Computational geophysics

We'll look at 2 equations:

- heat diffusion
- wave propagation

using different numerical methods to solve

- finite-difference
- pseudo-spectral

$\Big\}$ use discretization of differential operator

- finite-element
- spectral-element

$\Big\}$ use weak form & approximate integral

seismology in a nutshell: "do simulations & compare synthetics with data"

Conservation laws

- mass
- linear momentum *
- angular momentum
- energy **

these are (all) the physical laws we are solving

* wave propagation

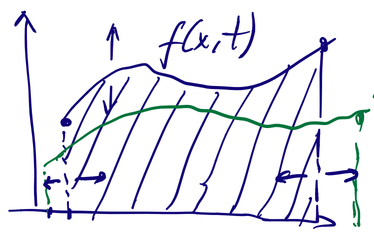
** heat equation

\+ constitutive relation

That's all we do in computational seismology / geophysics.

Conservation of mass

Example : rate of change for area



$$\text{area} = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

is the area under the curve $f(x,t)$.

What is the rate of change of this area?

$$\frac{d}{dt}\,\text{area} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big[\underbrace{\int_{a(t+\Delta t)}^{b(t+\Delta t)} f(x, t+\Delta t)\,dx}_{\text{area at time } t+\Delta t} - \underbrace{\int_{a(t)}^{b(t)} f(x,t)\,dx}_{\text{area at time } t}\Big]$$

$$= \int_{a(t)}^{b(t)} \lim \frac{1}{\Delta t}\left[f(x,t+\Delta t) - f(x,t)\right] dx$$

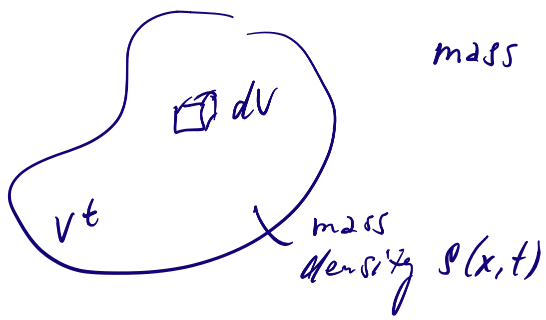
$$+ f(b(t),t) \lim \frac{1}{\Delta t}\left[b(t+\Delta t) - b(t)\right]$$

$$- f(a(t),t) \lim \frac{1}{\Delta t}\left[a(t+\Delta t) - a(t)\right]$$

$$= \int_{a(t)}^{b(t)} \underbrace{\partial_t}_{\partial/\partial t} f(x,t)\,dx + f(b(t),t)\overbrace{\frac{d}{dt} b(t)}^{\dot b} - f(a(t),t)\overbrace{\frac{d}{dt}a(t)}^{\dot a}$$

$$= \int_{a(t)}^{b(t)} \partial_t f(x,t)\,dx + \underbrace{\dot b(t) f(b(t),t) - \dot a(t) f(a(t),t)}_{f(x,t)\, v\cdot n \,\big|_a^b \quad \text{flux in \& out}}$$

conservation of mass



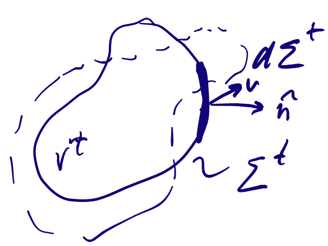
$$\text{mass} = \int_{V^t} \underbrace{\rho(x,t)}_{\text{density}} \, d^3x$$

conservation of mass, that is the rate of change of mass is zero:

$$\frac{d}{dt}\,\text{mass} = \frac{d}{dt}\int_{V^t} \rho(x,t)\, d^3x \underset{\substack{\uparrow \\ \text{physics}}}{=} 0$$

mass balance

$$\frac{d}{dt}\int_{V^t} \rho(x,t)\,dV = \underbrace{\int_{V^t} \partial_t \rho(x,t)\, dV}_{\substack{\text{fixed volume} \\ \text{change in mass}}} + \underbrace{\int_{\Sigma^t} \rho\, \underline{v}\cdot\underline{\hat{n}}\, d\Sigma}_{\text{flux through surface}}$$



Gauss' theorem $\displaystyle\int_S \underline{\hat{n}}\cdot\underline{u}\, dS = \int_V \underline{\nabla}\cdot\underline{u}\, dV$

Thus,

$$\int_{\Sigma^t} \rho\, \underline{v} \cdot \underline{\hat{n}}\, d\Sigma = \int_{V^t} \underline{\nabla} \cdot (\rho \underline{v})\, dV$$

and

$$\frac{d}{dt} \int_{V^t} \rho(x,t)\, dt = \int_{V^t} [\partial_t \rho + \underline{\nabla} \cdot (\rho \underline{v})]\, dV \overset{!}{=} 0$$

valid for any volume

Therefore

$$\boxed{\partial_t \rho + \nabla \cdot (\rho \underline{v}) = 0}$$

continuity equation

We define the material derivative as

$$\underbrace{D_t}_{} = \partial_t + \underline{v} \cdot \underline{\nabla}$$

time derivative in Lagrangian description

measures the rate of change as you move with the flow ($\underline{v}$)

(connection between Lagrangian & Eulerian description)

We can rewrite

$$\partial_t \rho + \nabla \cdot (\rho \underline{v}) = \underbrace{\partial_t \rho + \underline{v} \cdot \underline{\nabla} \rho}_{D_t \rho} + \rho\, \underline{\nabla} \cdot \underline{v}$$

$$D_t \rho + \rho\, \underline{\nabla} \cdot \underline{v} \overset{!}{=} 0$$

and find

$$\boxed{\frac{1}{\rho} D_t \rho = - \underline{\nabla} \cdot \underline{v}}$$

The rate of change of density is then equal to the opposite as the divergence of the velocity field incompressible flow $\underline{\nabla} \cdot \underline{v} = 0 \longrightarrow D_t \rho = 0$

incompressible fluid : $D_t \rho \overset{!}{=} 0$

water incompressible,

air nearly