Chapter 4 – Hydrostatics

4.1 Restoring Forces for Underwater Vehicles
4.2 Restoring Forces for Surface Vessels
4.3 Load Conditions and Natural Periods
4.4 Seakeeping Analysis
4.5 Ballast Systems

Archimedes (287-212 BC) derived the basic laws of fluid statics which are the fundamentals of hydrostatics today.

In hydrostatic terminology, the gravitational and buoyancy forces are called restoring forces, and they are equivalent to the spring forces in a mass-damper-spring system.

Archimedes (287-212 BC)
Wikipedia Commons

Chapter Goals

- Understand that the restoring forces behave like spring forces in 2nd-order systems and that they are only present in heave, roll and pitch.
- Be able to compute the restoring forces for both floating and submerged vehicles and understand the differences.
- Be able to explain what the "Metacenter" is and explain what we mean by metacentric stability.
- Be able to define the center of flotation and pivot point and explain which point a vehicle roll, pitch and yaw about.
- Understand how load conditions affect hydrostatic quantities such as heave, roll and pitch periods.
- Understand the concepts of manually and automatic pretrimming.

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_o = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}}$$

The main goal of this chapter is to understand how the restoring forces and ballast terms in the equations of motion are modeled.

Chapter 4 – Hydrostatics

In the derivation of the restoring forces and moments we will distinguish between two cases:

- **Section 4.1** Underwater vehicles (ROVs, AUVs and submarines)
- **Section 4.2** Surface vessels (ships, semisubmersibles and high-speed craft)

6-DOF equations of motion

$$\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\eta}) + \mathbf{g}_o = \boldsymbol{\tau} + \boldsymbol{\tau}_{wind} + \boldsymbol{\tau}_{wave}$$

Gravitational/buoyancy terms **Ballast control**

$\mathbf{M} = \mathbf{M}_{RB} + \mathbf{M}_A$	- system inertia matrix (including added mass)
$\mathbf{C}(\boldsymbol{\nu}) = \mathbf{C}_{RB}(\boldsymbol{\nu}) + \mathbf{C}_A(\boldsymbol{\nu})$	- Coriolis-centripetal matrix (including added mass)
$\mathbf{D}(\boldsymbol{\nu})$	- damping matrix
$\mathbf{g}(\boldsymbol{\eta})$	- vector of gravitational/buoyancy forces and moments
$\mathbf{g}_o$	- vector used for pretrimming (ballast control)
$\boldsymbol{\tau}$	- vector of control inputs
$\boldsymbol{\tau}_{wind}$	- vector of wind loads
$\boldsymbol{\tau}_{wave}$	- vector of wave loads

Archimedes (287-212 BC)
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4.1 Restoring Forces for Underwater Vehicles

Underwater Vehicles

According to the SNAME (1950) it is standard to express the submerged *weight* of the body and *buoyancy force* as

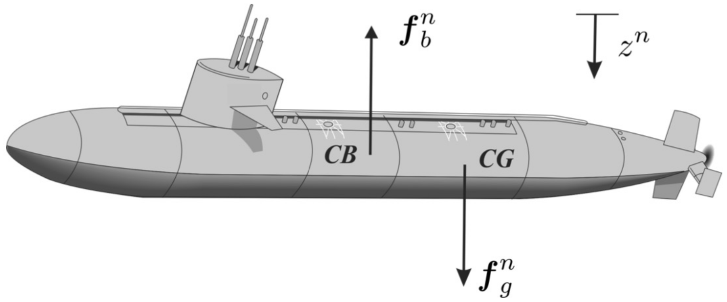
$$W = mg, \qquad B = \rho g \nabla$$

ρ = water density

∇ = volume of fluid displaced by the vehicle

m = mass of the vessel including water in free flooding space

g = acceleration of gravity



$$\mathbf{f}_g^n = \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix} \qquad \mathbf{f}_b^n = -\begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix}$$

The weight and buoyancy force can be transformed from NED to BODY by

$$\boldsymbol{f}_g^b = \boldsymbol{R}^\top(\boldsymbol{\Theta}_{nb})\boldsymbol{f}_g^n$$

$$\boldsymbol{f}_b^b = \boldsymbol{R}^\top(\boldsymbol{\Theta}_{nb})\boldsymbol{f}_b^n$$

4.1 Hydrostatics of Submerged Vehicles

The sign of the restoring forces and moments $\mathbf{f}_i^b$ and $\mathbf{m}_i^b = \mathbf{r}_i^b \times \mathbf{f}_i^b$ must be changed when moving these terms to the left-hand side of Newton's 2nd law, e.g. $ma = f \Longrightarrow ma - f = 0$

We denote the generalized restoring forces $\mathbf{g}(\boldsymbol{\eta})$. Notice that the force and moment vectors are multiplied with -1. Consequently, the generalized restoring force in BODY with coordinate origin CO becomes:

$$\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\eta}) + \mathbf{g}_o = \boldsymbol{\tau} + \boldsymbol{\tau}_{wind} + \boldsymbol{\tau}_{wave}$$

$$\boxed{\boldsymbol{g}(\boldsymbol{\eta}) = -\begin{bmatrix} \boldsymbol{f}_g^b + \boldsymbol{f}_b^b \\ \boldsymbol{r}_{bg}^b \times \boldsymbol{f}_g^b + \boldsymbol{r}_{bb}^b \times \boldsymbol{f}_b^b \end{bmatrix}}$$

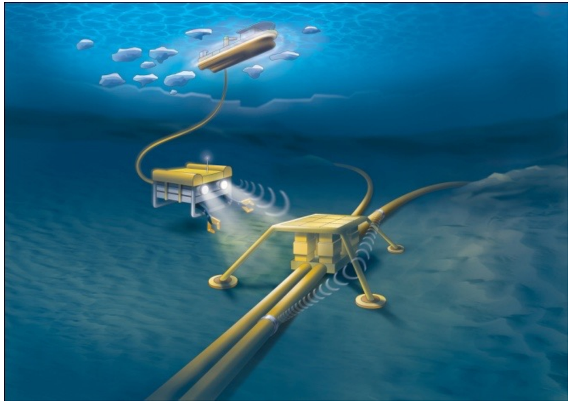
where

$\boldsymbol{r}_{bg}^b := [x_g, y_g, z_g]^\top$ center of buoyancy with respect to the CO

$\boldsymbol{r}_{bb}^b := [x_b, y_b, z_b]^\top$ center of gravity with respect to the CO

4.1 Hydrostatics of Submerged Vehicles

Main Result: Underwater Vehicles



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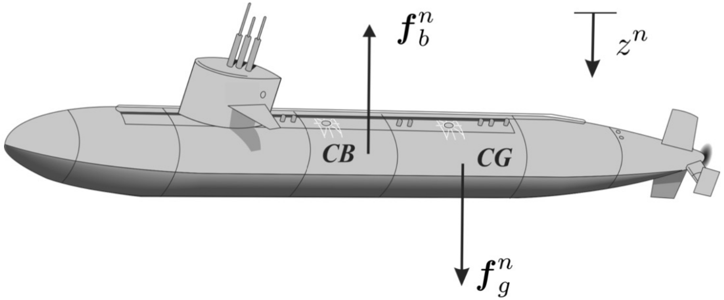


The 6-DOF gravity and buoyancy forces and moments about the CO are given by

$$\boldsymbol{g}(\boldsymbol{\eta}) = \begin{bmatrix} (W-B)\sin(\theta) \\ -(W-B)\cos(\theta)\sin(\phi) \\ -(W-B)\cos(\theta)\cos(\phi) \\ -(y_gW-y_bB)\cos(\theta)\cos(\phi)+(z_gW-z_bB)\cos(\theta)\sin(\phi) \\ (z_gW-z_bB)\sin(\theta)+(x_gW-x_bB)\cos(\theta)\cos(\phi) \\ -(x_gW-x_bB)\cos(\theta)\sin(\phi)-(y_gW-y_bB)\sin(\theta) \end{bmatrix}$$

4.1 Hydrostatics of Submerged Vehicles

Matlab:

The generalized restoring force $g(\eta)$ for an underwater vehicle can be computed with the CO as coordinate origin by using the MSS toolbox commands:

```
r_g = [0, 0, 0];      % location of the CG relative to the CO
r_b = [0, 0, -10];    % location of the CB relative to the CO
m = 1000;             % mass
g = 9.81;             % acceleration of gravity
W = m * g;            % weight
B = W;                % buoyancy

% pitch and roll angles
theta = 20 * (pi/180); phi = 30 * (pi/180);

% g vector expressed in the CO
g = gvect(W, B, theta, phi, r_g, r_b)

g =
   1.0e+04 *
         0
         0
         0
    4.6092
    3.3552
         0
```

4.1 Hydrostatics of Submerged Vehicles

Example 4.1 (Neutrally Buoyant Underwater Vehicles)

Let the distance between the center of gravity *CG* and the center of buoyancy *CB* be defined by the vector:

$$\boldsymbol{r}_{bg}^{b} := [\mathrm{BG}_x,\ \mathrm{BG}_y,\ \mathrm{BG}_z]^\top = [x_g - x_b,\ y_g - y_b,\ z_g - z_b]^\top$$

For neutrally buoyant vehicles $W = B$, and this simplifies to:

$$\boldsymbol{g}(\boldsymbol{\eta}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\mathrm{BG}_y W\cos(\theta)\cos(\phi) + \mathrm{BG}_z W\cos(\theta)\sin(\phi) \\ \mathrm{BG}_z W\sin(\theta) + \mathrm{BG}_x W\cos(\theta)\cos(\phi) \\ -\mathrm{BG}_x W\cos(\theta)\sin(\phi) - \mathrm{BG}_y W\sin(\theta) \end{bmatrix}$$

An even simpler representation is obtained for vehicles where the *CG* and *CB* are located vertically on the *z*-axis, that is $x_b = x_g$ and $y_g = y_b$. This yields

$$\boldsymbol{g}(\boldsymbol{\eta}) = [\ 0,\ \ 0,\ \ 0,\ \ \mathrm{BG}_z W\cos(\theta)\sin(\phi),\ \ \mathrm{BG}_z W\sin(\theta),\ \ 0\ ]^\top$$

```matlab
% exAUVHydrostatics is compatible with MATLAB and GNU Octave (www.octave.org)
% This script calculates various AUV parameters including dimensions,
% physical properties, center of gravity (CG), center of buoyancy (CB),
% moments of inertia, metacentric heights, and the g vector based on
% Fossen (2021, Chapter 4.1).
%
% References:
%     T. I. Fossen (2021). Handbook of Marine Craft Hydrodynamics and
%          Motion Control. 2nd Edition, Wiley. URL: www.fossen.biz/wiley

% AUV main characteristics
L_auv = 1.6;                % Length (m)
D_auv = 0.19;               % Cylinder diameter (m)

% Physical constants
rho = 1025;                 % Density of water (kg/m^3)
g = 9.81;                   % Gravitational acceleration (m/s^2)

% CG location relative to the midships coordinate origin (CO)
r_bg = [0 0 0.02]';         % CG position (m)

% Center of buoyancy (CB) location relative to the coordinate origin (CO)
r_bb = [0 0 0]';            % CB position (m)

% Rigid-body mass, Fossen (2021, Ch. 8.4.2)
a = L_auv / 2;              % Spheroid semi-axes a and b
b = D_auv / 2;
[MRB, CRB] = spheroid(a, b, [0 0 0]', r_bg);
m = MRB(1,1);
nabla = m / rho;  % Calculate volume displacement
W = m * g;        % Calculate the weight
B = W;            % Buoyancy = Weight

% User input for phi and theta
phi = input('Enter roll angle in degrees: ');
theta = input('Enter pitch in degrees: ');
theta = deg2rad(theta);
phi = deg2rad(phi);

% Compute the restoring forces and moments in the CO
gVect = gvect(W, B, theta, phi, r_bg, r_bb);
```

```
>> exAUVHydrostatics
Enter roll angle in degrees: 1
Enter pitch in degrees: 1
--------------------------------------------------------------------
AUV HYDROSTATICS
--------------------------------------------------------------------
Length (L):                                 1.60 m
Cylinder diameter (d):                      0.19 m
Mass (m):                                  31.03 kg
Density of water (rho):                  1025.00 kg/m^3
Volume displacement (nabla):                0.03 m^3
Weight (W):                               304.40 N
Buoyancy (B):                             304.40 N
Center of gravity (r_bg):              [0.0 0.0 0.0] m
Center of buoyancy (r_bb):             [0.0 0.0 0.0] m
--------------------------------------------------------------------
g VECTOR
--------------------------------------------------------------------
g =
         0
         0
         0
    0.1062
    0.1062
         0
```

4.2 Restoring Forces for Surface Vessels

For surface vessels, the restoring forces will depend on the craft's *metacentric height*, the location of the *CG* and the *CB* as well as the shape and size of the water plane. Let A_{wp} denote the water plane area and:

GM_T = transverse metacentric height (m)

GM_L = longitudinal metacentric height (m)

The metacentric height GM_i where $i = \{T,L\}$ is the distance between the metacenter M_i and CG.

