README.md

FEM Fatale

A jello simulation using the Finite Element Method

Demo video link: https://youtu.be/rfcBbrljoug

Contributors

Connie Chang - Collisions and rendering
Marissa Como - I/O, tetgen, test cases, debugging
Zach Corse - Force calculations, K matrix calculations
Anantha Srinivas - Explicit and implicit integrators

Implementation details

- Fixed Corotated elastic model
- Tetgen file input reader
- Even distribution of mass between tetrahedron vertices
- Collisions using signed distance functions
- Forward Euler integrator
- (Work in progress) Backward Euler integrator
- OBJ output for rendering
- Rendering in Houdini

Implicit Integration

$$\sigma = -V_r \frac{\partial \Psi}{\partial \bar{X}}$$

$$f_{pi} = \sigma_{pi} = -V_r \frac{\partial \Psi}{\partial F_{jk}} \frac{\partial F_{jk}}{\partial x_{pi}}$$

$$K_{pi\,qr} = \frac{\partial f_{pi}}{\partial x_{qr}} = -V_r \underbrace{\frac{\partial^2 \Psi}{\partial F_{jk} \partial F_{mn}}}_{A} \underbrace{\frac{\partial F_{jk}}{\partial x_{pi}}}_{B} \underbrace{\frac{\partial F_{mn}}{\partial x_{qr}}}_{B}$$

(B) $$F_{jk} = \sum_{l=1}^{3} (D_s)_{jl} (D_m^{-1})_{lk}$$

$$\frac{\partial F_{jk}}{\partial x_{pi}} = \sum_{l=1}^{3} \frac{\partial (D_s)_{jl}}{\partial x_{pi}} (D_m^{-1})_{lk} = \begin{cases} \sum_{l=1}^{3} (D_m^{-1})_{lk} \, \delta_{l0} \, \delta_{ij} & p = 0 \\ \sum_{l=1}^{3} (D_m^{-1})_{lk} \, \delta_{l1} \, \delta_{ij} & p = 1 \\ \sum_{l=1}^{3} (D_m^{-1})_{lk} \, \delta_{l2} \, \delta_{ij} & p = 2 \\ \sum_{l=1}^{3} (D_m^{-1})_{lk} (-\delta_{ij}) & p = 3 \end{cases}$$

(A) $$\frac{\partial^2 \Psi}{\partial F_{jk} \partial F_{mn}} = \frac{\partial P_{jk}}{\partial F_{mn}}$$

$$P = 2\mu (F - R) + \lambda (J-1)(J F^{-T})$$

$$P_{jk} = 2\mu (F_{jk} - R_{jk}) + \lambda (J-1)(J F^{-T})_{jk}$$

$$\frac{\partial P_{jk}}{\partial F_{mn}} = 2\mu \left(\underbrace{\frac{\partial F_{jk}}{\partial F_{mn}}}_{1} - \underbrace{\frac{\partial R_{jk}}{\partial F_{mn}}}_{2} \right) + \lambda \left[(J F^{-T})_{mn} (J F^{-T})_{jk} + (J-1) \underbrace{\frac{\partial H_{jk}}{\partial F_{mn}}}_{3} \right]$$

(1) $$\frac{\partial F_{jk}}{\partial F_{mn}} = \delta_{jm} \delta_{kn}$$

(3) $$H_{jk} = (J F^{-T})_{jk} \quad \$ \quad H = \begin{bmatrix} F_{11}F_{22} - F_{12}F_{21} & F_{12}F_{20} - F_{10}F_{22} & F_{10}F_{21} - F_{11}F_{20} \\ F_{02}F_{21} - F_{01}F_{22} & F_{00}F_{22} - F_{02}F_{20} & F_{01}F_{20} - F_{00}F_{21} \\ F_{01}F_{12} - F_{02}F_{11} & F_{02}F_{10} - F_{00}F_{12} & F_{00}F_{11} - F_{01}F_{10} \end{bmatrix}$$

