The Ulam sequence and related phenomena

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1 Introduction

The Ulam sequence is a sequence of positive integers that is defined in a recursive way that sounds like it should make it difficult to compute. It starts with $a_1 = 1$, $a_2 = 2$, and then for $n > 2$, a_n is the integer satisfying:

1. It is expressible as a sum of distinct previous terms in exactly one way: There is exactly one pair of $i < j$ with $a_i + a_j = a_n$.

2. It is larger than the previous element of the sequence: $a_n > a_{n-1}$.

3. It is the smallest positive integer with the above two properties.

Thus the first few terms can be computed:

$$1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, 62, 69, \ldots$$

In particular, there are two ways a number could fail to be Ulam: Either it has a representation as a sum of two distinct smaller Ulam numbers in more than one way (such as $5 = 4 + 1 = 2 + 3$), or it has no representations as a sum of distinct smaller Ulam numbers at all (such as 23).

One thing that makes the sequence interesting is that it seems historically to have been very difficult to prove anything about it. We know, for example, that it must be infinite, since, given the first n elements $a_1, \dots, a_n$, we can always find at least one number that satisfies both the criteria above, namely $a_{n-1}+a_n$. Thus there must be a smallest such number, which is therefore the next Ulam number.

We also know that if we use the same definition but start with different initial values, we can get sequences that we can analyse very easily indeed: If the (u, v)-Ulam sequence is the sequence with $a_1 = u, a_2 = v$, and a_n (for $n > 3$) defined exactly as before, then by theorems of Finch [2] and Schmerl and Speigel [14], we know that the $(2, v)$-Ulam sequence, in the case where v is odd and at least 5, is regular in the following sense:

Definition 1. An increasing, infinite sequence $\{a_i\}$ of positive integers is **regular** if the sequence $\{b_i = a_i - a_{i-1} : i > 1\}$ is eventually periodic.

Such sequences are very easy to describe–we could specify them (after some initial segment) by a set of congruence classes modulo some (possibly large) m. In particular, a regular Ulam-like sequence will be far easier to compute than the definition of an Ulam sequence would naively suggest.

There are other initial values that are variously known to and believed to give rise to regular sequences, also. See, for example, [6]. That said, many Ulam-type sequences appear not to be regular, among them the $(1, 2)$- and $(2, 3)$-Ulam sequences. So one might ask some questions:

- What is it that causes some initial conditions to be regular and not others (if indeed they are not)?
- Is there any perhaps more general notion of regularity that even the irregular-looking sequences do satisfy?

In looking for hidden regularity, one might take a signal processing approach to the sequence and try, for example, to Fourier transform the indicator function of the sequence and see if the spectrum has any interesting features. In [15], Stefan Steinerberger did exactly that and behold, the spectrum has a large spike exactly only at some $\alpha \in \mathbb{R}/\mathbb{Z}$ (and at its harmonics), and seemingly nowhere else.

More precisely:

Definition 2. If $f : [N] \to \mathbb{C}$, recall the the **Fourier transform** of f is a function $\widehat{f}$ defined by the formula:

$$\widehat{f}(x) = \frac{1}{N} \sum_{t=0}^{N-1} f(t)e(-tx)$$

where $e(x) = e_N(x) = e^{2\pi i x/N}$. Thus $\widehat{f}$ is a function defined on all of $\mathbb{R}/\mathbb{Z}$. N will often be omitted from the notation and understood from context. If we wish to make N explicit in the notation for the Fourier transform itself, we will denote it as $\mathcal{F}_N f$ rather than $\widehat{f}$.

This definition satisfies many properties, which are standard from Fourier analysis and additive combinatorics [17]:

Proposition 1.1. *If $f : [N] \to \mathbb{C}$, then:*

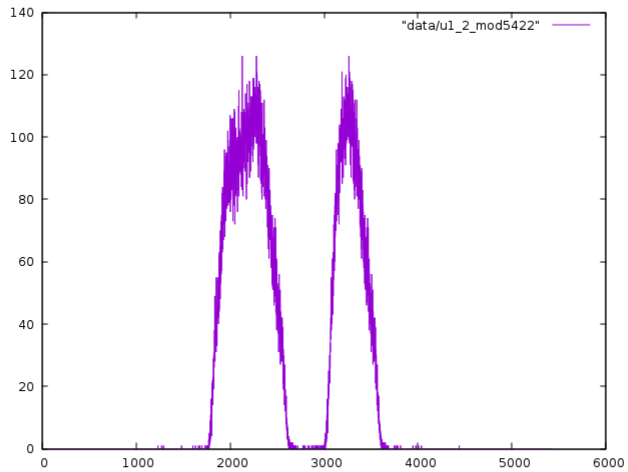
- *If in fact f takes values in $\mathbb{R}$, then $\widehat{f}(-x) = \overline{\widehat{f}(x)}$ for all $x \in \mathbb{R}/\mathbb{Z}$.*
- *$\widehat{\widehat{f}}(x) = f(-x)$ for all $x \in [N]$.*
- *$\widehat{fg}(x) = (\widehat{f} * \widehat{g})(x)$ for all $x \in \mathbb{R}/\mathbb{Z}$.*
- *If in fact f is the indicator function of a set $A \subseteq [N]$, then $\widehat{f}(0) = \frac{|A|}{N}$.*

So if A is a set of positive integers (say, the Ulam sequence), and 1_A is the indicator function of A, then we might define $\widehat{1_A}(x) = \lim_{N\to\infty} \widehat{1_{A_N}}(x)$, where as usual, $A_N = A \cap [N]$ is the truncation of A at N. Then in the case of the Ulam sequence, what is observed numerically is that for one particular value of $\alpha \in \mathbb{R}/\mathbb{Z}$ (namely $\alpha = \ldots$), that $\widehat{1_A}(\alpha) \approx 0.8$, and for $k \in \mathbb{Z}$, $\widehat{1_A}(k\alpha)$ is also some non-zero value that shrinks with k. For example, for $N = 100000$, we compute this for a few values of k (noting that of course the values for $-k$ are just the conjugates of these:

k	$\widehat{A}(k\alpha)$
1	0.7985467537954992
2	0.32061814928309407
3	0.30359418284609546
4	0.6019015738751037
5	0.6004862934665662
6	0.3699258112430088
7	0.1279822946231241
8	0.1443860719363926
9	0.14047291569572581

and as N gets large, it appears that $\widehat{1_{A_N}}(\beta) \to 0$ as $N \to \infty$ for all other $\beta \notin \alpha\mathbb{Z}$.

From a signal processing perspective, this might suggest that the set A has some periodicity mod $\frac{1}{\alpha} \approx \ldots$. Using $\ldots$ as a rational approximation to this, we can plot the distribution of A_N for, say, $N = 100000$ modulo this number:



This has some notable features:

- From the value of $\widehat{1_A}(0)$, it looks like the Ulam sequence has small but nonzero density (in fact, around 0.07).
- As noted in [15] it looks like as we increase N that this is converging to an actually continuous distribution.
- It looks at a glance like this distribution is supported on the middle third of the interval $[0, \frac{1}{\alpha}]$. This is not literally the case, but in [7] there is a conjecture in this direction.

TODO: More questions–do the easy-to-explain-now highlights of the later "Questions" section.

1.1 Results

TODO: Highlight the major results (whatever those end up being)

2 Background

We will start by giving an overview of some known results that should lead us in the right direction. No arguments in this section are original unless noted otherwise.

2.1 Known regularity results

If we want to prove this kind of generalised regularity statement, it might help to understand existing proofs of regularity (i.e. that consecutive differences are eventually periodic). We discuss two such results in this section.

2.1.1 Finch's criterion for regularity

In [6], Finch proves:

Theorem 2.1. *If $A = U(a, b)$ is a 1-additive set containing finitely many even elements, then A is regular.*

The idea of the proof is that if there are finitely many evens, say $e_1 < \ldots < e_s$, then every term n after the last even must be odd. Since it can be written as sum of two earlier terms, and it is odd, one of its summands must be even. And since it can be written in such a sum in a unique way, this is saying that $n - e_i$ is in the sequence for a unique i from 1 to s. This is finitely many things to check.

More precisely:

Proof. If x_n is the number of representations of n as a sum of two elements of A and n is odd, then because an odd number that is a sum of two smaller elements of A must have an even summand and we have only finitely many evens $e_1 < \ldots < e_s$, we can write a finite recurrence:

$$x_n = \sum_{i=1}^{s} 1(x_{n-e_i})$$

where $1(x)$ is 0 unless $x = 1$, in which case $1(x) = 1$. In particular, $0 < x_n \leq s$ for all odd $n > e_s$. Note also that x_n depends on a finite range of earlier x_i's: $x_{n-2}, x_{n-4}, \ldots, x_{n-e_s}$. Call this sequence B_n. Each of the $e_s/2$ values in B_n is between 1 and s, so there are finitely many possible such sequences. Thus, for some N and n, we must have $B_n = B_{n+N}$. But since x_n and x_{n+N} only depend on B_n and B_{n+N} respectively, this means $x_n = x_{n+N}$.

And further, x_{n+2} and x_{n+N+2} only depend on the sequences B_{n+2} and B_{n+N+2}, respectively. But

$$\begin{aligned}
B_{n+N+2} &= (x_{n+N}, x_{n+N-2}, \ldots, x_{n+N+2-e_s}) \text{ by definition} \\
&= (x_{n+N}, x_{n-2}, \ldots, x_{n+2-e_s}) \text{ because } B_n = B_{n+N} \\
&= (x_n, x_{n-2}, \ldots, x_{n+2-e_s}) \text{ as noted above} \\
&= B_{n+2}
\end{aligned}$$

So in fact $B_{n+N+2} = B_{n+2}$ and we can proceed by induction to show the B_n are periodic with period N. Since the x_n are a function of the B_n, x_n is therefore also periodic with period N. ∎

Using empirical computations inspired by this criterion, Finch conjectures [6] that the following is the complete list of (a, b) with $U(a, b)$ regular:

Conjecture 2.2 (Finch)**.** *$U(a, b)$ has only finitely many even terms if and only if (a, b) is in the following list:*

- $(2, v)$ *for* $v \geq 5$ *and odd.*
- $(5, 6)$.
- (u, v) *for* $u \geq 4$ *and even.*
- (u, v) *for* $u \geq 7$ *and odd if* v *is even.*

In particular the conjecture implies that all of these sequences are regular, although it may not be the complete list of regular $U(a, b)$.

2.1.2 Regularity of $U(2, 2n + 3)$

Using the above criterion, Schmerl and Speigel in [14] prove: regularity for the 1-additive sets $U(2, 2n + 3)$:

Theorem 2.3. *The 1-additive sets* $U(2, v)$ *for* $v > 5$ *and odd are regular.*

Since they use Finch's criterion, this boils down to showing that each of these sets has finitely many evens. Specifically:

Lemma 2.4. *The only even elements in the 1-additive set* $U(2, v)$ *(with* $v > 5$ *odd) are 2 and* $2v + 2$.

Proof. The proof goes by supposing that x is the next even element of $U(2, v)$ after $2v+2$ and using an exhaustive knowledge of small elements of the sequence (up to about $5v$) to write $x = a + b$ for smaller $a, b \in U(2, v)$ in more than one way. To do this, we have to understand the small elements of the sequence and the elements just before x.

We leave out the computation of the small elements and simply state the result:

Lemma 2.5. *The elements of* $U(2, v)$ *up to* $5v + 10$ *are:*

- 2
- $2v + 2$
- *All odds between* v *and* $3v$*, inclusive.*
- $3v+4i$ *for* $0 < i \leq \frac{v+1}{2}$ *(that is, every other odd from* $3v$ *to* $5v+2$ *inclusive)*
- $5v + 4$
- $5v + 10$

To use these to express our supposed next even element x as a sum of elements of $U(2, v)$ in multiple ways, we also need to understand the elements immediately leading up to x.

Lemma 2.6. *There is no gap of length* $2v$ *in the odd numbers in the sequence up to* $x - 2v$*. More precisely, if* r *is any odd number less than* $x - 2v$*, then one of* $r, r + 2, \ldots, r + 2v$ *is in* $U(2, v)$.

Proof. If we take r to be the minimal counterexample to this, then $r-2$ is in $U(2,v)$, else $r-2$ would be a smaller counterexample (note that 1 is manifestly not a counterexample, so $r-2>0$).

But then $r+2v=(r-2)+(2v+2)$ expresses $r+2v$ as a sum of elements of $U(2,v)$, so the only way it can fail to be in $U(2,v)$ is if there is another such expression. But $r+2v$ is odd, so any other expression of it as $a+b$ for $a,b\in U(2,v)$ requires that one of a and b be even. And $r+2v<x$, so the only choice other than $2v+2$ (which we have already used) is 2. So this means $r+2v=2+(r+2v-2)$ is the other such expression. But for this to be such an expression, $r+2v-2$ must be in $U(2,v)$, and we are done. $\square$

Corollary 2.6.1. *It follows from the proof that for any odd $r<x-2v$ $r\in U(2,v)$ if and only if exactly one of $r+2v+2$ and $r+2v$ is in $U(2,v)$.*

This will allow us to, for example, find several elements of $U(2,v)$ between $x-3v$ and x. We already know that we have a lot of elements between v and $3v$, so this gives us a good chance of expressing x as a sum of elements of $U(2,v)$ in multiple ways.

For example, the second lemma tells us that some odd between $x-3v$ and $x-v$, say $x-v-2i$ (for some $0\le i\le v$) is in $U(2,v)$. But we know everything of the form $v+2i$ with $0\le i\le v$ is in $U(2,v)$ as well, so:

$$x=(x-v-2i)+(v+2i)$$

is the qualifying expression for x as a sum of smaller elements. Since this expression must be unique, we also know that $x-v-2j$ for $0\le j\le v$ and $j\neq i$ cannot be in $U(2,v)$.

To get a second such expression (and therefore a contradiction), we will look also at the odd elements from $x-5v$ to $x-3v$, using our knowledge of the odd elements of $U(2,v)$ from $3v$ to $5v$.

After some casework, this will end up giving a second qualifying expression for x, thereby disqualifying it. We refer to [14] for the details.

$\square$

2.2 Sum-free sets

The set of Ulam numbers A has the property that for each $a\in A$, there is exactly one solution to $x+y=a$ with $x<y$ in A. The condition that $x<y$ is a little hard to capture using standard techniques, but, for example, this entails that the number of solutions to $x+y=a$ with $x,y\in A$ is at most 3 (namely, the unique solution $x+y=a$ above, then also $y+x=a$, and then possibly some other $z+z=a$, since the definition of the Ulam numbers does not consider this. For example, 4 is Ulam, and its unique representation is $1+3=4$, but it also happens that $2+2=4$ and 2 is also Ulam).

In particular, this implies that if A_N is again the set of Ulam numbers up to N, then A_N has at most $3|A_N|$ solutions to $x+y=z$ with $x,y,z\in A_N$.

We might ask how special such a condition is on sets of integers. For instance, suppose we take the integers up to N and we generate a random subset by including each one with probability p. The size of set we expect to get is pN. The number of pairs x, y is $(pN)^2$, and of these, we expect p of them have $x+y$ in the set, so we expect p^3N^2 solutions to $x+y=z$. In particular, an arbitrary set of density p we expect to have $O(N^2)$ solutions. Since the Ulam numbers appear to have density around 0.07 but by construction have only $O(N)$ solutions to $x+y=z$, they are already somewhat special.

In the interest of understanding what precisely is happening with the Ulam numbers, then, we might turn our attention to the more extreme situation of sets with no solutions to this equation at all: So-called "sum-free sets".

Definition 3. A subset A of an abelian group is **sum-free** if the equation $x+y=z$ has no solutions with $x, y, z \in A$.

Example 1.

1. The odd positive integers are sum-free.
2. More generally, if $A \subset \mathbb{Z}/m$ is sum-free, then the set of integers x that reduce to an element of A modulo m is also sum-free.
3. Even more generally, for any homomorphism $\pi : \mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, if A is a sum-free subset of $\mathbb{R}/\mathbb{Z}$, then $\pi^{-1}(A)$ is a sum-free set of integers.
4. Any subset of a sum-free set is sum-free also.

When we think about generalising the particular notion of "regularity" above for the purpose of the Ulam sequence or for sum-free sets, the basic idea is that a set should be "regular" if it has some correlation with a set of the form in example 3.

2.2.1 Decision sequences

It turns out there is a construction that bijects sum-free sets of positive integers with infinite binary sequences. In words, the construction is simple: Take the positive integers in turn starting with 1. Flip a coin. If it's heads, include it in the set and erase all integers that are sums of elements in the set thus far (as these could not be in the set if it is to be sum-free). If tails, do not include that integer in the set. Then move on to the next integer that has not been included, excluded, or disqualified.

