuvering theory to describe the motions of marine craft in surge, sway and yaw.

In maneuvering theory, frequency-dependent added mass and potential damping are approximated by constant values and thus it is not necessary to compute the fluid memory effects.

Chapter Goals

- Understand the rigid-body kinetics and zero-frequency potential coefficients used in maneuvering theory.
- Explain what the hydrodynamic added mass matrix is and how it contributes to the marine craft equations of motion.
- Understand why there exist Coriolis and centripetal matrices due to rigid-body and hydrodynamic added mass. This should be related to using an approximately inertial frame NED and a BODY frame rotating about the NED coordinate frame.
- Have an overview of the different maneuvering models that are in use, including
  - 3-DOF ship maneuvering models (surge, sway, and yaw)
  - 4-DOF ship maneuvering models (surge, sway, roll and yaw)
  - 3-DOF stationkeeping models (surge, sway, and yaw)
- Understand how to include ocean currents using relative velocity in the equations of motion:

$$\underbrace{\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{rigid-body forces}} + \underbrace{\boldsymbol{M}_A\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r}_{\text{hydrodynamic forces}}$$

$$+ \underbrace{\boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_o}_{\text{hydrostatic forces}} = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}}$$

Chapter 6 – Maneuvering Models (cont.)

Maneuvering equations of motion (no ocean currents)

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_o = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}}$$

In the case of ocean currents

$\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_c \qquad \boldsymbol{\nu}_c =$ ocean current velocity

$$\underbrace{\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{rigid-body forces}} + \underbrace{\boldsymbol{M}_A\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r}_{\text{hydrodynamic forces}} + \underbrace{\boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_o}_{\text{hydrostatic forces}} = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}}$$

The ocean currents are assumed to be <u>constant</u> and <u>irrotational</u> in NED such that (see next page)

$$\boldsymbol{\nu}_c = [u_c,\ v_c,\ w_c,\ 0,\ 0,\ 0]^\top \qquad \dot{\boldsymbol{\nu}}_c = \begin{bmatrix} -\boldsymbol{S}(\boldsymbol{\omega}_{nb}^b) & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \end{bmatrix} \boldsymbol{\nu}_c$$

Hence, it is possible to represent the equations of motion by relative velocities only

$$\boldsymbol{M}\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_o = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}}$$

For a marine craft exposed to irrotational ocean currents, it follows from Property 10.1 in Section 10.3 that

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} \equiv \boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}_{RB}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r$$

$$\mathbf{M} = \mathbf{M}_{RB} + \mathbf{M}_A$$

$$\mathbf{C}(\boldsymbol{\nu}_r) = \mathbf{C}_{RB}(\boldsymbol{\nu}_r) + \mathbf{C}_A(\boldsymbol{\nu}_r)$$

Chapter 6 – Maneuvering Models (cont.)

Relative and Absolute Velocity Models for Ocean Currents

State-space model for relative velocity

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}_\Theta(\boldsymbol{\eta})\,(\boldsymbol{\nu}_r + \boldsymbol{\nu}_c)$$
$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{M}^{-1}\,(\boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} + \boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \boldsymbol{g}(\boldsymbol{\eta}) - \boldsymbol{g}_0)$$

$$\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_c$$
$$\boldsymbol{\nu}_c = [\underbrace{u_c,\ v_c,\ w_c}_{\boldsymbol{v}_c^b},\ 0,\ 0,\ 0]^\top$$

State-space model for absolute velocity

$$\dot{\boldsymbol{\eta}} = \boldsymbol{J}_\Theta(\boldsymbol{\eta})\boldsymbol{\nu}$$
$$\dot{\boldsymbol{\nu}} = \begin{bmatrix} -\boldsymbol{S}(\boldsymbol{\omega}_{nb}^b)\boldsymbol{v}_c^b \\ \boldsymbol{0}_{3\times1} \end{bmatrix} + \boldsymbol{M}^{-1}\,(\boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} - \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \boldsymbol{g}(\boldsymbol{\eta}) - \boldsymbol{g}_0)$$

$$\boldsymbol{v}_c^b = [u_c,\ v_c,\ w_c]^\top$$

$$\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_c$$

$$\boxed{\boldsymbol{v}_c^n = \boldsymbol{R}(\boldsymbol{\Theta}_{nb})\boldsymbol{v}_c^b}$$

Definition 10.1 (Irrotational Constant Ocean Current)
An irrotational constant ocean current expressed in $\{n\}$ is defined by

$$\dot{\boldsymbol{v}}_c^n = \dot{\boldsymbol{R}}(\boldsymbol{\Theta}_{nb})\boldsymbol{v}_c^b + \boldsymbol{R}(\boldsymbol{\Theta}_{nb})\dot{\boldsymbol{v}}_c^b := \boldsymbol{0}$$

where

$$\dot{\boldsymbol{R}}(\boldsymbol{\Theta}_{nb}) = \boldsymbol{R}(\boldsymbol{\Theta}_{nb})\boldsymbol{S}(\boldsymbol{\omega}_{nb}^b)$$

Consequently,

$$\dot{\boldsymbol{v}}_c^b = -\boldsymbol{S}(\boldsymbol{\omega}_{nb}^b)\boldsymbol{v}_c^b$$

MSS Toolbox

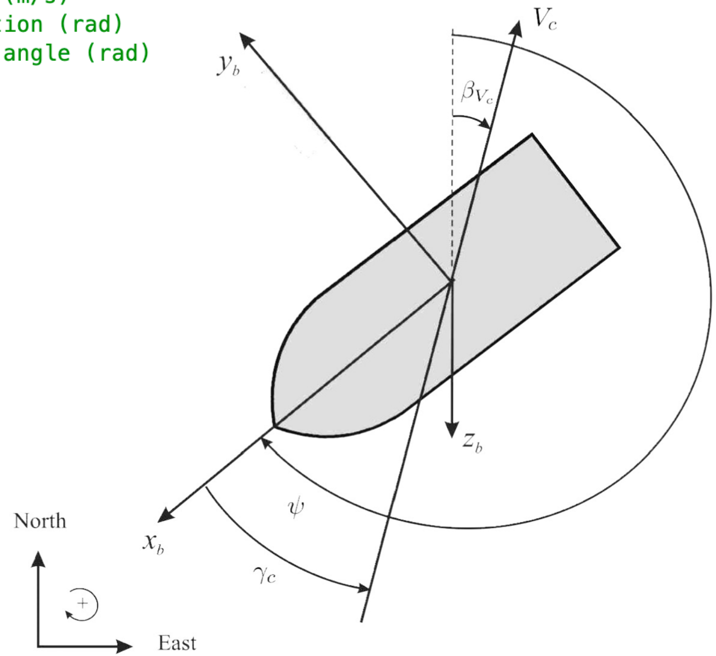
```
>> help Rzyx
  R = Rzyx(phi,theta,psi) computes the Euler angle
  rotation matrix R in SO(3) using the zyx convention
```

Computation of current velocities expressed in BODY

We assume that the current is irrotational and constant in NED. Consequently,

```
% constant 2-D ocean current expressed in NED
Vc = 1;                                     % current speed (m/s)
betaVc = deg2rad(30);                       % current direction (rad)
psi = deg2rad(10);                          % craft heading angle (rad)

% current velocities expressed in NED
vc_n = [Vc * cos(betaVc) Vc * sin(betaVc) 0]'

% current velocities expressed in BODY
vc_b = Rzyx(0,0,psi)' * vc_n
```

```
vc_n =
    0.8660
    0.5000
         0
vc_b =
    0.9397
    0.3420
         0
```



6.1 Rigid-Body Kinetics

Rigid-Body Kinetics (see Chapter 3)

$$\boxed{\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}}$$

$$\boldsymbol{\nu} = [u, v, w, p, q, r]^\top$$

It is common to assume that conventional marine craft has homogeneous mass distribution and *xz*-plane symmetry so that

$$I_{xy} = I_{yz} = 0$$

Let the {b}-frame coordinate origin be set in the centerline of the craft in the point CO, such that $y_g = 0$. Under the previously stated assumptions the 3-DOF model satisfies

$$\mathbf{M}_{RB} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & mx_g \\ 0 & mx_g & I_z \end{bmatrix}, \quad \mathbf{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & -mr & -mx_g r \\ mr & 0 & 0 \\ mx_g r & 0 & 0 \end{bmatrix}$$

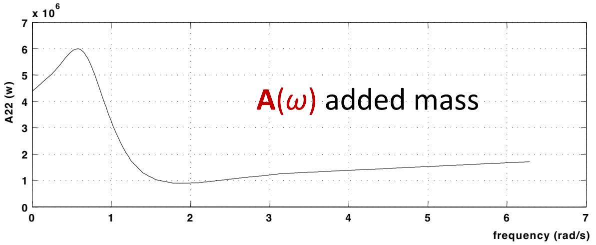
Linear Theory ($u = U$ = constant, $v = 0$ and $r = 0$)

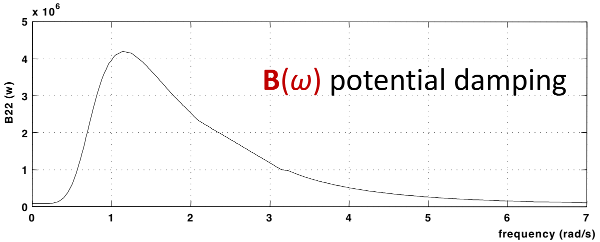
$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}^*_{RB}\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

$$\boldsymbol{C}^*_{RB} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & mU \\ 0 & 0 & mx_g U \end{bmatrix}$$

6.2 Potential Coefficients

Potential coefficients for a ship





The potential coefficients are usually represented as frequency-dependent matrices for 6-DOF motions. The matrices are

- **A(ω) added mass**
- **B(ω) potential damping**

where ω is the wave excitation frequency.

For underwater vehicles operating at water depths below the wave-affected zone (*20 m*), the potential coefficients will be independent of the wave excitation frequency

$$\begin{aligned} \mathbf{A}(\omega) &= \text{constant} \quad \forall \omega \\ \mathbf{B}(\omega) &= \mathbf{0} \end{aligned}$$

$$[\boldsymbol{M}_{RB} + \boldsymbol{A}(\omega)]\ddot{\boldsymbol{\xi}} + [\boldsymbol{B}(\omega) + \boldsymbol{B}_V(\omega)]\dot{\boldsymbol{\xi}} + \boldsymbol{C}\boldsymbol{\xi} = \delta\boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}}$$

As stressed in Chapter 5 the seakeeping model should be represented by Cummins equation in the time domain to avoid the frequency-dependent matrices.

This introduces fluid memory effects which can be interpreted as filtered potential damping forces. It is standard to include the fluid memory effects in seakeeping models while 3-DOF classical maneuvering theory neglects the fluid memory by using a zero-frequency assumption.

6.2 Potential Coefficients

Definition 6.1 (Zero-Frequency Models for Surge, Sway and Yaw)
The horizontal motions (surge, sway and yaw) of a marine craft moving at forward speed can be described by a zero-frequency model where:

$$\boldsymbol{M}_A = \boldsymbol{A}^{\{1,2,6\}}(0) = \begin{bmatrix} A_{11}(0) & 0 & 0 \\ 0 & A_{22}(0) & A_{26}(0) \\ 0 & A_{62}(0) & A_{66}(0) \end{bmatrix}$$

$$\boldsymbol{D}_p = \boldsymbol{B}^{\{1,2,6\}}(0) = \boldsymbol{0}$$

are constant matrices.

Maneuvering theory: Use the added mass and potential damping at one frequency, but which one?

The zero-frequency assumption is valid for surge, sway, and yaw since the natural frequency of a PD-controlled ship (closed loop) will be close to zero. The natural period will typically be in the 100 s to 150 s range. Since $\omega = 2\pi/T$ the minimum natural frequency is in the range of 0.03–0.06 rad/s, which confirms that we could use the zero-wave excitation frequency.

Maneuvering model based on the linear seakeeping model in Chapter 5, when using the zero-frequency assumption

$$(\boldsymbol{M}_{RB} + \boldsymbol{M}_A)\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}^{*}\boldsymbol{\nu} + \boldsymbol{C}_A^{*}\boldsymbol{\nu} + \boldsymbol{D}\boldsymbol{\nu} = \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} + \boldsymbol{\tau}$$

Not that the single frequency assumption implies that the fluid memory effects $\mu = 0$

$$\boldsymbol{L}^{\{1,2,6\}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{M}_A = \boldsymbol{A}^{\{1,2,6\}}(0)$$
$$\boldsymbol{C}_A^{*} = U\boldsymbol{A}^{\{1,2,6\}}(0)\boldsymbol{L}^{\{1,2,6\}}$$
$$\boldsymbol{D} = \boldsymbol{B}_V^{\{1,2,6\}}(0)$$

6.2 Extension to 6-DOF models

One limitation of the *zero-frequency assumption* is that it cannot be applied to heave, roll, and pitch. For 2nd-order mass-damper-spring systems the dominating frequencies are the natural frequencies.

Solution: Formulate frequency-independent models in heave, pitch, and roll at their respective natural frequencies and <u>not</u> the zero frequency.

$$\boxed{\begin{aligned} \omega_3 &= \sqrt{\frac{C_{33}^{CF}}{m + A_{33}^{CF}(\omega_3)}} \\ \omega_4 &= \sqrt{\frac{C_{44}^{CF}}{I_x^{CF} + A_{44}^{CF}(\omega_4)}} \\ \omega_5 &= \sqrt{\frac{C_{55}^{CF}}{I_y^{CF} + A_{55}^{CF}(\omega_5)}} \end{aligned}}$$

$$\boldsymbol{D}_p \approx \begin{bmatrix} 0 & 0 & & \cdots & & 0 \\ 0 & 0 & & & & 0 \\ & \cdots & B_{33}(\omega_3) & 0 & 0 & \cdots \\ & & 0 & B_{44}(\omega_4) & 0 & \\ & & 0 & 0 & B_{55}(\omega_5) & \\ 0 & 0 & & \cdots & & 0 \end{bmatrix}$$

$$\boldsymbol{D}_V \approx \mathrm{diag}\{B_{11v}, B_{22v}, B_{33v}, B_{44v}, B_{55v}, B_{66v}\}$$

Key assumption: No coupling between the surge-sway-yaw and the heave-roll-pitch subsystems.

$$\boldsymbol{M}_A \approx \begin{bmatrix} A_{11}(0) & 0 & & \cdots & & 0 \\ 0 & A_{22}(0) & & & & A_{26}(0) \\ & & A_{33}(\omega_3) & 0 & 0 & \\ & \cdots & 0 & A_{44}(\omega_4) & 0 & \cdots \\ & & 0 & 0 & A_{55}(\omega_5) & \\ 0 & A_{62}(0) & & \cdots & & A_{66}(0) \end{bmatrix}$$

6.3 Added Mass Forces in a Rotating Coordinate System

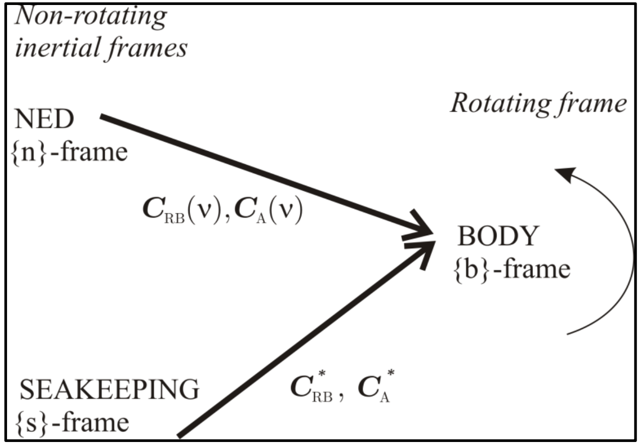
Added Mass: Hydrodynamic added mass can be seen as a virtual mass added to a system because an accelerating or decelerating body must move some volume of the surrounding fluid as it moves through it. Moreover, the object and fluid cannot occupy the same physical space simultaneously.