Definition 4.1 (Metacenter)

The theoretical point M_i at which an imaginary vertical line through the CB intersects another imaginary vertical line through a new CB_1 created when the body is displaced, or tilted, in the water.

4.2 Hydrostatics of Floating Vessels

For a floating vessel at rest, buoyancy and weight are in balance such that

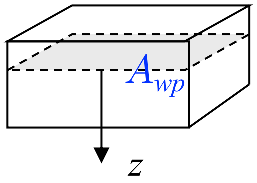
$$mg = \rho g \nabla$$

z displacement in heave

$z = 0$ is the equilibrium position

The hydrostatic force in heave is recognized as the difference of the gravitational and buoyancy forces

$$\begin{aligned} Z &= mg - \rho g(\nabla + \delta\nabla(z)) \\ &= -\rho g \delta\nabla(z) \end{aligned}$$

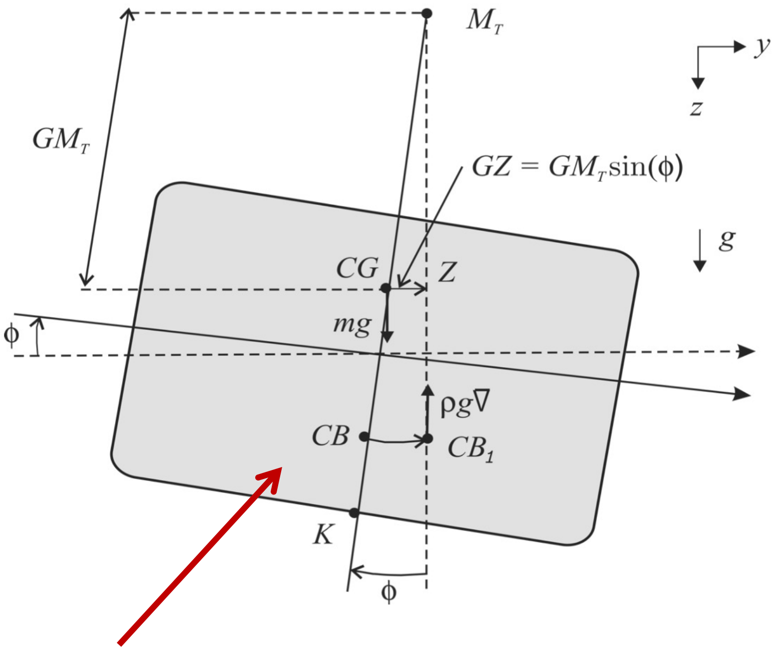


where the change in displaced water is

$$\delta\nabla(z) = \int_0^z A(\zeta)\mathrm{d}\zeta$$

$A(\zeta)$ is the waterplane area of the vessel as a function of the heave position

$A(0) := A_{wp}$



CB moves to CB_1 when the hull is rotated a roll angle ϕ.
CG is fixed for rigid bodies.

4.2 Hydrostatics of Floating Vessels

For conventional rigs and ships with box-shaped walls it can be assumed that

$A(0) := A_{wp}$

A is constant for small perturbations in *z*. Hence, the restoring force *Z* will be <u>linear</u> in *z*

$$Z_{\text{hs}} \approx -\rho g A_{wp} z := Z_z z$$

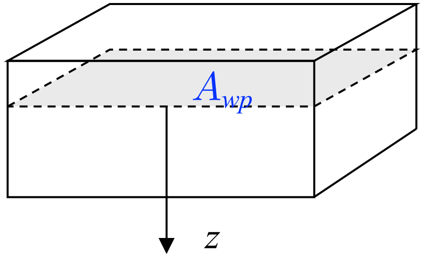
This is physically equivalent to a *mass—damper—spring* system.

$$\dot{z}^n = w$$

$$(m - Z_{\dot{w}})\dot{w} - Z_w w - Z_z z^n = Z_{\text{ext}}$$

The restoring forces and moments decomposed in BODY

$$\delta \boldsymbol{f}_b^n = \begin{bmatrix} 0 \\ 0 \\ -\rho g \int_0^z A(\zeta)\mathrm{d}\zeta \end{bmatrix} \qquad \boldsymbol{f}_g^n = -\boldsymbol{f}_b^n$$



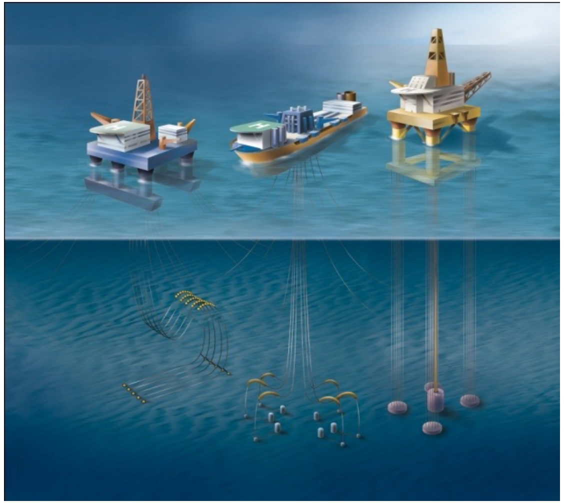
$$\boxed{\begin{aligned} \boldsymbol{f}_r^b &= \boldsymbol{R}^\top(\boldsymbol{\Theta}_{nb})\left(\boldsymbol{f}_g^n + \boldsymbol{f}_b^n + \delta\boldsymbol{f}_b^n\right) \\ &= -\rho g \begin{bmatrix} -\sin(\theta) \\ \cos(\theta)\sin(\phi) \\ \cos(\theta)\cos(\phi) \end{bmatrix} \int_0^z A(\zeta)\mathrm{d}\zeta \end{aligned}}$$

4.2 Hydrostatics of Floating Vessels

The moment arms in roll and pitch are $GM_T \sin\phi$ and $GM_L \sin\theta$, respectively.
Weight and buoyancy $W = B = \rho g \nabla$ act in the *z*-direction and they form a force pair. Hence,

$$\boldsymbol{r}^b_{\mathrm{GM}} = \begin{bmatrix} -\mathrm{GM}_L \sin(\theta) \\ \mathrm{GM}_T \sin(\phi) \\ 0 \end{bmatrix}$$

$$\boldsymbol{f}^b_b = \boldsymbol{R}^\top(\boldsymbol{\Theta}_{nb}) \begin{bmatrix} 0 \\ 0 \\ -\rho g \nabla \end{bmatrix} = -\rho g \nabla \begin{bmatrix} -\sin(\theta) \\ \cos(\theta)\sin(\phi) \\ \cos(\theta)\cos(\phi) \end{bmatrix}$$



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Neglecting the moment contribution due to $\delta\mathbf{f}^b_r$ (only considering $\mathbf{f}^b_r$) implies that the restoring moment becomes:

$$\boxed{\begin{aligned} \boldsymbol{m}^b_r &= \boldsymbol{r}^b_{\mathrm{GM}} \times \boldsymbol{f}^b_b \\ &= -\rho g \nabla \begin{bmatrix} \mathrm{GM}_T \sin(\phi)\cos(\theta)\cos(\phi) \\ \mathrm{GM}_L \sin(\theta)\cos(\theta)\cos(\phi) \\ -\mathrm{GM}_L \cos(\theta) + \mathrm{GM}_T \sin(\phi)\sin(\theta) \end{bmatrix} \end{aligned}}$$

$$\boxed{\boldsymbol{g}(\boldsymbol{\eta}) = -\begin{bmatrix} \boldsymbol{f}^b_r \\ \boldsymbol{m}^b_r \end{bmatrix}}$$

4.2 Hydrostatics of Floating Vessels

Main Result: Surface Vessels

6-DOF generalized gravity and buoyancy forces

$$\boldsymbol{g}(\boldsymbol{\eta}) = \begin{bmatrix} -\rho g \int_0^z A(\zeta)\mathrm{d}\zeta \ \sin(\theta) \\ \rho g \int_0^z A(\zeta)\mathrm{d}\zeta \ \cos(\theta)\sin(\phi) \\ \rho g \int_0^z A(\zeta)\mathrm{d}\zeta \ \cos(\theta)\cos(\phi) \\ \rho g \nabla \mathrm{GM}_T \sin(\phi)\cos(\theta)\cos(\phi) \\ \rho g \nabla \mathrm{GM}_L \sin(\theta)\cos(\theta)\cos(\phi) \\ \rho g \nabla(-\mathrm{GM}_L \cos\theta + \mathrm{GM}_T)\sin(\phi)\sin(\theta) \end{bmatrix}$$

4.2 Linear Small-Angle Theory for Boxed-Shaped Vessels

Linear Small-Angle Theory for Boxed-Shaped Vessels

Assumes that ϕ, θ, z are small such that

$$\boldsymbol{g}(\boldsymbol{\eta}) \approx \begin{bmatrix} -\rho g A_{wp} z\theta \\ \rho g A_{wp} z\phi \\ \rho g A_{wp} z \\ \rho g \nabla \mathrm{GM}_T \phi \\ \rho g \nabla \mathrm{GM}_L \theta \\ \rho g \nabla(-\mathrm{GM}_L + \mathrm{GM}_T)\phi\theta \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0 \\ \rho g A_{wp} z \\ \rho g \nabla \mathrm{GM}_T \phi \\ \rho g \nabla \mathrm{GM}_L \theta \\ 0 \end{bmatrix}$$

$$\sin\theta \approx \theta, \qquad \cos\theta \approx 1$$
$$\sin\phi \approx \phi, \qquad \cos\phi \approx 1$$
$$\int_0^z A(\zeta)\mathrm{d}\zeta \approx A_{wp} z$$

Equations of motion expressed in the CO

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_0 = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}}$$

$$\boldsymbol{g}(\boldsymbol{\eta}) \approx \boldsymbol{G}\boldsymbol{\eta}$$

If the G matrix is computed in the center of floatation (CF)

$$\boldsymbol{G}^{CF} = \mathrm{diag}\{0, 0, \rho g A_{wp}, \rho g \nabla \mathrm{GM}_T, \rho g \nabla \mathrm{GM}_L, 0\}$$

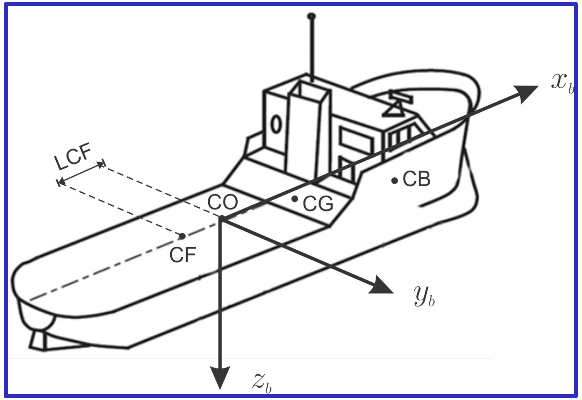
it is necessary to transform G^{CF} to the CO.