More formally:

Definition 4. Define the function $\theta : \{0,1\}^{\mathbb{N}} \to \{f : \mathbb{N} \to \{0,1\}\}$ from binary sequences to sum-free sets of natural numbers (or, in this case, their indicator functions) as follows: If $s \in \{0,1\}$ is a binary sequence, then using s, we will actually define three disjoint sets that partition the natural numbers: The target set A, the excluded set E, and the disqualified set D. For each $n \in \mathbb{N}$, iteratively select a set for n as follows:

$$\begin{cases} n \in A+A & \implies n \in D \\ n \notin A+A \text{ and } s_k = 1 & \implies n \in A \\ n \notin A+A \text{ and } s_k = 0 & \implies n \in E \end{cases}$$

where, at each stage, $k = |A| + |E| + 1$ is the index of the first element of s that we have yet to consult.

If S is a sequence and A is a sum-free set with $\theta(S) = A$, then S is called the **decision sequence** for A.

Example 2. For example, let us compute $\theta(111111111...)$: We start with 1 and flip a coin and get heads, so we include 1 in the set A. This automatically disqualifies 2 as $2 = 1 + 1$. The next possible candidate is 3, so we flip another coin and get heads, and so we include 3. This automatically disqualifies 4 ($4 = 1 + 3$) and 6 ($6 = 3 + 3$). Continuing in this way, it is clear we will never get a chance to include an even number and will always include the odd numbers, so in the end, the result is the set of odd positive integers.

It is also possible to reverse this construction. In words: Say we start with A a sum-free set. We again walk through the positive integers starting at 1. For each n there are three possibilities: Either $n \in A$, $n \in A + A$, or neither. If $n \in A$, then it got there by a coin landing heads, so we write down a 1. If $n \in A + A$, then n was disqualified from being in A not by a coin flip, but by being a sum of elements of A, so we write down nothing. If $n \notin A$ and also $n \notin A + A$ then n could have been included in A, but was excluded simply because of a coin flip, so we write down a 0.

Formally, we write down the sequence $s = \theta^{-1}(A)$ by writing down first the string s' whose nth character is:

$$s'_n = \begin{cases} \text{`A'} & \text{if } n \in A \\ \text{`D'} & \text{if } n \in A+A \\ \text{`E'} & \text{if } n \notin A \cup (A+A) \end{cases}$$

(So all the 'A's are elements of A, all the 'D's are automatically excluded from A by being sums of prior elements of A, and all the 'E's are things that are excluded from A despite the fact that their inclusion would not violate the sum-free property.) Then the decision sequence s of A is got by starting with s' and deleting all Ds, replacing all As with 1, and replacing all Es with 0.

There are many questions about this construction. For example, it is known that if a sum-free set A is regular (as defined above–i.e. its sequence of successive differences is ultimately periodic), then its decision sequence $\theta^{-1}(A)$ must also be ultimately periodic [1]. The converse is believed to be false, with one of the simplest apparent counterexamples being $\theta(\overline{01001})$ ($\overline{01001}$ meaning the binary sequence that repeats the pattern 01001 forever). This is a set $\{2, 6, 9, 14, 19, 26, 29, 36, 39, 47, 54, 64, 69, 79, 84, 91, 96, \ldots\}$ that has been computed extensively and for which no period has been identified. There is other computational evidence that this sequence may not be periodic beyond just brute force attempts to compute a period found in [10]. Nevertheless, there is no known example of an ultimately periodic decision sequence for which we can prove its corresponding sum-free set is non-regular.

2.2.2 Density and regularity

We start with the observation that if A is a sum-free set and $a \in A$, then A and $A + a$ are disjoint sets of integers. This automatically guarantees that a sum-free set cannot have density in the integers of more than $\frac{1}{2}$. Specifically:

Definition 5. A subset $A \subset \mathbb{Z}^+$ has **density** δ if $\lim_{N\to\infty} \frac{|A_N|}{N}$ exists and is equal to δ.

Since this may not always exist, we might work with another number that always will exist and that, in cases when the density does not exist, provides what should be thought of as at least an upper bound:

Definition 6. A subset $A \subset \mathbb{Z}^+$ has **upper density** δ if $\limsup_{N\to\infty} \frac{|A_N|}{N} = \delta$.

As we have noted, then, the maximal upper density a sum-free set can have is $\frac{1}{2}$, which is realised by the example of the odd positive integers. Luczak has given a sort of converse to this example, proving in [9] the following:

Theorem 2.7 (Luczak)**.** *If A is a sum-free set of positive integers and there is at least one even integer in A, then the upper density of A is bounded above by $\frac{2}{5}$.*

The proof is short, but a little delicate, and we shall recall a version of it in this section.

The basic idea of the proof is to find disjoint subsets of $[N]$ that are the same size as A_N, or of a size related to A_N. For example, if $a \in A_N$ is any element, then because A is sum-free, A_N and $A_N + a$ are disjoint in $[N + a]$, but have the same size, and thus $2|A_N| \leq N + a$, i.e. $|A_N|/N \leq \frac{1}{2} + \frac{a}{2N}$. Taking the limit as $N \to \infty$, we again deduce our earlier statement about A having density bounded by $\frac{1}{2}$.

Proof. Note first that if A is all even elements, then $\frac{1}{2}A$ is also sum-free, and therefore with density $\leq \frac{1}{2}$, and so A has density $\leq \frac{1}{4}$ and the result is automatic, so without loss we may assume A has at least one odd element in addition to its at least one even element.

With this in mind, the proof breaks up into two cases: Where A contains consecutive elements and where it does not.

Case 1: A has no consecutive elements In the case where A has an even element but no two consecutive elements, let t be the minimal odd positive element of $A - A$ which does exist using the odd and even elements, and is not 1, since there are no consecutive elements. Also fix $x, y \in A$ with $t = x - y$.

This means that if $a \in A$, then $a+t-2$ cannot be in A (else $t-2$ would be a smaller odd positive difference than the minimal odd difference t). Put another way, if a and $a+2$ are both in A, then $a+t$ cannot be in A. Put another way, if B is the set of $a \in A$ with $a+2$ also in A, then $B+t$ and A are disjoint. Of course, we already know that finding two disjoint subsets of size even as large as $|A|$ is already easy, however this lets us in fact find three: Since $t = x - y$, this means $B + x - y$ and A are disjoint, meaning $B + x$ and $A + y$ are disjoint. But both of these are contained in $A + A$, so they are both also disjoint from A. Thus we have A, $A + y$, and $B + x$ all disjoint. If we truncate A to A_N, then these are all disjoint subsets of $[N + x]$, and so

$$2|A_N| + |B_N| \leq N + x$$

.

So if we can relate $|B|$ to $|A|$ (for the moment using the shorthand $B = B_N$, $A = A_N$), then we are done.

But by the definition of B, we have two cases for an element of A:

- $a \in B$, in which case $a + 1$ is not in A.
- $a \in A \setminus B$, in which case we know $a + 1$ is not in A (since A has no consecutive elements) and $a + 2$ is not in A, (since otherwise a would be in B).

So we have the five sets: $B, B + 1, A \setminus B, A \setminus B + 1, A \setminus B + 2$, and these are all pairwise disjoint in $[N + 2]$. (The only one that might not be clear is $B + 1 \cap A \setminus B + 2$, but if $a \in A \setminus B$ and $b \in B$ with $a + 2 = b + 1$, then $a + 1 = b$, giving two consecutive elements of A which does not happen.)

Thus $2|B_N| + 3(|A_N| - |B_N|) \leq N + 2$, i.e.

$$|B_N| \geq 3|A_N| - N - 2$$

Now we have a relationship between $|B|$ and $|A|$, so we can pair this with our earlier inequality relating the two of them to N and find:

$$2|A_N| + 3|A_N| - N - 2 \leq N + x$$

or

$$\frac{|A_N|}{N} \leq \frac{2}{5} + o(1)$$

as we wanted.

Case 2: A has consecutive elements: In the case where A has d consecutive elements $a, a+1, \ldots, a+d-1$, say, the argument is similar in flavour to the above, but the technical details are all slightly different. We will first need a t to serve the role of our t in case 1. But now, the minimal odd difference is simply 1. So we do something slightly different: This time, we let t be any positive element of $A - A$ for which $t+1, \ldots, t+d$ are all not in $A - A$.

Lemma 2.8. *Such t does exist*

Proof. Since $a, a+1 \in A$, we know $1 \in A - A$. Then let t be the maximum of $1, \ldots, a-1$ that is in $A - A$, so nothing from t to $a-1$ is in $A - A$ (by definition), and nothing from a to $a+d-1$ is in $A - A$ either (since these are all in A), so at least d elements (and possibly more) immediately after t are not in $A - A$). $\square$

Again, write $t = x - y$ for some fixed $x, y \in A$.

We proceed broadly as before on the two-step plan:

1. Find a set B of elements that gives rise to many disjoint subsets of $[N]$ and deduce a bound relating $|A_N|$ and $|B_N|$ to N.

2. Upper-bound $|B_N|$ in terms of $|A_N|$ and N, and plug this into the previous bound to get a bound on $|A_N|$ in terms of N.

Step 1: Let B be the set of elements b for which $b+1, \ldots, b+d-1$ are all not in A. Then certainly the sets $A, B+1, \ldots, B+d-1$, are all disjoint. In fact, we can get one more than this: We can shift all these sets by a and they are still disjoint: $A+a, B+a+j$ $(j = 1, \ldots, d-1)$. But now since the $a+j$ are all in A, these sets are all themselves disjoint from A (since they are all subsets of $A+A$). Thus, again truncating at N, we have two sets of size $|A_N|$ and $d-1$ sets of size $|B_N|$ all disjoint and inside $[N+a+d-1]$. Thus:

$$2|A_N| + (d-1)|B_N| \le N + a + d - 1$$

Step 2: So again, we need control over the size of $|B_N|$ in terms of $|A_N|$ and we will be done. But this time, we note that if $z \in A$, it is possible that $z + t$ could be in A, but that then because of the definition of t, none of $z+t+1, \ldots, z+t+(d-1)$ can be in A (lest one of $t+1, \ldots, t+(d-1)$ lie in $A - A$). Thus elements of $A + t$ that lie in A in fact must lie in B. Put another way, $A + t$ and $A \setminus B$ are disjoint. Again, this is only two sets, but we can use the same trick as before to make it three: Since $t = x - y$, we can equally say $A + x$ and $(A \setminus B) + y$ are disjoint, at which point these are also disjoint from A (again, being subsets of $A + A$). So we have three disjoint subsets $A + x$, $A \setminus B + y$, and A of $[N + x]$, with sizes $|A_N|$, $|A_N|$, and $|A_N| - |B_N|$, respectively. This gives $|A_N| + |A_N| + (|A_N| - |B_N|) \le N + x$ or:

$$|B_N| \ge 3|A_N| - N - x$$

Dropping this into the first inequality and rearranging, we get:

$$2|A_N| + (d-1)(3|A_N| - N - x) \le N + a + d -$$

which simplifies to:

$$\frac{|A_N|}{N} \le \frac{d}{3d-1} + o(1)$$

Since $d \ge 2$ (as we are assuming we have at least two consecutive elements), this is again bounded by $\frac{2}{5}$ in the limit, so the claimed bound follows. $\square$

2.2.3 Aperiodic sum-free sets

A construction of Erdos in [5] supplies an example of a sum-free set with density $\frac{1}{3}$ that has no periodicity, namely: Take $\alpha \in \mathbb{R}$ irrational, and let A_α be the set of integers n such that $n \mod \alpha$ lies in $\left(\frac{\alpha}{3}, \frac{2\alpha}{3}\right)$. A_α is clearly sum-free, since it is sum-free modulo α, but for irrational α, A_α is also not periodic. That is, for every modulus m and every residue class k, there is an element of A_α congruent to k mod m.

Indeed, equidistribution results for irrational numbers tell us that there the integers are equidistributed modulo any irrational. For example, there is at least one n that reduces to the interval $\left(\frac{\alpha}{3m} - k, \frac{2\alpha}{3m} - k\right)$ modulo the irrational number $\frac{\alpha}{m}$. Then it is clear that $mn + k$ will reduce to $\left(\frac{\alpha}{3}, \frac{2\alpha}{3}\right)$ mod α, meaning that $mn + k \in A_\alpha$ as desired.

2.3 Abelian arithmetic regularity

2.4 Roth's theorem

Roth's theorem is about the number 3-term arithmetic progressions x, y, z in a set $A \subseteq \mathbb{Z}^+$. Specifically:

Theorem 2.9 (Roth's theorem)**.** *Let $A \subseteq Z^+$ be a set of positive integers with positive upper density. Then A contains infinitely many arithmetic progressions $a, a+d, a+2d$ of length 3.*

Equivalently, such an A always has at least one solution to $x + z = 2y$ (whereupon x, y, z is an arithmetic progression of length 3). A sum-free set A instead has are no solutions to $x + z = y$ (swapping around variable names to highlight the similarity), so if we have a sum-free set that we believe has positive density, we might wonder what the proof of Roth's theorem has to say about it. (After all, in the case of the slightly different equation $x + z = 2y$ it says that the set A cannot exist.)

As it turns out, many new techniques in additive combinatorics cut their teeth on Roth's theorem, and so there are many proofs, from those that use probabilistic techniques to ergodic theory. We will discuss two in particular: The density increment and energy increment proofs. We will not give the complete proofs in either case, but will simply work through the steps that we shall return to later and outline the rest.

2.4.1 Density increment proof

Proofs of Roth's theorem often work with a finitary version of the statement, which we make now:

Theorem 2.10 (Roth's theorem)**.** *For every $\delta > 0$, there is an $N_0 > 0$ such that for every $N > N_0$, every $A \subseteq [N]$ with $|A| > \delta N$ contains a solution to $x + z = 2y$.*

One strategy of proof goes via Fourier analysis, saying that if A has no large Fourier coefficients, then A is guaranteed to behave "pseudorandomly" in some sense, and computes that such sets must automatically have many length-3 arithmetic progressions, and we are done already.

If, on the other hand, A does have some large Fourier coefficient, then one can find a long arithmetic progression that has large intersection with A, and on which A in fact has higher density than it had originally. We can repeat this step (the "density increment") as often as needed until either our intersected A has no large Fourier coefficient (in which case we are done as above) or else A's density in the arithmetic progression increases to 1. If we are careful about it, we can ensure that at least 3 elements will still remain by the time we get to this point.

Proof of Roth's theorem via density increment. Rather than working on the set $[N]$, we shall work with the group $\mathbb{Z}/N$, noting that if A only contains elements smaller than $N/2$, then a solution to $x + z = 2y$ in $\mathbb{Z}/N$ is an honest solution to $x + z = 2y$ in A viewed as a subset of $\mathbb{Z}$.

If A is a set of density δ in $\mathbb{Z}/N$, then the number S of solutions to $x+z = 2y$ is counted by

$$\begin{aligned}
S &= \frac{1}{N}\sum_{t=0}^{N-1} \hat{1}_A(t)\hat{1}_g A(t)\hat{1}_A(-2t) \\
S &= \frac{1}{N}|A|^3 + \frac{1}{N}\sum_{t=1}^{N-1} \hat{1}_A(t)\hat{1}_g A(t)\hat{1}_A(-2t) \\
&= \delta^3 N^2 + \frac{1}{N}\sum_{t=0}^{N-1} \hat{1}_A(t)^2\hat{1}_A(-2t) \\
&\geq \delta^3 N^2 - \sup_t |\hat{1}_A(-2t)|\frac{1}{N}\sum_{t=0}^{N-1} |\hat{1}_A(t)|^2 \\
&= \delta^3 N^2 - \sup_t |\hat{1}_A(-2t)|\frac{1}{N}\sum_{t=0}^{N-1} |1_A(t)|^2 \\
&= \delta^3 N^2 - \sup_t |\hat{1}_A(-2t)||A| \\
&= \delta^3 N^2 - \sup_k |\hat{1}_A(k)|\delta N
\end{aligned}$$

So if there is no large Fourier coefficient–that is, every Fourier coefficient is $\leq \epsilon N$, then

$$S \geq (\delta^3 - \delta\epsilon)N^2$$

So if $\epsilon < \delta^2$, then $S > 0$, at which point there is at lest one solution, as desired.

