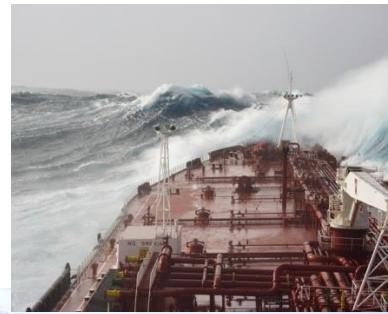


# Chapter 10 - Environmental Forces and Moments

- 10.1 Wind Forces and Moments
- 10.2 Wave Forces and Moments
- 10.3 Ocean Current Forces and Moments



# Chapter Goals

- Understand the principle for linear **superposition** of **wind** and **wave forces**
- Understand the **equations of relative motion** for **ocean currents**
- Be able to **compute the wind loads** on a marine craft using **wind coefficient** look-up tables and **projected areas** found from general arrangement drawings as required by the MSS functions: *Blendermann94* and *Isherwood72*.
- Understand how statistical descriptions can be used to define the **sea state** in terms of parameters such as wave period, wind speed, etc. for different regions on the Earth.
- Explain what we mean with regular and irregular waves, wave spectrum and directional wave spectrum.
- Be able to compute and simulate the **wave elevation** of regular and irregular waves for a given sea state and a wave spectrum.
- Understand the three methods for simulation and computation of wave loads:
  - **Motion RAOs**
  - **Force RAOs**
  - **Simplified transfer functions for WF motions**
- Be able to explain what we mean by **first- and second-order wave loads** and how these relate to hydrodynamic data computed in Wamit and ShipX.
- Be able to simulate ocean currents (magnitude and direction) in 2D and 3D and transform the results to current velocities.

# Chapter 10 - Environmental Forces and Moments

## Superposition of Wind and Wave Disturbances

For control systems design it is common to assume the principle of superposition when considering wind and wave forces:

$$\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\eta}) + \mathbf{g}_0 = \underbrace{\boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}}}_{\mathbf{w}} + \boldsymbol{\tau}$$

## Ocean Currents

The effect due to ocean currents is simulated by introducing the relative velocity vector:

$$\underbrace{\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{rigid-body terms}} + \underbrace{\mathbf{M}_A\dot{\boldsymbol{\nu}}_r + \mathbf{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \mathbf{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r}_{\text{hydrodynamic terms}} + \underbrace{\mathbf{g}(\boldsymbol{\eta}) + \mathbf{g}_0}_{\text{hydrostatic terms}} = \boldsymbol{\tau} + \mathbf{w}$$



$$\mathbf{x}_c \begin{bmatrix} u_c \\ v_c \\ w_c \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



# 10.1 Wind Forces and Moments on Marine Craft at Rest

Generalized force using area-based wind coefficients:

$$X_{\text{wind}} = q C_X \mathbf{A}_{Fw} \mathbf{U}_{Fw}$$

$$Y_{\text{wind}} = q C_Y \mathbf{A}_{Fw} \mathbf{U}_{Lw}$$

$$Z_{\text{wind}} = q C_Z \mathbf{A}_{Fw} \mathbf{U}_{Fw}$$

$$K_{\text{wind}} = q C_K \mathbf{A}_{Fw} \mathbf{U}_{Lw} H_{Lw}$$

$$M_{\text{wind}} = q C_M \mathbf{A}_{Fw} \mathbf{U}_{Fw} H_{Fw}$$

$$N_{\text{wind}} = q C_N \mathbf{A}_{Fw} \mathbf{U}_{Lw} L_{oa}$$

$$\gamma_w = \psi - \beta_{Vw} - \pi$$

wind angle of attack

Wind loads cannot be measured. The standard measurements are wind speed and direction,  $V_w$  and  $\beta_{Vw}$ .

Hence, you need the wind coefficients  $C_x, C_y, C_z, C_k, C_M$  and  $C_N$  to compute the wind loads.

Dynamic pressure

$$q = \frac{1}{2} \rho V_w^2$$

$\rho_a$  = air density

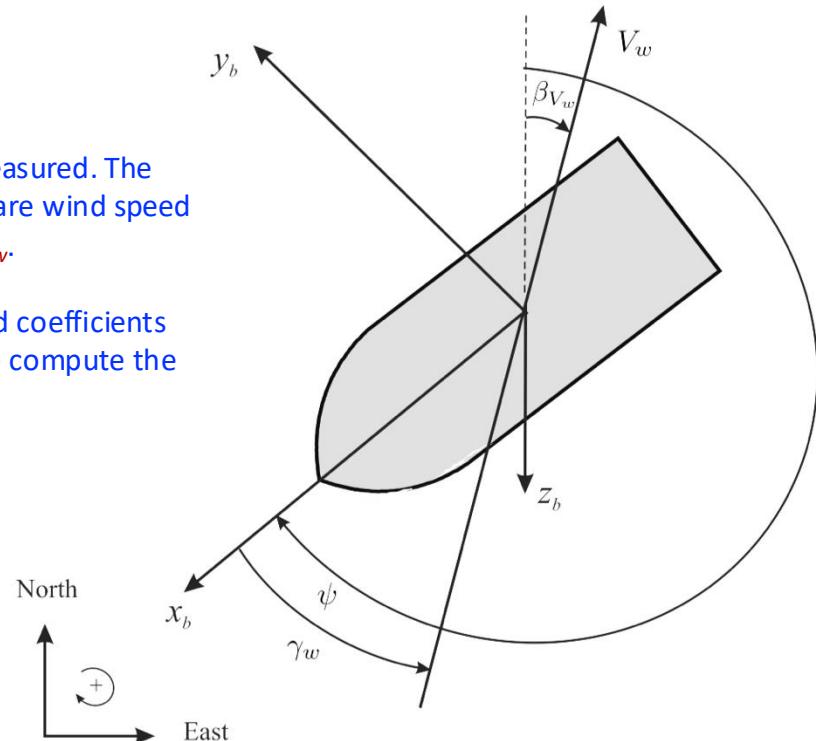
$A_{Fw}$  = frontal projected area

$A_{Lw}$  = lateral projected area

$H_{Fw}$  = centroid of  $A_{Fw}$  above the water line

$H_{Lw}$  = centroid of  $A_{Lw}$  above the water line

$L_{oa}$  = length over all



# 10.1 Wind Forces and Moments on Marine Craft at Rest

The wind speed is usually specified in terms of Beaufort numbers (Price and Bishop 1974)

Beaufort number	Description of wind	Wind speed (knots)
0	Calm	0–1
1	Light air	2–3
2	Light breeze	4–7
3	Gentle breeze	8–11
4	Moderate breeze	12–16
5	Fresh breeze	17–21
6	Strong breeze	22–27
7	Moderate gale	28–33
8	Fresh gale	34–40
9	Strong gale	41–48
10	Whole gale	49–56
11	Storm	57–65
12	Hurricane	More than 65

1 knots = 0.51 m/s



Air density as a function of temperature

°C	Air density, $\rho_a$ (kg/m <sup>3</sup> )
-10	1.342
-5	1.317
0	1.292
5	1.269
10	1.247
15	1.225
20	1.204
25	1.184
30	1.165

# Experiments with Ship Models in a Wind Tunnel



Force Technology: <https://forcetechnology.com/no/alla-faciliteter/wind-tunnel-facility>

Youtube: [https://www.youtube.com/watch?time\\_continue=1&v=NyiaMRSGPg8&feature=emb\\_logo](https://www.youtube.com/watch?time_continue=1&v=NyiaMRSGPg8&feature=emb_logo)

# 10.1 Wind Forces and Moments on Marine Craft at Rest

## Wind Coefficient Approximation for Symmetrical Ships

For ships that are symmetrical with respect to the xz- and yz-planes, the wind coefficients for horizontal-plane motions can be approximated by

$$C_x \approx 0.25 c_x \cos \theta_w$$

$$C_y \approx 0.25 c_y \sin \theta_w$$

$$C_n \approx 0.25 c_n \sin \theta_w$$

which are convenient formulae for computer simulations.

Experiments indicate that:

$$c_x \in [0.50, 0.90]$$

$$c_y \in [0.70, 0.95]$$

$$c_n \in [0.05, 0.20]$$

These values should be used with care.



# 10.1 Wind Forces and Moments on Moving Marine Craft

Generalized force in terms of relative angle of attack  $\gamma_{rw}$  and relative wind speed  $V_{rw}$

$$\tau_{\text{wind}} = \frac{1}{2} \rho_a V_{rw}^2 \begin{bmatrix} C_X(\gamma_{rw}) A_{Fw} \\ C_Y(\gamma_{rw}) A_{Lw} \\ C_Z(\gamma_{rw}) A_{Fw} \\ C_K(\gamma_{rw}) A_{Lw} H_{Lw} \\ C_M(\gamma_{rw}) A_{Fw} H_{Fw} \\ C_N(\gamma_{rw}) A_{Lw} L_{oa} \end{bmatrix}$$

$$V_{rw} = \sqrt{u_{rw}^2 + v_{rw}^2}$$

$$\gamma_{rw} = \text{atan2}(v_{rw}, u_{rw})$$

$$\begin{aligned} u_{rw} &= u - u_w & u_w &= V_w \cos(\beta_{V_w} - \psi) \\ v_{rw} &= v - v_w & v_w &= V_w \sin(\beta_{V_w} - \psi) \end{aligned}$$

The wind speed  $V_w$  and its direction  $\beta_{V_w}$  can be measured by a wind sensor (anemometer)



The wind coefficients (look-up tables) are computed numerically or by experiments in wind tunnels.

# 10.1 Wind Coefficients Based on Flow over a Helmholtz-Kirchhoff Plate

Reference: Blndermann (1986, 1994)

The load functions are parameterized in terms of four primary wind load parameters:

$$C_X(\gamma_w) = -\underbrace{\text{CD}_l \frac{A_{L_w}}{A_{F_w}}}_{\text{CD}_{lAF}} \frac{\cos(\gamma_w)}{1 - \frac{\delta}{2} \left(1 - \frac{\text{CD}_l}{\text{CD}_t}\right) \sin^2(2\gamma_w)}$$

$$C_Y(\gamma_w) = \text{CD}_t \frac{\sin(\gamma_w)}{1 - \frac{\delta}{2} \left(1 - \frac{\text{CD}_l}{\text{CD}_t}\right) \sin^2(2\gamma_w)}$$

$$C_K(\gamma_w) = \kappa C_Y(\gamma_w)$$

$$C_N(\gamma_w) = \left[ \frac{s_L}{L_{oa}} - 0.18 \left( \gamma_w - \frac{\pi}{2} \right) \right] C_Y(\gamma_w)$$

$$\text{CD}_l = \text{CD}_{lAF}(\gamma_w) \frac{A_{F_w}}{A_{L_w}}$$

Type of vessel	$\text{CD}_t$	$\text{CD}_{lAF}(0)$	$\text{CD}_{lAF}(\pi)$	$\delta$	$\kappa$
1. Car carrier	0.95	0.55	0.60	0.80	1.2
2. Cargo vessel, loaded	0.85	0.65	0.55	0.40	1.7
3. Cargo vessel, container on deck	0.85	0.55	0.50	0.40	1.4
4. Container ship, loaded	0.90	0.55	0.55	0.40	1.4
5. Destroyer	0.85	0.60	0.65	0.65	1.1
6. Diving support vessel	0.90	0.60	0.80	0.55	1.7
7. Drilling vessel	1.00	0.70–1.00	0.75–1.10	0.10	1.7
8. Ferry	0.90	0.45	0.50	0.80	1.1
9. Fishing vessel	0.95	0.70	0.70	0.40	1.1
10. Liquefied natural gas tanker	0.70	0.60	0.65	0.50	1.1
11. Offshore supply vessel	0.90	0.55	0.80	0.55	1.2
12. Passenger liner	0.90	0.40	0.40	0.80	1.2
13. Research vessel	0.85	0.55	0.65	0.60	1.4
14. Speed boat	0.90	0.55	0.60	0.60	1.1
15. Tanker, loaded	0.70	0.90	0.55	0.40	3.1
16. Tanker, in ballast	0.70	0.75	0.55	0.40	2.2
17. Tender	0.85	0.55	0.55	0.65	1.1

Matlab MSS toolbox

[w\_wind,CX,CY,CK,CN] = blndermann94(gamma\_r,V\_r,AFw,ALw,sH,sL,Loa,vessel\_no)

# MSS Toolbox

```

gam_deg = (0:10:180)';
gamma_r = (pi/180)*gam_deg;
V_r = 20;

% *****
% CASE 1 Blidnermann (1994)
% *****

CDt      = 0.85;
CDL_AF{1} = 0.55;      % gamma_r = 0
CDL_AF{2} = 0.65;      % gamma_r = pi
delta    = 0.60;
kappa    = 1.4;
AFw      = 160.7;
ALw      = 434.8;
Loa      = 55.0;
sL       = 1.48;
sH       = 5.1;
vessel_no = 13;

[tau_w,CX,CY,CK,CN] = blidnermann94(gamma_r,V_r,AFw,ALw,sH,sL,Loa,vessel_no);

```

From table of ships

Lookalike ship

```

% vessel_no =
% 1. Car carrier
% 2. Cargo vessel, loaded
% 3. Cargo vessel, container on deck
% 4. Container ship, loaded
% 5. Destroyer
% 6. Diving support vessel
% 7. Drilling vessel
% 8. Ferry
% 9. Fishing vessel
% 10. Liquified natural gas tanker
% 11. Offshore supply vessel
% 12. Passenger liner
% 13. Research vessel
% 14. Speed boat
% 15. Tanker, loaded
% 16. Tanker, in ballast
% 17. Tender

```

The ship data are computed using a general arrangement drawing (manually or digitally)

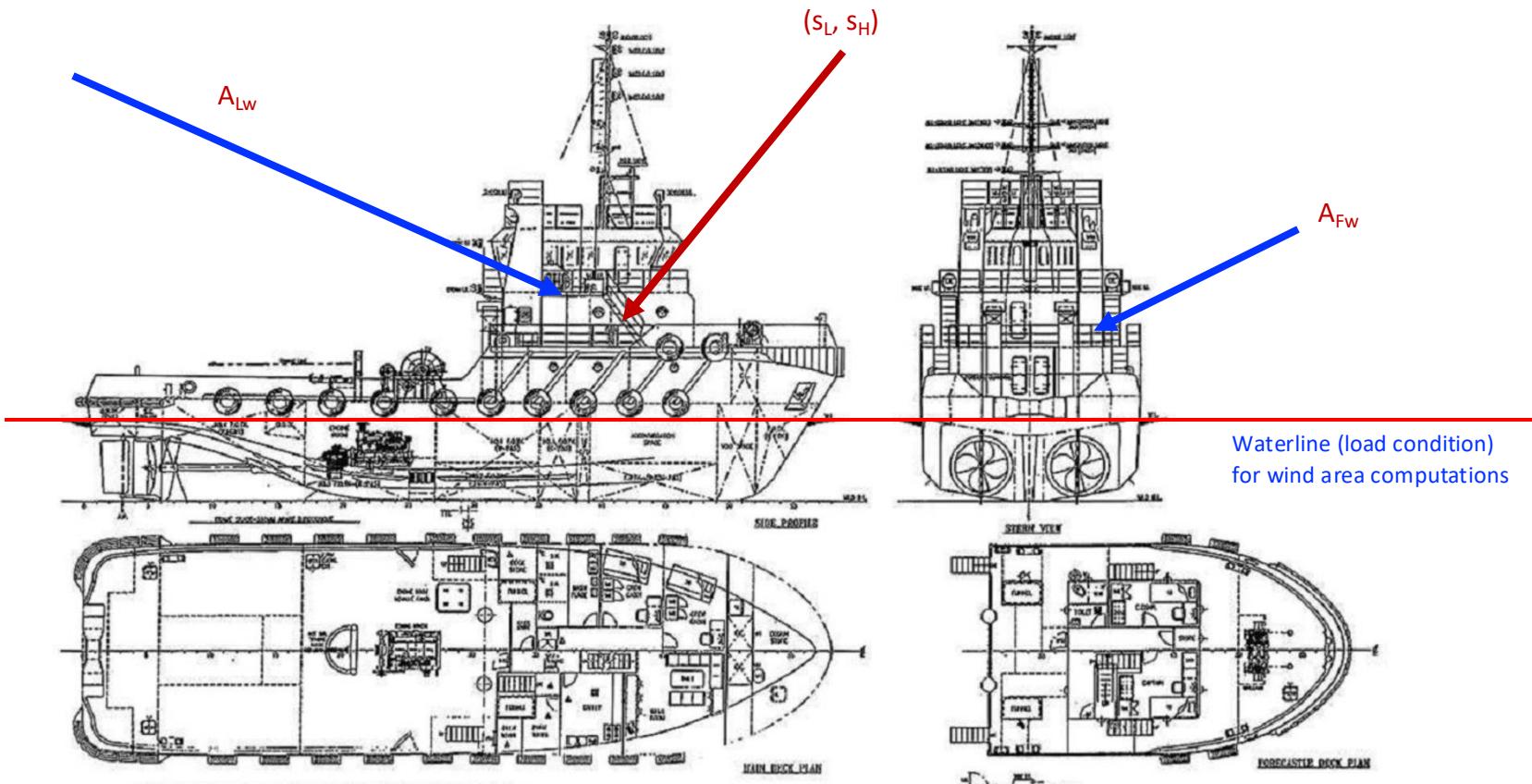
$A_{Fw}$  = Frontal projected area

$A_{Lw}$  = Lateral (transverse) projected area

Loa = Length overall

( $s_L$ ,  $s_H$ ) =  $x_b$  and  $y_b$  coordinates of the centroid of  $A_{Lw}$

