The Ulam sequence and related phenomena

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Chapter 1

Introduction

The Ulam sequence, defined by Stanislaw Ulam in [22], is a sequence a_n of positive integers that is given by the folowing recursive definition: It starts with $a_1 = 1$, $a_2 = 2$, and then for $n > 2$, a_n is the integer satisfying:

1. It is expressible as a sum of distinct previous Ulam numbers in exactly one way: There is exactly one pair of $0 < i < j < n$ with $a_i + a_j = a_n$.
2. It is larger than the previous element of the sequence: $a_n > a_{n-1}$.
3. It is the smallest positive integer with the above two properties.

Thus the first few terms can be computed:

$$1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, 62, 69, \ldots$$

In particular, there are two ways a number could fail to be Ulam: Either it has a representation as a sum of two distinct smaller Ulam numbers in more than one way (such as $5 = 4 + 1 = 2 + 3$), or it has no representations as a sum of distinct smaller Ulam numbers at all (such as 23).

One thing that makes the sequence interesting is that it seems historically to have been very difficult to prove anything about it. We know, for example, that it must be infinite: given the first n elements $a_1, \ldots, a_n$, we can always find at least one number that satisfies the first two criteria above, namely $a_{n-1} + a_{n-2}$. Thus there must be a smallest such number, which is therefore the next Ulam number. However we do not know whether this sequence has positive density in any sense.

We also know that if we use the same definition but start with different initial values, we can get sequences that we can analyse very easily indeed: If the "(u, v)-Ulam sequence", denoted $U(u, v)$, is the sequence with $a_1 = u, a_2 = v$, and a_n (for $n > 3$) defined exactly as before, then by a theorem of Schmerl and Speigel [18] we know that the $(2, v)$-Ulam sequence, in the case where v is odd and at least 5, is regular in the following sense:

Definition 1.0.1. *An increasing, infinite sequence $\{a_i\}$ of positive integers is* ***regular*** *if the sequence $\{b_i = a_i - a_{i-1} : i > 1\}$ is eventually periodic.*

Regular sequences are very easy to describe–we could specify them (after some initial segment) by a set of congruence classes modulo some (possibly large) m. In particular, a regular sequence $U(u, v)$ will be far easier to compute than the definition would naively suggest.

There are other initial values that are variously known to or believed to give rise to regular sequences, also. See, for example, [9]. That said, many Ulam-type sequences appear not to be regular, among them $U(1, 2)$ and $U(2, 3)$. So we might wonder if these exhibit some other kind of similar pattern, though perhaps not as rigid as that for, say, $U(2, 5)$.

In looking for hidden regularity, one might take a signal processing approach to the sequence and try, for example, to Fourier transform the indicator function of the sequence and see if the spectrum has any interesting features. In [19], Stefan Steinerberger does exactly that and finds that the spectrum has a large spike at 0 (suggesting some flavour of positive density) as well as another at some $\alpha \in \mathbb{R}/\mathbb{Z}$ (and also, therefore, at its harmonics $n\alpha$ for $n \in \mathbb{Z}, n \neq 0$), and seemingly nowhere else.

More precisely if A is a subset of the natural numbers, and by abuse of notation we also use A to denote the indicator function of the set A, then we can define a "Fourier transform" by:

$$f_N(x) = \frac{1}{N}\sum_{t=1}^{N} A(t)e(tx)$$

where $e(x) = e^{ix}$. In the case of where $A = U(1, 2)$, what is observed numerically is that $f_N(0)$ approaches the density of the sequence $\delta \approx 0.07$ as $N \to \infty$. Looking at the definition of f_N, it is clear that $f_N(x)$ cannot ever be larger than this δ. However, for one particular value of $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ (namely $\alpha = 2.571447\ldots$), we find $f_N(\alpha) \approx 0.8\delta$, even as $N \to \infty$, and for $k \in \mathbb{Z}$, $f_N(k\alpha)$ is also some non-zero value that shrinks with k. For example, for $N = 100000$, we compute this for a few values of k (noting that of course the values for $-k$ are just the conjugates of these). The output is in table 1.1.

As N gets large, it appears that $f_N(\beta) \to 0$ as $N \to \infty$ for all other $\beta \notin \alpha\mathbb{Z}$.

From a signal processing perspective, this might suggest that the set A has some periodicity mod $\frac{2\pi}{\alpha} \approx 2.443442\ldots$. Using 5422/2219 as a rational approximation to this, we can plot the distribution of the first 10^8 elements of A modulo this number. This is done in figure 1.1.

This has some notable features:

- From the value of $f_N(0)$, it looks like the Ulam sequence has small but nonzero density (in fact, around 0.07).
- As noted in [19] it looks like as we increase N that this is converging to an actually continuous distribution.

Table 1.1: Fourier coefficients of $U(1,2)$

k	$\lvert f_N(k\alpha)\rvert$
0	δ
1	0.79854
2	0.32061
3	0.30359
4	0.60190
5	0.60048
6	0.36992
7	0.12798
8	0.14438
9	0.14047

- It looks at a glance like this distribution is supported on the middle third of the interval $[0, \frac{2\pi}{\alpha}]$. This is not literally the case, but in [10] there is a conjecture in this direction.

This phenomenon is actually apparent even after plotting a much smaller number of Ulam numbers–say 10^3, and does not appear to weaken even after other people have computed many more Ulam numbers.

The striking nature and numerical strength of this phenomenon naturally leads to many questions. We will list a larger number of these in section 3, but we give a sampling now:

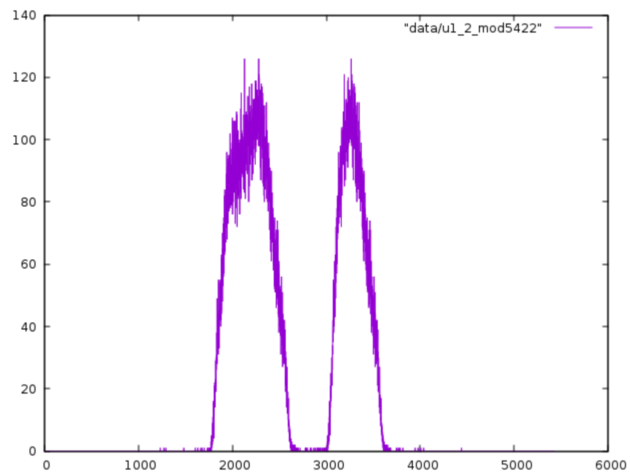
1. What is the α that appears to account for the entire spectrum of the Ulam numbers? Is it irrational? Transcendental?
2. What feature of the Ulam numbers gives rise to this distribution? For example, can we write down a general class of sets with this behaviour?
3. What is it that causes some initial conditions to be regular and not others (if indeed they are not)?
4. Can we distill this phenomenon into a more general notion of regularity that even the irregular-looking sequences do satisfy?

1.0.1 Notation

Before we get to our results, we provide a few pieces of notation that we shall use throughout this document:

- Denote by $[N]$ the set $\{1, \dots, N\}$.
- If $A \subseteq \mathbb{N}$, let A_N denote $A \cap [N]$.
- If $f(N)$ is a function:

Figure 1.1: Distribution of A modulo $\frac{2\pi}{\alpha}$



  - f is $O(g(N))$ if there is a constant $C > 0$ such that $f(N) \leq Cg(N)$ for N sufficiently large.
  - f is $\Omega(g(N))$ if there is a constant $C > 0$ such that $f(N) \geq Cg(N)$ for N sufficiently large.
  - f is $\Theta(g(N))$ if there are constants $C, C' > 0$ such that $Cg(N) \leq f(N) \leq C'g(N)$.
- For $x \in \mathbb{R}$, let $e(x)$ denote e^{ix}.

1.1 Results

In this document, we do not provide a complete explanation for the phenomenon, but we do focus attention on a class of sets that all seem to exhibit simlar behaviour, namely "almost sum-free sets": The idea is that the set A of Ulam numbers has the property that the number of pairs $(x, y) \in A^2$ with $x + y \in A$ is unusually small. For example, in a randomly generated set B of density δ, we might expect that B_N has about δN elements, and hence $\delta^2 N^2$ pairs of elements. Of these pairs, we expect δ to be the proportion of them whose sums are also in A. That is, we expect about $\delta^3 N^2$, or, generally, $\Theta(N^2)$ solutions to $x + y = z$ with all $x, y, z \in A$. If we have fewer of these, say $O(N^{2-\epsilon})$ for some $\epsilon > 0$, then that would suggest that the set cannot be purely random. So we define:

Definition 1.1.1. *Call a set A of positive integers **almost sum-free** if there is a constant $C \geq 0$ and $\epsilon > 0$ such that if $T(A_N) = |\{(x, y) \in A_N^2 : x + y \in A_N\}|$, then $T(A_N) \leq CN^{2-\epsilon}$ for all N.*

Example 1. *Any (a, b)-Ulam sequence is almost sum-free because for each Ulam number $x < N$, the number of representations x has as a sum of other Ulam numbers is at most 3: The sum that qualifies it to be an Ulam number in the first place, that same sum with the order reversed, and possible $\frac{x}{2} + \frac{x}{2}$ if $\frac{x}{2}$ happens to also be in the sequence. Thus such a sequence has $T(A_N) \leq 3N$, and thus is almost sum-free with $C = 3$ and $\epsilon = 1$.*

Example 2. *A set A is called "sum-free" if for every N, $T(A_N) = 0$. These sets are thus also almost sum-free and indeed will provide examples of the same phenomena that are in some ways simpler and more extreme than the Ulam numbers.*

As we discussed above, these sets are defined in a way that guarantees they behave differently from truly random sets. One might wonder, then, what structure this definition actually guarantees for such sets.

We suspect that the nature of this structure is roughly "correlation with a periodic sum-free set". More precisely, that there should be a $\lambda \in \mathbb{R}$ such that if $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the map $x \mapsto \lambda x \mod 1$, then there is an actually sum-free subset $E \subseteq \mathbb{R}/\mathbb{Z}$ such that A and $\pi^{-1}(E)$ are in some sense strongly correlated.

In this document, we come to a first result in this direction: theorem 6.3.1, which gives such a λ and a set $E \subseteq \mathbb{R}/\lambda\mathbb{Z}$ that is "more correlated than expected" with A. We also show (theorem 6.3.2) that regular Ulam sequences are eventually equivalent to lifts of a sum-free subset of some $\mathbb{Z}/m$.

To get such results, we first prove a theorem (theorem 5.1.2) that almost sum-free sets are guaranteed (in a certain sense) a large Fourier coefficient α in their spectrum. We also show in theorem 6.2.2, granting certain statements about the Fourier spectrum that we could not prove, that certain features of this distribution should follow from arguments in the vein of the circle method.

Finally, turning our attention from the general study of almost sum-free sets to the Ulam numbers in particular, we consider the combinatorial structure of the Ulam numbers that results from their definition. For example, we know that every Ulam number a has a unique pair of Ulam numbers $x < y$ with $x + y = a$. We can ask questions about the distribution of such x and y–for example, does every Ulam number show up as a summand of other Ulam numbers equally often? Further, x and y are also Ulam numbers so they can be decomposed likewise until every (u, v)-Ulam number is expressed as a sum of us and vs. The distributions of these also makes for interesting study. And lastly, we might ask what kind of structure would give rise to an "irrationally regular" sequence with the additive structure that the Ulam numbers have. To this end, we propose in proposition 7.2.2 a construction that is not regular in the earlier sense, but that does have some non-trivial irrational periodicity.

Chapter 2

Background

We will start by giving an overview of some known results that should lead us in the right direction. No arguments in this section are original.

2.1 Fourier Transform

The definition of the Fourier transform-like operator that seems experimentally to be the one that applies is the following:

Definition 2.1.1. *For $A \subseteq \mathbb{Z}^+$ a set of positive integers, define for any $x \in \mathbb{R}/2\pi\mathbb{Z}$:*

$$\widehat{A_N}(x) = \frac{1}{N} \sum_{t=0}^{N-1} A(x)e(tx)$$

Also define

$$\widehat{A}(x) = \lim_{N \to \infty} \widehat{A_N}(x)$$

if this limit converges.

So, for example, if A is the Ulam numbers, $\widehat{A}(0)$ would be the so-called natural density δ (which may be around 0.07) if it exists. Then the observation of Steinerberger is that $\widehat{A}(x) = 0$ unless $x \in \alpha\mathbb{Z}$ for the above particular choice of α, and $\widehat{A}(\alpha) \approx 0.8\delta \approx 0.056$, for example.

Of course, it is not clear that this definition ever converges or what properties it might satisfy, so we shall in this document always work only with truncated sequences $A_N \subseteq [N] = \{1, \ldots, N\}$, where we can view $[N]$ as $\mathbb{Z}/N$ and apply a more standard definition:

Definition 2.1.2. *For $f : \mathbb{Z}/N \to \mathbb{C}$, let the* ***discrete Fourier transform*** *of f be a function $\mathscr{F}_N f$ defined for $k \in \mathbb{Z}/N$ by:*

$$(\mathscr{F}_N f)(k) = \frac{1}{N} \sum_{t=0}^{N-1} f(t) e(-\frac{2\pi kt}{N})$$

This definition satisfies many properties, which are standard from Fourier analysis and additive combinatorics [21]:

Proposition 2.1.3. *If $f : \mathbb{Z}/N \to \mathbb{C}$, then:*

- *If in fact f takes values in $\mathbb{R}$, then $(\mathscr{F}_N f)(-x) = \overline{(\mathscr{F}_N f)(x)}$ for all $x \in \mathbb{Z}/N$.*
- *If in fact f is the indicator function of a set $A \subseteq \mathbb{Z}/N$ with $|A| = \delta N$, then $(\mathscr{F}_N f)(0) = \delta$, and $|(\mathscr{F}_N f)(x)| \leq \delta$ for all $x \in \mathbb{Z}/N$.*
- *(Fourier inversion): $(\mathscr{F}_N^{-1} f)(x) = N(\mathscr{F}_N f)(-x)$ for all $x \in \mathbb{Z}/N$.*
- *(Convolution formula): For two functions $f, g : \mathbb{Z}/N \to \mathbb{C}$, we have*

  $$(\mathscr{F}_N(f * g))(x) = N(\mathscr{F}_N f)(x)(\mathscr{F}_N g)(x)$$

  for all $x \in \mathbb{Z}/N$, where $(F * G)(x) = \sum_{t=0}^{N-1} F(t)G(x-t)$ is the convolution of F and G.
- *(Cross-correlation formula): For two functions $f, g : \mathbb{Z}/N \to \mathbb{C}$, we have*

  $$(\mathscr{F}_N(f \star g))(x) = N(\mathscr{F}_N f)(-x)(\mathscr{F}_N g)(x)$$

  for all $x \in \mathbb{Z}/N$, where $(F \star G)(x) = \sum_{t=0}^{N-1} F(t)G(t+x)$ is the cross-correlation of F and G.
- *(Parseval's identity): For $f : \mathbb{Z}/N \to \mathbb{C}$, we have*

  $$\sum_{t=0}^{N-1} |(\mathscr{F}_N f)(t)|^2 = \frac{1}{N} \sum_{t=0}^{N-1} |f(t)|^2$$

2.2 Known regularity results

If we want to prove some kind of generalised regularity statement, it might help to understand existing proofs of regularity (i.e. that consecutive differences are eventually periodic) in cases where this is known to hold. We discuss two such results in this section.

2.2.1 Finch's criterion for regularity

In [9], Finch proves:

Theorem 2.2.1. *If $A = U(a,b)$ is an Ulam sequence containing finitely many even elements, then A is regular.*

The idea of the proof is that if there are finitely many evens, say $e_1 < \ldots < e_s$, then every term n after the last even must be odd. Since it can be written as sum of two earlier terms, and it is odd, one of its summands must be even. And since it can be written in such a sum in a unique way, this is saying that $n - e_i$ is in the sequence for a unique i from 1 to s. This is finitely many things to check, from which regularity should follow.

More precisely:

Proof. If x_n is the number of representations of n as a sum of two elements of A and n is odd, then because an odd number that is a sum of two smaller elements of A must have an even summand and we have only finitely many evens $e_1 < \ldots < e_s$, we can write a finite recurrence:

$$x_n = \sum_{i=1}^{s} 1(x_{n-e_i})$$

where $1(x)$ is 0 unless $x = 1$, in which case $1(x) = 1$. In particular, $0 < x_n \leq s$ for all odd $n > e_s$. Note also that x_n depends on a finite range of earlier x_i's: $x_{n-2}, x_{n-4}, \ldots, x_{n-e_s}$. Call this sequence B_n. Each of the $e_s/2$ values in B_n is between 1 and s, so there are finitely many possible such sequences. Thus, for some N and n, we must have $B_n = B_{n+N}$. But since x_n and x_{n+N} only depend on B_n and B_{n+N} respectively, this means $x_n = x_{n+N}$.

And further, x_{n+2} and x_{n+N+2} only depend on the sequences B_{n+2} and B_{n+N+2}, respectively. But

$$\begin{aligned} B_{n+N+2} &= (x_{n+N}, x_{n+N-2}, \ldots, x_{n+N+2-e_s}) \text{ by definition} \\ &= (x_{n+N}, x_{n-2}, \ldots, x_{n+2-e_s}) \text{ because } B_n = B_{n+N} \\ &= (x_n, x_{n-2}, \ldots, x_{n+2-e_s}) \text{ as noted above} \\ &= B_{n+2} \end{aligned}$$

So in fact $B_{n+N+2} = B_{n+2}$ and we can proceed by induction to show the B_n are periodic with period N. Since the x_n are determined by the B_n, x_n is therefore also periodic with period N. □

Using numerical computations inspired by this criterion, Finch conjectures [9] that the following $U(a,b)$ are regular:

Conjecture 2.2.2 (Finch)**.** *$U(a,b)$ has only finitely many even terms if and only if (a,b) is in the following list:*

- $(5, 6)$.
- $(2, v)$ *for* $v \geq 5$ *and odd.*
- (u, v) *for* $u \geq 4$ *and even.*
- (u, v) *for* $u \geq 7$ *and odd if* v *is even.*

In particular this would imply that all of these sequences are regular, although the above may not be the complete list of regular $U(a, b)$.

2.2.2 Regularity of $U(2, 2n + 3)$

Using the above criterion, Schmerl and Speigel in [18] prove:

Theorem 2.2.3. *The sets* $U(2, v)$ *for* $v > 5$ *and odd are regular.*

Since they use Finch's criterion, this boils down to showing that each of these sets has finitely many evens. Specifically:

Lemma 2.2.4. *The only even elements in the 1-additive set* $U(2, v)$ *(with* $v > 5$ *odd) are 2 and* $2v + 2$.

Proof. The proof goes by supposing that x is the next even element of $U(2, v)$ after $2v+2$ and using an exhaustive knowledge of small elements of the sequence (up to about $5v$) to write $x = a + b$ for smaller $a, b \in U(2, v)$ in more than one way. To do this, we have to understand the small elements of the sequence and the elements just before x.

We leave out the computation of the small elements and simply state the result:

Lemma 2.2.5. *The elements of* $U(2, v)$ *up to* $5v + 10$ *are:*

- 2
- $2v + 2$
- *All odds between* v *and* $3v$*, inclusive.*
- $3v+4i$ *for* $0 < i \leq \frac{v+1}{2}$ *(that is, every other odd from* $3v$ *to* $5v+2$ *inclusive)*
- $5v + 4$
- $5v + 10$

To use these to express our supposed next even element x as a sum of elements of $U(2, v)$ in multiple ways, we also need to understand the elements immediately leading up to x.

Lemma 2.2.6. *There is no gap of length* $2v$ *in the odd numbers in the sequence up to* $x - 2v$*. More precisely, if* r *is any odd number less than* $x - 2v$*, then one of* $r, r + 2, \ldots, r + 2v$ *is in* $U(2, v)$.

Proof. If we take r to be the minimal counterexample to this, then $r-2$ is in $U(2,v)$, else $r-2$ would be a smaller counterexample (note that 1 is manifestly not a counterexample, so $r-2>0$).

But then $r+2v=(r-2)+(2v+2)$ expresses $r+2v$ as a sum of elements of $U(2,v)$, so the only way it can fail to be in $U(2,v)$ is if there is another such expression. But $r+2v$ is odd, so any other expression of it as $a+b$ for $a,b\in U(2,v)$ requires that one of a and b be even. And $r+2v<x$, so the only choice other than $2v+2$ (which we have already used) is 2. So this means $r+2v=2+(r+2v-2)$ is the other such expression. But for this to be such an expression, $r+2v-2$ must be in $U(2,v)$, and we are done. ☐

Corollary 2.2.6.1. *It follows from the proof that for any odd $r<x-2v$ $r\in U(2,v)$ if and only if exactly one of $r+2v+2$ and $r+2v$ is in $U(2,v)$.*

This will allow us to, for example, find several elements of $U(2,v)$ between $x-3v$ and x. We already know that we have a lot of elements between v and $3v$, so this gives us a good chance of expressing x as a sum of elements of $U(2,v)$ in multiple ways.

For example, the second lemma tells us that some odd between $x-3v$ and $x-v$, say $x-v-2i$ (for some $0\leq i\leq v$) is in $U(2,v)$. But we know everything of the form $v+2i$ with $0\leq i\leq v$ is in $U(2,v)$ as well, so

$$x=(x-v-2i)+(v+2i)$$

is the qualifying expression for x as a sum of smaller elements. Since this expression must be unique, we also know that $x-v-2j$ for $0\leq j\leq v$ and $j\neq i$ cannot be in $U(2,v)$.

To get a second such expression (and therefore a contradiction), we will look also at the odd elements from $x-5v$ to $x-3v$, using our knowledge of the odd elements of $U(2,v)$ from $3v$ to $5v$.

After some casework, this will end up giving a second qualifying expression for x, thereby disqualifying it. We refer to [18] for the details.

