The Ulam sequence and related phenomena

Daniel Ross

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1 Introduction

The Ulam sequence is a sequence of positive integers that is defined in a recursive way that sounds like it should make it difficult to compute. It starts with $a_1 = 1$, $a_2 = 2$, and then for $n > 2$, a_n is the integer satisfying:

1. It is expressible as a sum of distinct previous terms in exactly one way: There is exactly one pair of $i < j$ with $a_i + a_j = a_n$.

2. It is larger than the previous element of the sequence: $a_n > a_{n-1}$.

3. It is the smallest positive integer with the above two properties.

Thus the first few terms can be computed:

$$1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, 62, 69, \ldots$$

In particular, there are two ways a number could fail to be Ulam: Either it has a representation as a sum of two distinct previous smaller Ulam numbers in more than one way (such as $5 = 4 + 1 = 2 + 3$), or it has no representations as a sum of distinct smaller Ulam numbers at all (such as 23).

One thing that makes the sequence interesting is that it seems historically to have been very difficult to prove anything about it. We know, for example, that it must be infinite, since, given the first n elements $a_1, \ldots, a_n$, we can always find at least one number that satisfies both the criteria above, namely $a_{n-1}+a_n$. Thus there must be a smallest such number, which is the next Ulam number.

We also know that if we use the same definition but start with different initial values, we can get sequences that we can analyse very easily indeed: If the (u, v)-Ulam sequence is the sequence with $a_1 = u, a_2 = v$, and a_n (for $n > 3$) defined exactly as before, then by theorems of Finch [3] and Schmerl and Speigel [4], we know that the $(2, v)$-Ulam sequence, in the case where v is odd and at least 5, is regular in the following sense:

Definition 1. An increasing, infinite sequence $\{a_i\}$ of positive integers is **regular** if the sequence $\{b_i = a_i - a_{i-1} : i > 1\}$ is eventually periodic.

Such sequences are very easy to describe–we could specify them (after some initial segment) by a set of congruence classes modulo some (possibly large) m. In particular, a regular Ulam-like sequence will be far easier to compute than the definition of an Ulam sequence would naively suggest.

There are other initial values that are variously known to and believed to give rise to regular sequences, also. See, for example, [2]. That said, many

Ulam-type sequences appear not to be regular, among them the $(1,2)$- and $(2,3)$-Ulam sequences. So one might ask some questions:

- What is it that causes some initial conditions to be regular and not others (if indeed they are not)?
- Is there any perhaps more general notion of regularity that even the irregular-looking sequences do satisfy?

In looking for hidden regularity, one might take a signal processing approach to the sequence and try, for example, to Fourier transform the indicator function of the sequence and see if the spectrum has any interesting features. In [1], Stefan Steinerberger did exactly that and behold, the spectrum has a large spike exactly only at some $\alpha \in \mathbb{R}/\mathbb{Z}$ (and at its harmonics), and seemingly nowhere else.

More precisely:

Definition 2. If $f : [N] \to \mathbb{C}$, recall the the **Fourier transform** of f is a function $\widehat{f}$ defined by the formula:

$$\widehat{f}(x) = \frac{1}{N} \sum_{t=0}^{N-1} f(t) e(-tx)$$

where $e(x) = e_N(x) = e^{2\pi i x/N}$. Thus $\widehat{f}$ is a function defined on all of $\mathbb{R}/\mathbb{Z}$.

N will often be omitted from the notation and understood from context. If we wish to make N explicit in the notation for the Fourier transform itself, we will denote it as $\mathcal{F}_N f$ rather than $\widehat{f}$.

This definition satisfies many properties, which are standard from Fourier analysis and additive combinatorics [12]:

Proposition 1.1. *If $f : [N] \to \mathbb{C}$, then:*

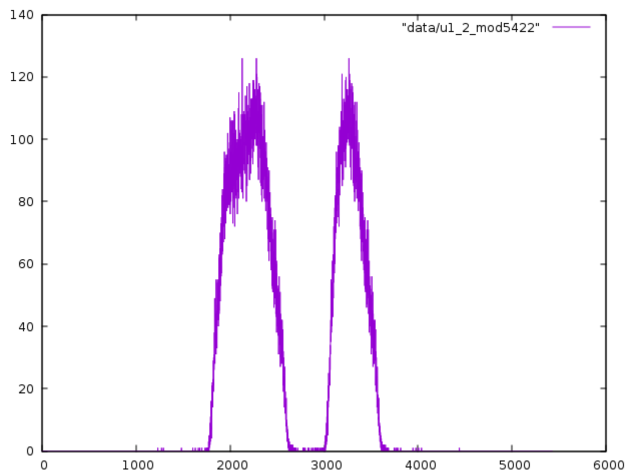
- *If in fact f takes values in $\mathbb{R}$, then $\widehat{f}(-x) = \overline{\widehat{f}(x)}$ for all $x \in \mathbb{R}/\mathbb{Z}$.*
- *$\widehat{\widehat{f}}(x) = f(-x)$ for all $x \in [N]$.*
- *$\widehat{fg}(x) = (\widehat{f} * \widehat{g})(x)$ for all $x \in \mathbb{R}/\mathbb{Z}$.*
- *If in fact f is the indicator function of a set $A \subseteq [N]$, then $\widehat{f}(0) = \frac{|A|}{N}$.*

So if A is a set of positive integers (say, the Ulam sequence), and 1_A is the indicator function of A, then we might define $\widehat{1_A}(x) = \lim_{N\to\infty} \widehat{1_{A_N}}(x)$, where as usual, $A_N = A \cap [N]$ is the truncation of A at N. Then in the case of the Ulam sequence, what is observed numerically is that for one particular value of $\alpha \in \mathbb{R}/\mathbb{Z}$ (namely $\alpha = \ldots$), that $\widehat{1_A}(\alpha) \approx 0.8$, and for $k \in \mathbb{Z}$, $\widehat{1_A}(k\alpha)$ is also some non-zero value that shrinks with k. For example, for $N = 100000$, we compute this for a few values of k (noting that of course the values for $-k$ are just the conjugates of these:

k	$\widehat{1_{A_N}}(k\alpha)$
1	0.7985467537954992
2	0.32061814928309407
3	0.30359418284609546
4	0.6019015738751037
5	0.6004862934665662
6	0.3699258112430088
7	0.1279822946231241
8	0.1443860719363926
9	0.14047291569572581

and as N gets large, it appears that $\widehat{1_{A_N}}(\beta) \to 0$ as $N \to \infty$ for all other $\beta \notin \alpha\mathbb{Z}$.

From a signal processing perspective, this might suggest that the set A has some periodicity mod $\frac{1}{\alpha} \approx \ldots$. Using $\ldots$ as a rational approximation to this, we can plot the distribution of A_N for, say, $N = 100000$ modulo this number:



This has many notable features:

- From the value of $\widehat{1_A}(0)$, it looks like the Ulam sequence has small but nonzero density (in fact, around 0.07).
- As noted in [1] it looks like as we increase N that this is converging to an actually continuous distribution.
- It looks at a glance like this distribution is supported on the middle third of the interval $[0, \frac{1}{\alpha}]$. This is not literaly the case, but in [10] there is a conjecture in this direction.
- While the obvious peaks seem somewhat irregular, the major bumps look kind of like they might be a sum of normal or other nice distributions.

1.1 Results

TODO

2 Background

2.1 Known regularity results

If we want to prove this kind of generalised regularity statement, it might help to understand existing proofs of regularity (i.e. that consecutive differences are eventually periodic). We discuss two such results in this section.

2.1.1 Finch's criterion for regularity

In [3], Finch proves:

Theorem 2.1. *If $A = U(a, b)$ is a 1-additive set containing finitely many even elements, then A is regular.*

The idea of the proof is that if there are finitely many evens, say $e_1 < \ldots < e_s$, then every term n after the last even must be odd. Since it can be written as sum of two earlier terms, and it is odd, one of its summands must be even. And since it can be written in such a sum in a unique way, this is saying that $n - e_i$ is in the sequence for a unique i from 1 to s. This is finitely many things to check.

More precisely:

Proof. If x_n is the number of representations of n as a sum of two elements of A and n is odd, then because an odd number that is a sum of two smaller elements of A must have an even summand and we have only finitely many evens $e_1 < \ldots < e_s$, we can write a finite recurrence:

$$x_n = \sum_{i=1}^{s} 1(x_{n-e_i})$$

where $1(x)$ is 0 unless $x = 1$, in which case $1(x) = 1$. In particular, $0 < x_n \leq s$ for all odd $n > e_s$. Note also that x_n depends on a finite range of earlier x_i's: $x_{n-2}, x_{n-4}, \ldots, x_{n-e_s}$. Call this sequence B_n. Each of the $e_s/2$ values in B_n is between 1 and s, so there are finitely many possible such sequences. Thus, for some N and n, we must have $B_n = B_{n+N}$. But since x_n and x_{n+N} only depend on B_n and B_{n+N} respectively, this means $x_n = x_{n+N}$.