The fluid kinetic energy can be used in an energy formulation to derive the expressions for the Coriolis and centripetal matrix $\mathbf{C}_A(\boldsymbol{\nu}_r)$. We will assume that the SEAKEEPING reference frame is non-rotating (approximative inertial frame).

Note that we use different approximative inertial reference frames in seakeeping and maneuvering theory.

The expression for $\mathbf{C}_A(\boldsymbol{\nu}_r)$ depends on which reference frames that are considered:

1) Maneuvering Theory considers the motion of a rotating frame {b} with respect to {n} using Langrangian mechanics.
2) Seakeeping Theory, the body frame {b} rotates about {s}. It is common to consider only the linearized version denoted by $\mathbf{C}_A^*$.



6.3 Added Mass Forces in a Rotating Coordinate System

Alternative approach to the Newton-Euler formulation: *Lagrangian mechanics*

Euler-Lagrange's Equation (only for generalized coordinates)

$$L = T - V$$

Difference between kinetic and potential energy

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\eta}}}\right) - \frac{\partial L}{\partial \boldsymbol{\eta}} = \mathbf{J}^{-\top}(\boldsymbol{\eta})\boldsymbol{\tau}$$

Generalized coordinates = 6 DOF

$$\boldsymbol{\eta} = [x^n, y^n, z^n, \phi, \theta, \psi]^\top$$

Quaternions are not generalized coordinates

$$\boldsymbol{\eta} = [x^n, y^n, z^n, \eta, \varepsilon_1, \varepsilon_2, \varepsilon_3]^\top$$

Body velocities are not generalized coordinates

$$\boldsymbol{\nu} = [u, v, w, p, q, r]^\top$$

Kirchhoff's Equations in Vector Form (uses only kinetic energy / velocity)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \boldsymbol{\nu}_1}\right) + \mathbf{S}(\boldsymbol{\nu}_2)\frac{\partial T}{\partial \boldsymbol{\nu}_1} = \boldsymbol{\tau}_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \boldsymbol{\nu}_2}\right) + \mathbf{S}(\boldsymbol{\nu}_2)\frac{\partial T}{\partial \boldsymbol{\nu}_2} + \mathbf{S}(\boldsymbol{\nu}_1)\frac{\partial T}{\partial \boldsymbol{\nu}_1} = \boldsymbol{\tau}_2$$

$$\boldsymbol{\nu}_1 = [u, v, w]^\top \qquad \boldsymbol{\tau}_1 = [X, Y, Z]^\top$$

$$\boldsymbol{\nu}_2 = [p, q, r]^\top \qquad \boldsymbol{\tau}_2 = [K, M, N]^\top$$

$$T = \frac{1}{2}\boldsymbol{\nu}^\top \mathbf{M}\boldsymbol{\nu}$$

Kinetic energy

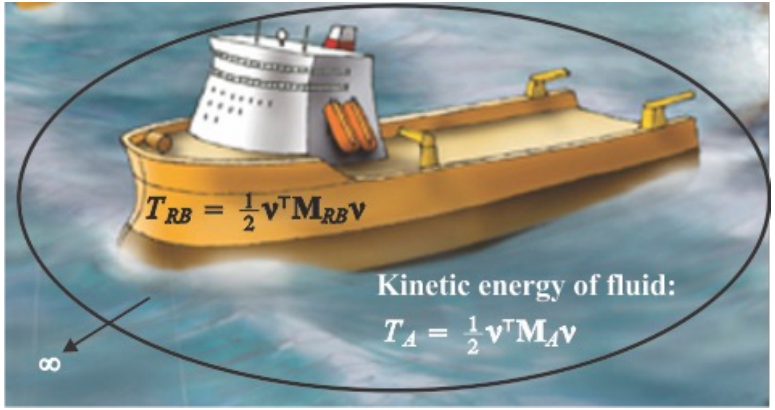
6.3 Added Mass Forces in a Rotating Coordinate System

The matrix $\mathbf{C}_A^*$ represents linearized forces due to a rotation of {b} about the seakeeping frame {s}.

Instead of using {s} as the inertial frame, we will assume that {n} is the inertial frame and that {b} rotates about {n}.

The added mass matrix in {b} satisfies

$$T_A = \frac{1}{2}\boldsymbol{\nu}^\top \boldsymbol{M}_A \boldsymbol{\nu}, \qquad \dot{\boldsymbol{M}}_A = \mathbf{0}$$



Fluid Kinetic Energy

- The concept of fluid kinetic energy can be used to derive the added mass terms.
- Any motion of the vessel will induce a motion in the otherwise stationary fluid. In order to allow the vessel to pass through the fluid, it must move aside and then close behind the vessel.
- Consequently, the fluid motion possesses kinetic energy that it would lack otherwise (Lamb 1932).

$$\mathbf{M}_A = -\begin{bmatrix} X_{\dot u} & X_{\dot v} & X_{\dot w} & X_{\dot p} & X_{\dot q} & X_{\dot r} \\ Y_{\dot u} & Y_{\dot v} & Y_{\dot w} & Y_{\dot p} & Y_{\dot q} & Y_{\dot r} \\ Z_{\dot u} & Z_{\dot v} & Z_{\dot w} & Z_{\dot p} & Z_{\dot q} & Z_{\dot r} \\ K_{\dot u} & K_{\dot v} & K_{\dot w} & K_{\dot p} & K_{\dot q} & K_{\dot r} \\ M_{\dot u} & M_{\dot v} & M_{\dot w} & M_{\dot p} & M_{\dot q} & M_{\dot r} \\ N_{\dot u} & N_{\dot v} & N_{\dot w} & N_{\dot p} & N_{\dot q} & N_{\dot r} \end{bmatrix}$$

6.3 Added Mass Forces in a Rotating Coordinate System

Kirchhoff's Equations

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \mathbf{v}_1}\right) + \mathbf{S}(\mathbf{v}_2)\frac{\partial T}{\partial \mathbf{v}_1} &= \boldsymbol{\tau}_1 \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \mathbf{v}_2}\right) + \mathbf{S}(\mathbf{v}_2)\frac{\partial T}{\partial \mathbf{v}_2} + \mathbf{S}(\mathbf{v}_1)\frac{\partial T}{\partial \mathbf{v}_1} &= \boldsymbol{\tau}_2 \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial T_A}{\partial u} &= r\frac{\partial T_A}{\partial v} - q\frac{\partial T_A}{\partial w} - X_A \\ \frac{d}{dt}\frac{\partial T_A}{\partial v} &= p\frac{\partial T_A}{\partial w} - r\frac{\partial T_A}{\partial u} - Y_A \\ \frac{d}{dt}\frac{\partial T_A}{\partial w} &= q\frac{\partial T_A}{\partial u} - p\frac{\partial T_A}{\partial v} - Z_A \\ \frac{d}{dt}\frac{\partial T_A}{\partial p} &= w\frac{\partial T_A}{\partial v} - v\frac{\partial T_A}{\partial w} + r\frac{\partial T_A}{\partial q} - q\frac{\partial T_A}{\partial r}e - K_A \\ \frac{d}{dt}\frac{\partial T_A}{\partial q} &= u\frac{\partial T_A}{\partial w} - w\frac{\partial T_A}{\partial u} + p\frac{\partial T_A}{\partial r} - r\frac{\partial T_A}{\partial p} - M_A \\ \frac{d}{dt}\frac{\partial T_A}{\partial r} &= v\frac{\partial T_A}{\partial u} - u\frac{\partial T_A}{\partial v} + q\frac{\partial T_A}{\partial p} - p\frac{\partial T_A}{\partial q} - N_A \end{aligned}$$

Property 6.1 (Hydrodynamic System Inertia Matrix) For a rigid-body at rest or moving at forward speed $U \geq 0$ in ideal fluid, the infinity frequency hydrodynamic system inertia matrix is positive definite

$$\mathbf{M}_A = \mathbf{M}_A^\top > 0$$

$$\mathbf{M}_A = -\begin{bmatrix} X_{\dot{u}} & X_{\dot{v}} & X_{\dot{w}} & X_{\dot{p}} & X_{\dot{q}} & X_{\dot{r}} \\ Y_{\dot{u}} & Y_{\dot{v}} & Y_{\dot{w}} & Y_{\dot{p}} & Y_{\dot{q}} & Y_{\dot{r}} \\ Z_{\dot{u}} & Z_{\dot{v}} & Z_{\dot{w}} & Z_{\dot{p}} & Z_{\dot{q}} & Z_{\dot{r}} \\ K_{\dot{u}} & K_{\dot{v}} & K_{\dot{w}} & K_{\dot{p}} & K_{\dot{q}} & K_{\dot{r}} \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & M_{\dot{p}} & M_{\dot{q}} & M_{\dot{r}} \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & N_{\dot{p}} & N_{\dot{q}} & N_{\dot{r}} \end{bmatrix}$$

$$T_A = \frac{1}{2}\boldsymbol{\nu}^\top \mathbf{M}_A \boldsymbol{\nu}$$

6.3 Added Mass Forces in a Rotating Coordinate System

Hydrodynamic added mass forces and moments in 6 DOFs

The expressions are complicated and not to suited for control design.

Hydrodynamic software programs such as WAMIT and ShipX can be used to compute the added mass terms

The model can be more compactly written using the *added mass system inertia matrix* $\mathbf{M}_A$ and the *added mass Coriolis and centripetal matrix* $\mathbf{C}_A(\boldsymbol{\nu})$

$$
\begin{aligned}
X_A = {} & X_{\dot{u}}\dot{u} + X_{\dot{w}}(\dot{w} + uq) + X_{\dot{q}}\dot{q} + Z_{\dot{w}}wq + Z_{\dot{q}}q^2 \\
& + X_{\dot{v}}\dot{v} + X_{\dot{p}}\dot{p} + X_{\dot{r}}\dot{r} - Y_{\dot{v}}vr - Y_{\dot{p}}rp - Y_{\dot{r}}r^2 \\
& - X_{\dot{v}}ur - Y_{\dot{w}}wr \\
& + Y_{\dot{w}}vq + Z_{\dot{p}}pq - (Y_{\dot{q}} - Z_{\dot{r}})qr \\
Y_A = {} & X_{\dot{v}}\dot{u} + Y_{\dot{w}}\dot{w} + Y_{\dot{q}}\dot{q} \\
& + Y_{\dot{v}}\dot{v} + Y_{\dot{p}}\dot{p} + Y_{\dot{r}}\dot{r} + X_{\dot{v}}vr - Y_{\dot{w}}vp + X_{\dot{r}}r^2 + (X_{\dot{p}} - Z_{\dot{r}})rp - Z_{\dot{p}}p^2 \\
& - X_{\dot{w}}(up - wr) + X_{\dot{u}}ur - Z_{\dot{w}}wp \\
& - Z_{\dot{q}}pq + X_{\dot{q}}qr \\
Z_A = {} & X_{\dot{w}}(\dot{u} - wq) + Z_{\dot{w}}\dot{w} + Z_{\dot{q}}\dot{q} - X_{\dot{u}}uq - X_{\dot{q}}q^2 \\
& + Y_{\dot{w}}\dot{v} + Z_{\dot{p}}\dot{p} + Z_{\dot{r}}\dot{r} + Y_{\dot{v}}vp + Y_{\dot{r}}rp + Y_{\dot{p}}p^2 \\
& + X_{\dot{v}}up + Y_{\dot{w}}wp \\
& - X_{\dot{v}}vq - (X_{\dot{p}} - Y_{\dot{q}})pq - X_{\dot{r}}qr \\
K_A = {} & X_{\dot{p}}\dot{u} + Z_{\dot{p}}\dot{w} + K_{\dot{q}}\dot{q} - X_{\dot{v}}wu + X_{\dot{r}}uq - Y_{\dot{w}}w^2 - (Y_{\dot{q}} - Z_{\dot{r}})wq + M_{\dot{r}}q^2 \\
& + Y_{\dot{p}}\dot{v} + K_{\dot{p}}\dot{p} + K_{\dot{r}}\dot{r} + Y_{\dot{w}}v^2 - (Y_{\dot{q}} - Z_{\dot{r}})vr + Z_{\dot{p}}vp - M_{\dot{r}}r^2 - K_{\dot{q}}rp \\
& + X_{\dot{w}}uv - (Y_{\dot{v}} - Z_{\dot{w}})vw - (Y_{\dot{r}} + Z_{\dot{q}})wr - Y_{\dot{p}}wp - X_{\dot{q}}ur \\
& + (Y_{\dot{r}} + Z_{\dot{q}})vq + K_{\dot{r}}pq - (M_{\dot{q}} - N_{\dot{r}})qr \\
M_A = {} & X_{\dot{q}}(\dot{u} + wq) + Z_{\dot{q}}(\dot{w} - uq) + M_{\dot{q}}\dot{q} - X_{\dot{w}}(u^2 - w^2) - (Z_{\dot{w}} - X_{\dot{u}})wu \\
& + Y_{\dot{q}}\dot{v} + K_{\dot{q}}\dot{p} + M_{\dot{r}}\dot{r} + Y_{\dot{p}}vr - Y_{\dot{r}}vp - K_{\dot{r}}(p^2 - r^2) + (K_{\dot{p}} - N_{\dot{r}})rp \\
& - Y_{\dot{w}}uv + X_{\dot{v}}vw - (X_{\dot{r}} + Z_{\dot{p}})(up - wr) + (X_{\dot{p}} - Z_{\dot{r}})(wp + ur) \\
& - M_{\dot{r}}pq + K_{\dot{q}}qr \\
N_A = {} & X_{\dot{r}}\dot{u} + Z_{\dot{r}}\dot{w} + M_{\dot{r}}\dot{q} + X_{\dot{v}}u^2 + Y_{\dot{w}}wu - (X_{\dot{p}} - Y_{\dot{q}})uq - Z_{\dot{p}}wq - K_{\dot{q}}q^2 \\
& + Y_{\dot{r}}\dot{v} + K_{\dot{r}}\dot{p} + N_{\dot{r}}\dot{r} - X_{\dot{v}}v^2 - X_{\dot{r}}vr - (X_{\dot{p}} - Y_{\dot{q}})vp + M_{\dot{r}}rp + K_{\dot{q}}p^2 \\
& - (X_{\dot{u}} - Y_{\dot{v}})uv - X_{\dot{w}}vw + (X_{\dot{q}} + Y_{\dot{p}})up + Y_{\dot{r}}ur + Z_{\dot{q}}wp \\
& - (X_{\dot{q}} + Y_{\dot{p}})vq - (K_{\dot{p}} - M_{\dot{q}})pq - K_{\dot{r}}qr
\end{aligned}
$$

6.3 Added Mass Forces in a Rotating Coordinate System

The *added mass Coriolis and centripetal matrix* is found by collecting all terms that are not functions of body accelerations.