4.2 Linear Small-Angle Theory for Boxed-Shaped Vessels

The diagonal G^{CF} matrix assumes that CF is the coordinate origin.

If G^{CF} is transformed to the CO, two additional coupling terms $G_{35} = G_{53}$ appears

$$\mathbf{G} = \mathbf{G}^\top = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -Z_z & 0 & -Z_\theta & 0 \\ 0 & 0 & 0 & -K_\phi & 0 & 0 \\ 0 & 0 & -M_z & 0 & -M_\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} > 0$$



The coupling terms depend on the location of CF with respect to the CO: $\boldsymbol{r}^b_{bf} = [\mathrm{LCF}, 0, 0]^\top$

$$\boldsymbol{G} = \boldsymbol{H}^\top(\boldsymbol{r}^b_{bf})\boldsymbol{G}^{\mathrm{CF}}\boldsymbol{H}(\boldsymbol{r}^b_{bf})$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho g A_{wp} & 0 & -\rho g A_{wp}\,\mathrm{LCF} & 0 \\ 0 & 0 & 0 & \rho g \nabla \mathrm{GM}_T & 0 & 0 \\ 0 & 0 & -\rho g A_{wp}\,\mathrm{LCF} & 0 & \rho g \left(A_{wp}\,\mathrm{LCF}^2 + \nabla \mathrm{GM}_L \right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.2 Linear Small-Angle Theory for Boxed-Shaped Vessels

The first moment of areas are zero in the CF since the integrals are computed about the centroid (geometric center) of A_{wp}

$$\frac{1}{A_{wp}} \iint_{A_{wp}} x \mathrm{d}A = 0, \quad \frac{1}{A_{wp}} \iint_{A_{wp}} y \mathrm{d}A = 0$$

The second moment of areas are both positive

$$I_L := \iint_{A_{wp}} x^2 \mathrm{d}A, \quad I_T := \iint_{A_{wp}} y^2 \mathrm{d}A$$

The moment of areas are usually computed numerically by a 3-D hydrostatic program or approximative formulas.

For conventional ships, the CG and the CB lies on the same vertical line ($x_b = x_g$ and $y_b = y_g$) such that (Newman 1977)

$$\mathrm{GM}_T = \frac{I_T}{\nabla} + z_g - z_b$$
$$\mathrm{GM}_L = \frac{I_L}{\nabla} + z_g - z_b$$

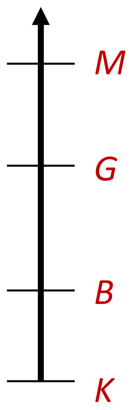
The GM-values can be computed using the moment of areas, CG, and CB.

Newman, J. N. (1977). Marine Hydrodynamics. MIT Press. Cambridge, MA.

4.2 Computation of Metacenter Height for Surface Vessels

The metacenter height *M* can be computed by using basic hydrostatics

$$\mathrm{GM}_T = \mathrm{BM}_T - \mathrm{BG}, \quad \mathrm{GM}_L = \mathrm{BM}_L - \mathrm{BG}$$



For small roll and pitch angles the transverse and longitudinal radius of curvature can be approximated by

$$\mathrm{BM}_T = \frac{I_T}{\nabla}, \quad \mathrm{BM}_L = \frac{I_L}{\nabla}$$

Moments of area about the waterplane

$$I_L = \iint_{A_{wp}} x^2 dA, \qquad I_T = \iint_{A_{wp}} y^2 dA$$

For conventional ships, an upper bound on these integrals can be found by considering a rectangular waterplane area $A_{wp} = BL$ where B and L are the beam and length of the hull upper bounded by

$$I_L < \frac{1}{12} L^3 B, \qquad I_T < \frac{1}{12} B^3 L$$

Munro-Smith formula

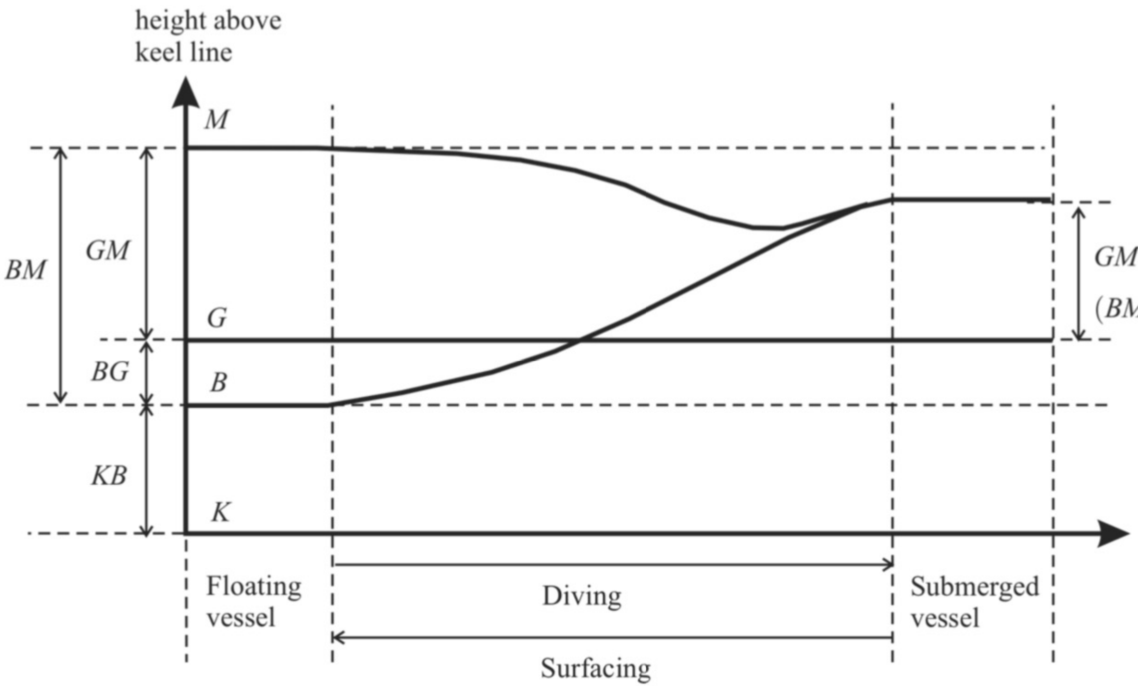
$$I_T = \frac{1}{12} B^3 L \frac{6C_w^3}{(1 + C_w)(1 + 2C_w)}$$

$C_w = A_{wp}/(LB)$ is the waterplane area coefficient

Morrish's formula

$$\mathrm{KB} = \frac{1}{3}\left(\frac{5T}{2} - \frac{\nabla}{A_{wp}}\right)$$

4.2 Computation of Metacenter Height for Surface Vessels



Metacenter *M*, center of gravity *G*, and center of buoyancy *B* for a submerged and a floating vessel.
The reference line is the keel line *K*.

4.2 Computation of Metacenter Height for Surface Vessels

Definition 4.2 (Metacentric Stability)
A floating vessel is said to be:

Transverse metacentrically stable if $GM_T \geq GM_{T,min} > 0$

Longitudinal metacentrically stable if $GM_L \geq GM_{L,min} > 0$

The metacentric stability of a vessel is concerned with its ability to return to an upright position after being tilted or heeled by external forces, such as waves or wind.

The lateral requirement is an important design criterion used to prescribe sufficient stability in **roll** to avoid that the vessel does not roll around. The vessel must also have damage stability (stability margins) in case of accidents.

A trade-off between stability and comfort should be made since a large stability margin will result in large restoring forces which can be quite uncomfortable for passengers (the mechanical equivalent is a stiff spring).

The longitudinal stability requirement in **pitch** is easy to satisfy for ships since the pitching motion is limited. This corresponds to a large GM_L value, typically $GM_{L,min} > 100.0$ m.

The lateral stability requirement in **roll** depends on the vessel type. The minimum GM_T value needed to perform an operations is specified by the classification societies such as DNV, ABS and Lloyds. The minimum value is $GM_{T,min} > 0.5$ m.

```matlab
% exShipHydrostatics is compatible with MATLAB and GNU Octave (www.octave.org).
% This script calculates various ship parameters including dimensions,
% physical properties, center of gravity (CG), center of buoyancy (CB),
% moments of inertia, metacentric heights, and the G matrix based on
% Fossen (2021 Chapter 4.2).
%
% References:
%      T. I. Fossen (2021). Handbook of Marine Craft Hydrodynamics and
%           Motion Control. 2nd. Edition, Wiley. URL: www.fossen.biz/wiley

% Ship main characteristics
L = 92;                      % Length (m)
B = 21;                      % Beam (m)
T = 6;                       % Draft (m)

% Physical constants
rho = 1025;                  % Density of water (kg/m^3)

% Mass properties and block coefficient: Cb = nabla / (L * B * T)
Cb = 0.75;                   % Block coefficient, dimensionless
nabla = Cb * L * B * T;      % Volume displacement (m^3)
m = rho * nabla;             % Mass (kg)

% CG location relative to the midships coordinate origin (CO)
r_bg = [-0.5 0 -1]';         % CG position (m)

% Waterplane area coefficient: Cw = Awp / (L * B)
Cw = 0.85;                   % Waterplane area coefficient, dimensionless
Awp = Cw * B * L;            % Waterplane area (m^2)

% Calculating KB (height of the center of buoyancy above keel)
KB = (1/3) * (5*T/2 - nabla/Awp);     % Equation (4.38)

% Munro-Smith coefficient calculation
k_munro_smith =  (6 * Cw^3) / ((1+Cw) * (1+2*Cw));     % Equation (4.37)

% Center of buoyancy (CB) location relative to the coordinate origin (CO)
r_bb = [-0.5 0 T-KB]';       % CB position (m)

% Vertical distance between CG and CB
BG = r_bb(3) - r_bg(3);      % Vertical distance (m)

% Moments of inertia
I_T = k_munro_smith * (B^3 * L) / 12;   % Transverse moment of inertia (m^4)
I_L = 0.7 * (L^3 * B) / 12;             % Longitudinal moment of inertia (m^4)

% Metacentric heights
BM_T = I_T / nabla;          % Transverse metacentric radius (m)
BM_L = I_L / nabla;          % Longitudinal metacentric radius (m)

% Metacentric heights (should be between 0.5 and 1.5 m for transverse)
GM_T = BM_T - BG;            % Transverse metacentric height (m)
GM_L = BM_L - BG;            % Longitudinal metacentric height (m)

% G matrix calculation
LCF = -0.5;                  % x-distance from the CO to the center of Awp
r_bp = [0 0 0]';             % Zero vector for G matrix computation

% Compute the G matrix in the CO
G = Gmtrx(nabla, Awp, GM_T, GM_L, LCF, r_bp);
```

```
-----------------------------------------------------------------
SHIP HYDROSTATICS
-----------------------------------------------------------------
Length (L):                                    92.00 m
Beam (B):                                      21.00 m
Draft (T):                                      6.00 m
Mass (m):                                 8911350.00 kg
Density of water (rho):                     1025.00 kg/m^3
Volume displacement (nabla):                8694.00 m^3
Block coefficient (C_b):                        0.75
Waterplane area coefficient (C_w):              0.85
Center of gravity (r_bg):                [-0.5 0.0 -1.0] m
Transverse metacentric height (GM_T):           2.26 m
Longitudinal metacentric height (GM_L):       105.95 m
-----------------------------------------------------------------
G MATRIX
-----------------------------------------------------------------
G =
   1.0e+09 *
         0         0         0         0         0         0
         0         0         0         0         0         0
         0         0    0.0165         0    0.0083         0
         0         0         0    0.1975         0         0
         0         0    0.0083         0    9.2666         0
         0         0         0         0         0         0
```

4.3 Load Conditions and Natural Periods

The load condition will determine the heave, roll, and pitch periods of a marine craft. The load condition varies over time (due to loading, offloading, fuel burning, water tanks, etc.)

$$(\boldsymbol{M}_{RB} + \boldsymbol{A}(\omega))\,\ddot{\boldsymbol{\xi}} + (\boldsymbol{B}(\omega) + \boldsymbol{B}_V(\omega))\,\dot{\boldsymbol{\xi}} + \boldsymbol{C}\boldsymbol{\xi} = \boldsymbol{\tau}_{\text{ext}}$$

In a linear system, the natural periods will be independent on the coordinate origin if they are computed using the 6-DOF coupled equations of motion. This is because the eigenvalues of a linear system do not change when applying a similarity transformation.