And further, x_{n+2} and x_{n+N+2} only depend on the sequences B_{n+2} and B_{n+N+2}, respectively. But

$$
\begin{aligned}
B_{n+N+2} &= (x_{n+N}, x_{n+N-2}, \ldots, x_{n+N+2-e_s}) \text{ by definition} \\
&= (x_{n+N}, x_{n-2}, \ldots, x_{n+2-e_s}) \text{ because } B_n = B_{n+N} \\
&= (x_n, x_{n-2}, \ldots, x_{n+2-e_s}) \text{ as noted above} \\
&= B_{n+2}
\end{aligned}
$$

So in fact $B_{n+N+2} = B_{n+2}$ and we can proceed by induction to show the B_n are periodic with period N. Since the x_n are a function of the B_n, x_n is therefore also periodic with period N. □

2.1.2 Regularity of $U(2, 2n+3)$

Using the above criterion, Schmerl and Speigel in [4] prove: regularity for the 1-additive sets $U(2, 2n+3)$:

Theorem 2.2. *The 1-additive sets $U(2, v)$ for $v > 5$ and odd are regular.*

Since they use Finch's criterion, this boils down to showing that each of these sets has finitely many evens. Specifically:

Lemma 2.3. *The only even elements in the 1-additive set $U(2, v)$ (with $v > 5$ odd) are 2 and $2v + 2$.*

Proof. The proof goes by supposing that x is the next even element of $U(2, v)$ after $2v+2$ and using an exhaustive knowledge of small elements of the sequence (up to about $5v$) to write $x = a + b$ for smaller $a, b \in U(2, v)$ in more than one way. To do this, we have to understand the small elements of the sequence and the elements just before x.

We leave out the computation of the small elements and simply state the result:

Lemma 2.4. *The elements of $U(2, v)$ up to $5v + 10$ are:*

- 2
- $2v + 2$
- *All odds between v and $3v$, inclusive.*
- $3v+4i$ *for* $0 < i \leq \frac{v+1}{2}$ *(that is, every other odd from* $3v$ *to* $5v+2$ *inclusive)*
- $5v + 4$
- $5v + 10$

To use these to express our supposed next even element x as a sum of elements of $U(2, v)$ in multiple ways, we also need to understand the elements immediately leading up to x.

Lemma 2.5. *There is no gap of length $2v$ in the odd numbers in the sequence up to $x-2v$. More precisely, if r is any odd number less than $x-2v$, then one of $r, r+2, \ldots, r+2v$ is in $U(2,v)$.*

Proof. If we take r to be the minimal counterexample to this, then $r-2$ is in $U(2,v)$, else $r-2$ would be a smaller counterexample (note that 1 is manifestly not a counterexample, so $r-2>0$).

But then $r+2v=(r-2)+(2v+2)$ expresses $r+2v$ as a sum of elements of $U(2,v)$, so the only way it can fail to be in $U(2,v)$ is if there is another such expression. But $r+2v$ is odd, so any other expression of it as $a+b$ for $a,b \in U(2,v)$ requires that one of a and b be even. And $r+2v<x$, so the only choice other than $2v+2$ (which we have already used) is 2. So this means $r+2v=2+(r+2v-2)$ is the other such expression. But for this to be such an expression, $r+2v-2$ must be in $U(2,v)$, and we are done. □

Corollary 2.5.1. *It follows from the proof that for any odd $r<x-2v$ $r \in U(2,v)$ if and only if exactly one of $r+2v+2$ and $r+2v$ is in $U(2,v)$.*

This will allow us to, for example, find several elements of $U(2,v)$ between $x-3v$ and x. We already know that we have a lot of elements between v and $3v$, so this gives us a good chance of expressing x as a sum of elements of $U(2,v)$ in multiple ways.

For example, the second lemma tells us that some odd between $x-3v$ and $x-v$, say $x-v-2i$ (for some $0 \leq i \leq v$) is in $U(2,v)$. But we know everything of the form $v+2i$ with $0 \leq i \leq v$ is in $U(2,v)$ as well, so:

$$x=(x-v-2i)+(v+2i)$$

is the qualifying expression for x as a sum of smaller elements. Since this expression must be unique, we also know that $x-v-2j$ for $0 \leq j \leq v$ and $j \neq i$ cannot be in $U(2,v)$.

To get a second such expression (and therefore a contradiction), we will look also at the odd elements from $x-5v$ to $x-3v$, using our knowledge of the odd elements of $U(2,v)$ from $3v$ to $5v$.

After some casework, this will end up giving a second qualifying expression for x, thereby disqualifying it. We refer to [4] for the details.

□

2.2 Sum-free sets

The set of Ulam numbers A has the property that for each $a \in A$, there is exactly one solution to $x+y=a$ with $x<y$ in A. The condition that $x<y$ is a little hard to capture using standard techniques, but, for example, this entails that the number of solutions to $x+y=a$ with $x,y \in A$ is at most 3 (namely, the unique solution $x+y=a$ above, then also $y+x=a$, and then possibly some other $z+z=a$, since the definition of the Ulam numbers does not consider this. For example, 4 is Ulam, and its unique representation is $1+3=4$, but it also happens that $2+2=4$ and 2 is also Ulam).

In particular, this implies that if A_N is again the set of Ulam numbers up to N, then A_N has at most $3|A_N|$ solutions to $x+y=z$ with $x,y,z \in A_N$.

We might ask how special such a condition is on sets of integers. For instance, suppose we take the integers up to N and we generate a random subset by including each one with probability p. The size of set we expect to get is pN. The number of pairs x,y is $(pN)^2$, and of these, we expect p of them have $x+y$ in the set, so we expect p^3N^2 solutions to $x+y=z$. In particular, an arbitrary set of density p we expect to have $O(N^2)$ solutions. Since the Ulam numbers appear to have density around 0.07 but by construction have only $O(N)$ solutions to $x+y=z$, they are already somewhat special.

In the interest of understanding what precisely is happening with the Ulam numbers, then, we might turn our attention to the more extreme situation of sets with no solutions to this equation at all: So-called "sum-free sets".

Definition 3. A subset A of an abelian group is **sum-free** if the equation $x+y=z$ has no solutions with $x,y,z \in A$.

Example 1.

1. The odd positive integers are sum-free.
2. More generally, if $A \subset \mathbb{Z}/m$ is sum-free, then the set of integers x that reduce to an element of A modulo m is also sum-free.
3. Even more generally, for any homomorphism $\pi : \mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, if A is a sum-free subset of $\mathbb{R}/\mathbb{Z}$, then $\pi^{-1}(A)$ is a sum-free set of integers.
4. Any subset of a sum-free set is sum-free also.

When we think about generalising the particular notion of "regularity" above for the purpose of the Ulam sequence or for sum-free sets, the basic idea is that a set should be "regular" if it has some correlation with a set of the form in example 3.

2.2.1 Decision sequences

It turns out there is a construction that bijects sum-free sets of positive integers with infinite binary sequences. In words, the construction is simple: Take the positive integers in turn starting with 1. Flip a coin. If it's heads, include it in the set and erase all integers that are sums of elements in the set thus far (as these could not be in the set if it is to be sum-free). If tails, do not include that integer in the set. Then move on to the next integer that has not been included, excluded, or disqualified.

More formally:

Definition 4. Define the function $\theta : \{0,1\}^{\mathbb{N}} \to \{f : \mathbb{N} \to \{0,1\}\}$ from binary sequences to sum-free sets of natural numbers (or, in this case, their indicator functions) as follows: If $s \in \{0,1\}$ is a binary sequence, then using s, we will actually define three disjoint sets that partition the natural numbers: The target set A, the excluded set E, and the disqualified set D. For each $n \in \mathbb{N}$, iteratively select a set for n as follows:

$$\begin{cases} n \in A + A & \implies n \in D \\ n \notin A + A \text{ and } s_k = 1 & \implies n \in A \\ n \notin A + A \text{ and } s_k = 0 & \implies n \in E \end{cases}$$

where, at each stage, $k = |A| + |E| + 1$ is the index of the first element of s that we have yet to consult.

If S is a sequence and A is a sum-free set with $\theta(S) = A$, then S is called the **decision sequence** for A.

Example 2. For example, let us compute $\theta(111111111...)$: We start with 1 and flip a coin and get heads, so we include 1 in the set A. This automatically disqualifies 2 as $2 = 1 + 1$. The next possible candidate is 3, so we flip another coin and get heads, and so we include 3. This automatically disqualifies 4 $(4 = 1 + 3)$ and 6 $(6 = 3 + 3)$. Continuing in this way, it is clear we will never get a chance to include an even number and will always include the odd numbers, so in the end, the result is the set of odd positive integers.

It is also possible to reverse this construction. In words: Say we start with A a sum-free set. We again walk through the positive integers starting at 1. For each n there are three possibilities: Either $n \in A$, $n \in A + A$, or neither. If $n \in A$, then it got there by a coin landing heads, so we write down a 1. If $n \in A + A$, then n was disqualified from being in A not by a coin flip, but by being a sum of elements of A, so we write down nothing. If $n \notin A$ and also $n \notin A + A$ then n could have been included in A, but was excluded simply because of a coin flip, so we write down a 0.

Formally, we write down the sequence $s = \theta^{-1}(A)$ by writing down first the string:

$$s'_n = \begin{cases} \text{"A" if} & n \in A \\ \text{"D" if} & n \in A + A \\ \text{"E" if} & n \notin A \cup A + A \end{cases}$$

Then s is got by deleting all Ds, replacing all As with 1, and Es with 0.