Property (Hydrodynamic Coriolis and centripetal matrix) For a rigid-body moving through an ideal fluid the hydrodynamic Coriolis and centripetal matrix can always be parameterized such that it is skew-symmetric

$$\mathbf{C}_A(\boldsymbol{\nu}) = -\mathbf{C}_A^\top(\boldsymbol{\nu}), \qquad \forall \boldsymbol{\nu} \in \mathbb{R}^6$$

by defining

$$\mathbf{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} \mathbf{0}_{3\times3} & -\mathbf{S}(\mathbf{A}_{11}\boldsymbol{\nu}_1 + \mathbf{A}_{12}\boldsymbol{\nu}_2) \\ -\mathbf{S}(\mathbf{A}_{11}\boldsymbol{\nu}_1 + \mathbf{A}_{12}\boldsymbol{\nu}_2) & -\mathbf{S}(\mathbf{A}_{21}\boldsymbol{\nu}_1 + \mathbf{A}_{22}\boldsymbol{\nu}_2) \end{bmatrix}$$

Example

$$\mathbf{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} 0 & 0 & 0 & 0 & -a_3 & a_2 \\ 0 & 0 & 0 & a_3 & 0 & -a_1 \\ 0 & 0 & 0 & -a_2 & a_1 & 0 \\ 0 & -a_3 & a_2 & 0 & -b_3 & b_2 \\ a_3 & 0 & -a_1 & b_3 & 0 & -b_1 \\ -a_2 & a_1 & 0 & -b_2 & b_1 & 0 \end{bmatrix}$$

$$\begin{aligned} a_1 &= X_{\dot u}u + X_{\dot v}v + X_{\dot w}w + X_{\dot p}p + X_{\dot q}q + X_{\dot r}r \\ a_2 &= Y_{\dot u}u + Y_{\dot v}v + Y_{\dot w}w + Y_{\dot p}p + Y_{\dot q}q + Y_{\dot r}r \\ a_3 &= Z_{\dot u}u + Z_{\dot v}v + Z_{\dot w}w + Z_{\dot p}p + Z_{\dot q}q + Z_{\dot r}r \\ b_1 &= K_{\dot u}u + K_{\dot v}v + K_{\dot w}w + K_{\dot p}p + K_{\dot q}q + K_{\dot r}r \\ b_2 &= M_{\dot u}u + M_{\dot v}v + M_{\dot w}w + M_{\dot p}p + M_{\dot q}q + M_{\dot r}r \\ b_3 &= N_{\dot u}u + N_{\dot v}v + N_{\dot w}w + N_{\dot p}p + N_{\dot q}q + N_{\dot r}r \end{aligned}$$

Sagatun, S. I. and T. I. Fossen (1991). Lagrangian Formulation of Underwater Vehicles' Dynamics. *IEEE International Conference on Systems, Man, and Cybernetics*, Charlottesville, VA, USA, IEEE Xplore, pp. 1029-1034.

MSS Toolbox

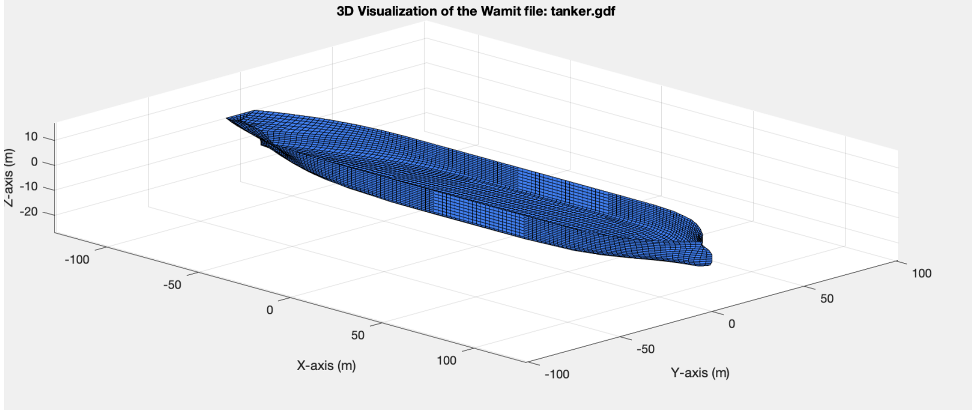
m2c.m

```
function C = m2c(M,nu)
% C = m2c(M,nu) computes the 6x6 Coriolis-centripetal matrix C(nu) from the
% the 6x6 system inertia matrix M > 0 using the velocity vector
% nu = [u, v, w, p, q, r]'.
%
% Examples:   C_RB = m2c(M_RB,nu)
%             C_A  = m2c(M_A, nu)
%
% See also: [M_RB,C_RB] = rb(m,RX,RY,RZ,nu2,r_p)
%
% Author:    Thor I. Fossen
% Date:      14 June 2001
% Revisions: 26 June 2002, M21 = M12 is corrected to M12'
%            10 Jan 2004, the computation of C is generalized to a nonsymmetric M > 0

% Kinetic energy:
%    T = 1/2 (v'Mv) = 1/2 v'(1/2(M + M')+1/2(M-M')) v =  1/2 v' Msym v
%    Msym = 1/2 (M+M')
Msym = 0.5*(M+M');
M = Msym;

M11 = M(1:3,1:3);
M12 = M(1:3,4:6);
M21 = M12';
M22 = M(4:6,4:6);

nu1 = nu(1:3);
nu2 = nu(4:6);
dt_dnu1 = M11*nu1 + M12*nu2;
dt_dnu2 = M21*nu1 + M22*nu2;

C = [  zeros(3,3)      -Smtrx(dt_dnu1)
      -Smtrx(dt_dnu1)  -Smtrx(dt_dnu2) ];
```

MSS Toolbox

```
>> load tanker
>> vessel
vessel =
  struct with fields:

          main: [1×1 struct]
    velocities: 0
      headings: [1×36 double]
           MRB: [6×6 double]
             C: [6×6×60 double]
         freqs: [1×60 double]
             A: [6×6×60 double]
             B: [6×6×60 double]
     motionRAO: [1×1 struct]
      forceRAO: [1×1 struct]
      driftfrc: [1×1 struct]
            Bv: [6×6 double]
>> vessel.freqs
ans =
  Columns 1 through 8
         0    0.0383    0.0424    0.0468    0.0517    0.0571    0.0631    0.0697
  Columns 9 through 16
    0.0770    0.0850    0.0939    0.1037    0.1145    0.1264    0.1395    0.1540
  Columns 17 through 24
    0.1700    0.1876    0.2070    0.2284    0.2519    0.2779    0.3063    0.3378
  Columns 25 through 32
    0.3722    0.4101    0.4517    0.4971    0.5473    0.6018    0.6621    0.7281
  Columns 33 through 40
    0.7994    0.8775    0.9622    1.0560    1.1571    1.2668    1.3840    1.5104
  Columns 41 through 48
    1.6491    1.8003    1.9574    2.1299    2.3100    2.5033    2.7083    2.9361
  Columns 49 through 56
    3.1574    3.3963    3.6530    3.9270    4.1888    4.4880    4.7600    5.0671
  Columns 57 through 60
    5.3702    5.7120    5.9840   10.0000
```

```
>> plot_wamitgdf('tanker')
```



Zero-frequency added mass matrix

```
>> MA = vessel.A(:,:,1)
MA =
   1.0e+11 *
    0.0000         0   -0.0000         0   -0.0128         0
         0    0.0005         0    0.0030         0    0.0029
   -0.0000         0    0.0039         0    0.0096         0
         0    0.0030         0    0.1103         0   -0.0419
   -0.0128         0    0.0096         0    9.0021         0
         0    0.0029         0   -0.0420         0    1.7383
```

```
>> nu = [10 0 0 0 0 0]';
>> CA = m2c(MA,nu)
CA =
   1.0e+10 *
         0         0         0         0   -0.0032         0
         0         0         0    0.0032         0    0.0041
         0         0         0         0   -0.0041         0
         0   -0.0032         0         0         0    1.2758
    0.0032         0    0.0041         0         0         0
         0   -0.0041         0   -1.2758         0         0
```

6.3 Added Mass Forces in a Rotating Coordinate System

Example 6.1 (Added Mass for Surface Vessels)

For surface ships such as tankers, cargo ships and cruise-liners it is common to decouple the surge mode from the steering dynamics due to xz-plane symmetry. Similarly, the heave, pitch, and roll modes are neglected under the assumption that these motion variables are small. Hence, $\boldsymbol{\nu}_r = [u_r, v_r, r]^\top$ implies that the added mass derivatives for a surface ship are

$$\boldsymbol{M}_A = \boldsymbol{M}_A^\top = -\begin{bmatrix} X_{\dot{u}} & 0 & 0 \\ 0 & Y_{\dot{v}} & Y_{\dot{r}} \\ 0 & Y_{\dot{r}} & N_{\dot{r}} \end{bmatrix} \qquad (N_{\dot{v}} = Y_{\dot{r}})$$

$$\boldsymbol{C}_A(\boldsymbol{\nu}_r) = -\boldsymbol{C}_A^\top(\boldsymbol{\nu}_r) = \begin{bmatrix} 0 & 0 & Y_{\dot{v}} v_r + Y_{\dot{r}} r \\ 0 & 0 & -X_{\dot{u}} u_r \\ -Y_{\dot{v}} v_r - Y_{\dot{r}} r & X_{\dot{u}} u_r & 0 \end{bmatrix}$$

The Coriolis and centripetal forces are recognized as

$$\boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r = \begin{bmatrix} Y_{\dot{v}} v_r r + Y_{\dot{r}} r^2 \\ -X_{\dot{u}} u_r r \\ \underbrace{(X_{\dot{u}} - Y_{\dot{v}}) u_r v_r}_{\text{Munk moment}} - Y_{\dot{r}} u_r r \end{bmatrix}$$

where the first term in the yaw moment is the nonlinear Munk moment, which is known to have destabilizing effects.

6.3 Added Mass Forces in a Rotating Coordinate System

Example 6.2 (Added Mass for Underwater Vehicles)
In general, the motion of an underwater vehicle moving in 6 DOF at high speed will be highly nonlinear and coupled. However, in many AUV and ROV applications the vehicle will only be allowed to move at low speed. If the vehicle also has three planes of symmetry, this suggests that the contribution from the off-diagonal elements in the matrix $\boldsymbol{M}_A$ can be neglected. Hence, the following simple expressions for the matrices $\boldsymbol{M}_A$ and $\boldsymbol{C}_A$ are obtained:

$$\boldsymbol{M}_A = \boldsymbol{M}_A^\top = -\mathrm{diag}\{X_{\dot{u}}, Y_{\dot{v}}, Z_{\dot{w}}, K_{\dot{p}}, M_{\dot{q}}, N_{\dot{r}}\}$$

$$\boldsymbol{C}_A(\boldsymbol{\nu}_r) = -\boldsymbol{C}_A^\top(\boldsymbol{\nu}_r) = \begin{bmatrix} 0 & 0 & 0 & 0 & -Z_{\dot{w}}w_r & Y_{\dot{v}}v_r \\ 0 & 0 & 0 & Z_{\dot{w}}w_r & 0 & -X_{\dot{u}}u_r \\ 0 & 0 & 0 & -Y_{\dot{v}}v_r & X_{\dot{u}}u_r & 0 \\ 0 & -Z_{\dot{w}}w_r & Y_{\dot{v}}v_r & 0 & -N_{\dot{r}}r & M_{\dot{q}}q \\ Z_{\dot{w}}w_r & 0 & -X_{\dot{u}}u_r & N_{\dot{r}}r & 0 & -K_{\dot{p}}p \\ -Y_{\dot{v}}v_r & X_{\dot{u}}u_r & 0 & -M_{\dot{q}}q & K_{\dot{p}}p & 0 \end{bmatrix}$$

The diagonal structure is often used since it is time consuming to determine the off-diagonal elements from experiments as well as theory. In practice, the diagonal approximation is found to be quite good for many applications. This is due to the fact that the off-diagonal elements of a positive inertia matrix will be much smaller than their diagonal counterparts.

6.4 Dissipative Forces

Potential theory codes assume that the fluid is inviscid. Water is assumed to be an inviscid fluid, in which the viscosity of the fluid is equal to zero. When viscous forces are neglected, such as the case of inviscid flow, the Navier-Stokes equation can be simplified to a form known as the Euler equation.

Consequently, potential codes only compute:

Potential Damping: Dissipative force in an inviscid fluid. The contribution from the potential damping terms compared to other dissipative terms like viscous damping is usually negligible.

d'Alembert's paradox (from Wikipedia)

d'Alembert's paradox (or the hydrodynamic paradox) is a contradiction reached in 1752 by the French mathematician Jean le Rond d'Alembert.

D'Alembert proved that – for incompressible and inviscid potential flow – the drag force is zero on a body moving with constant velocity relative to the fluid. Zero drag is in direct contradiction to the observation of substantial drag on bodies moving relative to fluids, such as air and water; especially at high velocities corresponding with high Reynolds numbers.

As of consequence of this, we need to add viscous damping forces (such as quadratic drag and lift) to the seakeeping and maneuvering models.

Jean le Rond d'Alembert
(1717-1783)

6.4 Dissipative Forces

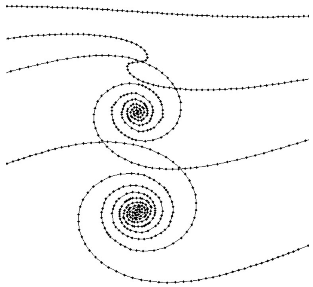
Hydrodynamic viscous damping for marine craft is mainly caused by

- ✓ **Skin friction:** Linear skin friction is caused by laminar boundary layer theory and pressure variations. Important when considering the low-frequency motion of the craft. This again affects how we design the motion control system.

  In addition to linear skin friction, there will be a high-frequency contribution due to a turbulent boundary layer (*quadratic or nonlinear skin friction*).

- ✓ **Wave Drift Damping:** Wave drift damping can be interpreted as added resistance for surface craft advancing in waves. This type of damping is derived from 2nd-order wave theory. Wave drift damping is the most important damping contribution to surge for higher sea states. This is since the wave drift forces are proportional to the square of the significant wave height H_s.

- ✓ **Damping Due to Vortex Shedding:** This is commonly referred to as interference drag. It arises due to the shedding of vortex sheets (surface across which there is a discontinuity in fluid velocity) at sharp edges.



- ✓ **Lifting Forces** Hydrodynamic lift forces arise from two physical mechanisms. The first is due to the linear circulation of water around the hull. The second is a nonlinear effect, commonly called cross-flow drag, which acts from a momentum transfer from the body to the fluid.

- ✓ **… and other viscous effects**

Unfortunately, it is very hard to model all these affects and collect the terms into a common nonlinear damping matrix $\mathbf{D}(\boldsymbol{\nu})$

6.4 Linear Viscous Damping

Property 6.3 (Hydrodynamic Damping Matrix)
For a rigid-body moving through an ideal fluid the hydrodynamic damping matrix

$$\mathbf{D}(\boldsymbol{\nu}_r) > 0, \quad \forall\, \boldsymbol{\nu} \in \mathbb{R}^6$$

will be real, nonsymmetric and strictly positive damping matrix

$$\mathbf{x}^\top \mathbf{D}(\boldsymbol{\nu}_r)\mathbf{x} = \frac{1}{2}\mathbf{x}^\top[\mathbf{D}(\boldsymbol{\nu}_r) + \mathbf{D}(\boldsymbol{\nu}_r)^\top]\mathbf{x} > 0 \quad \forall\ \mathbf{x} \neq \mathbf{0}$$

Property 6.3 applies for linear and nonlinear damping.