1-DOF Decoupled Analysis (Natural Periods)

The decoupled natural periods should be computed in the CF using the decoupled equations of motion. If not, the results can be very wrong since the eigenvalues of the decoupled equations depend on the coordinate origin as opposed to the 6-DOF coupled system

$$\boxed{\begin{aligned}
\left(m + A_{33}^{\mathrm{CF}}(\omega_3)\right)\ddot{z} + \left(B_{33}^{\mathrm{CF}}(\omega_3) + B_{v,33}^{\mathrm{CF}}(\omega_3)\right)\dot{z} + C_{33}^{\mathrm{CF}}\,z &= 0\\
\left(I_x^{\mathrm{CF}} + A_{44}^{\mathrm{CF}}(\omega_4)\right)\ddot{\phi} + \left(B_{44}^{\mathrm{CF}}(\omega_4) + B_{v,44}^{\mathrm{CF}}(\omega_4)\right)\dot{\phi} + C_{44}^{\mathrm{CF}}\phi &= 0\\
\left(I_y^{\mathrm{CF}} + A_{55}^{\mathrm{CF}}(\omega_5)\right)\ddot{\theta} + \left(B_{55}^{\mathrm{CF}}(\omega_5) + B_{v,55}^{\mathrm{CF}}(\omega_5)\right)\dot{\theta} + C_{55}^{\mathrm{CF}}\theta &= 0
\end{aligned}}$$

4.3 Load Conditions and Natural Periods

$$\omega_3 = \sqrt{\frac{C_{33}^{\mathrm{CF}}}{m + A_{33}^{\mathrm{CF}}(\omega_3)}}, \qquad T_3 = \frac{2\pi}{\omega_3}$$

$$\omega_4 = \sqrt{\frac{C_{44}^{\mathrm{CF}}}{I_x^{\mathrm{CF}} + A_{44}^{\mathrm{CF}}(\omega_4)}}, \qquad T_4 = \frac{2\pi}{\omega_4}$$

$$\omega_5 = \sqrt{\frac{C_{55}^{\mathrm{CF}}}{I_y^{\mathrm{CF}} + A_{55}^{\mathrm{CF}}(\omega_5)}}, \qquad T_5 = \frac{2\pi}{\omega_5}$$

The implicit equations for natural frequency must be solved by iteration. Note that added mass is a function of frequency.

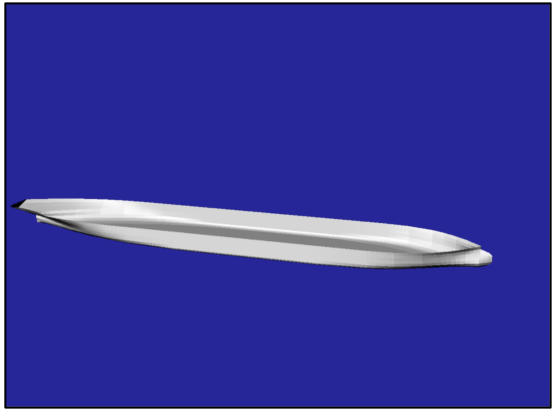
Matlab: fsolve.m (requires optimization toolbox), alternatively fzero.m

Matlab:
1-DOF decoupled analysis for computation of natural periods for the MSS tanker and supply vessel. The hydrodynamic data have been computed using WAMIT and ShipX.

```
% w_n = natfrequency(vessel,dof,w_0,speed,LCF)
% vessel = MSS vessel data (computed in CO)
% dof = degree of freedom (3,4,5), use -1 for 6-DOF analysis
% w_0 = initial natural frequency (typical 0.5)
% speed = index 1,2,3... for hydrodynamic data set
% LCF = (optionally) x coordinate from CO to CF (negative)

load tanker;    % WAMIT data file
T_3 = 2 * pi / natfrequency(vessel,3,0.5,1)
T_4 = 2 * pi / natfrequency(vessel,4,0.5,1)
T_5 = 2 * pi / natfrequency(vessel,5,0.5,1)

T_3 =
    9.6814
T_4 =
   12.5074
T_5 =
    9.0851
```



```
load supply;    % shipX data file
T_3 = 2 * pi / natfrequency(vessel,3,0.5,1)
T_4 = 2 * pi / natfrequency(vessel,4,0.5,1)
T_5 = 2 * pi / natfrequency(vessel,5,0.5,1)

T_3 =
    6.3617
T_4 =
   10.8630
T_5 =
    6.0988
```

4.3 Computation of Natural Periods in a 6-DOF Coupled System

6-DOF Coupled Analysis (Natural Periods)

$$(\boldsymbol{M}_{RB} + \boldsymbol{A}(\omega))\,\ddot{\boldsymbol{\xi}} + (\boldsymbol{B}(\omega) + \boldsymbol{B}_V(\omega) + \boldsymbol{K}_d)\,\dot{\boldsymbol{\xi}} + (\boldsymbol{C} + \boldsymbol{K}_p)\,\boldsymbol{\xi} = \boldsymbol{0}$$

$$\boldsymbol{K}_p = \mathrm{diag}(K_{p_{11}}, K_{p_{22}}, 0, 0, 0, K_{p_{66}})$$
$$\boldsymbol{K}_d = \mathrm{diag}(K_{d_{11}}, K_{d_{22}}, 0, 0, 0, K_{d_{66}})$$

A frequency-dependent modal analysis can be used to compute the natural frequencies

$$\boxed{\boldsymbol{M}(\omega)\ddot{\boldsymbol{\xi}} + \boldsymbol{D}(\omega)\dot{\boldsymbol{\xi}} + \boldsymbol{G}\boldsymbol{\xi} = \boldsymbol{0}}$$

$$\begin{aligned} \mathbf{M}(\omega) &= \mathbf{M}_{RB} + \mathbf{A}(\omega) \\ \mathbf{D}(\omega) &= \mathbf{B}(\omega) + \mathbf{B}_V(\omega) + \mathbf{K}_d \\ \mathbf{G} &= \mathbf{C} + \mathbf{K}_p \end{aligned}$$

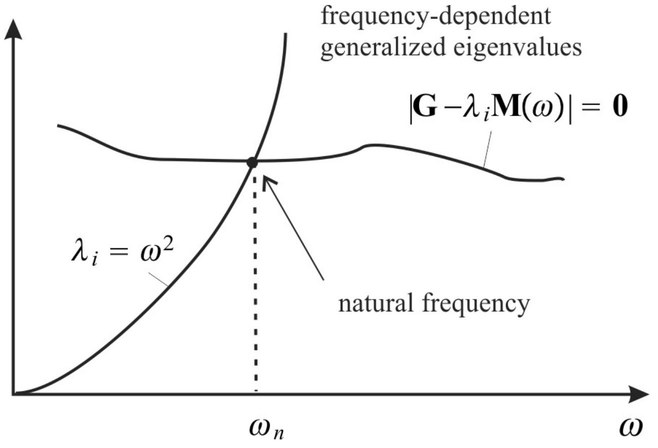
Assume that the floating vessel carries out harmonic oscillations such that

$$[\mathbf{G} - \omega^2\mathbf{M}(\omega) - j\omega\mathbf{D}(\omega)]\mathbf{a} = \mathbf{0}$$

The natural frequencies are computed for the undamped system $\mathbf{D}(\omega) = \mathbf{0}$ which gives

$$\|\boldsymbol{G} - \lambda_i \boldsymbol{M}(\omega)\| = \boldsymbol{0} \qquad \lambda_i = \omega^2$$

This is a frequency-dependent eigenvalue problem



The eigenvalues must be computed for all frequencies

4.3 Computation of Natural Periods in a 6-DOF Coupled System

Matlab:
The 6-DOF coupled eigenvalue analysis in Section 4.3.2 is implemented in the MSS toolbox. The natural periods for tanker is computed by using the following commands:

```
dof = -1;          % use -1 for 6-DOF analysis
load tanker;       % load WAMIT tanker data
T = 2 * pi ./ natfrequency(vessel, dof, 0.5, 1)

T =
    9.8261
   12.4543
    8.9536

load supply;       % load ShipX supply ship data
T = 2 * pi ./ natfrequency(vessel, dof, 0.5, 1)

T =
    6.5036
   10.4205
    6.0210
```

4.3 Load Conditions and Natural Periods

Radius of gyration

Offshore vessels

$$R_{44} \approx 0.37B$$
$$R_{55} = R_{66} \approx 0.25L_{pp}$$

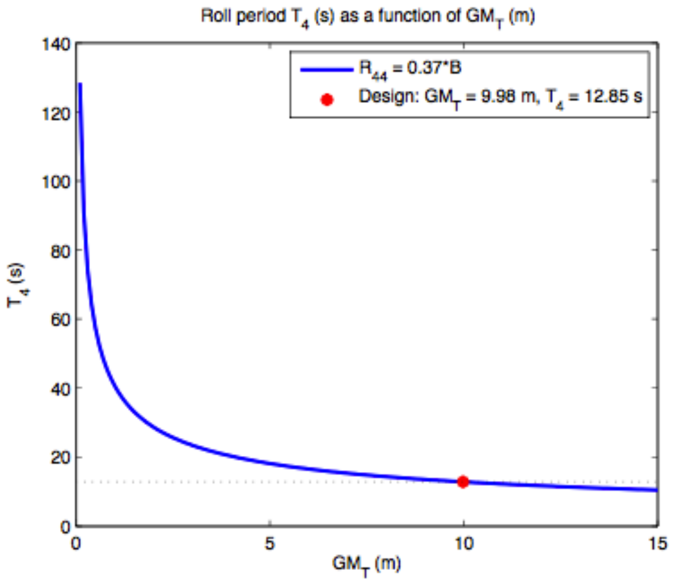
Tankers:

$$R_{44} \approx 0.35B$$
$$R_{55} = R_{66} \approx 0.27L_{pp}$$

$$I_x^{\mathrm{CG}} = mR_{44}^2$$
$$I_y^{\mathrm{CG}} = mR_{55}^2$$

$$I_x^{\mathrm{CF}} = mR_{44}^2 + mz_g^2 := m\left(R_{44}^{\mathrm{CF}}\right)^2$$
$$I_y^{\mathrm{CF}} = mR_{55}^2 + m\left((x_g - \mathrm{LCF})^2 + z_g^2\right) := m\left(R_{55}^{\mathrm{CF}}\right)^2$$

$$T_3 = 2\pi\sqrt{\frac{m + A_{33}^{\mathrm{CF}}(\omega_3)}{\rho g A_{wp}}}$$
$$T_4 = 2\pi\sqrt{\frac{I_x^{\mathrm{CF}} + A_{44}^{\mathrm{CF}}(\omega_4)}{\rho g \nabla \mathrm{GM}_T}}$$
$$T_5 = 2\pi\sqrt{\frac{I_y^{\mathrm{CF}} + A_{55}^{\mathrm{CF}}(\omega_5)}{\rho g \nabla \mathrm{GM}_L}}$$



The roll period clearly depend on the load condition, i.e., added moment of inertia A_{44}, mass m radius of gyration R_{44} and metacentric height $\mathrm{GM_T}$

