Finite-element method

The FE method is a weak form method where the operators, i.e., (time &) spatial derivatives, are exact, but the solution basis functions are approximated.

Weak form (or variational statement):

Let's consider the following 1D partial differential equation for $u \in [0,1]$

$$\boxed{\partial_x^2 u + f = 0}$$

steady-state diffusion

with boundary conditions

$$u(1) = q$$ Dirichlet boundary

$$\partial_x u(0) = -h$$ Neumann boundary

The weak form is derived by introduction an arbitrary "test" function $w(x)$ and

1.) the PDE is dotted with the test function

$$w(x)\,\partial_x^2 u(x) + w(x)\,f(x) = 0$$

2.) the new PDE is integrated over the problem domain

$$\int_0^1 w(x)\,\partial_x^2 u(x)\,dx + \int_0^1 w(x)\,f(x)\,dx = 0$$

$$\overbrace{\text{using integration parts}}$$

$$-\int_0^1 \partial_x w \, \partial_x u \, dx + \underbrace{\int_0^1 \partial_x (w \partial_x u) \, dx}_{w \partial_x u |_0^1}$$

$$\partial_x (w \partial_x u) = w \partial_x^2 u + \partial_x w \, \partial_x u$$

Thus, the weak form of the steady-state diffusion equation can be written

$$\boxed{-\int_0^1 \partial_x w \, \partial_x u \, dx + w \partial_x u |_0^1 + \int_0^1 w f \, dx = 0}$$

With boundary conditions:

- the arbitrary test function with $\boxed{w(1) = 0}$

$$w \partial_x u |_0^1 = \underbrace{w(1)}_{=0} \partial_x u(1) - w(0) \underbrace{\partial_x u(0)}_{-h}$$

$$= + h \, w(0)$$

- the weak form becomes

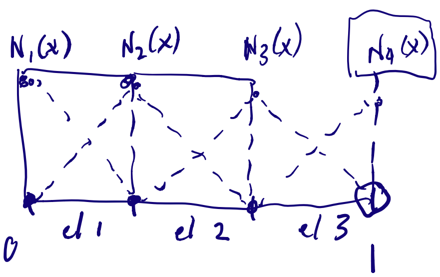
$$\boxed{\underbrace{-\int_0^1 \partial_x w \, \partial_x u \, dx}_{\text{stiffness term}} + \underbrace{h \, w(0)}_{\text{boundaries}} + \underbrace{\int_0^1 w f \, dx}_{\text{forcing}} = 0}$$

$$\int_0^1 \partial_x f \, \partial_x g \, dx = a(f, g) \quad \text{bilinear functional}$$

$$\int_0^1 f g \, dx = (f, g) \quad \text{linear functional}$$

The finite-element method is solving the problem locally on an element, by introducing a discretization scheme and finite-dimensional approximation.
The discretization is done by an approximation of the solution by interpolating basis functions.

Galerkin's method: We define shape functions $N_A(x)$ as basis functions on which we expand our functions:



$$\boxed{\begin{aligned} w(x) &= \sum_{A=1}^{n} c_A N_A(x) \\ u(x) &= \sum_{B=1}^{n} d_B N_B(x) \end{aligned}}$$

$N_A(x)$: shape ("nodal") functions

Galerkin's approach: $N_A(x) \hat{=} N_B(x)$

with boundary conditions: use $\underline{N_A(1)=0}$ since $\underline{w(1)=0}$

$$w(x) = \sum_{A=1}^{n} c_A N_A(x)$$

$$u(x) = \sum_{B=1}^{n} d_B N_A(x) + q \underbrace{N_{n+1}(x)}$$

with $N_{n+1}(1)=1$ since $u(1)=q$

n : number of elements
$n+1$: number of shape functions (or number of nodes)

where the coefficients c_A & d_B are the unknowns to solve for.

Weak form

$$-\int_0^1 \partial_x w \, \partial_x u \, dx + \int_0^1 w f \, dx + h w(0) = 0$$

replacing into the weak form leads

$$-\int_0^1 \underbrace{\sum_A c_A \partial_x N_A}_{\partial_x w} \underbrace{\sum_B d_B \partial_x N_B}_{\partial_x u} dx - \int_0^1 \underbrace{\sum_A c_A \partial_x N_A}_{\partial_x w} q \partial_x N_{n+1} + \int_0^1 \underbrace{\sum_A c_A N_A}_{w} f \, dx$$

$$+ h \underbrace{\sum_A c_A N_A(0)}_{w(0)} = 0$$

valid for any coefficient c_A (test function).

Zeroing out all but one put to 1, we find

$$\boxed{-\sum_B \int_0^1 \partial_x N_A \boxed{d_B} \partial_x N_B \, dx - \int_0^1 \partial_x N_A \, q \, \partial_x N_{n+1} \, dx + \int_0^1 N_A f \, dx + h N_A(0) = 0}$$

In matrix form $\underbrace{K_{AB}}_{\text{stiffness matrix}} := \int_0^1 \partial_x N_A \, \partial_x N_B \, dx = a(N_A, N_B)$

$n \times n$ - matrix

$\underbrace{F_A}_{\text{force vector}} := \int_0^1 N_A f \, dx + N_A(0) h - q \int_0^1 \partial_x N_A \, \partial_x N_{n+1} \, dx$

n-vector

We can rewrite it as

$$K_{AB}\, d_B = F_A \qquad \text{with} \quad \begin{array}{l} A = 1, \ldots, n \\ B = 1, \ldots, n \end{array}$$

The solution for unknowns d_B is simply

$$d_B = \underline{\underline{K}}^{-1}\, \underline{F}$$