☐

2.3 Sum-free sets

The set of Ulam numbers A has the property that for each $a\in A$, there is exactly one solution to $x+y=a$ with $x<y$ in A. The condition that $x<y$ is a little hard to capture using standard techniques, but, for example, this entails that the number of solutions to $x+y=a$ with $x,y\in A$ is at most 3 (namely, the unique solution $x+y=a$ above, then also $y+x=a$, and then at most one other solution of the form $z+z=a$, since the definition of the Ulam numbers does not consider this. For example, 4 is Ulam, and its unique representation is $1+3=4$, but it also happens that $2+2=4$ and 2 is also Ulam).

In particular, this implies that if A_N is again the set of Ulam numbers up to N, then A_N has at most $3|A_N|$ solutions to $x+y=z$ with $x,y,z\in A_N$.

In the interest of understanding what precisely is happening with the Ulam numbers, then, we might turn our attention to the more extreme situation of sets with no solutions to this equation at all: So-called “sum-free sets”.

Definition 2.3.1. *A subset A of an abelian group is **sum-free** if the equation $x + y = z$ has no solutions with $x, y, z \in A$.*

Example 3.

1. *The odd positive integers are sum-free.*
2. *More generally, if $A \subset \mathbb{Z}/m$ is sum-free, then the set of integers x that reduce to an element of A modulo m is also sum-free.*
3. *Even more generally, for any homomorphism $\pi : \mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, if A is a sum-free subset of $\mathbb{R}/\mathbb{Z}$, then $\pi^{-1}(A)$ is a sum-free set of integers.*
4. *Any subset of a sum-free set is sum-free also.*

When we think about generalising the particular notion of “regularity” above for the purpose of the Ulam sequence or for sum-free sets, the basic thought is that a set should be “regular” in a more general sense if it has some non-trivial correlation with a set of the form in example 3.

2.3.1 Decision sequences

It turns out there is a construction that bijects sum-free sets of positive integers with infinite binary sequences. In words, the construction is simple: Take the positive integers in turn starting with 1. For each integer, flip a coin. If the coin shows heads, include that integer in the set and erase all integers that are sums of pairs of elements in the set thus far (as these cannot be in the set if the set is to be sum-free). If tails, do not include that integer in the set. Then move on to the next integer that has not already been included, excluded by a “tails” result, or disqualified by being erased.

More formally:

Definition 2.3.2. *Define the function $\theta : \{0,1\}^{\mathbb{N}} \to \{f : \mathbb{N} \to \{0,1\}\}$ from binary sequences to sum-free sets of natural numbers (or, in this case, their indicator functions) as follows: If $s \in \{0,1\}$ is a binary sequence, then using s, we will actually define three disjoint sets that partition the natural numbers: The target set A, the excluded set E, and the disqualified set D. For each $n \in \mathbb{N}$, iteratively select a set for n as follows:*

$$\begin{cases} n \in A + A & \implies n \in D \\ n \notin A + A \text{ and } s_k = 1 & \implies n \in A \\ n \notin A + A \text{ and } s_k = 0 & \implies n \in E \end{cases}$$

where, at each stage, $k = |A| + |E| + 1$ is the index of the first element of s that we have yet to consult.

*If S is a sequence and A is a sum-free set with $\theta(S) = A$, then S is called the **decision sequence** for A.*

Example 4. *For example, let us compute $\theta(111111111...)$: We start with 1 and flip a coin and get heads, so we include 1 in the set A. This automatically disqualifies 2 as $2 = 1 + 1$. The next possible candidate is 3, so we flip another coin and get heads, and so we include 3. This automatically disqualifies 4 ($4 = 1 + 3$) and 6 ($6 = 3 + 3$). Continuing in this way, it is clear we will never get a chance to include an even number and will always include the odd numbers, so in the end, the result is the set of odd positive integers.*

It is also possible to reverse this construction. In words: Say we start with A a sum-free set. We again walk through the positive integers starting at 1. For each n there are three possibilities: Either $n \in A$, $n \in A + A$, or neither. If $n \in A$, then it got there by a coin landing heads, so we write down a 1. If $n \in A + A$, then n was disqualified from being in A not by a coin flip, but by being a sum of elements of A, so we write down nothing. If $n \notin A$ and also $n \notin A + A$ then n could have been included in A, but was excluded simply because of a coin flip, so we write down a 0.

Formally, we write down the sequence $s = \theta^{-1}(A)$ by writing down first the string s' whose nth character is:

$$s'_n = \begin{cases} \text{`A'} & \text{if } n \in A \\ \text{`D'} & \text{if } n \in A + A \\ \text{`E'} & \text{if } n \notin A \cup (A + A) \end{cases}$$

(So all the 'A's are elements of A, all the 'D's are automatically excluded from A by being sums of prior elements of A, and all the 'E's are things that are excluded from A despite the fact that their inclusion would not violate the sum-free property.) Then the decision sequence s of A is got by starting with s' and deleting all Ds, replacing all As with 1, and replacing all Es with 0.

There are many questions about this construction. For example, it is known that if a sum-free set A is regular (as defined above–i.e. its sequence of successive differences is ultimately periodic), then its decision sequence $\theta^{-1}(A)$ must also be ultimately periodic [3]. Conversely, many periodic decision sequences give rise to provably regular sum-free sets. For example, say for a finite string s that $\overline{s}$ denotes the infinite string got by repeating s forever. Then all s of length five are known to have $\theta(\overline{s})$ regular except $s = 01001$, $s = 01010$, and $s = 10010$. For example, $\theta(\overline{01001})$ (which we will somewhat abusively refer to as $\theta(01001)$) starts $\{2, 6, 9, 14, 19, 26, 29, 36, 39, 47, 54, 64, 69, 79, 84, 91, 96, \ldots\}$ and has been computed extensively with no period being identified to date. Beyond the failure of brute force attempts to find a period for it, there is other computational evidence in [2] that this sequence may not be periodic. Nevertheless, there is no known example of an ultimately periodic decision sequence for which we can actually prove its corresponding sum-free set is non-regular.

2.3.2 Density and regularity

In the world of sum-free sets, the odd integers provide an example of a very large sum-free set. In fact, a theorem of Luczak that we will discuss in this section shows that they are the largest example: Any sum-free set that has an even number has density bounded by $\frac{2}{5}$. Before we can discuss this, however, we will dwell for a brief moment on what we mean by "density":

Definition 2.3.3. *A subset $A \subset \mathbb{Z}^+$ has* ***density*** *δ if $\lim_{N\to\infty} \frac{|A_N|}{N}$ exists and is equal to δ.*

Since this may not always exist, we might work with another quantity that will always exist and that, in cases when the density does not exist, provides what should be thought of as at least an upper bound:

Definition 2.3.4. *A subset $A \subset \mathbb{Z}^+$ has* ***upper density*** *δ if $\limsup_{N\to\infty} \frac{|A_N|}{N} = \delta$.*

As we have noted, then, the maximal upper density a sum-free set can have is $\frac{1}{2}$, which is realised by the example of the odd positive integers. Luczak has given a sort of converse to this example, proving in [14] the following:

Theorem 2.3.5 (Luczak)**.** *If A is a sum-free set of positive integers and there is at least one even integer in A, then the upper density of A is bounded above by $\frac{2}{5}$.*

The proof is short, but a little delicate, and we shall recall a version of it in this section.

The basic idea of the proof is to find disjoint subsets of $[N]$ that are the same size as A_N, or of a size related to A_N. For example, if $a \in A_N$ is any element, then because A is sum-free, A_N and $A_N + a$ are disjoint in $[N + a]$, but have the same size, and thus $2|A_N| \leq N + a$, i.e. $|A_N|/N \leq \frac{1}{2} + \frac{a}{2N}$. Taking the limit as $N \to \infty$, we again deduce our earlier statement about A having density bounded by $\frac{1}{2}$.

Proof. Note first that if A is all even elements, then $\frac{1}{2}A$ is also sum-free, and therefore with density $\leq \frac{1}{2}$, and so A has density $\leq \frac{1}{4}$ and the result is automatic, so without loss we may assume A has at least one odd element in addition to its at least one even element.

With this in mind, the proof breaks up into two cases: Where A contains consecutive elements and where it does not.

Case 1: A has no consecutive elements In the case where A has an even element but no two consecutive elements, let t be the minimal odd positive element of $A - A$ which does exist (using the fact that A has both odd and even elements), and is not 1 (since there are no consecutive elements). Also fix $x, y \in A$ with $t = x - y$.

This means that if $a \in A$, then $a+t-2$ cannot be in A (else $t-2$ would be a smaller odd positive difference than the minimal odd difference t). Put another way, if a and $a + 2$ are both in A, then $a + t$ cannot be in A. Put another way, if B is the set of $a \in A$ with $a + 2$ also in A, then $B + t$ and A are disjoint. Of course, we already know that finding two disjoint subsets of size even as large as $|A|$ is already easy, however this lets us in fact find three: Since $t = x - y$, this means $B+x-y$ and A are disjoint, meaning $B+x$ and $A+y$ are disjoint. But both of these are contained in $A+A$, so they are both also disjoint from A. Thus we have A, $A+y$, and $B+x$ all disjoint. If we truncate A to A_N, then A_N, A_N+y, and B_N+x are all disjoint subsets of $[N+x]$, and so

$$2|A_N| + |B_N| \leq N + x$$

.

So if we can relate $|B|$ to $|A|$ (for the moment using the shorthand $B = B_N$, $A = A_N$), then we are done.

But by the definition of B, we have two cases for an element of A:

- $a \in B$, in which case $a+1$ is not in A.
- $a \in A \setminus B$, in which case we know $a+1$ is not in A (since A has no consecutive elements) and $a+2$ is not in A, (since otherwise a would be in B).

So we have the five sets: $B, B+1, A \setminus B, (A \setminus B)+1, (A \setminus B)+2$, and these are all pairwise disjoint in $[N+2]$. (The only one that might not be clear is $(B+1) \cap ((A \setminus B)+2)$, but if $a \in A \setminus B$ and $b \in B$ with $a+2 = b+1$, then $a+1 = b$, giving two consecutive elements of A which does not happen.)

Thus $2|B_N| + 3(|A_N| - |B_N|) \leq N+2$, i.e.

$$|B_N| \geq 3|A_N| - N - 2$$

Now we have a relationship between $|B|$ and $|A|$, so we can pair this with our earlier inequality relating the two of them to N and find:

$$2|A_N| + 3|A_N| - N - 2 \leq N + x$$

or

$$\frac{|A_N|}{N} \leq \frac{2}{5} + o(1)$$

as we wanted.

Case 2: A **has consecutive elements:** In the case where A has d consecutive elements $a, a+1, \ldots, a+d-1$, say, the argument is similar in flavour to the above, but the technical details are all slightly different. We will first need a t to serve the role of our t in case 1. But now, the minimal odd difference is simply 1. So we do something slightly different: This time, we let t be any positive element of $A-A$ for which $t+1, \ldots, t+d$ are all not in $A-A$.

Lemma 2.3.6. *Such t does exist.*

Proof. Since $a, a+1 \in A$, we know $1 \in A-A$. Then let t be the maximum of $1, \ldots, a-1$ that is in $A-A$, so nothing from t to $a-1$ is in $A-A$ (by definition), and nothing from a to $a+d-1$ is in $A-A$ either (since these are all in A), so at least d elements (and possibly more) immediately after t are not in $A-A$). □

Again, write $t = x - y$ for some fixed $x, y \in A$.

We proceed broadly as before on the two-step plan:

1. Find a set B of elements that gives rise to many disjoint subsets of $[N]$ and deduce a bound relating $|A_N|$ and $|B_N|$ to N.

2. Upper-bound $|B_N|$ in terms of $|A_N|$ and N, and plug this into the previous bound to get a bound on $|A_N|$ in terms of N.

Step 1: Let B be the set of elements b for which $b+1, \ldots, b+d-1$ are all not in A. Then certainly the sets $A, B+1, \ldots, B+d-1$, are all disjoint. In fact, we can get one more than this: We can shift all these sets by a and they are still disjoint: $A+a, B+a+j$ ($j = 1, \ldots, d-1$). But now since the $a+j$ are all in A, these sets are all themselves disjoint from A (since they are all subsets of $A+A$). Thus, again truncating at N, we have two sets of size $|A_N|$ and $d-1$ sets of size $|B_N|$ all disjoint and inside $[N+a+d-1]$. Thus:

$$2|A_N| + (d-1)|B_N| \leq N + a + d - 1$$

Step 2: So again, we need control over the size of $|B_N|$ in terms of $|A_N|$ and we will be done. But this time, we note that if $z \in A$, it is possible that $z+t$ could be in A, but that then because of the definition of t, none of $z+t+1, \ldots, z+t+(d-1)$ can be in A (lest one of $t+1, \ldots, t+(d-1)$ lie in $A-A$). Thus elements of $A+t$ that lie in A in fact must lie in B. Put another way, $A+t$ and $A \setminus B$ are disjoint. Again, this is only two sets, but we can use the same trick as before to make it three: Since $t = x - y$, we can equally say $A+x$ and $(A \setminus B)+y$ are disjoint, at which point these are also disjoint from A (again, being subsets of $A+A$). So we have three disjoint subsets $A+x$, $A \setminus B + y$, and A of $[N+x]$, with sizes $|A_N|$, $|A_N|$, and $|A_N| - |B_N|$, respectively. This gives $|A_N| + |A_N| + (|A_N| - |B_N|) \leq N + x$ or:

$$|B_N| \geq 3|A_N| - N - x$$

Dropping this into the first inequality and rearranging, we get:

$$2|A_N| + (d-1)(3|A_N| - N - x) \leq N + a + d - 1$$

which simplifies to:

$$\frac{|A_N|}{N} \leq \frac{d}{3d-1} + o(1)$$

Since $d \geq 2$ (as we are assuming we have at least two consecutive elements), this is again bounded by $\frac{2}{5}$ in the limit, so the claimed bound follows. $\square$

2.3.3 Aperiodic sum-free sets

A construction of Erdos in [8] supplies an example of a sum-free set with density $\frac{1}{3}$ that is provably not regular, namely: Take $\alpha \in \mathbb{R}$ irrational, and let A_α be the set of integers n such that $n \mod \alpha$ lies in $\left(\frac{\alpha}{3}, \frac{2\alpha}{3}\right)$. A_α is clearly sum-free, since it is sum-free modulo α, but for irrational α, A_α is also not periodic. That is, for every modulus m and every residue class k, there is an element of A_α congruent to k mod m.

Indeed, equidistribution results for irrational numbers tell us that there the integers are equidistributed modulo any irrational. For example, there is at least one n that is equivalent to an integer in the interval $\left(\frac{\alpha}{3m} - k, \frac{2\alpha}{3m} - k\right)$ modulo the irrational number $\frac{\alpha}{m}$. Then it is clear that $mn + k$ will reduce to an element of $\left(\frac{\alpha}{3}, \frac{2\alpha}{3}\right)$ mod α, meaning that $mn + k \in A_\alpha$ as desired.

2.4 Roth's theorem

Roth's theorem is about the number of 3-term arithmetic progressions x, y, z in a set $A \subseteq \mathbb{Z}^+$. Specifically:

Theorem 2.4.1 (Roth's theorem)**.** *Let $A \subseteq Z^+$ be a set of positive integers with positive upper density. Then A contains infinitely many arithmetic progressions $a, a+d, a+2d$ of length 3.*

Equivalently, such an A always has at least one solution to $x + z = 2y$ (whereupon x, y, z is an arithmetic progression of length 3). A sum-free set A instead has are no solutions to $x + z = y$ (swapping around variable names to highlight the similarity), so if we have a sum-free set that we believe has positive density, we might wonder what the proof of Roth's theorem has to say about it. (After all, in the case of the slightly different equation $x + z = 2y$, it says that the set A cannot exist.)

As it turns out, many new techniques in additive combinatorics cut their teeth on Roth's theorem, and so there are many proofs, from those that use probabilistic techniques to ergodic theory. We will discuss one in particular: The density increment proof. We will not give the complete proof, but will simply work at a high level through the part that is relevant to our study and will outline the rest.

2.4.1 Density increment proof

Proofs of Roth's theorem often work with a finitary version of the statement, which we make now:

Theorem 2.4.2 (Roth's theorem)**.** *For every $\delta > 0$, there is an $N_0 > 0$ such that for every $N > N_0$, every $A \subseteq [N]$ with $|A| > \delta N$ contains a solution to $x + z = 2y$.*

One strategy of proof goes via Fourier analysis, saying that if A has no large Fourier coefficients, then A is guaranteed to behave "pseudorandomly" in some sense, and computes that such sets must automatically have many length-3 arithmetic progressions, and we are done already.

If, on the other hand, A does have some large Fourier coefficient, then one can find a long arithmetic progression that has large intersection with A, and on which A in fact has higher density than it had originally. We can repeat this step (the "density increment") as often as needed until either our intersected A has no large Fourier coefficient (in which case we are done as before) or else A's density in the arithmetic progression increases to 1. And if we are careful about it, we can ensure that at least 3 elements will still remain by the time we get to this point.

Proof of Roth's theorem via density increment. Rather than working on the set $[N]$, we shall work with the group $\mathbb{Z}/N$, noting that if A only contains elements smaller than $N/2$, then a solution to $x + z = 2y$ in $\mathbb{Z}/N$ is an honest solution to $x + z = 2y$ in A viewed as a subset of $\mathbb{Z}$.

If A is a set of density δ in $\mathbb{Z}/N$, with all elements of A less than $N/2$, then the number S of solutions to $x + z = 2y$ is counted by:

$$S = \sum_{x,y,z=0}^{N-1} A(x)A(y)A(z)\delta_{x+y-z}$$

Where $\delta_x = 1$ if $x = 0$, else $\delta_x = 0$. Then, writing

$$\delta_x = \frac{1}{N}\sum_{t=0}^{N-1} e(\frac{2\pi tx}{N}) = \widehat{1}$$

we can substitute this into our expression for S and rearrange, where for the sake of brevity (and only for the duration of this proof), we depart from our previous notation and denote by $\widehat{A}$ the discrete Fourier transform

$$\widehat{A}(x) = \frac{1}{N}\sum_{t=0}^{N-1} e(\frac{2\pi tx}{N})$$

On doing so, we find:

$$S = N^2 \sum_{t=0}^{N-1} \widehat{A}(t)\widehat{A}(t)\widehat{A}(-2t)$$

The idea will be to pull out the $t = 0$ term of this sum, note that this is large and that if all other terms are small, than it dominates and guarantees $S > 0$. Precisely:

$$\begin{aligned}
S &= N^2 \sum_{t=0}^{N-1} \widehat{A}(t)\widehat{A}(t)\widehat{A}(-2t) \\
S &= N^2\delta^3 + N^2 \sum_{t=1}^{N-1} \widehat{A}(t)\widehat{A}(t)\widehat{A}(-2t) \\
&= \delta^3 N^2 + N^2 \sum_{t=0}^{N-1} \widehat{A}(t)^2 \widehat{A}(-2t) \\
&\geq \delta^3 N^2 - \sup_t |\widehat{A}(-2t)| N^2 \sum_{t=0}^{N-1} |\widehat{A}(t)|^2 \\
&= \delta^3 N^2 - \sup_t |\widehat{A}(-2t)| N \sum_{t=0}^{N-1} |A(t)|^2 \\
&= \delta^3 N^2 - \sup_t |\widehat{A}(-2t)| N|A| \\
&= \delta^3 N^2 - \sup_k |\widehat{A}(k)| \delta N^2
\end{aligned}$$

So if there is no large Fourier coefficient–that is, every Fourier coefficient is $\leq \epsilon N$, then

$$S \geq (\delta^3 - \delta\epsilon)N^2$$

In particular, if $\epsilon < \delta^2$, then $S > 0$, at which point there is at least one solution, as desired.

If, on the other hand, there is a k such that $|\widehat{A}(k)| \geq \delta^2 N$, then this argument does not guarantee a solution. However, in that case, let $P = d[1, L]$ be the arithmetic progression of length L and difference d $\{d, 2d, \ldots, Ld\}$ (d to be chosen later). We want an arithmetic progression in which A has higher density than it has in $\mathbb{Z}/N$ at large. In other words, we want to find an a that makes $Q(a) = |A \cap (P + a)| = (A \star P)(a)$ large. But this we can analyse using Fourier analysis:

$$\widehat{Q}(s) = \widehat{A}(s)\overline{\widehat{P}(s)}$$

Further, we know that for all $s \neq 0$, $\sum_a Q(a) \geq |\widehat{Q}(s)|$ (looking at the definition of the Fourier transform and using the triangle inequality). So in particular, for $s = k$ (the large Fourier coefficient):

$$\begin{aligned}
\sum_a Q(a) &\geq |\widehat{Q}(k)| \\
&= |\widehat{A}(k)||\widehat{P}(k)| \\
&\geq \epsilon N |\widehat{P}(k)|
\end{aligned}$$

Thus for some a, $Q(a)/N \geq \delta^2|\widehat{P}(k)|$. We can select d and L such that $|\widehat{P}(k)| \geq L/2$, so for some A, $Q(a)/N \geq \epsilon L/2$. In particular, A intersected with an arithmetic progression of length L has density $\delta + \epsilon/2$, meaning we have increased the density, whereupon we can repeat the argument.

The details (such as actually selecting the correct d and L, as well as properly transitioning from $\mathbb{Z}/N$ back to $\mathbb{Z}$), are covered in many places, for example [17]. $\square$

2.5 Quantitative bounds in finite fields

There have been several recent developments in a finite field setting on analogous problems (specifically, the work of Croot, Lev, and Pach [5] on length-3 arithmetic progression-free sets in $\mathbb{F}_4^n$ and subsequent work by others [7] pushing it to $\mathbb{F}_3^n$).