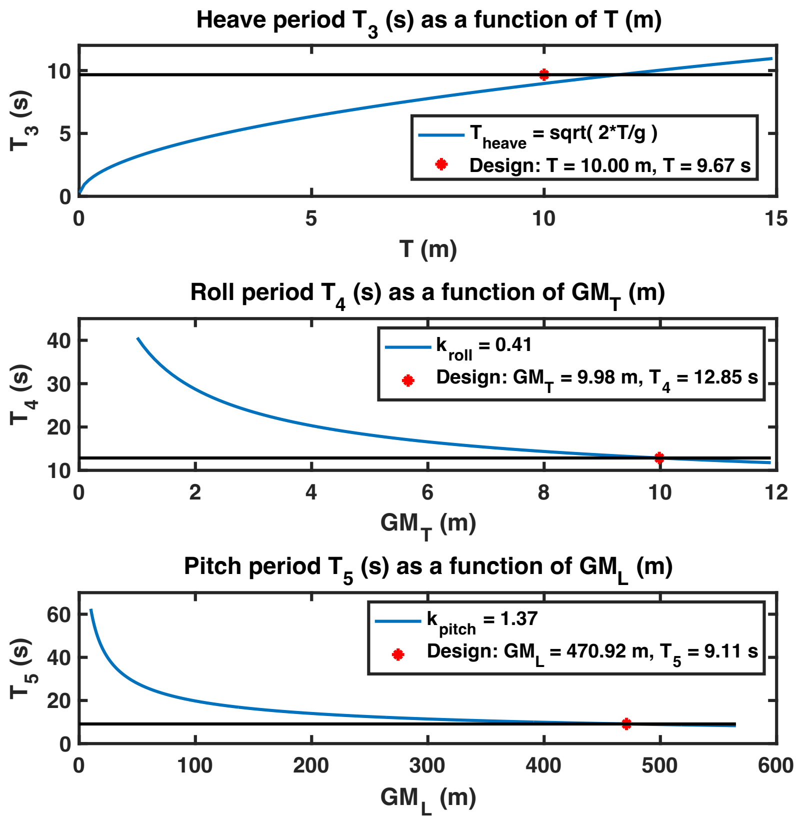
4.3 Load Conditions and Natural Periods

$$T_3 = 2\pi\sqrt{\frac{m + A_{33}^{\mathrm{CF}}(\omega_3)}{\rho g A_{wp}}}$$

$$T_4 = 2\pi\sqrt{\frac{I_x^{\mathrm{CF}} + A_{44}^{\mathrm{CF}}(\omega_4)}{\rho g \nabla \mathrm{GM}_T}}$$

$$T_5 = 2\pi\sqrt{\frac{I_y^{\mathrm{CF}} + A_{55}^{\mathrm{CF}}(\omega_5)}{\rho g \nabla \mathrm{GM}_L}}$$

$$\Rightarrow \quad T_3 \approx 2\pi\sqrt{\frac{2T}{g}}$$



Heave, roll and pitch periods for varying draft and metacenter

The design values (asterisks) are computed using the WAMIT data for the operational condition.

Matlab:

```
load tanker;           % load WAMIT tanker data
loadcond(vessel);      % periods as a function of load condition
```

4.3 Free-Surface Effects

Many ships are equipped liquid tanks like ballast and anti-roll tanks. A partially filled tank is known as a slack tank and in these tanks the liquid can move and endanger the ship stability.

The reduction of metacentric height GM_T caused by the liquids in slack tanks is known as the free-surface effect.

The mass of the liquid or the location of the tanks play no role, only the moment of inertia of the surface affects stability.

The effective metacentric height corrected for slack tanks filled with sea water is

$$\boxed{\mathrm{GM}_{T,\mathrm{eff}} = \mathrm{GM}_T - \mathrm{FSC}}$$

The free surface effect is a mechanism which can cause a watercraft to become unstable and capsize.

Free-surface-correction (FSC) for N tanks

$$FSC = \sum_{r=1}^{N} \frac{\rho}{m} i_r$$

i_r is the moment of inertia of the water surface

$$i_r = \frac{lb^3}{12}$$

Rectangular tank with length *l* in the x direction and width *b* in the y direction

4.3 Payload Effects

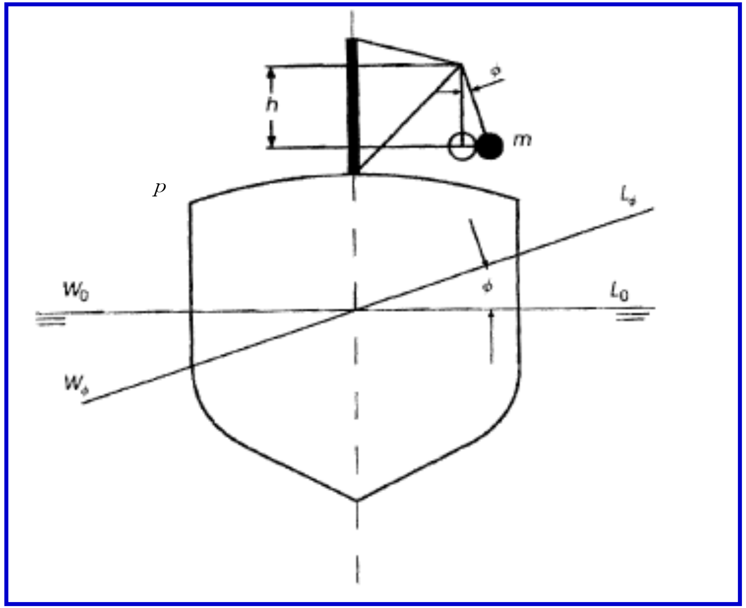
The metacentric height GM_T is reduced on board a ship if a payload with mass m_p is lifted and suspended at the end of a rope of length h.

The effective metacentric height is

$$\mathrm{GM}_{T,\mathrm{eff}} = \mathrm{GM}_T - h\frac{m_p}{m}$$

where m *is the mass of the vessel.*

The destabilizing effect appears immediately after raising the load sufficiently to let it move freely.

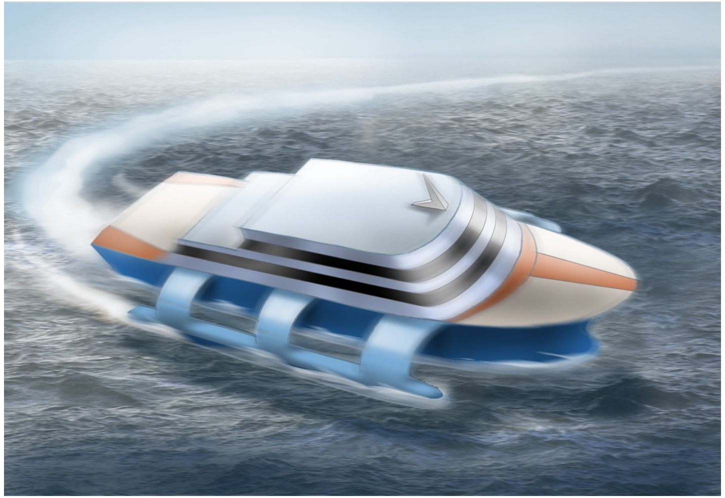


4.4 Seakeeping Analysis

In the design of ships and ocean structures, the wave-induced motions are of great importance to the assessment of the comfort and safety of the crew and the passengers.

Seakeeping analyses should be performed to estimate seakeeping ability or seaworthiness, that is how well-suited a marine craft is to conditions when underway.

This section presents methods for computation of the heave, Roll, and pitch responses in regular waves as well as resonance analyses.



4.4 Harmonic Oscillator without Forcing

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = 0$$

For ships we add waves as forcing

$$m\ddot{z} + d\dot{z} + \quad k\ z \quad = 0$$

Solution (no forcing)

$$\boxed{z(t) = Ae^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right)}$$

Natural frequency (undamped angular frequency)

$$\omega_n = \sqrt{\frac{k}{m}}$$

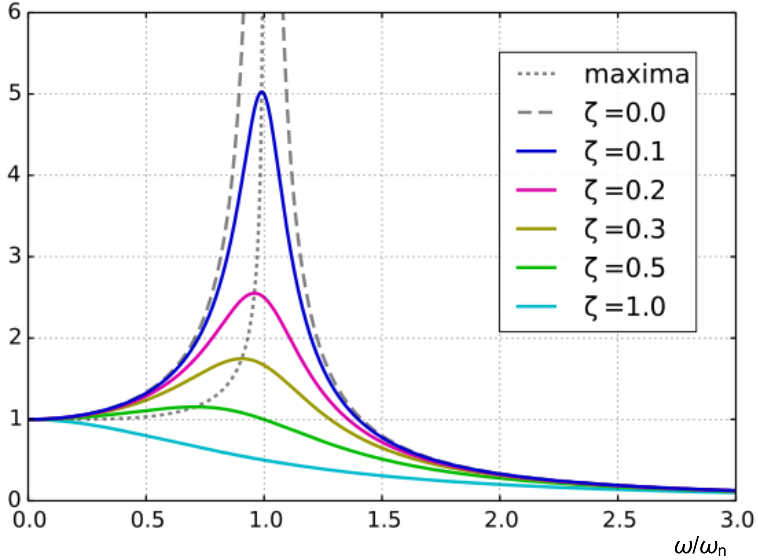
Relative damping ratio

$$\zeta = \frac{d}{2\sqrt{mk}}$$

1) Will a ship oscillate at the natural frequency ω_n when excited by regular waves?
2) Will the ship oscillate at 6 different frequencies when moving in 6 DOFs?

4.4 Harmonic Oscillator with Sinusoidal Forcing

$$m\ddot{x} + d\dot{x} + kx = F\sin(\omega t)$$



Steady-state variation of amplitude with relative frequency ω/ω_n and damping ζ of a forced harmonic oscillator (resonance at $\omega/\omega_n = 1.0$)

$$x = \frac{F}{mZ_m\omega}\sin(\omega t + \varepsilon)$$

Impedance and phase

$$Z_m = \sqrt{(2\zeta\omega_n)^2 + \frac{1}{\omega^2}(\omega_n^2 - \omega^2)^2}$$

$$\varepsilon = \mathrm{atan}\left(\frac{2\zeta\omega_n\omega}{\omega^2 - \omega_n^2}\right)$$

Solving Z_m = 0 gives the resonant frequency

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

Answer to questions:

Marin craft oscillates at ω and not ω_n in all DOFs <u>when sufficient excited</u>, i.e., for a fully developed sea with peak frequency *ω.*

Before “fully excited” the craft can oscillate with different frequencies in all 6 DOFs.

4.4 Steady-State Heave, Roll and Pitch Responses in Regular Waves

$$z = \frac{F_3}{\left(m + A_{33}^{\mathrm{CF}}(\omega_3)\right) Z_{m,3}\, \omega_e} \cos(\omega_e t + \varepsilon_3)$$

$$\phi = \frac{F_4}{\left(I_x^{\mathrm{CF}} + A_{44}^{\mathrm{CF}}(\omega_4)\right) Z_{m,4}\, \omega_e} \cos(\omega_e t + \varepsilon_4)$$

$$\theta = \frac{F_5}{\left(I_y^{\mathrm{CF}} + A_{55}^{\mathrm{CF}}(\omega_5)\right) Z_{m,5}\, \omega_e} \sin(\omega_e t + \varepsilon_5)$$

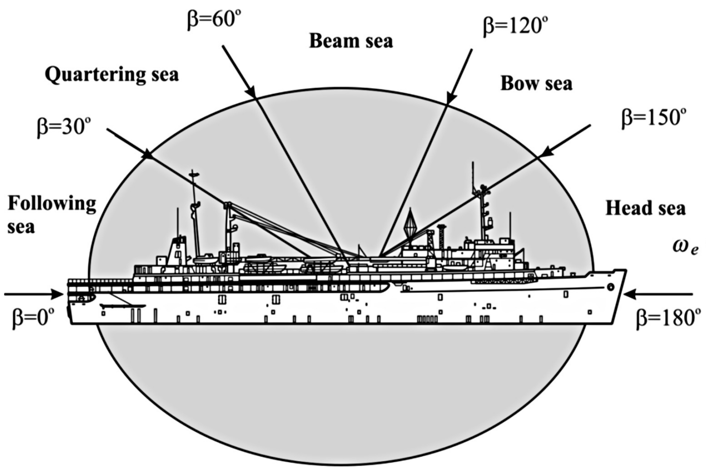
Encounter frequency

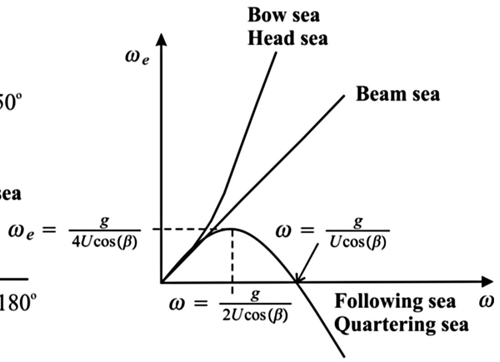
$$\boxed{\omega_e = \omega - kU\cos(\beta)}$$