If, on the other hand, there is a k such that $|\hat{1}_A(k)| \geq \delta^2 N$, then this argument does not guarantee a solution. However, in that case, let $P = d[1, L]$ be the arithmetic progression of length$L$ and difference d $\{d, 2d, \ldots, Ld\}$ (d to be chosen later). We want an arithmetic progression in which A has higher density than it has in $\mathbb{Z}/N$ at large. In other words, we want to find an a that makes $Q(a) = |A \cap (P + a)| = 1_A \star 1_P(a)$ large. But this we can analyse using Fourier analysis:

$$\hat{Q}(s) = \hat{1}_A(s)\overline{\hat{1}_P(s)}$$

Further, we know that for all $s \neq 0$, $\sum_a Q(a) \geq |\hat{Q}(s)|$ (looking at the definition of the Fourier transform and using the triangle inequality). So in particular, for $s = k$ (the large Fourier coefficient):

$$\begin{aligned}\sum_a Q(a) &\geq |\hat{Q}(k)| \\ &= |\hat{1}_A(k)||\hat{1}_P(k)| \\ &\geq \epsilon N|\hat{1}_P(k)|\end{aligned}$$

Thus for some a, $Q(a)/N \geq \delta^2|\hat{1}_P(k)|$. We can select d and L such that $|\hat{1}_P(k)| \geq L/2$, so for some A, $Q(a)/N \geq \epsilon L/2$. In particular, A intersected with an arithmetic progression of length L has density $\delta + \epsilon/2$, meaning we have increased the density, whereupon we can repeat the argument.

The details (such as actually selecting the correct d and L, as well as properly transitioning from $\mathbb{Z}/N$ back to $\mathbb{Z}$), are covered in many places, for example [13].

□

2.4.2 Proof via the regularity theorem

The density increment proof exemplifies a kind of dichootomy between structure and randomness in its two cases: Where A behaves pseudorandomly, in which density controls all the counting expressions, and where A has structure, which we can use exploit to iterate the argument.

The pseudorandomness argument will carry over nicely to the sum-free and Ulam cases, but second part of the argument relies on the fact that the structure of interest–in the case of Roth, arithmetic progressions, and in our case, additive triangles–can be found equally in any arithmetic progression. For progressions, this is clear, but for triangles, this is simply false: The very dense progression of all odd numbers already contains zero additive triangles.

As this part of the proof seems unlikely to generalise to our situation, we explore a second avenue of proof for Roth's theorem, namely using the arithmetic regularity and counting lemmas of the earlier section. This comes from [16]

Proof of Roth's theorem via arithmetic regularity. For a function $f : [N] \to \{0,1\}$, define $T(f) = \sum_{x,y\in[N]} f(x)f(x+y)f(x+2y)$ to be the function that counts 3-term progressions if f is thought of as an indicator function. So our goal is to prove that for $A \subseteq [N]$ of density δ (for N sufficiently large relative to δ) that $T(1_A) > 0$.

As in the statement of the regularity lemma, write $1_A = f_{str} + f_{sml} + f_{unf}$ with parameters M and ϵ. . □

2.5 Quantitative bounds in finite fields

There have been several recent developments in a finite field setting on analogous problems (specifically, the work of Croot, Lev, and Pach [3] on length-3 arithmetic progression-free sets in $\mathbb{F}_4^n$ and subsequent work by others [4] pushing it to $\mathbb{F}_3^n$.

We will recall the method used here by outlining the proof in [4], in view of the possibility of later asking about Ulam-like sequences in the same context.

Theorem 2.11 (Ellenberg-Gijswijt)**.** *Let α, β, γ be elements of $\mathbb{F}_q$ such that $\alpha + \beta + \gamma = 0$ and $\gamma \neq 0$. Let A be a subset of $\mathbb{F}_q^n$ such that the equation $\alpha a_1 + \beta a_2 + \gamma a_3 = 0$ has no solutions $(a_1, a_2, a_3) \in A^3$ apart from $a_1 = a_2 = a_3$. Then $|A| = o(2.756^n)$.*

Proof. Let S^d be the space of all polynomial functions on $\mathbb{F}_q^n$ of degree d (that is, polynomials of total degree d where each of the n variables shows up with degree less than q). Let m_d be the dimension of this space, and let V_d be the subspace of polynomial functions vanishing on the complement of $2A$ (this is more or less a trick). Then

$$\dim(V_d) >= m_d - (q^n - |A|)$$

(since the requirement to vanish on the complement of $2A$ is at most $q^n - |A|$ conditions).

It turns out that we can actually get a polynomial P_d in V_d with support of size exactly $\dim(V_d)$, and so this polynomial has:

$$|\text{supp}(P_d)| >= m_d - q^n + |A|$$

Now for the last bit: If we have a degree-d polynomial P vanishing on the complement of $2A$, then we can form the $|A|$ by $|A|$ matrix M whose i, j entry is $P(a_i + a_j)$ where a_i are the elements of A. First of all, because for i and j different, $a_i + a_j$ is never in $2A$, the off-diagonal terms all vanish, whereas because the diagonal terms are $P(2a_i)$, they may or may not vanish.

We can brutally expand this polynomial into a sum of monomials:

$$P(a_i + a_j) = \sum_{\text{monomials m,m' of degree d or less}} c_{m,m'} m(a_i) m'(a_j)$$

Further, in each term at least one of m and m' has degree at most d/2, so we can sum over

$$P(a_i + a_j) = \sum_{\text{monomials m of degree d/2 or less}} c_m m(a_i) F_m(a_j) + c'_m m(a_j) G_m(a_i)$$

So M is a linear combiantion of $2m_{d/2}$ matrices $(m(a_i)F_m(a_j))$ each of which, as the exterior product of two vectors, has rank 1. Thus the rank of M is at most $2m_{d/2}$. And since M is diagonal, this means that in fact on $2A$, P has only $2m_{d/2}$ non-zero points. So the support of P is bounded above by $2m_{d/2}$. Since the support of P_d was already bounded below by $m_d - q^n + |A|$ we can apply this argument to P_d and conclude that

$$2m_{d/2} \geq m_d - q^n + |A|$$

i.e.

$$|A| \leq 2m_{d/2} - m_d + q^n$$

Choosing a particular value of d and bounding these quantities is all that remains. In [4] they take $d = 2(q-1)n/3$ and use Cramer's theorem to bound m_d and related quantities in terms of the claimed exponential. We refer to the paper for details. □

3 Questions and Definitions

Bearing in mind this landscape of ideas, theorems, and techniques, we now raise some questions in the particular context of the Ulam numbers and related sequences, and the various phenomena that we observe around them. In addition, we shall give definitions in this section to allow us to formulate our questions precisely.

3.1 Ulam numbers and other 1-additive sequences

Definition 7. The **Ulam sequence** starting with positive integers a, b is denoted $U(a, b)$ and is the sequence with $a_1 = a$, $a_2 = b$, and, for $n > 2$, $a_n > a_{n-1}$ is the integer satisfying:

1. 1-additivity: There is exactly one pair of $0 < i < j < n$ with $a_i + a_j = a_n$.
2. Greediness: a_n is the smallest positive integer with the above two properties.

One of the first questions that was asked by Ulam himself about the Ulam sequence $U(1, 2)$ was:

Question 3.1. *Does $U(1, 2)$ have positive uppper density?*

For all examples of Ulam-like sequences where we know the answer to this question, the way we know is by first establishing a regularity result, at which point positive density is immediate. Given the known regularity results concerning 1-additive sequences, a very basic question about the Ulam numbers then would be:

Question 3.2. *Can we prove the Ulam numbers are not regular (in the sense of definition 1)?*

Supposing we could do so, we might then ask:

Question 3.3. *Is there a notion of "regularity" that generalises 1 and that captures the behaviour observed in [15] and that we can prove?*

Supposing that in some way Steinerberger's constant α will come into this definition, we might wonder about what it is specifically:

Question 3.4. *Is α irrational? Algebraic? What about $\frac{2\pi}{\alpha}$?*

Beyond just asking about α, we can ask about other features of the spectrum:

Question 3.5. *Are there other nonzero Fourier coefficients not in $\alpha\mathbb{Z}$?*

Question 3.6. *How quickly does $\widehat{1_A}(k\alpha)$ decay with k?*

Moving on from $U(1,2)$, we can also ask about similar sequences:

Question 3.7. *How does α behave for other non-regular-looking Ulam-like sequences? For example, supposing $\alpha_n \in (0, \pi]$ is the maximal Fourier coefficient associated with $U(1,n)$ (supposing there even is a unique such Fourier coefficient), what is the behaviour of α_n as n grows?*

Separately, in light of the triangle removal lemma, there is a set of questions we might ask regarding the additive structure of the Ulam sequence:

Question 3.8. *What is the minimal subset $X \subseteq A$ that we might remove so that A becomes sum-free? (For example, the set such that X_N is minimal among all such possible X for each N.)*

A very similar question that gets more precisely at such a set X in terms of the actual definition of A: We know that each element $a \in A$ is written uniquely as $x + y$ for $x < y$ elements of A. Certainly the set $S = \{x \in A : x + y \in A \text{ for some } y > x \text{ in } A\}$ of "small summands" is a candidate for such an X in the previous question, but S itself might be of interest even if it ends up not being minimal (though one might reasonably expect that it would be).

Question 3.9. *Can we characterise the elements of S? What is the growth rate of $|S_N|$ as N grows?*

Finally, we can ask about the distribution that Steinerberger observes for the Ulam sequence modulo $\frac{2\pi}{\alpha}$, starting with a question from [15]:

Question 3.10. *Does the distribution of A_N mod $\frac{2\pi}{\alpha}$ converge to a continuous distribution? More precisely, let $\lambda = \frac{2\pi}{\alpha}$. Then if for each $M > 0$ we cut up the interval $[0, \lambda]$ into M equal intervals and define a step function $f_{M,N}(x)$ to be the proportion of Ulam numbers up to N that lie in the same one of the M intervals as x, then as M and N go to infinity, does $f_{M,N}$ converge to a continuous function on $\mathbb{R}/\lambda\mathbb{Z}$?*

We can ask a lot more than just about the distribution's continuity, however. The distributions particularly for other Ulam-like sequences such as $U(2, 3)$ look like they have some further internal structure as a sum of perhaps smaller, more regular-looking peaks. So we can ask somewhat broadly about this also:

Question 3.11. *What gives this distribution its particular shape? For example, what about the shape of the distribution can be deduced from the knowledge of the spectrum alone?*

3.2 Sum-free sets

We start noting the same dichotomy that existed with 1-additive sets seems present for sum-free sets as well: Many sum-free sets with easy-to-describe decision sequences (much as the procedure for generating Ulam numbers was in algorithmic) are provably regular, but for some we do not know. For instance, we might start with the aforementioned "smallest" three examples:

Question 3.12. *Are any of the sets $\theta(01001), \theta(01010)$, or $\theta(10010)$ regular?*

Supposing once again (as is suggested in [10]) that the answer is no, we might try to ask similar questions with the these sets:

Question 3.13. *What does the spectrum of these sets look like? Is there a mapping to $\mathbb{R}/\mathbb{Z}$ under which the indicator functions of these sets approach continuous-looking distributions?*

Question 3.14. *Does whatever notion of regularity applies to the Ulam sequence apply here as well?*

Question 3.15. *What is the density of these sets? Is there a statement relating the density of 1s in the decision sequence with the density of the resulting sum-free set?*

In the sum-free case, the work of Luczak outlined above gives some relationship between the regularity of a sum-free set and its density (saying in his case that a sum-free set that contained an even number (i.e. whose image mod 2 was everything) had density bounded by $\frac{2}{5}$–a meaningful improvement from the automatic bound of $\frac{1}{2}$ on the density of an arbitrary sum-free set.

On the other hand, the construction of Erdos tells us that there exist -sum-free sets with density $\frac{1}{3}$ whose image mod m, for all m, is everything. We might ask if there is any condition we can prove in the gap between $\frac{1}{3}$ and $\frac{2}{5}$.

Question 3.16. *If m is a positive integer, what is the maximal density d_m of a sum-free set that hits every congruence class modulo m?*

For example,

- $d_1 = \frac{1}{2}$ (upper bound is by the argument $(A + a) \cap A = \emptyset$ for any $a \in A$, and lower bound comes from the example of the odd numbers).
- $d_2 = \frac{2}{5}$ (upper bound is by [9], and the lower bound is established by the example from the same paper of integers congruent to 2 or 3 mod 5).
- $d_3 = \frac{1}{2}$ using the same argument as for d_1.
- $d_4 = \frac{2}{5}$ by the same argument as for d_2, since $(2 + 5\mathbb{Z}) \cup (3 + 5\mathbb{Z})$ covers every congruence class mod 4.

Lastly, thinking back on the arithmetic regularity lemma, we consider the decpomosition $1_A = f_{str} + f_{sml} + f_{unf}$ for A the various sum-free or nearly sum-free sets (e.g. the Ulam sequence) under consideration.

Question 3.17. *What can we deduce about the structured component from our knowledge of the structure of A? For example, f_{str} comes with bounded Fourier complexity. How does the spectrum of f_{str} relate to the structure and/or density of A? For example, if f_{str} has spectrum consisting of $\alpha\mathbb{Z}$ and $\beta\mathbb{Z}$ for two independent irrational α, β, does this guarantee some bound on the density of A?*

4 Strategy

4.1 Overview

In this document we shall unfortunately not answer all these questions, nor supply a complete understanding of the observed phenomena. We do, however, propose a strategy that we hope will lead to such an understanding, and we shall partially execute certain components of that strategy.

Broadly, we will first try to understand the Fourier spectrum, and then determine what this says about the distrubution. This happens in four steps:

1. Prove the existence of a large Fourier coefficient at some α.
2. Prove that the spectrum of A is supported in $\alpha\mathbb{Z}$.
3. Prove that the Fourier coefficients $\widehat{A}(k\alpha)$ decay fast enough as $k \to \infty$.
4. Deduce features of the distribution of A modulo $2\pi/\alpha$.

Of this programme, we will provide results in the direction of steps 1 and 4, and computational and heuristic evidence in favour of the others.

4.2 Outline

Before we embark on this journey, we provide a little fuller picture of what we intend to actually do towards each of the steps and in the remainder of this document:

1. First, we study the large Fourier coefficient α and corresponding period $\lambda = \frac{2\pi}{\alpha}$ of various Ulam sequences $U(a,b)$ and sum-free $\theta(s)$ (collectively, "almost sum-free sets"). Specifically:

   (a) We use a computer program to estimate the maximal Fourier coefficient of many such sequences.

   (b) We use a computer program to compute continued fraction convergents and to attempt to compute minimal polynomials for certain λ to understand whether they are rational or at least algebraic. We also study the constraints that would be imposed on a $U(a,b)$ if the corresponding λ were to be rational with numerator 3.

   (c) We prove the existence of an α for almost sum-free sets A (truncated at N) at which the Fourier transform has size comparable to $|A_N|$. In fact, we show that the real part of the Fourier transform itself must be large compared to $|A_N|$.

2. We then study the complete spectrum of various almost sum-free sets, specifically:

   (a) We then study the complete spectrum of almost sum-free sets A, computing via computer program any other nonzero values of the respective Fourier transforms. We will find that the Fourier transform for the sets we consider appears to be supported only on $\alpha\mathbb{Z}$.

   (b) We construct an example of a sum-free set with a Fourier spectrum not supported only on some $\alpha\mathbb{Z}$.

   (c) For some almost sum-free sets with spectrum $\alpha\mathbb{Z}$, we use a computer to enumerate the values of $\widehat{f}(k\alpha)$ as k grows and see how these evolve.

   (d) We give an argument that suggests how one might prove the observed decay of these values.

3. Having some picture of the spectrum, we will then study the distribution of A modulo λ and see what information we can deduce from the definition as well as from the spectral information we have gathered in the previous steps. Specifically:

   (a) First the definition of such A means that whether a given x is in A is controlled largely by $r_{A+A}(x)$–the number of representations of x as a sum of elements of A. So we start by studying the distribution of this. We use a computer to generate plots of this that suggest a high degree of regularity in the behaviour of this function, and we show that an estimate of the function using the "major arcs" coming from the Fourier spectrum bear this out, which will give us at least a very plausible conjecture about the behaviour of this function.

   (b) Having come to some understanding of r_{A+A} using the Fourier spectrum, we will use this to attempt to deduce features of the distribution itself, ultimately proving a mild theorem that the distribution is at least non-uniform.

4. Finally, we will focus specifically on $U(1,2)$ and turn to more combinatorial analysis of the structure of the sequence. Specifically:

   (a) By the definition of $U(1,2)$, every element of the sequence has a unique expression $x+y$ for $x, y \in U(1,2)$, $x < y$. So we might ask which x's actually show up in the Ulam sequence. For the provably regular Ulam sequences, the analogous consideration reveals that there are only finitely many values of x that are used in forming the sequence. In the case of $U(1,2)$, we observe a highly skewed distribution of the "small summands" x, and make similar observations about the distribution of y that show up and about the pairs (x, y) as well.