We will recall the method used here by outlining the proof in [7], in view of the possibility of later asking about Ulam-like sequences in the same context.

Theorem 2.5.1 (Ellenberg-Gijswijt)**.** *Let α, β, γ be elements of $\mathbb{F}_q$ such that $\alpha + \beta + \gamma = 0$ and $\gamma \neq 0$. Let A be a subset of $\mathbb{F}_q^n$ such that the equation $\alpha a_1 + \beta a_2 + \gamma a_3 = 0$ has no solutions $(a_1, a_2, a_3) \in A^3$ apart from $a_1 = a_2 = a_3$. Then $|A| = o(2.756^n)$.*

Proof. Let S^d be the space of all polynomial functions on $\mathbb{F}_q^n$ of degree d (that is, polynomials of total degree d where each of the n variables shows up with degree less than q). Let m_d be the dimension of this space, and let V_d be the subspace of polynomial functions vanishing on the complement of $2A$ (this is more or less a trick). Then

$$\dim(V_d) >= m_d - (q^n - |A|)$$

(since the requirement to vanish on the complement of $2A$ is at most $q^n - |A|$ conditions).

It turns out that we can actually get a polynomial P_d in V_d with support of size exactly $\dim(V_d)$, and so this polynomial has:

$$|\mathrm{supp}(P_d)| >= m_d - q^n + |A|$$

Now for the last bit: If we have a degree-d polynomial P vanishing on the complement of $2A$, then we can form the $|A|$-by-$|A|$ matrix M whose i, j entry is $P(a_i + a_j)$ where a_i are the elements of A. First of all, because for i and j different, $a_i + a_j$ is never in $2A$, the off-diagonal terms all vanish, whereas because the diagonal terms are $P(2a_i)$, they may or may not vanish.

We can brutally expand this polynomial into a sum of monomials:

$$P(a_i + a_j) = \sum_{\text{monomials } m, m' \text{ of degree } d \text{ or less}} c_{m,m'} m(a_i) m'(a_j)$$

Further, in each term at least one of m and m' has degree at most $d/2$, so we can sum over

$$P(a_i + a_j) = \sum_{\text{monomials } m \text{ of degree } d/2 \text{ or less}} c_m m(a_i) F_m(a_j) + c'_m m(a_j) G_m(a_i)$$

So M is a linear combiantion of $2m_{d/2}$ matrices $(m(a_i)F_m(a_j))$ each of which, as the exterior product of two vectors, has rank 1. Thus the rank of M is at most $2m_{d/2}$. And since M is diagonal, this means that in fact on $2A$, P has only $2m_{d/2}$ non-zero points. So the size of the support of P is bounded above by $2m_{d/2}$. Since the size of the support of P_d was already bounded below by $m_d - q^n + |A|$ we can apply this argument to P_d and conclude that

$$2m_{d/2} \geq m_d - q^n + |A|$$

i.e.

$$|A| \leq 2m_{d/2} - m_d + q^n$$

Choosing a particular value of d and bounding these quantities is all that remains. In [7] they take $d = 2(q-1)n/3$ and use Cramer's theorem to bound m_d and related quantities to get the claimed exponential bound on $|A|$. We refer to the paper for details. $\square$

Chapter 3

Questions

Bearing in mind this landscape of ideas, theorems, and techniques, we now raise some questions in the particular context of the Ulam numbers and related sequences, and the various phenomena that we observe around them.

3.1 Ulam sequences

Recall the definition of an Ulam sequence:

Definition 3.1.1. *The* ***Ulam sequence*** *starting with positive integers* a, b *is denoted* $U(a,b)$ *and is the sequence with* $a_1 = a$, $a_2 = b$, *and, for* $n > 2$, $a_n > a_{n-1}$ *is the integer satisfying:*

1. *1-additivity: There is exactly one pair of* $0 < i < j < n$ *with* $a_i + a_j = a_n$.
2. *Greediness:* a_n *is the smallest positive integer with the above two properties.*

One of the first questions that was asked by Ulam himself about the Ulam sequence $U(1,2)$ was:

Question 3.1.2. *Does* $U(1,2)$ *have positive (uppper) density?*

For all examples of Ulam-like sequences where we know the answer to this question, the way we do so is by first establishing a regularity result, at which point positive density is immediate. Given the known regularity results concerning Ulam sequences, a very basic question about the Ulam numbers then would be:

Question 3.1.3. *Can we prove the Ulam numbers are not regular (in the sense of definition 1.0.1)?*

Supposing we could do so, we might then ask:

Question 3.1.4. *Is there a notion of "regularity" that generalises 1.0.1 and that captures the behaviour observed in [19] and that we can prove?*

Supposing that in some way Steinerberger's constant α will come into this definition, we might wonder about what it is specifically:

Question 3.1.5. *Is α irrational? Algebraic? What about $\frac{2\pi}{\alpha}$?*

We will see later that mod $\frac{2\pi}{\alpha}$ Ulam sequences appear to be controlled by a few $x \in A = U(a,b)$ that form the summands of further elements. That is, there is a small set $S \subseteq A$ such that any $z \in A$ is in fact in $S + A$. These $x \in S$ also end up being close to 0 mod $\frac{2\pi}{\alpha}$, so we might wonder whether these will help to compute α:

Question 3.1.6. *Is there a way of using knowledge of the set S for a given A to compute the corresponding α for that A?*

Beyond just asking about α, we can ask about other features of the spectrum:

Question 3.1.7. *Are there other nonzero Fourier coefficients not in $\alpha\mathbb{Z}$?*

Question 3.1.8. *How quickly does $\widehat{A}(k\alpha)$ decay with k?*

Moving on from $U(1,2)$, we can also ask about similar sequences:

Question 3.1.9. *How does α behave for other non-regular-looking Ulam-like sequences? For example, supposing $\alpha_n \in (0,\pi]$ is the maximal Fourier coefficient associated with $U(1,n)$ (supposing there even is a unique such Fourier coefficient), what is the behaviour of α_n as n grows?*

Separately, in light of the triangle removal lemma, there is a set of questions we might ask regarding the additive structure of the Ulam sequence:

Question 3.1.10. *What is the minimal subset $X \subseteq A$ that we might remove so that A becomes sum-free? (For example, the set such that X_N is minimal among all such possible X for each N.)*

A very similar question that gets more precisely at such a set X in terms of the actual definition of A: We know that each element $a \in A$ is written uniquely as $x + y$ for $x < y$ elements of A. Certainly the set $S = \{x \in A : x + y \in A \text{ for some } y > x \text{ in } A\}$ of "small summands" is a candidate for such an X in the previous question, but S itself might be of interest even if it ends up not being minimal (though one might reasonably expect that it would be).

Question 3.1.11. *Can we characterise the elements of S? What is the growth rate of $|S_N|$ as N grows?*

Finally, we can ask about the distribution that Steinerberger observes for the Ulam sequence modulo $\frac{2\pi}{\alpha}$, starting with a question from [19]:

Question 3.1.12. *Does the distribution of A_N mod $\frac{2\pi}{\alpha}$ converge to a continuous distribution? More precisely, let $\lambda = \frac{2\pi}{\alpha}$. Then if for each $M > 0$ we cut up the interval $[0, \lambda]$ into M equal intervals and define a step function $f_{M,N}(x)$ to be the proportion of Ulam numbers up to N that lie in the same one of the M intervals as x, then as M and N go to infinity, does $f_{M,N}$ converge to a continuous function on $\mathbb{R}/\lambda\mathbb{Z}$?*

We can ask a lot more than just about the distribution's continuity, however. The distributions particularly for other Ulam-like sequences such as $U(2,3)$ look like they have some further internal structure as a sum of perhaps smaller, more normal-looking peaks. So we can ask somewhat broadly about this also:

Question 3.1.13. *What gives this distribution its particular shape? For example, what about the shape of the distribution can be deduced from the knowledge of the spectrum alone?*

3.2 Sum-free sets

We start noting the same dichotomy that existed with Ulam sequences seems present for sum-free sets as well: Many sum-free sets with easy-to-describe decision sequences are provably regular, but others we do not know whether or not they are regular. For instance, we might start with the aforementioned "smallest" three examples:

Question 3.2.1. *Are any of the sets $\theta(01001), \theta(01010)$, or $\theta(10010)$ regular?*

Supposing once again (as is suggested in [2]) that the answer is no, we might try to ask similar questions with the these sets:

Question 3.2.2. *What does the spectrum of these sets look like? Is there a mapping to $\mathbb{R}/\mathbb{Z}$ under which the indicator functions of these sets approach continuous-looking distributions?*

Question 3.2.3. *Does whatever notion of regularity applies to the Ulam sequence apply here as well?*

Question 3.2.4. *What is the density of these sets? Is there a statement relating the density of 1s in the decision sequence with the density of the resulting sum-free set?*

In the sum-free case, the work of Luczak outlined earlier gives some relationship between the regularity of a sum-free set and its density (saying in his case that a sum-free set that contained an even number (i.e. a sum-free set whose image mod 2 was all of $\mathbb{Z}/2$) had density bounded by $\frac{2}{5}$–a meaningful improvement from the automatic bound of $\frac{1}{2}$ on the density of an arbitrary sum-free set.

On the other hand, the construction of Erdos tells us that there exist sum-free sets with density $\frac{1}{3}$ whose image mod m, for all m, is everything. We might ask if there is any condition we can prove in the gap between $\frac{1}{3}$ and $\frac{2}{5}$.

Question 3.2.5. *If m is a positive integer, what is the maximal density d_m of a sum-free set that hits every congruence class modulo m?*

For example,

- $d_1 = \frac{1}{2}$ (upper bound is by the argument $(A + a) \cap A = \emptyset$ for any $a \in A$, and lower bound comes from the example of the odd numbers).
- $d_2 = \frac{2}{5}$ (upper bound is by [14], and the lower bound is established by the example from the same paper of integers congruent to 2 or 3 mod 5).
- $d_3 = \frac{1}{2}$ using the same argument as for d_1.
- $d_4 = \frac{2}{5}$ by the same argument as for d_2, since $(2 + 5\mathbb{Z}) \cup (3 + 5\mathbb{Z})$ covers every congruence class mod 4.

Lastly, thinking back on our definition of almost sum-free sets, all our examples have $\epsilon \geq 1$. We might wonder what we would observe in a case that is closer to the case of random sets would look like:

Question 3.2.6. *Is there anything we can say about about a set $A \subseteq \mathbb{N}$ where $C'N^{2-\epsilon'} \leq T(A_N) \leq CN^{2-\epsilon}$ for some $\epsilon < \epsilon' < 1$ and constants $C, C' > 0$, say? For example, do the phenomena we have observed here continue for such sets even as ϵ' gets small?*

Chapter 4

Strategy

4.1 Overview

In this document we shall unfortunately not answer all the questions from the previous section, nor supply a complete understanding of the observed phenomena. We do, however, propose a strategy that we hope will lead to such an understanding, and we shall partially execute certain components of that strategy.

Broadly, we will first try to understand the Fourier spectrum, and then determine what this says about the distrubution. This happens in four steps:

1. Prove the existence of a large Fourier coefficient at some α.
2. Prove that the spectrum of A is supported in $\alpha\mathbb{Z}$.
3. Prove that the Fourier coefficients $\widehat{A}(k\alpha)$ decay fast enough as $k \to \infty$.
4. Deduce features of the distribution of A modulo $2\pi/\alpha$.

Of this programme, we will provide results in the direction of steps 1 and 4, and computational and heuristic evidence in favour of the others.

4.2 Outline

Before we embark on this journey, we provide a somewhat fuller picture of what we intend to actually do towards each of the steps and in the remainder of this document:

1. First, we study the large Fourier coefficient α and corresponding period $\lambda = \frac{2\pi}{\alpha}$ of various Ulam sequences $U(a, b)$ and sum-free $\theta(s)$. Specifically:
   (a) We use a computer program to estimate the maximal Fourier coefficient of many such sequences.

(b) We use a computer program to compute continued fraction convergents and to attempt to compute minimal polynomials for certain λ to understand whether they are rational or at least algebraic. We also study the constraints that would be imposed on a $U(a, b)$ if the corresponding λ were to be rational with numerator 3.

(c) We prove the existence of an α for almost sum-free sets A (truncated at N) at which the Fourier transform has size comparable to $|A_N|$. In fact, we show that even the real part of the Fourier transform must itself be large compared to $|A_N|$.

2. We then study the complete spectrum of various almost sum-free sets, specifically:

   (a) We compute via computer program any other nonzero values of the respective Fourier transforms. We will find that the Fourier transform for the sets we consider appears to be supported only on $\alpha\mathbb{Z}$.

   (b) We construct an example of a sum-free set with a Fourier spectrum not supported only on some $\alpha\mathbb{Z}$.

   (c) For some almost sum-free sets with spectrum $\alpha\mathbb{Z}$, we use a computer to enumerate the values of $\widehat{f}(k\alpha)$ as k grows and see how these evolve.

   (d) We give an argument that suggests how one might prove the observed decay of these values.

3. Having some picture of the spectrum, we will then study the distribution of A modulo λ and see what information we can deduce from the definition as well as from the spectral information we have gathered in the previous steps. Specifically:

   (a) The definition of such A means that whether a given x is in A is controlled largely by $r_{A+A}(x)$–the number of representations of x as a sum of elements of A. So we start by studying the distribution of this function. We use a computer to generate plots of this that suggest a high degree of regularity in its behaviour, and we show that an estimate of the function using the "major arcs" coming from the Fourier spectrum bears this out.

   (b) Having come to some understanding of r_{A+A} using the Fourier spectrum, we will use this to attempt to deduce features of the distribution itself, ultimately proving a mild theorem that, supposing our computational understanding of the spectrum is correct, the distribution is at least non-uniform.

4. Finally, we will focus specifically on $U(1, 2)$ and turn to more combinatorial analysis of the structure of the sequence. Specifically:

   (a) By the definition of $U(1, 2)$, every element of the sequence has a unique expression $x + y$ for $x, y \in U(1, 2)$, $x < y$. So we might ask which x's actually show up in the Ulam sequence. For the provably regular Ulam sequences, the analogous consideration reveals that there are only finitely many values of x that are used in forming the sequence. In the case of $U(1,2)$, we observe a highly skewed distribution of the "small summands" x, and make similar observations about the distribution of y that show up and about the pairs (x, y) as well.

(b) From these observations, we will start to get a picture of what combinatorial phenomenon might underlie the Ulam sequence, and will consider the implications in particular for the density of the $U(1,2)$.

4.3 About the computations

When we refer to computations done by computer, we will mention the code that was use to execute them by referring to various programs in the repository [15]. In most cases, we will be referring to the particular file `/experiments.py`. For example, when we talk about `experiment17`, we mean the function `experiment17()` in the `experiments.py` file in that repository. Sometimes we will refer to code in other files from the same repository, and will mention where the code in question is located in each case.

Chapter 5

Spectrum

The first three steps of our strategy are about understanding the Fourier spectrum of the various almost sum-free sets under consideration.

5.1 Large Fourier coefficient

The initial observation of [19] is that the Fourier transform of the indicator function of $U(1,2)$ has a large value at some α. That is, $\widehat{A_N}(\alpha) = \frac{1}{N}\sum_{t=1}^{N} A(t)e^{-it\alpha}$ is large relative to $|A_N|$. This suggests that t being in A is correlated with $t+\frac{2\pi}{\alpha}k$ being in A for various integer k. In other words, $\lambda = \frac{2\pi}{\alpha}$ behaves somewhat like a period for A.

5.1.1 Computing λ

We will start by computing this period for several $U(a,b)$ which are not believed to have only finitely many even numbers (and hence nothing about their regularity is known). Specifically, we will look at:

- $(1,v)$ for $v = 2,\ldots,10$.
- $(2,3)$.
- $(3,v)$ for $v = 4,\ldots,10$, $3 \nmid v$.
- $(5,v)$ for $v = 7,\ldots,9$.

We do this by running `experiment1`, which computes $\alpha_{a,b}$ to around 5-6 decimal places using the first 10^4 elements of each of these sequences. We get, letting $\rho_{a,b} = \widehat{1_{a,b}}(\alpha_{a,b})$, the results summarised in table 5.1.

There are a few possible patterns observed:

1. The fractional part of $|\rho_{1,n}|$ remains roughly constant at around 0.417 for $4 \le n$.

Table 5.1: α for various Ulam sequences

a	b	$\alpha_{a,b}$	$\lambda_{a,b}$	$\|\rho_{a,b}\|$	$\delta_{a,b}$	$\|\rho_{a,b}\|/\delta_{a,b}$	$\rho_{a,b}$
1	2	2.5714477	0.0600580	2.4434427	0.0753	0.797	-7950.91 + 629.89i
1	3	2.8334973	0.1008799	2.2174664	0.1268	0.795	-7954.28 + 39.70i
1	4	0.5060131	0.1383183	12.4170387	0.1612	0.857	-8579.77 + -42.90i
1	5	0.4075476	0.1413112	15.4170585	0.1657	0.852	-8527.86 + 96.69i
1	6	0.3411608	0.1404853	18.4170739	0.1656	0.848	-8481.21 + 139.21i
1	7	0.2933728	0.1401227	21.4170681	0.1658	0.845	-8451.66 + 82.39i
1	8	0.2573278	0.1418084	24.4170405	0.1681	0.843	-8433.19 + -16.00i
1	9	0.2291699	0.1448657	27.4171464	0.1720	0.842	-8419.45 + 197.20i
2	3	1.1650122	0.0763468	5.3932355	0.0921	0.828	-8274.76 + 354.04i
3	4	2.2090393	0.1032148	2.8443067	0.1202	0.858	-8580.75 + 189.01i
3	5	2.0048486	0.0976665	3.1339948	0.1132	0.862	-8620.02 + 276.09i
3	7	2.1653662	0.1139341	2.9016732	0.1328	0.857	-8575.18 + 83.86i
3	8	2.0338232	0.1302185	3.0893467	0.1518	0.857	-8574.62 + -4.90i
3	10	2.1437414	0.1231638	2.9309436	0.1429	0.861	-8609.81 + -330.86i
5	7	3.2044799	0.0953540	1.9607503	0.1092	0.872	-8700.91 + -685.218i
5	8	1.2287890	0.1074708	5.1133148	0.1229	0.874	-8742.48 + -68.70i
5	9	2.4845837	0.1060436	2.5288683	0.1215	0.872	-8721.63 + 319.69i

Table 5.2: α for sum-free sets

s	α_s	λ_s	$\|\rho_s\|$	δ_s	$\|\rho_s\|/\delta_s$	ρ_s
10010	1.9559313	0.0700865	3.2123750	0.0802	0.8735	-6973.19 + -463.86i
01001	2.5086193	0.0848948	2.5046387	0.0970	0.8750	-8741.87 + 392.95i
01010	1.8018310	0.0859267	3.4871112	0.0966	0.8893	-7114.61 + -74.04i

2. None of the λs we have computed appear to be obviously rational. (By contrast, when we repeat this in `experiment1A` with a known regular Ulam sequence such as $U(2, 5)$, we get λ that is within 10^{-6} of 2).

3. Though we have only a small amount of data, the first few $\rho_{3,n}$ look like they might alternate around and converge to some value close to 3.

Similarly, we can compute the α maximising $\widehat{A}(\alpha)$ for various sum-free sets A–for example, those that are believed to be non-periodic. In particular, we will look at $s = 01001$, $s = 01010$, and $s = 10010$ by running `experiment2`, with results in table 5.2.

5.1.2 Is λ rational? Algebraic?

As mentioned earlier, the existence of a large Fourier coefficient α for one of these sets A sugggests some kind of pattern or bias modulo $\lambda = \frac{2\pi}{\alpha}$. If this λ were rational, say $\frac{m}{k}$, then this would mean that kA is non-uniformly distributed modulo m. In particular, we might wonder whether λ is well-approximated by

Table 5.3: Continued fractions for α_A for various A

$U(1,2)$	$[2; 2, 3, 1, 11, 1, 1, 4, 1, \ldots]$
$U(1,3)$	$[2; 4, 1, 1, 2, 24, 1, 3, 10, 1, \ldots]$
$U(2,3)$	$[5; 2, 1, 1, 5, 3, 5, 4, 1, 1, 1, \ldots]$
$\theta(01001)$	$[2; 1, 1, 53, 2, 1, 1, 5, 1, 17, 2, \ldots]$
$\theta(01010)$	$[3; 2, 18, 1, 8, 1, 2, 6, 4, 12, 1, 1, \ldots]$
$\theta(10010)$	$[3; 4, 1, 2, 2, 3, 5, 5, 3, 1, 21, 7, \ldots]$

Table 5.4: Convergents for α_A for various A

A	$\frac{P_i}{Q_i}$								
01001	2	$\frac{3}{1}$	$\frac{5}{2}$	$\frac{268}{107}$	$\frac{541}{216}$	$\frac{809}{323}$	$\frac{809}{323}$	$\frac{7559}{3018}$	$\frac{8909}{3557}$
01010	3	$\frac{7}{2}$	$\frac{129}{37}$	$\frac{136}{39}$	$\frac{1217}{349}$	$\frac{1353}{388}$	$\frac{1353}{388}$		
10010	3	$\frac{13}{4}$	$\frac{16}{5}$	$\frac{45}{14}$	$\frac{106}{33}$	$\frac{363}{113}$	$\frac{363}{113}$	$\frac{9968}{3103}$	
$U(1,2)$	2	$\frac{5}{2}$	$\frac{17}{7}$	$\frac{22}{9}$	$\frac{259}{106}$	$\frac{281}{115}$	$\frac{281}{115}$	$\frac{2441}{999}$	$\frac{2981}{1220}$
$U(1,3)$	2	$\frac{9}{4}$	$\frac{11}{5}$	$\frac{20}{9}$	$\frac{51}{23}$	$\frac{1244}{561}$	$\frac{1244}{561}$	$\frac{5129}{2313}$	
$U(2,3)$	5	$\frac{11}{2}$	$\frac{16}{3}$	$\frac{27}{5}$	$\frac{151}{28}$	$\frac{480}{89}$	$\frac{480}{89}$		

a rational number or, if not, whether it is at least annihilated by some low-complexity polynomial.