There are many questions about this construction. For example, it is known that if a sum-free set A is regular (as defined above–i.e. its sequence of successive differences is ultimately periodic), then its decision sequence $\theta^{-1}(A)$ must also be ultimately periodic [7]. The converse is believed to be false, with one of the simplest apparent counterexamples being $\theta(\overline{01001})$ ($\overline{01001}$ meaning the binary sequence that repeats the pattern 01001 forever). This is a set $\{2, 6, 9, 14, 19, 26, 29, 36, 39, 47, 54, 64, 69, 79, 84, 91, 96, \ldots\}$ that has been computed extensively and for which no period has been identified. There is other computational evidence that this sequence may not be periodic beyond just brute force attempts to compute a period found in [11]. Nevertheless, there is no known example of an ultimately periodic decision sequence for which we can prove its corresponding sum-free set is non-regular.

2.2.2 Density and regularity

We start with the observation that if A is a sum-free set and $a \in A$, then A and $A + a$ are disjoint sets of integers. This automatically guarantees that a sum-free set cannot have density in the integers of more than $\frac{1}{2}$. Specifically:

Definition 5. A subset $A \subset \mathbb{Z}^+$ has **density** δ if $\lim_{N\to\infty} \frac{|A_N|}{N}$ exists and is equal to δ.

Since this may not always exist, we might work with another number that always will exist and that, in cases when the density does not exist, provides what should be thought of as at least an upper bound:

Definition 6. A subset $A \subset \mathbb{Z}^+$ has **upper density** δ if $\limsup_{N\to\infty} \frac{|A_N|}{N} = \delta$.

As we have noted, then, the maximal upper density a sum-free set can have is $\frac{1}{2}$, which is realised by the example of the odd positive integers. Luczak has given a sort of converse to this example, proving in [9] the following:

Theorem 2.6 (Luczak)**.** *If A is a sum-free set of positive integers and there is at least one even integer in A, then the upper density of A is bounded above by $\frac{2}{5}$.*

The proof is short, but a little delicate, and we shall recall a version of it in this section.

The basic idea of the proof is to find disjoint subsets of $[N]$ that are the same size as A_N, or of a size related to A_N. For example, if $a \in A_N$ is any element, then because A is sum-free, A_N and $A_N + a$ are disjoint in $[N + a]$, but have the same size, and thus $2|A_N| \leq N + a$, i.e. $|A_N|/N \leq \frac{1}{2} + \frac{a}{2N}$. Taking the limit as $N \to \infty$, we again deduce our earlier statement about A having density bounded by $\frac{1}{2}$.

Proof. Note first that if A is all even elemenets, then $\frac{1}{2}A$ is also sum-free, and therefore with density $\leq \frac{1}{2}$, and so A has density $\leq \frac{1}{4}$ and the result is automatic, so without loss we may assume A has at least one odd element in addition to its at least one even element.

With this in mind, the proof breaks up into two cases: Where A contains consecutive elements and where it does not.

Case 1: A has no consecutive elements In the case where A has an even element but no two consecutive elements, let t be the minimal odd positive element of $A - A$ which does exist using the odd and even elements, and is not 1, since there are no consecutive elements. Also fix $x, y \in A$ with $t = x - y$.

This means that if $a \in A$, then $a+t-2$ cannot be in A (else $t-2$ would be a smaller odd positive difference than the minimal odd difference t). Put another way, if a and $a + 2$ are both in A, then $a + t$ cannot be in A. Put another way, if B is the set of $a \in A$ with $a + 2$ also in A, then $B + t$ and A are disjoint. Of course, we already know that finding two disjoint subsets of size even as large as $|A|$ is already easy, however this lets us in fact find three: Since $t = x - y$, this means $B + x - y$ and A are disjoint, meaning $B + x$ and $A + y$ are disjoint.

But both of these are contained in $A+A$, so they are both also disjoint from A. Thus we have A, $A+y$, and $B+x$ all disjoint. If we truncate A to A_N, then these are all disjoint subsets of $[N+x]$, and so

$$2|A_N| + |B_N| \leq N + x$$

.

So if we can relate $|B|$ to $|A|$ (for the moment using the shorthand $B = B_N$, $A = A_N$), then we are done.

But by the definition of B, we have two cases for an element of A:

- $a \in B$, in which case $a+1$ is not in A.
- $a \in A \setminus B$, in which case we know $a+1$ is not in A (since A has no consecutive elements) and $a+2$ is not in A, (since otherwise a would be in B).

So we have the five sets: $B, B+1, A \setminus B, A \setminus B+1, A \setminus B+2$, and these are all pairwise disjoint in $[N+2]$. (The only one that might not be clear is $B+1 \cap A \setminus B+2$, but if $a \in A \setminus B$ and $b \in B$ with $a+2 = b+1$, then $a+1 = b$, giving two consecutive elements of A which does not happen.)

Thus $2|B_N| + 3(|A_N| - |B_N|) \leq N+2$, i.e.

$$|B_N| \geq 3|A_N| - N - 2$$

Now we have a relationship between $|B|$ and $|A|$, so we can pair this with our earlier inequality relating the two of them to N and find:

$$2|A_N| + 3|A_N| - N - 2 \leq N + x$$

or

$$\frac{|A_N|}{N} \leq \frac{2}{5} + o(1)$$

as we wanted.

Case 2: A has consecutive elements: In the case where A has d consecutive elements $a, a+1, \ldots, a+d-1$, say, the argument is similar in flavour to the above, but the technical details are all slightly different. We will first need a t to serve the role of our t in case 1. But now, the minimal odd difference is simply 1. So we do something slightly different: This time, we let t be any positive element of $A-A$ for which $t+1, \ldots, t+d$ are all not in $A-A$.

Lemma 2.7. *Such t does exist*

Proof. Since $a, a+1 \in A$, we know $1 \in A-A$. Then let t be the maximum of $1, \ldots, a-1$ that is in $A-A$, so nothing from t to $a-1$ is in $A-A$ (by definition), and nothing from a to $a+d-1$ is in $A-A$ either (since these are all in A), so at least d elements (and possibly more) immediately after t are not in $A-A$). □

Again, write $t = x - y$ for some fixed $x, y \in A$.

We proceed broadly as before on the two-step plan:

1. Find a set B of elements that gives rise to many disjoint subsets of $[N]$ and deduce a bound relating $|A_N|$ and $|B_N|$ to N.

2. Upper-bound $|B_N|$ in terms of $|A_N|$ and N, and plug this into the previous bound to get a bound on $|A_N|$ in terms of N.

Step 1: Let B be the set of elements b for which $b+1, \dots, b+d-1$ are all not in A. Then certainly the sets $A, B+1, \dots, B+d-1$, are all disjoint. In fact, we can get one more than this: We can shift all these sets by a and they are still disjoint: $A+a, B+a+j$ ($j = 1, \dots, d-1$). But now since the $a+j$ are all in A, these sets are all themselves disjoint from A (since they are all subsets of $A+A$). Thus, again truncating at N, we have two sets of size $|A_N|$ and $d-1$ sets of size $|B_N|$ all disjoint and inside $[N+a+d-1]$. Thus:

$$2|A_N| + (d-1)|B_N| \leq N + a + d - 1$$

Step 2: So again, we need control over the size of $|B_N|$ in terms of $|A_N|$ and we will be done. But this time, we note that if $z \in A$, it is possible that $z+t$ could be in A, but that then because of the definition of t, none of $z+t+1, \dots, z+t+(d-1)$ can be in A (lest one of $t+1, \dots, t+(d-1)$ lie in $A - A$). Thus elements of $A+t$ that lie in A in factmust lie in B. Put another way, $A+t$ and $A \setminus B$ are disjoint. Again, this is only two sets, but we can use the same trick as before to make it three: Since $t = x - y$, we can equally say $A + x$ and $(A \setminus B) + y$ are disjoint, at which point these are also disjoint from A (again, being subsets of $A + A$). So we have three disjoint subsets $A + x$, $A \setminus B + y$, and A of $[N+x]$, wtih sizes $|A_N|$, $|A_N|$, and $|A_N| - |B_N|$, respectively. This gives $|A_N| + |A_N| + (|A_N| - |B_N|) \leq N + x$ or:

$$|B_N| \geq 3|A_N| - N - x$$

Dropping this into the first inequality and rearranging, we get:

$$2|A_N| + (d-1)(3|A_N| - N - x) \leq N + a + d-$$

which simplifies to:

$$\frac{|A_N|}{N} \leq \frac{d}{3d-1} + o(1)$$

Since $d \geq 2$ (as we are assuming we have at least two consecutive elements), this is again bounded by $\frac{2}{5}$ in the limit, so the claimed bound follows. $\square$

2.2.3 Aperiodic sum-free sets

A construction of Erdos in [8] supplies an example of a sum-free set with density $\frac{1}{3}$ that has no periodicity, namely: Take $\alpha \in \mathbb{R}$ irrational, and let A_α be the set of integers n such that $n \pmod{\alpha}$ lies in $\left(\frac{\alpha}{3}, \frac{2\alpha}{3}\right)$. A_α is clearly sum-free, since it is sum-free modulo α, but for irrational α, A_α is also not periodic. That is, for every modulus m and every residue class k, there is an element of A_α congruent to k mod m.