   (b) From these observations, we will start to get a picture of what combinatorial phenomenon might underlie the Ulam sequence, and will consider the implications in particular for the density of the $U(1,2)$.

4.3 About the computations

When we refer to computations done by computer, we will mention the code that was use to execute them by referring to various programs in the repository [11]. In most cases, we will be referring to the particular file `/experiments.py`. For example, when we talk about `experiment17`, we mean the function `experiment17()` in the `experiments.py` file in that repository. Sometimes we will refer to code in other files as well from the same repository, and will mention where the code in question is located in each case.

5 Spectrum

The first three steps of our strategy are about understanding the Fourier spectrum of the various almost sum-free sets under consideration.

5.1 Large Fourier coefficient

The initial observation of [15] is that the Fourier transform of the indicator function of $U(1,2)$ has takes a large value at some α. That is, $\widehat{A_N}(\alpha) = \frac{1}{N}\sum_{t=1}^{N} A(t)e^{-it\alpha}$ is large relative to $|A_N|$. This suggests that t being in A is correlated with $t + \frac{2\pi}{\alpha}k$ being in A for various integer k. In other words, $\lambda = \frac{2\pi}{\alpha}$ behaves somewhat like a period for A.

5.1.1 Computing α

We will start by computing this period for several $U(a,b)$ which are not believed to be regular by virtue of having only finitely many even numbers. Specifically, we will look at:

- $(1,v)$ for $v = 2,\dots,10$.
- $(2,3)$.
- $(3,v)$ for $v = 4,\dots,10$, $3 \nmid v$.
- $(5,v)$ for $v = 7,\dots,9$.

We do this by running `experiment1`, which computes $\alpha_{a,b}$ to around 6 decimal places using the first 10^5 elements of each of these sequences. We get, letting $\rho_{a,b} = \widehat{1_{a,b}}(\alpha_{a,b})$:

a	b	$\alpha_{a,b}$	$\lambda_{a,b}$	$\lvert\rho_{a,b}\rvert$	$\delta_{a,b}$	$\lvert\rho_{a,b}\rvert/\delta_{a,b}$	$\rho_{a,b}$
1	2	2.5714477	0.0600580	2.4434427	0.0753	0.797	-7950.910 + 629.898i
1	3	2.8334973	0.1008799	2.2174664	0.1268	0.795	-7954.286 + 39.703i
1	4	0.5060131	0.1383183	12.4170387	0.1612	0.857	-8579.779 + -42.905i
1	5	0.4075476	0.1413112	15.4170585	0.1657	0.852	-8527.866 + 96.697i
1	6	0.3411608	0.1404853	18.4170739	0.1656	0.848	-8481.219 + 139.212i
1	7	0.2933728	0.1401227	21.4170681	0.1658	0.845	-8451.664 + 82.392i
1	8	0.2573278	0.1418084	24.4170405	0.1681	0.843	-8433.192 + -16.009i
1	9	0.2291699	0.1448657	27.4171464	0.1720	0.842	-8419.458 + 197.206i
2	3	1.1650122	0.0763468	5.3932355	0.0921	0.828	-8274.765 + 354.046i
3	4	2.2090393	0.1032148	2.8443067	0.1202	0.858	-8580.751 + 189.010i
3	5	2.0048486	0.0976665	3.1339948	0.1132	0.862	-8620.021 + 276.095i
3	7	2.1653662	0.1139341	2.9016732	0.1328	0.857	-8575.189 + 83.865i
3	8	2.0338232	0.1302185	3.0893467	0.1518	0.857	-8574.628 + -4.904i
3	10	2.1437414	0.1231638	2.9309436	0.1429	0.861	-8609.816 + -330.867i
5	7	3.2044799	0.0953540	1.9607503	0.1092	0.872	-8700.912 + -685.2187i
5	8	1.2287890	0.1074708	5.1133148	0.1229	0.874	-8742.485 + -68.702i
5	9	2.4845837	0.1060436	2.5288683	0.1215	0.872	-8721.638 + 319.693i

There are a few possible patterns here:

1. The fractional part of $\lambda_{1,n}$ remains roughly constant at around 0.416 for $4 \le n \le 10$, and then alternates between 0.708 and 0.208 thereafter. We also note that $2 \times 0.708 = 1.416$.
2. Indeed, there is a marked change in the value of the maximal Fourier coefficient for $U(1,n)$ between $4 \le n \le 10$ and $n > 10$.
3. None of the λs we have computed appear to be obviously rational. (By contrast, when we repeat this in `experiment1A` with a known regular Ulam sequence such as $U(2,5)$, we get λ that is within 10^{-6} of 2).
4. Though we have only a small amount of data, the first few $\lambda_{3,n}$ look like they might alternate around and converge to some value close to 3.

Similarly, we can compute the α maximising $\widehat{1_A}(\alpha)$ for various sum-free sets A–for example, those that are believed to be non-periodic. In particular, we will look at $s = 01001, s = 01010$, and $s = 10010$ by running `experiment2`:

s	α_s	λ_s	$\lvert\widehat{1_s}(\alpha_s)\rvert$	δ_s	$\lvert\widehat{1_s}(\alpha_s)\rvert/\delta_s$	$\widehat{1_s}(\alpha_s)$
10010	1.9559313	0.0700865	3.2123750	0.0802	0.8735761	-6973.198 + -463.862i
01001	2.5086193	0.0848948	2.5046387	0.0970	0.8750702	-8741.875 + 392.958i
01010	1.8018310	0.0859267	3.4871112	0.0966	0.8893744	-7114.610 + -74.047i

5.1.2 Is λ rational? Algebraic?

As mentioned earlier, the existence of a large Fourier coefficient α for one of these sets A sugggests some kind of pattern or bias modulo $\lambda = \frac{2\pi}{\alpha}$. If this λ were rational, say $\frac{m}{k}$, then this would mean that kA is very non-uniformly distributed modulo m. In particular, we might wonder whether λ is well-approximated by a rational number or, if not, whether it is at least annihilated by some low-complexity polynomial.

To get rational approximations for λ, we run the program `cf_alpha.py` in SAGE to compute continued fractions and convergents:

[TABLE]

(Since we only have believe our λ up to 10^{-6}, we have reason to mistrust convergents with denominator larger than 10^6, so we have left these out.)

One indicator that λ might be quite near a rational number would be if the continued fraction had a single anomalously large coefficient. This is not observed in any of these examples. Further, if we examine the non-uniformity of, say, $U(1, 2)$ modulo m where m runs through the numerators of the convergents (where we measure non-uniformity simply by the variance σ_m in the number of elements that lie in each congruence class modulo m, as computed in `experiment2`:

[TABLE]

Indeed, the variance is quite large for these moduli, but for nearby moduli tends to be far less pronounced, as we compute in `experiment2A`:

[TABLE]

We might wonder: All of the known regularity results that we mentioned earlier were showed that various $U(a, b)$ were biased modulo 2. However there are no such results even for modulo 3, and we do not see any examples of sequences that appear to be heavily biased modulo 3. We leave exploration of why this might be to the future, but we do consider one special case: What would happen if we had an Ulam sequence $U(a, b)$ whose elements were eventually all $1 \mod 3$?

We suspect such sequences might be rare as a consequence of the greediness of the definition of Ulam sequences. For example, the sequence $3, 4, 7, 10, 13, \ldots$ is a sequence that is eventually all congruent to $1 \mod 3$ and that is Ulam-like in that every element bigger than 4 is a sum of distinct smaller elements in a unique way. It fails however to be an actual Ulam sequence because it is not "greedy" in the way that the definition requires. For example, $U(3, 4)$ would indeed start $3, 4, 7, 10$, but then $4 + 7 = 11$ would be next, since that is the smallest number with only one such representation.

In general, the greediness condition on Ulam numbers means that if $x < y$ are in $U(a, b)$, then either $x+y \in U(a, b)$ or else there exist $x' < y'$ in $U(a, b)$ with $x' + y' = x + y$. Thus, if we were to have such a sequence A with only elements that are 1 mod 3, say all elements above some M are $x_1, x_2, \ldots$, then if these are an arithmetic progression, say, then they can ensure that excepting $x_1 + x_2$, every sum of thes elements $x_i + x_j$ can also be expressed as either $x_{i-1} + x_{j+1}$ or $x_{i+1} + x_{j-1}$ so that all these sums are excluded from the sequence. To additionally exclude $x_1 + x_2$, we will need some further elements of the sequence that sum to this quantity as well (which is why the above example failed).

But now, to ensure that x_i are all in A, we will need some elements that are 0 mod 3 to be the small summands of the x_i. Say $y_1, \ldots, y_n$ are all the elements of A that are 0 mod 3, say in increasing order. But then $y_i + y_n$ must not be in the sequence for any i, whcih this means that we need some further elements a_i, b_i that are 1 and 2 mod 3 respectively to be in the sequence and with $a_i + b_i = y_i + y_n$. And we can continue like this, where the more elements that we find must be in the set, the more elements we have to add to the set in order to exclude certain sums of these elements. Perhaps this process does terminate, for some x_i in an actual set A of the form $U(a, b)$, or perhaps if one follows it to its conclusion as in [14], one might find there is in fact no such set. We do not pursue this project here, however.

Leaving aside the question of rationality for the moment, even if λ is irrational, we might still wonder whether it is algebraic. We can use LLL to hunt for the minimal polynomial for various λ. In our case, we use SAGE's LLL code in `alg_lambda.py` to compute. It should be noted that $f(\lambda) << 10^{-10}$ experimentally computed is what is needed to be convincing that $f(\lambda)$ might be actually zero is actually zero, since that is the chosen precision for our lattice reduction.

-4.88604712245 43e-11	$5*X^7-9*X^6-8*X^5-4*X^4+6*X^3+6*X^2+3*X+24)$
1.1938777481609e-10	$(-1)*X*(5*X^7-9*X^6-8*X^5-4*X^4+6*X^3+6*X^2+3*X+24))$
-6.4223470985780e-10	$22*X^5-27*X^4-47*X^3-22*X^2-49*X-17)$
1.3065359905085e-9	$X^9-6*X^8+6*X^7+8*X^6-4*X^5-X^4+5*X^3+6*X^2-12*X+1)$
2.0213413165493e-9	$(-1)*(4*X^6+10*X^5-39*X^4-24*X^3-2*X^2+18*X-14))$
2.9011744118179e-9	$(-1)*(28*X^4-13*X^3-91*X^2-95*X-33))$
3.7695372157032e-8	$(-1)*(25*X^3-62*X^2-123*X+306))$
2.5785948309931e-7	$(-1)*(509*X^2-947*X-725))$
1.8155628112027e-9	$X^8-11*X^5-13*X^4-9*X^3+8*X^2-6*X+9)$
1.8155628112027e-9	$X^8-11*X^5-13*X^4-9*X^3+8*X^2-6*X+9)$
-1.8250148059451e-9	$X^6+6*X^5-22*X^4+7*X^3+X^2-34*X-40)$
-2.3348913913424e-9	$(-1)*(3*X^7-3*X^6-15*X^5-2*X^4+21*X^3+4*X^2+13*X-6))$
3.9355683156828e-9	$27*X^4-92*X^3+40*X^2+25*X+55)$
6.7800982606059e-9	$3*X^5+22*X^4-54*X^3-32*X^2-55*X-28)$
-1.7553418274474e-8	$(-1)*(7*X-18)^2)$
-1.7553418274474e-8	$(-1)*(7*X-18)^2)$

In particular, there is no indication that an of these λ has small degree, if they are algebraic at all.

5.1.3 Existence of α

The common thread with all these almost sum-free sets is that they have few solutions to $x+y=z$ in them. As we talk about why this gives us large Fourier coefficients, the following definition will be helpful:

Definition 8. For $A \subseteq [N]$, define

$$T(A) = \left|\{(x,y,z) \in A^3 : x+y=z\}\right|$$

More generally, for $f : [N] \to [0,1]$, define

$$T(f) = \sum_{x,y \in [N]} f(x)f(y)f(x+y)$$

(So in particular, $T(A) = T(1_A)$.)

Now, the statement that A has a Large Fourier coefficient could be thought of conceptually as saying that A cannot be too random. To get a heuristic for what this would mean, suppose A were random in the sense that we construct it by going through each element of $[N]$ and including it in A with some probability p. Then we expect to have $(pN)^2$ pairs $(x,y) \in A^2$. For any such pair, $x+y$ will be in A with probability p, so the number of pairs $(x,y,x+y) \in A^3$ will be around p^3N^2 as N grows. So for a random set of density δ, we might then expect $T(A_N) \approx \delta^3 N^2$ as N grows.

But by definition, if A is sum-free, $T(A_N) = 0$ for all N, and $T(U(a,b)_N) \leq 3N$ (since each $z \in U(a,b)$ has at most 3 representations as $z = x+y$ for $x,y \in A$, namely, the one with $x < y$ that qualifies it to be in $U(a,b)$, the same one in reverse ($z = y+x$), and possibly $z = \frac{z}{2} + \frac{z}{2}$ if $\frac{z}{2} \in U(a,b)$ (since the definition does not exclude that possibiility). In particular, these sets do not behave like truly random sets would be expected to.

We give a result in this direction saying that if $T(A_N)$ is small relative to N (i.e. is less than N^2), but the set itself is reasonably large, then there must be a Fourier coefficient that explains this:

Theorem 5.1. *If $A \subseteq [N]$ is a set of size δN such that $T(A)$ is bounded by $cN^{2-\epsilon}$ for some constants $c > 0, \epsilon > 0$, then there is an $k \in [2N]$ such that $\widehat{A}(\frac{2\pi k}{2N}) \geq \frac{\delta^2}{2} - \frac{c}{\delta N^\epsilon}$.*

Proof. Denote the discrete Fourier transform by $(\mathscr{F}_N A)(k) = \sum_{t=0}^{N-1} A(t)e(\frac{-2\pi kt}{N})$. Note that this really is a function on $\mathbb{Z}/N$, rather than just $[N]$. Since we want to relate $\mathscr{F}A$ to $T(A)$, we would like to compute $T(A)$ in terms of $\mathscr{F}A$. The following standard trick allows us to do this:

$$T(A) = \sum_{x,y,z=0}^{N-1} A(x)A(y)A(z)\delta_{x+y-z}$$

Where $\delta_0 = 1$ and $\delta_x = 0$ for $x \neq 0$. Then the trick is to write

$$\delta_x = \frac{1}{N}\sum_{t=0}^{N-1} e(\frac{2\pi xt}{N}) = \mathscr{F}_N 1$$

However, we note that this only tests for $x = 0 \mod N$. Thus if we substitute this into our expression for $T(A)$, then we will only be counting solutions to $x + y = z$ modulo N. However, because $A \subseteq [N]$, solving $x + y = z$ in $\mathbb{Z}$ is the same as solving it modulo $2N$ (which we will fix later). Thus we can use the formula:

$$\delta_x = \frac{1}{2N}\sum_{t=0}^{2N-1} e(\frac{2\pi xt}{2N}) = \mathscr{F}_{2N} 1$$

to compute:

$$\begin{aligned} T(A) &= \frac{1}{2N}\sum_{x,y,z=0}^{N-1} A(x)A(y)A(z)\sum_{t=0}^{2N-1} e(\frac{2\pi(x+y-z)t}{2N}) \\ &= \frac{1}{2N}\sum_{x,y,z=0}^{2N-1} A(x)A(y)A(z)\sum_{t=0}^{2N-1} e(\frac{2\pi(x+y-z)t}{2N}) \\ &= 4N^2\sum_{t=0}^{2N-1}\sum_{x=0}^{2N-1}\frac{1}{2N}A(x)e(\frac{2\pi xt}{2N})\sum_{y=0}^{2N-1}\frac{1}{2N}A(y)e(\frac{2\pi yt}{2N})\sum_{z=0}^{2N-1}\frac{1}{2N}A(z)e(\frac{-2\pi zt}{2N}) \\ &= 4N^2\sum_{t=0}^{2N-1}(\mathscr{F}_{2N}A)(-t)(\mathscr{F}_{2N}A)(-t)(\mathscr{F}_{2N}A)(t) \\ &= 4N^2\sum_{t=0}^{2N-1}|(\mathscr{F}_{2N}A)(t)|^2(\mathscr{F}_{2N}A)(-t) \end{aligned}$$

where the second equality we get by extending A by zero.