# General Arrangement Drawing

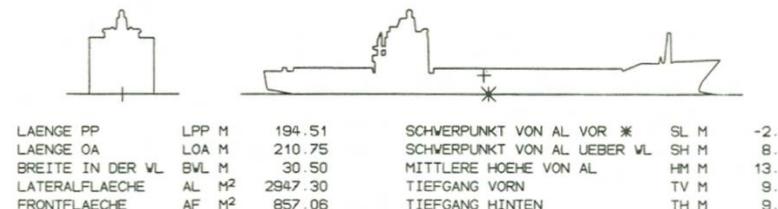
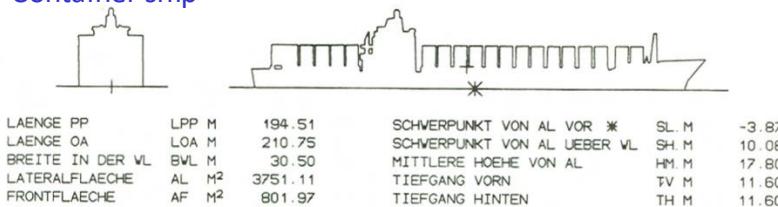


# Blendermann (1986, 1994)

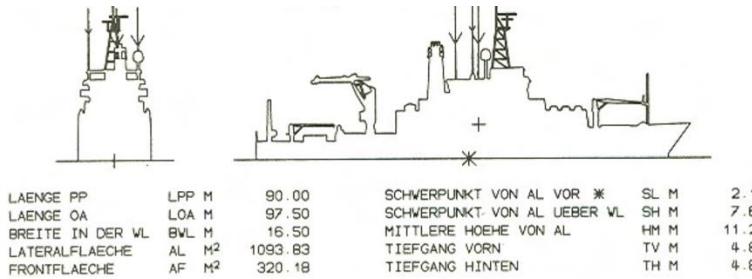
## Car carrier



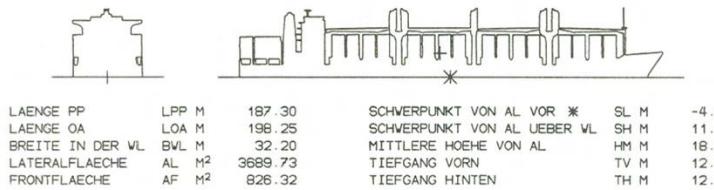
## Container ship



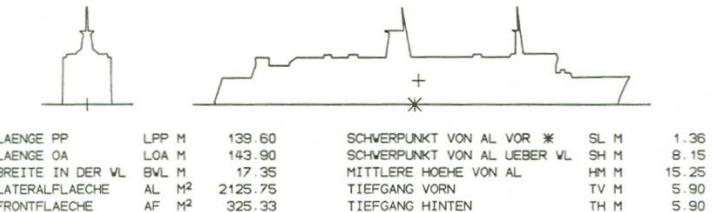
## Research vessel



## Cargo ship

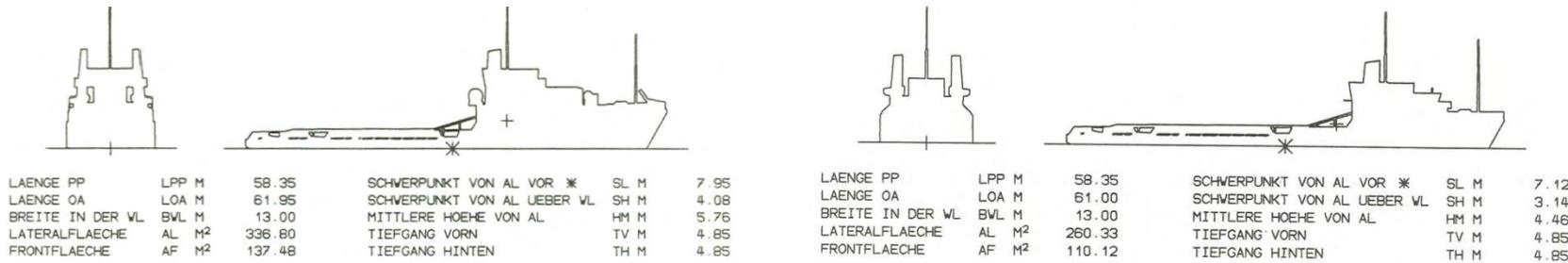


## Ferry

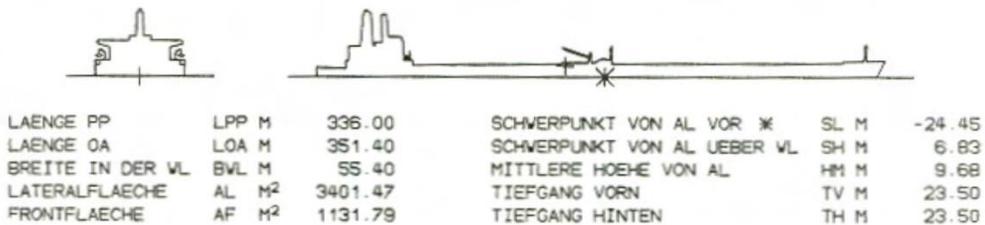


# Blendermann (1986, 1994)

## Supply vessel



## Tanker



# MSS Toolbox

Research vessel #13

$$\tau_{\text{wind}} = \frac{1}{2} \rho_a V_{rw}^2 \begin{bmatrix} C_X(\gamma_{rw}) A_{Fw} \\ C_Y(\gamma_{rw}) A_{Lw} \\ \hline C_Z(\gamma_{rw}) A_{Fw} \\ C_K(\gamma_{rw}) A_{Lw} H_{Lw} \\ \hline C_M(\gamma_{rw}) A_{Fw} H_{Fw} \\ C_N(\gamma_{rw}) A_{Lw} L_{oa} \end{bmatrix}$$

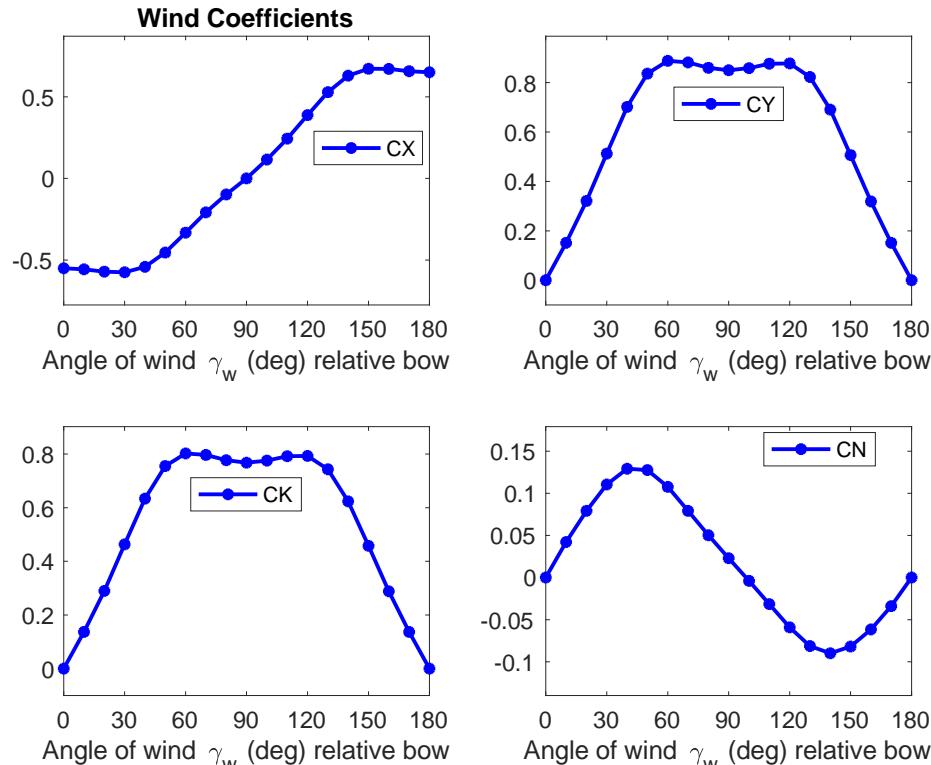
```

CDt      = 0.85;
CDl_AF{1} = 0.55;          % gamma_r = 0
CDl_AF{2} = 0.65;          % gamma_r = pi
delta     = 0.60;
kappa     = 1.4;
AFw       = 160.7;
ALw       = 434.8;
Loa       = 55.0;
sL        = 1.48;
sH        = 5.1;
vessel_no = 13;

```

Ship data

```
[tau_w, CX, CY, CK, CN] = blendermann94(gamma_r, V_r, AFw, ALw, sH, sL, Loa, vessel_no);
```



# 10.1 Wind Coefficients for Merchant Ships (Isherwood 1972)

Isherwood (1972) has derived a set of wind coefficients by using **multiple regression techniques** to fit experimental data of **merchant ships**. The wind coefficients are parameterized in terms of eight parameters.

$L_{oa}$	length overall
$B$	beam
$A_{Lw}$	lateral projected area
$A_{Tw}$	transverse projected area
$A_{SS}$	lateral projected area of superstructure
$S$	length of perimeter of lateral projection of model excluding water line and slender bodies such as masts and ventilators
$C$	distance from bow of centroid of lateral projected area
$M$	number of distinct groups of masts or king posts seen in lateral projection; king posts close against the bridge front are not included

Typical merchant ships are:

- Bulk carriers for transport of raw materials
- Container ships
- Tankers
- Ferries and cruise ships

From regression analyses it was concluded that the measured data were best fitted to the following three equations:

$$C_X = - \left( A_0 + A_1 \frac{2A_L}{L^2} + A_2 \frac{2A_T}{B^2} + A_3 \frac{L}{B} + A_4 \frac{S}{L} + A_5 \frac{C}{L} + A_6 M \right)$$

$$C_Y = B_0 + B_1 \frac{2A_L}{L^2} + B_2 \frac{2A_T}{B^2} + B_3 \frac{L}{B} + B_4 \frac{S}{L} + B_5 \frac{C}{L} + B_6 \frac{A_{SS}}{A_L}$$

$$C_N = C_0 + C_1 \frac{2A_L}{L^2} + C_2 \frac{2A_T}{B^2} + C_3 \frac{L}{B} + C_4 \frac{S}{L} + C_5 \frac{C}{L}$$

Matlab MSS toolbox

`[w_wind,CX,CY,CN] = isherwood72(gamma_r,V_r,Loa,B,AFw,ALw,A_SS,S,C,M)`

# 10.1 Wind Coefficients for Merchant Ships (Isherwood 1972)

$\gamma_w$ (deg)	$A_0$	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$	$A_6$	S.E.
0	2.152	-5.00	0.243	-0.164	—	—	—	0.086
10	1.714	-3.33	0.145	-0.121	—	—	—	0.104
20	1.818	-3.97	0.211	-0.143	—	—	0.033	0.096
30	1.965	-4.81	0.243	-0.154	—	—	0.041	0.117
40	2.333	-5.99	0.247	-0.190	—	—	0.042	0.115
50	1.726	-6.54	0.189	-0.173	0.348	—	0.048	0.109
60	0.913	-4.68	—	-0.104	0.482	—	0.052	0.082
70	0.457	-2.88	—	-0.068	0.346	—	0.043	0.077
80	0.341	-0.91	—	-0.031	—	—	0.032	0.090
90	0.355	—	—	—	-0.247	—	0.018	0.094
100	0.601	—	—	—	-0.372	—	-0.020	0.096
110	0.651	1.29	—	—	-0.582	—	-0.031	0.090
120	0.564	2.54	—	—	-0.748	—	-0.024	0.100
130	-0.142	3.58	—	0.047	-0.700	—	-0.028	0.105
140	-0.677	3.64	—	0.069	-0.529	—	-0.032	0.123
150	-0.723	3.14	—	0.064	-0.475	—	-0.032	0.128
160	-2.148	2.56	—	0.081	—	1.27	-0.027	0.123
170	-2.707	3.97	-0.175	0.126	—	1.81	—	0.115
180	-2.529	3.76	-0.174	0.128	—	1.55	—	0.112
Mean S.E.								0.103

$\gamma_w$ (deg)	$B_0$	$B_1$	$B_2$	$B_3$	$B_4$	$B_5$	$B_6$	S.E.
10	0.096	0.22	—	—	—	—	—	0.015
20	0.176	0.71	—	—	—	—	—	0.023
30	0.225	1.38	—	0.023	—	-0.29	—	0.030
40	0.329	1.82	—	0.043	—	-0.59	—	0.054
50	1.164	1.26	0.121	—	-0.242	-0.95	—	0.055
60	1.163	0.96	0.101	—	-0.177	-0.88	—	0.049
70	0.916	0.53	0.069	—	—	-0.65	—	0.047
80	0.844	0.55	0.082	—	—	-0.54	—	0.046
90	0.889	—	0.138	—	—	-0.66	—	0.051
100	0.799	—	0.155	—	—	-0.55	—	0.050
110	0.797	—	0.151	—	—	-0.55	—	0.049
120	0.996	—	0.184	—	-0.212	-0.66	0.34	0.047
130	1.014	—	0.191	—	-0.280	-0.69	0.44	0.051
140	0.784	—	0.166	—	-0.209	-0.53	0.38	0.060
150	0.536	—	0.176	-0.029	-0.163	—	0.27	0.055
160	0.251	—	0.106	-0.022	—	—	—	0.036
170	0.125	—	0.046	-0.012	—	—	—	0.022
Mean S.E.								0.044

Isherwood (1972) regression tables  
for merchant ships

Used by the isherwood72.m  
MSS Matlab function

$\gamma_w$ (deg)	$C_0$	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	S.E.	
10	0.0596	0.061	—	—	—	-0.074	0.0048	
20	0.1106	0.204	—	—	—	-0.170	0.0074	
30	0.2258	0.245	—	—	—	-0.380	0.0105	
40	0.2017	0.457	—	0.0067	—	-0.472	0.0137	
50	0.1759	0.573	—	0.0118	—	-0.523	0.0149	
60	0.1925	0.480	—	0.0115	—	-0.546	0.0133	
70	0.2133	0.315	—	0.0081	—	-0.526	0.0125	
80	0.1827	0.254	—	0.0053	—	-0.443	0.0123	
90	0.2627	—	—	—	—	-0.508	0.0141	
100	0.2102	—	-0.0195	—	0.0335	-0.492	0.0146	
110	0.1567	—	-0.0258	—	0.0497	-0.457	0.0163	
120	0.0801	—	-0.0311	—	0.0740	-0.396	0.0179	
130	-0.0189	—	-0.0488	0.0101	0.1128	-0.420	0.0166	
140	0.0256	—	-0.0422	0.0100	0.0889	-0.463	0.0162	
150	0.0552	—	-0.0381	0.0109	0.0689	-0.476	0.0141	
160	0.0881	—	-0.0306	0.0091	0.0366	-0.415	0.0105	
170	0.0851	—	-0.0122	0.0025	—	-0.220	0.0057	
Mean S.E.								0.0127

# MSS Toolbox

Merchant ship with masts/king posts

$$\tau_{\text{wind}} = \frac{1}{2} \rho_a V_{rw}^2 \begin{bmatrix} C_X(\gamma_{rw}) A_{Fw} \\ C_Y(\gamma_{rw}) A_{Lw} \\ \hline C_Z(\gamma_{rw}) A_{Fw} \\ \hline C_K(\gamma_{rw}) A_{Lw} H_{Lw} \\ \hline C_M(\gamma_{rw}) A_{Fw} H_{Fw} \\ \hline C_N(\gamma_{rw}) A_{Lw} L_{oa} \end{bmatrix}$$

Ship data

```

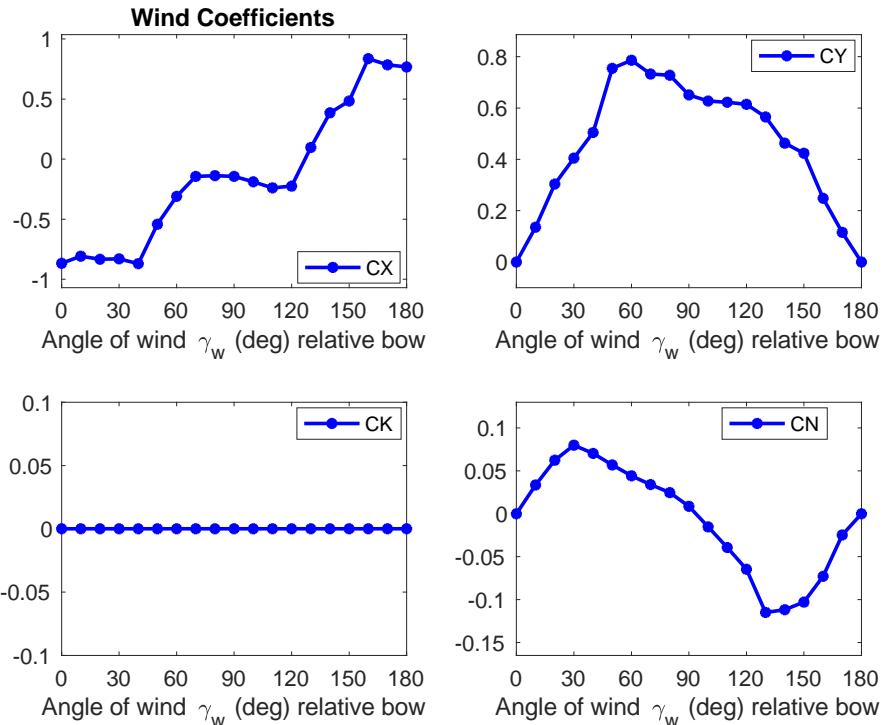
Loa      = 100; % length overall (m)
B        = 30;  % beam (m)
ALw     = 900; % lateral projected area (m^2)
AFw     = 300; % transverse projected area (m^2)
A_SS    = 100; % lateral projected area of superstructure (m^2)
S        = 100; % length of perimeter of lateral projection of model (m)
              % excluding waterline and slender bodies such as masts and ventilators (m)
C        = 50;  % distance from bow of centroid of lateral projected area (m)
M        = 2;   % number of distinct groups of masts or king posts seen in lateral
              % projection; king posts close against the bridge front are not included

```

```

[tau_wind,CX,CY,CN] = isherwood72(gamma_r,V_r,Loa,B,ALw,AFw,A_SS,S,C,M);
CK = 0*CX;

```



# 10.1 Wind Coefficients – Other Useful Methods and References

## Wind Resistance of Very Large Crude Carriers (OCIMF, 1977)

Wind loads on very large crude carriers (VLCCs) in the range 150 000 to 500 000 dwt can be computed by applying the results of (OCIMF 1977).