Linear Damping

$$\mathbf{D} = \mathbf{D}_P + \mathbf{D}_V$$

$$= -\begin{bmatrix} X_u & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_v & 0 & Y_p & 0 & Y_r \\ 0 & 0 & Z_w & 0 & Z_q & 0 \\ 0 & K_v & 0 & K_p & 0 & K_r \\ 0 & 0 & M_w & 0 & M_q & 0 \\ 0 & N_v & 0 & N_p & 0 & N_r \end{bmatrix}$$

The linear damping matrix is the sum of potential damping and viscous damping matrices.

For linear maneuvering theory, the zero-frequency assumption implies that $\mathbf{D}_P = \mathbf{0}$. Hence, we need to find a way to compute the viscous damping matrix $\mathbf{D}_V$ e.g., the approach discussed in Section 5.3.2.

6.4 Linear Viscous Damping

Example 6.3 (Linear Damping for Surface Vessels)
In maneuvering theory, the heave, pitch, and roll modes are neglected under the assumption that these motion variables are small. Hence, the linear damping matrix for a surface vessel becomes

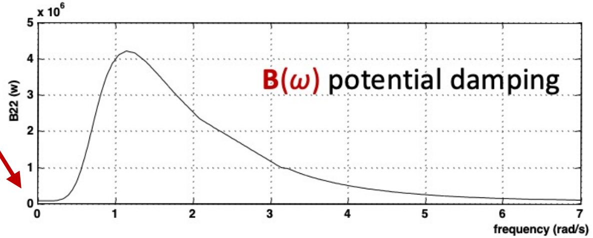
$$\boldsymbol{D} = -\begin{bmatrix} X_u & 0 & 0 \\ 0 & Y_v & Y_r \\ 0 & N_v & N_r \end{bmatrix}$$

The diagonal terms relate to seakeeping theory according to

$$-X_u = B_{11v}(0)$$
$$-Y_v = B_{22v}(0)$$
$$-N_r = B_{66v}(0)$$

The zero-frequency terms are due to viscous damping.

Potential damping at the zero frequency is zero.



where the expressions for B_{iiv} (0) for i = 1, 2, 6 are recognized as the viscous damping terms used in seaekeeping theory (see Section 5.3.2)

6.4 Linear Viscous Damping

Example 6.4 (Linear Damping for Underwater Vehicles)
In general, the motion of an underwater vehicle moving in 6 DOF at high speed will be highly nonlinear and coupled. However, in many AUV and ROV applications the vehicle will only be allowed to move at low speed. If the vehicle also has three planes of symmetry, this suggests that the contribution from the off-diagonal elements in the matrix $\boldsymbol{D}$ *can be neglected. Consequently,*

$$\boldsymbol{D} = -\mathrm{diag}\{X_u, Y_v, Z_w, K_p, M_q, N_r\}$$

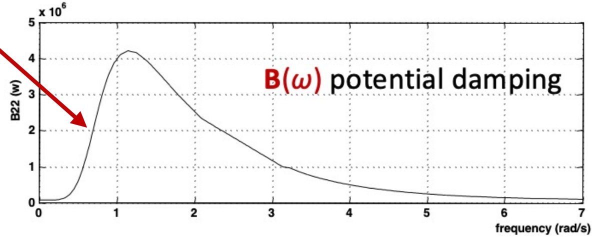
The diagonal structure is often used since it is time consuming to determine the off-diagonal elements from experiments as well as theory.

For heave, roll and pitch it is common to approximate

The viscous damping terms are computed at the natural frequencies.

$$\begin{aligned} -Z_w &= B_{33v}(\omega_3) \\ -K_p &= B_{44v}(\omega_4) \\ -M_q &= B_{55v}(\omega_5) \end{aligned}$$

The potential damping terms are zero for submerged vehicles, but they should be included for surface craft since they are nonzero at the natural frequencies > 0



using (5.60) or by specifying the relative damping ratios ζ_3, ζ_4 and ζ_5 in the formula

$$D_{ii} = 2\zeta_i \omega_i M_{ii}$$

for $i = 3, 4, 5$. Here M_{ii} denotes the sum of rigid-body and hydrodynamic added mass while ω_i are given by (6.18)–(6.20).

We know mass and added mass quite well, then we could specify the relative damping factors and compute the LINEAR damping terms in heave, roll, and pitch

On the next pages, we will discuss methods for nonlinear viscous damping.

6.4 Nonlinear Surge Damping

ITTC resistance

$$X = -\frac{1}{2}\rho S(1+k)C_f(u_r)|u_r|u_r$$

$$C_f(u_r) = \underbrace{\frac{0.075}{(\log_{10}(\mathrm{R}_n)-2)^2}}_{C_F} + C_R$$

$$\begin{aligned} u_r &= u - u_c \\ &= u - V_c\cos(\beta_c - \psi) \end{aligned}$$

where

- ρ density of water
- S wetted surface of the hull
- k form factor giving a viscous correction (typically 0.1 for ships in transit)
- C_F flat plate friction from the ITTC (1957) line
- C_R residual friction due to hull roughness, pressure resistance, wave making resistance

Reynolds number:

$$\mathrm{R}_n = \frac{L_{pp}}{\nu}|u_r| \geq 0$$

$\nu = 1 \cdot 10^{-6} m/s^2$ is the kinematic viscosity at 20°C

Notation of SNAME (1950)

$$X = X_{|u|u} u_r |u_r$$

$$X_{|u|u} = -\frac{1}{2}\rho S(1+k)C_f < 0$$

6.4 Nonlinear Surge Damping

Modified ITTC resistance

$$X_{|u|u|} = -\frac{1}{2}\rho S(1+k)C_f < 0$$

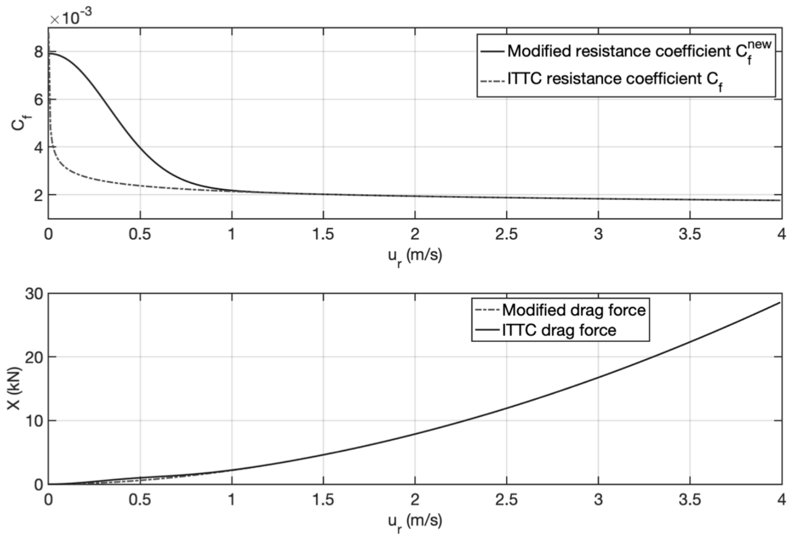
$$X = X_{|u|u|} u_r |u_r$$

gives to little damping compared to the quadratic drag formula

$$X = \frac{1}{2}\rho A_{F_c} C_X(\gamma_c) V_c^2$$

where $C_X > 0$ is the current coefficient and A_{Fc} is the frontal project area. Hence,

$$C_f(0) = -\frac{A_{F_c}}{S(1+k)} C_X(0)$$



Modified resistance curve $C_f^{new}(u_r)$ and $C_f(u_r)$ as a function of u_r when $C_R = 0$ and $\sigma(u_r) = \exp(-5\, u_r^2)$. The zero-speed value $C_f^{new}(0) = -(A_{Fc}/S(1+k))C_X = 0.08$ where $C_X = 0.16$ is the current coefficient.

Modified resistance curve and blending function

$$C_f^{\text{new}}(u_r) = (1-\sigma(u_r))\, C_f(u_r) + \sigma(u_r)\left(-\frac{A_{F_c}}{S(1+k)} C_X(0)\right)$$

$$\sigma(u_r) = e^{-\alpha u_r^2}$$

with $\alpha > 0$ (typically 5.0).

6.4 Cross-Flow Drag Principle

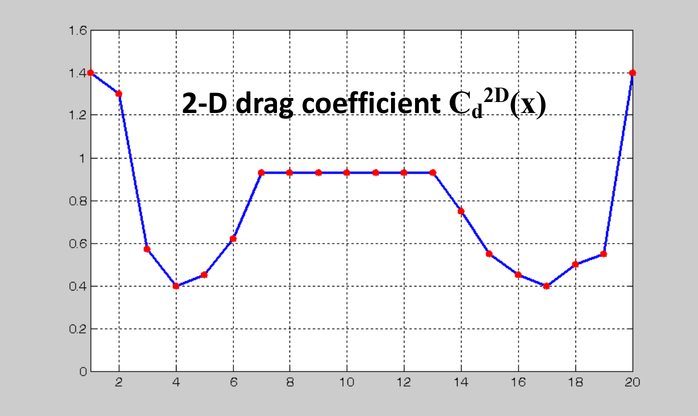
For relative current angles $|\beta_c - \psi| \gg 0$, where β_c is the current direction, the cross-flow drag principle may be applied to calculate the nonlinear damping force in sway and the yaw moment (Faltinsen 1990)

$$Y = -\frac{1}{2}\rho \int_{-\frac{L_{pp}}{2}}^{\frac{L_{pp}}{2}} T(x) C_d^{2D}(x) |v_r + xr|(v_r + xr)dx$$

$$N = -\frac{1}{2}\rho \int_{-\frac{L_{pp}}{2}}^{\frac{L_{pp}}{2}} T(x) C_d^{2D}(x) x |v_r + xr|(v_r + xr)dx$$

$$\begin{aligned} v_r &= v - v_c \\ &= v - V_c \sin(\beta_c - \psi) \end{aligned}$$

where $C_d^{2D}(x)$ is the 2-D sectional drag coefficient and $T(x)$ is the draft.



Strip theory means that the ship is cut in section along the x-axis. The integral is evaluated by summing up the contribution from each section

Faltinsen, O. M. (1990). Sea Loads on Ships and Offshore Structures. Cambridge University Press.

6.4 Cross-Flow Drag Principle

Hoerner's Approximation (2-D theory using constant drag coefficients)

$$Y = -\frac{1}{2}\rho T C_d^{2D} \int_{-\frac{L_{pp}}{2}}^{\frac{L_{pp}}{2}} |v_r + xr|(v_r + xr)dx$$

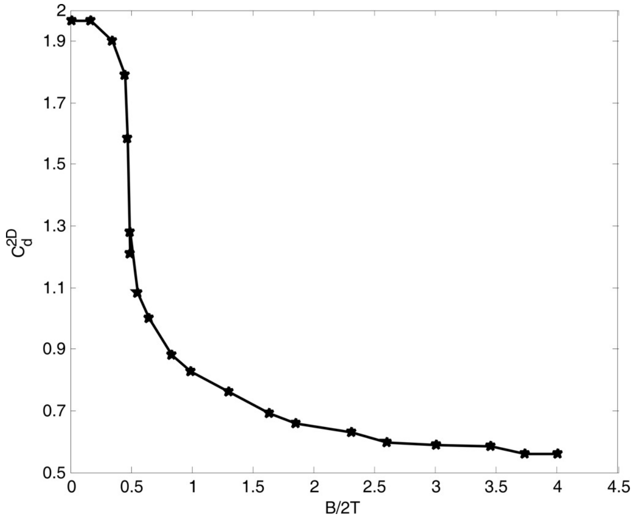
$$N = -\frac{1}{2}\rho T C_d^{2D} \int_{-\frac{L_{pp}}{2}}^{\frac{L_{pp}}{2}} x|v_r + xr|(v_r + xr)dx$$

where C_d^{2D} is the constant 2-D drag coefficient and T is the draft.

The 2-D drag coefficients C_d^{2D} can be computed as a function of beam B and length T using Hoerner's curve.

Matlab:
The 2-D drag coefficients C_d^{2D} can be computed as a function of beam B and length T using Hoerner's curve. This is implemented in the Matlab MSS toolbox as

```
Cd = Hoerner(B,T)
```



Dr. Sighard F. Hoerner (1906-1971) is the author of two famous compendiums in aerodynamics entitled, Fluid-Dynamic Drag and Fluid-Dynamic Lift.

MSS Toolbox

$$Y = -\frac{1}{2}\rho T C_d^{2D} \int_{-\frac{L_{pp}}{2}}^{\frac{L_{pp}}{2}} |v_r + xr|(v_r + xr)dx$$

$$N = -\frac{1}{2}\rho T C_d^{2D} \int_{-\frac{L_{pp}}{2}}^{\frac{L_{pp}}{2}} x|v_r + xr|(v_r + xr)dx$$

```
>> L = 100; B = 10; T = 8;
>> nu_r = [1 0.2 0 0 0 0]';
>> Cd_2D = Hoerner(B,T)
Cd_2D =
    1.0193
>> tau_crossflow = crossFlowDrag(L,B,T,nu_r)
tau_crossflow =
   1.0e+04 *
         0
   -1.7570
         0
         0
         0
   -0.0000
>>
```

```
function tau_crossflow = crossFlowDrag(L,B,T,nu_r)
% tau_crossflow = crossFlowDrag(L,B,T,nu_r) computes the cross-flow drag
% integrals for a marine craft using strip theory. Application:
%
%  M d/dt nu_r + C(nu_r)*nu_r + D*nu_r + g(eta) = tau + tau_crossflow
%
% Inputs: L:  length
%         B:  beam
%         T:  draft
%         nu_r = [u-u_c, v-v_c, w-w_c, p, q, r]': relative velocity vector
%
% Output: tau_crossflow = [0 Yh 0 0 0 Nh]:  cross-flow drag in sway and yaw
%
% Author:     Thor I. Fossen
% Date:       25 Apr 2021
% Revisions:

rho = 1026;               % density of water
n = 20;                   % number of strips

dx = L/20;
Cd_2D = Hoerner(B,T);   % 2D drag coefficient based on Hoerner's curve

Yh = 0;
Nh = 0;
for xL = -L/2:dx:L/2
    v_r = nu_r(2);             % relative sway velocity
    r = nu_r(6);               % yaw rate
    Ucf = abs(v_r + xL * r) * (v_r + xL * r);
    Yh = Yh - 0.5 * rho * T * Cd_2D * Ucf * dx;         % sway force
    Nh = Nh - 0.5 * rho * T * Cd_2D * xL * Ucf * dx;    % yaw moment
end

tau_crossflow = [0 Yh 0 0 0 Nh]';

end
```

6.4 Cross-Flow Drag Principle

3-D Theory (Curve Fitting to Maneuvering Coefficients)

A 3-D representation of

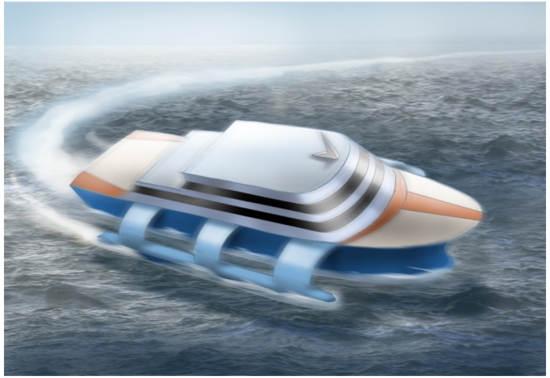
$$Y = -\frac{1}{2}\rho\int_{-\frac{L_{pp}}{2}}^{\frac{L_{pp}}{2}} T(x)C_d^{2D}(x)|v_r + xr|(v_r + xr)dx$$

$$N = -\frac{1}{2}\rho\int_{-\frac{L_{pp}}{2}}^{\frac{L_{pp}}{2}} T(x)C_d^{2D}(x)x|v_r + xr|(v_r + xr)dx$$

eliminating the integrals can be found by curve fitting these expressions to 2nd-order modulus terms to obtain a maneuvering model in sway and yaw (Fedyaevsky and Sobolev 1963)

$$Y = Y_{|v|v}|v_r|v_r + Y_{|v|r}|v_r|r + Y_{v|r|}v_r|r| + Y_{|r|r}|r|r$$

$$N = N_{|v|v}|v_r|v_r + N_{|v|r}|v_r|r + N_{v|r|}v_r|r| + N_{|r|r}|r|r$$



Fedyaevsky, K. K. and G. V. Sobolev (1963). Control and Stability in Ship Design. State Union Shipbuilding House.

