Problem Set 11

Finite-volume solution to 1D wave equation

Use the Finite-Volume method (FVM) to solve the 1D wave equation to find the displacement $s(x,t)$ for $x \in [0, L = 100]$ such that (strong form)

$$\rho\,\partial_t^2 s = \partial_x(\mu\,\partial_x s)$$

where ρ is the medium density and μ is the shear modulus, with the following initial & boundary conditions:

$$(a)\ \ Dirichlet \qquad \begin{cases} s(x,0) &=& f(x) \\ s(L,t) &=& 0 \\ s(0,t) &=& 0 \end{cases}$$

and

$$(b)\ \ Neumann \qquad \begin{cases} s(x,0) &=& f(x) \\ \partial_x s(L,t) &=& 0 \\ \partial_x s(0,t) &=& 0 \end{cases}$$

Problem:

Follow these steps to solve the problems (a) and (b):

- write the integral form of the wave equation
- discretize the mesh: $\Omega = [0, L] = \bigcup_e \Omega_e$
- calculate the mass and stiffness matrix contributions
- impose the boundary conditions for (a) and (b)
- consider the initial condition with $f(x) = \exp[-(x-50)^2 * 0.1]$
  and media properties $\{\rho = 1$ and $\mu = 1\}$ and/or combined with a heterogeneous one $\{\rho = 1$ and $\mu = 2\}$. Note that for an elastic solid-solid interface, the stresses (in normal direction) should be kept continuous.

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Time scheme:

Use the following Newmark algorithm to march in time:

- Predictor:

$$\begin{aligned} d_{n+1} &= d_n + \Delta t \, v_n + \frac{1}{2} \Delta t^2 \, a_n \\ v_{n+1} &= v_n + \frac{1}{2} \Delta t \, a_n \\ a_{n+1} &= 0 \quad \text{(initialization at the beginning of each time step)} \end{aligned}$$

- Solve:

$$\begin{aligned} F_{n+1} &= K \, d_{n+1} \\ a_{n+1} &= M^{-1} \, F_{n+1} \end{aligned}$$

- Corrector:

$$v_{n+1} = v_{n+1} + \frac{1}{2} \Delta t \, a_{n+1}$$

Use different numbers of finite-volume grid cells ($N_{el} = 100$ or $N_{el} = 500$) and plot several time steps. For the heterogeneous case, you can compare the FVM solutions against pre-computed SEM solutions in folder figures/.