## Wind Resistance of Large Tankers and Medium Sized Ships

For wind resistance on large tankers in the 100 000 to 500 000 dwt class the reader is advised to consult Van Berlekom et al. (1974).

Medium sized ships of the order 600 to 50 000 dwt is discussed by Wagner (1967).

## Wind Resistance of Moored Ships and Floating Structures

Wind loads on moored ships are discussed by De Kat and Wijchers (1991) while an excellent reference for huge pontoon type floating structures is Kitamura et al. (1997).

## 10.2 Wave Forces and Moments

A marine control system can be simulated under influence of wave-induced forces by separating the 1st-order and 2nd-order effects:

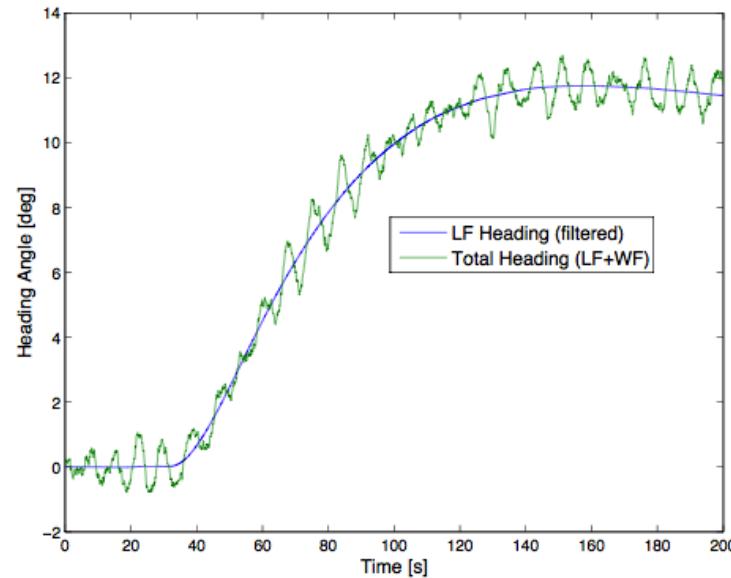
- **1st-order wave-induced forces (wave-frequency motion)**  
Zero-mean oscillatory motions
- **2nd-order wave-induced forces (wave drift forces)**  
Nonzero slowly-varying component

Wave forces are observed as a mean slowly-varying component referred to as **Low- Frequency (LF) motions** and an oscillatory component called **Wave-Frequency (WF) motions** which must be compensated for by the feedback control system

- 1st-order wave-induced forces are usually removed by using a **wave filter** to filter out the WF component (to be designed later)
- 2nd-order wave-induced forces (drift) are removed by using **integral action** in the control law.

$$M\dot{\nu} + C(\nu)\nu + D(\nu)\nu + g(\eta) + g_0 = \tau_{\text{wind}} + \tau_{\text{wave}} + \tau$$

$\eta_{\text{wave}}$   $\eta_{\text{wave1}}$   $\eta_{\text{wave2}}$



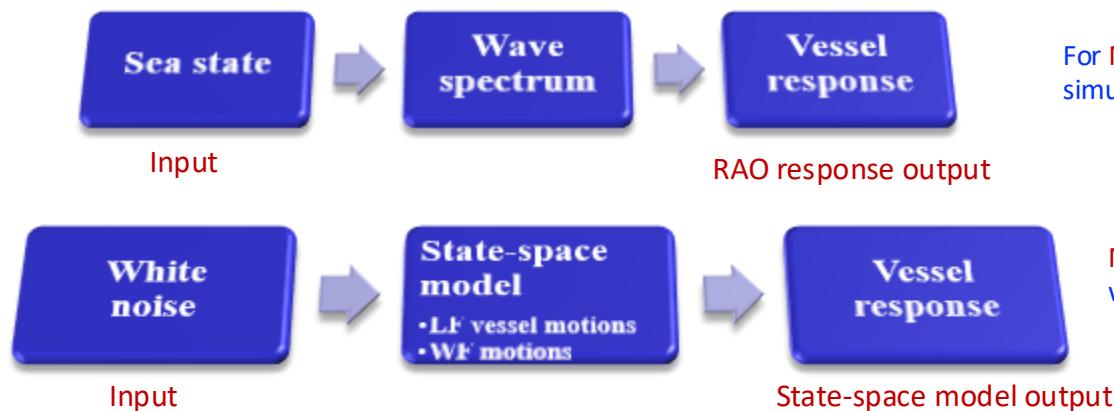
LF and WF heading angles

## 10.2 Wave Forces and Moments

### Wave Response Models:

1. Force RAOs
2. Motion RAOs
3. Linear state-space models (WF models)

The first two methods require that the RAO tables are computed using a hydrodynamic program (the RAO tables depend on the vessel's geometry)



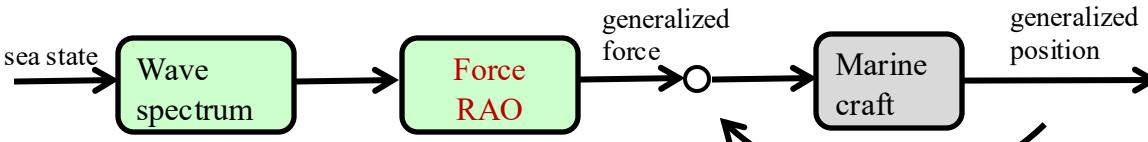
For Methods 1 and 2, the vessel response is simulated using three blocks in a cascaded structure

Method 3 is a state-space model driven by white noise (no sea state description needed)

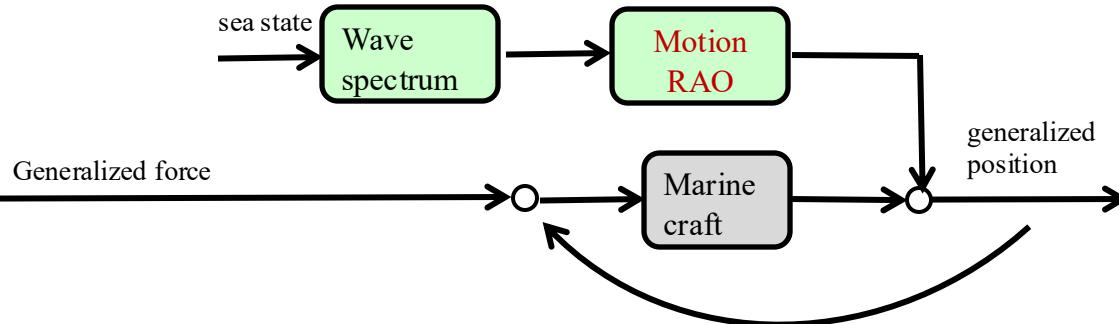
The last method is frequently used due to simplicity, but it is only intended for testing of robustness and performance of control systems, that is closed-loop analysis.

# 10.2 Wave Forces and Moments

## 1. Force RAOs

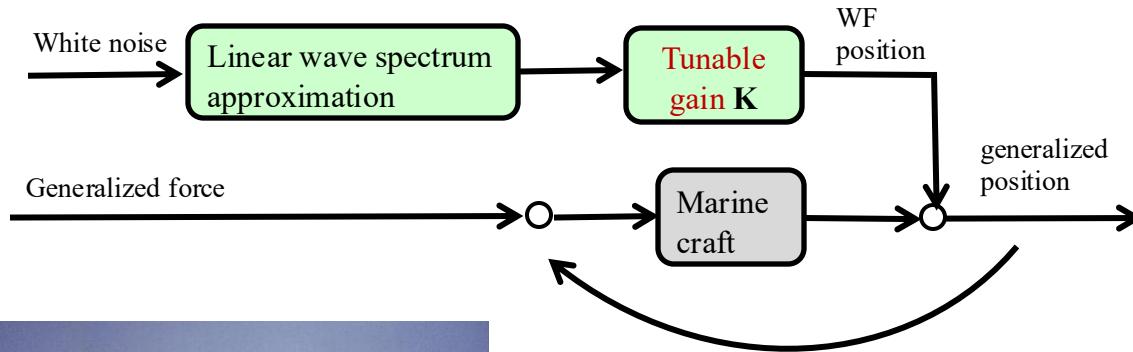


## 2. Motion RAOs



## 10.2 Wave Forces and Moments

### 3. Linear State-Space Models (WF models)



## 10.2 Sea State Descriptions

For marine craft, the sea can be characterized by the following wave spectrum parameters:

- The significant wave height  $H_s$

The mean wave height of the one third highest waves, also denoted as  $H_{1/3}$

- One of the following wave periods:

- The average wave period,  $T_1$
- Average zero-crossing wave period,  $T_z$
- Peak period,  $T_p$  (this is equivalent to the modal period,  $T_o$ )

$$T_z = 0.710T_0 = 0.921T_1$$

Sea state code	Description of sea	Wave height observed (m)	Percentage probability		
			World wide	North Atlantic	Northern North Atlantic
0	Calm (glassy)	0			
1	Calm (rippled)	0–0.1	11.2486	8.3103	6.0616
2	Smooth (wavelets)	0.1–0.5			
3	Slight	0.5–1.25	31.6851	28.1996	21.5683
4	Moderate	1.25–2.5	40.1944	42.0273	40.9915
5	Rough	2.5–4.0	12.8005	15.4435	21.2383
6	Very rough	4.0–6.0	3.0253	4.2938	7.0101
7	High	6.0–9.0	0.9263	1.4968	2.6931
8	Very high	9.0–14.0	0.1190	0.2263	0.4346
9	Phenomenal	Over 14.0	0.0009	0.0016	0.0035



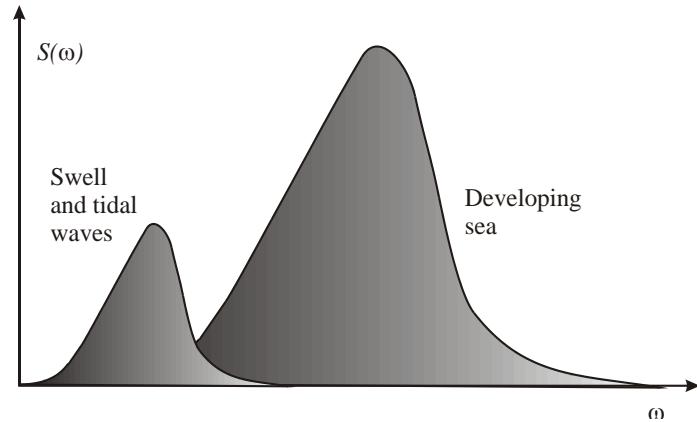
## 10.2 Wave Spectra

The process of wave generation due to wind starts with small wavelets appearing on the water surface. This increases the drag force, which in turn allows short waves to grow. These short waves continue to grow until they finally break, and their energy is dissipated.

A **developing sea**, or storm, starts with high frequencies creating a spectrum with a **peak at a relative high frequency**.

A storm which has lasted for a long time is said to create a **fully developed sea**.

After the wind has stopped, a low -frequency decaying sea or **swell** is formed. These long waves form a wave spectrum with a **low peak frequency**.



If the swell from one storm interacts with the waves from another storm, a wave spectrum with **two-peak frequencies** may be observed.

In addition, **tidal waves** will generate a peak at a low frequency. Hence, the resulting wave spectrum might be quite complicated in cases where the weather changes rapidly

# 10.2 Wave Spectra

## Neumann Spectrum (1952)

One-parameter spectrum

$$S(\omega) = C\omega^{-6} \exp(-2g^2\omega^{-2}V^{-2})$$

$$\lim_{\omega \gg 1} S(\omega) = \alpha g^2 \omega^{-5}$$

## Bretschneider Spectrum (1959)

Two-parameter spectrum

$$S(\omega) = 1.25 \frac{\omega_0^4 H_s^2}{4} \omega^{-5} \exp\left[-1.25 (\omega_0/\omega)^4\right]$$

## Pierson-Moskowitz Spectrum (1964)

Two-parameter spectrum

$$S(\omega) = A\omega^{-5} \exp(-B\omega^{-4})$$

$$A = 8.1 \times 10^{-3} g^2 = \text{constant}$$

$$B = 0.74 \left(\frac{g}{V_{19.5}}\right)^4 = \frac{3.11}{H_s^2}$$

### Matlab:

The Bretschneider and PM spectra are implemented in the MSS toolbox as wave spectra 1 and 2

```
S = wavespec(1, [A, B], w, 1);
S = wavespec(2, V20, w, 1);
```

where A and B are the spectrum parameters, V20 is wind speed at 20 m height and w is the wave frequency vector.

## 10.2 Wave Spectra

### Modified Pierson-Moskowitz (MPM) Spectrum (1964)

For prediction of responses of marine craft and offshore structures in open sea, the International Ship and Offshore Structures Congress and the International Towing Tank Conference (ITTC), have recommended the use of a modified version of the PM-spectrum where

$$S(\omega) = A\omega^{-5} \exp(-B\omega^{-4}) \quad A = \frac{4\mathcal{V}H_s^2}{T_z^4}, \quad B = \frac{16\mathcal{V}}{T_z^4}$$

$$H_s = \frac{2.06}{g^2} V_{19.5}^2$$

#### Matlab:

The modified PM spectrum is implemented in the MSS toolbox as wave spectra 3 to 5

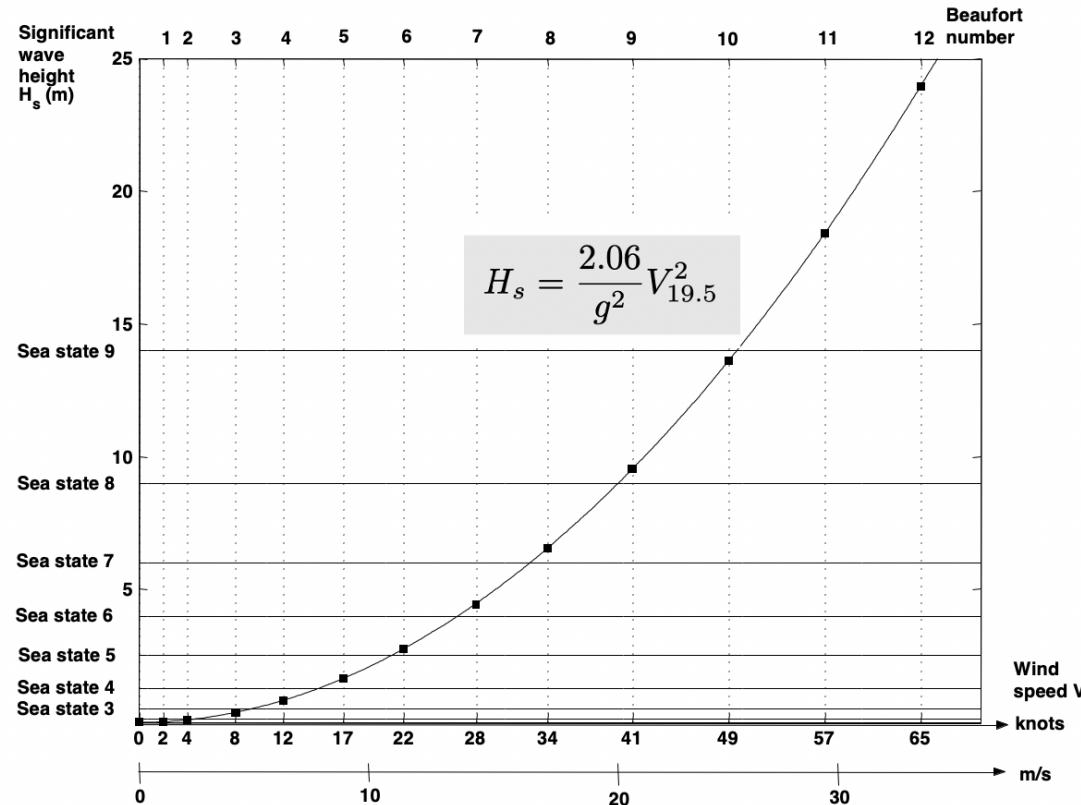
```
S = wavespec(3, [Hs, T0], w, 1);
S = wavespec(4, [Hs, T1], w, 1);
S = wavespec(5, [Hs, Tz], w, 1);
```

where  $H_s$  is the significant wave height,  $T_0$ ,  $T_1$  and  $T_z$  are the peak, average and average zero-crossing wave periods, respectively, while  $w$  is the wave frequency vector.

- The average wave period,  $T_1$
- Average zero-crossing wave period,  $T_z$
- Peak period,  $T_p$  (this is equivalent to the modal period,  $T_0$ )

$$T_z = 0.710T_0 = 0.921T_1$$

## 10.2 Wave Spectra



Significant wave height as a function of wind speed

Why 19.5 meters above the surface level?

This is the height of the anemometers on the weather ships used by Pierson and Moskowitz in 1964.

