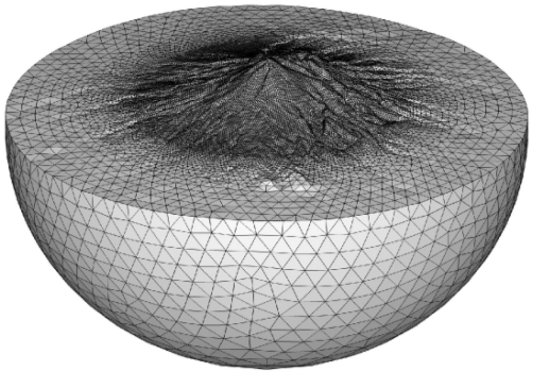

Discontinuous Galerkin method

Lecture notes

The Discontinuous Galerkin (DG) method is a higher-order weak-form method, like the Spectral-Element method (SEM), but allowing for more general and flexible piecewise continuous functions. The method combines the accuracy of high-order methods with the concept of fluxes met in finite-volume methods.

Weak form or variational statement

Let us consider the 1-D steady-state diffusion equation in the strong form to find the temperature field $T(x,t)$ with $x \in [0, L]$ such that

$$\partial_x^2 T + f = 0$$

where f is some heat source.

As previously shown with the FEM, the weak form is obtained by dotting the strong form with a test function $\mathrm{w}(x)$ and integrating over the whole domain. Thus, the weak form writes as

$$\int_0^L \mathrm{w}\, \partial_x^2 T \,\mathrm{d}x + \int_0^L \mathrm{w}\, f \,\mathrm{d}x = 0$$

Now let us define the 1D finite-elements:

$$\Omega = \bigcup_e \Omega_e \text{ , with elements } \Omega_e = [x_e, x_{e+1}]$$

The weak form can thus be written as a sum over local elements

$$\sum_e \int_e \mathrm{w}\, \partial_x^2 T \,\mathrm{d}x + \sum_e \int_e \mathrm{w}\, f \,\mathrm{d}x = 0$$

To ensure the locality of the scheme, we duplicate the degrees of freedom at the nodes x_e to write the discretized temperature field T as a vector

$$T = [T_1,\, T_2,\, T_2,\, T_3,\, T_3,\, ..]^T$$

and allow the field to be approximated in each element by polynomial functions. Thus, the functional space of the solution field becomes a piecewise polynomial space defined on a local level of

degrees of freedom. Note that this deviates from the previous finite-element and spectral-element approaches we looked at, where the solution field T would be defined on a global level. For the DG approach to find a global solution to our problem, we have to connect the local elemental solutions between neighboring elements. This will be achieved through the concept of fluxes, known from finite-volume methods.

For simplicity, let us focus on the diffusion term and consider only the integral for a single element e. Using integration by parts on an elemental level, we see that

$$\begin{aligned}\int_e \mathsf{w}\,\partial_x^2 T\,\mathrm{d}x &= \int_e \{-\partial_x \mathsf{w}\,\partial_x T + \partial_x\,(\mathsf{w}\,\partial_x T)\}\,\mathrm{d}x \\ &= -\int_e \partial_x \mathsf{w}\,\partial_x T\,\mathrm{d}x + \left[\mathsf{w}(x_{e+1})\,\partial_x T(x_{e+1}) \;-\; \mathsf{w}(x_e)\,\partial_x T(x_e)\right] \\ &= -\int_e \partial_x \mathsf{w}\,\partial_x T\,\mathrm{d}x + \mathsf{w}\,\partial_x T\Big|_{x_e}^{x_{e+1}}\end{aligned}$$

In a more general form, Gauss' theorem turns the volume integral over the divergence to surface integrals with the contributions along the surface normals $\hat{n}$

$$\int_e \nabla\cdot(\mathsf{w}\,\nabla T)\,\mathrm{d}V \;= \int_{\partial e} \mathsf{w}\,\nabla T\cdot\hat{n}\,\mathrm{d}S$$

For the 1D problem above, this would look like:

$$\int_e \partial_x\,(\mathsf{w}\,\partial_x T)\,\mathrm{d}x \;= \int_{\partial e} \mathsf{w}\,\partial_x T\cdot\hat{n}\,\mathrm{d}x = \mathsf{w}\,\partial_x T\Big|_{x_e}^{x_{e+1}}$$

where the normal $\hat{n}$ is outward pointing and either $+1$ or -1 for the right and left interface, respectively.

These surface integrals lead to the concept of "fluxes" which help connecting the solution between elements. The numerical flux, in the case above we can define it as $\mathcal{F}^* = \nabla T^* = \partial_x T^*$, is the unique solution used at the interface, combining information from neighboring elements. The schemes to do so are known from the finite-volume method, for example we could use an upwinding scheme to determine the fluxes.

Choice of basis functions

We still have to define the polynomial basis to approximate the field within each element, together with a way to compute volume and surface integrals, e.g., by a quadrature rule to approximate them.

For the function approximations, the DG method uses a local polynomial basis of order $N = N_p - 1$ and chooses between a *modal* or *nodal* representation:

- the *modal* form expresses the local solution within element e using the modes $\psi_n(x)$

$$u^e(x,t) = \sum_{n=1}^{N_p} \hat{u}_n^e(t)\,\psi_n(x)$$

where the local polynomial basis can have a simple example $\psi_n(x) = x^{n-1}$.

- the *nodal* representation uses a number of local grid (nodal) points x_i^e together with, e.g., associated Lagrange polynomials $l_i^e(x)$

$$u^e(x,t) = \sum_{i=1}^{N_p} \hat{u}^e(x_i^e, t)\, l_i^e(x)$$

The global solution $u(x,t)$ is then approximated by the piecewise N-th order polynomial approximations $u^e(x,t)$, combining them accordingly at interface points,

$$u(x,t) \simeq \bigoplus_e u^e(x,t)$$

Note that in case the *nodal* representation uses Lagrange polynomials, the integral approximations become very similar to the spectral-element method. We can choose the same mapping techniques as seen before to map between physical and reference elements, as well as using a GLL quadrature rule to take advantage of the orthogonal property of GLL collocation points and Lagrange polynomials. The *modal* representation is usually choosing orthogonal polynomials as basis functions and pre-computes corresponding integral terms analytically.

Furthermore, there is no assembly of global mass-, flux- and stiffness matrices needed as the method is completely local. This will make the DG approach a very efficient one, highly parallelizable, and with the flexibility of choosing different sizes and polynomial degrees N on each element (so called *hp*-adaptivity) in a non-conforming mesh.

Finally, choosing different element shapes other than quadrangles or hexahedrals, for example triangles or tetrahedrals, will make meshing much more flexible. Unfortunately, Lagrange polynomials and GLL collocation points for such reference elements will loose the orthogonal property, and therefore we would either choose different polynomials and quadrature rules or resort to a *modal* representation. A popular choice for seismic waves propagation with triangles is to use Dubiner polynomials as in Käser & Dumbser (2006) [1] due their orthogonal properties and a *modal* representation approach. The corresponding integrals over basis functions in mass-, flux- and stiffness matrices can then be pre-computed analytically (or numerically) for the reference element, making it a quadrature-free scheme.

[1] Käser M., Dumbser M., 2006. *An arbitrary high order discontinuous Galerkin method for elastic waves on unstructured meshes - I. The two-dimensional isotropic case with external source terms*, Geophys. J. Int., **166**, 855–877.