To get rational approximations for λ, we run the program `cf_alpha.py` in SAGE to compute continued fractions and convergents, found in tables 5.3 and 5.4, respectively.

(Since we only have believe our values of λ up to 10^{-5}, we have reason to mistrust convergents with larger heights anyways, so we only list those with denominators up to 10^4.)

One indicator that λ might be quite near a rational number would be if the continued fraction had a single anomalously large coefficient. This is not observed in any of these examples. Further, if we examine the non-uniformity of, say, $U(1,2)$ modulo m where m runs through the numerators of the convergents (where we measure non-uniformity simply by the variance σ_m in the number of elements that lie in each congruence class modulo m, as computed in `experiment3`, with results in table 5.5.

Indeed, the variance is quite large for these moduli, but for nearby moduli tends to be far less pronounced, as we compute in `experiment3A`, with results in table 5.6

We might wonder: All of the known regularity results that we mentioned earlier were showed that various $U(a,b)$ were biased modulo 2. However there

Table 5.5: Variance in Ulam numbers modulo numerators of λ

m	σ_m
2	1798281.0
5	22177.839999999997
17	37704.93425605536
22	48301.867768595046
259	2233.0616418956242
281	2723.807474576056
540	3113.9309876543243
2441	2543.052250061714
2981	11485.177502016855

Table 5.6: Variance in Ulam numbers modulo other m

m	σ_m
537	1116.920445678975
538	1224.5307969762737
539	1272.699942517064
540	3113.9309876543243
541	1246.7075553247405
542	1190.8150079655782
543	1238.8901641179054
2438	281.8043293195446
2439	282.66582245158276
2440	283.49388538027415
2441	2543.052250061714
2442	272.53664877743483
2443	265.38973022070724
2444	268.4666754616002

are no such results even for modulo 3, and we do not see any examples of sequences that appear to be heavily biased modulo 3. We leave exploration of why this might be to the future, but we do consider one special case: What would happen if we had an Ulam sequence $U(a, b)$ whose elements were eventually all $1 \mod 3$?

We suspect such sequences might be rare as a consequence of the greediness of the definition of Ulam sequences. For example, the sequence $3, 4, 7, 10, 13, \ldots$ is a sequence that is eventually all congruent to $1 \mod 3$ and that is Ulam-like in that every element bigger than 4 is a sum of distinct smaller elements in a unique way. It fails however to be an actual Ulam sequence because it is not "greedy" in the way that the definition requires. For example, $U(3, 4)$ would indeed start $3, 4, 7, 10$, but then $4 + 7 = 11$ would be next, since that is the smallest number with only one such representation.

In general, the greediness condition on Ulam numbers means that if $x < y$ are in $U(a, b)$, then either $x+y \in U(a, b)$ or else there exist $x' < y'$ in $U(a, b)$ with $x' + y' = x + y$. Thus, if we were to have such a sequence A with only elements that are 1 mod 3, say all elements above some M are $x_1, x_2, \ldots$, then if these are an arithmetic progression, say, then they can ensure that excepting $x_1 + x_2$, every sum of these elements $x_i + x_j$ can also be expressed as either $x_{i-1} + x_{j+1}$ or $x_{i+1} + x_{j-1}$ so that all these sums are excluded from the sequence. To additionally exclude $x_1 + x_2$, we will need some further elements of the sequence that sum to this quantity as well (which is why the above example failed).

But now, to ensure that x_i are all in A, we will need some elements that are 0 mod 3 to be the small summands of the x_i. Say $y_1, \ldots, y_n$ are all the elements of A that are 0 mod 3, say in increasing order. But then $y_i + y_n$ must not be in the sequence for any i, which means that we need some further elements a_i, b_i that are 1 and 2 mod 3 respectively to be in the sequence and with $a_i + b_i = y_i + y_n$. And we can continue like this, where the more elements that we find must be in the set, the more elements we have to add to the set in order to exclude certain sums of these elements. Perhaps this process does terminate, for some x_i in an actual set A of the form $U(a, b)$, or perhaps if one follows it to its conclusion as in [18], one might find there is in fact no such set. We do not pursue this project here, however.

Even if λ is irrational, we might still wonder whether it is algebraic. We can use LLL to hunt for the minimal polynomial for various λ. This is a standard technique: We pick a degree d and a large N and let v_i (for $0 \leq i \leq d$) be the $d + 1$-dimensional vector with 1 in the ith slot, and $\lfloor N\lambda^i \rfloor$ in the $d + 1$th slot:

$$v_i = \left(0, \ldots, 1, \ldots, 0, N\lambda^i\right)$$

We can then use the Lenstra-Lenstra-Lovasz (or "LLL") algorithm to find a small integer linear combination $w = \sum_{i=0}^{d} c_i v_i$ that has small norm (say, small relative to N). And since the last coordinate of w will be $N \sum_i c_i \lambda^i$, the only way the norm of can be close to zero is if $\sum_i c_i \lambda^i$ is close to zero, so this could be a reasonable guess for the minimal polynomial provided the value we get from this polynomial is significantly less than $\frac{1}{N}$.

In our case, we use SAGE's LLL code in `alg_lambda.py` to compute. In this case, we use $N = 10^{10}$, so if we get a polynomial f out then we will want $f(\lambda) \ll 10^{-10}$ for f to be a good candidate for minimal polynomial. The results are in table 5.7.

In particular, there is no indication that any of these λ has small degree, if it is algebraic at all.

5.1.3 Existence of α

The common thread with all these almost sum-free sets is that they have few solutions to $x+y=z$ in them. As we talk about why this gives us large Fourier coefficients, the following definition will be helpful:

Definition 5.1.1. *For $A \subseteq [N]$, define*

$$T(A) = \left|\{(x,y,z) \in A^3 : x+y=z\}\right|$$

More generally, for $f : [N] \to [0,1]$, define

$$T(f) = \sum_{x,y \in [N]} f(x)f(y)f(x+y)$$

(So in particular, $T(A) = T(A)$, where the left side denotes T of the set A, and the right side denotes T of the indicator function of that set.)

Now, the statement that A has a large Fourier coefficient could be thought of conceptually as saying that A cannot be too random. To get a heuristic for what this would mean, suppose A were random in the sense that we construct it by going through each element of $[N]$ and including it in A with some probability p. Then we expect to have $(pN)^2$ pairs $(x,y) \in A^2$. For any such pair, $x+y$ will be in A with probability p, so the number of pairs $(x,y,x+y) \in A^3$ will be around p^3N^2 as N grows. So for a random set of density δ, we might then expect $T(A_N) \approx \delta^3 N^2$ as N grows.

But by definition, if A is sum-free, $T(A_N) = 0$ for all N, and $T(U(a,b)_N) \leq 3N$ (since each $z \in U(a,b)$ has at most 3 representations as $z = x+y$ for $x, y \in A$, namely, the one with $x < y$ that qualifies it to be in $U(a,b)$, the same one in reverse ($z = y + x$), and possibly $z = \frac{z}{2} + \frac{z}{2}$ if $\frac{z}{2} \in U(a,b)$ (since the definition does not exclude that possibiility). In particular, these sets do not behave like truly random sets would be expected to.

We give a result in this direction saying that if $T(A_N)$ is small relative to N (i.e. is less than N^2), but the set itself is reasonably large, then there must be a Fourier coefficient that explains this:

Theorem 5.1.2. *If $A \subseteq [N]$ is a set of size δN such that $T(A)$ is bounded by $cN^{2-\epsilon}$ for some constants $c > 0, \epsilon > 0$, then there is an $k \in [2N]$ such that $\widehat{A_N}(\frac{2\pi k}{2N}) \geq \frac{\delta^2}{2} - \frac{c}{\delta N^\epsilon}$.*

Table 5.7: Candidate minimal polynomials for λ_A

A	$f(\lambda)$	$f(X)$
$U(1,2)$	-6.742e-10	$-6X^5+4X^4+23X^3-2X^2+28X-12$
	-6.742e-10	$-6X^5+4X^4+23X^3-2X^2+28X-12$
	-6.742e-10	$-6X^5+4X^4+23X^3-2X^2+28X-12$
	1.2583e-9	$X^8-2X^7-9X^6+18X^5-3X^4+11X^3+10X^2+4X-6$
	1.4451e-9	$X^9-12X^7+12X^6+10X^5-8X^4+3X^3-5X^2-7X-1$
	-8.806e-9	$-34X^4+39X^3+149X^2-87X-34$
	5.2857e-8	$3X^3-109X^2+212X+89$
	2.4059e-7	$-215X^2+1359X-2037$
$U(1,3)$	-1.089e-10	$X^8-4X^7+6X^6-7X^5+X^4+X^3+9X^2+21X+6$
	2.3621e-10	$-10X^7+11X^6+23X^5+5X^4-2X^3-7X^2+8X+13$
	2.6459e-10	$-5X^9+11X^8+4X^7-6X^6-7X^5-2X^4+8X^3+7X^2+4X+3$
	-1.579e-9	$10X^5-22X^4-9X^3+10X^2-15X+78$
	-1.694e-9	$-4X^6-10X^5+31X^4+24X^3+2X^2+3X-16$
	1.2805e-8	$-5X^4+24X^3-2X^2-55X-9$
	1.4511e-8	$59X^3-62X^2-209X+125$
	-2.285e-7	$244X^2-629X+195$
$U(1,9)$	1.3553e-8	$16X^4-428X^3-278X^2-369X-903$
	-2.551e-8	$-5X^6+143X^5-166X^4+109X^3-99X^2+81X-110$
	-3.505e-8	$X^5-40X^4+357X^3-330X^2+11X+156$
	-1.167e-7	$-21X^3+468X^2+2930X+671$
	-8.036e-7	$-79X^2+2175X-248$
	-2.669e-6	$3X^7-81X^6-33X^5-38X^4+57X^3-9X^2-157X-176$
	-0.00002	$-2X^8+53X^7+55X^6-127X^5-55X^4-75X^3+10X^2+17X-70$
	0.001502	$X^9-26X^8-41X^7+57X^6+50X^5+8X^4-8X^3+136X^2+79$
$U(2,3)$	-6.626e-11	$3X^8-15X^7+X^6-36X^5-20X^4+2X^3-8X^2-19X+26$
	-8.778e-10	$2X^9-11X^8+2X^7-4X^6-6X^5+16X^4-6X^3+18X^2+17X+3$
	-1.407e-9	$-7X^7+34X^6+22X^5-11X^4+4X^3+16X^2+38X-27$
	4.8321e-9	$10X^6-53X^5-4X^4-7X^3+8X^2+11X-64$
	7.1473e-9	$-10X^5+56X^4-27X^3+78X^2+52X-63$
	-1.300e-8	$33X^4-163X^3-77X^2-7X-72$
	-2.169e-8	$-3X^3-85X^2+412X+721$
	-1.977e-7	$674X^2-3366X-1451$
$U(3,4)$	4.4999e-10	$-5X^5+13X^4-X^3+5X^2+16X+17$
	4.4999e-10	$-5X^5+13X^4-X^3+5X^2+16X+17$
	4.4999e-10	$-5X^5+13X^4-X^3+5X^2+16X+17$
	4.4999e-10	$-5X^5+13X^4-X^3+5X^2+16X+17$
	4.4999e-10	$-5X^5+13X^4-X^3+5X^2+16X+17$
	-3.030e-9	$-20X^4+83X^3-23X^2-136X-28$
	-2.031e-8	$-53X^3+82X^2+153X+121$
	-4.504e-8	$588X^2-1850X+505$

Proof. Denote the discrete Fourier transform by $(\mathscr{F}_N A)(k) = \sum_{t=0}^{N-1} A(t)e(\frac{-2\pi kt}{N})$. Note that this really is a function on $\mathbb{Z}/N$, rather than just $[N]$. Since we want to relate $\mathscr{F}A$ to $T(A)$, we would like to compute $T(A)$ in terms of $\mathscr{F}A$. The following standard trick allows us to do this:

$$T(A) = \sum_{x,y,z=0}^{N-1} A(x)A(y)A(z)\delta_{x+y-z}$$

where $\delta_0 = 1$ and $\delta_x = 0$ for $x \neq 0$. Then the trick is to write

$$\delta_x = \frac{1}{N}\sum_{t=0}^{N-1} e(\frac{2\pi xt}{N}) = \mathscr{F}_N 1$$

However, we note that this only tests for $x = 0 \mod N$. Thus if we substitute this into our expression for $T(A)$, then we will only be counting solutions to $x + y = z$ modulo N. However, because $A \subseteq [N]$, solving $x + y = z$ in $\mathbb{Z}$ is the same as solving it modulo $2N$. Thus we can use the formula:

$$\delta_x = \frac{1}{2N}\sum_{t=0}^{2N-1} e(\frac{2\pi xt}{2N}) = \mathscr{F}_{2N} 1$$

to compute:

$$\begin{aligned}
T(A) &= \frac{1}{2N}\sum_{x,y,z=0}^{N-1} A(x)A(y)A(z)\sum_{t=0}^{2N-1} e(\frac{2\pi(x+y-z)t}{2N}) \\
&= \frac{1}{2N}\sum_{x,y,z=0}^{2N-1} A(x)A(y)A(z)\sum_{t=0}^{2N-1} e(\frac{2\pi(x+y-z)t}{2N}) \\
&= 4N^2\sum_{t=0}^{2N-1}\sum_{x=0}^{2N-1}\frac{1}{2N}A(x)e(\frac{2\pi xt}{2N})\sum_{y=0}^{2N-1}\frac{1}{2N}A(y)e(\frac{2\pi yt}{2N})\sum_{z=0}^{2N-1}\frac{1}{2N}A(z)e(\frac{-2\pi zt}{2N}) \\
&= 4N^2\sum_{t=0}^{2N-1}(\mathscr{F}_{2N}A)(-t)(\mathscr{F}_{2N}A)(-t)(\mathscr{F}_{2N}A)(t) \\
&= 4N^2\sum_{t=0}^{2N-1}|(\mathscr{F}_{2N}A)(t)|^2(\mathscr{F}_{2N}A)(-t)
\end{aligned}$$

where the second equality we get by extending A by zero.

So by assumption,

$$cN^{2-\epsilon} \geq 4N^2\sum_{t=0}^{2N-1}|(\mathscr{F}_{2N}A)(t)|^2(\mathscr{F}_{2N}A)(-t)$$

We can pull out the $t = 0$ term which is $\frac{\delta^3}{8}$. Then we can bound the remaining sum from below by replacing one of the three $(\mathscr{F}_{2N}A)(t)$ terms inside the sum with $-\max_{t\neq 0}|(\mathscr{F}_{2N}A)(t)| = -\rho$:

$$cN^{2-\epsilon} \geq N^2 \frac{\delta^3}{2} - \rho \cdot 4N^2 \sum_{t=1}^{2N-1} |(\mathscr{F}_{2N}A)(t)|^2$$

Now, by Plancherel we know that

$$\sum_{t=0}^{2N-1} |(\mathscr{F}_{2N}A)(t)|^2 = \frac{1}{2N} \sum_{t=0}^{2N-1} |A(t)|^2 = \frac{1}{2N}|A| = \frac{\delta}{2}$$

so:

$$cN^{2-\epsilon} \geq N^2 \frac{\delta^3}{2} - \rho \cdot 4N^2 \frac{\delta}{2} = N^2 \left(\frac{\delta^3}{2} - \rho \cdot 2\delta \right)$$

Or, rearranging,

$$\rho \geq \frac{\delta^2}{4} - \frac{c}{2\delta N^{\epsilon}}$$

Thus for if k is the value of t that realises the maximum, then we have shown that:

$$\left| \frac{1}{2N} \sum_{t=0}^{2N-1} A(t) e(\frac{-2\pi k t}{2N}) \right| \geq \frac{\delta^2}{4} - \frac{c}{2\delta N^{\epsilon}}$$

But comparing the left side to

$$\widehat{A_N}(\frac{2\pi k}{2N}) = \frac{1}{N} \sum_{t=0}^{N-1} A_N(t) e(\frac{-2\pi k t}{2N}) = \frac{1}{N} \sum_{t=0}^{2N-1} A_N(t) e(\frac{-2\pi k t}{2N})$$

we get that for some $k \neq 0$

$$|\widehat{A_N}(\frac{2\pi k}{2N})| \geq \frac{\delta^2}{2} - \frac{c}{\delta N^{\epsilon}}$$

as claimed. □

We noticed earlier that the actual complex value of the Fourier transform of A for most of the A had large negative real part and relatively small imaginary part. If we look at the proof of the theorem, we notice that we can use the fact that $(\mathscr{F}_{2N}A)(t) = \overline{(\mathscr{F}_{2N}A)(2N-t)}$ to rewrite:

$$\begin{aligned} T(A) &= 4N^2 \sum_{t=0}^{2N-1} |(\mathscr{F}_{2N}A)(t)|^2 (\mathscr{F}_{2N}A)(-t) \\ &= 4N^2 \sum_{t=0}^{2N-1} |(\mathscr{F}_{2N}A)(t)|^2 \Re((\mathscr{F}_{2N}A)(t)) \end{aligned}$$

Then we can use the same exact proof as before, letting instead

$$\rho' = \min_{t \neq 0} \Re((\mathscr{F}_{2N}A)(t))$$

which gives us:

$$cN^{2-\epsilon} \geq N^2 \left(\frac{\delta^3}{2} + \rho' \cdot 2\delta \right)$$

Thus we end up with

$$\rho' \leq - \left(\frac{\delta^2}{4} - \frac{c}{2\delta N^\epsilon} \right)$$

Finally, comparing once again the discrete Fourier transform to the definition of $\widehat{A_N}$, we conclude that just as the magnitude of some Fourier coefficent must be large, so too must the real part also be large and negative (as observed earlier in table 5.1):

Corollary 5.1.2.1. *If A is as in the theorem, then there is a $k \in [2N]$ such that $\Re(\widehat{A_N}(\frac{2\pi k}{2N})) \leq -\left(\frac{\delta^2}{2} - \frac{c}{\delta N^\epsilon}\right)$.*

The theorem gives us a non-zero Fourier coefficient α_N for A_N. As N grows, we get a sequence of such $\alpha_N \in \mathbb{R}/2\pi\mathbb{Z}$. If the density eventually stays above some fixed δ and if the α_N have a non-zero limit point α, then we can show that in fact this α would indeed be a non-zero Fourier coefficient for the whole set A in the sense of definition 2.1.1:

Theorem 5.1.3. *If $A \subseteq \mathbb{N}$ is a sequence of positive integers of density $\delta > 0$ such that for all N, $T(A_N)$ is bounded by $cN^{2-\epsilon}$ for some constants $c > 0, \epsilon > 0$, and suppose further that α is a nonzero limiting point of the α_N that the theorem guarantees us. Then $\widehat{A}(\alpha) \geq \frac{\delta^2}{2}$, supposing that the limit $\widehat{A}(\alpha)$ does in fact exist.*

Example 5. *For example, in the Ulam sequence we know by construction that $T(A_N) \leq 3|A| \leq 3N$, and we believe that the Ulam sequence has density around* 0.07*, so this theorem would guarantee that if we had a non-zero Fourier coefficient, it would have size at least* 0.00245*. This is a bit off our numerical value of* $0.8\delta \approx 0.8 \times 0.07 = 0.056$*, but it is a start.*

Proof. Since A has density δ, for any $\epsilon > 0$, there is an $N > 0$ with $\frac{|A_N|}{N} > \delta - \epsilon$. So as $j \to \infty$, for each j we can find N_j with $\frac{|A_{N_j}|}{N_j} > \delta - \frac{1}{j}$.

Thus, by the theorem, there is an $\alpha_j = \frac{2\pi k_j}{2N_j}$ with $\widehat{A_{N_j}}(\alpha_j) \geq \frac{\delta^2}{2} - \frac{\delta}{j} - \frac{1}{2j^2}$.

Now, the $\alpha_j \in [0, 2\pi]$ necessarily have a convergent subsequence. Replacing the sequence by this subsequence, we can say α_j converges to some α as $j \to \infty$, and the limiting value of $\widehat{A_{N_j}}(\alpha_j)$ is greater than $\frac{\delta^2}{2}$ (or, if there is no limiting value, take a convergent subsequence of these values). So then we would like to conclude that $\widehat{A}(\alpha) \geq \frac{\delta^2}{2}$. However, this claim involves swapping some limits, so let us be more careful: We have that $\lim_{j\to\infty} \widehat{A_{N_j}}(\alpha_j) \geq \frac{\delta^2}{2}$, but we need to show that in fact the same is true for $\lim_{j\to\infty} \widehat{A_{N_j}}(\alpha)$.

For any j, say $\alpha = \alpha_j + \epsilon_j$, with $\epsilon_j \to 0$ as $j \to \infty$. We want to show that in fact $\widehat{A_{N_j}}(\alpha) \geq \frac{\delta^2}{2} - o(j) - O(\epsilon_j)$. But now we can compute using Taylor's theorem, with some β_j between α_j and α, and using the fact that

$\left|\sum_{t=0}^{N-1} te(-tx)\right| = \left|\frac{d}{dx}\sum_{t=0}^{N-1} e(-tx)\right| = \left|\frac{d}{dx}\frac{e(-Nx)-1}{e(-x)-1}\right| \leq \frac{2+2N}{C^2} \leq \frac{3N}{C^2}$ if $|e(-x)-1| \geq C$ and if $N \geq 2$:

$$\begin{aligned}
\widehat{A_{N_j}}(\alpha) &= \frac{1}{N_j}\sum_{t=0}^{N-1} A(t)e(-t(\alpha_j+\epsilon_j)) \\
&= \frac{1}{N_j}\sum_{t=0}^{N-1} A(t)(e(-t\alpha_j) - t\epsilon_j e(-t\beta_j)) \\
&\geq \frac{\delta^2}{2} - \frac{1}{j} - \frac{\epsilon_j}{N_j}\sum_{t=0}^{N-1} te(-t\beta_j) \\
&\geq \frac{\delta^2}{2} - \frac{1}{j} - \frac{\epsilon_j}{N_j}\frac{3N_j}{C^2} \\
&= \frac{\delta^2}{2} - \frac{1}{j} - \frac{3\epsilon_j}{C^2}
\end{aligned}$$

provided $|e(\alpha_j)-1| \geq C$ for all j (which is possible provided $\alpha \neq 0$ and we take j large enough).