Indeed, equidistribution results for irrational numbers tell us that there the integers are equidistributed modulo any irrational. For example, there is at least one n that reduces to the interval $\left(\frac{\alpha}{3m} - k, \frac{2\alpha}{3m} - k\right)$ modulo the irradional number $\frac{\alpha}{m}$. Then it is clear that $mn + k$ will reduce to $\left(\frac{\alpha}{3}, \frac{2\alpha}{3}\right)$ mod α, meaning that $mn + k \in A_\alpha$ as desired.

2.3 Abelian arithmetic regularity

If A is any set, then an easy probablistic argument shows that there is a sum-free subset of A of size $|A|/3$. Indeed, if for any real α, we define A_α to be set the set of elements x of A with

$$A_\alpha = \{x \in A : 1/3 < \alpha x \pmod 1 \leq 2/3$$

then the expected size of A_α is $|A|/3$ as α varies over $[0, 1]$, and so there must be at least one a for which A_α has at least this size.

In [13] prove that for every e ¿ 0, there is a set A such that A has no sum-free set larger than $(1/3 + \epsilon)|A|$, so this bound is in fact sharp (to first order). (Apropos of nothing, I find this paper's organisation and general exposition to be highly user-friendly and otherwise excellent.)

In our case, if A is the Ulam sequence up to N, then for any $\epsilon > 0$ we believe that for N large enough, we could generate a sum-free subset of size $(1 - \epsilon)|A|$–kind of the opposite extreme.

Nevertheless, the argument follows a kind of local-to-global flow that might make it worth understanding, so we work through at least the basic idea here:

...

2.4 Roth's theorem

Roth's theorem is about the number 3-term arithmetic progressions x, y, z in a set $A \subseteq \mathbb{Z}^+$. Specifically:

Theorem 2.8 (Roth's theorem)**.** *Let $A \subseteq Z^+$ be a set of positive integers with positive upper density. Then A contains infinitely many arithmetic progressions $a, a + d, a + 2d$ of length 3.*

Equivalently, such an A alwayshas at least one solution to $x + z = 2y$ (whereupon x, y, z is an arithmetic progression of length 3). A sum-free set A instead has are no solutions to $x + z = y$ (swapping around variable names to highlight the similarity), so if we have a sum-free set that we believe has positive density, we might wonder what the proof of Roth's theorem has to say about it. (After all, in the case of the slightly different equation $x + z = 2y$ it says that the set A cannot exist.)

As it turns out, many new techniques in additive combinatorics cut their teeth on Roth's theorem, and so there are many proofs, from those that use probabilistic techniques to ergodic theory. We will discuss two in particular: The density increment and energy increment proofs. We will not give the complete proofs in either case, but will simply work through the steps that we shall return to later and outline the rest.

2.4.1 Density increment proof

Proofs of Roth's theorem often work with a finitary version of the statement, which we make now:

Theorem 2.9 (Roth's theorem)**.** *For every $\delta > 0$, there is an $N_0 > 0$ such that for every $N > N_0$, every $A \subseteq [N]$ with $|A| > \delta N$ contains a solution to $x + z = 2y$.*

One strategy of proof goes via Fourier analysis, saying that if A has no large Fourier coefficients, then A is guaranteed to behave "pseudorandomly" in some sense, and computes that such sets must automatically have many length-3 arithmetic progressions, and we are done already.

If, on the other hand, A does have some large Fourier coefficient, then one can find a long arithmetic progression that has large intersection with A, and on which A in fact has higher density than it had originally. We can repeat this step (the "density increment") as often as needed until either our intersected A has no large Fourier coefficient (in which case we are done as above) or else A's density in the arithmetic progression increases to 1. If we are careful about it, we can ensure that at least 3 elements will still remain by the time we get to this point.

Proof of Roth's theorem via density increment. Rather than working on the set $[N]$, we shall work with the group $\mathbb{Z}/N$, noting that if A only contains elements smaller than $N/2$, then a solution to $x + z = 2y$ in $\mathbb{Z}/N$ is an honest solution to $x + z = 2y$ in A viewed as a subset of $\mathbb{Z}$.

If A is a set of density δ in $\mathbb{Z}/N$, then the number S of solutions to $x+z = 2y$ is counted by

$$
\begin{aligned}
S &= \frac{1}{N}\sum_{t=0}^{N-1}\hat{1}_A(t)\hat{1}_gA(t)\hat{1}_A(-2t) \\
S &= \frac{1}{N}|A|^3+\frac{1}{N}\sum_{t=1}^{N-1}\hat{1}_A(t)\hat{1}_gA(t)\hat{1}_A(-2t) \\
&= \delta^3N^2+\frac{1}{N}\sum_{t=0}^{N-1}\hat{1}_A(t)^2\hat{1}_A(-2t) \\
&\geq \delta^3N^2-\sup_t|\hat{1}_A(-2t)|\frac{1}{N}\sum_{t=0}^{N-1}|\hat{1}_A(t)|^2 \\
&= \delta^3N^2-\sup_t|\hat{1}_A(-2t)|\frac{1}{N}\sum_{t=0}^{N-1}|\hat{1}_A(t)|^2 \\
&= \delta^3N^2-\sup_t|\hat{1}_A(-2t)||A| \\
&= \delta^3N^2-\sup_k|\hat{1}_A(k)|\delta N
\end{aligned}
$$

So if there is no large Fourier coefficient–that is, every Fourier coefficient is $\leq \epsilon N$, then

$$S \geq (\delta^3-\delta\epsilon)N^2$$

So if $\epsilon < \delta^2$, then $S > 0$, at which point there is at lest one solution, as desired.

If, on the other hand, there is a k such that $|\hat{1}_A(k)| \geq \delta^2 N$, then this argument does not guarantee a solution. However, in that case, let $P = d[1, L]$ be the arithmetic progression of length$L$ and difference d $\{d, 2d, \ldots, Ld\}$ (d to be chosen later). We want an arithmetic progression in which A has higher density than it has in $\mathbb{Z}/N$ at large. In other words, we want to find an a that makes $Q(a) = |A \cap (P+a)| = 1_A \star 1_P(a)$ large. But this we can analyse using Fourier analysis:

$$\hat{Q}(s) = \hat{1}_A(s)\overline{\hat{1}_P(s)}$$

Further, we know that for all $s \neq 0$, $\sum_a Q(a) \geq |\hat{Q}(s)|$ (looking at the definition of the Fourier transform and using the triangle inequality). So in particular, for $s = k$ (the large Fourier coefficient):

$$
\begin{aligned}
\sum_a Q(a) &\geq |\hat{Q}(k)| \\
&= |\hat{1}_A(k)||\hat{1}_P(k)| \\
&\geq \epsilon N|\hat{1}_P(k)|
\end{aligned}
$$

Thus for some a, $Q(a)/N \geq \delta^2 |\hat{1}_P(k)|$. We can select d and L such that $|\hat{1}_P(k)| \geq L/2$, so for some A, $Q(a)/N \geq \epsilon L/2$. In particular, A intersected with an arithmetic progression of length L has density $\delta + \epsilon/2$, meaning we have increased the density, whereupon we can repeat the argument.

The details (such as actually selecting the correct d and L, as well as properly transitioning from $\mathbb{Z}/N$ back to $\mathbb{Z}$), are covered in many places, for example [14]. □

2.4.2 Proof via the regularity theorem

Proof of Roth's theorem via energy increment. □

2.5 Quantitative bounds in finite fields

There have been several recent developments in a finite field setting on analogous problems (specifically, the work of Croot, Lev, and Pach [6] on length-3 arithmetic progression-free sets in $\mathbb{F}_4^n$ and subsequent work by others [5] pushing it to $\mathbb{F}_3^n$.

We will recall the method used here by outlining the proof in [5], in view of the possibility of later asking about Ulam-like sequences in the same context.

Theorem 2.10 (Ellenberg-Gijswijt)**.** *Let α, β, γ be elements of $\mathbb{F}_q$ such that $\alpha + \beta + \gamma = 0$ and $\gamma \neq 0$. Let A be a subset of $\mathbb{F}_q^n$ such that the equation $\alpha a_1 + \beta a_2 + \gamma a_3 = 0$ has no solutions $(a_1, a_2, a_3) \in A^3$ apart from $a_1 = a_2 = a_3$. Then $|A| = o(2.756^n)$.*

Proof. Let S^d be the space of all polynomial functions on $\mathbb{F}_q^n$ of degree d (that is, polynomials of total degree d where each of the n variables shows up with degree less than q). Let m_d be the dimension of this space, and let V_d be the subspace of polynomial functions vanishing on the complement of $2A$ (this is more or less a trick). Then

$$\dim(V_d) >= m_d - (q^n - |A|)$$

(since the requirement to vanish on the complement of $2A$ is at most $q^n - |A|$ conditions).

It turns out that we can actually get a polynomial P_d in V_d with support of size exactly $\dim(V_d)$, and so this polynomial has:

$$|\mathrm{supp}(P_d)| >= m_d - q^n + |A|$$

Now for the last bit: If we have a degreee-d polynomial P vanishing on the complement of $2A$, then we can form the $|A|$ by $|A|$ matrix M whose i, j entry is $P(a_i + a_j)$ where a_i are the elements of A. First of all, because for i and j different, $a_i + a_j$ is never in $2A$, the off-diagonal terms all vanish, whereas because the diagonal terms are $P(2a_i)$, they may or may not vanish.

