Exercises
Computational Intelligence Lab
SS 2017

Machine Learning Institute
Dept. of Computer Science, ETH Zürich
Prof. Dr. Thomas Hofmann
Web http://cil.inf.ethz.ch/

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(Sparse Coding and Wavelets)

Problem 1 (1D Compression and Orthonormal Basis):

Let $x(t)$ be a function expressed as a weighted sum of K basis functions $u_1(t), \ldots, u_K(t)$:

$$x(t) = \sum_{k=1}^{K} z_k u_k(t)$$

For the sake of simplicity, we will consider orthonormal basis functions (e.g. normalized Haar Wavelets) that take discrete values.Thus, we can express these basis functions as vectors:

$$\mathbf{x} = \sum_{k=1}^{K} z_k \mathbf{u}_k = \mathbf{U}\mathbf{z}$$

For a fixed basis, we want to find a good approximation for $\mathbf{x}$ using only $\tilde{K}$ coefficients, where $\tilde{K} < K$. We denote the approximation $\hat{\mathbf{x}}$:

$$\hat{\mathbf{x}} = \sum_{k=1}^{\tilde{K}} z_k \mathbf{u}_k$$

If we consider only orthonormal bases, we can formulate the compression problem as picking from the original coefficients $z_1, \ldots, z_K$ a subset $\tilde{K}$ of them which minimize the approximation error.

Let σ be a permutation of indices $\{1, \ldots, K\}$ and $\hat{\mathbf{x}}_\sigma$ the function that uses the coefficients corresponding to the first $\tilde{K}$ indices of the permutation σ :

$$\hat{\mathbf{x}}_\sigma = \sum_{k=1}^{\tilde{K}} z_{\sigma(k)} \mathbf{u}_{\sigma(k)}$$

Question: Find the permutation σ^{min} which minimizes the L^2 approximation error $\left\|\mathbf{x} - \hat{\mathbf{x}}_\sigma\right\|_2^2$:

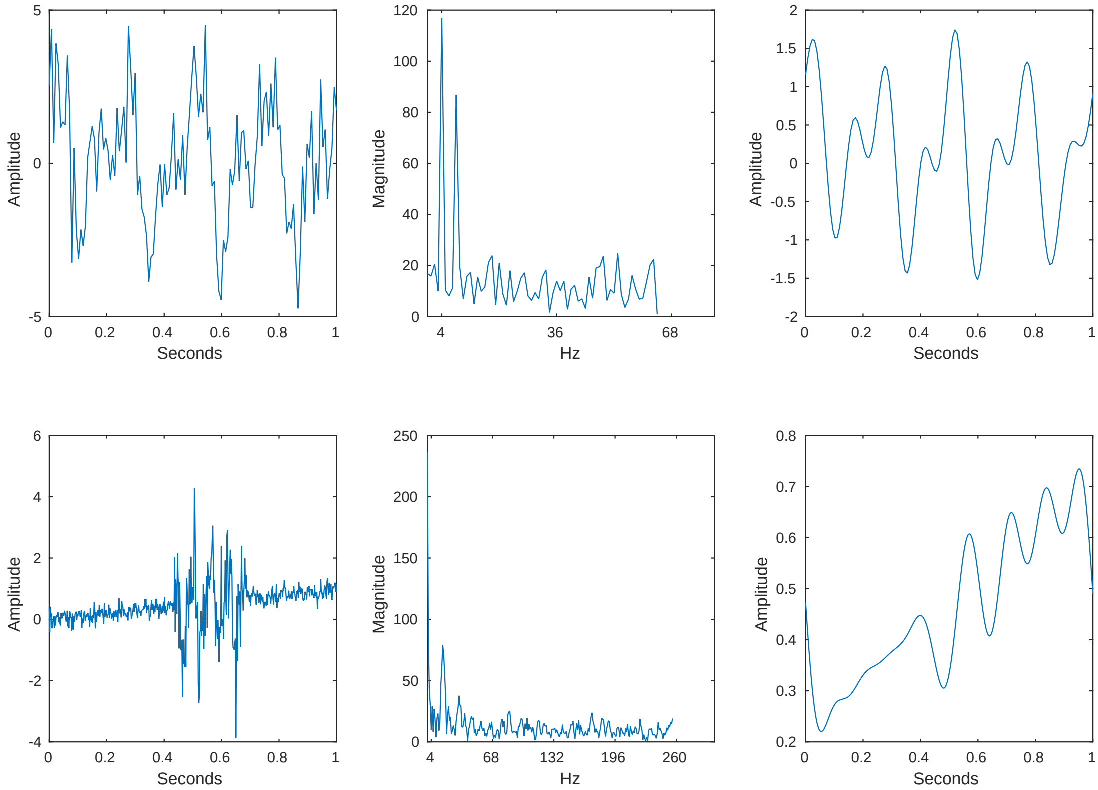
$$\sigma^{min} = \underset{\sigma}{\operatorname{argmin}} \left\|\mathbf{x} - \hat{\mathbf{x}}_\sigma\right\|_2^2$$

Hint: Remember that the L^2 norm can be written with the help of an inner product as

$$\left\|\mathbf{x} - \hat{\mathbf{x}}_\sigma\right\|_2^2 = \langle \mathbf{x} - \hat{\mathbf{x}}_\sigma, \mathbf{x} - \hat{\mathbf{x}}_\sigma \rangle$$

Also keep in mind that orthonormal basis means that the vectors comprising the basis are mutually orthogonal (zero inner product) and have unit length:

$$\langle \mathbf{u}_k, \mathbf{u}_l \rangle = 0, \qquad k \neq l$$
$$\langle \mathbf{u}_k, \mathbf{u}_k \rangle = 1$$



The figure above shows two different 1-D signals (left column) with their corresponding spectrum obtained using the FFT (middle column). In the right column, we show a signal obtained by discarding part of the frequencies in the spectrum.

- (1) Write down the formula to obtain the spectrum in the middle column of the previous figure, in terms of linear transformation or change of basis (assuming a given basis $\mathbf{U}$) applied to the original signal $\mathbf{x}$.
  (2) Write down the inverse formula to obtain the reconstructed signal in the right column in terms of linear transformation (change of basis) applied to the filtered spectrum $\hat{\mathbf{z}}$.
- What part of the signal would you discard to obtain the reconstructed signal? Draw a rectangle on each spectrum in the middle column where everything inside the rectangle is kept for the reconstruction.
- **true/false** The Wavelet transform is a better choice than Fourier for the first signal (top row).
- **true/false** The Wavelet transform is a better choice than Fourier for the second signal (bottom row).
- Looking at the middle figure in the top row, what do the first peaks in the spectrum correspond to?

Problem 2 (Fourier Transform and Low-pass Filtering):

Please find the iPython notebook `sparse_coding.ipynb` from

github.com/dalab/lecture_cil_public/tree/master/exercises/ex10,

and answer the questions in `sparse_coding.ipynb`.