Thus indeed, $\lim_{j\to\infty} \widehat{A_{N_j}}(\alpha) \geq \frac{\delta^2}{2}$, which, because we are assuming that the actual limit $\widehat{A}(\alpha)$ exist, gives that $\widehat{A}(\alpha) \geq \frac{\delta^2}{2}$, as desired. □

Some remarks:

Remark 1. As we mentioned before, this bound for the Ulam sequence, which works out to around 0.00245 is not anywhere near as good as the computed estimate of 0.056. However, this method finds a rational k/N where the Fourier transform is large for *every* N, whereas experimentally the large value of around 0.056 only occurs actually at α and can only be observed at rational k/N that are good approximations to α. In particular, we also observe that for some N, the largest Fourier coefficient might honestly only be as large as $\frac{\delta^2}{2}$. So any approach that gives a large Fourier coefficient in $\left(\frac{2\pi}{N}\mathbb{Z}\right)/\mathbb{Z}$ for all N should not give a Fourier coefficient as large as we expect.

Remark 2. It is interesting to note that this argument does not provide an obvious way to take advantage of the uniformity with which solutions to $x+y=z$ occur in the Ulam case. For example, it also applies to a sequence where $a_{2^i+1},\ldots,a_{2^{i+1}-1}$ have no representations but a_{2^i} has 2^{i-1} representations for each i (in which case the number of representations is not bounded above, but is growing, albeit sort of slowly and non-uniformly).

5.2 The complete spectrum of A

Knowing that the spectrum of A has some nonzero α, this suggests at least that $\widehat{A}(n\alpha)$ would also be nonzero for $n \in \mathbb{Z}$ (again, ignoring convergence issues).

Table 5.8: Spectrum of $U(1,2)$

x with $N\widehat{A_N} > \sqrt{2N}$	$N\widehat{A_N}(x)$	$k\alpha \approx \pm x \mod 2\pi$
0	5000.0	0α
2.5715	2316.833319	1α
2.2805	678.751099	4α
0.291	567.681578	5α
1.722	525.609312	8α
0.5585	324.005045	12α
3.13	281.976985	11α
1.431	281.803113	3α

Further, we also have the theorem of Weyl from [16] that tells us that for any infinite sequence a_n, the set of $x \in \mathbb{R}/\mathbb{Z}$ with $\widehat{A}(x) \neq 0$ has measure zero. We have found one such set of x, namely $\alpha\mathbb{Z}$. We now investigate whether for our particular almost sum-free sets there are any others.

5.2.1 Spectral complexity

There is some computational evidence that suggests that the spectrum of $U(1,2)$ should consist only of $\alpha\mathbb{Z}$. We can brute-force compute $\widehat{A_N}(t)$ for many values of t between 0 and π. Plancherel tells us that over $x \in \mathbb{Z}/N$, the average value of

$$\left|\sum_{t=0}^{N-1} A_N(t)e(-\frac{2\pi xt}{N})\right|$$

will be around $\sqrt{|A_N|}$, so in this computation, we want to look at the values that are significantly larger than $\sqrt{|A_N|}$. However, if we are looking at finite N, then while this is the average size we still expect some random variation to put some values over this mark. To account for this, then, we look at values above $\sqrt{2N}$ in the first 5000 values of each sequence. We do this in `experiment7` and search for corresponding multiples of α using `experiment7A`, with output found in table 5.8.

Here, it appears that the only large Fourier coefficients are those in $\alpha\mathbb{Z}$. However, if we run the same computations for other sequences such as $U(1,3)$, $U(1,4)$, $U(1,9)$, $U(2,3)$ or the various $\theta(s)$, we see some values that we cannot account for by multiples of the corresponding α, found in tables 5.9–5.13. We give here only the top 20 elements of each spectrum (as some sequences, such as $U(1,3)$ and $U(1,9)$ had many elements that were somewhat large, while others, such as the $\theta(s)$, had very few).

We should point out that simply being almost sum-free is not enough to guarantee the spectrum is of the form $\alpha\mathbb{Z}$ only. For example, take two irrational numbers α and β that are not rational multiples of each other (such as $\sqrt{2}$ and

Table 5.9: Spectrum of $U(1,3)$

x with $N\widehat{A_N} > \sqrt{2N}$	$N\widehat{A_N}(x)$	$k\alpha \approx \pm x \mod 2\pi$
0	5000.0	0α
2.8335	3970.6103	1α
1.601	938.111303	5α
1.2325	883.00755	4α
1.8485	811.10209	6α
0.2475	764.72957	11α
3.081	696.841779	10α
2.586	405.192550	12α
0.062	337.314378	$?\alpha$
1.909	288.594562	$?\alpha$
2.7715	274.70726	$?\alpha$
1.4785	266.48125	15α
1.662	262.855597	$?\alpha$
2.8955	255.87568	$?\alpha$
1.7225	235.36338	25α
1.5415	234.36759	$?\alpha$
1.694	231.416004	$?\alpha$
0.338	230.309317	$?\alpha$
1.5085	229.67156	$?\alpha$
1.355	228.435939	16α

Table 5.10: Spectrum of $U(2,3)$

x with $N\widehat{A_N} > \sqrt{2N}$	$N\widehat{A_N}(x)$	$k\alpha \approx \pm x \mod 2\pi$
0	5000.0	0α
1.165	4046.51672	1α
2.33	1897.442650	2α
1.2045	706.58017	28α
0.0395	685.86680	27α
2.3695	473.30677	$?\alpha$
1.1255	435.47060	26α
3.037	312.437616	8α
1.271	309.489276	$?\alpha$
1.8325	300.43267	20α
2.9975	292.10900	19α
2.436	282.381451	$?\alpha$
0.0605	278.69216	$?\alpha$
2.5	264.10806	41α
2.5365	263.75355	14α
2.6835	259.84376	$?\alpha$
1.2255	254.17553	$?\alpha$
2.6385	253.11488	$?\alpha$
0.0205	253.09852	$?\alpha$
1.2595	237.28295	$?\alpha$

Table 5.11: Spectrum of $U(1,9)$

x with $\widehat{NA_N} > \sqrt{2N}$	$\widehat{NA_N}(x)$	$k\alpha \approx \pm x \mod 2\pi$
0	5000.0	0α
0.229	1001.16658	1α
0.4585	729.79222	2α
1.146	380.393081	5α
0.018	374.496531	192α
0.012	340.513667	$?\alpha$
0.032	332.931953	$?\alpha$
0.2555	329.91899	$?\alpha$
0.028	310.803845	$?\alpha$
0.2525	291.89760	$?\alpha$
0.0385	291.20968	$?\alpha$
0.0245	286.37647	$?\alpha$
1.8335	283.13708	8α
0.0415	278.34269	$?\alpha$
0.0465	275.05065	$?\alpha$
0.271	260.280839	$?\alpha$
0.2105	260.14071	191α
0.184	245.639695	$?\alpha$
0.1745	241.64022	$?\alpha$
0.0635	240.31454	$?\alpha$

Table 5.12: Spectrum of $\theta(01010)$

x with $\widehat{NA_N} > \sqrt{2N}$	$\widehat{NA_N}(x)$	$k\alpha \approx \pm x \mod 2\pi$
0	5000.0	0α
2.6795	2810.46101	2α
1.802	1028.28906	1α
2.726	384.98962	5α
0.8775	327.55811	3α

Table 5.13: Spectrum of $\theta(10010)$

x with $\widehat{NA_N} > \sqrt{2N}$	$\widehat{NA_N}(x)$	$k\alpha \approx \pm x \mod 2\pi$
0	5000.0	0α
1.956	1677.650536	1α
1.5405	603.318237	4α
2.3715	390.730699	2α

Table 5.14: Values of $\widehat{A_N}(\beta)$ for $N \in 1000 \times \{1, \dots, 10\}$

A	β	$\widehat{A_{1000k}}(\beta)$ for $k = 2, \dots, 10$
$U(1,3)$	1.909	0.025 0.0403 0.0557 0.0577 0.0528 0.0458 0.0481 0.0436 0.0435
$U(1,3)$	2.7715	0.0105 0.0471 0.0583 0.0549 0.048 0.052 0.0485 0.0401 0.0359
$U(1,3)$	1.662	0.0377 0.0516 0.0555 0.0526 0.045 0.0418 0.0412 0.0372 0.0458
$U(1,9)$	0.005	0.0681 0.0677 0.0542 0.0428 0.0459 0.0533 0.0495 0.0514 0.0476
$U(1,9)$	0.0271	0.0796 0.0684 0.0799 0.0588 0.0588 0.0472 0.0369 0.0301 0.0335
$U(1,9)$	0.247	0.0442 0.0395 0.0438 0.0433 0.0605 0.0569 0.0524 0.0542 0.0472
$U(1,9)$	0.2672	0.0717 0.0475 0.0418 0.0487 0.0358 0.0245 0.0212 0.0212 0.0218
$U(2,3)$	2.3694	0.0558 0.0626 0.0673 0.0691 0.0665 0.0677 0.0639 0.0568 0.0484
$U(2,3)$	1.2743	0.035 0.0277 0.0311 0.0349 0.0465 0.0466 0.0457 0.0422 0.0417
$U(2,3)$	2.4358	0.0852 0.0551 0.0661 0.0607 0.0592 0.0589 0.047 0.0431 0.0436
$U(2,3)$	0.0599	0.0484 0.0155 0.025 0.0432 0.0478 0.0536 0.0535 0.0481 0.0437

$\sqrt{3}$) and let $A = A_\alpha \cap A_\beta$. Then because $A \subseteq A_\alpha$, e.g., A is certainly sumfree, however it will not be equidistributed modulo either $\frac{2\pi}{\alpha}$ or $\frac{2\pi}{\beta}$, giving it nonzero Fourier coefficients on $\alpha\mathbb{Z}$ and on $\beta\mathbb{Z}$.

The question is whether the values β observed in the above computations that we could not match with an element of $\alpha\mathbb{Z}$ actually represent another component in the spectrum. In other words, we want to check whether the growth of $N\widehat{A_N}(\beta)$ as $N \to \infty$ is $o(N)$, and we are only seeing an unusually large initial value.

To this end, we pick a few such values and increase N and see how they evolve. We do this in `experiment7D` and the results are in table 5.14.

The results suggest a slow decay, but are not entirely convincing either way, so we leave this as a question for further investigation.

5.2.2 Decay of Fourier coefficients

Recall from the background section our intuition on why smoother functions have better decay in their Fourier coefficients: The Fourier coefficient $\widehat{f}(\xi) = \int_0^1 f(t)e^{2\pi it\xi}dt$ can be integrated by parts, which will replace $f(t)$ with $f'(t)$ (which, if f is well-behaved, should be bounded or otherwise controlled), and $e^{2\pi it\xi}$ with the integral of this, which as ξ grows, will wind faster and faster around the unit circle, giving this integral more and more cancellation and therefore making it smaller. In particular, the antiderivative of $e^{2\pi it\xi}$ involves a $\frac{1}{\xi}$, so the decay should go like $\frac{1}{\xi}$.

Though turning this intuition into a proof in our context has not yet succeeded, we give a conjecture in this direction and some computational evidence in its favour:

Conjecture 5.2.1. *If A is a almost sum-free set of positive integers, and $\widehat{A}(\alpha) \neq 0$, then for $d \in \mathbb{Z}$ with $|d| \leq \sqrt{N}$ with $d \neq 0$, we have*

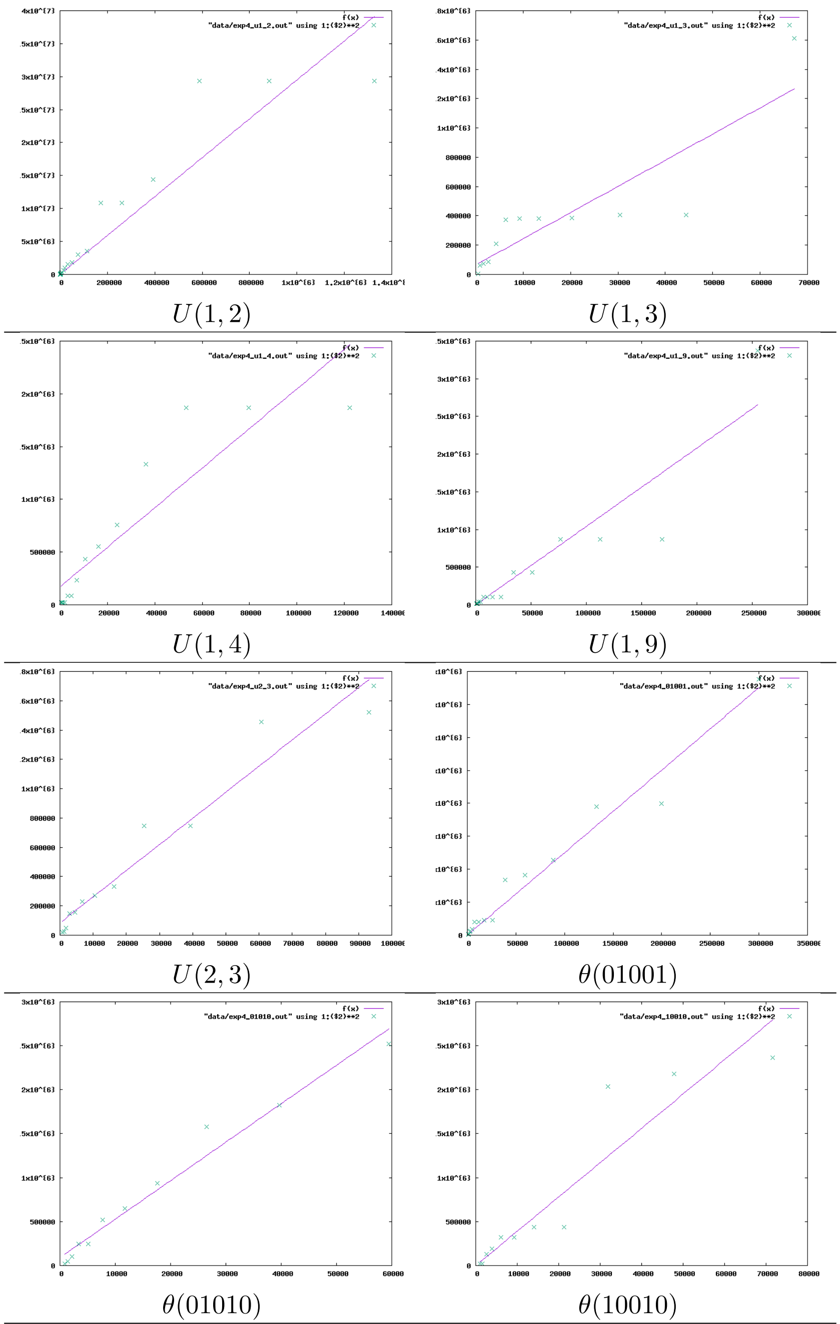
$$\widehat{A_N}(d\alpha) \leq \frac{C}{d}$$

for some constant C independent of N.

To support this conjecture, we compute $d\widehat{A_N}(d\alpha)$ for increasing d increasing until we see $d\widehat{A_N}(d\alpha) > 4$. This is done in `experiment4` for various N. If we call the minimum such d for a given N to be d_N, then the points (N, d_N^2) (along with the best fitting line) are plotted in figure 5.1.

These suggest, if somewhat loosely, that the $d_N = \Theta(\sqrt{N})$, (with some oscillation around the line of best fit). In particular, there is some c with $d_N \leq c\sqrt{N}$.

Figure 5.1: Plots of first $d_N \alpha_A$ to violate conjectured decay rate.



Chapter 6

Distribution

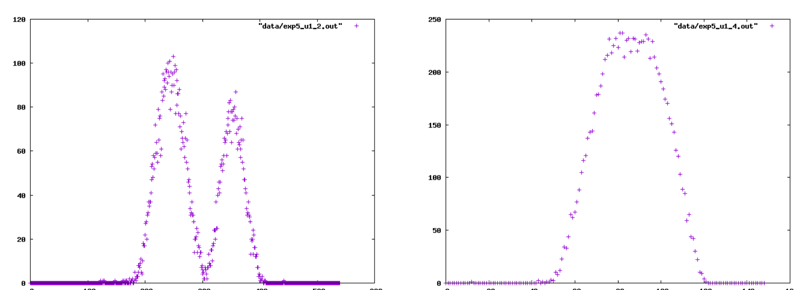
As we have remarked before, a sequence A having a large Fourier coefficient at α suggests A should not be equidistributed modulo $\lambda = \frac{2\pi}{\alpha}$. Indeed, we have experimentally found large Fourier coefficients for many of the sequences we have considered, and if we take the distribution of these sequences modulo the corresponding values of λ, we get non-uniform, continuous-looking distributions as in figure 6.1.

In this section, we will examine how what we have learned about the spectrum might control various features about the corresponding distributions.

6.1 Distrubution of r_{A+A}

The common feature of the almost sum-free sets that we have been considering is that whether x is in one of these sets A is determined by how many ways we can write x as a sum $x = a + b$ for $a, b \in A$. To study this, we make the following definitions:

Figure 6.1: Some distribuions of A mod λ_A



$U(1,2)$ $\quad$ $U(1,3)$

Definition 6.1.1. *For $A \subseteq \mathbb{N}$, define the **sum representation counting function** r_{A+A} by*

$$r_{A+A}(x) = \left|\{(a,b) \in A^2 : a+b = x\}\right|$$

*Also define, the **modified sum representation counting function***

$$r^*_{A+A}(x) = \left|\{(a,b) \in A^2 : a+b = x; a < b\}\right|$$

*Finally, define the **difference representation counting function***

$$r_{A-A}(x) = \left|\{(a,b) \in A^2 : a-b = x\}\right|$$

So $r_{A+A}(x) = 0$ is the necessary condition for x to lie in a sum-free set A, as is $r_{A-A}(x) = 0$, whereas $r^*_{A+A}(x) = 1$ is the condition for being in an Ulam sequence A. Likewise, for x in an Ulam sequence A, $r_{A-A}(x)$ is the number of times x is used as a summand in A. (We note that r_{A-A} may be infinite if A is infinite, but it will make sense when we truncate A.)

But we can write formulae for $r_{A+A}(x)$ and $r_{A-A}(x)$ in terms of the indicator function of A:

$$r_{A+A}(x) = \sum_{0<y<x} A(y)A(x-y)$$

$$r_{A-A}(x) = \sum_{0<y} A(y)A(x+y)$$

which are exactly the definition of the convolution $(A*A)(x)$ and cross-correlation $(A \star A)(x)$ respectively.

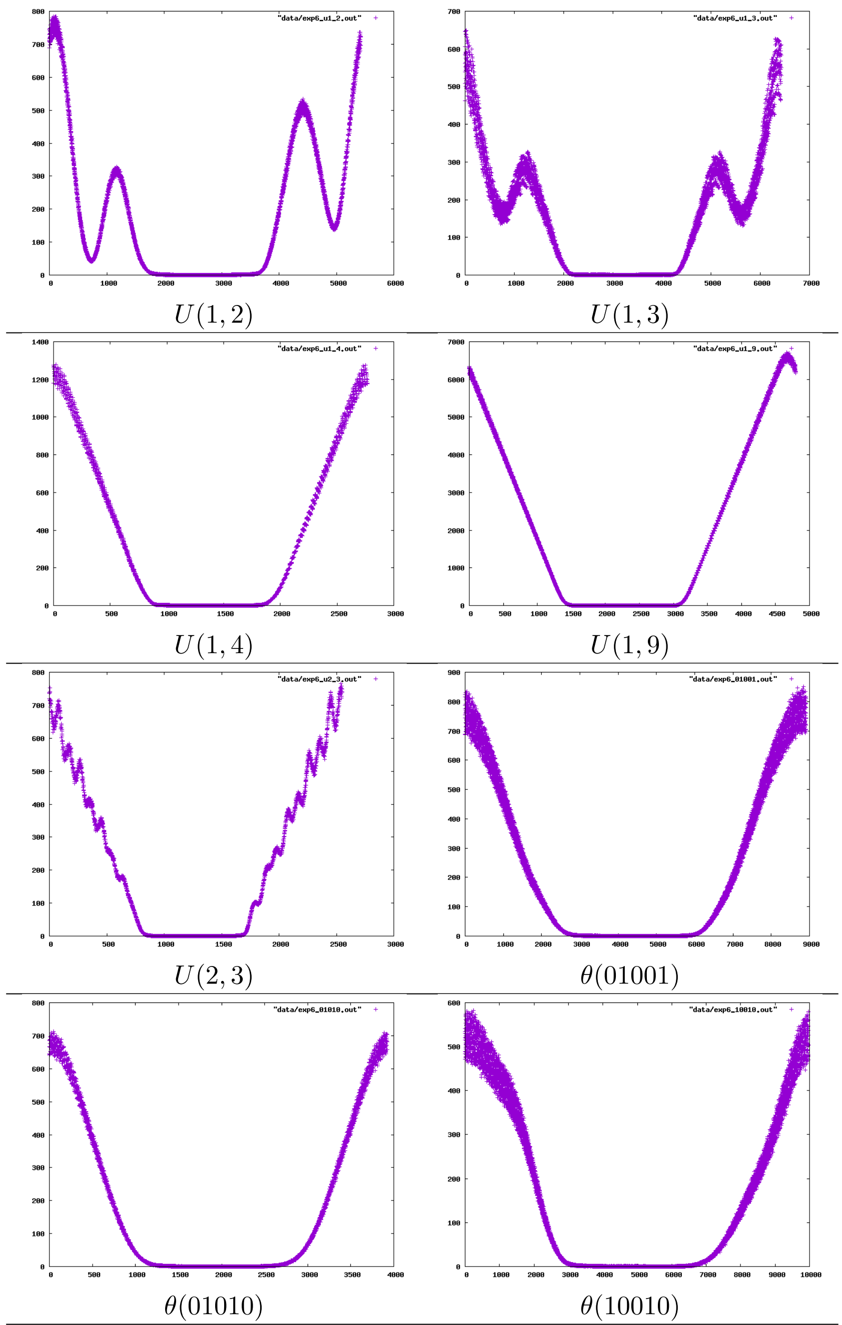
r^*_{A+A} is less clean, but we can write, $2r^*_{A+A}(x) = r_{A+A}(x) - A(x/2)$. So if we get bounds on $r_{A+A}(x)$, we can expect to transfer these to bounds of $r^*_{A+A}(x)$. Thus we expect that understanding r_{A+A} will be sufficient to allow us to understand Ulam sequences as well as sum-free sets.