So by assumption,

$$cN^{2-\epsilon} \geq 4N^2\sum_{t=0}^{2N-1}|(\mathscr{F}_{2N}A)(t)|^2(\mathscr{F}_{2N}A)(-t)$$

We can pull out the $t = 0$ term which is $\frac{\delta^3}{8}$. Then we can bound the remaining sum from below by replacing one of the three $(\mathscr{F}_{2N}A)(t)$ terms inside the sum with $-\max_{t\neq 0}|(\mathscr{F}_{2N}A)(t)| = -\rho$:

$$cN^{2-\epsilon} \geq N^2\frac{\delta^3}{2} - \rho 4N^2\sum_{t=1}^{2N-1}|(\mathscr{F}_{2N}A)(t)|^2$$

Now, by Plancherel we know that, $\sum_{t=0}^{2N-1}|(\mathscr{F}_{2N}A)(t)|^2 = \frac{1}{2N}\sum_{t=0}^{2N-1}|A(t)|^2 = \frac{1}{2N}|A| = \frac{\delta}{2}$, so:

$$cN^{2-\epsilon} \geq N^2\frac{\delta^3}{2} - \rho \cdot 4N^2\frac{\delta}{2} = N^2\left(\frac{\delta^3}{2} - \rho \cdot 2\delta\right)$$

Or, rearranging,

$$\rho \geq \frac{\delta^2}{4} - \frac{c}{2\delta N^\epsilon}$$

Thus for if k is the value of t that realises the maximum, then we have shown that:

$$\left|\frac{1}{2N}\sum_{t=0}^{2N-1} A(t)e(\frac{-2\pi kt}{2N})\right| \geq \frac{\delta^2}{4} - \frac{c}{2\delta N^\epsilon}$$

But comparing the left side to

$$\widehat{A_N}(\frac{2\pi k}{2N}) = \frac{1}{N}\sum_{t=0}^{N-1} A_N(t)e(\frac{-2\pi kt}{2N}) = \frac{1}{N}\sum_{t=0}^{2N-1} A_N(t)e(\frac{-2\pi kt}{2N})$$

we get that for some $k \neq 0$

$$|\widehat{A_N}(\frac{2\pi k}{2N})| \geq \frac{\delta^2}{2} - \frac{c}{\delta N^\epsilon}$$

as claimed. □

In particular, it follows that if ultimately the upper density of any of the almost sum-free sequences is positive, then we are guaranteed a non-zero Fourier coefficient in $\mathbb{R}/\mathbb{Z}$:

Corollary 5.1.1. *If $A \subseteq \mathbb{N}$ is a sequence of positive integers of upper density density $\delta > 0$ such that for all N, $T(A_N)$ is bounded by $cN^{2-\epsilon}$ for some constants $c > 0, \epsilon > 0$, then there is an $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ with $\alpha \neq 0$ such that $\widehat{A}(\alpha) \geq \frac{\delta^2}{2}$.*

Example 3. For example, in the Ulam sequence we know by construction that $T(A_N) \leq 3|A| \leq 3N$, and we believe that the Ulam sequence has density around 0.07, so this theorem would guarantee us a non-zero Fourier coefficient of size at least 0.00245. This is a bit off our numerical value of $0.8\delta \approx 0.8 \times 0.07 = 0.056$, but it is a start.

Proof. Since A has upper density δ, for any $\epsilon > 0$, there is an $N > 0$ with $\frac{|A_N|}{N} > \delta - \epsilon$. So as $j \to \infty$, for each j we can find N_j with $\frac{|A_{N_j}|}{N_j} > \delta - \frac{1}{j}$.

Thus, by the theorem, there is an $\alpha_j = \frac{2\pi k_j}{2N}$ with $\widehat{A_{N_j}}(\alpha_j) \geq \frac{\delta^2}{2} - \frac{\delta}{j} - \frac{1}{2j^2}$.

Now, the $\alpha_j \in [0, 2\pi]$ necessarily have a convergent subsequence. Replacing the sequence by this subsequence, we can say α_j converges to some α as $j \to \infty$, and the limiting value of $\widehat{A_{N_j}}(\alpha_j)$ is greater than $\frac{\delta^2}{2}$ (or, if there is no limiting value, take a convergent subsequence of these values). So then we would like to conclude that $\alpha \neq 0$ in $\mathbb{R}/2\pi\mathbb{Z}$ and that $\widehat{A}(\alpha) \geq \frac{\delta^2}{2}$, the former claim is requires some work and the latter involves swapping some limits, so let us be more careful:

First, why is $\alpha \neq 0$:

...

Now, why does it follow that $\widehat{A}(\alpha) \geq \frac{\delta^2}{2}$? That is, we have that $\lim_{j\to\infty} \widehat{A_{N_j}}(\alpha_j) \geq \frac{\delta^2}{2}$, but we need to show that in fact the same is true for $\lim_{j\to\infty} \widehat{A_{N_j}}(\alpha)$.

For any j, say $\alpha = \alpha_j + \epsilon_j$, with $\epsilon_j \to 0$ as $j \to \infty$. We want to show that in fact $\widehat{A_{N_j}}(\alpha) \geq \frac{\delta^2}{2} - o(j) - O(\epsilon_j)$. But now we can compute using Taylor's theorem, with someβ_j between α_j and α, and using the fact that $\left|\sum_{t=0}^{N-1} te(-tx)\right| = \left|\frac{d}{dx}\sum_{t=0}^{N-1} e(-tx)\right| = \left|\frac{d}{dx}\frac{e(-Nx)-1}{e(-x)-1}\right| \leq \frac{2+2N}{C^2} \leq \frac{3N}{C^2}$ if $|e(-x)-1| \geq C$ and if $N \geq 2$:

$$
\begin{aligned}
\widehat{A_{N_j}}(\alpha) &= \frac{1}{N_j}\sum_{t=0}^{N-1} A(t)e(-t(\alpha_j+\epsilon_j)) \\
&= \frac{1}{N_j}\sum_{t=0}^{N-1} A(t)(e(-t\alpha_j) - t\epsilon_j e(-t\beta_j)) \\
&\geq \frac{\delta^2}{2} - \frac{1}{j} - \frac{\epsilon_j}{N_j}\sum_{t=0}^{N-1} te(-t\beta_j) \\
&\geq \frac{\delta^2}{2} - \frac{1}{j} - \frac{\epsilon_j}{N_j}\frac{3N_j}{C^2} \\
&= \frac{\delta^2}{2} - \frac{1}{j} - \frac{3\epsilon_j}{C^2}
\end{aligned}
$$

Provided $|e(\alpha_j) - 1| \geq C$ for all j (which is possible provided $\alpha \neq 0$ and we take j large enough).

Thus indeed, $\lim_{j\to\infty} \widehat{A_{N_j}}(\alpha) \geq \frac{\delta^2}{2}$, which means exactly that $\widehat{A}(\alpha) \geq \frac{\delta^2}{2}$, as desired. □

We noticed earlier that the actual complex value of the Fourier transform of A for most of the A had large real part and small imaginary part. If we look at the proof of the theorem, we notice that we can use the fact that $(\mathscr{F}_{2N}A)(t) = \overline{(\mathscr{F}_{2N}A)(2N-t)}$ to rewrite:

$$
\begin{aligned}
T(A) &= 4N^2\sum_{t=0}^{2N-1} |(\mathscr{F}_{2N}A)(t)|^2(\mathscr{F}_{2N}A)(-t) \\
&= 4N^2\sum_{t=0}^{2N-1} |(\mathscr{F}_{2N}A)(t)|^2\Re((\mathscr{F}_{2N}A)(t))
\end{aligned}
$$

Then we can use the same exact proof as before, letting instead

$$\rho = \max_{t \neq 0} |\Re((\mathscr{F}_{2N} A)(t))|$$

to conclude that just as the magnitude of some Fourier coefficent must be large, so too must the real part also be large:

Corollary 5.1.2. *If A is as in the theorem, then there is a $k \in [2N]$ such that $|\Re(\widehat{A}(\frac{2\pi k}{2N}))| \geq \frac{\delta^2}{2} - \frac{c}{\delta N^\epsilon}$.*

Some remarks:

Remark 1. As we mentioned before, this bound for the Ulam sequence, which works out to around 0.00245 is not anywhere near as good as the computed estimate of 0.056. However, this finds a rational k/N where the Fourier transform is large for *every* N, whereas experimentally the large value of around 0.056 only occurs actually at α and can only be observed at rational k/N that are good approximations to α. In particular, we also observe that for some N, the largest Fourier coefficient might honestly only be as large as $\frac{\delta^2}{2}$. In particular, an approach that gives a large Fourier coefficient for all.

Remark 2. It is interesting to note that this argument does not provide an obvious way to take advantage of the uniformity with which solutions to $x+y = z$ occur in the Ulam case. For example, it also applies to a sequence where $a_{2^i+1}, \ldots, a_{2^(i+1)-1}$ have no representations but a_{2^i} has 2^{i-1} representations for each i (in which case the number of representations is not bounded above, but is growing, albeit sort of slowly and non-uniformly).

5.2 The complete spectrum of A

Knowing that the spectrum of A has some nonzero α, this suggests at least that $\widehat{A}(n\alpha)$ would also be nonzero for $n \in \mathbb{Z}$.

Further, we also have the theorem of Weyl from [12] that tells us that for any infinite sequence a_n, the set of $x \in \mathbb{R}/\mathbb{Z}$ with $\widehat{A}(x) \neq 0$ has measure zero. We have found one such set of x, namely $\alpha\mathbb{Z}$. We now investigate whether for our particular almost sum-free sets there are any others.

5.2.1 Spectral complexity

There are two pieces of computational evidence that suggest that the spectrum of $U(1,2)$ should consist only of $\alpha\mathbb{Z}$: First, we can brute-force compute $\widehat{A_N}(t)$ for many values of t between 0 and π, and the only time we find values larger than $\sqrt{N}$ are:

[COMPUTATION]

Similar observations can be made for other almost sum-free sets, such as our various $\theta(s)$ and $U(2,3)$ and $U(1,3)$.

Secondly, Plancherel tells us that

[COMPUTATION]

We should point out, however, that simply being almost sum-free is not enough to guarantee the spectrum is of the form $\alpha\mathbb{Z}$ only. For example, take two irrational numbers α and β that are not rational multiples of each other (such as $\sqrt{2}$ and $\sqrt{3}$) and let $A = A_\alpha \cap A_\beta$. Then because $A \subseteq A_\alpha$, e.g., A is certainly sumfree, however it will not be equidistributed modulo either $\frac{2\pi}{\alpha}$ or $\frac{2\pi}{\beta}$, giving it nonzero Fourier coefficients on $\alpha\mathbb{Z}$ and on $\beta\mathbb{Z}$.

5.2.2 Decay of Fourier coefficients

Recall from the background section our intuition on why smoother functions have better decay in their Fourier coefficients: The Fourier coefficient $\widehat{f}(\xi) = \int_0^1 f(t)e^{2\pi i t\xi}dt$ can be integrated by parts, which will replace $f(t)$ with $f'(t)$ (which if f is well-behaved should be bounded or otherwise controlled), and $e^{2\pi i t\xi}$ with the integral of this, which as ξ grows, will wind faster and faster around the unit circle, giving this integral more and more cancellation and therefore making it smaller. In particular, the antiderivative of $e^{2\pi i t\xi}$ involves a $\frac{1}{\xi}$, so the decay should go like $\frac{1}{\xi}$.

Though turning this intuition into a proof in our context has not yet succeeded, we give a conjecture in this direction and some computational evidence in its favour:

Conjecture 5.2. *If A is a almost sum-free set of positive integers, and $\widehat{A}(\alpha) \neq 0$, then for $d \in \mathbb{Z}$ with $d \neq 0$, we have*

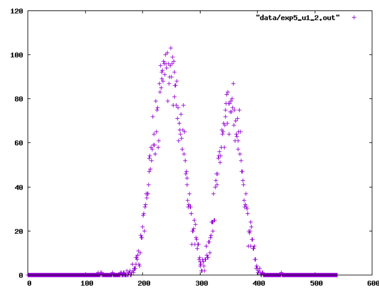
$$\widehat{f}(d\alpha) \leq \frac{C}{d}$$

for some constant C.

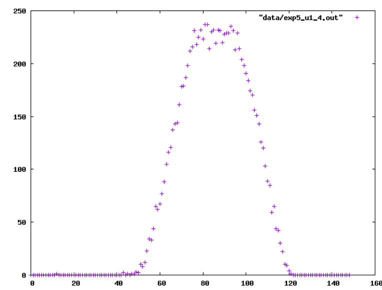
To support this conjecture, we compute $d\widehat{A_N}(d\alpha)$ for d up to $\sqrt{N}$ and observe that in all cases, d appears to be bounded by 4.

6 Distribution

As we have remarked before, a sequence A having a large Fourier coefficient at α suggests A should not be equidistributed modulo $\lambda = \frac{2\pi}{\alpha}$. Indeed, we have experimentally found large Fourier coefficients for many of the sequences we have considered, and if we take the distribution of these sequences modulo the corresponding λ values, we get non-uniform, continuous-looking distributions:



$U(1,2)$



$U(1,3)$

In this section, we will examine how what we have learned about the spectrum might control various features about the corresponding distributions.

6.1 Distrubution of r_{A+A}

The first observation is that the common feature of the almost sum-free sets is that being in x being in such a set A is a function of how many ways we can write x as a sum $x = a + b$ for $a, b \in A$. To study this, we make the following definitions:

Definition 9. Define the **sum representation counting function** r_{A+A} by

$$r_{A+A}(x) = \left|\{(a,b) \in A^2 : a + b = x\}\right|$$

Also define, the **modified sum representation counting function**

$$r^*_{A+A}(x) = \left|\{(a,b) \in A^2 : a + b = x; a < b\}\right|$$

Finally, define the **difference representation counting function**

$$r_{A-A}(x) = \left|\{(a,b) \in A^2 : a - b = x\}\right|$$

So $r_{A+A}(x) = 0$ is the necessary condition for x to lie in a sum-free set A, as is $r_{A-A}(x) = 0$, whereas $r^*_{A+A}(x) = 1$ is the condition for being in an Ulam sequence A. Likewise, for x in an Ulam sequence A, $r_{A-A}(x)$ is the number of times x is used as a summand in A. (We note that r_{A-A} may be infinite if A is infinite, but it will make sense when we truncate A.)

But we can write a formulae for $r_{A+A}(x)$ and $r_{A-A}(x)$ in terms of the indicator function of A:

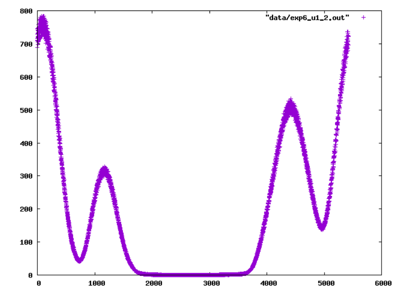
$$r_{A+A}(x) = \sum_{0<y<x} A(y)A(x-y)$$

$$r_{A-A}(x) = \sum_{0<y} A(y)A(x+y)$$

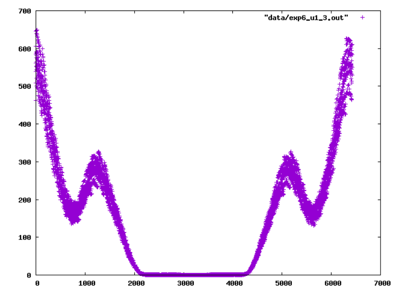
which are exactly the definition of the convolution $(A * A)(x)$ and cross-correlation $(A \star A)(x)$ respectively.

r^*_{A+A} is less clean, but we can write, $2r^*_{A+A}(x) = r_{A+A}(x) - A(x/2)$. So if we get bounds on $r_{A+A}(x)$, we can expect to transfer these to bounds of $r^*_{A+A}(x)$. Thus we expect that understanding r_{A+A} will be sufficient to allow us to understand Ulam sequences as well as sum-free sets.