# 10.2 Wave Spectra

## JONSWAP Spectrum

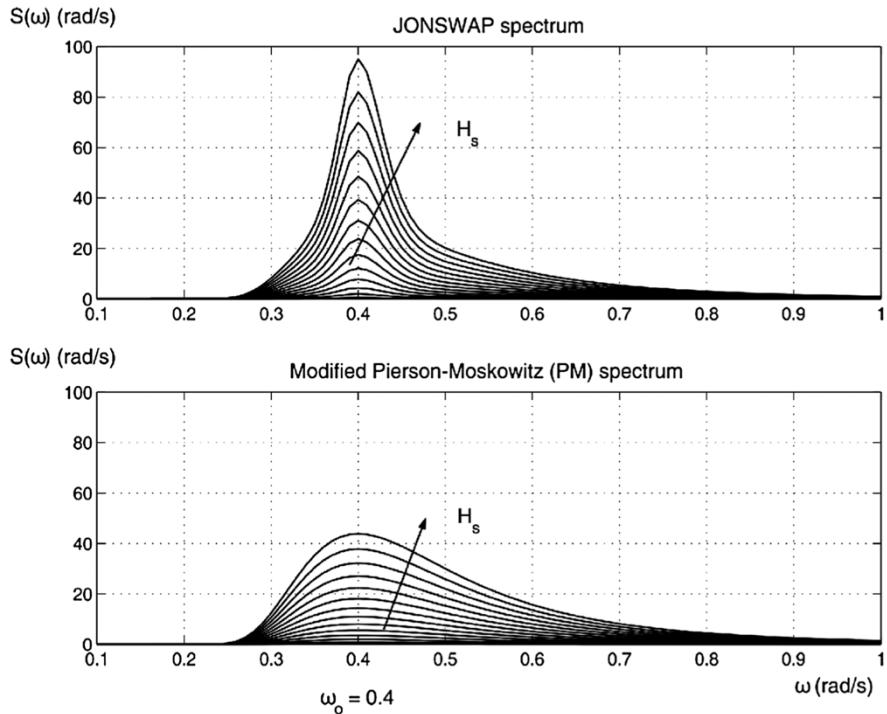
In 1968-1969 an extensive measurement program was carried out in the North Sea, between the island Sylt in Germany and Iceland. The measurement program is known as the Joint North Sea Wave Project (JONSWAP) and the results from these investigations have been adopted as an ITTC standard in 1984.

$$S(\omega) = 155 \frac{H_s^2}{T_1^4} \omega^{-5} \exp\left(\frac{-944}{T_1^4} \omega^{-4}\right) \gamma^Y$$

Peak enhancement factor  $\gamma^Y$

$$Y = \exp\left[-\left(\frac{0.191 \frac{H_s}{T_1} \omega_0}{\sqrt{2}}\right)^2\right]$$

$$\gamma = \begin{cases} 0.07 & \text{for } \frac{H_s}{T_1} \leq 5.24 \\ 0.09 & \text{for } \frac{H_s}{T_1} > 5.24 \end{cases}$$



### Matlab:

The JONSWAP spectrum is included in the MSS toolbox as wave spectra 6 and 7

```
S = wavespec(6, [V10, fetch], w, 1);
S = wavespec(7, [Hs, w0, gamma], w, 1);
```

where V10 is the wind speed at 10 m height, Hs is the significant wave height, w0 is peak frequency and w is the wave frequency vector.

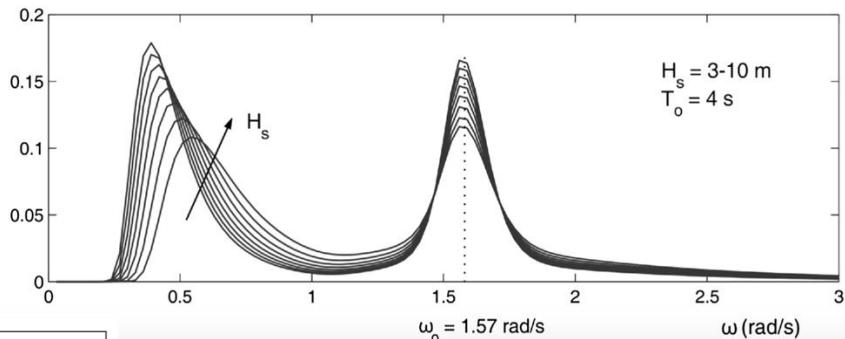
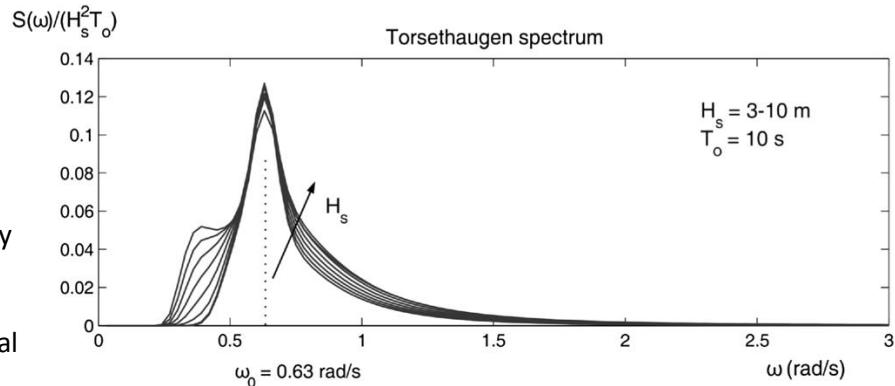
# 10.2 Wave Spectra

## Torsethaugen Spectrum

The Torsethaugen spectrum (1996) is an empirical, two-peaked spectrum. Includes the effect of swell (low-frequency peak) and newly developed waves (high-frequency peak).

The spectrum is developed using curve fitting of experimental data from the North Sea.

**Figure:** Upper plot shows only one peak at  $\omega_0 = 0.63$  rad/s representing swell and developing sea while the lower plot shows low-frequency swell and newly developing sea with peak frequency  $\omega_0 = 1.57$  rad/s.



### Matlab:

The Torsethaugen spectrum is included in the MSS toolbox as wave spectrum 7

```
S = wavespec(7, [Hs, w0], w, 1);
```

where  $H_s$  is the significant wave height,  $w_0$  is peak frequency and  $w$  is the wave frequency vector.

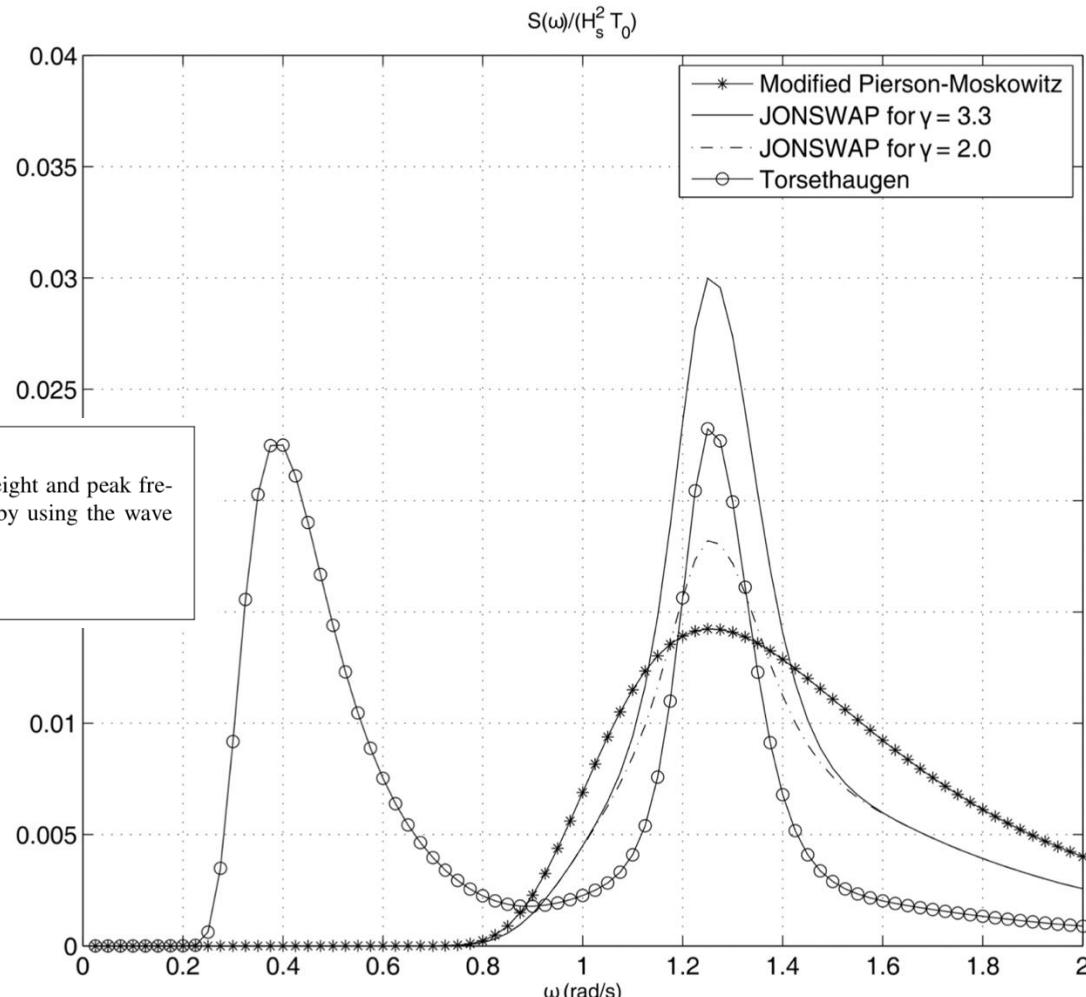
## 10.2 Wave Spectra

Wave spectra plotted for same wave height and peak frequency.

**Matlab:**

The different wave spectra when plotted for the same wave height and peak frequency are shown in Figure 10.11. The plots are generated by using the wave demo option in the MSS toolbox:

```
gncdemo;
```



# 10.2 Second-Order Wave Transfer Function Approximation

Transfer function

$$h(s) = \frac{\xi}{w}(s) = \frac{K_w s}{s^2 + 2\lambda\omega_0 s + \omega_0^2}$$

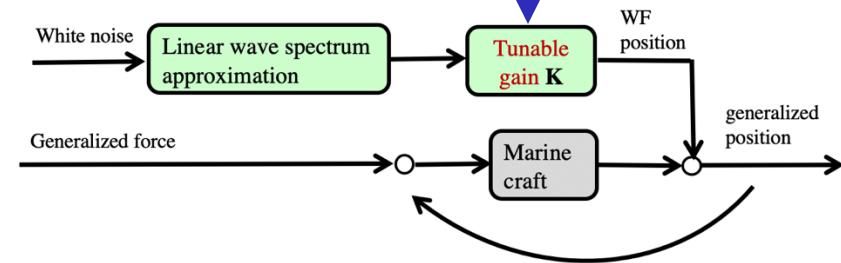
$$K_w = 2\lambda\omega_0\sigma$$

$$\sigma = \sqrt{\frac{A}{\omega_0^5} \exp\left(-\frac{B}{\omega_0^4}\right)}$$

Power spectral density

$$h(j\omega) = \frac{j 2(\lambda\omega_0\sigma)\omega}{(\omega_0^2 - \omega^2) + j 2\lambda\omega_0\omega}$$

$$P_{\xi\xi}(\omega) = |h(j\omega)|^2 P_{ww}(\omega) = |h(j\omega)|^2$$



### Matlab:

Power spectral density function:

```
function Pyy = Slin(lambda,w)
% Pyy = Slin(lambda,w) 2nd-order linear PSD function
% w = wave spectrum frequency (rad/s)
% lambda = relative damping factor

global sigma wo
Pyy = 4*(lambda*wo*sigma)^2*w.^2 ./ ( (wo^2-w.^2).^2 + ...
4*(lambda*wo.*w).^2 );
```

$$P_{\xi\xi}(\omega_0) = S(\omega_0)$$

⇓

$$\sigma^2 = \max_{0 < \omega < \infty} S(\omega)$$

# 10.2 Nonlinear Least-Squares Optimization of Linear Spectra

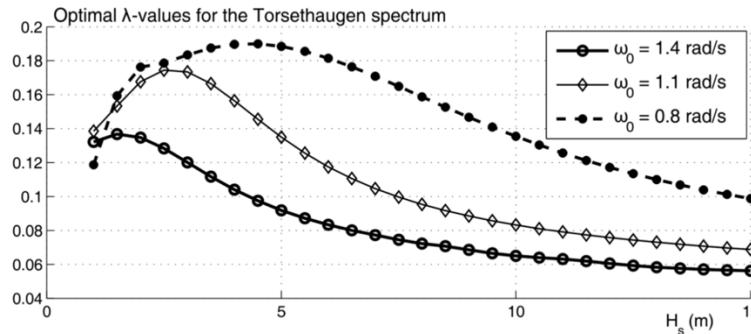
## Matlab:

Nonlinear least-squares:

```
% Script for plotting the nonlinear least-squares fit
% see ExLinspec.m
global sigma wo

wo = 0.8; Hs = 10; wmax = 3;
To = 2*pi/wo;
w = (0.0001:0.01:wmax)';

% Modified PM
subplot(311)
S = wavespec(3,[Hs,To],w,1); sigma = sqrt(max(S));
lambda = lsqcurvefit('Slin', 0.1, w, S)
hold on; plot(w,Slin(lambda,w), 'linewidth',2); hold off;
legend('Modified PM spectrum','Linear approximation')
```



```
% JONSWAP
subplot(312)
S = wavespec(7,[Hs,wo,3.3],w,1); sigma = sqrt(max(S));
lambda = lsqcurvefit('Slin', 0.1, w, S)
hold on; plot(w,Slin(lambda,w), 'linewidth',2); hold off;
legend('JONSWAP spectrum','Linear approximation')

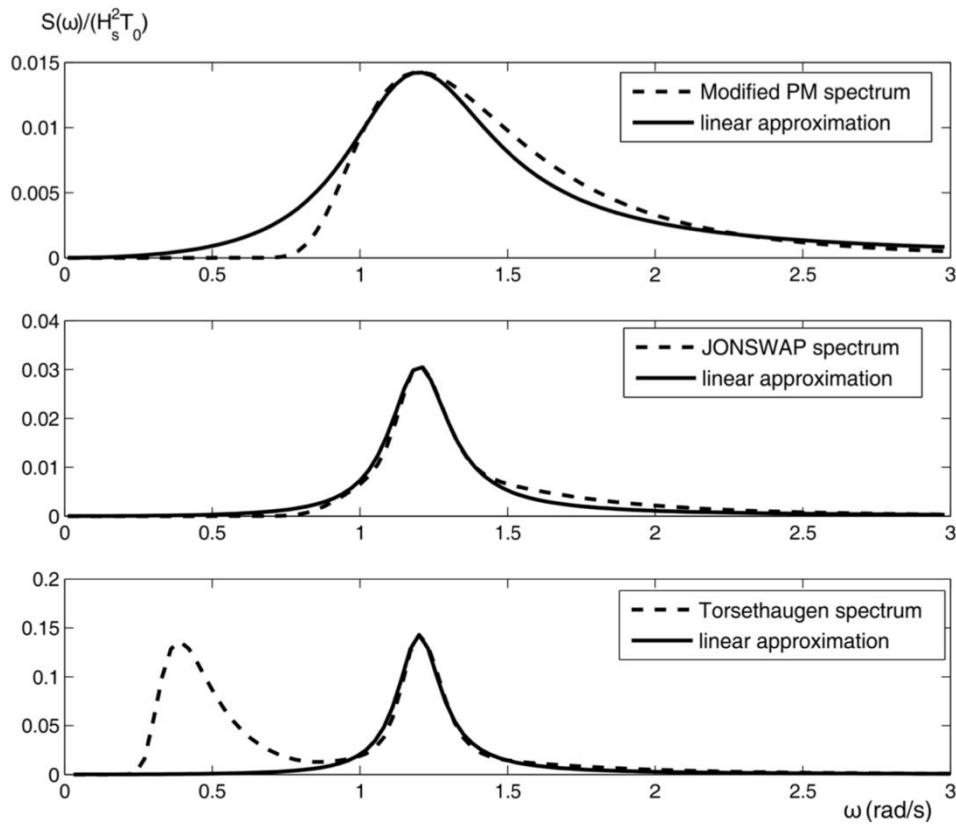
% Torsethaugen (only one peak is fitted)
subplot(313)
S = wavespec(8,[Hs,wo],w,1); sigma = sqrt(max(S));
lambda = lsqcurvefit('Slin', 0.1, w, S)
hold on; plot(w,Slin(lambda,w), 'linewidth',2); hold off;
legend('Torsethaugen spectrum','Linear approximation')
```

We can fit the nonlinear spectra to a linear model and find the optimal relative damping factor  $\lambda$  for varying frequencies  $\omega_0$

$$h(s) = \frac{\xi}{w}(s) = \frac{K_w s}{s^2 + 2\lambda\omega_0 s + \omega_0^2}$$

$\omega_0$	0.5	0.8	1.1	1.4	Recommended value
$\lambda$ (MPM)	0.2565	0.2573	0.2588	0.2606	0.26
$\lambda$ (JONSWAP)	0.1017	0.1017	0.1017	0.1017	0.10

## 10.2 Nonlinear Least-Squares Optimization of Linear Spectra



The nonlinear spectra compared with the transfer function model (linear approximation).

Quite good results are obtained.

One peak of the Torsethaugen spectrum is obviously missed (lower plot) by the linear approximation. However, we can use two transfer functions to correct this if needed.

## 10.2 Wave Amplitude Response Model

The relationship between the wave spectrum  $S(\omega_k)$  and the wave amplitude  $A_k$  for a wave component  $k$  is (Faltinsen 1990)

$$\frac{1}{2} A_k^2 = S(\omega_k) \Delta\omega$$

where  $\Delta\omega$  is a constant difference between the frequencies.

This relationship can be used to compute wave-induced responses in the time domain.

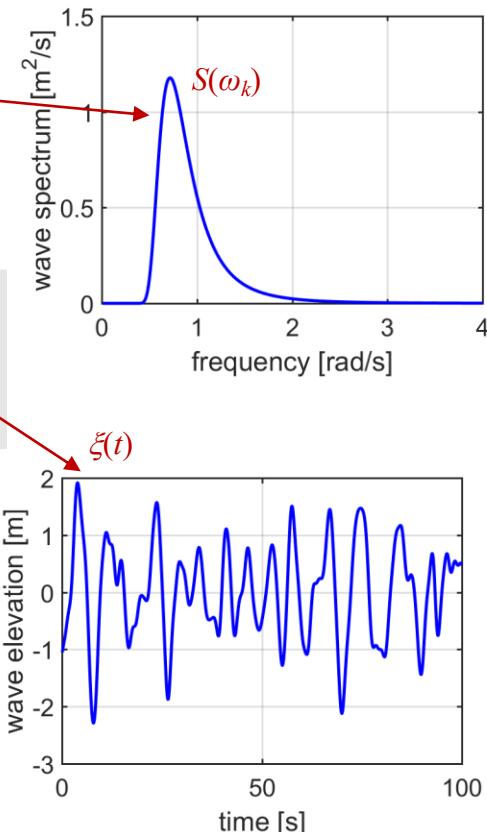
### Long-Crested Irregular Sea

The wave elevation of a long-crested irregular sea in the origin of  $\{s\}$  under the assumption of zero speed can be written as the sum of  $N$  harmonic components.

$$\begin{aligned} \xi &= \sum_{k=1}^N A_k \cos(\omega_k t + \epsilon_k) \\ &= \sum_{k=1}^N \sqrt{2S(\omega_k) \Delta\omega} \cos(\omega_k t + \epsilon_k) \end{aligned}$$

Here  $\epsilon_k$  is the random phase angle of wave component number  $k$ . Since this expression repeats itself after a time  $2\pi/\Delta\omega$  many wave components  $N$  are needed. However, a practical way to avoid this is to choose  $\omega_k$  randomly in the interval

$$\left[ \omega_k - \frac{\Delta\omega}{2}, \omega_k + \frac{\Delta\omega}{2} \right]$$



## 10.2 Wave Amplitude Response Model

### Short-Crested Irregular Sea

The most likely situation encountered at sea is short-crested or confused waves. This is observed as irregularities along the wave crest at right angles to the direction of the wind.