We can brutally expand this polynomial into a sum of monomials:

$$P(a_i + a_j) = \sum_{\text{monomials m,m' of degree d or less}} c_{m,m'} m(a_i) m'(a_j)$$

Further, in each term at least one of m and m' has degree at most d/2, so we can sum over

$$P(a_i + a_j) = \sum_{\text{monomials m of degree d/2 or less}} c_m m(a_i) F_m(a_j) + c'_m m(a_j) G_m(a_i)$$

So M is a linear combiantion of $2m_{d/2}$ matrices $(m(a_i)F_m(a_j))$ each of which, as the exterior product of two vectors, has rank 1. Thus the rank of M is at most $2m_{d/2}$. And since M is diagonal, this means that in fact on $2A$, P has only $2m_{d/2}$ non-zero points. So the support of P is bounded above by $2m_{d/2}$. Since the support of P_d was already bounded below by $m_d - q^n + |A|$ we can apply this argument to P_d and conclude that

$$2m_{d/2} \geq m_d - q^n + |A|$$

i.e.

$$|A| \leq 2m_{d/2} - m_d + q^n$$

Choosing a particular value of d and bounding these quantities is all that remains. In [5] they take $d = 2(q-1)n/3$ and use Cramer's theorem to bound m_d and related quantities in terms of the claimed exponential. We refer to the paper for details. $\square$

3 Observations

3.1 Studying the large Fourier coefficient α

3.1.1 For other initial conditions

We compute these in experiment1 and experiment2

a	b	$\alpha_{a,b}$	$\lvert\widehat{1_A}(\alpha_{a,b})\rvert^2$
1	2	2.5716	23348.35
1	3	2.8334	24293.72
1	4	2.0944	39007.46
1	5	1.7648	26597.04
1	6	4.8332	32338.77
1	7	0.3266	24192.13
1	8	0.2736	25928.99
1	9	6.0380	26372.97
1	10	6.0616	25717.80
1	11	6.0832	25743.73
1	12	6.1015	25737.80
1	13	6.1154	25926.87
1	14	6.1244	25983.88
1	15	6.1356	25647.87
1	16	6.1453	26048.46
1	17	6.1550	26086.33
1	18	6.1580	26167.08
1	19	6.1660	26600.04
1	20	6.1740	25678.08
1	2	2.5716	23348.35
2	3	1.1841	16192.09
3	4	5.3809	26414.80
4	5	5.6000	21640.56
5	6	2.0588	22783.57
6	7	2.6889	19502.05
7	8	3.9426	23910.35
8	9	3.4903	19172.22
9	10	5.9557	13422.67
10	11	3.4270	17537.29
11	12	6.0091	14069.21
12	13	3.1416	17955.65
13	14	2.9106	18768.82
14	15	3.1416	19043.70
15	16	2.9401	15916.09
16	17	3.1416	23103.74
17	18	2.9634	17551.75
18	19	3.1416	21903.76
19	20	6.1231	15870.33

So for example, when $a_1 = 12$, $a_2 = 13$, it looks like $\alpha = \pi$, which seems to be confirmed for this particular instance by computing more terms and searching with more precision. This is saying that the a_i are very biased mod $2\pi/\pi = 2$, which seems experimentally to be very much the case even out to thousands of terms (for this particular example): the a_i are over 80% odd. Proving something in this direction seems accessible, but some first efforts were unfruitful.

3.1.2 Rationality

It is more likely that there is a sequence of m mod which the congruence classes of a_i are increasingly clustered. The continued fraction of $2\pi/\alpha$, which we're imagining is m/k, doesn't have some really large coefficient where we would obviously truncate it. Instead, for $2\pi/\alpha_{1,2}$ it is just

$$[2; 2, 3, 1, 11, 1, 1, 4, 1, 1, 7, 2, 2, 6, 5, 3, 1, 3, 1, 2, 1, 3, 2, 1, 14, 2, 5, 3, 2, 3, 1, 2, 13, 2]$$

(For most of the $\alpha_{a,b}$ that we computed to any meaningful precision, either this "not obviously a rational number" continues to be true, except when there is a very small obvious modulus like 2.)

This gives rational approximations:

$$5/2, 17/7, 22/9, 259/106, 281/115, 540/221, 2441/999, 2981/1220, 5422/2219, 40935/16753, 87292/35725$$

These suggest, for example, that for $m = 540$, there should be substantial bias in which congruence classes show up in the Ulam sequence.

This is borne out in very crude measurement by taking the first 100000 terms of the Ulam sequence and computing them mod, e.g. 540, and asking how often each confruence class mod 540 shows up and computing the standard deviation of all these numbers.

For the first few moduli coming from the convergents of the confinued fraction, we get:

5	139.9757121789348
17	263.62138626089813
22	298.6996058675916
259	341.73231186554915
281	274.8670335289345
540	664.2715810068448
2441	3022.3025069077416
2981	3009.780526078754
5422	2580.6984215609386
40935	970.8607009744287
87292	690.3748130781282
215519	482.8027781782595
1380406	304.5611017423058

Note, however, that while it looks like the bias starts falling off at 40935, in fact we only know alpha to within $10^(- 10)$ or so, and for p/q convergents from the continued fraction, $|\alpha - p/q| < 1/q^2$. So being confident about alpha to within 10^{-10} suggests that we should only trust convergents up to 5-6 digits.

Moreover, we note that if we take fewer terms, then fewer of the terms will be less than the modulus, so we may see less of the bias even if there is some. For example, the same calculation with only the first 10000 terms looks like:

5	21.633307652783937
17	61.68515885956209
22	72.60478321332931
259	89.57462754381896
281	193.62880436289325
540	682.2864609640275
2441	382.62668898244124
2981	348.9472882781135
5422	263.99360062887
40935	122.81328398767178
87292	105.0829179694691
215519	97.65246431214172
1380406	99.63712935143478

Also of note is that if we repeat the computation with $N = 100000$ with other random moduli, then we don't see numbers of that magnitude at all:

...	...
538	266.52186279126255
539	255.35109519675422
540	664.2715810068448
541	258.6315258698218
542	263.9800814357665
...	...
2439	264.9962194257936
2440	255.43572224088751
2441	3022.3025069077416
2442	258.0622079927702
2443	255.08506362547774
...	...

One takeaway from this study of increasing moduli is the following: earlier we discussed the possibility of the behaviour indicting bias mod some m. In fact, there may not be a single m with the most bias, but an increasing sequence of m's with progressively more bias. For example, one could imagine a sequence that is slightly biased to being odd, say 60% are 1 mod 2. But then in fact it turns out that mod 4, it is more strongly biased, with 65% being only 2 or 3 mod 4. And maybe in fact mod 12, 80% of terms are only ever 2, 3, 6, or 8 mod 12, and maybe in fact 99% are 2, 3, 6, 8, or 1 mod 48, and maybe you can catch more and more of the sequence with a slowly expanding set of congruence classes modulo quickly growing modulus. If there is a "bias mod m" thing happening, this is probably the flavour it takes, but I'm happy to try to treat the approximation to alpha as indicating an "at least some bias toward some congruence classes mod some fixed m" phenomenon.

3.1.3 Algebraicity

My guess is that since apparently alpha=pi sometimes, alpha should not be expected to be algebraic, but really $2pi/alpha$ is the relevant quantity anyways.

At any rate, we tried some tests on both using LLL to hunt for the minimal polynomial of $b = 2pi/alpha$ and b = alpha. It should be noted that $f(b) <<$ $10^(- 10)$ is what is needed to be convincing that f(b) is actually zero. Also, I am not sure what effects result from the lack of precision in our knowledge of alpha.

For what it's worth, then, here is the basic computation (done in Sage) found in appendix A, and with output:

-4.8860471224543e-11	$5 * X^7 - 9 * X^6 - 8 * X^5 - 4 * X^4 + 6 * X^3 + 6 * X^2 + 3 * X + 24)$
1.1938777481609e-10	$(-1) * X * (5 * X^7 - 9 * X^6 - 8 * X^5 - 4 * X^4 + 6 * X^3 + 6 * X^2 + 3 * X + 24))$
-6.4223470985780e-10	$22 * X^5 - 27 * X^4 - 47 * X^3 - 22 * X^2 - 49 * X - 17)$
1.3065359905085e-9	$X^9 - 6 * X^8 + 6 * X^7 + 8 * X^6 - 4 * X^5 - X^4 + 5 * X^3 + 6 * X^2 - 12 * X + 1)$
2.0213413165493e-9	$(-1) * (4 * X^6 + 10 * X^5 - 39 * X^4 - 24 * X^3 - 2 * X^2 + 18 * X - 14))$
2.9011744118179e-9	$(-1) * (28 * X^4 - 13 * X^3 - 91 * X^2 - 95 * X - 33))$
3.7695372157032e-8	$(-1) * (25 * X^3 - 62 * X^2 - 123 * X + 306))$
2.5785948309931e-7	$(-1) * (509 * X^2 - 947 * X - 725))$
1.8155628112027e-9	$X^8 - 11 * X^5 - 13 * X^4 - 9 * X^3 + 8 * X^2 - 6 * X + 9)$
1.8155628112027e-9	$X^8 - 11 * X^5 - 13 * X^4 - 9 * X^3 + 8 * X^2 - 6 * X + 9)$
-1.8250148059451e-9	$X^6 + 6 * X^5 - 22 * X^4 + 7 * X^3 + X^2 - 34 * X - 40)$
-2.3348913913424e-9	$(-1) * (3 * X^7 - 3 * X^6 - 15 * X^5 - 2 * X^4 + 21 * X^3 + 4 * X^2 + 13 * X - 6))$
3.9355683156828e-9	$27 * X^4 - 92 * X^3 + 40 * X^2 + 25 * X + 55)$
6.7800982606059e-9	$3 * X^5 + 22 * X^4 - 54 * X^3 - 32 * X^2 - 55 * X - 28)$
-1.7553418274474e-8	$(-1) * (7 * X - 18)^2)$
-1.7553418274474e-8	$(-1) * (7 * X - 18)^2)$

All told, both look like they don't have small degree if algebraic at all...