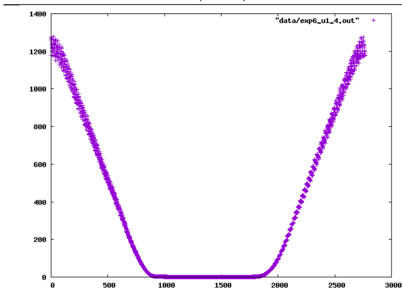
So if our assertion is that whether a given x is in A is strongly affected by the value of x modulo the associated λ_A, then we might wonder whether it appears that $r_{A+A}(x)$ is affected. We can do this by taking the various congruence classes a modulo λ and computing the average value of $r_{A+A}(x)$ for x ranging over integers congruent to a modulo λ. Plotting these for the first 10^5 terms of various sequences we get the plots:



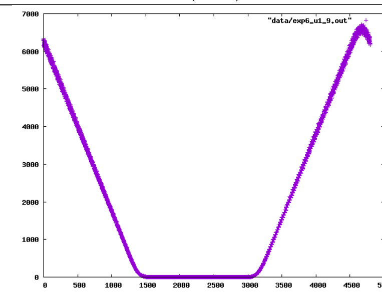
$U(1,2)$



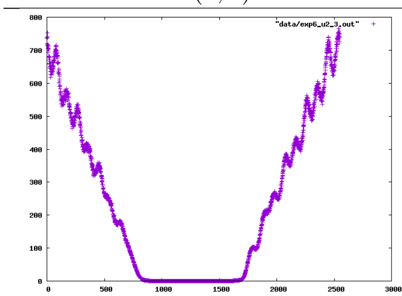
$U(1,3)$



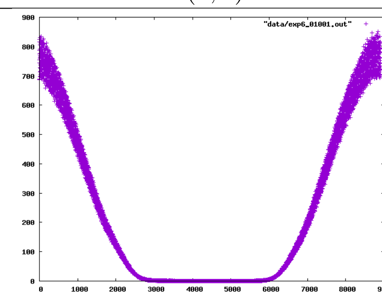
$U(1,4)$



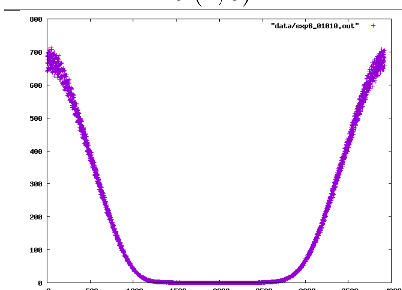
$U(1,9)$



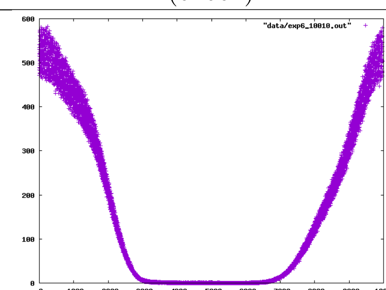
$U(2,3)$



$\theta(01001)$



$\theta(01010)$



$\theta(10010)$

The general pattern here seems to be that near 0, all these distributions are large, and within in the middle third, they are all close to zero, and then in between, we can get substantial fluctuations, from the relatively extreme example of $U(2,3)$, to the very sedate examples provided by the sum-free sets, or by $U(1,4)$.

These two observations would suggest that near 0, we should not have any elements of these sequences, and that most of the elements should lie in the middle third. Further, for Ulam sequences, this also suggests that any x that fails to be in the sequence because $r^*_{A+A}(x) = 0$ should live in the middle third as well, possibly explaining the gap in the distribution of elements of $U(1,2)$ in the very middle.

We shall address these observations in the following sections, but we give a word first about the strategy: The benefit of studying $r_{A+A}(x)$ is that, being a convolution, we understand its Fourier transform. More precisely, truncate A at N and view $r_{A_N+A_N}(x)$ and $r_{A_N-A_N}(x)$ as functions on $\mathbb{Z}/2N$. Then $(\mathscr{F}_{2N}r_{A_N+A_N}) = (\mathscr{F}_{2N}A_N)^2$ and $(\mathscr{F}_{2N}r_{A_N-A_N}) = |\mathscr{F}_{2N}A_N|^2$. Using Fourier inversion, then, we can compute $r_{A_N+A_N}$ and $r_{A_N-A_N}$ in terms of the Fourier coefficients of A:

$$r_{A_N+A_N}(x) = \sum_{t=0}^{2N-1} ((\mathscr{F}_{2N}A)(t))^2 e(\frac{2\pi t}{2N}x) \tag{6.1}$$

$$r_{A_N-A_N}(x) = \sum_{t=0}^{2N-1} |(\mathscr{F}_{2N}A)(t)|^2 e(\frac{2\pi t}{2N}x) \tag{6.2}$$

Since our observations suggest that the spectrum of A is simply $\alpha\mathbb{Z}$, in particular for A_N we have $\widehat{A_N}(k\alpha) \leq \frac{C}{k}$ for k up to around $\sqrt{N}$ and we have lower bounds for $(\mathscr{F}_{2N}A)(\alpha)$. We can then approximate this sum using these large Fourier coefficients and lumping the rest into an error term R_N that we expect (but were unable to prove) is small enough to ignore:

$$r_{A_N+A_N}(x) = \sum_{-\sqrt{N}<k<\sqrt{N}} (\mathscr{F}_{2N}A)^2(k\alpha)e(k\alpha x) + R_N \tag{6.3}$$

$$r_{A_N-A_N}(x) = \sum_{-\sqrt{N}<k<\sqrt{N}} |(\mathscr{F}_{2N}A)(k\alpha)|^2 e(k\alpha x) + R_N \tag{6.4}$$

This is what we shall attempt to exploit in the following section, but before then, a word about the actual distributions of r_{A+A} that we plotted above:

Proposition 6.1. *Let A be one of our almost sum-free sets, and let $\frac{m}{k}$ be a rational approximation to λ_A, so α is approximated by $\widetilde{\alpha} = \frac{2\pi k}{m}$. Then $f_N(x)$ be the function from $\mathbb{Z}/m \to \mathbb{R}$ that averages the values of $r_{A_N+A_N}(t)$ for $t = x$ mod λ, i.e. $kt = x \mod m$.*

That is,

$$f_N(x) = \frac{m}{N} \sum_{kt=x\ (m), t<N} r_{A+A}(t)$$

Then we can express:

$$f_N(x) = \sum_{l=0}^{m-1} e(\frac{-2\pi xl}{m})\widehat{A_N}(l\widetilde{\alpha})^2$$

Proof. This follows by the usual trick of expressing the indicator function of a congruence class mod m using an exponential sum. In our case:

$$1(kt = x \bmod m) = \frac{1}{m}\sum_{l=0}^{m-1} e(\frac{2\pi(kt-x)}{m}l)$$

So if we simply do this substitution and reverse the order of summation, we get:

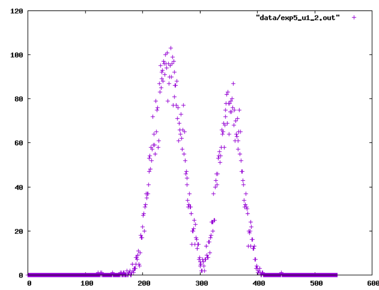
$$\begin{aligned}
f_N(x) &= \frac{m}{N}\sum_{kt=x\ (m),t<N} r_{A+A}(t) \\
&= \frac{m}{N}\sum_{t=0}^{N-1} r_{A+A}(t)1(kt = x \bmod m) \\
&= \frac{m}{N}\sum_{t=0}^{N-1} r_{A+A}(t)\frac{1}{m}\sum_{l=0}^{m-1} e(\frac{2\pi(kt-x)}{m}l) \\
&= \sum_{l=0}^{m-1}\frac{1}{N}\sum_{t=0}^{N-1} r_{A+A}(t)e(\frac{2\pi(kt-x)}{m}l) \\
&= \sum_{l=0}^{m-1} e(\frac{-2\pi xl}{m})\frac{1}{N}\sum_{t=0}^{N-1} r_{A+A}(t)e(\frac{2\pi klt}{m}) \\
&= \sum_{l=0}^{m-1} e(\frac{-2\pi xl}{m})\widehat{r_{A+A}}(l\widetilde{\alpha}) \\
&= \sum_{l=0}^{m-1} e(\frac{-2\pi xl}{m})\widehat{A_N}(l\widetilde{\alpha})^2
\end{aligned}$$

□

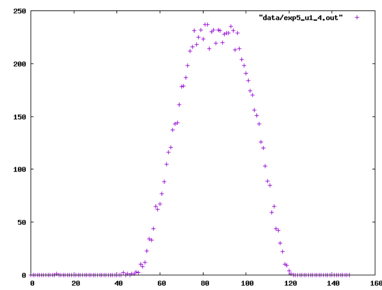
Thus if we had more precise knowledge of the argumnet of $\widehat{A_N}(\alpha)$ we could deduce, for example, that when x is close to zero, that this is large (since it would be dominated by the first two terms), or that when x is close to $\frac{m}{2}$, this should be small since it is an alternating sum.

6.2 Distrubution mod λ

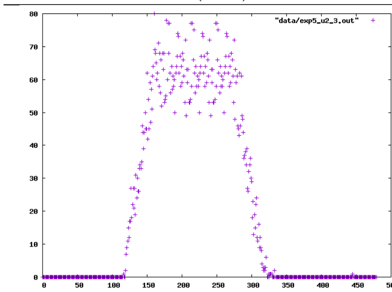
As mentioned, the distributions of various $U(a, b)$ and $\theta(s)$ modulo their respective λ values. are non-uniform, and seem to have most (but not all) of their support in the middle third of the interval $[0, \lambda]$. For example, if we take the first 10^5 elements of each sequence and take a rational approximation $\lambda \approx \frac{m}{k}$ and plot, for each congruence class r mod m how many $a \in A$ have $ka \cong r$ mod m, then we get the following plots:



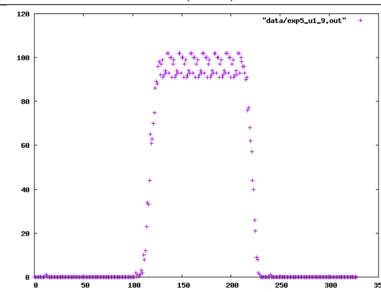
$U(1,2)$



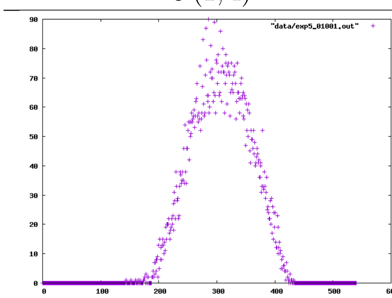
$U(1,3)$



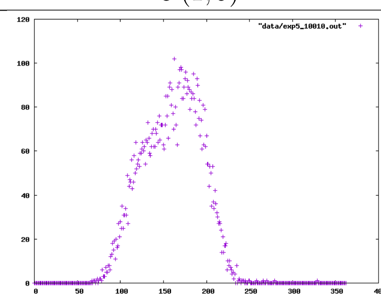
$U(1,4)$



$U(1,9)$



$\theta(01010)$



$\theta(10010)$

In this section, we will focus on the three observations we mentioned in the previous section, namely:

1. There seem to be few elements of each A in some interval around 0 modulo λ.
2. The support of these distributions seems to usually contain the middle third modulo λ.
3. For Ulam sequences, the x that fail to be in the sequence on account of $r^*_{A+A}(x) = 0$ all lie close to $\frac{\lambda}{2}$ modulo λ.

As a useful piece of notation for expressing the idea of intervals such as "the middle third modulo λ", we will define as is standard:

Definition 10. For $x \in \mathbb{R}$, define $\|x\|_{\mathbb{R}/\mathbb{Z}}$ to be the minimal distance between x and an integer: $\|x\|_{\mathbb{R}/\mathbb{Z}} = \min_{n\in\mathbb{Z}} |x - n|$.

Thus $\|x\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{2}$ always, and x is in the "middle third", i.e $[\lambda/3, 2\lambda/3]$ mod λ precisely if $\|x/\lambda\|_{\mathbb{R}/\mathbb{Z}} \geq \frac{1}{3}$. Likewise, an interval, (say of radius η) around zero (mod λ) is expressed by the condition $\|x/\lambda\|_{\mathbb{R}/\mathbb{Z}} < \eta$.

6.2.1 No elements close to 0 mod λ

So we move first, as promised, to investigate why it appears that there are no elements of A close to integer multiples of λ.

The distributions of r_{A+A} would suggest that the reason is that x that are close to $\lambda\mathbb{Z}$ have $r_{A+A}(x)$ large.

As we saw in equation 6.1, we can express this in terms of the spectrum of A, and hopefully can therefore extract some information. In particular, if $\|x/\lambda\|_{\mathbb{R}/\mathbb{Z}} < \eta$, then for $\frac{m}{2N} \approx \lambda$, $e(\frac{2\pi k}{2N}x)$ will be on the arc between $e(-\eta)$ and $e(\eta)$, making it very close to 1. So, particularly if we use the approximation given in equation 6.3, we get:

$$\begin{aligned}
r_{A_N+A_N}(x) &= 2N\left(\sum_{-\sqrt{N}<k<\sqrt{N}} (\mathscr{F}_{2N}A)(k\alpha)^2 e(k\alpha x) + R_N\right) \\
&\approx 2N\left(\frac{\delta^2}{2} + 2\Re((\mathscr{F}_{2N}A)(\alpha)^2 e(\eta)) + \sum_{k>1} 2\Re((\mathscr{F}_{2N}A)(k\alpha)^2 e(k\eta)) + R_N\right)
\end{aligned}$$

And so if the argument of $(\mathscr{F}_{2N}A)(\alpha)$ is close to π and η is small enough, then hopefully this first term is (or perhaps the first few are) larger than the error and the remaining sums and we get that $r_{A_N+A_N}(x) = O(N)$. However this requires very precise control over the argument of $(\mathscr{F}_{2N}A)(k\alpha)$ that we do not currently have.

However, at least for truly sum-free sets, we can still pull out a theorem (as always, conditional on small error) of this kind by using the fact that for a sum-free set, an element can be in the set A only if $r_{A-A}(x) = 0$ also. And, recalling 6.1 and 6.3, this we can compute using only the magnitudes of the Fourier coefficients, allowing us to ignore the argument issue completely:

Theorem 6.2. *Let A is a almost sum-free set of positive integers with $|A_N| = \delta N$. Suppose that A has spectrum $\alpha\mathbb{Z}$, that it satisfes 5.2, and that the error in 6.3 in fact satisfies $R_N = o(1/N)$. Then for some $\eta > 0$, $\|x/\lambda\| < \eta$ implies $r_{A_N-A_N}(x) = O(N)$.*

Proof. Take x with $\|x/\lambda\|_{\mathbb{R}/\mathbb{Z}} < \eta$ for η to be chosen later. Viewing A_N as usual inside $\mathbb{Z}/2N$, we have $x \in A_N - A_N$ viewed inside $\mathbb{Z}/2N$ iff $x \in A_N - A_N$ in $\mathbb{Z}$. So we can use equation 6.3. In preparation for doing this, let $|\widehat{A_N}(\alpha)| = \rho$ (we know $\rho > \frac{\delta^2}{2} - O(1/N)$ by 5.1). Finally, let k_0 be such that

$$\delta^2 + \rho^2 - 4C^2 \sum_{k=k_0}^{\infty} \frac{1}{k^2} > 0$$

(which we know is possible since $\delta > 0, \rho > 0$, and since the the sum converges, and so approaches 0 as $k_0 \to 0$). Then if we guarantee that $\|k_0 x/\lambda\|_{\mathbb{R}/\mathbb{Z}} < \frac{1}{4}$ (so that $\Re(e(k\alpha x)) > 0$ for all $k < k_0$), then as N gets large, we have:

$$\begin{aligned} r_{A_N - A_N}(x) &= 2N\left(\sum_{-\sqrt{N}<k<\sqrt{N}} (\mathscr{F}_{2N}A)(k\alpha)^2 e(k\alpha x) + R_N\right) \\ &\geq N\delta^2 + 2N|\widehat{1_A}(\alpha)|^2 \Re(e_N(\alpha x)) - 2N\sum_{|k|\geq k_0} |\widehat{1_A}(k\alpha)|^2 \end{aligned}$$

where here, we are using that $\Re(|\widehat{1_A}(k\alpha)|^2 e(k\alpha x)) \geq 0$ for $1 < k < k_0$ by choice of η, and that $\Re(|\widehat{1_A}(k\alpha)|^2 e(k\alpha x)) \geq -|\widehat{1_A}(k\alpha)|^2$ for $k \geq k_0$. So:

$$\begin{aligned} r_{A_N - A_N}(x) &\geq \frac{\delta^2}{2}N + \frac{\rho^2}{2}N - 2N\sum_{|k|\geq k_0} |\widehat{1_A}(k\alpha)|^2 \\ &\geq \frac{\delta^2}{2}N + \frac{\rho^2}{2}N - 2C^2 N\sum_{|k|\geq k_0} \frac{1}{k^2} \end{aligned}$$

Using theorem 5.2 to conclude $|\widehat{1_A}(k\alpha)| \leq \frac{C}{k}$ for some constant C.

$$\geq \frac{N}{2}\left(\delta^2 + \rho^2 - 4C^2 \sum_{|k|\geq k_0} \frac{1}{k^2}\right)$$

Which we know by choice of k_0 is $O(N)$, as desired.