Wave frequency

$$\boxed{\omega^2 = kg\tanh(kd)}$$

k is the wave number and d is the water depth

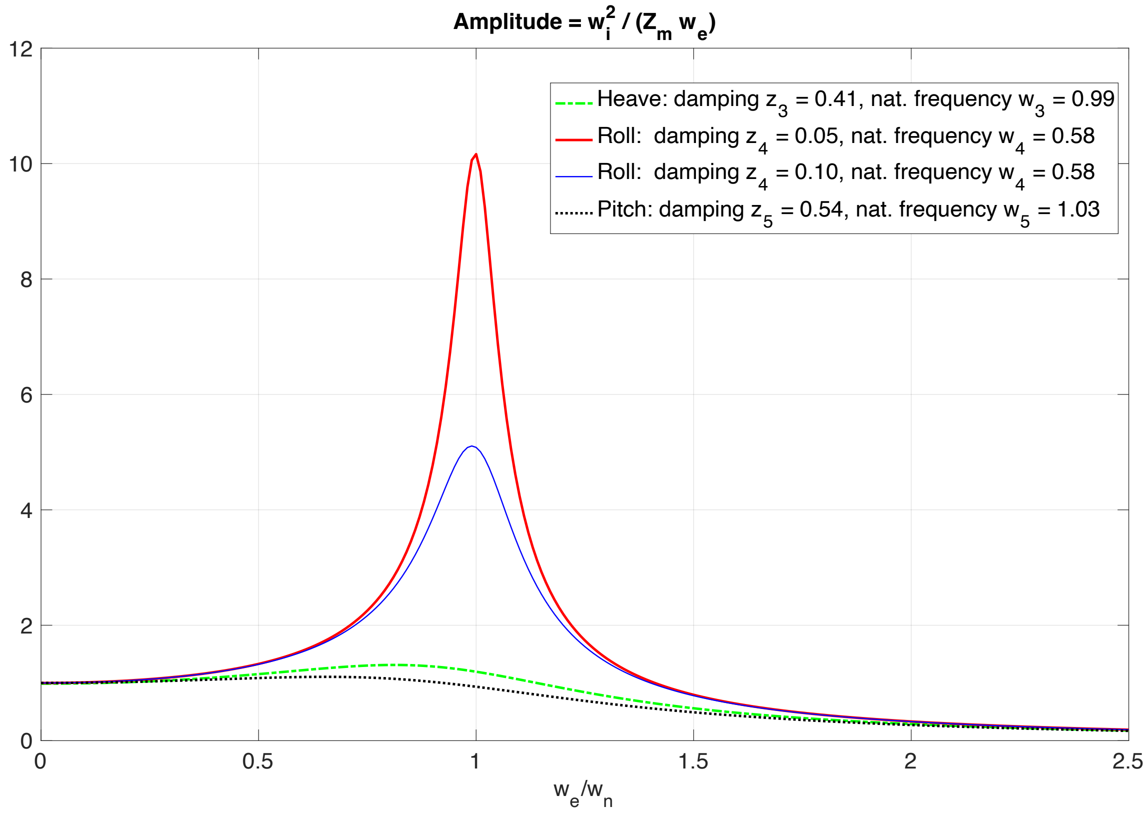




4.4 Steady-State Heave, Roll and Pitch Responses in Regular Waves

MSS Matlab script:
exResonance.m

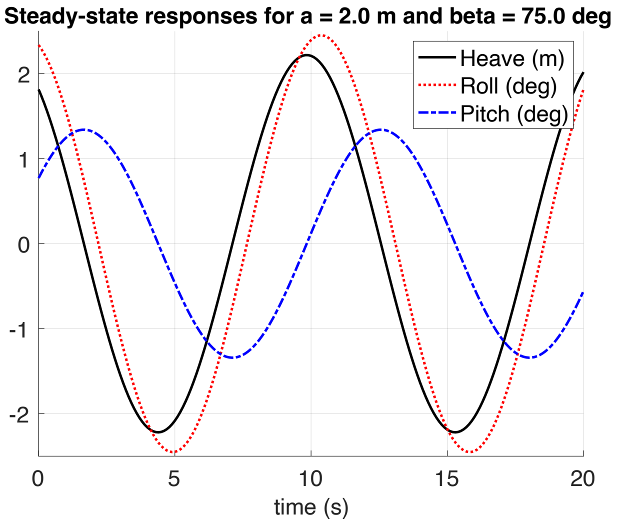
Data set:
>> load supply



As expected, roll is the critical DOF for which the amplitude of the roll angle φ is significantly amplified when $\omega_e/\omega_n = 1.0$. The natural frequency is $\omega_n = \omega_4 = 0.58$ rad/s. This corresponds to regular waves with a period $T_4 = 10.9$ s, which are likely to happen. The relative damping ratio $\zeta_4 = 0.05$ in roll for the MSS supply vessel is quite low, while $GM_T = 2.14$ m gives sufficient stability.

4.4 Steady-State Heave, Roll and Pitch Responses in Regular Waves

```
function waveresponse345(a, beta,T_0, zeta4,T4,GMT, Cb, U, L, B, T)
% waveresponse345(a, beta,T_0, zeta4,T4,GMT, Cb, U, L, B, T) computes and
% plots the steady-state heave, roll and pitch responses for a ship in
% regular waves using closed-form formulae.
%
% Ex: waveresponse345(2, 75*pi/180, 10, 0.2, 6, 1, 0.65, 5, 82.8, 19.2, 6)
%
% Inputs:
%
% a      Wave amplitude (m)
% beta   Wave direction (rad) where beta = pi (i.e. 180 deg) is head seas
% T_0    Wave periode (s) corresponding to the wave frequency w_0 = 2 pi/T_0
% zeta4  Relative damping factor in roll
% T4     Natural roll periode (s)
% GMT    Transver metacentric height (m)
% Cb     Block coefficeint
% U      Ship speed (m/s)
% L      Length (m)
% B      Breadth (m)
% T      Draught (m)
%
%
% Reference:
% J. Juncher Jensen, A. E. Mansour and A. S. Olsen. Estimation of
% Ship Motions using Closed-form Expressions. Ocean Eng. 31, 2004, pp.61-85
```



4.5 Ballast Systems

A floating or submerged vessel can be pretrimmed by pumping water between the ballast tanks of the vessel. This implies that the vessel can be trimmed in *heave*, *pitch,* and *roll.*

$$z = z_{d,} \quad \phi = \phi_d, \quad \theta = \theta_d$$

Heave, roll, and pitch setpoints

Steady-state solution

$$\mathbf{M}\cancel{\dot{\mathbf{v}}} + \mathbf{C}(\cancel{\mathbf{v}})\mathbf{v} + \mathbf{D}(\cancel{\mathbf{v}})\mathbf{v} + \mathbf{g}(\boldsymbol{\eta}) + \mathbf{g}_o = \boldsymbol{\tau} + \underbrace{\boldsymbol{\tau}_{wind} + \boldsymbol{\tau}_{wave}}_{\mathbf{w}}$$

$$\mathbf{g}(\boldsymbol{\eta}_d) + \mathbf{g}_o = \mathbf{w}$$

Steady-state ballast condition

Desired states $\boldsymbol{\eta}_d = [-,-,-,z_d,\phi_d,\theta_d,-]^\top$

The ballast vector $\mathbf{g}_o$ is computed by using hydrostatic analyses to satisfy the steady-state condition.

4.5 Ballast Systems

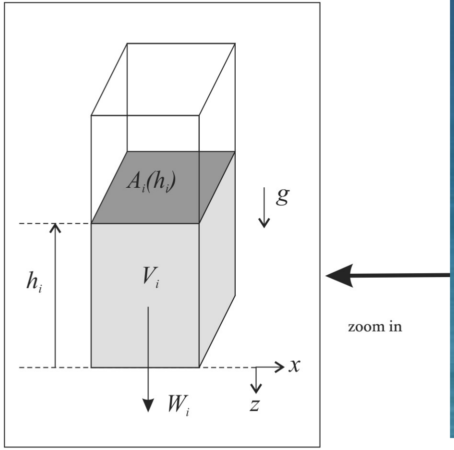
Consider a marine craft with n ballast tanks of volumes $V_i \leq V_{i,max}$ $(i=1,\ldots,n)$

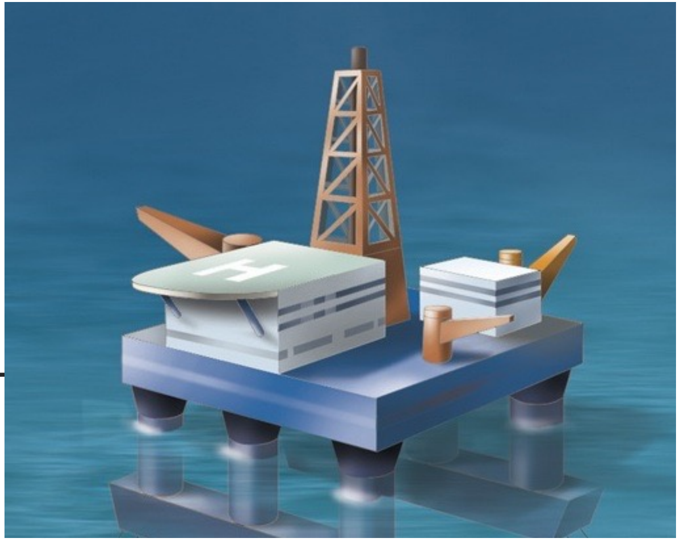
For each ballast tank the water volume is given by the integral

$$V_i(h_i) = \int_o^{h_i} A_i(h)dh \approx A_i h_i, \qquad (A_i(h) = \text{constant})$$

The gravitational force in heave due to W_i is

$$Z_{\text{ballast}} = \sum_{i=1}^{n} W_i = \rho g \sum_{i=1}^{n} V_i$$





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4.5 Ballast Systems

Ballast tank locations with respect to the CO $\boldsymbol{r}_{bi}^{b} = [x_i,\ y_i,\ z_i]^{\top} \quad (i = 1, \ldots, n)$

Restoring moments due to the heave force Z_{ballast}

$$\begin{aligned}\boldsymbol{m}_i^b &= \boldsymbol{r}_{bi}^b \times \boldsymbol{f}_i^b \\ &= \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \rho g V_i \end{bmatrix} \\ &= \begin{bmatrix} y_i \rho g V_i \\ -x_i \rho g V_i \\ 0 \end{bmatrix}\end{aligned}$$

$$K_{\text{ballast}} = \rho g \sum_{i=1}^{n} y_i V_i$$

$$M_{\text{ballast}} = -\rho g \sum_{i=1}^{n} x_i V_i$$

Resulting ballast model

$$\boldsymbol{g}_0 = \begin{bmatrix} 0 \\ 0 \\ -Z_{\text{ballast}} \\ -K_{\text{ballast}} \\ -M_{\text{ballast}} \\ 0 \end{bmatrix} = \rho g \begin{bmatrix} 0 \\ 0 \\ -\sum_{i=1}^{n} V_i \\ -\sum_{i=1}^{n} y_i V_i \\ \sum_{i=1}^{n} x_i V_i \\ 0 \end{bmatrix}$$