So if our assertion is that whether a given x is in A is strongly affected by the value of x modulo the associated λ_A, then we might wonder how $r_{A+A}(x)$ depends on the value of x modulo λ_A. We can do this by taking the various congruence classes a modulo λ and computing the average value of $r_{A+A}(x)$ for x ranging over integers congruent to a modulo λ. To do this in practice, we use a rational approximation to λ. Plotting these for the first 10^4 terms of various sequences we get the plots in figure 6.2.

The general pattern here seems to be that all these distributions are large near 0, all close to 0 within the middle third of the interval, and have some substantial fluctuations in between, from the relatively extreme example of $U(2,3)$, to the more sedate examples provided by the sum-free sets, or by $U(1,4)$.

These two observations would suggest that near 0, we should not have any elements of these sequences, and that most of the elements should lie in the middle third. Further, for Ulam sequences, this also suggests that any x that fails to be in the sequence because $r^*_{A+A}(x) = 0$ should live in the middle third

Figure 6.2: Plots of r_{A+A} modulo λ_A for various A



as well, possibly explaining the gap in the distribution of elements of $U(1,2)$ in the very middle.

We shall address these observations in the following sections, but we give a word first about the strategy: The benefit of studying $r_{A+A}(x)$ is that, being a convolution, we understand its Fourier transform. More precisely, truncate A at N and view (as before) $r_{A_N+A_N}(x)$ and $r_{A_N-A_N}(x)$ as functions on $\mathbb{Z}/2N$. Then by the convolution theorem (in proposition 2.1.3), we have $(\mathscr{F}_{2N} r_{A_N+A_N}) = (\mathscr{F}_{2N} A_N)^2$ and $(\mathscr{F}_{2N} r_{A_N-A_N}) = |\mathscr{F}_{2N} A_N|^2$. Using Fourier inversion, then, we can compute $r_{A_N+A_N}$ and $r_{A_N-A_N}$ in terms of the Fourier coefficients of A:

$$r_{A_N+A_N}(x) = 2N \sum_{t=0}^{2N-1} ((\mathscr{F}_{2N} A)(t))^2 e(\frac{2\pi t}{2N} x) \tag{6.1.1}$$

$$r_{A_N-A_N}(x) = 2N \sum_{t=0}^{2N-1} |(\mathscr{F}_{2N} A)(t)|^2 e(\frac{2\pi t}{2N} x) \tag{6.1.2}$$

Since our observations suggest that the spectrum of A is simply $\alpha\mathbb{Z}$, in particular for A_N we have $\widehat{A_N}(k\alpha) \leq \frac{C}{k}$ for k up to around $c\sqrt{N}$ for some constant c, and we have lower bounds for $(\mathscr{F}_{2N} A)(\alpha)$. We can then approximate this sum using these large Fourier coefficients and lump the rest into an error term R_N that we expect (but were unable to prove) is small enough to ignore. Precisely :

Conjecture 6.1.2. *For A a almost sum-free set with Fourier spectrum $\alpha\mathbb{Z}$, there are constants $c, c' > 0$ such that:*

$$r_{A_N-A_N}(x) = 2N \sum_{|k|<c\sqrt{N}} |(\mathscr{F}_{2N} A)(k\alpha)|^2 e(k\alpha x) + R_N \tag{6.1.3}$$

$$r_{A_N+A_N}(x) = 2N \sum_{|k|<c'\sqrt{N}} (\mathscr{F}_{2N} A)^2(k\alpha) e(k\alpha x) + R'_N \tag{6.1.4}$$

Where $|R_N|$ and $|R'_N|$ are both $o(N)$ as $N \to \infty$.

Computational evidence. We compute in `experiment7B` the values of

$$R'_N = r_{A_N+A_N}(x) - 2N \sum_{|k|<\sqrt{2N}} (\mathscr{F}_{2N} A)^2(k\alpha) e(k\alpha x)$$

for various random values of x and each of our various sequences A. We let $\widetilde{R}_N$ be the maximum value we see for this quantity over all the x we sample, and put into table 6.1. Already with the relatively conservative (relative to observed values of d_N) value of $c' = \sqrt{2}$, we find these to be small relative to N, suggesting that with careful tuning of this parameter we could estimate $r_{A+A}(x)$ as accurately as needed.

□

Table 6.1: Estimation of error in using $\alpha\mathbb{Z}$ to compute $r_{A+A}(x)$

A	N	$\widetilde{R}_N$	$\frac{\widetilde{R}_N}{N}$
$U(1,2)$	24588	-52.134	-0.0021203
$U(1,2)$	51022	-113.373	-0.00222204
$U(1,2)$	130434	-221.677	-0.00169953
$U(1,2)$	197476	-341.557	-0.00172961
$U(1,3)$	16256	54.32	0.00334154
$U(1,3)$	31184	97.948	0.00314097
$U(1,3)$	78328	99.275	0.00126743
$U(1,3)$	78346	99.417	0.00126895
$U(1,4)$	12480	44.821	0.00359143
$U(1,4)$	25046	-44.349	-0.0017707
$U(1,4)$	63338	-73.886	-0.00116654
$U(1,4)$	93412	141.062	0.00151011
$U(1,9)$	12034	-137.569	-0.0114317
$U(1,9)$	23544	91.041	0.00386685
$U(1,9)$	59084	161.87	0.00273966
$U(1,9)$	89730	-250.078	-0.00278701
$U(2,3)$	18104	-40.538	-0.00223917
$U(2,3)$	39614	-130.978	-0.00330636
$U(2,3)$	103858	-157.645	-0.00151789
$U(2,3)$	160638	-273.753	-0.00170416
$\theta(01001)$	20100	32.055	0.00159478
$\theta(01001)$	40788	-70.934	-0.00173909
$\theta(01001)$	102738	-118.029	-0.00114883
$\theta(01001)$	154408	-128.859	-0.000834536
$\theta(01010)$	20606	-38.63	-0.0018747
$\theta(01010)$	41374	-43.912	-0.00106134
$\theta(01010)$	103584	-80.162	-0.000773884
$\theta(01010)$	155298	-89.855	-0.000578597
$\theta(10010)$	24648	-52.342	-0.00212358
$\theta(10010)$	49930	-52.571	-0.00105289
$\theta(10010)$	124482	124.096	0.000996899
$\theta(10010)$	186906	-208.604	-0.00111609

This is what we shall attempt to exploit in the following section, but before then, a word about the actual distributions of r_{A+A} that we plotted above:

Proposition 6.1.3. *Let A be one of our almost sum-free sets, and let $\frac{m}{k}$ be a rational approximation to λ_A, so α is approximated by $\widetilde{\alpha} = \frac{2\pi k}{m}$. Let $f_N(x)$ be the function from $\mathbb{Z}/m \to \mathbb{R}$ that averages the values of $r_{A_N+A_N}(t)$ for $t = x$ mod λ, i.e. $kt = x \mod m$.*

That is,

$$f_N(x) = \frac{m}{N} \sum_{kt=x\ (m), t<N} r_{A+A}(t)$$

Then we can express:

$$f_N(x) = \sum_{l=0}^{m-1} e(\frac{-2\pi xl}{m})\widehat{A_N}(l\widetilde{\alpha})^2$$

Proof. This follows by the usual trick of expressing the indicator function of a congruence class mod m using an exponential sum. In our case:

$$1(kt = x \text{ mod } m) = \frac{1}{m}\sum_{l=0}^{m-1} e(\frac{2\pi(kt-x)}{m}l)$$

So if we simply do this substitution and reverse the order of summation, we get:

$$\begin{aligned}
f_N(x) &= \frac{m}{N} \sum_{kt=x\ (m), t<N} r_{A+A}(t) \\
&= \frac{m}{N}\sum_{t=0}^{N-1} r_{A+A}(t)1(kt = x \text{ mod } m) \\
&= \frac{m}{N}\sum_{t=0}^{N-1} r_{A+A}(t)\frac{1}{m}\sum_{l=0}^{m-1} e(\frac{2\pi(kt-x)}{m}l) \\
&= \sum_{l=0}^{m-1}\frac{1}{N}\sum_{t=0}^{N-1} r_{A+A}(t)e(\frac{2\pi(kt-x)}{m}l) \\
&= \sum_{l=0}^{m-1} e(\frac{-2\pi xl}{m})\frac{1}{N}\sum_{t=0}^{N-1} r_{A+A}(t)e(\frac{2\pi klt}{m}) \\
&= \sum_{l=0}^{m-1} e(\frac{-2\pi xl}{m})\widehat{r_{A+A}}(l\widetilde{\alpha}) \\
&= \sum_{l=0}^{m-1} e(\frac{-2\pi xl}{m})\widehat{A_N}(l\widetilde{\alpha})^2
\end{aligned}$$

□

Thus if we had more precise knowledge of the argument of $\widehat{A_N}(\alpha)$ we could perhaps deduce the features we want. For example, when x is close to zero, this sum would be dominated by the first two terms and so we should be able to show that it is large. Or when x is close to $\frac{m}{2}$, this sum should be small since it will then be an alternating sum.

6.2 Distribution mod λ

As mentioned, the distributions of various $U(a, b)$ and $\theta(s)$ modulo their respective λ values are non-uniform, and seem to have most (but not all) of their support in the middle third of the interval $[0, \lambda]$. For example, if we take the first 10^4 elements of each sequence and take a rational approximation $\lambda \approx \frac{m}{k}$ and plot, for each congruence class r mod m how many $a \in A$ have $ka \cong r$ mod m, then we get the plots in figure 6.3

There are many observations to be made, among which:

1. There seem to be few elements of each A in some interval around 0 modulo λ.
2. The support of these distributions seems to usually contain the middle third modulo λ.
3. The distribution of $U(1, n)$ seems like it might be converging to a multiple of the characteristic function of the middle third modulo λ.

Of these, we will examine in this section the first two.

As a useful piece of notation for expressing the idea of intervals such as "the middle third modulo λ", we will define as is standard:

Definition 6.2.1. *For $x \in \mathbb{R}$, define $\|x\|_{\mathbb{R}/\mathbb{Z}}$ to be the minimal distance between x and an integer: $\|x\|_{\mathbb{R}/\mathbb{Z}} = \min_{n\in\mathbb{Z}} |x - n|$.*

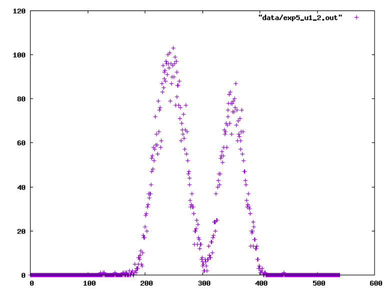
Thus $\|x\|_{\mathbb{R}/\mathbb{Z}} \leq \frac{1}{2}$ always, and x is in the "middle third", i.e $\left(\frac{\lambda}{3}, \frac{2\lambda}{3}\right)$ mod λ precisely if $\|x/\lambda\|_{\mathbb{R}/\mathbb{Z}} > \frac{1}{3}$. Likewise, an interval, (say of radius η) around zero (mod λ) is expressed by the condition $\|x/\lambda\|_{\mathbb{R}/\mathbb{Z}} < \eta$.

6.2.1 No elements close to 0 mod λ

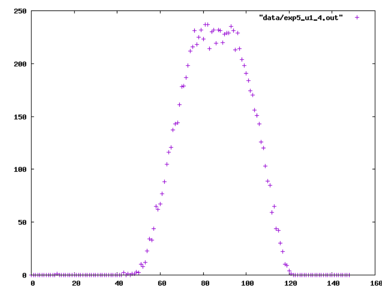
As promised, we move first to investigate why it appears that there are no elements of A close to integer multiples of λ. The distributions of r_{A+A} would suggest that the reason is that x that are close to $\lambda\mathbb{Z}$ have $r_{A+A}(x)$ large.

As we saw in equation 6.1.1, we can express this in terms of the spectrum of A, and hopefully can therefore extract some information. In particular, if $\|x/\lambda\|_{\mathbb{R}/\mathbb{Z}} < \eta$, then for $\frac{m}{2N} \approx \lambda$, $e(\frac{2\pi k}{2N}x)$ will be on the arc between $e(-\eta)$ and $e(\eta)$, making it very close to 1. So, particularly if we use the approximation given in equation 6.1.4, we get:

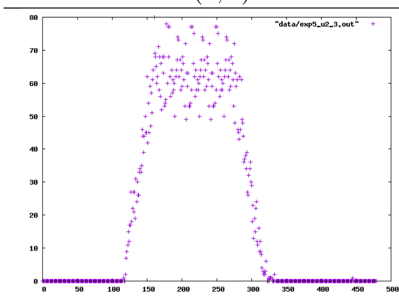
Figure 6.3: Distributions of various A modulo λ_A



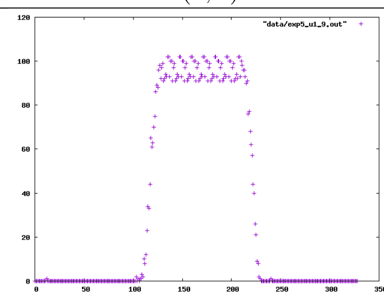
$U(1,2)$



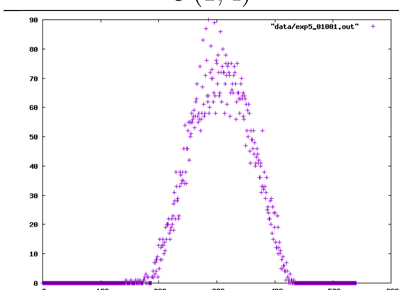
$U(1,3)$



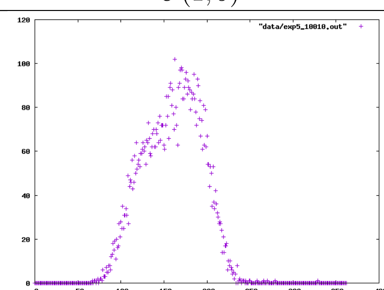
$U(1,4)$



$U(1,9)$



$\theta(01010)$



$\theta(10010)$

$$
\begin{aligned}
r_{A_N+A_N}(x) &= 2N \sum_{|k|<c\sqrt{N}} (\mathscr{F}_{2N}A)(k\alpha)^2 e(k\alpha x) + R_N \\
&\approx 2N \left(\frac{\delta^2}{2} + 2\Re((\mathscr{F}_{2N}A)(\alpha)^2 e(\eta)) + \sum_{k>1} 2\Re((\mathscr{F}_{2N}A)(k\alpha)^2 e(k\eta)) \right) + R_N
\end{aligned}
$$

And so if the argument of $(\mathscr{F}_{2N}A)(\alpha)$ is close to π and if η is small enough, then hopefully the term we've pulled out of the sum dominates this expression and we get that $r_{A_N+A_N}(x) = \Omega(N)$. But this requires very precise control over the argument of $(\mathscr{F}_{2N}A)(k\alpha)$ that we do not currently have.

However, at least for truly sum-free sets, we can still pull out a theorem (as always, conditional on various conjectures about the spectrum) of this kind by using the fact that for a sum-free set, an element can be in the set A only if $r_{A-A}(x) = 0$ also. And, recalling equations 6.1.1 and 6.1.4, this we can compute using only the magnitudes of the Fourier coefficients, allowing us to ignore the argument issue completely:

Theorem 6.2.2. *Let A be an almost sum-free set of positive integers with $|A_N| = \delta N$. Suppose that A has spectrum $\alpha\mathbb{Z}$, that it satisfes conjecture 5.2.1, and that the error in equation 6.1.4 in fact satisfies $|R_N| = o(N)$ (conjecture 6.1.2). Then for some $\eta > 0$, $\|x/\lambda\| < \eta$ implies $r_{A_N-A_N}(x) = \Omega(N)$.*

Proof. Take x with $\|x/\lambda\|_{\mathbb{R}/\mathbb{Z}} < \eta$ for η to be chosen later. Viewing A_N as usual inside $\mathbb{Z}/2N$, we have $x \in A_N - A_N$ viewed inside $\mathbb{Z}/2N$ iff $x \in A_N - A_N$ in $\mathbb{Z}$. So we can use equation 6.1.4. In preparation for doing this, let $|\widehat{A_N}(\alpha)| = \rho$ (we know $\rho > \frac{\delta^2}{2} - O(1/N)$ by 5.1.2), and let k_0 be such that

$$
\delta^2 + \rho^2 - 4C^2 \sum_{k=k_0}^{\infty} \frac{1}{k^2} > 0
$$

(which we know is possible since $\delta > 0, \rho > 0$, and since the the sum converges, and so approaches 0 as $k_0 \to \infty$). Then if we choose η to guarantee that $\|k_0 x/\lambda\|_{\mathbb{R}/\mathbb{Z}} < \frac{1}{4}$ (so that $\Re(e(k\alpha x)) > 0$ for all $k < k_0$), then as N gets large, we have:

$$
\begin{aligned}
r_{A_N-A_N}(x) &= 2N \sum_{|k|<c\sqrt{N}} (\mathscr{F}_{2N}A)(k\alpha)^2 e(k\alpha x) + R_N \\
&\geq N\delta^2 + 2N|\widehat{A_N}(\alpha)|^2 \Re(e_N(\alpha x)) - 2N \sum_{|k|\geq k_0} |\widehat{A_N}(k\alpha)|^2 - |R_N|
\end{aligned}
$$

where here, we are using that $\Re(|\widehat{A_N}(k\alpha)|^2 e(k\alpha x)) \geq 0$ for $1 < k < k_0$ by choice of η, and that $\Re(|\widehat{A_N}(k\alpha)|^2 e(k\alpha x)) \geq -|\widehat{A_N}(k\alpha)|^2$ for $k \geq k_0$. So:

$$\begin{aligned} r_{A_N - A_N}(x) &\geq \frac{\delta^2}{2}N + \frac{\rho^2}{2}N - 2N \sum_{|k| \geq k_0} |\widehat{A_N}(k\alpha)|^2 - |R_N| \\ &\geq \frac{\delta^2}{2}N + \frac{\rho^2}{2}N - 2C^2 N \sum_{|k| \geq k_0} \frac{1}{k^2} - |R_N| \end{aligned}$$

Using 5.2.1 to conclude $|\widehat{A_N}(k\alpha)| \leq \frac{C}{k}$ for some constant C. Thus:

$$r_{A_N - A_N}(x) \geq \frac{N}{2}\left(\delta^2 + \rho^2 - 4C^2 \sum_{|k| \geq k_0} \frac{1}{k^2} - \frac{2|R_N|}{N}\right)$$

which we know by choice of k_0 that for N large enough (so that conjecture 6.1.2 can kick in to allow us to ignore the R_N term) the value in parentheses is positive, and so the whole expression is $\Omega(N)$, as desired. $\square$

For sum-free A, this automatically guarantees that no integer within this interval can be in A, as for sum-free sets, $r_{A_N - A_N}(x) > 0$ for any N already implies $x \notin A$:

Corollary 6.2.2.1. *If A is a sum-free set satisfying all the conditions of the theorem, then there is an $\eta > 0$ such that $\|x/\lambda_A\|_{\mathbb{R}/\mathbb{Z}} < \eta \implies x \notin A$.*

We can make an analogous statement for Ulam sequences, noting that if x is in an Ulam sequence A and $r_{A_N - A_N}(x)$ is also large, then this means that x is a summand for many elements of A:

Corollary 6.2.2.2. *If A is an Ulam sequence, then there is an $\eta > 0$ such that $\|x/\lambda_A\|_{\mathbb{R}/\mathbb{Z}} < \eta$ implies that x appears as a summand for elements of A_N $\Omega(N)$ times.*

We will see examples of this observation in section 7.

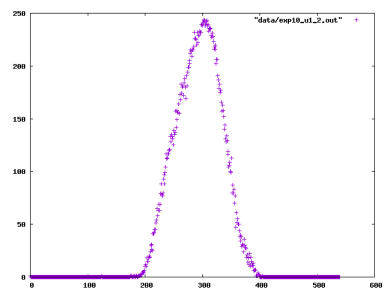
6.2.2 Numbers that are not sums of Ulam numbers close to middle mod λ

Another behaviour suggested by the distribution of r_{A+A} concerns non-Ulam numbers, and specifically numbers x that fail to be Ulam because $r^*_{A+A}(x) = 0$. The fact that it appears $r_{A+A}(x) > 0$ outside the middle third suggests that all of these would lie within the middle third.