3.2 Distribution of summands

3.2.1 Small summands

We note with interest the observation in the abstract of Steinerberger that $\cos(\alpha a_n) < 0$ for all a_n other than 2, 3, 47, and 69. In particular, thee were also the a_n that showed up most frequently as summands in our earlier computation.

So we compute which how often each a_n appears as the smaller summand of a later a_i and we compute $\cos(\alpha a_n)$ for each and sort by this quantity. We note what looks like a very strong correlation between how often a_n shows up as a summand and $\cos(\alpha a_n)$ in the resulting table, computed by experiment11. Define S_i to be the number of a_i such that a_n is the smaller summand of a_i

a_n	S_n	$\cos(\alpha_{1,2}a_n)$	$[a_i : a_i = a_n + a_?]$
2	3630	0.4173307	[6, 8, 13, 18, 28, 38, 99, 177, 182, 221, 238, ...
3	1356	0.1391857	[11, 16, 72, 102, 148, 180, 209, 241, 319, 412, ...
47	1190	0.0931494	[236, 253, 356, 363, 429, 456, 544, 720, 732, ...
69	999	0.0700500	[175, 258, 451, 483, 566, 820, 1018, 1052, 1101, ...
102	836	-0.0353342	[282, 441, 502, 585, 624, 646, 668, 949, 1125, ...
339	589	-0.071198	[695, 739, 751, 861, 905, 1186, 1230, 1770, 1853, ...
36	465	-0.1046811	[138, 309, 602, 927, 1191, 1550, 1682, 2090, 2288, ...
273	305	-0.140326	[612, 673, 685, 1164, 1296, 1308, 1428, 1660, 1765, ...
8	181	-0.1506519	[26, 36, 77, 114, 197, 324, 390, 991, 1470, 1602, ...
2581	85	-0.2874041	[5795, 7459, 8947, 9443, 9619, 9641, 9663, 10677, ...
400	55	-0.288437	[3214, 3605, 3991, 12763, 13562, 13799, 13931, 15160, ...
983	50	-0.316087	[2445, 2748, 5514, 9553, 16121, 17135, 19427, 21626, ...
97	47	-0.320453	[316, 370, 497, 1252, 2581, 3622, 4057, 10366, 13628, ...
356	21	-0.3324957	[983, 4118, 11226, 22676, 27817, 34104, 34969, 52789, ...
1155	16	-0.3466450	[4878, 9132, 13733, 16047, 27883, 30886, 38920, 40931, ...
206	33	-0.3521083	[522, 891, 1155, 1514, 2787, 4324, 9399, 11432, 20375, ...
53	35	-0.3640037	[155, 409, 1208, 3038, 5049, 8421, 14945, 16648, 19480, ...
1308	18	-0.369428	[3029, 8368, 10501, 20937, 29147, 34784, 37765, 61029, ...
9193	8	-0.3921254	[20419, 68914, 74099, 83323, 92073, 108317, 123718, 124864]
10831	4	-0.427570	[31878, 56503, 89101, 126493]
13	6	-0.427833	[82, 219, 273, 19642, 59734, 91748]
14892	2	-0.4327025	[41620, 84500]
13531	3	-0.437951	[47279, 61451, 83139]
23883	2	-0.440936	[65740, 106394]
10269	1	-0.446345	[44816]
8368	1	-0.449499	[20968]
20643	0	-0.4532970	[]
316	5	-0.457968	[2897, 9509, 37809, 44377, 45132]
30315	1	-0.460326	[64928]
3205	1	-0.4609725	[89057]
56437	0	-0.462393	[]
4118	0	-0.4642290	[]
10247	0	-0.4669642	[]
3038	2	-0.468304	[7156, 95616]
57	3	-0.4692568	[126, 339, 9250]
483	1	-0.471289	[80891]
60665	0	-0.471907	[]
63646	0	-0.4728583	[]
39912	1	-0.473267	[128969]
1023	3	-0.4733195	[2178, 20643, 65705]
69608	0	-0.474758	[]
47920	0	-0.475271	[]
123683	0	-0.4762679	[]
33274	1	-0.4817218	[97956]
11586	0	-0.482232	[]
75706	0	-0.48225	[]
128969	0	-0.4825153	[]
3723	1	-0.483432	[34038]
39653	1	-0.48445	[84469]
42217	0	-0.48455	[]

If nothing else, this might suggest to us how to compute more a_n more quickly: Rather than searching previous summands in order, search in the order given by using $\cos(\alpha a_n)$ as the index. Once we find a sum that is unique, we only have to search all smaller sums, again in this order.

3.2.2 Large summands

We've studied the smaller summands a bit–now the question is: What about the large summands?

We note first that if 2 or 3 is the small summand of a_{n+1}, then the large summand is necesarily a_n (if 2 is the small summand and a_n is not the large summand, then a_{n+1} would be a_n+1 which is impossible since this is a duplicate sum with $a_n + a_1$. If 3 is the small summand and a_n is not the large summand, then a_{n+1} is either $a_n + 1$ or $a_n + 2$, both of which would be duplicates.)

This means that over 50% of the time, the large summand will be the last thing in the list so far. When looking at the large summand, then, it seems more relevant to consider how many indices from the end it lives, rather than its actual value. We compute these in experiment13, generating output [FILE] of the form:

n	a_n	i	$n-j$ (with $a_i + a_j = a_n$ and $i < j$)
2	1	0	1
3	2	0	1
4	3	1	1
5	4	1	1
6	6	2	1
7	8	1	1
8	11	2	1
9	13	1	1
10	16	5	1
11	18	1	1

We can do some processing on these to figure out which $n-j$ are the most common:

results [FILE]. Note in particular that there are only 159 of them and (as suggested above) that n-j = 1 accounts for over 50% of them. This list seems to contain few surprises: Among all the values of $n-j$ that appear more than 10 times, nothing bigger than 34 shows up.

If we look at values that show up fewer than 10 times, then it looks like $n-j = 100$ and $185 < n-j < 205$ seem to be preferred, with many of these showing up 3 or more times, while all other values show up 2 or fewer times. This could be an artifact of not much data, however.

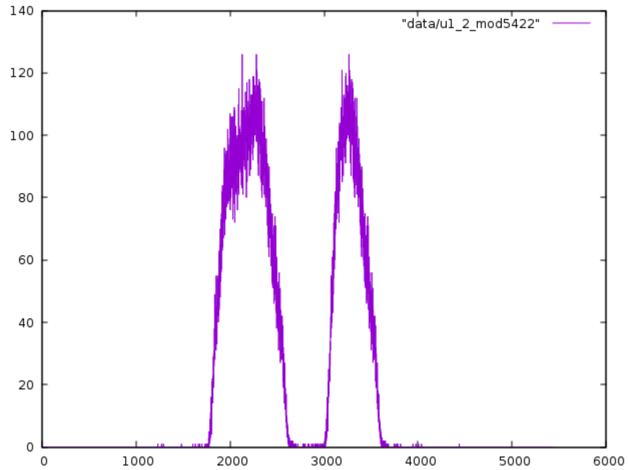
We might instead take a look instead at enumerating $(i, n-j)$ pairs, rather than just values of $n-j$:

with results in [FILE]. Note in particular that there are only 312 distinct such pairs, meaning that technically, to compute the first 10000 Ulam numbers, we only have to check 312 possibilities for each. If only there were a way of knowing ahead of time which 312 we had to check...

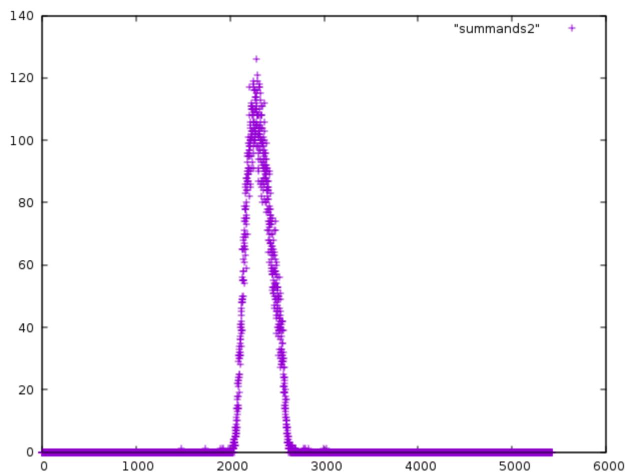
3.3 Distribution of complements

In the cases Steinerberger looks at, the resulting non-uniform distributions consist usually of multiple peaks. In the case of the 1,2 Ulam sequence one of these peaks looks a little misshapen, so we might reasonably wonder what each of these peaks actually is.

