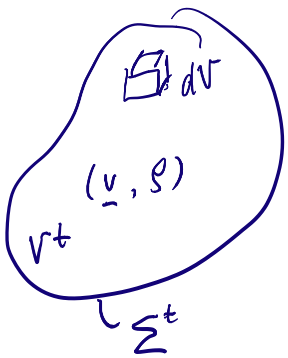
Conservation of energy



kinetic energy $\frac{1}{2} \int_{V^t} \rho \, \underline{v} \cdot \underline{v} \, dV$

rate of change $\frac{d}{dt} \int_{V^t} \frac{1}{2} \rho \, \underline{v} \cdot \underline{v} \, dV$

rate at which mechanical energy is added

$$\underbrace{\int_{V^t} \underline{v} \cdot \underline{f} \, dV}_{\text{rate of work by body forces}} + \underbrace{\int_{\Sigma^t} \underline{v} \cdot \underline{t} \, d\Sigma}_{\text{rate of work by surface tractions}}$$

traction $\underline{t} = \hat{\underline{n}} \cdot \underline{\underline{T}}$

and additional energies : - thermodynamic internal energy
- heat production (radioactive)

rate of change of thermodynamic internal energy

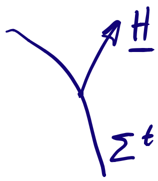
$$\frac{d}{dt} \int_{V^t} \rho \, \mathcal{U} \, dV$$

$\mathcal{U}$: internal energy (per unit mass)

rate of heat production : $\int_{V^t} \rho \, h \, dV$

h : heat production (per unit mass)

rate at which heat is conducted out of V^t:



$$\int_{\Sigma^t} \hat{\underline{n}} \cdot \underline{H} \, d\Sigma$$ $\underline{H}$: heat flux

Conservation of energy:

$$\frac{d}{dt} \int_{V^t} \Big(\underbrace{\tfrac{1}{2} \rho \underline{v} \cdot \underline{v}}_{\text{kinetic}} + \underbrace{\rho \mathcal{U}}_{\text{potential/internal}} \Big) dV = \int_{V^t} (\underline{v} \cdot \underline{f} + \rho h) dV$$

$$+ \int_{\Sigma^t} (\hat{\underline{n}} \cdot \underline{\underline{T}} \cdot \underline{v} - \hat{\underline{n}} \cdot \underline{H}) \, d\Sigma$$

$$= \int_{V^t} (\underline{v} \cdot \underline{f} + \rho h + \underline{\nabla} \cdot (\underline{\underline{T}} \cdot \underline{v} - \underline{H})) \, dV$$

valid for any volume V^t

After some math, we find using $\underline{\nabla} \cdot (\underline{\underline{T}} \cdot \underline{v}) = \underline{v} \cdot \underline{\nabla} \cdot \underline{\underline{T}} + \underline{\underline{T}} : (\underline{\nabla} \underline{v})$,

$$\rho D_t \left(\tfrac{1}{2} \underline{v} \cdot \underline{v} + \mathcal{U} \right) + \underline{\nabla} \cdot (\underline{H} - \underline{\underline{T}} \cdot \underline{v}) = \underline{v} \cdot \underline{f} + \tilde{h}$$

$$\underline{v} \cdot \underbrace{(\rho D_t \underline{v} - \underline{\nabla} \cdot \underline{\underline{T}} - \underline{f})}_{\substack{\text{conservation of momentum} \\ = 0}} + \rho D_t \mathcal{U} + \underline{\nabla} \cdot \underline{H} = \underline{\underline{T}} : (\underline{\nabla} \underline{v}) + \tilde{h}$$

$\tilde{h}$: heat production (per unit volume)

Using $\underline{\underline{T}} : \underline{\nabla} \underline{v} = T^{ij} \nabla_i v_j$

$$= T^{ij} \tfrac{1}{2} (\nabla_i v_j + \nabla_j v_i) = \underline{\underline{T}} : \underbrace{\dot{\underline{\underline{\varepsilon}}}}$$

strain rate

we find

$$\boxed{\rho D_t \mathcal{U} + \underline{\nabla} \cdot \underline{H} = \underline{\underline{T}} : \dot{\underline{\underline{\varepsilon}}} + h}$$ conservation of energy

Note: Due to conservation of energy, for an elastic medium $\underline{\underline{T}} = \underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}}$ and therefore

$$\rho D_t \mathcal{U} = \underline{\underline{T}} : \dot{\underline{\underline{\varepsilon}}} = \underbrace{\underline{\underline{\varepsilon}} : \underline{\underline{\underline{\underline{c}}}}}_{\underline{\underline{T}}} : \dot{\underline{\underline{\varepsilon}}}$$ in the absence of h and $\underline{\nabla} \cdot \underline{H} = 0$

it follows that $c_{ijkl} = c_{klij}$

Heat equation

$\mathcal{U}$ is a thermodynamic quantity which needs an equation of state $\mathcal{U} = \mathcal{U}(\hat{T})$ $\hat{T}$: temperature

$$D_t \mathcal{U} = \underbrace{\frac{\partial \mathcal{U}}{\partial \hat{T}}}_{= c_v} D_t \hat{T}$$

c_v specific heat at constant volume

Assume that heat flux $\underline{H}$ is proportional to the temperature gradient $\underline{\nabla} \hat{T}$, i.e., Fourier's law $\underline{H} = -\kappa \underline{\nabla} \hat{T}$

with κ conductivity.

Thus,

$$\rho c_v D_t \hat{T} + \underline{\nabla} \cdot (-\kappa \underline{\nabla} \hat{T}) = \hat{h}$$

or

$$\boxed{\rho c_v (\partial_t \hat{T} + \underline{v} \cdot \underline{\nabla} \hat{T}) = \underline{\nabla} \cdot (\kappa \underline{\nabla} \hat{T}) + \hat{h}}$$ heat equation

$\hat{h}$: heat source

ρ, c_v: constants material properties

κ: conductivity

1-D equation: We find

$$\rho c_v \underline{\partial_t \hat{T}} = \underline{\partial_x} (\kappa \underline{\partial_x} \hat{T}) + h \quad \text{heat equation}$$

$$\rho \underline{\underline{\partial_t^2}} s = \underline{\partial_x} (\mu \underline{\partial_x} s) + f \quad \text{wave equation}$$