Short-crested irregular sea can be modeled by a 2-D wave spectrum

$$S_M(\omega, \mu) = S(\omega)M(\mu)$$

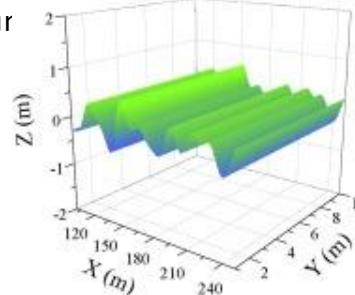
where  $\mu = 0$  corresponds to the main wave propagation direction while a nonzero  $\mu$  will spread the energy over a certain angle contained within  $[-\pi/2, \pi/2]$  from the wind direction.

### Spreading function

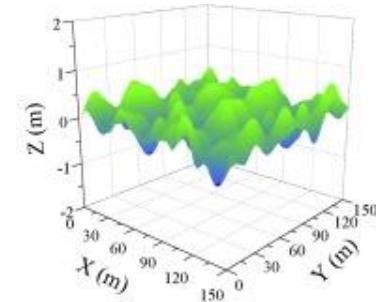
$$M(\mu) = \begin{cases} \frac{2}{\pi} \cos^2(\mu), & -\pi/2 \leq \mu \leq \pi/2 \\ 0, & \text{elsewhere} \end{cases}$$

### Wave elevation

$$\xi = \sum_{k=1}^N \sum_{i=1}^M \sqrt{2S_M(\omega_k, \beta_{\text{rel}} + \mu_i) \Delta\omega \Delta\mu} \cos(\omega_k t + \epsilon_k)$$

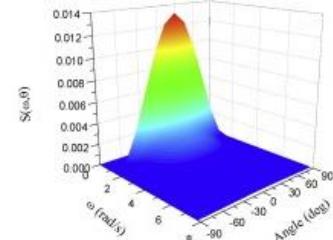


(a) Long-crested wave



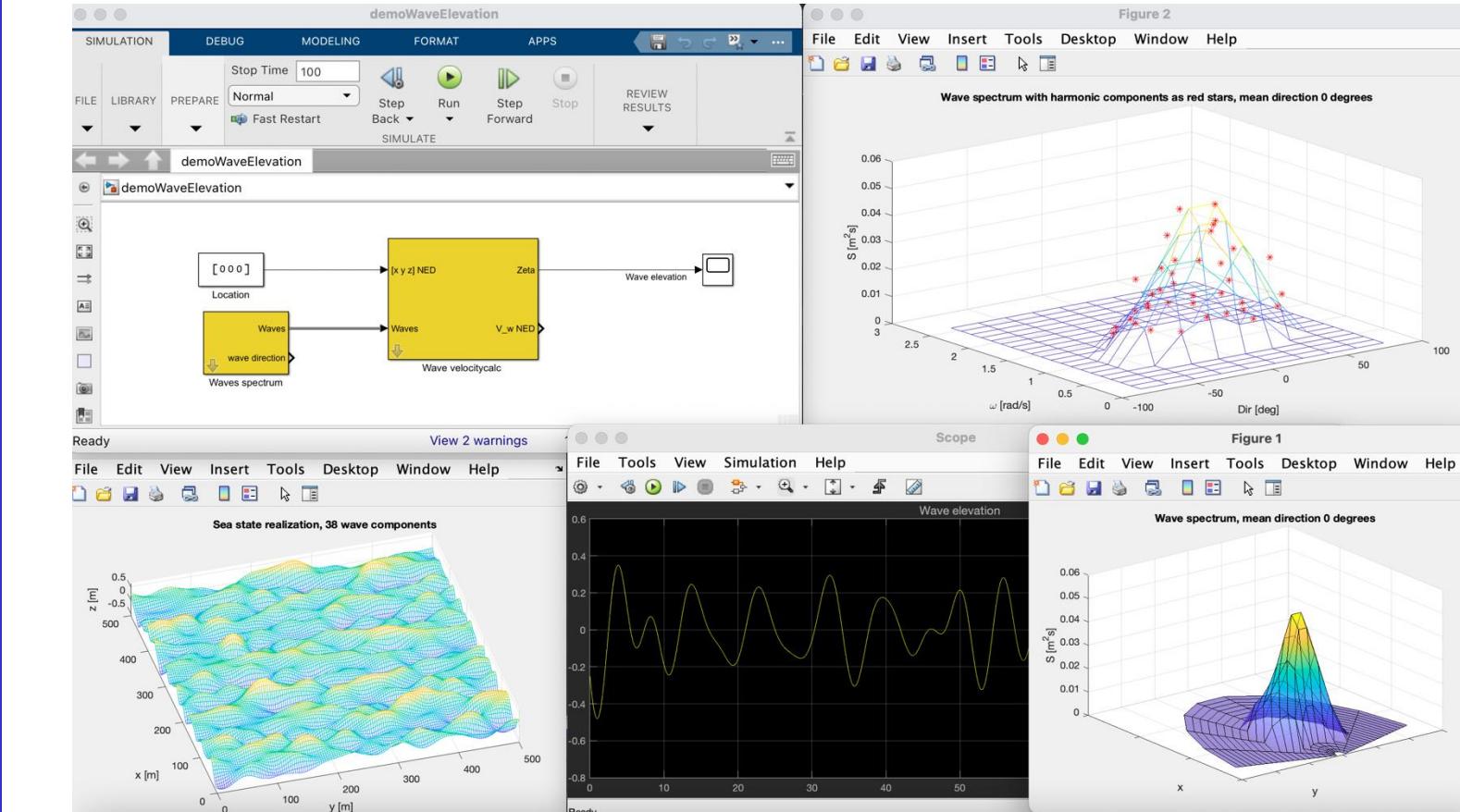
(b) Short-crested wave

$$\int_{-\pi/2}^{\pi/2} M(\mu) d\mu = 1$$



(c) Directional spectrum

# Simulink Demo: Wave Spectrum to Wave Elevation



# 10.2 Wave Amplitude Response Model

## Extension to Forward Speed using the Frequency of Encounter

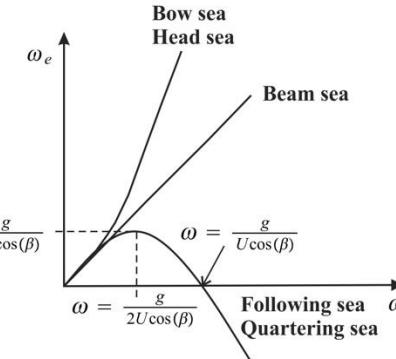
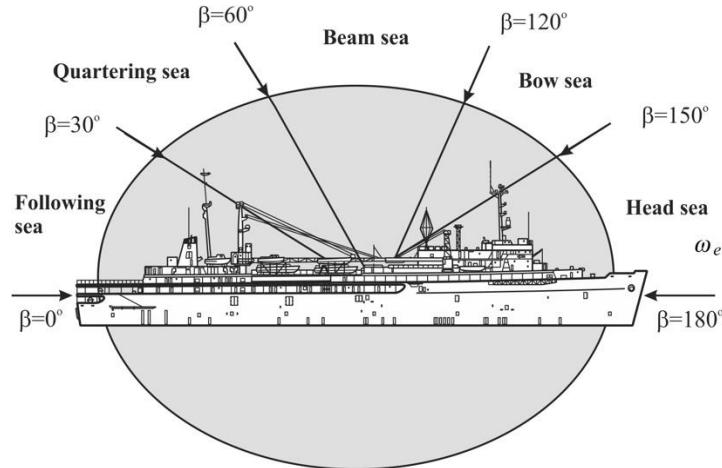
For a ship moving with forward speed  $U$ , the peak frequency of the spectrum  $\omega_0$  will be modified according to:

$$\omega_e(U, \omega_0, \beta_{\text{rel}}) = \left| \omega_0 - \frac{\omega_0^2}{g} U \cos(\beta_{\text{rel}}) \right|$$

$\omega_e$  is the frequency of encounter

$\beta_{\text{rel}}$  is the wave encounter angle relative the stern (not the sideslip angle!)

$\beta_{\text{rel}} = \beta - \psi$  is the relative wave encounter angle

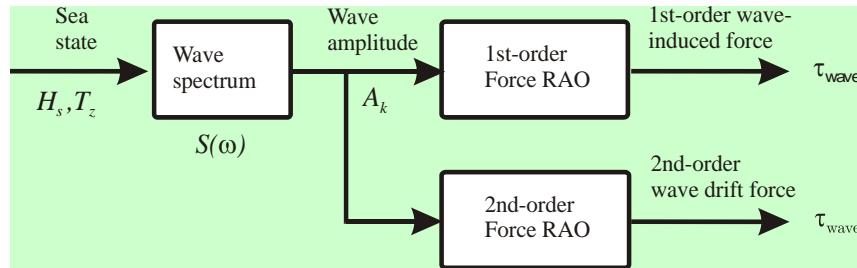


## Wave Elevation using Frequency of Encounter

$$\xi = \sum_{k=1}^N \sum_{i=1}^N \sqrt{2S_M(\omega_k, \mu_i - \beta)} \Delta\omega \Delta\mu \cos \left( \omega_k t - \frac{\omega_k^2}{g} U \cos(\mu_i - \beta) t + \epsilon_k \right)$$

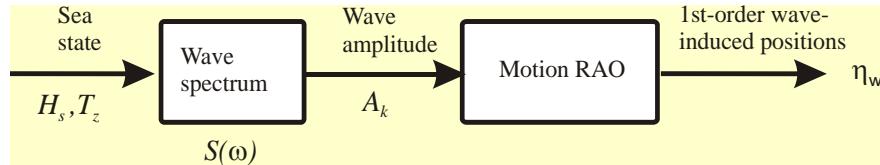
# 10.2 Wave Forces and Moments

## 1. Force RAOs



Generalized  
force

## 2. Motion RAOs



Generalized  
position

# 10.2 Wave Force Response Amplitude Operators

## Normalized Force RAOs

The 1st- and 2nd-order wave forces for varying wave directions  $\beta_i$  and wave frequencies  $\omega_k$  are

$$F_{\text{wave1}}^{\{\text{dof}\}}(\omega_k, \beta_i) = \left| \frac{\tilde{\tau}_{\text{wave1}}^{\{\text{dof}\}}(\omega_k, \beta_i)}{\rho g A_k} \right| e^{j\angle \tilde{\tau}_{\text{wave1}}^{\{\text{dof}\}}(\omega_k, \beta_i)}$$

$$F_{\text{wave2}}^{\{\text{dof}\}}(\omega_k, \beta_i) = \left| \frac{\tilde{\tau}_{\text{wave2}}^{\{\text{dof}\}}(\omega_k, \beta_i)}{\rho g A_k^2} \right| e^{j\angle \tilde{\tau}_{\text{wave2}}^{\{\text{dof}\}}(\omega_k, \beta_i)}$$

complex variables

The **Re-Im** parts of the RAOs (ASCII file generated by the hydrodynamic code) are processed in the MSS toolbox to obtain the **amplitudes** and **phases** of the forces.

### 1st-order wave-induced forces

$$\left| F_{\text{wave1}}^{\{\text{dof}\}}(\omega_k, \beta_i) \right| = \sqrt{(\text{ImRAO\_wave1}\{\text{dof}\}(k, i))^2 + (\text{ReRAO\_wave1}\{\text{dof}\}(k, i))^2}$$

$$\angle F_{\text{wave1}}^{\{\text{dof}\}}(\omega_k, \beta_i) = \text{atan2}(\text{ImRAO\_wave1}\{\text{dof}\}(k, i), \text{ReRAO\_wave1}\{\text{dof}\}(k, i))$$

### 2nd-order wave-induced forces (wave drift)

$$\left| F_{\text{wave2}}^{\{\text{dof}\}}(\omega_k, \beta_i) \right| = \text{ReRAO\_wave2}\{\text{dof}\}(k, i)$$

$$\angle F_{\text{wave2}}^{\{\text{dof}\}}(\omega_k, \beta_i) = 0$$

Matlab structures containing the amplitudes and phases

## 10.2 Wave Force Response Amplitude Operators

### Matlab:

The motion RAOs are processed in the MSS Hydro Matlab toolbox by using m-file commands:

```
wamit2vessel    % read and process WAMIT data  
veres2vessel    % read and process ShipX (Veres) data
```

The data are represented in the workspace as Matlab structures:

```
vessel.forceRAO.w(k)                      % frequencies  
vessel.forceRAO.amp{dof}(k,i,speed_no)      % amplitudes  
vessel.forceRAO.phase{dof}(k,i,speed_no)     % phases
```

where  $speed\_no = 1$  represents  $U = 0$ . For the mean drift forces only surge, sway and yaw are considered ( $dof \in \{1, 2, 6\}$  where the third component corresponds to yaw)

```
vessel.driftfrc.w(k)                      % frequencies  
vessel.driftfrc.amp{dof}(k,i,speed_no)      % amplitudes
```

It is possible to plot the force RAOs using

```
plotTF                      % plot transfer function  
plotWD                      % plot wave drift
```

There are no phase signal for the drift force, only amplitude is needed to simulate drift

How to use amplitude and phase to simulate the wave forces in Matlab? →

## 10.2 Wave Force Response Amplitude Operators

$$\begin{aligned}
 \tau_{\text{wave1}}^{\{\text{dof}\}} &= \sum_{k=1}^N \rho g \left| F_{\text{wave1}}^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}}) \right| A_k \cos \left( \omega_e(U, \omega_k, \beta_{\text{rel}})t + \angle F_{\text{wave1}}^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}}) + \epsilon_k \right) \\
 &= \sum_{k=1}^N \rho g \left| F_{\text{wave1}}^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}}) \right| \sqrt{2S(\omega_k)\Delta\omega} \\
 &\quad \cos \left( \omega_e(U, \omega_k, \beta_{\text{rel}})t + \angle F_{\text{wave1}}^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}}) + \epsilon_k \right)
 \end{aligned} \tag{10.93}$$

$$\begin{aligned}
 \tau_{\text{wave2}}^{\{\text{dof}\}} &= \sum_{k=1}^N \rho g \left| F_{\text{wave2}}^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}}) \right| A_k^2 \\
 &= \sum_{k=1}^N \rho g \left| F_{\text{wave2}}^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}}) \right| 2S(\omega_k)\Delta\omega
 \end{aligned} \tag{10.94}$$

where

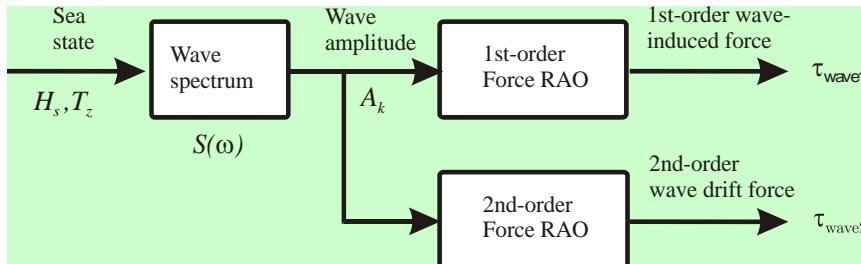
$$\omega_e(U, \omega_k, \beta_{\text{rel}}) = \omega_k - \frac{\omega_k^2}{g} U \cos(\beta_{\text{rel}}) \tag{10.95}$$

## 10.2 Wave Force Response Amplitude Operators

$$\begin{aligned} \tau_{\text{wave1}}^{\{\text{dof}\}} = & \sum_{k=1}^N \sum_{i=1}^M \rho g \left| F_{\text{wave1}}^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}} + \mu_i) \right| \sqrt{2S_M(\omega_k, \beta_{\text{rel}} + \mu_i) \Delta\omega \Delta\mu} \\ & \cos \left( \omega_e(U, \omega_k, \beta_{\text{rel}} + \mu_i) t + \angle F_{\text{wave1}}^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}} + \mu_i) + \epsilon_k \right) \end{aligned} \quad (10.96)$$

$$\tau_{\text{wave2}}^{\{\text{dof}\}} = \sum_{k=1}^N \sum_{i=1}^M \rho g \left| F_{\text{wave2}}^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}} + \mu_i) \right| 2S_M(\omega_k, \beta_{\text{rel}} + \mu_i) \Delta\omega \Delta\mu \quad (10.97)$$