6.5 Ship Maneuvering Models (3 DOFs)

Section 6.5 summarizes the linear and nonlinear maneuvering equations using the results in Sections 6.1–6.4.

$$\dot{\boldsymbol{\eta}} = \boldsymbol{R}(\psi)\boldsymbol{\nu}$$

$$\underbrace{\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{rigid-body forces}} + \underbrace{\boldsymbol{M}_{A}\dot{\boldsymbol{\nu}}_r + \boldsymbol{N}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r}_{\text{hydrodynamic forces}} = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}}$$

$$\boldsymbol{N}(\boldsymbol{\nu}_r) := \boldsymbol{C}_A(\boldsymbol{\nu}_r) + \boldsymbol{D} + \boldsymbol{D}_n(\boldsymbol{\nu}_r)$$

Several maneuvering models are included the MSS toolbox.

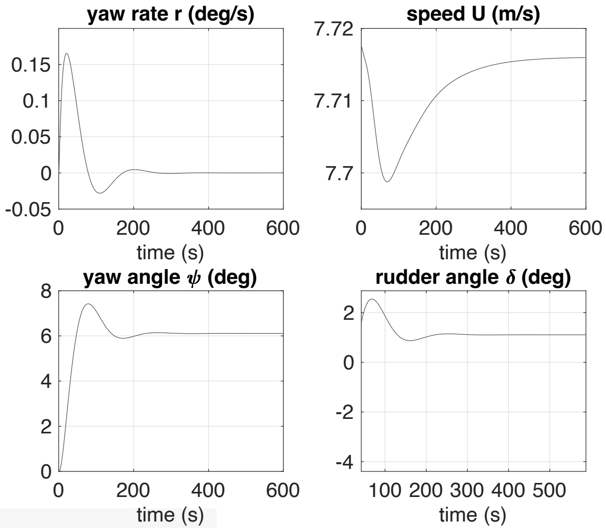
Matlab:
Several MSS maneuvering models are located under the toolbox catalog MSS/VESSELS:

```
[xdot,U] = container(x,tau)    % Container ship L = 175 m
[xdot,U] = Lcontainer(x,tau)   % Linearized container ship
[xdot,U] = mariner(x,ui,U0)    % Mariner class vessel L = 161 m
[xdot,U] = navalvessel(x,tau) % Naval vessel L = 51.5 m
[xdot,U] = tanker(x,ui)        % Tanker L = 304.8 m
```

The models can be simulated under feedback control using the user editable scripts

```
Sim<model name>.m
```

MSS Toolbox



Name

- documentation
- FDI
- GNC
- HYDRO
- mss_demos
  - Contents.m
  - GNCDemo.m
  - KinDemo.m
  - ManDemo.m
  - SimDemo.m
  - SimDemo1.m
  - SimDemo2.m
  - StabDemo.m
  - WaveDemo.m

```
% SIMDEMO1    User editable script for simulation of the
%              mariner class vessel under feedback control
%
% Calls:      mariner.m and euler2.m
%
% Author:     Thor I. Fossen
% Date:       19th July 2001
% Revisions:

t_f = 600;   % final simulation time (sec)
h   = 0.1;   % sample time (sec)

Kp = 1;      % controller P-gain
Td = 10;     % controller derivative time

% initial states:  x = [ u v r x y psi delta ]'
x = zeros(7,1);

% --- MAIN LOOP ---
N = round(t_f/h);                 % number of samples
xout = zeros(N+1,length(x)+2);    % memory allocation

for i=1:N+1,
    time = (i-1)*h;                         % simulation time in seconds

    r   = x(3);
    psi = x(6);

    % control system
    psi_ref = 5*(pi/180);                    % desired heading
    delta = -Kp*((psi-psi_ref)+Td*r);        % PD-controller

    % ship model
    [xdot,U] = mariner(x,delta);             % ship model, see .../gnc/VesselModels/

    % store data for presentation
    xout(i,:) = [time,x',U];

    % numerical integration
    x = euler2(xdot,x,h);                   % Euler integration
end
```

6.5 Nonlinear Maneuvering Equations

Model matrises

$$\boldsymbol{R}(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{M}_{RB} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & mx_g \\ 0 & mx_g & I_z \end{bmatrix}$$

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & -mr & -mx_g r \\ mr & 0 & 0 \\ mx_g r & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{M}_A = -\begin{bmatrix} X_{\dot{u}} & 0 & 0 \\ 0 & Y_{\dot{v}} & Y_{\dot{r}} \\ 0 & N_{\dot{v}} & N_{\dot{r}} \end{bmatrix} = \begin{bmatrix} A_{11}(0) & 0 & 0 \\ 0 & A_{22}(0) & A_{26}(0) \\ 0 & A_{26}(0) & A_{66}(0) \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{C}_A(\boldsymbol{\nu}_r) &= \begin{bmatrix} 0 & 0 & Y_{\dot{v}} v_r + Y_{\dot{r}} r \\ 0 & 0 & -X_{\dot{u}} u_r \\ -Y_{\dot{v}} v_r - Y_{\dot{r}} r & X_{\dot{u}} u_r & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -A_{22}(0) v_r - A_{26}(0) r \\ 0 & 0 & A_{11}(0) u_r \\ A_{22}(0) v_r + A_{26}(0) r & -A_{11}(0) u_r & 0 \end{bmatrix} \end{aligned}$$

$$\boldsymbol{D} = -\begin{bmatrix} X_u & 0 & 0 \\ 0 & Y_v & Y_r \\ 0 & N_v & N_r \end{bmatrix} \approx \begin{bmatrix} B_{11v}(0) & 0 & 0 \\ 0 & B_{22v}(0) & 0 \\ 0 & 0 & B_{66v}(0) \end{bmatrix}$$

$$\boldsymbol{D}_n(\boldsymbol{\nu}_r) = -\begin{bmatrix} X_{|u|u}|u_r| & 0 & 0 \\ 0 & Y_{|v|v}|v_r| + Y_{|r|v}|r| & Y_{|v|r}|v_r| + Y_{|r|r}|r| \\ 0 & N_{|v|v}|v_r| + N_{|r|v}|r| & N_{|v|r}|v_r| + N_{|r|r}|r| \end{bmatrix}$$

6.5 Nonlinear Maneuvering Equations

State-space model including ocean currents (relative velocity)

$$\dot{\boldsymbol{\eta}} = \boldsymbol{R}(\psi)\left(\boldsymbol{\nu}_r + \begin{bmatrix} u_c \\ v_c \\ 0 \end{bmatrix}\right)$$

$$\dot{\boldsymbol{\nu}}_r = \boldsymbol{M}^{-1}\left(\boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} - \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \boldsymbol{D}\boldsymbol{\nu}_r - \boldsymbol{D}_n(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r\right)$$

$$\begin{aligned} u_c &= V_c \cos(\beta_c - \psi) \\ v_c &= V_c \sin(\beta_c - \psi) \end{aligned}$$

$$\begin{aligned} \dot{u}_c &= rV_c \sin(\beta_c - \psi) = rv_c \\ \dot{v}_c &= -rV_c \cos(\beta_c - \psi) = -ru_c \end{aligned}$$

This model is based on the assumption that $C_{\text{RB}}(\boldsymbol{\nu})$ is parametrized independent of linear velocity (see Chapter 3)

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & -mr & -mx_g r \\ mr & 0 & 0 \\ mx_g r & 0 & 0 \end{bmatrix}$$

6.5 Nonlinear Maneuvering Equations

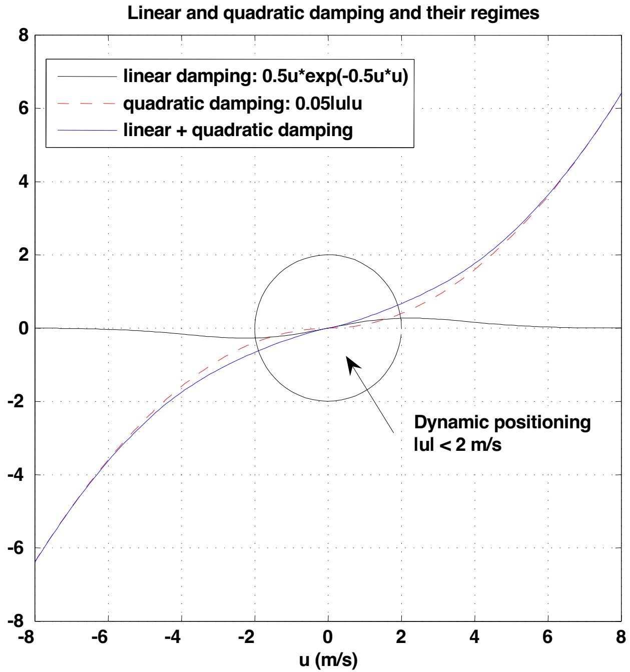
Speed Regimes – Linear and Nonlinear Damping

Dynamic Positioning (stationkeeping and low-speed maneuvering):

Linear damping dominates

Maneuvering (high speed):

Nonlinear damping dominates



The figure illustrates the significance of linear and quadratic damping for low-speed and high-speed applications.

6.5 Nonlinear Maneuvering Models based on Surge Resistance and Cross-Flow Drag

Expanding the 3-DOF N-matrix into three terms:

Added mass Coriolis-centripetal terms + linear damping + nonlinear damping

$$\boldsymbol{N}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r = \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}\boldsymbol{\nu}_r + \boldsymbol{d}(\boldsymbol{\nu}_r)$$

$$\mathbf{C}_A(\mathbf{v}_r) = \begin{bmatrix} 0 & 0 & Y_{\dot{v}} v_r + Y_{\dot{r}} r \\ 0 & 0 & X_{\dot{u}} u_r \\ -Y_{\dot{v}} v_r - Y_{\dot{r}} r & -X_{\dot{u}} u_r & 0 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} -X_u & 0 & 0 \\ 0 & -Y_v & -Y_r \\ 0 & -N_v & -N_r \end{bmatrix}$$

$$\mathbf{d}(\mathbf{v}_r) = \begin{bmatrix} \frac{1}{2}\rho S(1+k) C_f^{new}(u_r)|u_r|u_r \\ \frac{1}{2}\rho \int_{-L_{pp}/2}^{L_{pp}/2} T(x) C_d^{2D}(x)|v_r + xr|(v_r + xr)dx \\ \frac{1}{2}\rho \int_{-L_{pp}/2}^{L_{pp}/2} T(x) C_d^{2D}(x) x|v_r + xr|(v_r + xr)dx \end{bmatrix}$$

Added mass Coriolis-centripetal matrix

Linear damping

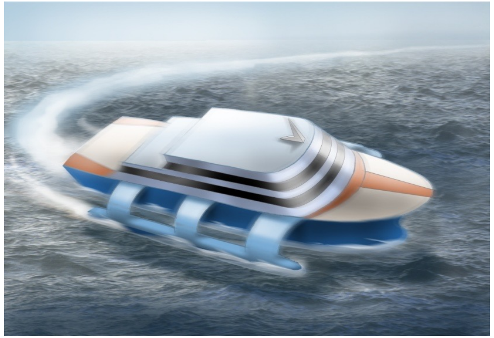
ITTC surge resistance + cross-flow drag

6.5 Nonlinear Maneuvering Models based on 2nd-order Modulus Functions

The idea of using 2nd-order modulus functions to describe the nonlinear terms in the N-matrix dates to Fedyaevsky and Sobolev (1963). This is motivated by first principles (quadratic drag).

A simplified form of Norrbin's (1970) nonlinear model which retains the most important terms for steering and propulsion loss assignment has been proposed by Blanke (1981). This model corresponds to fitting the cross-flow drag to 2nd-order modulus functions.

$$
\begin{aligned}
\mathbf{N}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r &= \mathbf{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \mathbf{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r \\
&= \begin{bmatrix} -Y_{\dot{v}} v_r r + Y_{\dot{r}} r^2 \\ X_{\dot{u}} u_r r \\ (Y_{\dot{v}} - X_{\dot{u}}) u_r v_r + Y_{\dot{r}} u_r r \end{bmatrix} \left(\text{alternatively:} \begin{bmatrix} X_{vr} v_r r + X_{rr} r^2 \\ Y_{ur} u_r r \\ N_{uv} u_r v_r + N_{ur} u_r r \end{bmatrix} \right) \\
&\quad + \begin{bmatrix} -X_{|u|u} |u_r| u_r \\ -Y_{|v|v} |v_r| v_r - Y_{|v|r} |v_r| r - Y_{v|r|} v_r |r| - Y_{|r|r} |r| r| \\ -N_{|v|v} |v_r| v_r - N_{|v|r} |v_r| r - N_{v|r|} v_r |r| - N_{|r|r} |r| r| \end{bmatrix}
\end{aligned}
$$



6.5 Nonlinear Maneuvering Models based on 2nd-order Modulus Functions

Maneuvering model in matrix form

$$
\mathbf{N}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r = \begin{bmatrix} -X_{|u|u}|u_r|u_r - Y_{\dot{v}} v_r r + Y_{\dot{r}} r^2 \\ X_{\dot{u}} u_r r - Y_{|v|v}|v_r|v_r - Y_{|v|r}|v_r|r - Y_{v|r|} v_r|r| - Y_{|r|r}|r|r \\ (Y_{\dot{v}} - X_{\dot{u}}) u_r v_r + Y_{\dot{r}} u_r r - N_{|v|v}|v_r|v_r - N_{|v|r}|v_r|r - N_{v|r|} v_r|r| - N_{|r|r}|r|r \end{bmatrix}
$$

$$
\mathbf{C}_A(\boldsymbol{\nu}_r) = \begin{bmatrix} 0 & 0 & Y_{\dot{v}} v_r + Y_{\dot{r}} r \\ 0 & 0 & X_{\dot{u}} u_r \\ -Y_{\dot{v}} v_r - Y_{\dot{r}} r & -X_{\dot{u}} u_r & 0 \end{bmatrix}
$$

$$
\mathbf{D}(\boldsymbol{\nu}_r) = \begin{bmatrix} -X_{|u|u}|u_r| & 0 & 0 \\ 0 & -Y_{|v|v}|v_r| - Y_{|r|v}|r| & -Y_{|v|r}|v_r| - Y_{|r|r}|r| \\ 0 & -N_{|v|v}|v_r| - N_{|r|v}|r| & -N_{|v|r}|v_r| - N_{|r|r}|r| \end{bmatrix}
$$

Recall that $\mathbf{D}(\boldsymbol{\nu}_r) = \mathbf{D} + \mathbf{D}_n(\boldsymbol{\nu}_r)$. Notice that linear potential damping and skin friction are neglected in this model ($\mathbf{D} = \mathbf{0}$) since the nonlinear quadratic terms $\mathbf{D}_n(\boldsymbol{\nu}_r)$ dominate at higher speeds

6.5 Nonlinear Maneuvering Models based on Odd Functions

Experimental data can be curve fitted to Taylor series of 1st and 3rd order (odd functions).