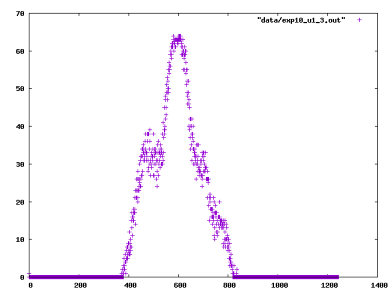
Indeed, if we take such numbers and plot their distribution mod λ, as we do in `experiment10`, we get figure 6.4.

These plots share many features with the distributions of the Ulam numbers modulo λ, bolstering the idea that it isn't the feature of "being an Ulam number" as such that creates the distribution, but simply the relationship between $r_{A+A}(x)$ and being in A. Specifically, for Ulam sequences A, x being in A is

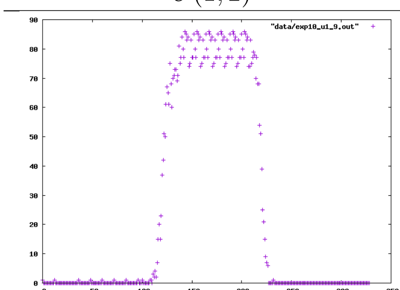
Figure 6.4: Numbers not in A nor in $A + A$ plotted modulo λ_A



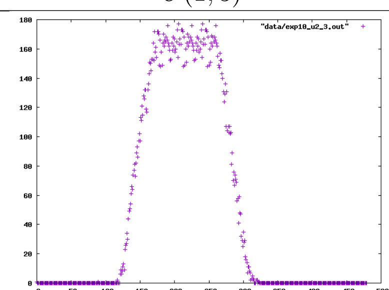
$U(1, 2)$



$U(1, 3)$



$U(1, 9)$



$U(2, 3)$

Table 6.2: Count of non-sums in A_N up to N

A	N (with $\|A_N\| = 10^4$)	$\|\{x < N : r^*_{A+A}(x) = 0\}\|$
$U(1,2)$	132788	24415
$U(1,3)$	78819	13445
$U(1,9)$	58114	7859
$U(2,3)$	108466	23052

related to $r_{A+A}(x)$ is small, whether it is 2 or 3 and so $x \in A$ or it is literally 0, and x is in the sets we're considering in this section, such x's still show the same bias modulo λ,

We end with a final remark that within the first 10^4 elements of each of these Ulam sequences, we have many elements that fail to qualify by having no representations, as enumerated in table 6.2.

6.3 Non-uniformity/Regularity

Without any unconditional result describing the distributions of our various A modulo λ_A, we might ask whether we can at least guarantee some kind of non-uniformity of these distributions. For example, can we find a set $E \subseteq \mathbb{R}/\mathbb{Z}$ such that if $\pi : Z \to \mathbb{R}/\mathbb{Z}$ is $x \mapsto x/\lambda \mod 1$, then $|A_N \cap \pi^{-1}(E)|/N$ is creater than what might be expected from the sizes of these sets alone, namely $\frac{|A_N|}{N}\frac{|\pi^{-1}(E)_N|}{N}$? In fact, we can do slightly better:

Theorem 6.3.1. *For A an almost sum-free set, and let α be the maximal Fourier coefficient, and define $E_t = \{n \in [N] : \Re(e(\alpha n)) \leq \eta\}$ (roughly, the set of integers that land in an interval of radius η centred at $\lambda/2$ modulo λ. Then there is a η such that $\langle A_N, E_{\eta,N}\rangle = |A_N \cap E_{\eta,N}| \geq \delta^2/4 + \delta\frac{|E_{\eta,N}|}{N}$.*

Proof. Let $f(t) = A(t) - \delta$ be the "balanced" indicator function of A. Then we know from 5.1.2.1 that $\frac{1}{N}\sum_{t=0}^{N-1} f(t)\Re(e(\alpha t)) \leq -\frac{\delta^2}{2}$. The key here is to write:

$$\Re(e(\alpha t)) = 1 - \int_{-1}^{1} E_\eta(t) d\eta$$

Then

$$\begin{aligned}
\frac{\delta^2}{2} &\leq -\frac{1}{N}\sum_{t=0}^{N-1} f(t)\Re(e(\alpha t)) \\
&= -\frac{1}{N}\sum_{t=0}^{N-1} f(t) + \frac{1}{N}\sum_{t=0}^{N-1}\int_{-1}^{1} f(t)E_\eta(t)d\eta \\
&= \frac{1}{N}\sum_{t=0}^{N-1}\int_{-1}^{1} f(t)E_\eta(t)d\eta \\
&= \int_{-1}^{1} \langle f, E_\eta\rangle \, d\eta
\end{aligned}$$

Thus $\langle f, E_\eta\rangle \geq \frac{\delta^2}{4}$ for some η. But $f = A - \delta$, so

$$\langle A, E_\eta\rangle \geq \frac{\delta^2}{4} + \langle \delta, E_\eta\rangle = \frac{\delta^2}{4} + \delta\frac{|E_{\eta,N}|}{N}$$

And this is what we wanted to show. □

So this is our first basic result in the direction we want to go, namely towards some statement that large almost sum-free sets are almost sum-free for local reasons. Another simpler example of this comes from thinking about regular such sequences:

Theorem 6.3.2. *If A is a regular 1-additive or sum-free set, then there is an $N > 0$, an m, and a sum-free $E \subseteq \mathbb{Z}/m$ such that if $\pi : \mathbb{Z} \to \mathbb{Z}/m$ is the quotient map, then A and E agree for all integers larger than N. In other words, such A eventually agree with a locally sum-free set.*

Proof. If A is regular, then there is an $N > 0$ and a modulus m with a list of congruence classes $a_1, \dots, a_n$ mod m such that for all $x > N$, $x \in A$ if and only if $x = a_i \mod m$ for some i.

If $E = \{a_1, \dots, a_n\} \subseteq \mathbb{Z}/m$ is not sum-free, then there are i, j, k with $a_i + a_j = a_k \mod m$. Supposing, as we may without loss, that $0 \leq a_r < m$ for all r, then $a_i + a_j = a_k + m\epsilon$ for $\epsilon = 0$ or $\epsilon = 1$.

Now, define five numbers x, y, z, w, c by $x = Nm + a_i$, $y = Nm + a_j$, $z = (N+1)m + a_i$, $w = (N+1)m + a_j$, and $c = (2N + 2 + \epsilon)m + a_k$. These are all distinct, since $0 \leq a_i, a_j, a_k < m$. They are also all in A, since they are in E modulo m, and are all greater than N. However, $x + w = c$ and $y + z = c$. So, c, an element of A, has two distinct representations as sums of distinct smaller elements of A, contradicting the 1-additivity of A.

Thus E had to have been sum-free, and then we have that for $x > N$, $x \in A \iff x \in \pi^{-1}(E)$, where $\pi : \mathbb{Z} \to \mathbb{Z}/m$ is the quotient map and $E \subseteq \mathbb{Z}/m$ is sum-free, as desired. □

Chapter 7

Structure of $U(1, 2)$

We now turn specifically to $A = U(1, 2)$, the original sequence of Ulam numbers. Much of the analysis in this section could be ported to other Ulam sequences, but we do not do so here.

7.1 Distribution of summands

We know that each element $a_n \in A$ is a sum $a_i + a_j$ of smaller elements with $a_i < a_j$. This gives us much structure to play with:

- For example, for each $x \in A$, let $S_x = \{y \in A : x + y \in A\}$. We might wonder about the sizes of S_x for various x–is it roughly constant among all x, or are there very few x that act as summands for elements of A?
- If we write each $a_n = a_i + a_j$ where $i < j$, what is the distribution of the values of i that show up? (That is, what is the distribution of small summands?) What about the values of j? Perhaps more reasonable would be to ask about the distribution of $n - j$ (so a question about the distribution of large summands).
- If we start by breaking up an $a \in A$ as $a = x + y$ for $x < y$ both in A, we can then break up x and y themselves into sums of Ulam numbers, and repeat until we get down to writing a as a sum of 1s and 2s. So we can wonder about the proportion of 1s and 2s in this factorisation.

7.1.1 Distribution of large summands

We note first that if 2 or 3 is the small summand of a_{n+1}, and if $a_n > 6$, then the large summand is necesarily a_n (if 2 is the small summand and a_n is not the large summand, then a_{n+1} would be $a_n + 1$ which is impossible since this would mean $a_n + a_1 = a_{n-1} + a_2$, which violates 1-additivity. If 3 is the small summand and a_n is not the large summand, then $a_{n+1} = a_{n-1} + 3$ or $a_{n+1} = a_{n-2} + 3$, either way giving $a_{n+1} = a_n + 1$ or $a_{n+1} = a_n + 2$, either of which would give an honestly distinct (since $a_n > 6$) second way of representing a_{n+1} as a sum of smaller Ulam numbers, again violating 1-additivity.)

This means that as often as 2 or 3 is the small summand, (which we will see is a large proportion of the time), the large summand will be the last thing in the list so far. When looking at the large summand, then, it seems like it may be interesting to consider how many indices from the end it lives, rather than its actual value. That is, for small summand a_i, we should consider for what values of j is $a_n = a_i + a_{n-j}$. We compute these in `experiment13`. Some post-processing of the output gives us table 7.1, which accounts for over 9000 of the a_n for $n \leq 10^4$.

All told there are 312 different pairs $(i, n-j)$ that appear in the first 10^4 Ulam numbers, including only 69 distinct values of i and 159 values distinct values of $n - j$.

Note in particular that there being only 312 distinct such pairs means that technically, to compute the first 10000 Ulam numbers, we would only have to check 312 possibilities for each, if we somehow knew which possibilities to check ahead of time.

7.1.2 Distribution of small summands

We note with interest the observation of Steinerberger [19] that $\cos(\alpha a_i) < 0$ for all a_i other than 2, 3, 47, and 69. In particular, these were also the a_i that showed up most frequently as summands in our earlier computation. We also note that $\cos(\alpha a_i) < 0$ is just the condition that $\|x/\lambda\|_{\mathbb{R}/\mathbb{Z}} < \frac{1}{4}$.

So we compute which how often each a appears as the smaller summand of a later a_i (that is, we compute $|S_a|$) and we compute $\|a/\lambda\|_{\mathbb{R}/\mathbb{Z}}$ for each and sort by this quantity. We note what looks like a very strong correlation between how often a shows up as a summand and $\|a/\lambda\|_{\mathbb{R}/\mathbb{Z}}$ in the resulting table 7.2, computed by `experiment12`.

7.1.3 Distribution of complements

In the cases Steinerberger looks at, the resulting non-uniform distributions consist usually of multiple peaks. In the case of $U(1, 2)$, one of these peaks looks a little misshapen, so we might reasonably wonder what each of these peaks actually is.

To get a handle on this, we take the Ulam sequence mod 5422, and multiply it by 2219 (5422/2219 being a good rational approximation to λ). Of course, this gives rise to the usual distribution we've come to expect in figure 7.1

Supposing we look instead only at a_n's for which 2, say is a summand–that is, the distribution of S_2. Then we get the nice picture in figure 7.2, and likewise for S_{47}, say, in figure 7.3.

These are relatively clean-looking distributions, by comparison. If we plot these graphs for all of the top 25 most common summands all in one picture,

Table 7.1: Large summands

$\lvert n : a_i + a_{n-j} = a_n \rvert$	a_i	i	$n - j$
3630	2	2	1
1356	3	3	1
382	47	15	3
266	69	20	5
192	69	20	6
178	47	15	4
176	47	15	6
163	47	15	5
161	36	13	4
161	102	27	9
146	47	15	2
138	69	20	4
137	102	27	7
136	102	27	10
121	69	20	3
119	102	27	8
113	36	13	2
111	8	6	1
109	69	20	7
89	47	15	1
85	36	13	3
79	36	13	5
70	8	6	2
69	102	27	11
65	102	27	6
63	339	59	25
62	339	59	26
61	69	20	8
51	339	59	27
49	339	59	24
48	339	59	28
47	339	59	29
44	102	27	5
43	47	15	7
43	339	59	30
42	339	59	23
39	69	20	1
39	339	59	22
39	273	53	23
39	102	27	4
37	273	53	20
33	69	20	2

Table 7.2: Small summands

a (with $\lvert S_a\rvert > 10$)	$\lvert S_a\rvert$	$\lVert a/\lambda\rVert_{\mathbb{R}/\mathbb{Z}}$
2	3631	0.18148283910656193
3	1357	0.2277757413401571
47	1192	0.2351532809957959
69	1006	0.23884205082361376
102	840	0.25562479443465946
339	592	0.2613412285622303
36	468	0.2666911039181148
273	307	0.27240753804569806
8	182	0.27406864357375227
2581	93	0.296396132981954
400	60	0.29656782131237946
983	54	0.301184579124822
97	49	0.3019176966682551
356	25	0.30394536096801517
1155	25	0.306339584039506
206	36	0.3072675720241307
53	37	0.3092952363238908
1308	22	0.31022322430851546
9193	19	0.31413004668820577
14892	10	0.32121998745969904

Figure 7.1: Distribution of $U(1,2)$ modulo λ

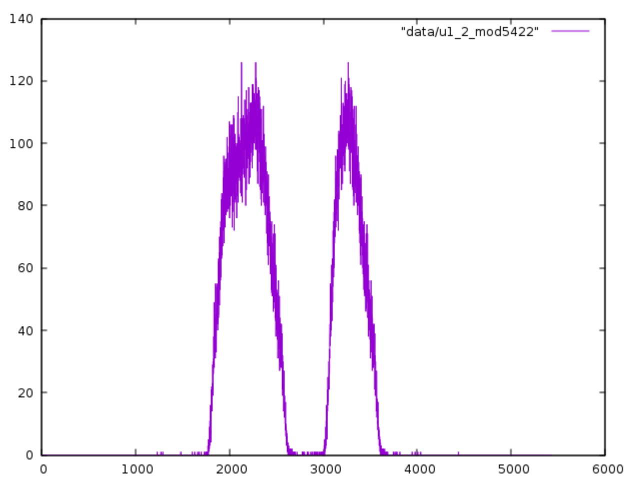


Figure 7.2: Distribution of S_2

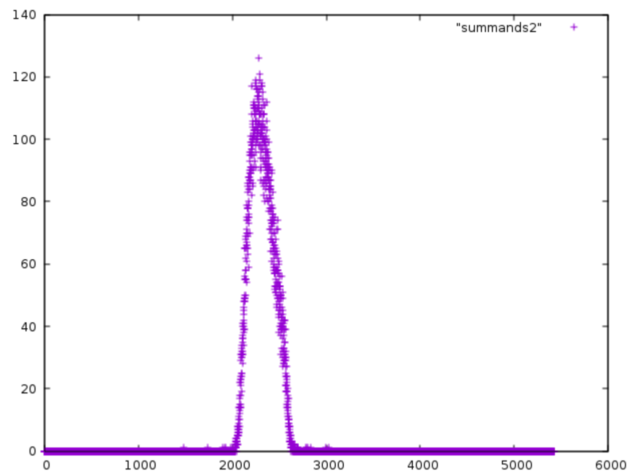


Figure 7.3: Distribution of S_{47}

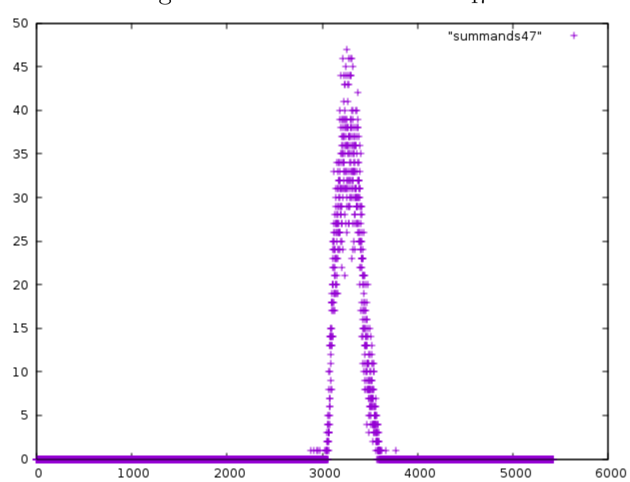
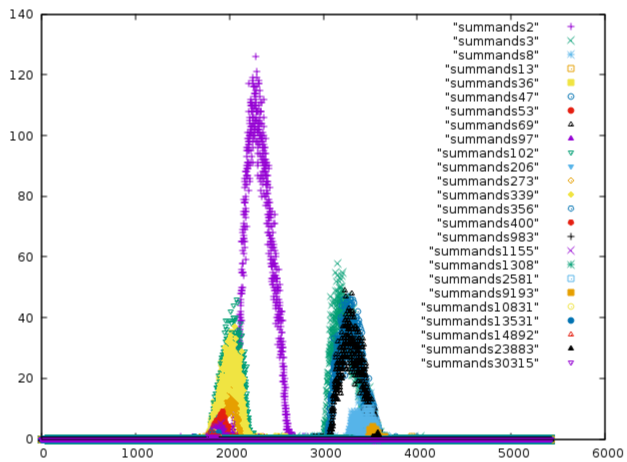


Figure 7.4: Distribution of all S_a for small summands a



we notice that these seem to be the components of the two observed peaks, illustrated in figure 7.4.

Since each of these seems to be instances of the same distribution with different parameters, we might be interested in computing the parameters of each, starting with the means. This gives us table 7.3.

Staring at that table for a minute, we notice that if we subtract the second column from the third, we seen to get roughly 2000 for the first 11 entries (those on the right end of the distribution). Likewise, those on the left end (rows 12-25) seem to have a similar pattern.

One possible reason for this is that the distribution we're taking the mean of in the first row, say, is of $2219a_n \mod 5422$ where 3 is a summand of a_n in the Ulam sequence. Since 3 is a summand of a_n in the sequence, we might instead look at the other summand of a_n, i.e. $a_n - 3$. This would lead to us not plotting $2219a_n \mod 5422$, but rather $2219(a_n - 3) \mod 5422$. We can compute these quickly and if we plot these, we get the plot in figure 7.5. That is, for each small summand a, we are plotting a histogram for the set of "complements" of a:

$$C_a = \{b \in A : a + b \in A\}$$

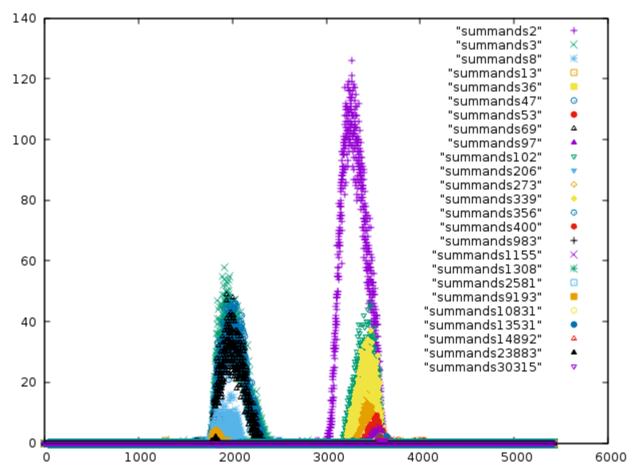
7.2 Density

One simple element of structure that we might hope to deduce from all of this is some kind of positive density result for sum-free and Ulam sequences. For example, we note that for some large proportion of all Ulam numbers x, then next Ulam number will be $x + 2$. For another slightly smaller proportion, the next will be $x + 3$. So one approach to studying the density would be to

Table 7.3: Means of complements of $x \in U(1,2)$

a	$2219a \mod 5422$	Mean of $2219S_a \mod 5422$
3	1235	3241.078
47	1275	3288.007
69	1295	3300.945
8	1486	3431.555
2581	1607	3485.878
983	1633	3503.280
206	1666	3518.475
1308	1682	3525.956
9193	1703	3541.352
13	1737	3551.591
23883	1749	3572.533
30315	3653	1818.700
13531	3675	1827.600
14892	3680	1833.363
10831	3685	1845.629
53	3745	1872.413
1155	3761	1883.377
356	3774	1878.850
97	3785	1891.297
400	3814	1912.785
273	3945	1984.791
36	3976	1995.708
339	4005	2013.333
102	4036	2027.299
2	4438	2319.242

Figure 7.5: Distribution of $S_a - a$



understand the structure of the Ulam numbers well enough to be able to bound how often any given d appears as a difference of consecutive Ulam numbers.

In this document we do not resolve this question either, but do give conjectures in this direction. However, one may note that the only reason we know other Ulam sequences such as $U(2, 5)$ have positive density is that we know they are regular. So in trying to get a handle on the distribution of, say, $U(1, 2)$, we expect that any actual positive density result will come from an understanding of its distribution, rather than vice-versa.

The first question is to attempt to compute the density of the various sum-free sets and Ulam sequences that we are studying.

7.2.1 Computations

The prescribed method for computing sum-free sets from a decision sequence is already decently fast, provided we keep track of the data appropriately. Recall, the approach is to track three sets: A, the actual sum-free set we are constructing, D, the set $A + A$ of things that are "disqualified" from appearing in A, and E, the set of things that are not disqualified by virtue of being sums but that, according to the decision sequence, we are nevertheless to exclude.

We thus store A as a list and D as a hashset, (and need not keep track of E). So the algorithm is to, for each x starting from 1 until we get bored:

1. Check if $x \in D$ (very fast, as D is a hashset).
2. If not, pop an item off the decision sequence.
3. If 1, append x in A and add $x + A$ to D ($|A|$ steps).

So this algorithm to compute the first N items will take $O(N \cdot |A_N|)$ steps.