4.5 Static Conditions for Trim and Heel

Trimming is usually done under the assumptions that ϕ_d and θ_d are small such

$$\mathbf{g}(\boldsymbol{\eta}_d) \approx \mathbf{G}\boldsymbol{\eta}_d$$

Reduced-order system (heave, roll, and pitch)

$$\mathbf{G}^{\{3,4,5\}} = \begin{bmatrix} -Z_z & 0 & -Z_\theta \\ 0 & -K_\phi & 0 \\ -M_z & 0 & -M_\theta \end{bmatrix} \qquad \mathbf{g}_o^{\{3,4,5\}} = \rho g \begin{bmatrix} -\sum_{i=1}^{n} V_i \\ -\sum_{i=1}^{n} y_i V_i \\ \sum_{i=1}^{n} x_i V_i \end{bmatrix} \qquad \begin{aligned} \boldsymbol{\eta}_d^{\{3,4,5\}} &= [z_d, \phi_d, \theta_d]^\top \\ \mathbf{w}^{\{3,4,5\}} &= [w_3, w_4, w_5]^\top \end{aligned}$$

Steady-state condition: $\mathbf{G}^{\{3,4,5\}}\boldsymbol{\eta}_d^{\{3,4,5\}} + \mathbf{g}_o^{\{3,4,5\}} = \mathbf{0}$

$$\Updownarrow$$

$$\begin{bmatrix} -Z_z & 0 & -Z_\theta \\ 0 & -K_\phi & 0 \\ -M_z & 0 & -M_\theta \end{bmatrix} \begin{bmatrix} z_d \\ \phi_d \\ \theta_d \end{bmatrix} + \rho g \begin{bmatrix} -\sum_{i=1}^{n} V_i \\ -\sum_{i=1}^{n} y_i V_i \\ \sum_{i=1}^{n} x_i V_i \end{bmatrix} = \mathbf{0}$$

This is a set of linear equations where the volumes V_i can be found by assuming that **w** = **0** (zero disturbances)

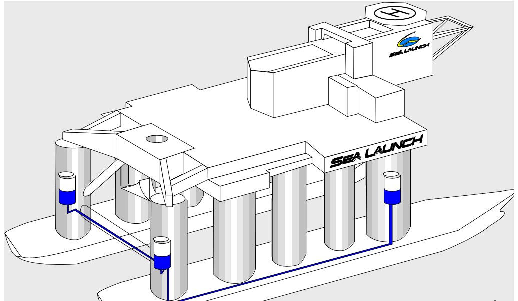
4.5 Static Conditions for Trim and Heel

Assume that the disturbances in heave, roll, and pitch have zero means. Consequently:

$$\mathbf{w}^{\{3,4,5\}} = [w_3, w_4, w_5]^\top = \mathbf{0}$$

and

$$\begin{bmatrix} -Z_z & 0 & -Z_\theta \\ 0 & -K_\phi & 0 \\ -M_z & 0 & -M_\theta \end{bmatrix} \begin{bmatrix} z_d \\ \phi_d \\ \theta_d \end{bmatrix} = \begin{bmatrix} \rho g \sum_{i=1}^{n} V_i + w_3 \\ \rho g \sum_{i=1}^{n} y_i V_i + w_4 \\ -\rho g \sum_{i=1}^{n} x_i V_i + w_5 \end{bmatrix}$$



Courtesy to SeaLaunch

$$\mathbf{H}\boldsymbol{\upsilon} = \mathbf{y}$$

$$\Updownarrow$$

$$\rho g \begin{bmatrix} 1 & \cdots & 1 & 1 \\ y_1 & \cdots & y_{n-1} & y_n \\ -x_1 & \cdots & -x_{n-1} & -x_n \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} -Z_z z_d - Z_\theta \theta_d \\ -K_\phi \phi_d \\ -M_z z_d - M_\theta \theta_d \end{bmatrix}$$

The water volumes V_i is found by using the pseudo-inverse

$$\boldsymbol{\upsilon} = \mathbf{H}^\dagger \mathbf{y} = \mathbf{H}^\top (\mathbf{H}\mathbf{H}^\top)^{-1} \mathbf{y}$$

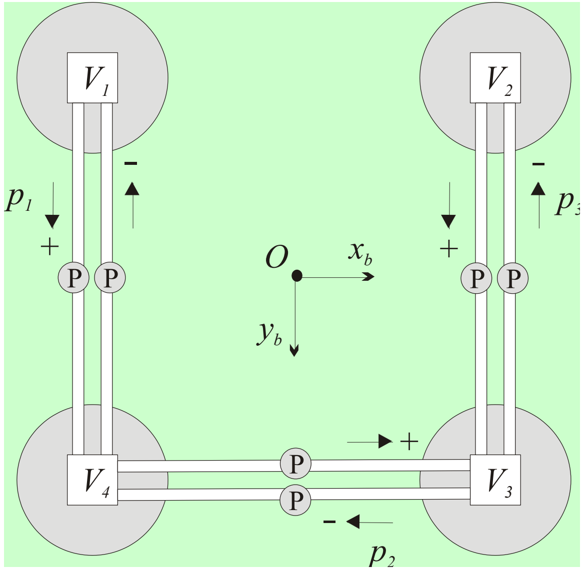
4.5 Static Conditions for Trim and Heel

Example 4.3(Semisubmersible Ballast Control) Consider a semi-submersible with 4 ballast tanks located at $\mathbf{r}_1^b = [-x,-y], \mathbf{r}_2^b = [x,-y], \mathbf{r}_3^b = [x,y], \mathbf{r}_4^b = [-x,y]$

In addition, *yz*-symmetry implies that $Z_\theta = M_z = 0$

$$\mathbf{H} = \rho g \begin{bmatrix} 1 & 1 & 1 & 1 \\ -y & -y & y & y \\ x & -x & -x & x \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} -Z_z z_d \\ -K_\phi \phi_d \\ -M_\theta \theta_d \end{bmatrix} = \begin{bmatrix} \rho g A_{wp}(0) z_d \\ \rho g \nabla \overline{GM}_T \phi_d \\ \rho g \nabla \overline{GM}_L \theta_d \end{bmatrix}$$



$$\boldsymbol{\upsilon} = \mathbf{H}^\dagger \mathbf{y} = \mathbf{H}^\top (\mathbf{H}\mathbf{H}^\top)^{-1} \mathbf{y}$$

$$\boldsymbol{\upsilon} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \frac{1}{4\rho g} \begin{bmatrix} 1 & -\frac{1}{y} & \frac{1}{x} \\ 1 & -\frac{1}{y} & -\frac{1}{x} \\ 1 & \frac{1}{y} & -\frac{1}{x} \\ 1 & \frac{1}{y} & \frac{1}{x} \end{bmatrix} \begin{bmatrix} \rho g A_{wp}(0) z_d \\ \rho g \nabla \overline{GM}_T \phi_d \\ \rho g \nabla \overline{GM}_L \theta_d \end{bmatrix}$$

Inputs: z_d, ϕ_d, θ_d

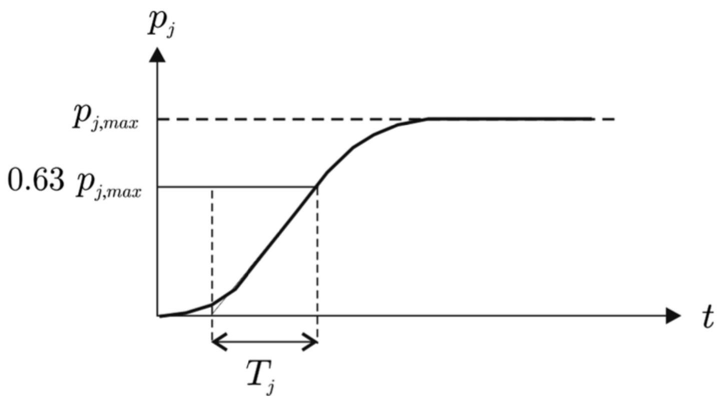
4.5 Automatic Ballast Control Systems

In the static analyses it was assumed that $\mathbf{w}^{\{3,4,5\}} = \mathbf{0}$. This assumption can be relaxed by using feedback.

The closed-loop dynamics of a PID controlled water pump can be described by a 1st-order model with amplitude saturation

$$T_j \dot{p}_j + p_j = \mathrm{sat}(p_{d_j})$$

T_j (s) is a positive time constant
p_j (m^3/s) is the volumetric flow rate pump j
p_d^j is the pump set-point.



The water pump capacity is different for positive and negative flow directions

$$\mathrm{sat}(p_{d_j}) = \begin{cases} p^+_{j,\max} & p_j > p^+_{j,\max} \\ p_{d_j} & p^-_{j,\max} \leq p_{d_j} \leq p^+_{j,\max} \\ p^-_{j,\max} & p_{d_j} < p^-_{j,\max} \end{cases}$$

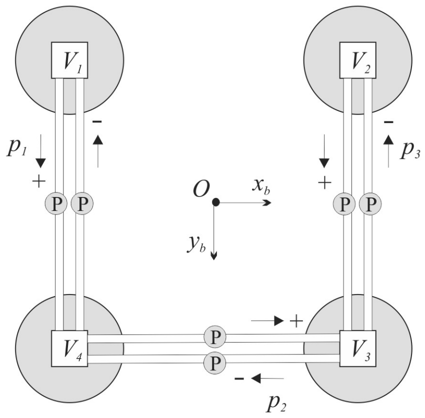
4.5 Automatic Ballast Control Systems

Example 4.4 (Semisubmersible Ballast Control):

The water flow model corresponding to the figure is:

$$
\begin{aligned}
\dot{V}_1 &= -p_1 \\
\dot{V}_2 &= -p_3 \\
\dot{V}_3 &= p_2 + p_3 \\
\dot{V}_4 &= p_1 - p_2
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{T}\dot{\mathbf{p}} + \mathbf{p} &= \mathrm{sat}(\mathbf{p}_d) \\
\dot{\boldsymbol{\upsilon}} &= \mathbf{L}\mathbf{p}
\end{aligned}
$$



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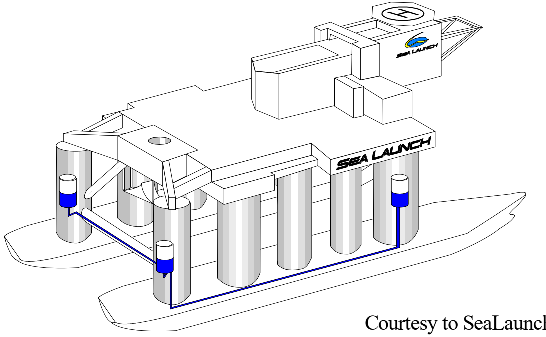
$$
\boldsymbol{\upsilon} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}, \quad
\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \quad
\mathbf{L} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}
$$

4.5 Automatic Ballast Control Systems

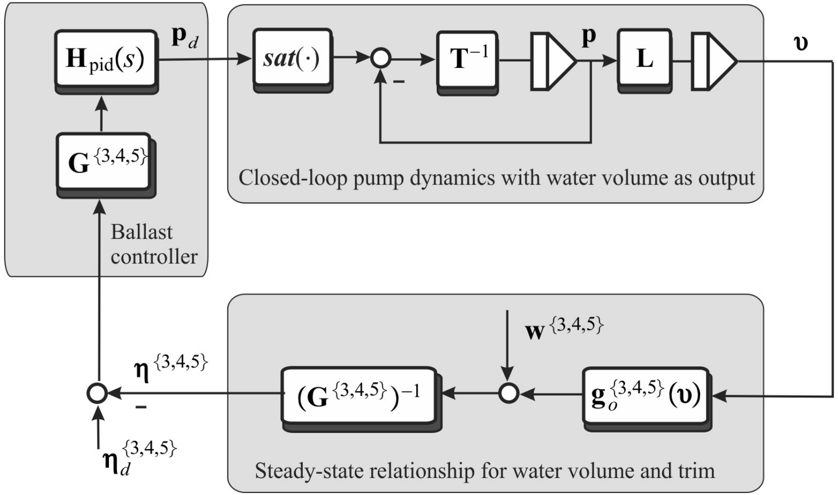


Courtesy to SeaLaunch

Feedback control system

$$\mathbf{p}_d = \mathbf{H}_{\text{pid}}(s)\mathbf{G}^{\{3,4,5\}}\left(\boldsymbol{\eta}_d^{\{3,4,5\}} - \boldsymbol{\eta}^{\{3,4,5\}}\right)$$

$$\mathbf{H}_{\text{pid}}(s) = \text{diag}\{h_{1,\text{pid}}(s), h_{2,\text{pid}}(s), \ldots, h_{m,\text{pid}}(s)\}$$