Consider the nonlinear rigid-body kinetics:

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

$$\mathbf{x} = [u, v, r, \dot{u}, \dot{v}, \dot{r}, \delta]^\top$$

$$\boldsymbol{\tau}_{RB} = \begin{bmatrix} X(\mathbf{x}) \\ Y(\mathbf{x}) \\ N(\mathbf{x}) \end{bmatrix}$$

3rd-order truncated Taylor-series expansion about $\mathbf{x}_o = [U, 0, 0, 0, 0, 0, 0]^\top$

$$X(\mathbf{x}) \approx X(\mathbf{x}_0) + \sum_{i=1}^{n} \left(\frac{\partial X(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}_0} \Delta x_i + \frac{1}{2} \frac{\partial^2 X(\mathbf{x})}{(\partial x_i)^2}\bigg|_{\mathbf{x}_0} \Delta x_i^2 + \frac{1}{6} \frac{\partial^3 X(\mathbf{x})}{(\partial x_i)^3}\bigg|_{\mathbf{x}_0} \Delta x_i^3 \right)$$

$$Y(\mathbf{x}) \approx Y(\mathbf{x}_0) + \sum_{i=1}^{n} \left(\frac{\partial Y(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}_0} \Delta x_i + \frac{1}{2} \frac{\partial^2 Y(\mathbf{x})}{(\partial x_i)^2}\bigg|_{\mathbf{x}_0} \Delta x_i^2 + \frac{1}{6} \frac{\partial^3 Y(\mathbf{x})}{(\partial x_i)^3}\bigg|_{\mathbf{x}_0} \Delta x_i^3 \right)$$

$$N(\mathbf{x}) \approx Z(\mathbf{x}_0) + \sum_{i=1}^{n} \left(\frac{\partial N(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}_0} \Delta x_i + \frac{1}{2} \frac{\partial^2 N(\mathbf{x})}{(\partial x_i)^2}\bigg|_{\mathbf{x}_0} \Delta x_i^2 + \frac{1}{6} \frac{\partial^3 N(\mathbf{x})}{(\partial x_i)^3}\bigg|_{\mathbf{x}_0} \Delta x_i^3 \right)$$

$$\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0 = [\Delta x_1, \Delta x_2, \ldots \Delta x_n]^\top$$

Abkowitz, M. A. (1964). Lectures on Ship Hydrodynamics – Steering and Maneuverability. Technical Report Hy-5. Hydro- and Aerodynamics Laboratory. Denmark Technical University, Lyngby, Denmark.

6.5 Nonlinear Maneuvering Models based on Odd Functions

Unfortunately, a 3rd-order Taylor series expansion results in a large number of terms. By applying some physical insight, the complexity of these expressions can be reduced.

Assumptions:

1. Most ship maneuvers can be described with a 3rd-order truncated Taylor expansion about the steady state condition $u = u_0$.
2. Only 1st-order acceleration terms are considered.
3. Standard port/starboard symmetry simplifications except terms describing the constant force and moment arising from single-screw propellers.
4. The coupling between the acceleration and velocity terms is negligible.

$$
\begin{aligned}
X &= X^* + X_{\dot{u}}\dot{u} + X_u\Delta u + X_{uu}\Delta u^2 + X_{uuu}\Delta u^3 + X_{vv}v^2 + X_{rr}r^2 + X_{\delta\delta}\delta^2 \\
&\quad + X_{rv\delta}rv\delta + X_{r\delta}r\delta + X_{v\delta}v\delta + X_{vvu}v^2\Delta u + X_{rru}r^2\Delta u + X_{\delta\delta u}\delta^2\Delta u \\
&\quad + X_{rvu}rvu + X_{r\delta u}r\delta\Delta u + X_{v\delta u}v\delta\Delta u \\
Y &= Y^* + Y_u\Delta u + Y_{uu}\Delta u^2 + Y_r r + Y_v v + Y_{\dot{r}}\dot{r} + Y_{\dot{v}}\dot{v} + Y_\delta\delta + Y_{rrr}r^3 + Y_{vvv}v^3 \\
&\quad + Y_{\delta\delta\delta}\delta^3 + Y_{rr\delta}r^2\delta + Y_{\delta\delta r}\delta^2 r + Y_{rrv}r^2 v + Y_{vvr}v^2 r + Y_{\delta\delta v}\delta^2 v + Y_{vv\delta}v^2\delta + Y_{\delta vr}\delta vr \\
&\quad + Y_{vu}v\Delta u + Y_{vuu}v\Delta u^2 + Y_{ru}r\Delta u + Y_{ruu}r\Delta u^2 + Y_{\delta u}\delta\Delta u + Y_{\delta uu}\delta\Delta u^2 \\
N &= N^* + N_u\Delta u + N_{uu}\Delta u^2 + N_r r + N_v v + N_{\dot{r}}\dot{r} + N_{\dot{v}}\dot{v} + N_\delta\delta + N_{rrr}r^3 + N_{vvv}v^3 \\
&\quad + N_{\delta\delta\delta}\delta^3 + N_{rr\delta}r^2\delta + N_{\delta\delta r}\delta^2 r + N_{rrv}r^2 v + N_{vvr}v^2 r + N_{\delta\delta v}\delta^2 v + N_{vv\delta}v^2\delta \\
&\quad + N_{\delta vr}\delta vr + N_{vu}v\Delta u + N_{vuu}v\Delta u^2 + N_{ru}r\Delta u + N_{ruu}r\Delta u^2 + N_{\delta u}\delta\Delta u \\
&\quad + N_{\delta uu}\delta\Delta u^2
\end{aligned}
$$

$$F \in \{X, Y, N\}$$

$$F^* = F(\mathbf{x}_0)$$

$$F_{x_i} = \left.\frac{\partial F(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}_0}$$

$$F_{x_i x_j} = \frac{1}{2}\left.\frac{\partial^2 F(\mathbf{x})}{\partial x_i \partial x_j}\right|_{\mathbf{x}_0}$$

$$F_{x_i x_j x_k} = \frac{1}{6}\left.\frac{\partial^3 F(\mathbf{x})}{\partial x_i \partial x_j \partial x_k}\right|_{\mathbf{x}_0}$$

6.5 Taylor Series Cubic Curves or 2nd-order Modulus Curves

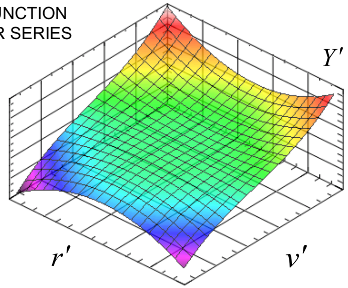
Taylor Series Cubic Curves

For a symmetric ship hydrodynamic force Y' and moment N' can be modelled by 1st and 3rd order terms using an odd function

$$Y' = Y'_v v' + Y'_r r' + Y'_{vvv} v'^3 + Y'_{vvr} v'^2 r' + Y'_{vrr} v' r'^2 + Y'_{rrr} r'^3$$

$$N' = N'_v v' + N'_r r' + N'_{vvv} v'^3 + N'_{vvr} v'^2 r' + N'_{vrr} v' r'^2 + N'_{rrr} r'^3$$

gives a smooth surface (physical)

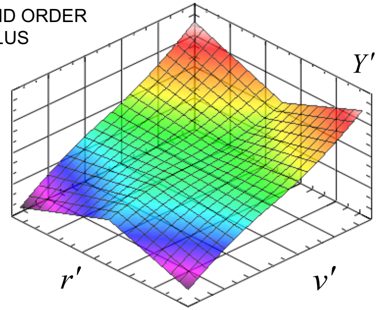


2nd-order Modulus Curves

$$Y' = Y'_v v' + Y'_r r' + Y'_{v|v|} v'|v'| + Y'_{v|r|} v'|r'| + Y'_{|v|r} |v'| r' + Y'_{r|r|} r'|r'|$$

$$N' = N'_v v' + N'_r r' + N'_{v|v|} v'|v'| + N'_{v|r|} v'|r'| + N'_{|v|r} |v'| r' + N'_{r|r|} r'|r'|$$

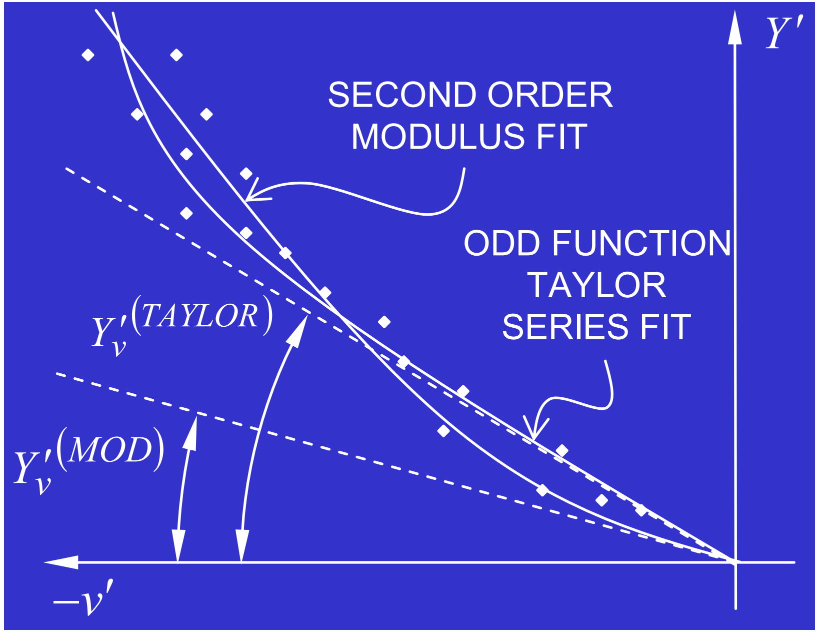
gives a flatter facetted surface (nonphysical)



6.5 Taylor Series Cubic Curves or 2nd-order Modulus Curves

We obtain different derivative values from cubic and 2nd-order modulus curve fit.

This is a big problem when comparing derivatives from different experimental facilities



6.5 Linearized Maneuvering Equations

The linearized maneuvering equations in surge, sway and yaw are

$$\underbrace{(\boldsymbol{M}_{RB}+\boldsymbol{M}_A)}_{\boldsymbol{M}}\dot{\boldsymbol{\nu}}_r+\underbrace{(\boldsymbol{C}^*_{RB}+\boldsymbol{C}^*_A+\boldsymbol{D})}_{\boldsymbol{N}}\boldsymbol{\nu}_r=\boldsymbol{\tau}+\boldsymbol{\tau}_{\text{wind}}+\boldsymbol{\tau}_{\text{wave}}$$

$$\boldsymbol{C}^*_{RB}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & mU\\ 0 & 0 & mx_gU\end{bmatrix}$$

$$\boldsymbol{C}^*_A=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -X_{\dot u}U\\ 0 & (X_{\dot u}-Y_{\dot v})U & -Y_{\dot r}U\end{bmatrix}$$

$$\boldsymbol{D}=-\begin{bmatrix}X_u & 0 & 0\\ 0 & Y_v & Y_r\\ 0 & N_v & N_r\end{bmatrix}$$

Linearization is performed under the assumption that the unknown current velocities are negligible, that is $u_c = v_c = 0$

$$\begin{bmatrix}m-X_{\dot u} & 0 & 0\\ 0 & m-Y_{\dot v} & mx_g-Y_{\dot r}\\ 0 & mx_g-N_{\dot v} & I_z-N_{\dot r}\end{bmatrix}\begin{bmatrix}\dot u_r\\ \dot v_r\\ \dot r\end{bmatrix}$$
$$+\begin{bmatrix}-X_u & 0 & 0\\ 0 & -Y_v & (m-X_{\dot u})U-Y_r\\ 0 & (X_{\dot u}-Y_{\dot v})U-N_v & (mx_g-Y_{\dot r})U-N_r\end{bmatrix}\begin{bmatrix}u_r\\ v_r\\ r\end{bmatrix}=\begin{bmatrix}\tau_1\\ \tau_2\\ \tau_6\end{bmatrix}$$

6.5 Linearized Maneuvering Equations

Surge Subsystem

$$(m - X_{\dot{u}})\dot{u}_r - X_u u_r = \tau_1$$

$$X_u = \frac{8A_1}{3\pi} X_{|u|u}$$

Sway-Yaw Subsystem

$$\boldsymbol{M}\dot{\boldsymbol{\nu}}_r + \boldsymbol{N}\boldsymbol{\nu}_r = \boldsymbol{b}\delta$$

$$\boldsymbol{M} = \begin{bmatrix} m - Y_{\dot{v}} & m x_g - Y_{\dot{r}} \\ m x_g - Y_{\dot{r}} & I_z - N_{\dot{r}} \end{bmatrix}$$

$$\boldsymbol{N} = \begin{bmatrix} -Y_v & (m - X_{\dot{u}})U - Y_r \\ (X_{\dot{u}} - Y_{\dot{v}})U - N_v & (m x_g - Y_{\dot{r}})U - N_r \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} -Y_\delta \\ -N_\delta \end{bmatrix}$$

$$\boldsymbol{C}_A^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -X_{\dot{u}}U \\ 0 & (X_{\dot{u}} - Y_{\dot{v}})U & -Y_{\dot{r}}U \end{bmatrix}$$

Notice: $C_A(\nu)$ includes the famous destabilizing Munk moment (from aerodynamics) and some other C_A-terms

6.6 Maneuvering Models including Roll (4 DOFs)

4-DOF models in Surge, Sway, Roll, and Yaw

The speed equation can be decoupled from the sway, roll, and yaw modes

$$u_o = \text{constant}$$

The resulting linear model takes the form

$$\dot{\phi} = p$$
$$\dot{\psi} = r$$

$$\boldsymbol{M}\dot{\boldsymbol{\nu}}_r + \boldsymbol{N}\boldsymbol{\nu}_r + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{\tau}$$

$$\boldsymbol{\nu} = [v, p, r]^\top$$
$$\boldsymbol{\eta} = [y^n, \phi, \psi]^\top$$

$$\mathbf{M} = \begin{bmatrix} m - Y_{\dot{v}} & -mz_g - Y_{\dot{p}} & mx_g - Y_{\dot{r}} \\ -mz_g - K_{\dot{v}} & I_x - K_{\dot{p}} & -I_{xz} - N_{\dot{p}} \\ mx_g - N_{\dot{v}} & -I_{xz} - N_{\dot{p}} & I_z - N_{\dot{r}} \end{bmatrix} \quad \mathbf{N}(u_o) = \begin{bmatrix} -Y_v & -Y_p & mu_0 - Y_r \\ -K_v & -K_p & -mz_g u_0 - K_r \\ -N_v & -N_p & mx_g u_0 - N_r \end{bmatrix}$$

$$\boldsymbol{G} = \text{diag}\{0, W\text{GM}_T, 0\}$$

6.6 Maneuvering Models including Roll (4 DOFs)

3-DOF and 4-DOF Maneuvering Models

The maneuvering models presented in Section 6.5 only describe the horizontal motions (surge, sway, and yaw) under a zero-frequency assumption. These models are intended for the design and simulation of DP systems, heading autopilots, trajectory-tracking and path-following control systems.