A similar algorithm can be implemented for the Ulam sequence, in fact: Now, we track the set A as a list, the hashset D of disqualified items (i.e. x for which $r^*_{A+A}(x) > 1$), the sorted list C of candidates (i.e. x bigger than every element of A with $r^*_{A+A}(x) = 1$), and a hashset C' that also contains the candidates. Then the algorithm is to initialise the following:

1. $A = [a, b]$
2. $C = [a + b]$
3. $C' = \{a + b\}$

And proceed thus:

1. Delete any inital elements of C that are smaller than the largest element of A. As we go, delete these elements from C' also.
2. Let x be the first element of C, and append x to A.

3. For each $a \in A$: Compute $x + a$ and, if it is in C', delete it from C' (fast, since C' is a hashset) and from C (where we can find it by bisection, since C is sorted).

There is a more advanced algorithm that leverages the apparent bias of such sequences as well, implemented in [13]. This speeds the basic algorithm up by, among other things, noting that if we track which elements of A are outside the middle third mod λ for a λ where there are few such elements, then when we're testing whether any new x within the middle third is actually a sum of smaller elements of A, we only have to look at whether $x - a$ is in A where a is one of the (hopefully few) elements of A outside the middle third.

In any case, the results of our computations give the estimates for the densities of these sets found in label 7.4.

Table 7.4: Densities

A	$\lvert A_N \rvert$	δ_N
$U(1,2)$	1000000	$0.07402 = \frac{1}{13.50907}$
$U(1,3)$	10000	$0.12687 = \frac{1}{7.8819}$
$U(1,4)$	20000	$0.15846 = \frac{1}{6.31065}$
$U(1,9)$	50000	$0.16909 = \frac{1}{5.91408}$
$U(2,3)$	10000	$0.09219 = \frac{1}{10.8466}$
01001	40000	$0.09686 = \frac{1}{10.32415}$
01010	8000	$0.09661 = \frac{1}{10.35037}$
10010	8000	$0.08023 = \frac{1}{12.46425}$

7.2.2 Constructions

Another line of thought is to note that that the Ulam numbers are, in some sense, as greedy as possible in their definition. And while, for example, $\theta(01001)$ is not maximally greedy, it is still greedy $\frac{2}{5}$ of the time. So in the family of Ulam-like sets or sum-free sets, if we have many positive-density examples, it seems unlikely (though not impossible, as we shall see) that these very greedy sets fail to be as high a density as possible.

We first start by noting that positive-density sum-free sets are abundant, as a result of the abundance of sum-free sets $A \subseteq \mathbb{R}/\mathbb{Z}$, coupled with the fact that if $\pi_\lambda : \mathbb{Z}^+ \to \mathbb{R}/\mathbb{Z}$ by $x \mapsto \frac{x}{\lambda} \mod 1$, the inverse image $\pi_\lambda^{-1}(A)$ is sum-free in the integers. For example, the set $A = \{1/2\}$ is sum-free in $\mathbb{R}/\mathbb{Z}$, and $\pi_2^{-1}(A)$ is the odd positive integers, which is sum-free. Likewise, the A_λ from earlier (for any irrational λ), where recall A_λ was the set of integers that, when reduced (in $\mathbb{R}$) modulo λ, land in the interval $(\frac{\lambda}{3}, \frac{2\lambda}{3})$, are also of this form. So this kind of example gives many sum-free sets, both regular and irregular, that all have positive density.

One might wonder whether we can similarly generate examples of Ulam-like sets of positive density. It turns out that one can do this using the basic idea behind the A_λ construction, but being more careful about it. But first, we will make precise what we mean by "Ulam-like":

Recall an Ulam sequence is an increasing sequence of positive integers that starts with some a and b and that continues by choosing integers according to the requirements of 1-additivity ("every element is uniquely a sum of previous elements") and greediness ("always choose the smallest such element available"). In some ways, it is the greediness that makes Ulam sequences hard to analyse. If we drop this condition, then we get a general class of sequences which contains the Ulam sequences, but also many others:

Definition 7.2.1. *For $S \subseteq \mathbb{Z}^+$ a finite set (say of size k), a* ***1-additive sequence with base*** *S is an infinite sequence of positive integers a_i such that $a_1 < \ldots < a_k$ are the elements of S, and, for $n > k$, a_n is greater than a_{n-1} and has a unique pair of integers i, j with $0 < i < j < n$ such that $a_n = a_i + a_j$.*

We may talk of simply a ***1-additive sequence****, by which we will mean a sequence of integers that is a 1-additive sequence with base S for some S.*

Example 6.

1. *Any Ulam sequence $U(a, b)$ is 1-additive with base a, b.*
2. *The Fibonacci numbers $1, 2, 3, 5, 8, \ldots$ are a 1-additive sequence with base $1, 2$.*
3. *The set $\{2, 3, 5, 7, 9, \ldots\} = 2, 2 + 3\mathbb{Z}^+$ is 1-additive with base $2, 3$.*
4. *More generally, for any $a < b$, the set $a, b, b + a, b + 2a, \ldots$ is 1-additive provided $b \neq 0 \mod a$.*

The last example gives plenty of examples of regular 1-additive sequences. (Indeed, $a = 2, b = 3$ provides an example of a set with density apparently higher than that of $U(2, 3)$ despite being less greedy: It goes $2, 3, 5, 7, 9, 11, \ldots$, whereas a greedy algorithm would include 8 as well. Nevertheless, $U(2, 3)$ appears to have density around $\frac{1}{10}$, whereas this less greedy set has density $\frac{1}{2}$.)

Nevertheless, all these examples are regular in the conventional "mod-m" sense. So we might wonder what the analogue of A_λ for *irrational* λ would be for 1-additive sets. Or, more simply, we might first ask whether there even is such a thing as an irregular 1-additive set of positive density. It turns out that there is:

Proposition 7.2.2. *There exists an irregular 1-additive set of positive density.*

The basic idea will be the following "ping-pong mod λ" construction: Take an irrational λ. Take a (the "left bat") just above 0 mod λ, and b (the "right bat") just below 0 mod λ, and a c (the "ball") just above $\lambda/3$ mod λ. The game will be to keep the ball in the set $T = (\frac{\lambda}{3}, \frac{2\lambda}{3})$ (the "table") mod λ by adding a

to it until it reaches the right side, then adding b to it until it reaches the left side, and repeating indefinitely.

Supposing we are a little careful about our choices of a, b, c, and λ, we should be able to show that this construction satisfies the required properties.

Proof. More precisely: start with an irrational λ, an a with $a \mod \lambda$ lying in $(0, \frac{\lambda}{12})$, and a $b > a$ with $b \mod \lambda$ in $(\frac{11\lambda}{12}, \lambda)$, and a $c > b$ with $c \mod \lambda$ in $(\frac{\lambda}{3}, \frac{\lambda}{2})$, and say further that $a \nmid b$ (for reasons that will become apparent later). Also, let $T = (\frac{\lambda}{3}, \frac{2\lambda}{3})$.

Then we will define a sequence of sets A_n which will comprise the sequence A, as follows: Let $c_1 = c$. For $n \geq 1$, let $A_n = \{c_n + ka : k \in \mathbb{Z}^+, c + ka \mod \lambda \in T\}$ if n is odd, and $A_n = \{c_n + kb : k \in \mathbb{Z}^+, c + ka \mod \lambda \in T\}$ if n even. Let c_{n+1} be the largest element of A_n (allowing the definition of A_{n+1} to make sense).

Then let $S = \{a, b, c\}$ and $A = S \cup \bigcup_{i=1}^{\infty} A_i$.

Now let us check the definition: Our base set is S. Every element a_i has $a_i < a_{i-1} + a + b$, so the set is has positive density. A is not dense mod λ (since all but three elements are in T) whereas any regular set would have to be (since λ is irrational), so A cannot be regular. It remains to check 1-additivity.

By construction, every element $a_n > c$ is either $a_{n-1} + a$ or $a_{n-1} + b$, so every element not in S is a sum of smaller elements in at least one way. Now suppose a_n is a sum of smaller elements in another way also. Then because a_n is in T, it cannot be a sum of two other elements in the middle third. Thus the only for a_n to be a sum of other elements of A in two different ways is if $a_n - a$ and $a_n - b$ are both in A.

But now, say $a_n = c + ax + by$, for some $x \geq 0, y \geq 0$. We know that $a_n \in A_i$ for some i. If i is even, then by definition, the previous element of A will be $a_n - b$, and $a_n - b < a_n - a < a_n$, so $a_n - a$ cannot be in A.

If instead i is odd, then $a_n - a \in A$. Then say $a_{n-r} = \max A_{i-1}$, so $a_n = a_{n-r} + ra$ (and we know $r \geq 1$, since $a \in A_i$ for odd i). Thus $a_{n-r} = c + a(x-r) + by$. Then the element before this in A would be $c + a(x-r) + b(y-1) < a_n - b$. Thus for $a_n - b$ to be in A, it would have to be an element after $c + a(x-r) + b(y-1)$. But all these elements are got by adding a to a_{n-r}. So $a_n - b$ has the form $c + a(x-s) + by$ for $0 \leq s \leq r$. But $c + a(x-s) + by = c + ax + by - b$ implies $as = b$, which contradicts $a \nmid b$, which was our condition on a and b.

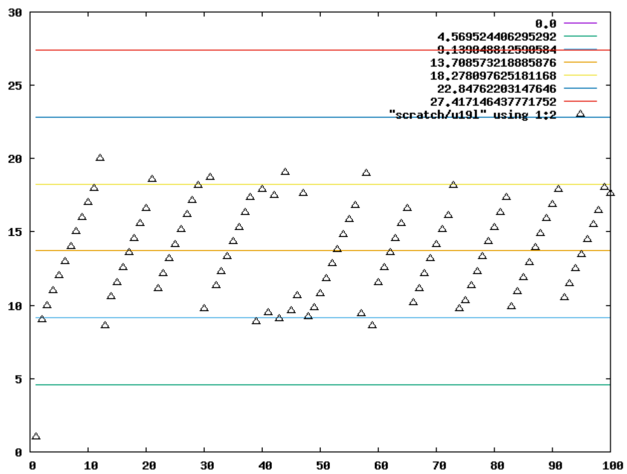
Thus $a_n - a$ and $a_n - b$ can never both be in A, making A also 1-additive. □

We note, however, that this construction gives us a 1-additive set with a base of size 3, whereas the Ulam numbers are a 1-additive set with a base of size 2. So we might wonder whether there is a similar (if more complicated) construction of this sort.

The following construction, we believe should work:

Conjecture 7.2.3. *There is an irregular, positive density 1-additive set with a base of size 2.*

Figure 7.6: $(n, a_n \mod \lambda)$ for $U(1,9)$ and $n \leq 100$



To try this, we will play the same game of ping-pong, but now we are only allowed to use two elements. We will again start with an irrational λ and $a \in \mathbb{Z}^+$ the "left bat" in the range $(\frac{\lambda}{6}, \frac{\lambda}{3}) \mod \lambda$. But now we will start with c being the "ball" inside the "table" $T = (\frac{\lambda}{3}, \frac{2\lambda}{3})$.

Then we will add a to c until it comes out on the right side as some $b \in (\frac{2\lambda}{3}, \frac{5\lambda}{6})$, and that will be our "right bat". Then we will do roughly the same thing as before, except now a and b have larger magnitude mod λ, so they may occasionally hit the ball off the table slightly. This will give us further bats with which to hit the ball–usually bats with even greater magnitude mod λ. The idea, then, will be to always hit the ball with the smallest available bat (in the sense of $\|x\|_{\mathbb{R}/\mathbb{Z}}$) that doesn't send it off the table (when possible), so that there is the most flexibility for the other side to ensure they are also able to hit the ball back onto the table.

If we plot the first 100 elements of $U(1,9)$ by plotting n on the x-axis, and $a_n \mod \lambda$ on the y-axis, then we get a picture that shows something of this kind happening in figure 7.6. In this picture, the horizontal lines are placed at $\frac{k\lambda}{6}$ for $k = 0, \ldots, 6$. So we see that 1 is very close to the left side mod λ, and so functions as the "left bat", while every time we go off the other side of the table, we get a new "right bat", which we can use to move things back to the left when they get too close to the edge.

7.2.3 Conjectures

Recall that sum-free sets of positive integers correspond bijectively with binary "decision" sequences. We know also that there are many sum-free sets of positive density. Further, we can easily see that if the 1s have zero density in the decision sequence, then the resulting sum-free set has zero density. So all the positive-density sum-free sets must have decision sequences with positive density.

Question 7.2.4. *For any decision sequence S with a positive density of 1s, the corresponding sum-free set $A = \theta(S)$ has positive upper density.*

Conjecture 7.2.5. *For any decision sequence S that is eventually periodic and for which the repeating pattern has at least one 1, then the corresponding sum-free set $A = \theta(S)$ has positive upper density.*

As we have said, it appears that the Ulam sequence has positive (upper) density around 0.07. This, together with the ultimate greediness of the Ulam sequence suggests for us the conjecture:

Conjecture 7.2.6. *The Ulam sequence $U(1, 2)$ has positive upper density.*

Chapter 8

Future Directions

In this section, we outline some possible future avenues of study, both for extending our main line of thought here as well as for answering separate, related questions.

8.1 Technology

There are a number of results and techniques that have seen application to related problems that we suspect may allow for further progress in future study of this problem. We briefly describe four of them in this section.

8.1.1 Triangle removal

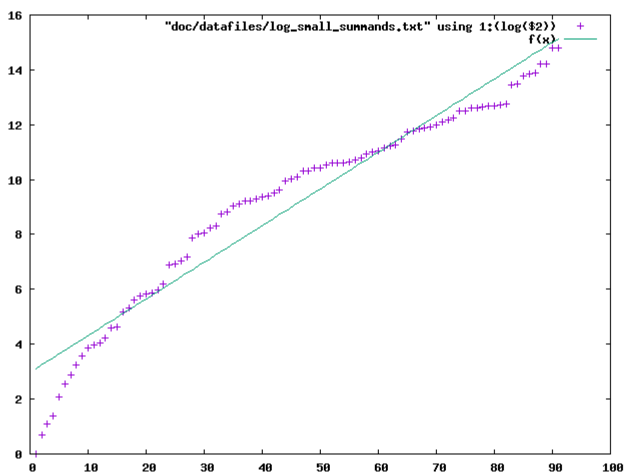
Throughout much of this study, we have been treating sum-free sets and other almost sum-free sets more or less simultaneously. However, in many ways sum-free sets are less rigid and are easier to reason about, whether via the correspondence with decision sequences or in the ease of constructing examples with certain desired features.

We have also noted that the Ulam sequence has few small summands. In particular, if S is the set of small summands, then we can partition the Ulam numbers A into $S \cup T$ where T is sum-free, and S seems to be small. A result about the size of S could allow us to reduce general questions about 1-additive and other almost sum-free sets to questions just about sum-free sets. One general result in this direction comes from Green [11] (Cor. 1.6) in the following theorem:

Theorem 8.1.1. *There is a universal function $f(\delta)$ with $f(\delta) \to 0$ as $\delta \to 0$ such that the following holds: For any $N > 0$ and any subset $A \subseteq [N]$ with δN^2 solutions to $x + y = z$. Then there is a partition of A as $A = B \cup C$ where B and C are disjoint, B is sum-free, C is small: $|C| \leq f(\delta)N$.*

In the case of the Ulam numbers, assuming they have positive density δ, then A_N hae at most $3\delta N$ solutions to $x + y = z$. That is, A_N has $\frac{3\delta}{N}N^2$ solutions,

Figure 8.1: $\log(s_i)$ for small summands s_i plotted against i



meaning this theorem gives us a $A_N = B_N \cup C_N$ where $|C_N| \le f(\frac{3\delta}{N})N$ and B_N is sum-free. However, this notation is slightly misleading: There isn't a guarantee that there is one whole set $C \subseteq \mathbb{N}$ such that $C_N = C \cap [N]$ as N grows. It also does not guarantee that C is in any way related to S, and so does not gives us a numerical bound on $|S_N|$ as N grows. Thus we have a question:

Question 8.1.2. *Can we estimate $|S_N|$ associated to a 1-additive set A? Say, can we prove $|S_N| = \Theta(\log(N))$? (For example, if we plot the logarithm of the ith small summand against i, we get figure 8.1.) Specifically, perhaps, can we take the proof of theorem 8.1.1 and in the particular case of the Ulam numbers get a good bound for f?*

8.1.2 Ultralimits

The phenomenon we have been addressing appears to be one that remains true in some kind of limit. However, the precise statement of this seems delicate. For example, trying to prove that there is a non-zero $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ with $\widehat{A}(\alpha) \neq 0$ in the sense of definition 2.1.1 is fraught with tricky convergence issues. In [20], Tao outlines an argument for Roth's theorem using ultralimits that manages to work entirely in the infinitary setting and deduce the finite structure (namely, 3-term arithmetic progressions) that is desired. It may be worth exploring whether this technique applies in the Ulam case as well, whether to simplify the arguments we have given or to prove more than we have managed here.

8.1.3 Energy increment

Considering the mileage we got out of imitating the classic "density increment" proof of Roth's theorem, we might turn to looking at other methods of proof and seeing what those tell us about Ulam sequences also. One such relies on the so-called Szemeredi regularity theorem, which is in turn proven by a technique that appears repeatedly in additive combinatorics: The "energy increment" argument.

Broadly, this type of argument (as described in [21]) goes by quantifying the presence of a certain kind of structure in a function by an "energy". Then, we approximate our given function with a low-complexity approximation function (for example, the indicator function of a set can be approximated at first by the constant function whose value is the density of the set). The argument then proceeds to use a dichotomy, much as in the density increment case: If the energy of this approximation is large, then there is a lot of the desired structure and we are done. If the energy is small, then this implies some bias (such as, in the case of Roth's theorem, a large Fourier coefficient) that we can use to find a better approximation with increased energy. Executing this enough times, we force our approximation into the large energy setting where we can win.

So if we could pin down a more general notion of regularity (such as a precise quantification of "bias modulo some λ"), then we might be able to follow a similar plan: Assign a notion of "energy modulo λ" to the distribution that is large if the distribution is very non-uniform, say. Then, approximate the indicator function A of $U(1,2)$ on $[N]$ by $\delta = \frac{|A_N|}{N}$. If this has low energy, prove that the difference between the function and our approximation has Fourier bias (as in 5.1.3) and therefore some correlation with a non-uniform distribution (as in 6.3.1), and use that to better approximate the indicator function with something of higher energy (i.e., more bias).

This plan would first require having a precise notion of "energy" that captures whatever notion of "regularity" we would ultimately hope to prove for $U(1,2)$.

8.1.4 Arithmetic regularity

The ultimate consequence of many energy increment type arguments is a very general sort of regularity theorem, from Szemeredi's regularity theorem to the more arithmetically flavoured regularity results of Green and Tao [12]. In that paper, the arithmetic regularity theorem in particular was used to prove Roth's theorem directly, and it has been used to study sum-free subsets of integers to great effect in [6]. In particular, [6] works with sum-free sets using a kind of "local-to-global" argument that has a similar flavour (if very different content) to what we are looking for.

A likely fruitful future direction, then, would be to understand the implications of such regularity results for Ulam numbers.

8.2 Variants of the Ulam problem

We have already described 1-additive sets (see definition 7.2.1) in greater generality than just the Ulam numbers. We could either modify the initial "seed set" or we could be less greedy about picking the next elements, but fundamentally, these sets are all almost sum-free and so we expect to be able to say the same things about them. There are, however, other modifications that may be worth considering for the purpose of being able to prove things, which we discuss in this section.

8.2.1 Sums of more than two previous elements

One modification is that our notion of 1-additive used only sums of two elements. We could equally define:

Definition 8.2.1. *A $(1, k)$-additive set is a set $A \subseteq \mathbb{N}$ with an N such that, for $x > N$, there is exactly one way to write $x = a_1 + \ldots + a_k$ for $a_i \in A$ and $a_1 < \ldots < a_k$.*

This we would expect to behave similarly, but one consideration draws us to this type of set specifically: When we were trying before to estimate $r_{A+A}(x)$ using the large spectrum $\alpha\mathbb{Z}$, this essentially amounted to using the circle method. In classical applications of the circle method (for example, the proof of Vinogradov's theorem) we find that the circle method is much better at counting representations of a given x by sums of k elements of a set A for larger k than for, say, $k = 2$ (which in the Vinogradov case would be the Goldbach conjecture).

This might lead us to suspect that $(1, k)$-additive sets may be even more susceptible to circle method analysis for larger k than $k = 2$, which is the case we have been dealing with.

8.2.2 Probabilistic versions

One thing we get from the bijection between sum-free sets and infinite binary sequences is a probability measure on all sum-free sets, taken from the measure on binary sequences that considers each entry as a flip of a fair coin (or, if desired, of a weighted coin). This leads to natural questions about the structure of random sum-free sets, which have been studied in [1] and [4].

Considering Ulam numbers as an extension of the idea of sum-free sets to considering sets A whose elements x have $r_{A+A}(x)$ small, we might try to define a random version of this. For example we can construct a random 1-additive set:

Definition 8.2.2. *Let A be a set consisting of a finite "seed set" S, and then being built by the following random process: Given the current set of elements in A, let A_1 be the set of elements $x \in \mathbb{N}, x > \max(A)$ such that $r^*_{A+A}(x) = 1$. Select an element uniformly at random from A_1 and include that element in A. Repeat forever.*

Or, we could define a random construction of a set that simply has small values for $r_{A+A}(x)$ for $x \in A$:

Definition 8.2.3. *Let A be a set consisting of a finite "seed set" S, and then being built by a random process that takes each x, computes $r_{A+A}(x)$ using the elements included in A thus far, and includes x with probability $\frac{1}{r_{A+A}(x)}$ (including it automatically if $r_{A+A}(x) = 0$, say.)*

Having such definitions in hand, we can then ask probabilistic versions of all the questions we have asked about density, Fourier spectrum, distribution, and structure that we have asked about the Ulam numbers.

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