□

For sum-free A, this automatically guarantees that no integer within this interval can be in A, as for sum-free sets, $r_{A_N - A_N}(x) > 0$ for any N already implies $x \notin A$:

Corollary 6.2.1. *If A is a sum-free set satisfying all the conditions of the theorem, then there is an $\eta > 0$ such that $\|x/\lambda_A\|_{\mathbb{R}/\mathbb{Z}} < \eta \implies x \notin A$.*

6.2.2 Few Ulam numbers outside middle third mod λ

A conjecture of Gibbs states...

6.2.3 Numbers that are not sums of Ulam numbers close to middle mod λ

Numbers x that fail to be Ulam because in fact $r^*_{A+A}(x) = 0$ all seem to lie within the middle third.

6.3 Non-uniformity/Regularity

Without any unconditional result describing the distributions of our various A modulo λ_A, we might ask whether we can at least guarantee some kind of non-uniformity of these distributions. For example, can we find a set $E \subseteq \mathbb{R}/\mathbb{Z}$ such that if $\pi : Z \to \mathbb{R}/\mathbb{Z}$ is $x \mapsto x/\lambda \mod 1$, then $|A_N \cap \pi^{-1}(E)|/N \geq \frac{|A_N|}{N}\frac{|\pi^{-1}(E)_N|}{N}$? In fact, we can do slightly better:

Theorem 6.3. *For A an almost sum-free set, and let α be the maximal Fourier coefficient, and define $E_t = \{n \in [N] : \Re(e(\alpha n)) \leq \eta\}$ (roughly, the set of integers that land in an interval of radius η centred at $\lambda/2$ modulo λ. Then there is a η such that $\langle A_N, E_{\eta,N}\rangle = |A_N \cap E_{\eta,N}| \geq \delta^2/4 + \delta\frac{|E_{\eta,N}|}{N}$.*

Proof. Let $f(t) = A(t) - \delta$ be the "balanced" indicator function of A. Then we know from 5.1.2 that $\frac{1}{N}\sum_{t=0}^{N-1} f(t)\Re(e(\alpha t)) \geq \frac{\delta^2}{2}$. The key here is to write:

$$\Re(e(\alpha t)) = 1 - \int_{-1}^{1} E_\eta(t)d\eta$$

Then

$$\begin{aligned}
\frac{\delta^2}{2} &\leq \frac{1}{N}\sum_{t=0}^{N-1} f(t)\Re(e(\alpha t)) \\
&= \frac{1}{N}\sum_{t=0}^{N-1} f(t) - \frac{1}{N}\sum_{t=0}^{N-1}\int_{-1}^{1} f(t)E_\eta(t)d\eta \\
&\leq \left|\int_{-1}^{1}\frac{1}{N}\sum_{t=0}^{N-1} f(t)E_\eta(t)\right| d\eta \\
&\leq \left|\int_{-1}^{1}\langle f, E_\eta\rangle\, d\eta\right|
\end{aligned}$$

Thus $\langle f, E_\eta\rangle \geq \frac{\delta^2}{4}$ for some η. But $f = A - \delta$, so

$$\langle A, E_\eta\rangle \geq \frac{\delta^2}{4} + \langle\delta, E_\eta\rangle = \frac{\delta^2}{4} + \delta\frac{|E_{\eta,N}|}{N}$$

And this is what we wanted to show. ☐

7 Structure of $U(1,2)$

7.1 Distribution of summands

7.1.1 Distribution of large summands

We've studied the smaller summands a bit–now the question is: What about the large summands?

We note first that if 2 or 3 is the small summand of a_{n+1}, then the large summand is necesarily a_n (if 2 is the small summand and a_n is not the large summand, then a_{n+1} would be a_n+1 which is impossible since this is a duplicate sum with $a_n + a_1$. If 3 is the small summand and a_n is not the large summand, then a_{n+1} is either $a_n + 1$ or $a_n + 2$, both of which would be duplicates.)

This means that over 50% of the time, the large summand will be the last thing in the list so far. When looking at the large summand, then, it seems more relevant to consider how many indices from the end it lives, rather than its actual value. We compute these in experiment13, generating output [FILE] of the form:

n	a_n	i	$n-j$ (with $a_i + a_j = a_n$ and $i < j$)
2	1	0	1
3	2	0	1
4	3	1	1
5	4	1	1
6	6	2	1
7	8	1	1
8	11	2	1
9	13	1	1
10	16	5	1
11	18	1	1

We can do some processing on these to figure out which $n-j$ are the most common:

results [FILE]. Note in particular that there are only 159 of them and (as suggested above) that n-j = 1 accounts for over 50% of them. This list seems to contain few surprises: Among all the values of $n-j$ that appear more than 10 times, nothing bigger than 34 shows up.

If we look at values that show up fewer than 10 times, then it looks like $n-j = 100$ and $185 < n-j < 205$ seem to be preferred, with many of these showing up 3 or more times, while all other values show up 2 or fewer times. This could be an artifact of not much data, however.

We might instead take a look instead at enumerating $(i, n-j)$ pairs, rather than just values of $n-j$:

with results in [FILE]. Note in particular that there are only 312 distinct such pairs, meaning that technically, to compute the first 10000 Ulam numbers, we only have to check 312 possibilities for each. If only there were a way of knowing ahead of time which 312 we had to check...

7.1.2 Distribution of small summands

We note with interest the observation in the abstract of Steinerberger that $\cos(\alpha a_n) < 0$ for all a_n other than 2, 3, 47, and 69. In particular, thee were also the a_n that showed up most frequently as summands in our earlier computation.

So we compute which how often each a_n appears as the smaller summand of a later a_i and we compute $\cos(\alpha a_n)$ for each and sort by this quantity. We note what looks like a very strong correlation between how often a_n shows up as a summand and $\cos(\alpha a_n)$ in the resulting table, computed by experiment11. Define S_i to be the number of a_i such that a_n is the smaller summand of a_i

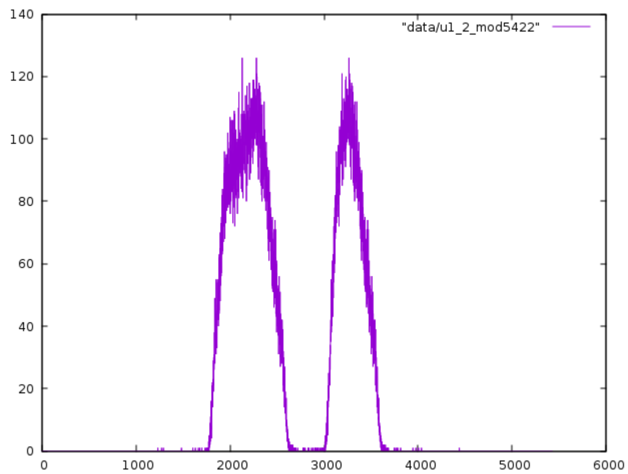
a_n	S_n	$\cos(\alpha_{1,2}a_n)$	$[a_i : a_i = a_n + a_?]$
2	3630	0.4173307	[6, 8, 13, 18, 28, 38, 99, 177, 182, 221, 238, ...
3	1356	0.1391857	[11, 16, 72, 102, 148, 180, 209, 241, 319, 412, ...
47	1190	0.0931494	[236, 253, 356, 363, 429, 456, 544, 720, 732, ...
69	999	0.0700500	[175, 258, 451, 483, 566, 820, 1018, 1052, 1101, ...
102	836	-0.0353342	[282, 441, 502, 585, 624, 646, 668, 949, 1125, ...
339	589	-0.071198	[695, 739, 751, 861, 905, 1186, 1230, 1770, 1853, ...
36	465	-0.1046811	[138, 309, 602, 927, 1191, 1550, 1682, 2090, 2288, ...
273	305	-0.140326	[612, 673, 685, 1164, 1296, 1308, 1428, 1660, 1765, ...
8	181	-0.1506519	[26, 36, 77, 114, 197, 324, 390, 991, 1470, 1602, ...
2581	85	-0.2874041	[5795, 7459, 8947, 9443, 9619, 9641, 9663, 10677, ...
400	55	-0.288437	[3214, 3605, 3991, 12763, 13562, 13799, 13931, 15160, ...
983	50	-0.316087	[2445, 2748, 5514, 9553, 16121, 17135, 19427, 21626, ...
97	47	-0.320453	[316, 370, 497, 1252, 2581, 3622, 4057, 10366, 13628, ...
356	21	-0.3324957	[983, 4118, 11226, 22676, 27817, 34104, 34969, 52789, ...
1155	16	-0.3466450	[4878, 9132, 13733, 16047, 27883, 30886, 38920, 40931, ...
206	33	-0.3521083	[522, 891, 1155, 1514, 2787, 4324, 9399, 11432, 20375, ...
53	35	-0.3640037	[155, 409, 1208, 3038, 5049, 8421, 14945, 16648, 19480, ...
1308	18	-0.369428	[3029, 8368, 10501, 20937, 29147, 34784, 37765, 61029, ...
9193	8	-0.3921254	[20419, 68914, 74099, 83323, 92073, 108317, 123718, 124864]
10831	4	-0.427570	[31878, 56503, 89101, 126493]
13	6	-0.427833	[82, 219, 273, 19642, 59734, 91748]
14892	2	-0.4327025	[41620, 84500]
13531	3	-0.437951	[47279, 61451, 83139]
23883	2	-0.440936	[65740, 106394]
10269	1	-0.446345	[44816]
8368	1	-0.449499	[20968]
20643	0	-0.4532970	[]
316	5	-0.457968	[2897, 9509, 37809, 44377, 45132]
30315	1	-0.460326	[64928]
3205	1	-0.4609725	[89057]
56437	0	-0.462393	[]
4118	0	-0.4642290	[]
10247	0	-0.4669642	[]
3038	2	-0.468304	[7156, 95616]
57	3	-0.4692568	[126, 339, 9250]
483	1	-0.471289	[80891]
60665	0	-0.471907	[]
63646	0	-0.4728583	[]
39912	1	-0.473267	[128969]
1023	3	-0.4733195	[2178, 20643, 65705]
69608	0	-0.474758	[]
47920	0	-0.475271	[]
123683	0	-0.4762679	[]
33274	1	-0.4817218	[97956]
11586	0	-0.482232	[]
75706	0	-0.48225	[]
128969	0	-0.4825153	[]
3723	1	-0.483432	[34038]
39653	1	-0.48445	[84469]
42217	0	-0.48455	[]

If nothing else, this might suggest to us how to compute more a_n more quickly: Rather than searching previous summands in order, search in the order given by using $\cos(\alpha a_n)$ as the index. Once we find a sum that is unique, we only have to search all smaller sums, again in this order.

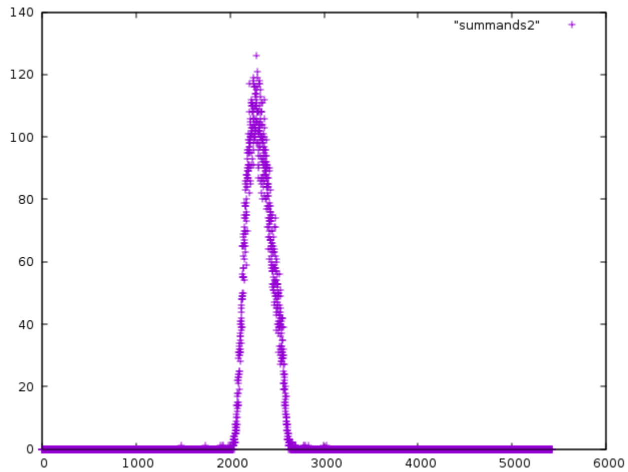
7.1.3 Distribution of complements

In the cases Steinerberger looks at, the resulting non-uniform distributions consist usually of multiple peaks. In the case of the 1,2 Ulam sequence one of these peaks looks a little misshapen, so we might reasonably wonder what each of these peaks actually is.

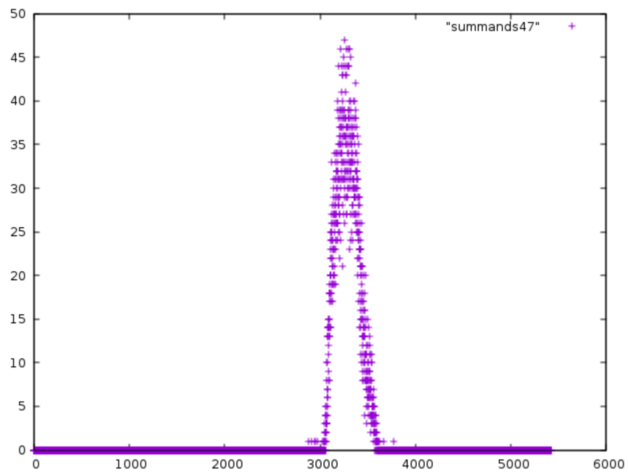
To get a handle on this, we take the Ulam sequence mod 5422, and multiply it by 2219 (5422/2219 being a good rational approximation to $2\pi/\alpha$). Of course, this gives rise to the usual distribution we've come to expect:



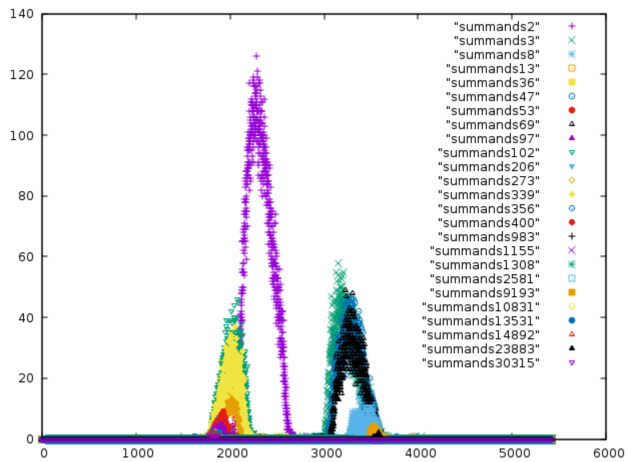
Supposing we look instead only at a_n's for which 2, say is a summand. Then we get this nice picture:



Likewise for 47:



These are relatively clean-looking distributions, by comparison. If we plot these graphs for all of the top 25 most common summands all in one picture, we notice that these seem to be the components of the two observed peaks:



Since each of these seems to be instances of the same distribution with different parameters, we might be interested in computing the parameters of each, starting with the means. We do this with a crunchy bash script

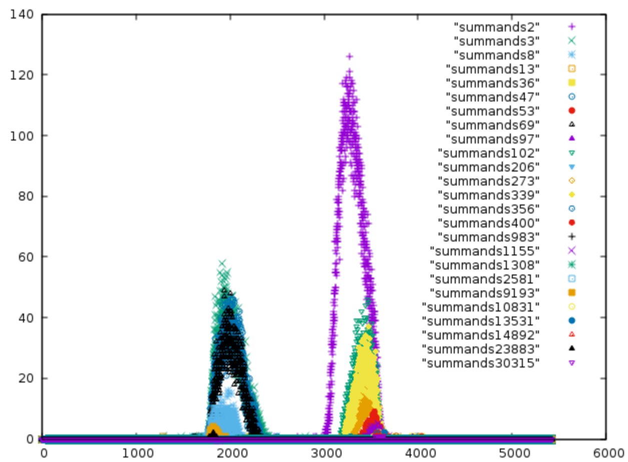
Which outputs the summand a_i, the quantity $2219 * a_i mod 5422$, and the calculated mean of the distribution of $2219 * a_n mod 5422$ for which a_i is a summand, we get:

3	1235	3241.07886089813800657174
47	1275	3288.00715563506261180679
69	1295	3300.94546174403642482430
8	1486	3431.55542168674698795180
2581	1607	3485.87893462469733656174
983	1633	3503.28070175438596491228
206	1666	3518.47580645161290322580
1308	1682	3525.95679012345679012345
9193	1703	3541.35227272727272727272
13	1737	3551.59183673469387755102
23883	1749	3572.53333333333333333333
30315	3653	1818.70000000000000000000
13531	3675	1827.60000000000000000000
14892	3680	1833.36363636363636363636
10831	3685	1845.62962962962962962962
53	3745	1872.41395348837209302325
1155	3761	1883.37745098039215686274
356	3774	1878.85087719298245614035
97	3785	1891.29746835443037974683
400	3814	1912.78585086042065009560
273	3945	1984.79197207678883071553
36	3976	1995.70821114369501466275
339	4005	2013.33377814845704753961
102	4036	2027.29907648591049017286
2	4438	2319.24248003248950859618

Staring at this for a minute, we notice that if we subtract the second column from the third, we seen to get roughly 2000 for the first 11 entries (those on the right end of the distribution). Likewise, those on the left end (rows 12-25) seem to have a similar pattern.