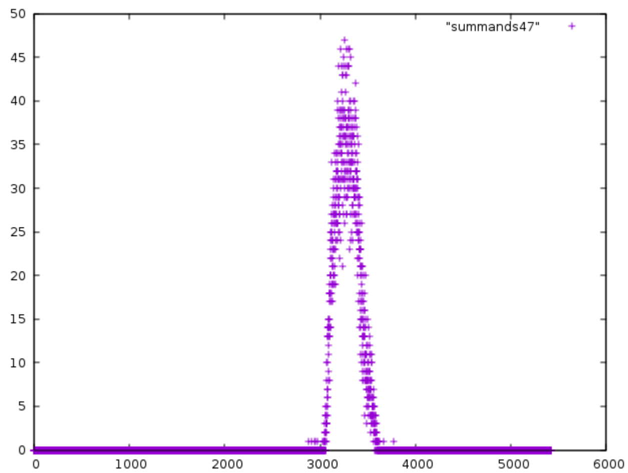
To get a handle on this, we take the Ulam sequence mod 5422, and multiply it by 2219 (5422/2219 being a good rational approximation to $2\pi/\alpha$). Of course, this gives rise to the usual distribution we've come to expect:



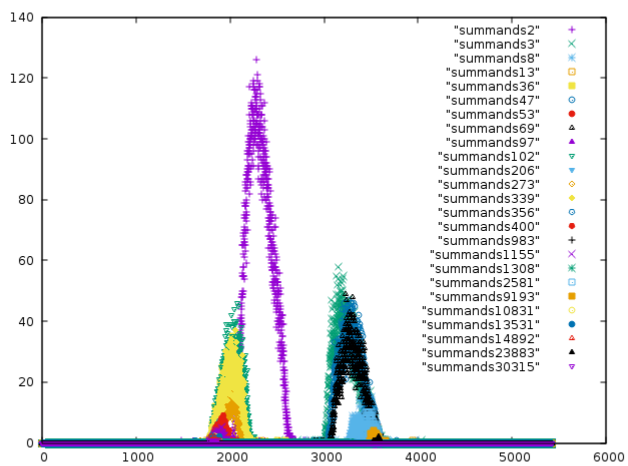
Supposing we look instead only at a_n's for which 2, say is a summand. Then we get this nice picture:



Likewise for 47:



These are relatively clean-looking distributions, by comparison. If we plot these graphs for all of the top 25 most common summands all in one picture, we notice that these seem to be the components of the two observed peaks:



Since each of these seems to be instances of the same distribution with different parameters, we might be interested in computing the parameters of each, starting with the means. We do this with a crunchy bash script

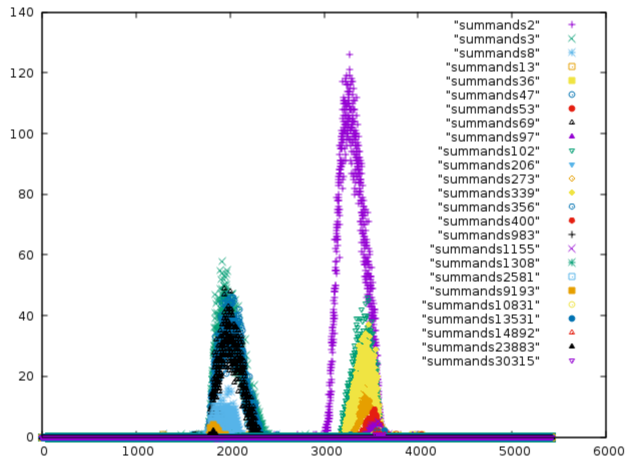
Which outputs the suqmmand a_i, the quantity $2219 * a_i mod 5422$, and the calculated mean of the distribution of $2219 * a_n mod 5422$ for which a_i is a summand, we get:

3	1235	3241.07886089813800657174
47	1275	3288.00715563506261180679
69	1295	3300.94546174403642482430
8	1486	3431.55542168674698795180
2581	1607	3485.87893462469733656174
983	1633	3503.28070175438596491228
206	1666	3518.47580645161290322580
1308	1682	3525.95679012345679012345
9193	1703	3541.35227272727272727272
13	1737	3551.59183673469387755102
23883	1749	3572.53333333333333333333
30315	3653	1818.70000000000000000000
13531	3675	1827.60000000000000000000
14892	3680	1833.36363636363636363636
10831	3685	1845.62962962962962962962
53	3745	1872.41395348837209302325
1155	3761	1883.37745098039215686274
356	3774	1878.85087719298245614035
97	3785	1891.29746835443037974683
400	3814	1912.78585086042065009560
273	3945	1984.79197207678883071553
36	3976	1995.70821114369501466275
339	4005	2013.33377814845704753961
102	4036	2027.29907648591049017286
2	4438	2319.24248003248950859618

Staring at this for a minute, we notice that if we subtract the second column from the third, we seen to get roughly 2000 for the first 11 entries (those on the right end of the distribution). Likewise, those on the left end (rows 12-25) seem to have a similar pattern.

One possible source for this is that the distribution we're taking the mean of in the first row, say, is of $2219 * a_n mod 5422$ where 3 is a summand of a_n in the Ulam sequence. Since 3 is a summand of a_n in the sequence, we might instead look at the other summand of a_n, i.e. $a_n - 3$. This would lead to us not plotting $2219 * a_n mod 5422$, but rather $2219 * (a_n - 3) mod 5422$. We can compute these quickly with another crunchy bash script [appendix A code 2]

And if we plot these, we get:



That's more like it. Running the same mean computation as above on this, we get:

3	1235	2006.0788
47	1275	2013.0071
69	1295	2005.9454
8	1486	1945.5554
2581	1607	1878.8789
983	1633	1870.2807
206	1666	1852.4758
1308	1682	1843.9567
9193	1703	1838.3522
13	1737	1814.5918
23883	1749	1823.5333
30315	3653	3587.7000
13531	3675	3574.6000
14892	3680	3575.3636
10831	3685	3582.6296
53	3745	3549.4139
1155	3761	3544.3774
356	3774	3526.8508
97	3785	3528.2974
400	3814	3520.7858
273	3945	3461.7919
36	3976	3441.7082
339	4005	3430.3337
102	4036	3413.2990
2	4438	3303.2424

We note also that the picture kind of suggests a binomial distribution because of the variance that appears to grow as the mean gets closer to the middle.

Indulging this hypothesis for just a moment, the actual graph (say for $a_i = 2$ specifically), is measuting "for each congruence class, how many complements of 2 in the sequence are in that congruence class?" The idea that this graph is a binomial distribution would be saying that we can perform 5422 independent trials (say one for each congruence class?) with identical success probabilities (one possible such test is pick randomly from the 100000 a_ns in that congruence class and ask whether 2 is ever a complement of that a_n), and the proportion of times 2 shows up as a complement of k is equal to the proportion of times we get exactly k successes in our trials.

4 Theory

4.1 Density

As we have said, it appears that the Ulam sequence has positive (upper) density around 0.07.

Conjecture 4.1. *The Ulam sequence has positive upper density.*

Conjecture 4.2. *For any decision sequence S with a positive density of 1s, the corresponding sum-free set $A = \theta(S)$ has positive upper density.*

For such sum-free sets A, there is kind of a battle: as we build A from the decision sequence, say we have built all elements up to some N, i.e. we have computed A_N. If A_N is very large, then $A_N + A_N$ will contain many elements and we will have to go far to find the next element of A. If on the other hand, A_N is not large, then $A_N + A_N$ will be sparse, which will make it easy to find the next few elements of A relatively quickly.

A first try at formalising this might go: $|A_N + A_N| \leq |A_N|^2$, and so if for all N, $|A_N|^2 \leq cN$ for some constant c, then

4.2 Circle method

Being in the Ulam sequence is determined by one property: x must be a sum of distinct Ulam numbers in exactly one way.

Definition 7. Define the **representation counting function** r_{A+A} by $r_{A+A}(x)$ to be the number of solutions to $a + b = x$ for $a, b \in A$. (Note that we do not necessarily require $a < b$ here.)

The point is that while this doesn't exactly capture the same notion as in the exact definition of the Ulam sequence, it is very close and has the advantage of being accessible from the Fourier-analytic perspective.

The main observation is that $r_{A+A} = 1_A * 1_A$, and so can be written as $\mathscr{F}^{-1}\mathscr{F}(1_A * 1_A) = \mathscr{F}^{-1}\widehat{1_A}^2$, or:

$$r_{A_{2N}+A_{2N}}(x) = \frac{1}{N}\sum_{t=0}^{N-1} \widehat{1_A}(t)^2 e(tx)$$

So we can conclude, for example, that $x \notin A$ if, say, $r_{A_{2N}+A_{2N}}(x) > 3$.

4.2.1 No Ulam numbers close to 0 mod λ

It appears that element in $[-\lambda/6, \lambda/6]$ are never in the Ulam sequence, and further, that the reason they have for not being in is that they are sums of smaller Ulam numbers in more than one way. We might take an $x \in \mathbb{N}$ with $x \pmod{\lambda} \in [-\lambda/6, \lambda/6]$ and see what we can make of it.