**MSS Toolbox function:** [waveForceRAO.m](#)



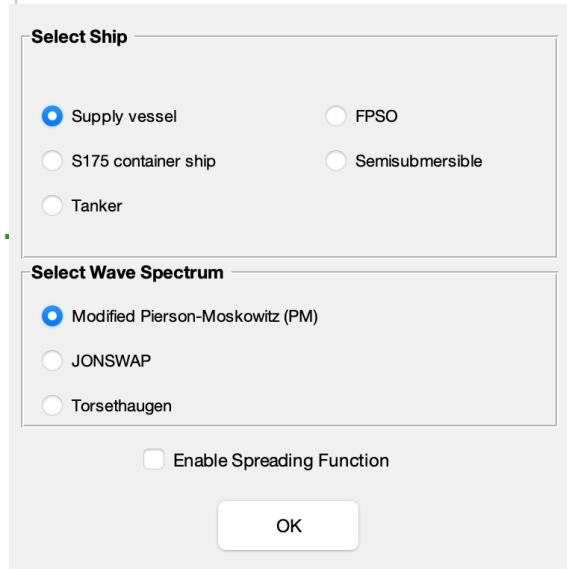
# MSS Toolbox

```

function [eta_WF, nu_WF, nudot_WF, waveElevation] = waveMotionRAO(
    t, S_M, Amp, Omega, mu, vessel, U, psi, beta_wave, numFreqIntervals)
% waveMotionRAO computes the wave elevation and the wave-frequency (WF)
% motion, eta_w, nu_w, and nu_dot_w on a ship using different wave spectra
% (Modified Pierson-Moskowitz, JONSWAP, Torsethaugen) and Response
% Amplitude Operators (RAOs) (Fossen 2021, Chapters 10.2.1 and 10.2.3).
% The total motion is:
%
%     y_eta    = eta + eta_w
%     y_nu     = nu  + nu_w
%     y_nudot = nudot + nudot_w
%
% where eta, nu, and nudot are the low-frequency (LF) ship model component.
% The real and imaginary parts of the RAO tables are interpolated in
% frequency and varying relative directions to compute the RAO amplitudes
% and phases. This approach avoids unwrapping problems and interpolation
% issues in RAO phase angles.
%
% INPUTS:
%     t          - Time vector (s)
%     S_M        - Spectral density matrix
%     Amp        - Wave amplitude matrix
%     Omega      - Wave frequencies (rad/s)
%     mu         - Wave spreading angles (radians)
%     vessel    - Vessel data structure containing RAO info
%     U          - Vessel speed (m/s)
%     psi        - Vessel heading angle (radians)
%     beta_wave - Wave direction, 0 following sea, pi head sea (radians)
%     numFreqIntervals - Number of frequency intervals (> 100)

```

**MSS Toolbox example script:**  
[exWaveForceRAO.m](#)



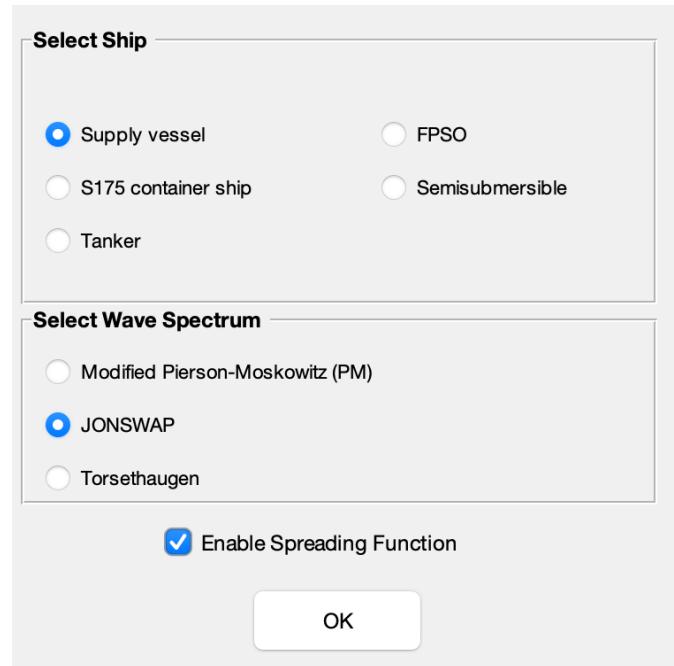
# MSS Toolbox

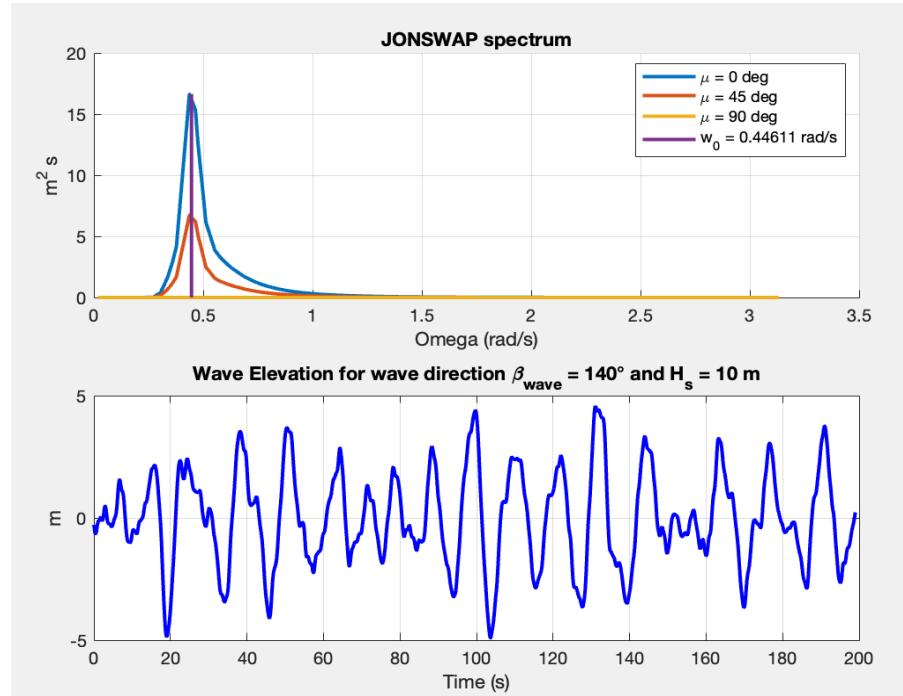
```
>> exWaveForceRAO
```

```
MSS toolbox: exWaveForceRAO.m:
```

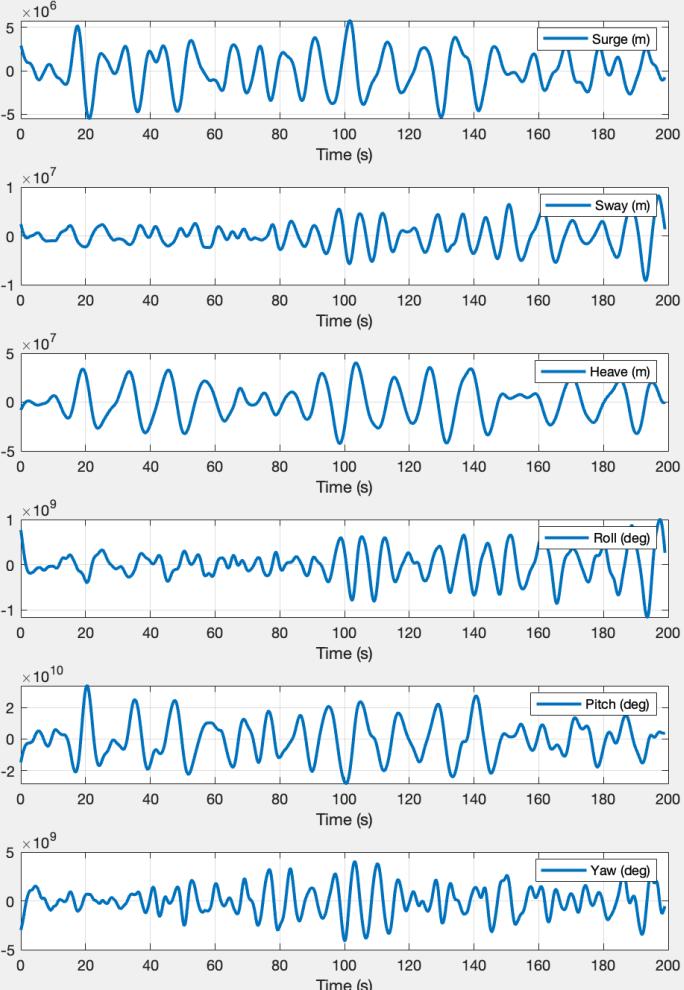
This function computes the 6-DOF generalized 1st-order wave forces,  $\tau_{\text{au\_wave1}}$ , on a marine craft using different wave spectra (Modified Pierson-Moskowitz, JONSWAP, and Torsethaugen) and Response Amplitude Operators (RAOs); see Fossen (2021, Chapters 10.2.1 and 10.2.4). The real and imaginary parts of the RAO tables are interpolated in frequency and varying wave directions to compute the RAO amplitudes and phases. This approach avoids unwrapping problems and interpolation issues in RAO phase angles.

```
Loaded the force RAO structure "vessel.forceRAO" (supply.mat)
Spectrum: JONSWAP with spreading function M(mu)
Significant wave height: Hs = 10 m
Zero crossing period: Tz = 10 s
Peak period: T0 = Tz / 0.710 = 14.0845 s
Peak frequency: w0 = 2*pi/T0 = 0.44611 rad/s
Number of wave frequencies: 100
Number of wave directions: 24
```





Generalized 1st-order Wave Forces for  $\beta_{\text{wave}} = 140^\circ$  and  $H_s = 10 \text{ m}$



# 10.2 Motion Response Amplitude Operators

Generalized Position (no spreading function)

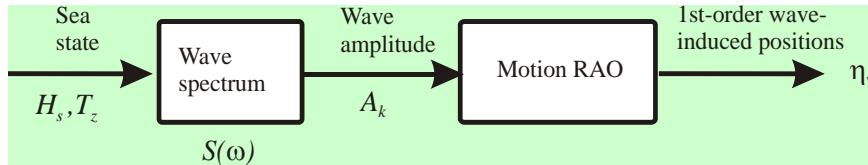
$$\begin{aligned}\eta_w^{\{\text{dof}\}} &= \sum_{k=1}^N \left| \eta_w^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}}) \right| A_k \cos \left( \omega_e(U, \omega_k, \beta_{\text{rel}})t + \angle \eta_w^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}}) + \epsilon_k \right) \\ &= \sum_{k=1}^N \left| \eta_w^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}}) \right| \sqrt{2S(\omega_k)\Delta\omega} \cos \left( \omega_e(U, \omega_k, \beta_{\text{rel}})t + \angle \eta_w^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}}) + \epsilon_k \right)\end{aligned}$$

where  $|\eta_w^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}})|$  and  $\angle \eta_w^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}})$  are the motion RAO amplitude and phase for frequency  $\omega_k$  and wave direction  $\beta_i$

Total Motion



Wave drift must be added manually to the equations of motion



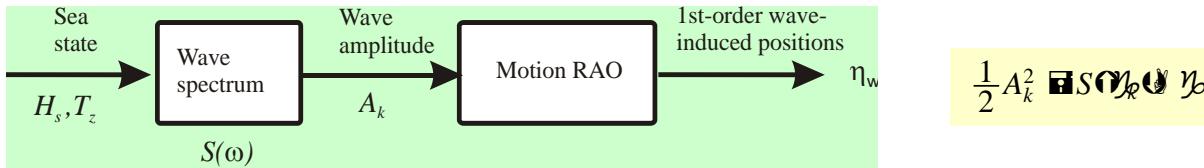
$$\frac{1}{2} A_k^2 \boxed{S(\omega) \eta_w^{\{\text{dof}\}}}$$

## 10.2 Motion Response Amplitude Operators

Generalized position (spreading function)

$$\eta_w^{\{\text{dof}\}} = \sum_{k=1}^N \sum_{i=1}^M \left| \eta_w^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}} + \mu_i) \right| \sqrt{2S_M(\omega_k, \beta_{\text{rel}} + \mu_i) \Delta\omega \Delta\mu} \cos \left( \omega_e(U, \omega_k, \beta_{\text{rel}} + \mu_i) t + \angle \eta_w^{\{\text{dof}\}}(\omega_k, \beta_{\text{rel}} + \mu_i) + \epsilon_k \right)$$

**MSS Toolbox function:** [waveMotionRAO.m](#)



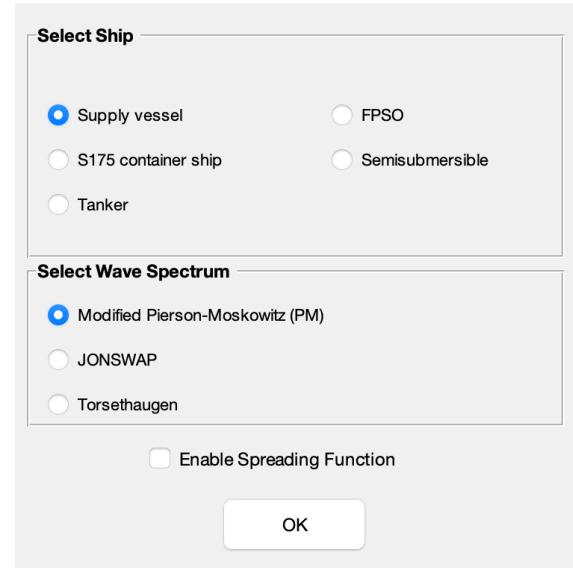
# MSS Toolbox

```

function [eta_WF, nu_WF, nudot_WF, waveElevation] = waveMotionRAO(...
    t, S_M, Amp, Omega, mu, vessel, U, psi, beta_wave, numFreqIntervals)
% waveMotionRAO computes the wave elevation and the wave-frequency (WF)
% motion, eta_w, nu_w, and nu_dot_w on a ship using different wave spectra
% (Modified Pierson-Moskowitz, JONSWAP, Torsethaugen) and Response
% Amplitude Operators (RAOs) (Fossen 2021, Chapters 10.2.1 and 10.2.3).
% The total motion is:
%
%     y_eta    = eta + eta_w
%     y_nu     = nu  + nu_w
%     y_nudot = nudot + nudot_w
%
% where eta, nu, and nudot are the low-frequency (LF) ship model component.
% The real and imaginary parts of the RAO tables are interpolated in
% frequency and varying relative directions to compute the RAO amplitudes
% and phases. This approach avoids unwrapping problems and interpolation
% issues in RAO phase angles.
%
% INPUTS:
%     t          - Time vector (s)
%     S_M        - Spectral density matrix
%     Amp        - Wave amplitude matrix
%     Omega      - Wave frequencies (rad/s)
%     mu         - Wave spreading angles (radians)
%     vessel     - Vessel data structure containing RAO info
%     U          - Vessel speed (m/s)
%     psi        - Vessel heading angle (radians)
%     beta_wave  - Wave direction, θ following sea, π head sea (radians)
%     numFreqIntervals - Number of frequency intervals (> 100)

```

**MSS Toolbox example script:**  
[exWaveMotionRAO.m](#)



# MSS Toolbox

>> exWaveMotionRAO

MSS toolbox: exWaveMotionRAO.m:

This function computes the wave elevation and the wave-frequency (WF) motion,  $\eta_w$ ,  $\nu_w$ , and  $\nu_{dot}w$  on a marine craft using different wave spectra (Modified Pierson-Moskowitz, JONSWAP, and Torsethaugen) and Response Amplitude Operators (RAOs); see Fossen (2021, Chapters 10.2.1 and 10.2.3). The total motion is:

```
y_eta = eta + eta_w
y_nu = nu + nu_w
y_nudot = nudot + nudot_w
```

where  $\eta$ ,  $\nu$ , and  $\nu_{dot}$  are the low-frequency (LF) ship model components. The real and imaginary parts of the RAO tables are interpolated in frequency and varying wave directions to compute the RAO amplitudes and phases. This approach avoids unwrapping problems and interpolation issues in RAO phase angles.

Loaded the motion RAO structure "vessel.motionRAO" (supply.mat)

Spectrum: JONSWAP (no spreading function)

Significant wave height:  $H_s = 10$  m

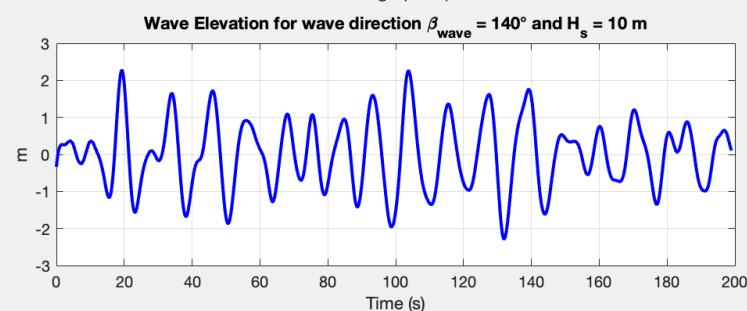
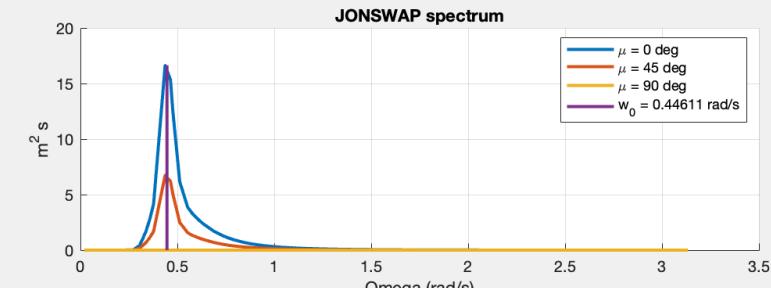
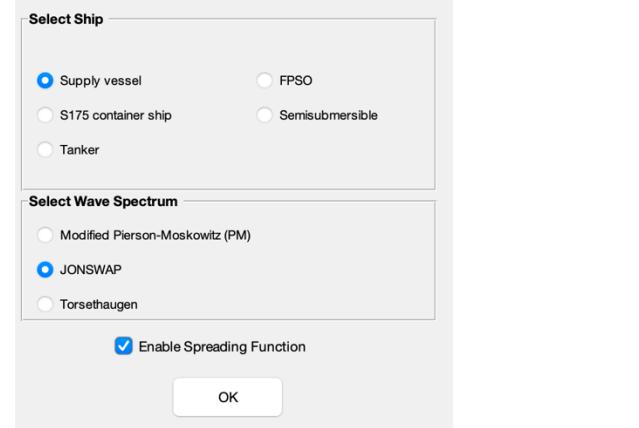
Zero crossing period:  $T_z = 10$  s