Equilibrium equation

$$\mathbf{G}^{\{3,4,5\}}\boldsymbol{\eta}^{\{3,4,5\}} = \mathbf{g}_o^{\{3,4,5\}}(\boldsymbol{\upsilon}) + \mathbf{w}^{\{3,4,5\}}$$

Dynamics

$$\mathbf{T}\dot{\mathbf{p}} + \mathbf{p} = \text{sat}(\mathbf{p}_d)$$

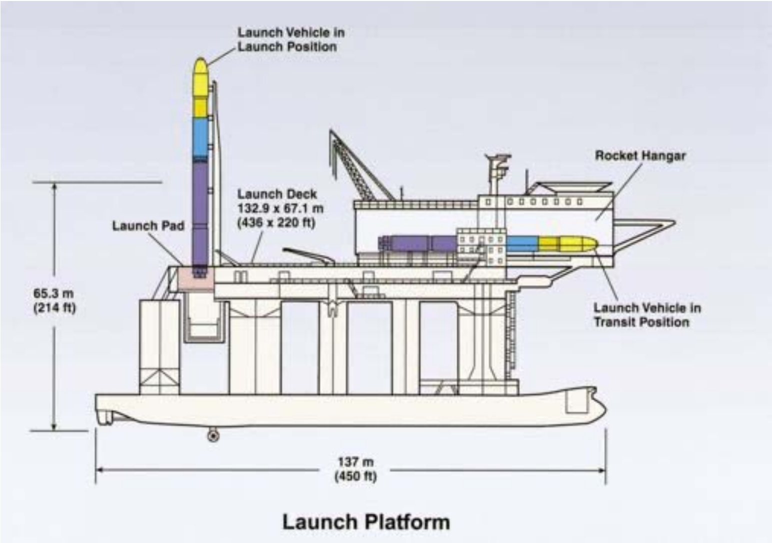
$$\dot{\boldsymbol{\upsilon}} = \mathbf{L}\mathbf{p}$$

4.5 Automatic Ballast Control Systems

SeaLaunch

An example of a highly sophisticated pretrimming system is the SeaLaunch *trim* and *heel correction system* (THCS)

This system is designed such that the platform maintains constant roll and pitch angles during changes in weight. The most critical operation is when the rocket is transported from the garage on one side of the platform to the launch pad. During this operation, the water pumps operate at their maximum capacity to counteract the shift in weight.



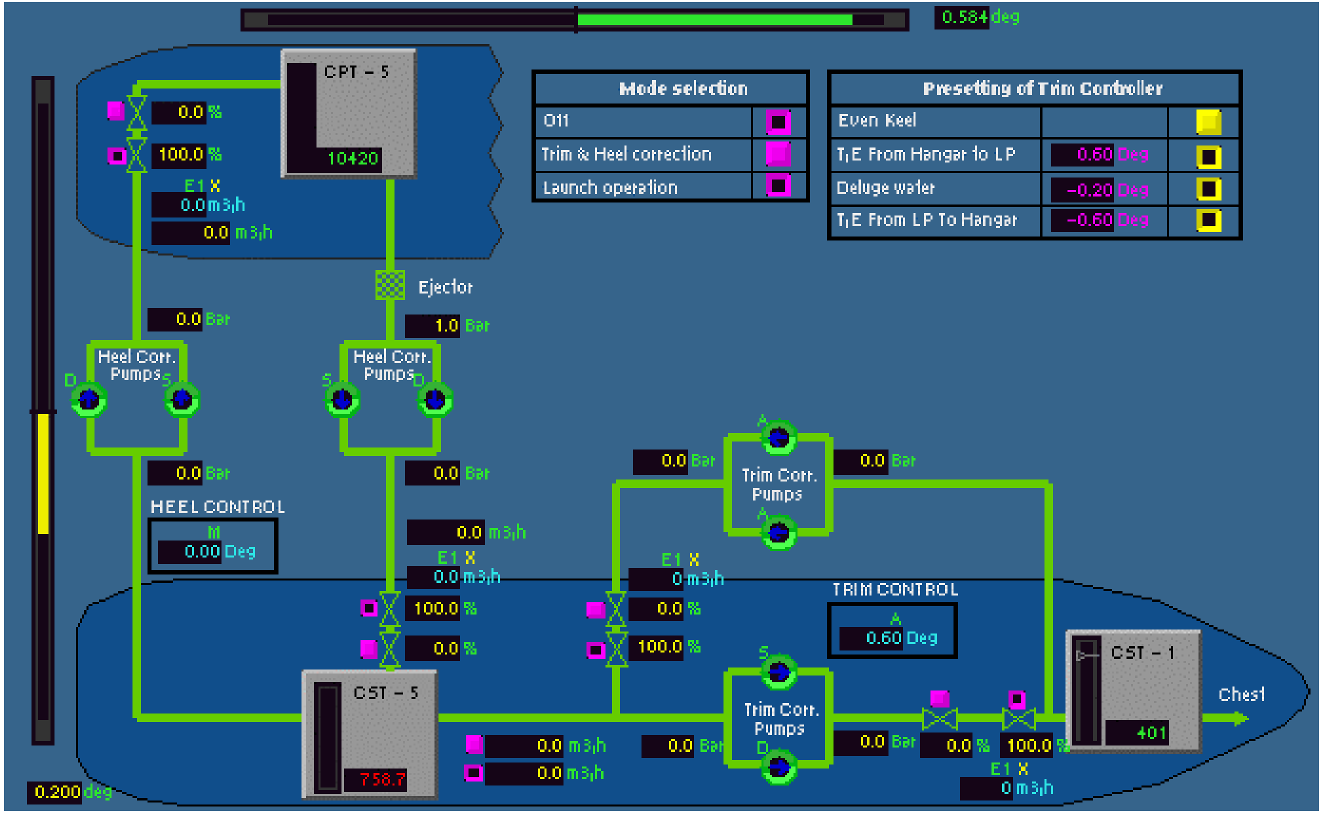
Courtesy to SeaLaunch

A feedback system controls the pumps to maintain the correct water level in each of the legs during transportation of the rocket

4.5 Automatic Ballast Control Systems

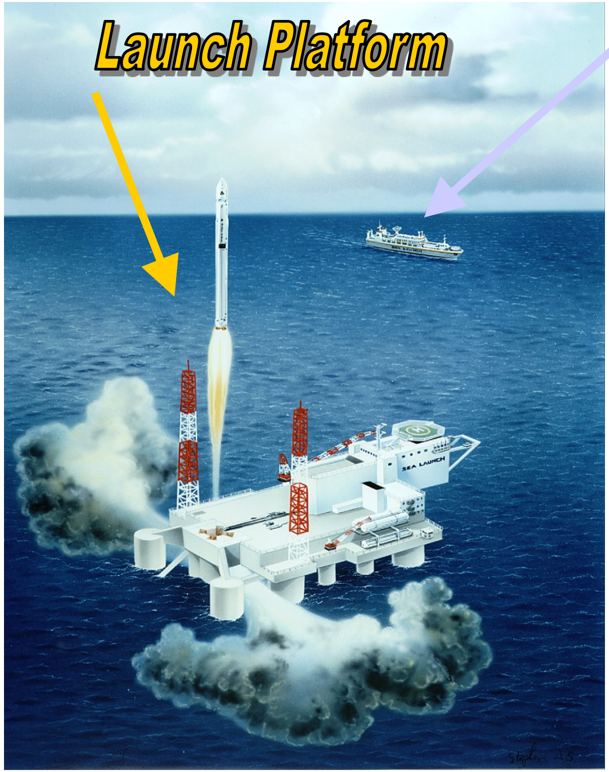
Courtesy to SeaLaunch

4.5 Automatic Ballast Control Systems



Courtesy to SeaLaunch

4.5 Automatic Ballast Control Systems



Assembly Command Ship

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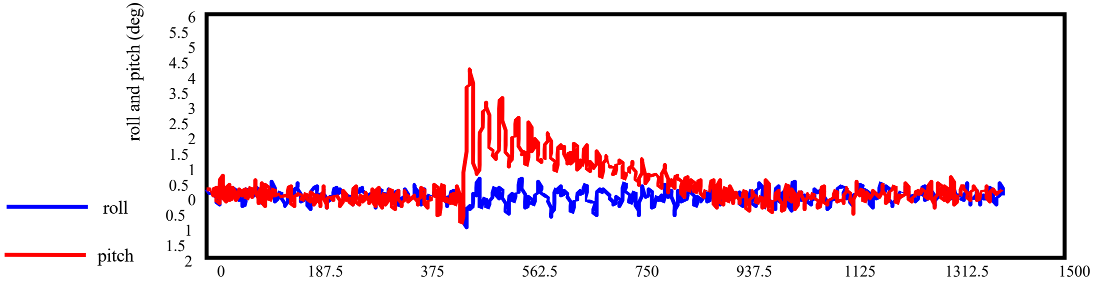
SEA LAUNCH SM
Marine Segment

4.5 Automatic Ballast Control Systems

Roll and pitch angles during lift-off



Roll and pitch during launch

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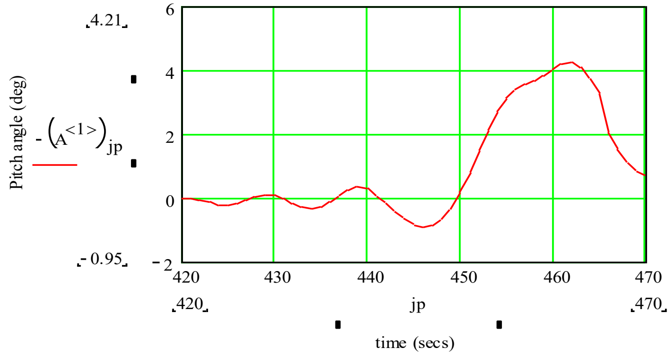
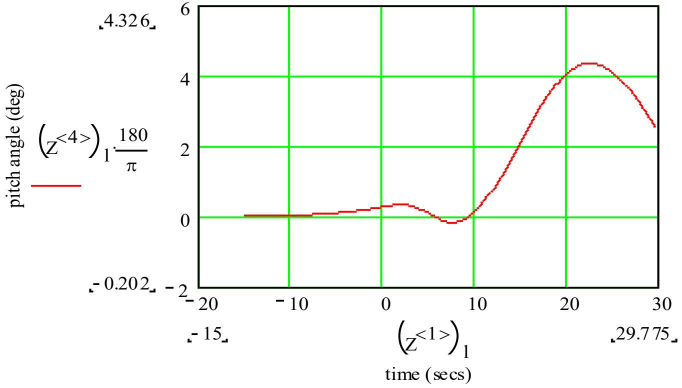
CNN 10th October 1999



Measured pitch during launch



Calculated pitch motions

Chapter Goals – Revisited

- Understand that the restoring forces behave like spring forces in 2nd-order systems and that they are only present in heave, roll and pitch.
- Be able to compute the restoring forces for both floating and submerged vehicles and understand the differences.
- Be able to explain what the "Metacenter" is and explain what we mean by metacentric stability.
- Be able to define the center of flotation and pivot point and explain which point a vehicle roll, pitch and yaw about.
- Understand how load conditions affect hydrostatic quantities such as heave, roll and pitch periods.
- Understand the concepts of manually and automatic pretrimming.

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_o = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}}$$

The main goal of this chapter is to understand how the restoring forces and ballast terms in the equations of motion are modeled.