[DATA]

With the assumption that $x \pmod{\lambda} \in [-\lambda/6, \lambda/6]$, we get that if k/N is approximately $1/\lambda$, then $e_N(kx)$ will have argument between $-\pi/3$ and $\pi/3$. In particular, the real part of $e_N(kx)$ will be at least $1/2$. Thus if we compute:

$$\begin{aligned}
r_{A+A}(x) &= \frac{1}{N}\sum_{t=0}^{N-1} \widehat{1_A}(t)^2 e(tx) \\
&= |A|/N + 2\Re(e(kx)\widehat{1_A}(k)^2) + \sum_{t\neq 0,k,N-k} \widehat{1_A}(t)^2 e(tx) \\
&\geq |A|/N + \Re(\widehat{1_A}(k)^2) + \sum_{t\neq 0,k,N-k} \widehat{1_A}(t)^2 e(tx)
\end{aligned}$$

And so if $\widehat{1_A}(k)$ is much larger than all the other Fourier coefficients, then we might be able to guarantee that this is positive. For example: If we know that the only Fourier coefficients that are nonzero (in the large N limit) are $k\alpha$, then we can write this as:

$$\begin{aligned}
r_{A+A}(x) &\geq |A|/N + \Re(\widehat{1_A}(\alpha)^2) + \sum_{t>1} \Re(\widehat{1_A}(t\alpha)^2 e(tx)) \\
&\geq |A|/N + \Re(\widehat{1_A}(\alpha)^2) - \sum_{t>1} |\widehat{1_A}(t\alpha)|^2
\end{aligned}$$

So if $|\widehat{1_A}(\alpha)|$ is large enough–say, c, and $|\widehat{1_A}(t\alpha)|$ decays appropriately as t grows–say is less than $\frac{A}{t}$, then we get:

$$\begin{aligned}
r_{A+A}(x) &\geq |A|/N + c^2 - \sum_{t>1} \frac{A^2}{t^2} \\
&= \delta + c^2 - A^2\left(\frac{\pi^2}{6} - 1\right) \\
&\geq \delta + c^2 - 0.644A^2
\end{aligned}$$

So depending on the precise constants involved, we might end up with a conclusion that every such x that lands close to 0 mod λ necessarily has many representations and is therefore disqualified from being an Ulam number for that reason.

4.2.2 Few Ulam numbers outside middle third mod λ

A conjecture of Gibbs states...

4.2.3 Numbers that are not sums of Ulam numbers close to middle mod λ

Numbers x that fail to be Ulam because in fact $r^*_{A+A}(x) = 0$ all seem to lie within the middle third.

4.2.4 Large Fourier coefficient

If instead of thinking about representations of any one element–that is, counting solutions to $x + y = z$ with $x, y \in A$ for a fixed z–we think about counting solutions to $x+y = z$ with $x, y, z \in A$ using the same Fourier analytic technique, then we can actually get a theorem:

Theorem 4.3. *If $A \subseteq \mathbb{N}$ is a sequence of positive integers of density δ such that $T(A_N)$–the number of additive triangles in A_N–is bounded by $cN^{2-\epsilon}$ for some constants $c > 0, \epsilon > 0$, then there is an $\alpha \in \mathbb{R}/\mathbb{Z}$ such that $\widehat{1_A}(\alpha) \geq \delta^2$.*

Example 3. For example, in the Ulam sequence we know by construction that $T(A_N) \leq 3|A| \leq 3N$, and we believe that the Ulam sequence has density around 0.07, so this theorem would guarantee us a non-zero Fourier coefficient of size at least 0.0049. This is a bit off our numerical value of 0.8, but it is a start.

Intuitively, this goes as follows: If this set had no large non-zero Fourier coefficient, then it would be in some sense "random". A random set of density δ has of course $\delta^2 N^2$ pairs (x, y) with $x, y \in A$. Supposing these sums are randomly distributed, with probability δ they will themselves lie in A, so we expect to have $\delta^3 N^2$ of them being in A, giving $T(A_N) \cong \delta^3 N^2$. If $T(A_N)$ grows more slowly than this, therefore, there must be a reason for it, in the form of a non-zero Fourier coefficient. Put another way "if a set is dense, then to avoid solutions to $x + y = z$ it needs to have some kind of pattern (such as being all odds)." The statement we have here is a first version of precisely the kind of "general regularity" result that we are gunning for.

Proof. As we've discussed many times,

$$T(A_N) = (1/N) \sum_{t=0..N-1} \widehat{1_{A_N}}(t)\widehat{1_{A_N}}(t)\widehat{1_{A_N}}(-t)$$

So by assumption,

$$cN^{2-\epsilon} \geq (1/N) \sum_{t=0..N-1} \widehat{1_{A_N}}(t)\widehat{1_{A_N}}(t)\widehat{1_{A_N}}(-t)$$

We can pull out the $t = 0$ term which is $|A|^3$, which, for N large enough, is close to $\delta^3 N^3$. Then we can bound the remaining sum by pulling out a $\widehat{1_A}(t)$ and replacing it with $-\max_{t\neq 0} \widehat{1_A}(t)$:

$$cN^{2-\epsilon} \geq \delta^3 N^2 - \max_{t=1..N-1}(\widehat{1_A}(t))(1/N) \sum_{t=1..N-1} |\widehat{1_A}(t)|^2$$

Now, by Plancherel we know that, $(1/N)\sum_{t=0..N-1} |\widehat{1_A}(t)|^2 = \sum_{t=0..N-1} 1_A(t) = |A|$, so:

$$cN^{2-\epsilon} \geq \delta^3 N^2 - \max_{t\neq 0}(\widehat{1_A}(t))|A| = \delta^3 N^2 - \max_{t=1..N-1}(\widehat{1_A}(t))\delta N$$

Thus if $\max_{t=1..N-1}(\widehat{1_A}(t)) \leq \epsilon N$, then

$$cN^{2-\epsilon} \geq N^2(\delta^3 - \delta\epsilon)$$

Or, rearranging,

$$\frac{c}{\delta} N^{-\epsilon} \geq \delta^2 - \epsilon$$

But the left side goes to zero as $N \to \infty$, so $\epsilon \geq \delta^2$, i.e. for some k in $\{1, \ldots, N-1\}$, we have

$$|\widehat{1_{A_N}}(k)| > \delta^2 N$$

Now, as we let N grow, we get a sequence of values of k_N with this property. On $\mathbb{R}/\mathbb{Z}$, the sequence $\frac{k_N}{N}$ for increasing N must have a convergent subsequence–say converging to some α. $\square$

Some remarks:

Remark 1. As we mentioned before, this bound for the Ulam sequence, which works out to around 0.0049 is not anywhere near as good as the computed estimate of 0.8. However, this finds a rational k/N where the Fourier transform is large for *every* N, whereas experimentally the large value of around 0.8 only occurs actually at α and can only be observed at rational k/N that are good approximations to α. In particular, for some N, the largest Fourier coefficient might honestly only be as large as δ^2.

Remark 2. In the proof of Roth's theorem, the existence of a large Fourier coefficient in A is somehow used to deduce the existence of an arithmetic progression P such that A intersected with P has higher density in P than A had in $\mathbb{Z}/N$. Roth's theorem concludes by making all this numerically precise to be able to say "if we repeat this often enough, either we'll eventually have small Fourier coefficient relative to the increased density, or we'll have density 1 in an arithmetic progression at which point...well...we will be guaranteed to contain an arithmetic progression!" In our case, we are always guaranteed a largeish Fourier coefficient by the above argument, so maybe we can always perform this "density increment" step until we are literally an arithmetic progression. Precisely what this implies about the Fourier coefficients of the original 1_A or whether this ensures us any global behaviour of the sequence A depends on precisely how the density-increment step goes. This will be a another thing to investigate shortly.

Remark 3. It is interesting to note that this argument does not provide an obvious way to take advantage of the uniformity with which solutions to $x+y=z$ occur in the Ulam case. For example, it also applies to a sequence where $a_{2^i+1}, \ldots, a_{2^{(i+1)}-1}$ have no representations but a_{2^i} has 2^{i-1} representations for each i (in which case the number of representations is not bounded above, but is growing, albeit sort of slowly and non-uniformly).

4.2.5 Variants of Ulam problem

Circle method gets better with more variables. For this reason, we might find it convenient to

4.3 Arithmetic regularity

4.4 Is there only one non-zero big Fourier coefficient?

5 Appendix A: Code

Code 1:

```
b2=2.57144749848
b=N(2*pi/2.57144749848)

def find_mp(x,deg=50,n=10^10):
    M = []
    for i in range(deg+1):
        M.append([1 if v == i else 0 for v in range(deg+1)]+[int(n*(x^i)+.5)])
    M = matrix(M)
    F=FP_LLL(M)
    F.LLL()
    l=F._sage_()[0][:-1]

    R = ZZ["X"]
    X=R.gen()
    fx = 0
    for i in range(len(l)):
        fx += X^i * l[i]
    ans = N(fx(x),50)
    return(fx, ans, fx.factor())

ms = [find_mp(b,i) for i in range(2,10)]
ms.sort(key=lambda x:abs(x[1]))
for x in ms:
    print(x)

print("")
ms = [find_mp(b2,i) for i in range(2,10)]
ms.sort(key=lambda x:abs(x[1]))
for x in ms:
    print(x)
```

Code 2:

```
cat data/indices_of_summands | cut -d " " -f 4 | sort -n|uniq -c|
    sort -n > data/n_minus_js
cat data/indices_of_summands | cut -d " " -f 3,4 | sort -n|uniq -c|
    sort -n > data/i_nmj
```

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