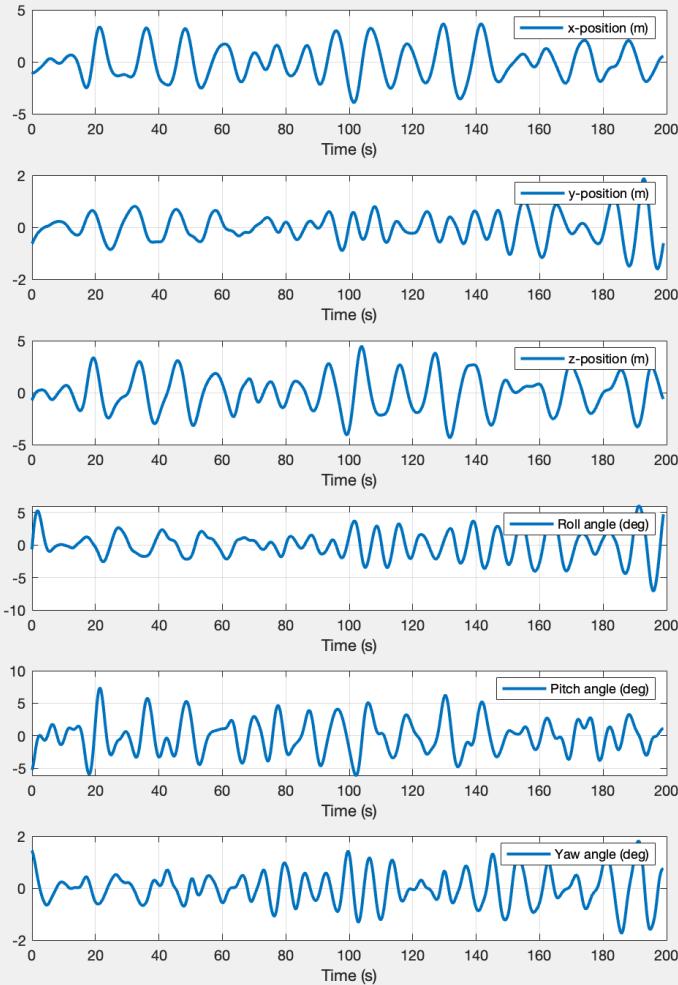
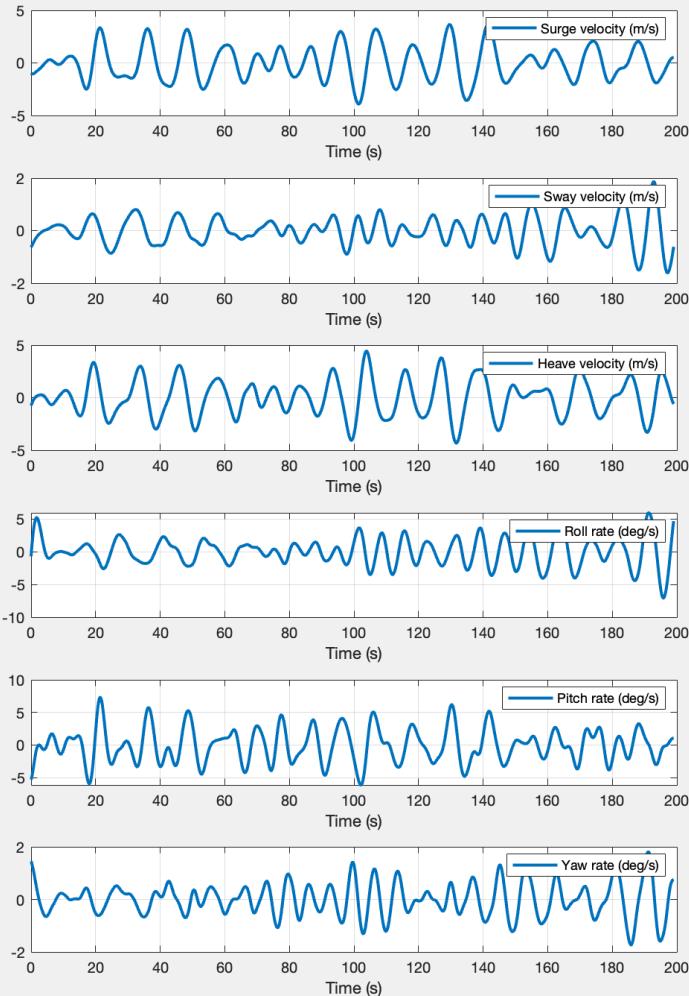
Peak period:  $T_0 = T_z / 0.710 = 14.0845$  s

Peak frequency:  $w_0 = 2\pi/T_0 = 0.44611$  rad/s

Number of wave frequencies: 100

Number of wave directions: 1



Wave-frequency positions and Euler angles for  $\beta_{\text{wave}} = 140^\circ$  and  $H_s = 10 \text{ m}$ Wave-frequency velocities for  $\beta_{\text{wave}} = 140^\circ$  and  $H_s = 10 \text{ m}$ 

# MSS Toolbox

The MSS functions `plotRAOamp` and `plotRAOphs` plots the motion and force RAOs using the data in the Matlab `vessel` structure.

```
load supply;

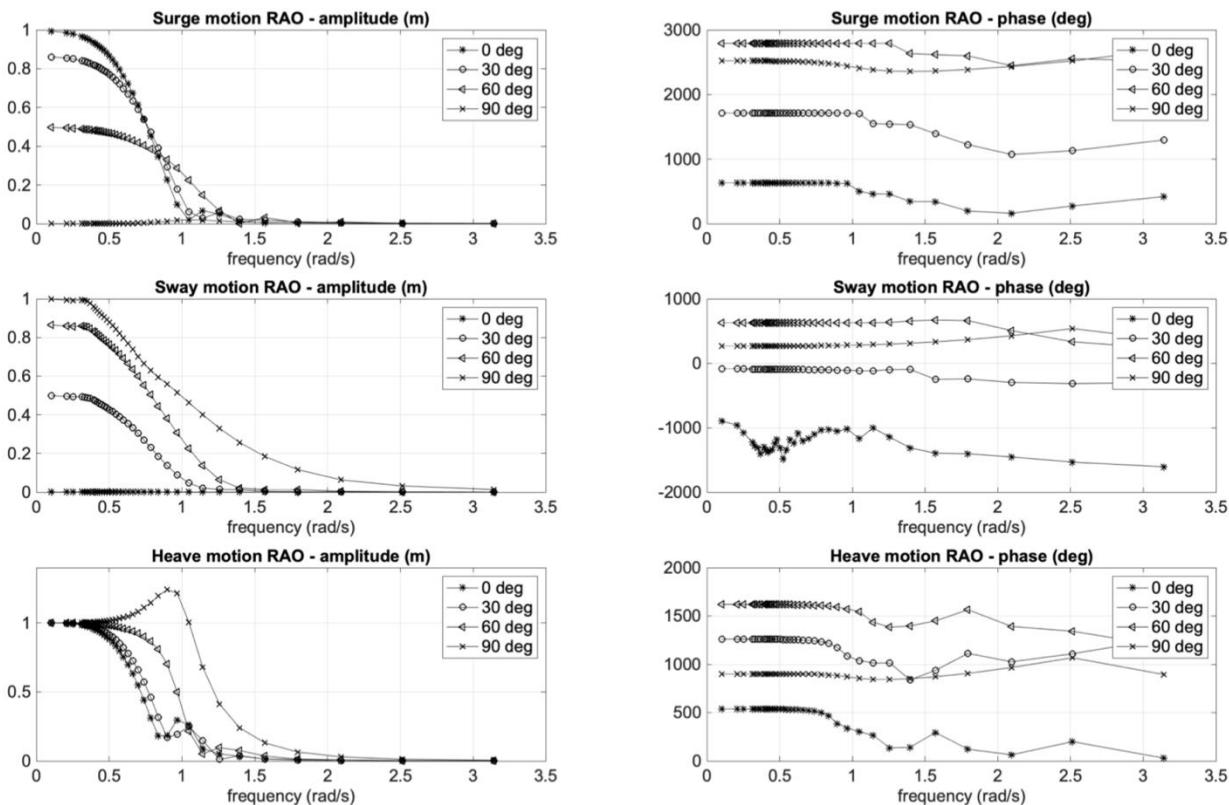
for DOF = 1:6
    figure(DOF);

    % Plot motion RAO
    w      = vessel.motionRAO.w;
    amp    = vessel.motionRAO.amp;
    phase  = vessel.motionRAO.phase;
    subplot(411); plotRAOamp(w,amp,DOF);
    subplot(412); plotRAOphs(w,phase,DOF);

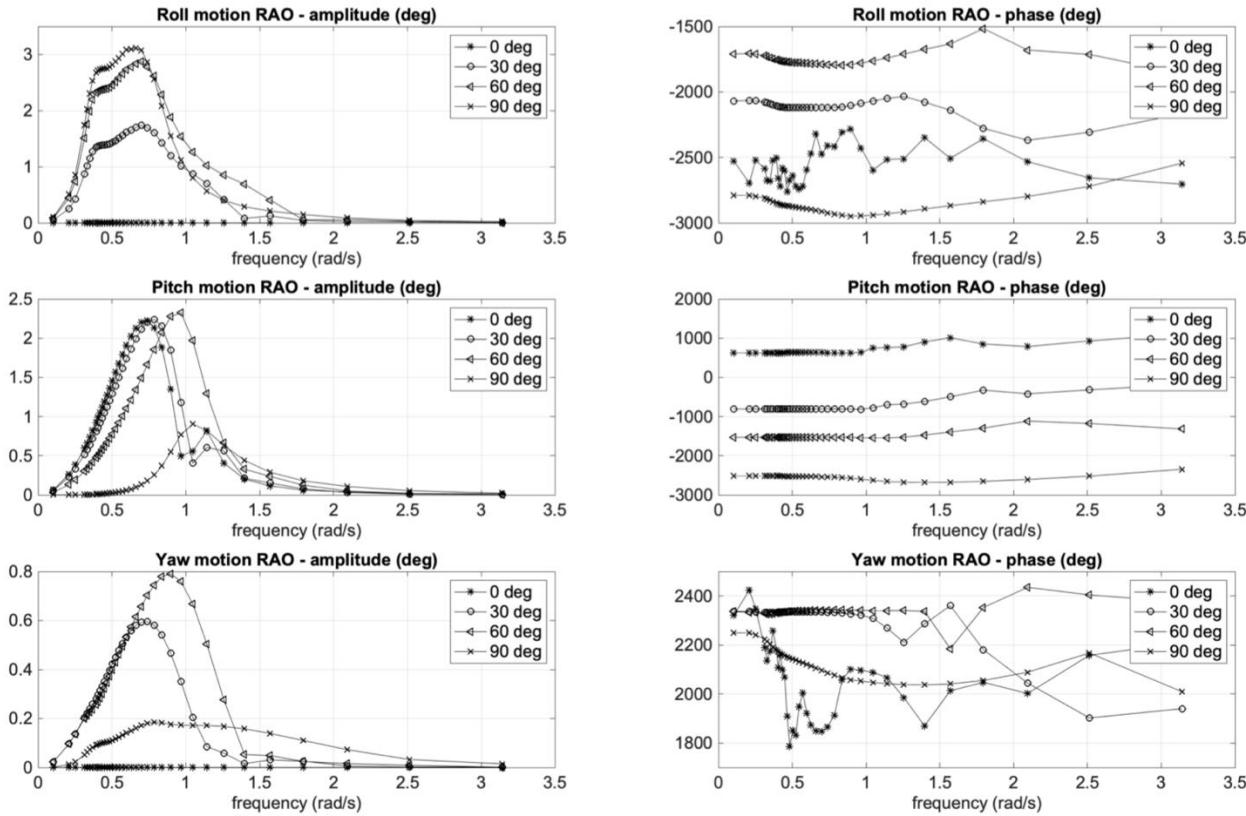
    % Plot force RAO
    w      = vessel.forceRAO.w;
    amp    = vessel.forceRAO.amp;
    phase  = vessel.forceRAO.phase;
    subplot(413); plotRAOamp(w,amp,DOF);
    subplot(414); plotRAOphs(w,phase,DOF);
end
```

```
function plotRAOamp(w,amp,DOF)
    velno = 1;
    arg = amp{DOF} (:,:,velno);
    plot(w,arg(:,1),'-*k',w, arg(:,4),'-ko',w, ...
          arg(:,7),'-k<',w, arg(:,10),'-kx');
    legend('0 deg','30 deg','60 deg','90 deg');
    xlabel('wave encounter frequency (rad/s)'), grid;
end

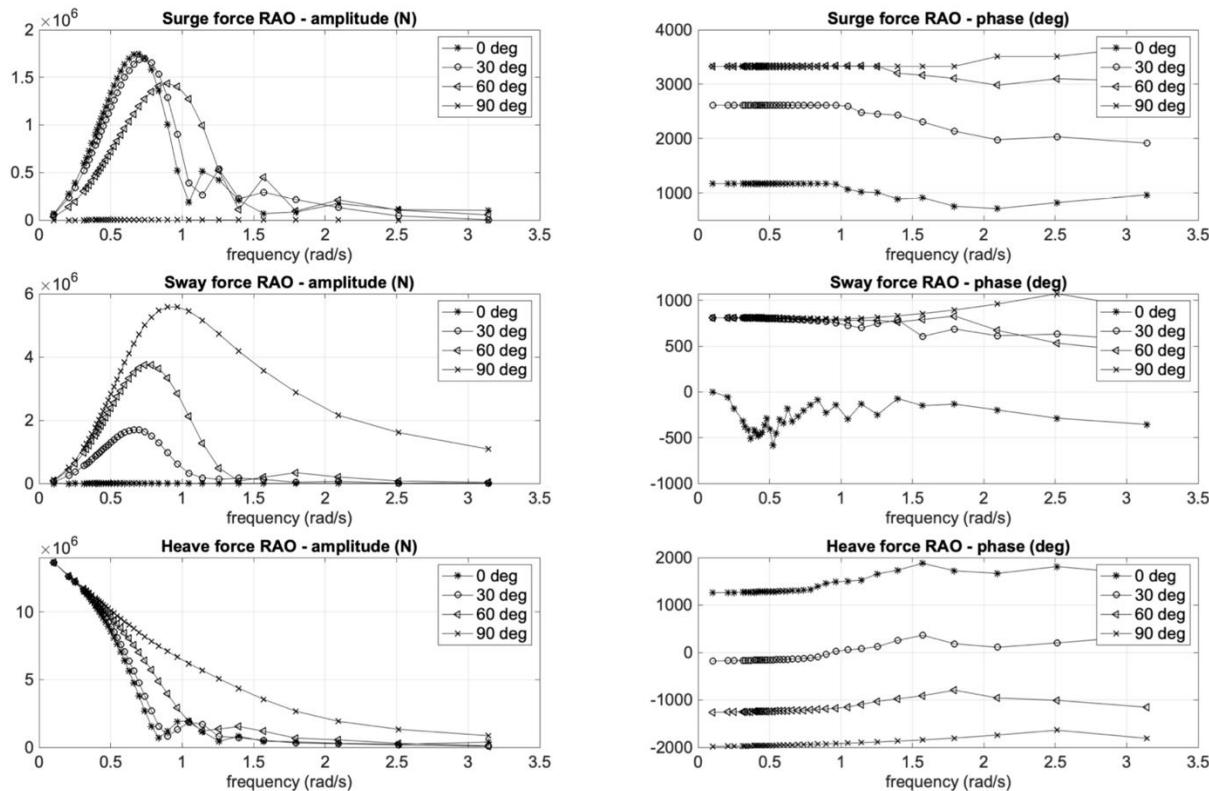
function plotRAOphs(w,phase,DOF)
    velno = 1;
    phs = (180/pi)*unwrap(phase{DOF} (:,:,velno));
    plot(w,phs(:,1),'-*k',w, phs(:,4),'-ko',w, ...
          phs(:,7),'-k<',w, phs(:,10),'-kx');
    legend('0 deg','30 deg','60 deg','90 deg');
    xlabel('wave encounter frequency (rad/s)'), grid;
end
```



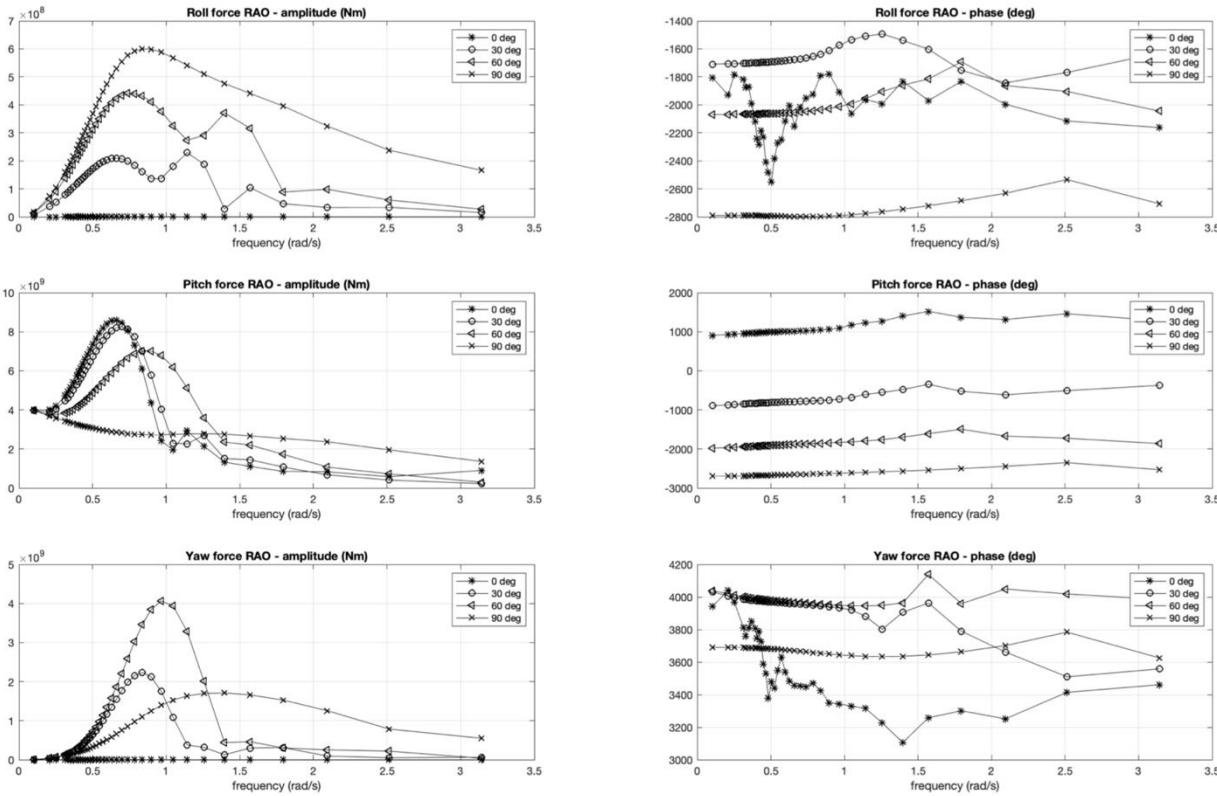
(a) Surge, sway and heave amplitudes (m) and phases (deg).