Many vessels, however, are equipped with actuators that can reduce the rolling motion. This could be anti-rolling tanks, rudders and fin stabilizers. In order to design a control system for roll damping, it is necessary to augment the roll equation to the horizontal plane model. Inclusion of roll means that the restoring moment due to buoyancy and gravity must be included.

The resulting model is a 4-DOF maneuvering model (surge, sway, roll, and yaw) that includes the rolling motion.

6.6 Maneuvering Models including Roll (4 DOFs)

State-Space Model

The linear model can be written in state-space model according to

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \mathbf{x} + \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ b_{31} & b_{32} & \cdots & b_{3r} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{\mathbf{B}} \mathbf{u}$$

$$\mathbf{x} = [v, p, r, \phi, \psi]^{\top}$$

Roll and yaw outputs

$$\phi = [0, 0, 0, 1, 0]\mathbf{x}$$
$$\psi = [0, 0, 0, 0, 1]\mathbf{x}$$

where the matrix elements are

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = -\mathbf{M}^{-1}\mathbf{N}(u_o)$$

$$\begin{bmatrix} * & a_{14} & * \\ * & a_{24} & * \\ * & a_{34} & * \end{bmatrix} = -\mathbf{M}^{-1}\mathbf{G}$$

b_{ij} are given by $\mathbf{B} = \mathbf{M}^{-1}\mathbf{T}\mathbf{K}$

$$\boldsymbol{\tau} = \mathbf{T}\mathbf{K}\mathbf{u}$$

6.6 Maneuvering Models including Roll (4 DOFs)

Decompositions in Roll and Sway-Yaw Subsystems

The state variables associated with steering and roll can be separated

$$\begin{bmatrix} \dot{v} \\ \dot{r} \\ \dot{\psi} \\ \hline \dot{p} \\ \dot{\phi} \end{bmatrix} = \left[\begin{array}{ccc|cc} a_{11} & a_{13} & 0 & a_{12} & a_{14} \\ a_{31} & a_{33} & 0 & a_{32} & a_{34} \\ 0 & 1 & 0 & 0 & 0 \\ \hline a_{21} & a_{23} & 0 & a_{22} & a_{24} \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} v \\ r \\ \psi \\ \hline p \\ \phi \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{31} & b_{32} & \cdots & b_{3r} \\ 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & b_{2r} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{u}$$

$$\begin{bmatrix} \dot{\mathbf{x}}_\psi \\ \dot{\mathbf{x}}_\phi \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{\psi\psi} & \mathbf{A}_{\psi\phi} \\ \mathbf{A}_{\phi\psi} & \mathbf{A}_{\phi\phi} \end{bmatrix} \begin{bmatrix} \mathbf{x}_\psi \\ \mathbf{x}_\phi \end{bmatrix} + \begin{bmatrix} \mathbf{B}_\psi \\ \mathbf{B}_\phi \end{bmatrix} \mathbf{u}$$

$$\mathbf{x}_\psi = [v, r, \psi]^\top$$

$$\mathbf{x}_\phi = [p, \phi]^\top$$

If the coupling matrices are small, we get:

$$\begin{bmatrix} \dot{v} \\ \dot{r} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{13} & 0 \\ a_{31} & a_{33} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ r \\ \psi \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{31} & b_{32} & \cdots & b_{3r} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{u}$$

Decoupled steering model (sway-yaw)

$$\begin{bmatrix} \dot{p} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} a_{22} & a_{24} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p \\ \phi \end{bmatrix} + \begin{bmatrix} b_{21} & b_{22} & \cdots & b_{2r} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{u}$$

Decoupled roll model

6.6 Maneuvering Models including Roll (4 DOFs)

Transfer Functions for Steering and Rudder-Roll Damping

The transfer functions for the decoupled models are

$$\frac{\phi}{\delta}(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \approx \frac{K_{\mathrm{roll}}\, \omega_{\mathrm{roll}}^2\, (1 + T_5 s)}{(1 + T_4 s)(s^2 + 2\zeta \omega_{\mathrm{roll}} s + \omega_{\mathrm{roll}}^2)}$$

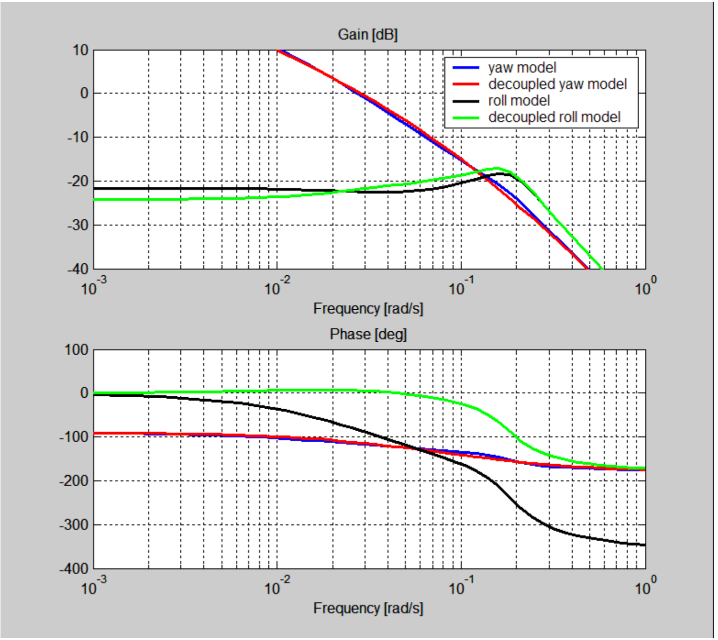
$$\frac{\psi}{\delta}(s) = \frac{c_3 s^3 + c_2 s^2 + c_1 s + c_0}{s(s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0)} \approx \frac{K_{\mathrm{yaw}}\, (1 + T_3 s)}{s(1 + T_1 s)(1 + T_2 s)}$$

This can be a rough approximation since interactions are neglected.

In particular, the roll mode is inaccurate as seen from the plots.

Matlab MSS toolbox: ExRRD1.m
Lcontainer.m

Plot showing reduced-order models and total model



Container ship: Son and Nomoto (1981)

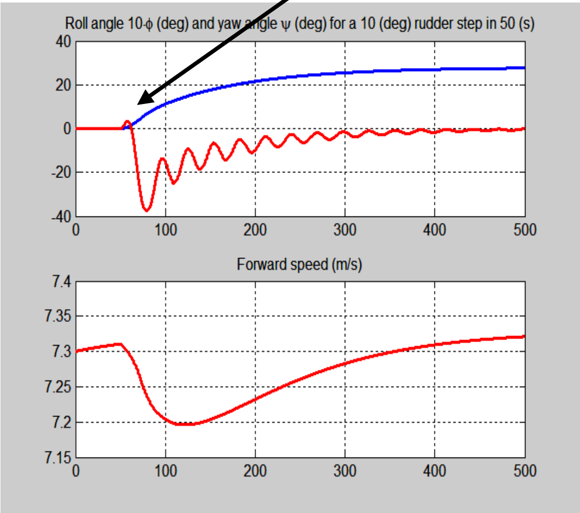
Son, K. H. and K. Nomoto (1981). On the Coupled Motion of Steering and Rolling of a High Speed Container Ship. Naval Architect of Ocean Engineering 20, 73–83. From J.S.N.A., Japan, Vol. 150, 1981.

6.6 Maneuvering Models including Roll (4 DOFs)

Length: L = 175 (m)
Displacement: 21,222 (m³)
Service speed u_0 = 7.0 (m/s)

Right-half-plane zero: $z = 0.036$ (rad/s)

Non-minimum phase property



$$\frac{\phi}{\delta}(s) = \frac{0.0032(s-0.036)(s+0.077)}{(s+0.026)(s+0.116)(s^2+0.136s+0.036)} \approx \frac{0.083(1+49.1s)}{(1+31.5s)(s^2+0.134s+0.033)}$$

$$\omega_{\text{roll}} = 0.189 \text{ (rad/s)}$$

$$\zeta = 0.36$$

$$\frac{\psi}{\delta}(s) = \frac{0.0024(s+0.0436)(s^2+0.162s+0.035)}{s(s+0.0261)(s+0.116)(s^2+0.136s+0.036)} \approx \frac{0.032(1+16.9s)}{s(1+24.0s)(1+9.2s)}$$

Matlab MSS toolbox*:* ExRRD3.m

6.6 The Nonlinear Model of Son and Nomoto

High-speed container ship given by

$$
\begin{aligned}
(m+m_x)\dot{u} - (m+m_y)vr &= X + \tau_1 \\
(m+m_y)\dot{v} + (m+m_x)ur + m_y\alpha_y\dot{r} - m_y l_y \dot{p} &= Y + \tau_2 \\
(I_x + J_x)\dot{p} - m_y l_y \dot{v} - m_x l_x ur &= K - W\mathrm{GM}_T\phi + \tau_4 \\
(I_z + J_z)\dot{r} + m_y\alpha_y\dot{v} &= N - x_g Y + \tau_6
\end{aligned}
$$

$$
\begin{aligned}
X &= X(u) + (1-t)T + X_{vr}vr + X_{vv}v^2 + X_{rr}r^2 + X_{\phi\phi}\phi^2 \\
&\quad + X_\delta \sin\delta + X_{\text{ext}} \\
Y &= Y_v v + Y_r r + Y_\phi \phi + Y_p p + Y_{vvv}v^3 + Y_{rrr}r^3 + Y_{vvr}v^2 r + Y_{vrr}vr^2 \\
&\quad + Y_{vv\phi}v^2\phi + Y_{v\phi\phi}v\phi^2 + Y_{rr\phi}r^2\phi + Y_{r\phi\phi}r\phi^2 + Y_\delta\cos\delta + Y_{\text{ext}} \\
K &= K_v v + K_r r + K_\phi \phi + K_p p + K_{vvv}v^3 + K_{rrr}r^3 + K_{vvr}v^2 r + K_{vrr}vr^2 \\
&\quad + K_{vv\phi}v^2\phi + K_{v\phi\phi}v\phi^2 + K_{rr\phi}r^2\phi + K_{r\phi\phi}r\phi^2 + K_\delta\cos\delta + K_{\text{ext}} \\
N &= N_v v + N_r r + N_\phi \phi + N_p p + N_{vvv}v^3 + N_{rrr}r^3 + N_{vvr}v^2 r + N_{vrr}vr^2 \\
&\quad + N_{vv\phi}v^2\phi + N_{v\phi\phi}v\phi^2 + N_{rr\phi}r^2\phi + N_{r\phi\phi}r\phi^2 + N_\delta\cos\delta + N_{\text{ext}}
\end{aligned}
$$

3rd-order Taylor-series expansion

Son, K. H. and K. Nomoto (1981). On the Coupled Motion of Steering and Rolling of a High Speed Container Ship. Naval Architect of Ocean Engineering 20, 73–83. From J.S.N.A., Japan, Vol. 150, 1981.

6.6 The Nonlinear Model of Son and Nomoto

Matlab:
The nonlinear container ship model is implemented in the MSS toolbox as

```
[xdot,U] = container(x,ui)
```

The linearized model for $U = U_0$ is accessed as

```
[xdot,U] = Lcontainer(x,ui,U0)
```

where `x = [u v r x y psi p phi delta]'` and `ui = [delta_c n_c]'`. In the linear case only one input, `delta_c`, is used since the forward speed `U0` is constant. For the nonlinear model, propeller rpm, `n_c`, should be positive.

Son, K. H. and K. Nomoto (1981). On the Coupled Motion of Steering and Rolling of a High Speed Container Ship. Naval Architect of Ocean Engineering 20, 73–83. From J.S.N.A., Japan, Vol. 150, 1981.

6.6 The Nonlinear Model of Blanke and Christensen

An alternative model formulation describing the steering and roll motions of ships has been proposed by Blanke and Christensen (1993).

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{\tau}_{\text{hyd}} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} + \boldsymbol{\tau}$$

$$\boldsymbol{M} = \begin{bmatrix} m - Y_{\dot{v}} & -mz_g - Y_{\dot{p}} & mx_g - Y_{\dot{r}} \\ -mz_g - K_{\dot{v}} & I_x - K_{\dot{p}} & 0 \\ mx_g - N_{\dot{v}} & 0 & I_z - N_{\dot{r}} \end{bmatrix}$$

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & 0 & mu \\ 0 & 0 & 0 \\ -mu & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{G} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & W\mathrm{GM}_T & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\nu} = [v, p, r]^\top$$

$$\boldsymbol{\tau}_{\text{hyd}} = [Y, K, N]^\top$$

Matlab:
A nonlinear naval ship model is implemented in the MSS toolbox as

```
[xdot,U] = navalvessel(x,tau)
```

where `x = [u v p r phi psi]'` and `tau = [X_ext Y_ext K_ext N_ext]'`.

2nd-order modulus model

$$\begin{aligned}
Y &= Y_{|u|v}|u|v + Y_{ur}ur + Y_{v|v|}|v|v + Y_{v|r|}v|r| + Y_{|v|r}|v|r \\
&\quad + Y_{\phi|uv|}\phi|uv| + Y_{\phi|ur|}\phi|ur| + Y_{\phi uu}\phi u^2 + Y_{\text{ext}} \\
K &= K_{|u|v}|u|v + K_{ur}ur + K_{v|v|}|v|v + K_{v|r|}v|r| + K_{|v|r}|v|r \\
&\quad + K_{\phi|uv|}\phi|uv| + K_{\phi|ur|}\phi|ur| + K_{\phi uu}\phi u^2 + K_{|u|p}|u|p \\
&\quad + K_{p|p|}p|p| + K_p p + K_{\phi\phi\phi}\phi^3 + K_{\text{ext}} \\
N &= N_{|u|v}|u|v + N_{|u|r}|u|r + N_{r|r|}|r|r + N_{v|r|}v|r| + N_{|v|r}|v|r \\
&\quad + N_{\phi|uv|}\phi|uv| + N_{\phi|ur|}\phi|ur| + N_p p + N_{|p|p}|p|p + N_{|u|p}|u|p \\
&\quad + N_{\phi u|u|}\phi|u|u| + N_{\text{ext}}
\end{aligned}$$

Blanke, M. and A. Christensen (1993). Rudder-Roll Damping Autopilot Robustness due to Sway-Yaw-Roll Couplings. 10th International Ship Control Systems Symposium (SCSS'93). Ottawa, Canada, pp. A.93–A.119.