One possible source for this is that the distribution we're taking the mean of in the first row, say, is of $2219 * a_n mod 5422$ where 3 is a summand of a_n in the Ulam sequence. Since 3 is a summand of a_n in the sequence, we might instead look at the other summand of a_n, i.e. $a_n - 3$. This would lead to us not plotting $2219 * a_n mod 5422$, but rather $2219 * (a_n - 3) mod 5422$. We can compute these quickly with another crunchy bash script [appendix A code 2]

And if we plot these, we get:



That's more like it. Running the same mean computation as above on this, we get:

3	1235	2006.0788
47	1275	2013.0071
69	1295	2005.9454
8	1486	1945.5554
2581	1607	1878.8789
983	1633	1870.2807
206	1666	1852.4758
1308	1682	1843.9567
9193	1703	1838.3522
13	1737	1814.5918
23883	1749	1823.5333
30315	3653	3587.7000
13531	3675	3574.6000
14892	3680	3575.3636
10831	3685	3582.6296
53	3745	3549.4139
1155	3761	3544.3774
356	3774	3526.8508
97	3785	3528.2974
400	3814	3520.7858
273	3945	3461.7919
36	3976	3441.7082
339	4005	3430.3337
102	4036	3413.2990
2	4438	3303.2424

We note also that the picture kind of suggests a binomial distribution because of the variance that appears to grow as the mean gets closer to the middle.

Indulging this hypothesis for just a moment, the actual graph (say for $a_i = 2$ specifically), is measuting "for each congruence class, how many complements of 2 in the sequence are in that congruence class?" The idea that this graph is a binomial distribution would be saying that we can perform 5422 independent trials (say one for each congruence class?) with identical success probabilities (one possible such test is pick randomly from the 100000 a_ns in that congruence class and ask whether 2 is ever a complement of that a_n), and the proportion of times 2 shows up as a complement of k is equal to the proportion of times we get exactly k successes in our trials.

7.2 Density

One simple element of structure that we might hope to deduce from all of this is some kind of positive density result for sum-free and Ulam sequences. For example, we note that for around 60% of all Ulam numbers x, then next Ulam number will be $x+2$. For another 30%, the next will be $x+3$. So one approach to studying the density would be to understand the structure of the Ulam numbers well enough to be able to bound how often any given d appears as a difference of consecutive Ulam numbers.

In this document we do not resolve this question either, but do give conjectures in this direction. However, one may note that the only reason we know other Ulam sequences such as $U(2,5)$ have positive density is that we know they are regular. So in trying to get a handle on the distribution of, say, $U(1,2)$, we expect that the actual positive density result will come from an understanding of its distribution, rather than vice-versa.

The first question is to attempt to compute the density of the various sum-free sets and Ulam sequences that we are studying.

7.2.1 Computations

The prescribed method for computing sum-free sets from a decision sequence is already decently fast, provided we keep track of the data appropriately. Recall, the approach is to track three sets: A, the actual sum-free set we are constructing, D, the set $A+A$ of things that are "disqualified" from appearing in A, and E, the set of things that are not disqualified by virtue of being sums but that, according to the decision sequence, we are nevertheless to exclude.

We thus store A as a list and D as a hashset, (and need not keep track of E). So the algorithm is to, for each x starting from 1 until we get bored:

1. Check if $x \in D$ (very fast, as D is a hashset).
2. If not, pop an item off the decision sequence.
3. If 1, append x in A and add $x + A$ to D ($|A|$ steps).

So this algorithm to compute the first N items will be roughly $O(N \cdot |A_N|)$

A similar algorithm can be implemented for the Ulam sequence, in fact: Now, we track the set A as a list, the hashset D of disqualified items (i.e. x for which $r^*_{A+A}(x) > 1$), the sorted list C of candidates (i.e. x bigger than every element of A with $r^*_{A+A}(x) = 1$), and a hashset C' that also contains the candidates. Then the algorithm is to initialise the following:

1. $A = [a, b]$
2. $C = [a + b]$
3. $C' = \{a + b\}$

And proceed thus:

1. Delete any inital elements of C that are smaller than the largest element of A. As we go, delete these elements from C' also.
2. Let x be the first element of C, and append x to A.
3. For each $a \in A$: Compute $x + a$ and, if it is in C', delete it from C' (fast, since C' is a hashset) and from C (where we can find it by bisection, since C is sorted).

There is a more advanced algorithm that leverages the apparent regularity of such sequences as well, implemented in [8], which speed this basic algorithm up by, among other things, noting that if we track which elements of A are outside the middle third mod λ for a λ where there are few such elements, then when we're testing whether any new x within the middle third is actually a sum of smaller elements of A, we only have to look at whether $x - a$ is in A where a is one of the (hopefully few) elements of A outside the middle third. In fact, one can be even more careful about it, which is done in [8].

In any case, the results of our computations give the following estimates for the densities of these sets:

[COMPUTATIONS]

7.2.2 Constructions

Another line of thought is to note that that the Ulam numbers are, in some sense, as greedy as possible in their definition. And while, for example, $\theta(01001)$ is not maximally greedy, it is still greedy $\frac{2}{5}$ of the time. So in the family of Ulam-like sets or sum-free sets, if we have many positive-density examples, it seems unlikely (though not impossible, as we shall see) that these very greedy sets fail to be positive-density.

We first start by noting that positive-density sum-free sets are abundant, as a result of the abundance of sum-free sets $A \subseteq \mathbb{R}/\mathbb{Z}$, coupled with the fact that if $\pi_\lambda : \mathbb{Z}^+ \to \mathbb{R}/\mathbb{Z}$ by $x \mapsto \frac{x}{\lambda} \mod 1$, the inverse image $\pi_\lambda^{-1}(A)$ is sum-free in the integers. For example, the set $A = \{1/2\}$ is sum-free in $\mathbb{R}/\mathbb{Z}$, and $\pi_2(A)$ is the odd positive integers, which is sum-free. Likewise, the A_λ from earlier (for any irrational λ), where recall A_λ was the set of integers that, when reduced (in $\mathbb{R}$) modulo λ, land in the interval $(\frac{\lambda}{3}, \frac{2\lambda}{3})$, are also of this form. So this kind of example gives many sum-free sets, both regular and irregular, that all have positive density.

One might wonder whether we can similarly generate examples of Ulam-like sets of positive density. It turns out that one can do this using the basic idea behind the A_λ construction, but being more careful about it. But first, we will make precise what we mean by "Ulam-like":

Recall an Ulam sequence is an increasing sequence of positive integers that starts with some a and b and that continues by choosing integers according to the requirements of 1-additivity ("every element is uniquely a sum of previous elements") and greediness ("always choose the smallest such element available"). In some ways, it is the greediness that makes Ulam sequences hard to analyse. If we drop this condition, then we get a general class of sequences to which belong the Ulam sequences, but also many others:

Definition 11. For $S \subseteq \mathbb{Z}^+$ a finite set (say of size k), a **1-additive sequence with base** S is an infinite sequence of positive integers a_i such that $a_1 < \ldots < a_k$ are the elements of S, and, for $n > k$, a_n is greater than a_{n-1} and has a unique pair of integers i, j with $0 < i < j < n$ such that $a_n = a_i + a_j$.

We may talk of simply a **1-additive sequence**, by which we will mean a sequence of integers that is a 1-additive sequence with base S for some S.

Example 4.

1. Any Ulam sequence $U(a, b)$ is 1-additive with base a, b.
2. The Fibonacci numbers $1, 2, 3, 5, 8, \ldots$ are a 1-additive sequence with base $1, 2$.
3. The set $\{2, 3, 5, 7, 9, \ldots\} = 2, 2 + 3\mathbb{Z}^+$ is 1-additive with base $2, 3$.
4. More generally, for any $a < b$, the set $a, b, b + a, b + 2a, \ldots$ is 1-additive provided $b \neq 0 \mod a$.

The last example gives plenty of examples of regular 1-additive sequences. (Indeed, $a = 3, b = 4$ provides an example of a set with density apparently higher than that of $U(1, 2)$ despite being less greedy: It goes $3, 4, 7, 10, 14, \ldots$, whereas a greedy algorithm would include 11 as well.)

Nevertheless, all these examples are regular in the conventional "mod-m" sense. So we might wonder what the analogue of A_λ for *irrational* λ would be for 1-additive sets. Indeed, these were our first examples of irregular sum-free sets of positive density, so we might first ask whether there even is such a thing for 1-additive sets. It turns out that there is:

Proposition 7.1. *There exists an irregular 1-additive set of positive density.*

Idea of proof. The basic idea will be the following "ping-pong mod λ" construction: Take an irrational λ. Take a (the "left bat") just above 0 mod λ, and b (the "right bat") just below 0 mod λ, and a c (the "ball") just above $\lambda/3$ mod λ. The game will be to keep the ball in the set $T = (\frac{\lambda}{3}, \frac{2\lambda}{3})$ (the "table") mod λ by adding a to it until it reaches the right side, then adding b to it until it reaches the left side, and repeating indefinitely.

Supposing we are a little careful about our choices of a, b, c, and λ, we should be able to show that this construction satisfies the required properties. □

Proof. More precisely: start with an irrational λ, an a with $a \mod \lambda$ lying in $(0, \frac{\lambda}{12})$, and a $b > a$ with $b \mod \lambda$ in $(\frac{11\lambda}{12}, \lambda)$, and a c with $c \mod \lambda$ in $(\frac{\lambda}{3}, \frac{\lambda}{2})$, and say further that $a \nmid b$ (for reasons that will become apparent later).

Then we will define a sequence of sets A_n which will comprise the sequence A, as follows: Let $c_1 = c$. For $n \geq 1$, let $A_n = \{c_n + ka : k \in \mathbb{Z}^+, c + ka \mod \lambda \in T\}$, and let c_{n+1} be the largest element of this set (allowing the definition of A_{n+1} to make sense).

Then let $S = \{a, b, c\}$ and $A = S \cup \bigcup_{i=1}^{\infty} A_i$.

Now let us check the definition: Our base set is S. Every element a_i has $a_i < a_{i-1} + a + b$, so the set is has positive density. This set is not dense mod λ, but any regular set would have to be (since λ is irrational), so A cannot be regular. It remains to check 1-additivity.

By construction, every element $a_n > c$ is either $a_{n-1} + a$ or $a_{n-1} + b$, so every element not in S is a sum of smaller elements in at least one way. Now suppose a_n is a sum of smaller elements in another way also. Then because a_n is in the middle third, it cannot be a sum of two other elements in the middle third. Thus the only for a_n to be a sum of other elements of A in two different ways is if $a_n - a$ and $a_n - b$ are both in A.

But now, say $a_n = c + ax + by$, for some $x \geq 0, y \geq 0$. We know either $a_n - a \in A$ or $a_n - b \in A$. If $a_n - b \in A$, Then by definition, the next element of A will be a_n, and $a_n - b < a_n - a < a_n$, so $a_n - a$ cannot be in A.

If instead $a_n - a \in A$, then say the place where we started adding as (before a_n) was r steps back ($r \geq 1$ since $a_n - a \in A$), i.e. at $c + a(x - r) + by$. Then the element before this in A would be $c + a(x-r) + b(y-1) < a_n - b$. Thus for $a_n - b$ to be in A, it would have to be an element after $c + a(x-r) + b(y-1)$, i.e. an element of the form $c + a(x-s) + by$. But $c + a(x-s) + by = c + ax + by - b$ implies $as = b$, which contradicts $a \nmid b$, which was our condition on a and b.

Thus $a_n - a$ and $a_n - b$ can never both be in A, making A also 1-additive. $\square$

We note, however, that this construction gives us a 1-additive set with a base of size 3, whereas the Ulam numbers are a 1-additive set with a base of size 2. So we might wonder whether there is a similar (if more complicated) construction of this sort.

The following construction, we believe should work:

Conjecture 7.2. *There is an irregular, positive density 1-additive set with a base of size 2.*

Idea. We will play the same game of ping-pong, but now we are only allowed to use two elements. We will again start with an irrational λ and $a \in \mathbb{Z}^+$ the "left bat" in the range $(\frac{\lambda}{6}, \frac{\lambda}{3})$ mod λ. But now we will start with c being the "ball" inside the "table" $T = (\frac{\lambda}{3}, \frac{2\lambda}{3})$.

Then we will add a to c until it comes out on the right side as some $b \in (\frac{2\lambda}{3}, \frac{5\lambda}{6})$, and that will be our "right bat". Then we will do roughly the same thing as before, except now a and b have larger magnitude mod λ, so they may occasionally hit the ball off the table slightly. This will give us further bats with which to hit the ball–usually bats with even greater magnitude mod λ. The idea, then, will be to always hit the ball with the largest available bat that doesn't send it off the table (when possible), so that there is the most flexibility for the other side to ensure they are also able to hit the ball back onto the table. $\square$

7.2.3 Conjectures

Recall that sum-free sets of positive integers correspond bijectively with binary "decision" sequences. We know also that there are many sum-free sets of positive density. Further, we can easily see that if the 1s have zero density in the decision sequence, then the resulting sum-free set has zero density. So all the positive-density sum-free sets must have decision sequences with positive density.

Question 7.3. *For any decision sequence S with a positive density of 1s, the corresponding sum-free set $A = \theta(S)$ has positive upper density.*

Conjecture 7.4. *For any decision sequence S that is eventually periodic and for which the repeating pattern has at least one 1, then the corresponding sum-free set $A = \theta(S)$ has positive upper density.*

As we have said, it appears that the Ulam sequence has positive (upper) density around 0.07. This, together with the ultimate greediness of the Ulam sequence suggests for us the conjecture:

Conjecture 7.5. *The Ulam sequence $U(1, 2)$ has positive upper density.*

8 Future Directions

8.1 Reducing 1-additive sets to sum-free sets

8.1.1 Via triangle removal

8.2 Technology

8.2.1 Arithmetic regularity

8.2.2 Ultralimits

8.3 Variants of the Ulam problem

8.3.1 Larger seed sets

8.3.2 Sums of more than two previous elements

Circle method gets better with more variables. For this reason, we might find it convenient to

8.3.3 Probabilistic versions

References

[1] P. Cameron. Portrait of a typical sum-free set. *Surveys in Combin.*, 123:13–42, 1987.

[2] J. Cassaigne and S. Finch. A class of 1-additive sequences and quadratic recurrences. *Experimental Mathematics*, 4:49–60, 1995.

[3] V. Lev E. Croot and P. Pach. Progression-free sets in $\mathbb{Z}_4^n$ are exponentially small. preprint.

[4] J. Ellenberg and D. Gijswijt. On large subsets of $\mathbb{F}_q^n$ with no three-term arithmetic progression. preprint.

[5] P. Erdös. Extremal problems in number theory. *Proc. Symp. Pure Math.*, 8:181–189, 1965.

[6] S. Finch. Patterns in 1-additive Sequences. *Experimental Mathematics*, 1(1):57–63, 1992.

[7] P. Gibbs. A conjecture for ulam sequences. preprint.

[8] D. Knuth. Algorithm for computing ulam numbers. Technical report, Stanford, 2016.

[9] T. Luczak. A note on the density of sum-free sets. *J. Combin. Theory Ser. A*, 70:334–336, 1995.

[10] T. Flowers N. Calkin, S. Finch. Difference density and aperiodic sum-free sets. *Integers*, 5(2), 2005.

[11] D. Ross. Experiments on ulam numbers. Technical report, UW–Madison, 2016.

[12] K. Roth. Uber die gleichverteilung von zahlen modulo eins. *Math. Ann.*, 77:313–352, 1916.

[13] K. Roth. On certain sets of integers. *J. London Math. Soc.*, 28:104–109, 1953.

[14] J. Schmerl and E. Spiegel. The Regularity of some 1-additive Sequences. *J. Combin. Theory Ser. A*, 66(1):57–63, 1994.

[15] S. Steinerberger. A Hidden Signal in the Ulam Sequence. preprint, 2016.

[16] T. Tao. *Higher Fourier Analysis*, volume 142 of *Graduate Studies in Mathematics*. American Mathematical Society, 2012.

[17] T. Tao and V. H. Vu. *Additive Combinatorics*. Cambridge University Press, 2006.