(b) Roll, pitch and yaw amplitudes (deg) and phases (deg)

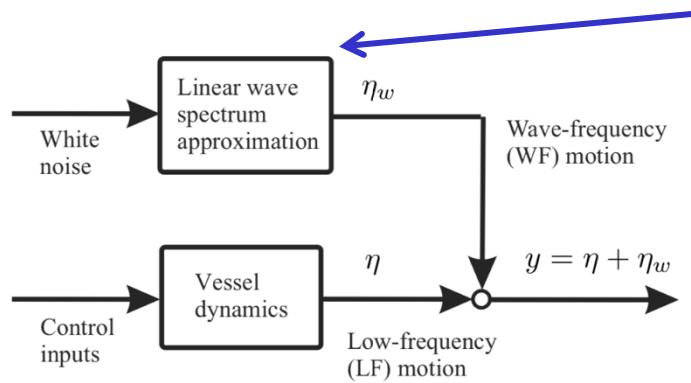


(a) Surge, sway and heave amplitudes (N) and phases (deg).



(b) Roll, pitch and yaw amplitudes (Nm) and phases (deg)

## 10.2 State-Space Models for Wave Response Simulation



$$\eta_{w,i}(s) = \frac{K_{w,i}s}{s^2 + 2\lambda\omega_e s + \omega_e^2} w_i(s)$$

$$\begin{aligned}\dot{\mathbf{x}}_{w,i} &= \begin{bmatrix} 0 & 1 \\ -\omega_e^2 & -2\lambda\omega_e \end{bmatrix} \mathbf{x}_{w,i} + \begin{bmatrix} 0 \\ K_{w,i} \end{bmatrix} w_i \\ \eta_{w,i} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}_{w,i}\end{aligned}$$

A linear wave response approximation for  $H(s)$  is usually preferred by ship control systems engineers. Easy to use and easy to test the performance and robustness of the control system.

No hydrodynamic SW is needed to the price of a tunable gain  $K_{w,i}$  which must be tuned manually in 6 DOFs to match the desired amplitudes of  $/_{w,i}$

## 10.2 State-Space Models for Wave Response Simulation

LF model:

$$\dot{\boldsymbol{\eta}} = \mathbf{J}_\Theta(\boldsymbol{\eta})(\boldsymbol{\nu}_r + \boldsymbol{\nu}_c)$$

$$\mathbf{M}\dot{\boldsymbol{\nu}}_r + \mathbf{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \mathbf{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \mathbf{g}(\boldsymbol{\eta}) + \mathbf{g}_0 = \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave2}} + \boldsymbol{\tau}$$

WF models:

$$\dot{\mathbf{x}}_{w,i} = \mathbf{A}_{w,i} \mathbf{x}_{w,i} + \mathbf{E}_{w,i} w_i, \quad (i = 1, 2, \dots, 6)$$

$$\eta_{w,i} = \mathbf{C}_{w,i} \mathbf{x}_{w,i}$$

Wave drift:

$$\dot{\mathbf{d}} = \mathbf{w}_d$$

$$\boldsymbol{\tau}_{\text{wave2}} = \mathbf{d}$$

Measurement:

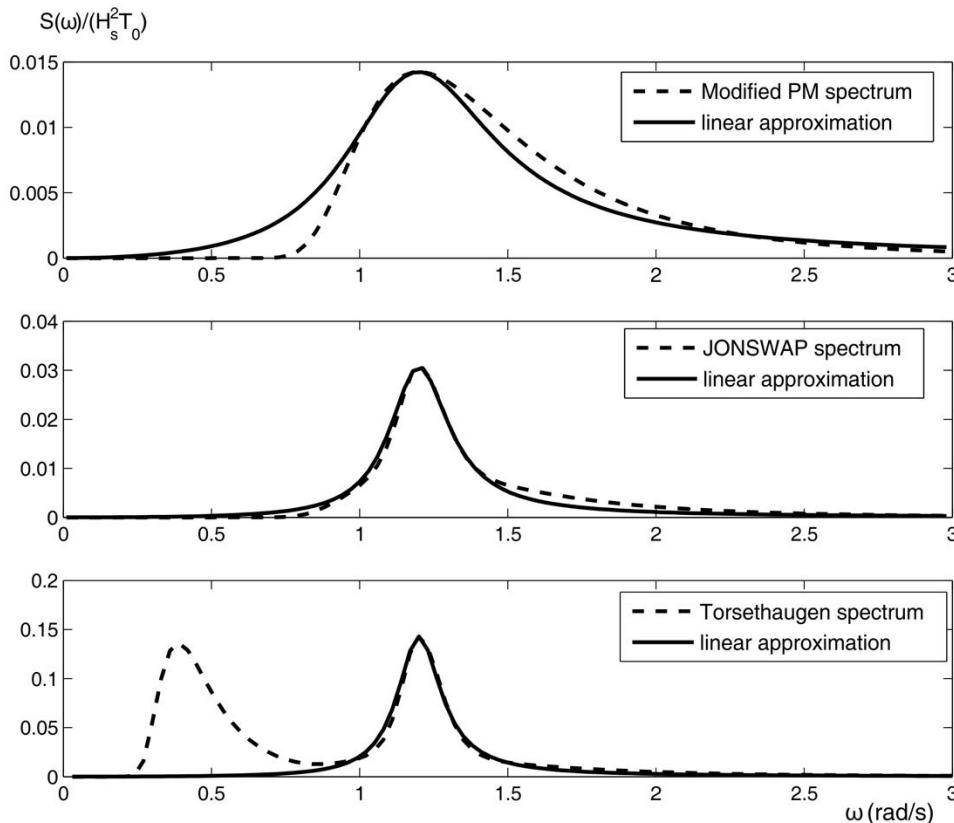
$$\mathbf{y} = \boldsymbol{\eta} + \boldsymbol{\eta}_w + \mathbf{v}$$

State-space model using 6 transfer functions for the WF motions and Wiener processes (random walks) for generalized wave drift forces

For surface craft only 3 transfer functions (surge, sway and yaw) are needed to simulate WF motions (6 ODEs) and wave drift can be represented by 3 ODEs, giving a total of 9 ODEs.



## 10.2 State-Space Models for Wave Responses



## 10.3 Ocean Current Forces and Moments

Ocean current forces and moments can be included in the equations of motion by replacing the absolute velocity terms with relative velocity (only the hydrodynamic terms)

Equations of relative motion

$$\underbrace{\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\eta}) + \mathbf{g}_0}_{\text{rigid-body and hydrostatic terms}} + \underbrace{\mathbf{M}_A\dot{\boldsymbol{\nu}}_r + \mathbf{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \mathbf{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r}_{\text{hydrodynamic terms}} = \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} + \boldsymbol{\tau}$$

**Definition 10.1 (Irrotational Constant Ocean Current)**  
 An irrotational constant ocean current in  $\{n\}$  is defined by

$$\dot{\boldsymbol{v}}_c^n = \dot{\mathbf{R}}(\boldsymbol{\Theta}_{nb})\boldsymbol{v}_c^b + \mathbf{R}(\boldsymbol{\Theta}_{nb})\dot{\boldsymbol{v}}_c^b := \mathbf{0}$$

where

$$\dot{\mathbf{R}}(\boldsymbol{\Theta}_{nb}) = \mathbf{R}(\boldsymbol{\Theta}_{nb})\mathbf{S}(\boldsymbol{\omega}_{nb}^b)$$

Consequently,

$$\dot{\boldsymbol{v}}_c^b = -\mathbf{S}(\boldsymbol{\omega}_{nb}^b)\boldsymbol{v}_c^b$$

$$\boldsymbol{\nu}_c = [\underbrace{u_c, v_c, w_c}_{\boldsymbol{v}_c^b}, 0, 0, 0]^\top$$

# 10.3 Ocean Current Forces and Moments

## Property 10.1 (Irrotational Constant Ocean Currents)

If the Coriolis and centripetal matrix  $C_{RB}(\nu_r)$  is parametrized independent of linear velocity  $\nu_1 = [u, v, w]^\top$ , for instance by using (3.63), and the ocean current is irrotational and constant (Definition 10.1), the rigid-body kinetics satisfies (Hegrenæs 2010)

$$M_{RB}\dot{\nu} + C_{RB}(\nu)\nu = M_{RB}\dot{\nu}_r + C_{RB}(\nu_r)\nu_r \quad (10.129)$$

with

$$\nu_r = \begin{bmatrix} v^b - v_c^b \\ \omega_{nb}^b \end{bmatrix} \quad (10.130)$$

Very useful result since it transforms the absolute velocity states to relative velocity states in the equations of motion

**Proof.** Since the Coriolis and centripetal matrix represented by (3.63) is independent of linear velocity  $\nu_1 = [u, v, w]^\top$ , it follows that  $C_{RB}(\nu_r) = C_{RB}(\nu)$ . The property

$$M_{RB}\dot{\nu}_c + C_{RB}(\nu_r)\nu_c = \mathbf{0} \quad (10.131)$$

is easily verified by expanding the matrices  $M_{RB}$  and  $C_{RB}(\nu_r)$  and corresponding acceleration and velocity vectors according to

$$\begin{bmatrix} mI_3 & -mS(r_{bg}^b) \\ mS(r_{bg}^b) & I_b \end{bmatrix} \begin{bmatrix} -S(\omega_{nb}^b)v_c^b \\ \mathbf{0}_{3 \times 1} \end{bmatrix} + \begin{bmatrix} mS(\omega_{nb}^b) & -mS(\omega_{nb}^b)S(r_{bg}^b) \\ mS(r_{bg}^b)S(\omega_{nb}^b) & -S(I_b\omega_{nb}^b) \end{bmatrix} \begin{bmatrix} v_c^b \\ \mathbf{0}_{3 \times 1} \end{bmatrix} = \mathbf{0}$$

Finally, it follows that

$$\begin{aligned} M_{RB}\dot{\nu} + C_{RB}(\nu)\nu &= M_{RB}[\dot{\nu}_r + \dot{\nu}_c] + C_{RB}(\nu_r)[\nu_r + \nu_c] \\ &= M_{RB}\dot{\nu}_r + C_{RB}(\nu_r)\nu_r \end{aligned}$$

$$\begin{aligned} &\underbrace{M_{RB}\dot{\nu} + C_{RB}(\nu)\nu + g(\eta) + g_0}_{\text{rigid-body and hydrostatic terms}} \\ &+ \underbrace{M_A\dot{\nu}_r + C_A(\nu_r)\nu_r + D(\nu_r)\nu_r}_{\text{hydrodynamic terms}} = \tau_{\text{wind}} + \tau_{\text{wave}} + \tau \end{aligned}$$

$$\begin{aligned} \dot{\eta} &= J_\Theta(\eta)(\nu_r + \nu_c) \\ \dot{\nu}_r &= M^{-1}(\tau_{\text{wind}} + \tau_{\text{wave}} + \tau - C(\nu_r)\nu_r - D(\nu_r)\nu_r - g(\eta) - g_0) \end{aligned}$$

$$\begin{aligned} M &= M_{RB} + M_A \\ C(\nu_r) &= C_{RB}(\nu_r) + C_A(\nu_r) \end{aligned}$$

# 10.3 Equations of Relative Motion

State-space model for relative velocity (6 DOF)

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= \mathbf{J}_\Theta(\boldsymbol{\eta}) [\boldsymbol{\nu}_r + \boldsymbol{\nu}_c] \\ \dot{\boldsymbol{\nu}}_r &= \mathbf{M}^{-1} (\boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} + \boldsymbol{\tau} - \mathbf{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \mathbf{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \mathbf{g}(\boldsymbol{\eta}) - \mathbf{g}_0)\end{aligned}$$

State-space model for absolute velocity (6 DOF)

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= \mathbf{J}_\Theta(\boldsymbol{\eta})\boldsymbol{\nu} & (10.141) \\ \dot{\boldsymbol{\nu}} &= \begin{bmatrix} -\mathbf{S}(\boldsymbol{\omega}_{nb}^b)\boldsymbol{v}_c^b \\ \mathbf{0}_{3 \times 1} \end{bmatrix} + \mathbf{M}^{-1} (\boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} - \mathbf{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \mathbf{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \mathbf{g}(\boldsymbol{\eta}) - \mathbf{g}_0)\end{aligned}$$

Surface craft (3 DOF)

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= \mathbf{R}(\psi)\boldsymbol{\nu} \\ \dot{\boldsymbol{\nu}} &= \begin{bmatrix} rv_c \\ -ru_c \\ 0 \end{bmatrix} + \mathbf{M}^{-1} (\boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} - \mathbf{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \mathbf{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r)\end{aligned}$$

Straight-line motion and stationkeeping (small turning rate,  $r = 0$ )

$$\dot{\boldsymbol{\nu}} = \mathbf{M}^{-1} (\boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} - \mathbf{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \mathbf{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r)$$

# 10.3 Current Velocity in NED and BODY

## Current Speed and Direction

The current speed is denoted by  $V_c$  while its direction relative to the moving craft is conveniently expressed in terms of two angles:  $\alpha_{V_c}$  and  $\beta_{V_c}$

For computer simulations the current velocity can be generated by using a 1st-order Gauss-Markov Process

$$\hat{V}_c \rightarrow V_c \rightarrow w$$

$$V_{\min} \hat{\rightarrow} V_c \hat{\rightarrow} V_{\max}$$

## Current velocities expressed in NED

$$\mathbf{v}_c^n = \begin{bmatrix} V_c \cos(\alpha_{V_c}) \cos(\beta_{V_c}) \\ V_c \sin(\beta_{V_c}) \\ V_c \sin(\alpha_{V_c}) \cos(\beta_{V_c}) \end{bmatrix}$$

## Current velocities expressed in BODY

$$\begin{bmatrix} u_c \\ v_c \\ w_c \end{bmatrix} = \mathbf{R}^\top(\boldsymbol{\Theta}_{nb}) \mathbf{v}_c^n$$

$$\mathbf{v}_c^n = \mathbf{R}_{y, \alpha_{V_c}}^\top \mathbf{R}_{z, -\beta_{V_c}}^\top \begin{bmatrix} V_c \\ 0 \\ 0 \end{bmatrix}$$

## Surface craft (2-D)

$$V_c = \sqrt{u_c^2 + v_c^2}$$

$$\mathbf{v}_c^n = \begin{bmatrix} V_c \cos(\beta_{V_c}) \\ V_c \sin(\beta_{V_c}) \\ 0 \end{bmatrix}$$

$$u_c = V_c \cos(\beta_{V_c} - \psi), \quad v_c = V_c \sin(\beta_{V_c} - \psi)$$



## 10.3 2-D Irrotational Ocean Current Model

Linear maneuvering model using relative sway velocity

$$\begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{v} - \dot{v}_c \\ \dot{r} \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} v - v_c \\ r \\ \psi \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} \delta + \begin{bmatrix} Y_{wind} \\ N_{wind} \\ 0 \end{bmatrix} + \begin{bmatrix} Y_{wave} \\ N_{wave} \\ 0 \end{bmatrix}$$

$u_c = V_c \cos(\beta_{V_c} - \psi)$   
 $v_c = V_c \sin(\beta_{V_c} - \psi)$   
 $\dot{v}_c = -ru_c$

Nonlinear state-space model where current speed is a state variable

$$\begin{bmatrix} \dot{v} \\ \dot{r} \\ \dot{\psi} \\ \dot{V}_c \end{bmatrix} = \begin{bmatrix} -rV_c \sin(\beta_{V_c} - \psi) \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{21} & m_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \left( \begin{bmatrix} b_1 \\ b_2 \\ 0 \\ 0 \end{bmatrix} \delta + \begin{bmatrix} Y_{wind} \\ N_{wind} \\ 0 \\ 0 \end{bmatrix} \right. \\
 \left. + \begin{bmatrix} Y_{wave} \\ N_{wave} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -d_{11} & -d_{12} & 0 & 0 \\ -d_{21} & -d_{22} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\mu \end{bmatrix} \begin{bmatrix} v \\ r \\ \psi \\ V_c \end{bmatrix} + \begin{bmatrix} d_{11}V_c \sin(\beta_{V_c} - \psi) \\ d_{21}V_c \sin(\beta_{V_c} - \psi) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w \right)$$



Note that the state-space model is nonlinear in  $\psi$ ,  $V_c$  and  $\beta_{V_c}$  even though the maneuvering model is linear

# Chapter Goals – Revisited

- Understand the principle for linear **superposition** of **wind** and **wave forces**
- Understand the **equations of relative motion** for **ocean currents**
- Be able to **compute the wind loads** on a marine craft using **wind coefficient** look-up tables and **projected areas** found from general arrangement drawings as required by the MSS functions: *Blendermann94* and *Isherwood72*.
- Understand how statistical descriptions can be used to define the **sea state** in terms of parameters such as wave period, wind speed, etc. for different regions on the Earth.
- Explain what we mean with regular and irregular waves, wave spectrum and directional wave spectrum.
- Be able to compute and simulate the **wave elevation** of regular and irregular waves for a given sea state and a wave spectrum.
- Understand the three methods for simulation and computation of wave loads:
  - **Motion RAOs**
  - **Force RAOs**
  - **Simplified transfer functions for WF motions**
- Be able to explain what we mean by **first- and second-order wave loads** and how these relate to hydrodynamic data computed in Wamit and ShipX.
- Be able to simulate ocean currents (magnitude and direction) in 2D and 3D and transform the results to current velocities.