6.7 Low-Speed Maneuvering Models for Dynamic Positioning (3 DOFs)

Speed Regimes – Hydrodynamic Methods (see Chapters 5 and 6)

Dynamic Positioning Models

- 3-D potential theory
- 2-D potential theory (strip theory)

Maneuvering Models

- 2-D potential theory (strip theory) up to *Froude numbers* of 0-3-0.4
- 2.5-D potential theory for high-speed craft



Low-speed maneuvering

Stationkeeping

Maneuvering at moderate speed (transit)

Maneuvering at high speed (high-speed craft)

Speed

0   1.5 m/s (3 knots)   …. $U = 0.3\sqrt{Lg}$ …..

6.7 Low-Speed Maneuvering Models for Dynamic Positioning (3 DOFs)

For low-speed applications such as DP, current forces and damping can be described by three area-based current coefficients C_X, C_Y and C_N. These can be experimentally obtained using scale models in wind tunnels.

The resulting forces are measured on the model which is restrained from moving ($U = 0$). The current coefficients can also be related to the surge resistance, cross-flow drag and the *Munk moment.*

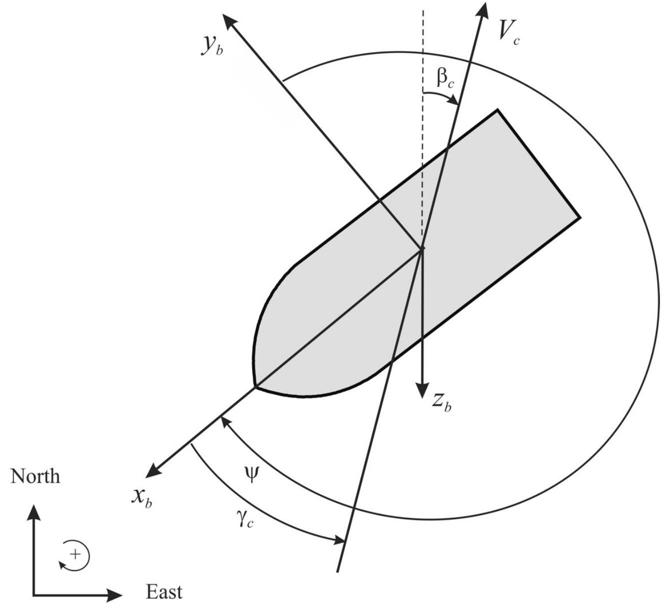
For a ship moving at forward speed $U > 0$, quadratic damping will be included in the current coefficients using relative speed.

Zero-Speed Representation

$$
\begin{aligned}
X_{\text{current}} &= \frac{1}{2}\rho A_{Fc} C_X(\gamma_c) V_c^2 \\
Y_{\text{current}} &= \frac{1}{2}\rho A_{Lc} C_Y(\gamma_c) V_c^2 \\
N_{\text{current}} &= \frac{1}{2}\rho A_{Lc} L_{oa} C_N(\gamma_c) V_c^2
\end{aligned}
$$

$$\gamma_c = \psi - \beta_c - \pi$$

$$
\begin{aligned}
u_c &= -V_c \cos(\gamma_c) \\
v_c &= V_c \sin(\gamma_c)
\end{aligned}
$$



γ_c Current direction (counter clock-wise rotation)
A_{Fc} Frontal projected current area
A_{Lc} Lateral projected current area
L_{oa} Length over all

6.7 Low-Speed Maneuvering Models for Dynamic Positioning (3 DOFs)

Forward Speed Representation

$$
\begin{aligned}
X_{\text{current}} &= \frac{1}{2}\rho A_{Fc} C_X(\gamma_{rc}) V_{rc}^2 \\
Y_{\text{current}} &= \frac{1}{2}\rho A_{Lc} C_Y(\gamma_{rc}) V_{rc}^2 \\
N_{\text{current}} &= \frac{1}{2}\rho A_{Lc} L_{oa} C_N(\gamma_{rc}) V_{rc}^2
\end{aligned}
$$

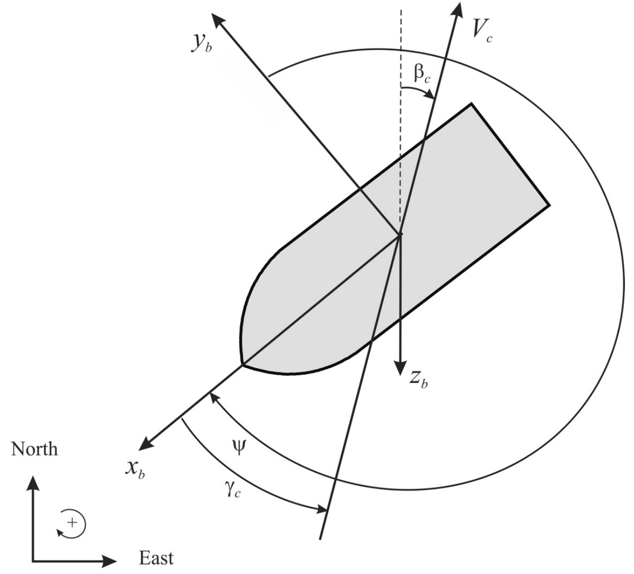
Relative speed and direction

$$
\begin{aligned}
V_{rc} &= \sqrt{u_{rc}^2 + v_{rc}^2} = \sqrt{(u-u_c)^2 + (v-v_c)^2} \\
\gamma_{rc} &= -atan2(v_{rc}, u_{rc})
\end{aligned}
$$

$$
\begin{aligned}
u_c &= V_c \cos(\beta_c - \psi) \\
v_c &= V_c \sin(\beta_c - \psi)
\end{aligned}
$$

The forward speed representation introduces quadratic damping using the concept of relative velocity.

This is **NOT** the case for the zero-speed representation.



6.7 Nonlinear DP Model using Current Coefficients

Relationship Between Current Coefficients and Surge Resistance/Cross-Flow Drag

The current coefficients can be related to the surge resistance and cross-flow drag coefficients by assuming low speed such that $u \approx 0$ and $v \approx 0$. This is a good assumption for DP.

$$
\begin{aligned}
u_r\,|u_r| &\approx -u_c\,|-u_c| \\
&= V_c^2\cos(\gamma_c)|\cos(\gamma_c)| \\
v_r\,|v_r| &\approx -v_c\,|-v_c| \\
&= -V_c^2\sin(\gamma_c)|\sin(\gamma_c)| \\
u_r v_r &\approx u_c v_c \\
&= -\frac{1}{2}V_c^2\sin(2\gamma_c)
\end{aligned}
$$

$$u = v = r = 0$$

$$
\begin{aligned}
X_{\text{current}} &= \frac{1}{2}\rho A_{Fc} C_X(\gamma_c) V_c^2 := X_{|u|u}\,|u_r|\,u_r \\
Y_{\text{current}} &= \frac{1}{2}\rho A_{Lc} C_Y(\gamma_c) V_c^2 := Y_{|v|v}|v_r|v_r \\
N_{\text{current}} &= \frac{1}{2}\rho A_{Lc} L_{oa} C_N(\gamma_c) V_c^2 := N_{|v|v}|v_r|v_r - \underbrace{(X_{\dot{u}} - Y_{\dot{v}})u_r v_r}_{\text{Munk moment}}
\end{aligned}
$$

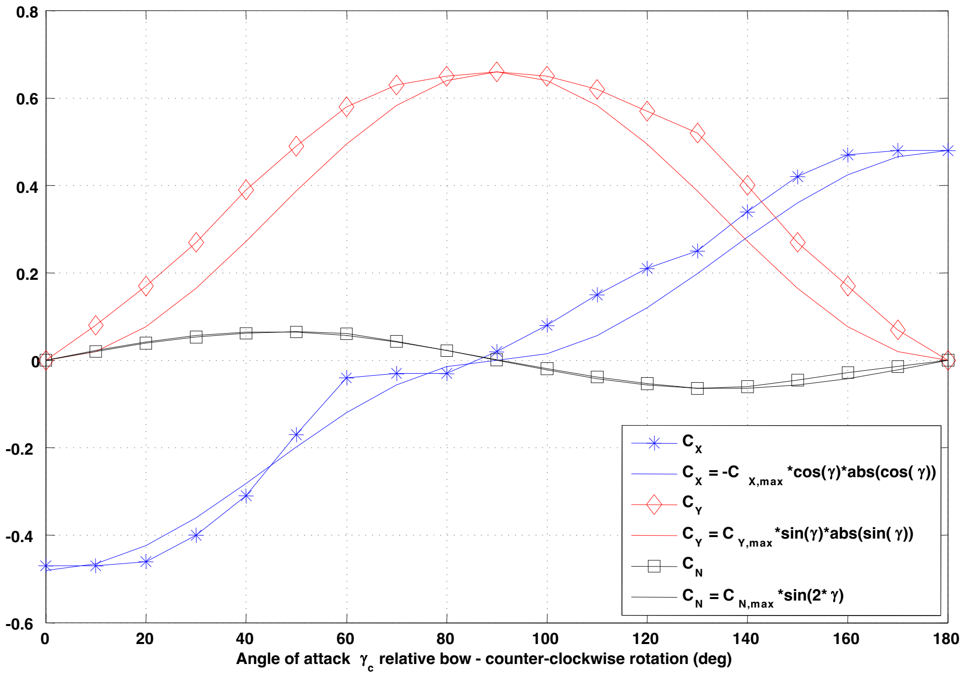
This gives the following analytical expressions for the area based current coefficients like Faltinsen (1990), pp. 187-188.

$$
\boxed{
\begin{aligned}
C_X(\gamma_c) &= -2\left(\frac{-X_{|u|u}}{\rho A_{Fc}}\right)\cos(\gamma_c)|\cos(\gamma_c)| \\
C_Y(\gamma_c) &= 2\left(\frac{-Y_{|v|v|}}{\rho A_{Lc}}\right)\sin(\gamma_c)|\sin(\gamma_c)| \\
C_N(\gamma_c) &= \frac{2}{\rho A_{Lc} L_{oa}}\left(-N_{|v|v}\sin(\gamma_c)|\sin(\gamma_c)| + \frac{1}{2}\underbrace{(X_{\dot{u}} - Y_{\dot{v}})}_{A_{22}-A_{11}}\sin(2\gamma_c)\right)
\end{aligned}
}
$$

Faltinsen, O. M. (1990). Sea Loads on Ships and Offshore Structures. Cambridge University Press.

6.7 Nonlinear DP Model using Current Coefficients

Current Coefficients



Experimental current coefficients for a tanker compared with analytical formulas.

6.7 Nonlinear DP Model using Current Coefficients

Nonlinear DP Model Based on Current Coefficients

$$\dot{\boldsymbol{\eta}} = \mathbf{R}(\psi)\boldsymbol{\nu}$$
$$\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}\exp(-\alpha V_{rc})\boldsymbol{\nu}_r + \mathbf{d}(V_{rc}, \gamma_{rc}) = \boldsymbol{\tau} + \boldsymbol{\tau}_{wind} + \boldsymbol{\tau}_{waves}$$

The model includes an optional linear damping matrix to ensure exponential convergence at low relative speed

$$\mathbf{d}(V_{rc}, \gamma_{cr}) = \begin{bmatrix} -\frac{1}{2}\rho A_{Fc} C_X(\gamma_{rc}) V_{rc}^2 \\ -\frac{1}{2}\rho A_{Lc} C_Y(\gamma_{rc}) V_{rc}^2 \\ -\frac{1}{2}\rho A_{Lc} L_{oa} C_N(\gamma_{rc}) V_{rc}^2 - N_{|r|r} r|r| \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} -X_u & 0 & 0 \\ 0 & -Y_v & -Y_r \\ 0 & -N_v & -N_r \end{bmatrix}$$

Optional quadratic damping coefficient used to counteract the destabilizing Munk moment since the current coefficients do not include damping in yaw

6.7 Linear DP Model

Kinematic Nonlinearity

$$\boldsymbol{R}(\psi(t)) := \boldsymbol{R}(t)$$

The kinematic nonlinearity due to the rotation matrix in yaw can be removed by assuming that the heading angle is measured. The measurement is a bounded smooth time-varying signal.

Linear Time-Varying (LTV) Model for DP for Controller-Observer Design

$$\begin{aligned}
\dot{\boldsymbol{\eta}} &= \boldsymbol{R}(t)\boldsymbol{\nu} \\
\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{D}\boldsymbol{\nu} &= \boldsymbol{R}^\top(t)\boldsymbol{b} + \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} \\
\dot{\boldsymbol{b}} &= \boldsymbol{0}
\end{aligned}$$

The bias $\boldsymbol{b}$ is assumed to be nearly constant in NED

$$\boldsymbol{\tau} = \boldsymbol{B}\boldsymbol{u}$$

$$\dot{\boldsymbol{x}} = \underbrace{\begin{bmatrix} \boldsymbol{0}_{3\times3} & \boldsymbol{R}(t) & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & -\boldsymbol{M}^{-1}\boldsymbol{D} & \boldsymbol{M}^{-1}\boldsymbol{R}^\top(t) \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \end{bmatrix}}_{\boldsymbol{A}(t)} \boldsymbol{x} + \underbrace{\begin{bmatrix} \boldsymbol{0}_{3\times r} \\ \boldsymbol{M}^{-1}\boldsymbol{B} \\ \boldsymbol{0}_{3\times r} \end{bmatrix}}_{\boldsymbol{B}} \boldsymbol{u} + \underbrace{\begin{bmatrix} \boldsymbol{0}_{3\times3} \\ \boldsymbol{M}^{-1} \\ \boldsymbol{0}_{3\times3} \end{bmatrix}}_{\boldsymbol{E}} \boldsymbol{w}$$

$\boldsymbol{x} = [\boldsymbol{\eta}^\top, \boldsymbol{\nu}^\top, \boldsymbol{b}^\top]^\top \qquad \boldsymbol{w} = \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}}$

6.7 Linear DP Model

Matlab:
A linear model of a supply vessel is included in the MSS toolbox

```
[xdot,U] = supply(x,tau)      % Supply vessel L = 76.2 m
```

Chapter Goals - Revisited

- Understand the rigid-body kinetics and zero-frequency potential coefficients used in maneuvering theory.
- Explain what the hydrodynamic added mass matrix is and how it contributes to the marine craft equations of motion.
- Understand why there exist Coriolis and centripetal matrices due to rigid-body and hydrodynamic added mass. This should be related to using an approximately inertial frame NED and a BODY frame rotating about the NED coordinate frame.
- Have an overview of the different maneuvering models that are in use, including
    - 3-DOF ship maneuvering models (surge, sway, and yaw)
    - 4-DOF ship maneuvering models (surge, sway, roll and yaw)
    - 3-DOF stationkeeping models (surge, sway, and yaw)
- Understand how to include ocean currents using relative velocity in the equations of motion:

$$\underbrace{\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{rigid-body forces}} + \underbrace{\boldsymbol{M}_{A}\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}_{A}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r}_{\text{hydrodynamic forces}} + \underbrace{\boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_o}_{\text{hydrostatic forces}} = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}}$$