Finite-difference method

Heat equation

	symbols	SI units
Temperature	$T(x,t)$	K
Specific heat at constant pressure	$c_p(x)$	J/kg/K
density	$\rho(x)$	$\mathrm{kg/m^3}$
thermal conductivity	$K(x)$	W/K/m
thermal diffusivity	$D(x)$	$\mathrm{m^2/s}$

1-D heat energy conservation: $\rho(x)c_p(x)\partial_t T(x,t) = -\partial_x[q(x,t)]$

where $q(x,t) = -K(x)\partial_x T(x,t)$ is the heat flux (Fourier's law)

=> $\rho(x)c_p(x)\partial_t T(x,t) = \partial_x[K(x)\partial_x T(x,t)]$

If considering constant thermal properties and introducing the thermal diffusivity:

$$D = \frac{K}{\rho c_p}$$

=> $$\boxed{\partial_t T(x,t) = D\partial_x^2 T(x,t)}$$

Forward scheme in time

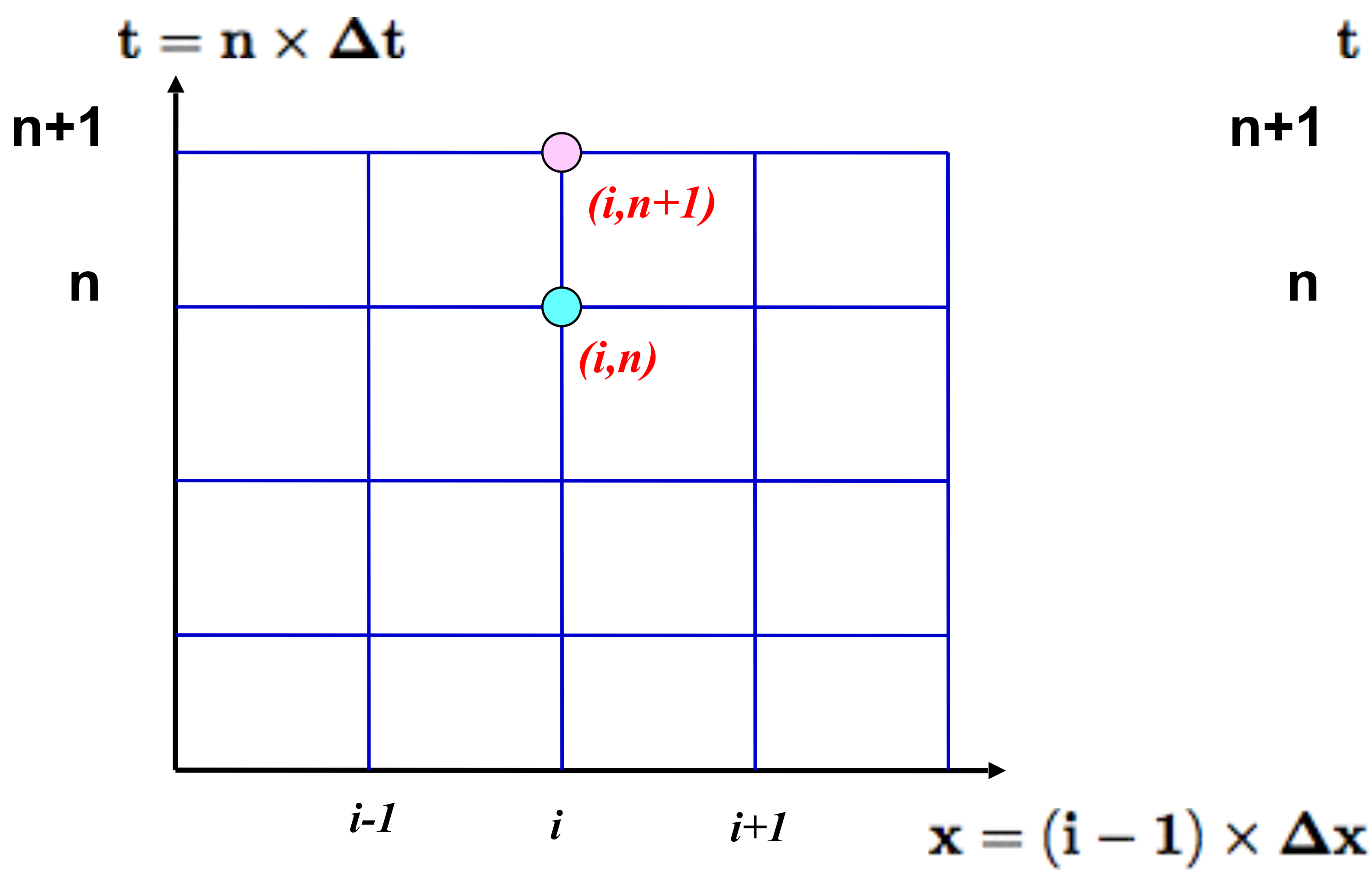
&

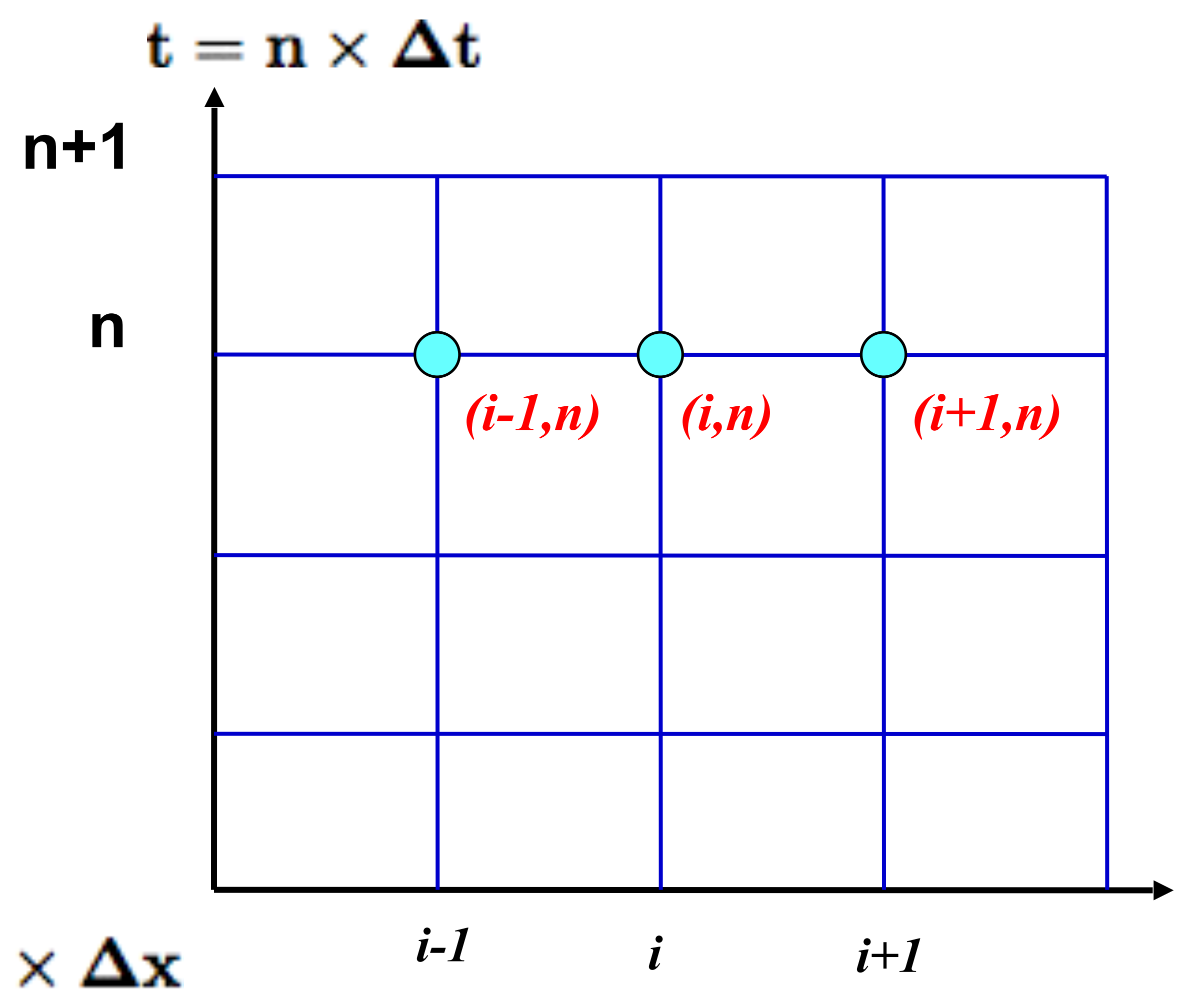
central difference in space

Forward scheme in time & central difference in space

$$\partial_t T(x,t) \approx \frac{T(x,t+\Delta t) - T(x,t)}{\Delta t}$$
$$\equiv \frac{T_i^{n+1} - T_i^n}{\Delta t}$$

$$\partial_x^2 T(x,t) \approx \frac{T(x+\Delta x,t) - 2T(x,t) + T(x-\Delta x,t)}{\Delta x^2}$$
$$\equiv \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}$$