$$\frac{\partial H}{\partial F_{00}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & F_{22} & -F_{21} \\ 0 & -F_{12} & F_{11} \end{bmatrix} \quad \frac{\partial H}{\partial F_{10}} = \begin{bmatrix} 0 & -F_{22} & F_{21} \\ 0 & 0 & 0 \\ 0 & F_{02} & -F_{01} \end{bmatrix} \quad \frac{\partial H}{\partial F_{20}} = \begin{bmatrix} 0 & F_{12} & F_{10} \\ 0 & -F_{02} & F_{01} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial H}{\partial F_{01}} = \begin{bmatrix} 0 & 0 & 0 \\ -F_{22} & 0 & F_{20} \\ F_{12} & 0 & -F_{10} \end{bmatrix} \quad \frac{\partial H}{\partial F_{11}} = \begin{bmatrix} F_{22} & 0 & -F_{20} \\ 0 & 0 & 0 \\ -F_{02} & 0 & F_{00} \end{bmatrix} \quad \frac{\partial H}{\partial F_{21}} = \begin{bmatrix} -F_{12} & 0 & F_{10} \\ F_{02} & 0 & -F_{00} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial H}{\partial F_{02}} = \begin{bmatrix} 0 & 0 & 0 \\ F_{21} & -F_{20} & 0 \\ -F_{11} & F_{10} & 0 \end{bmatrix} \quad \frac{\partial H}{\partial F_{12}} = \begin{bmatrix} -F_{21} & F_{20} & 0 \\ 0 & 0 & 0 \\ F_{01} & -F_{00} & 0 \end{bmatrix} \quad \frac{\partial H}{\partial F_{22}} = \begin{bmatrix} F_{11} & -F_{10} & 0 \\ -F_{01} & F_{00} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) $\frac{\partial R_{jk}}{\partial F_{mn}} = ?$

$F = U\Sigma V^T = RS$

$R = UV^T, \quad R^T R = I$

$S = V\Sigma V^T$

$\delta R^T R + R^T \delta R = \delta I = 0$

$\Rightarrow (R^T \delta R)^T + (R^T \delta R) = 0$

$\Rightarrow (R^T \delta R)$ is skew-symmetric

$$R^T \delta R = \Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \overset{\text{matrix } C_j}{C_j}\, \omega_j \quad \text{where } (C_j)_{ik} = \epsilon_{ijk}$$

ALSO,

$F = RS$

$R^T \delta F = R^T \delta R\, S + \delta S$

$$\underbrace{C_i : (R^T \delta F)}_{b_i} = C_i : ((C_j \omega_j) S) + \cancel{C_i : \delta S}^{\,0}$$

$$b_i = (\underbrace{C_i S^T : C_j}_{A_{ij}} \omega_j) \qquad \text{b/c} \quad A:(BC) = (AC^T : B)$$

$b_i = A_{ij}\omega_j$ where $A_{ij} = \sum_{\alpha,\beta,\gamma} \epsilon_{\alpha i \beta} S^T_{\beta\gamma} \epsilon_{\alpha j \gamma}$

$\Rightarrow \omega_j = (A^{-1})_{jk} b_k$ $\qquad = \sum_{\alpha,\beta,\gamma} \epsilon_{\alpha i \beta} S_{\gamma\beta} \epsilon_{\alpha j \gamma}$

FINALLY,

$\delta R = R\Omega = R C_j \omega_j$

$(\delta R)_{mn} = R_{ml} (C_j)_{ln} \omega_j$

$= R_{ml} (C_j)_{ln} (A^{-1})_{jk} b_k$

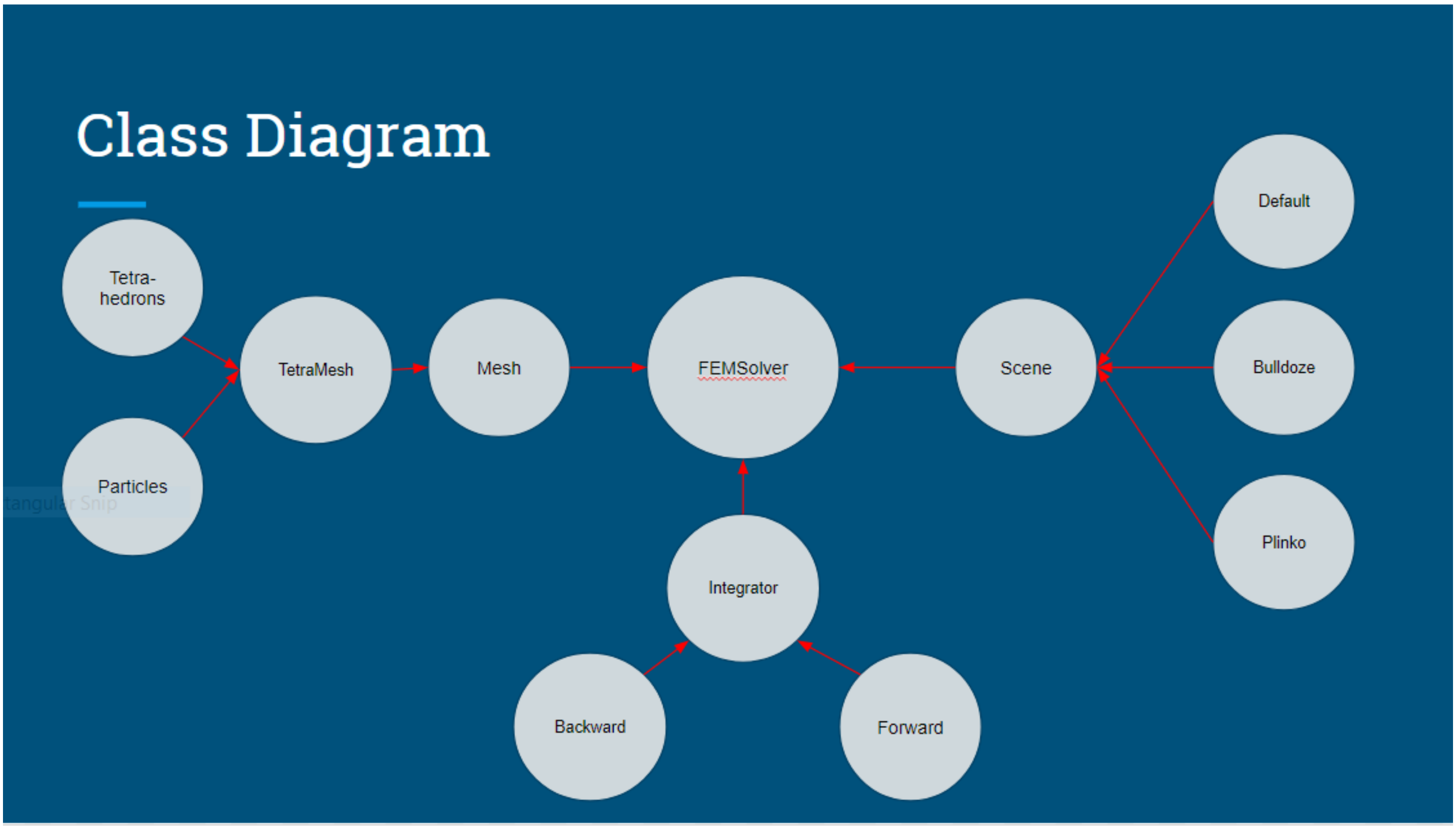
$= R_{ml} (C_j)_{ln} (A^{-1})_{jk} C_k : (R^T \delta F)$

$= R_{ml} (C_j)_{ln} (A^{-1})_{jk} (C_k)_{ab} R^T_{aq} (\delta F)_{qb}$

$= \sum_{l,a,b,q,j,k} R_{ml} \epsilon_{ljn} (A^{-1})_{jk} \epsilon_{akb} R_{qa} (\delta F)_{qb}$

$$\Rightarrow \frac{\partial R_{jk}}{\partial F_{mn}} = \sum_{l,a,\alpha,\beta} R_{jl} \epsilon_{l\alpha k} (A^{-1})_{\alpha\beta} \epsilon_{a\beta n} R_{ma}$$

Class Diagram



Limitations and Struggles

- Implicit solver does not work...yet
- Instability in explicit solver as simulation goes on (>3 bounces)

Resources

- 2012 SIGGRAPH course notes on FEM
- Professor Kavan's Youtube videos
- MPM Snow Simulation paper for implicit
- TA's Josh and Ziyin
- Andre
- Professor Jiang