Forward scheme in time & central difference in space

$$\partial_t T(x,t) \approx \frac{T(x,t+\Delta t) - T(x,t)}{\Delta t}$$

$$\partial_x^2 T(x,t) \approx \frac{T(x+\Delta x,t) - 2T(x,t) + T(x-\Delta x,t)}{\Delta x^2}$$

$$\boxed{\partial_t T(x,t) = D\partial_x^2 T(x,t)}$$

$$T_i^{n+1} = T_i^n + (T_{i+1}^n - 2T_i^n + T_{i-1}^n)\frac{D\Delta t}{\Delta x^2}$$

$$\begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{N-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_N \end{bmatrix} \begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \\ \dots \\ T_{N-1}^n \\ T_N^n \end{bmatrix} = \begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & 0 & a_N & b_N \end{bmatrix} \begin{bmatrix} T_1^{n-1} \\ T_2^{n-1} \\ T_3^{n-1} \\ \dots \\ T_{N-1}^{n-1} \\ T_N^{n-1} \end{bmatrix}$$

Backward scheme in time

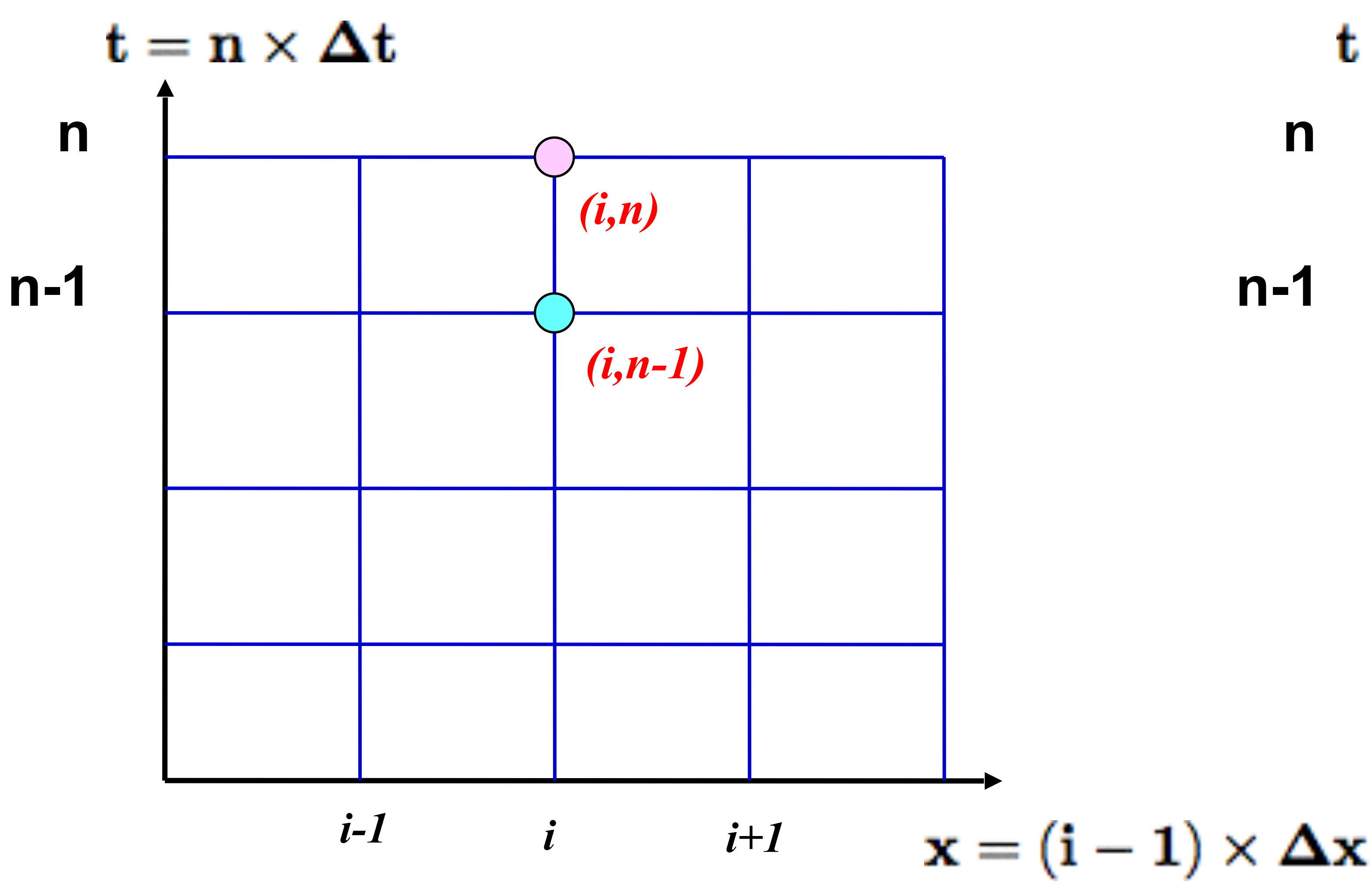
&

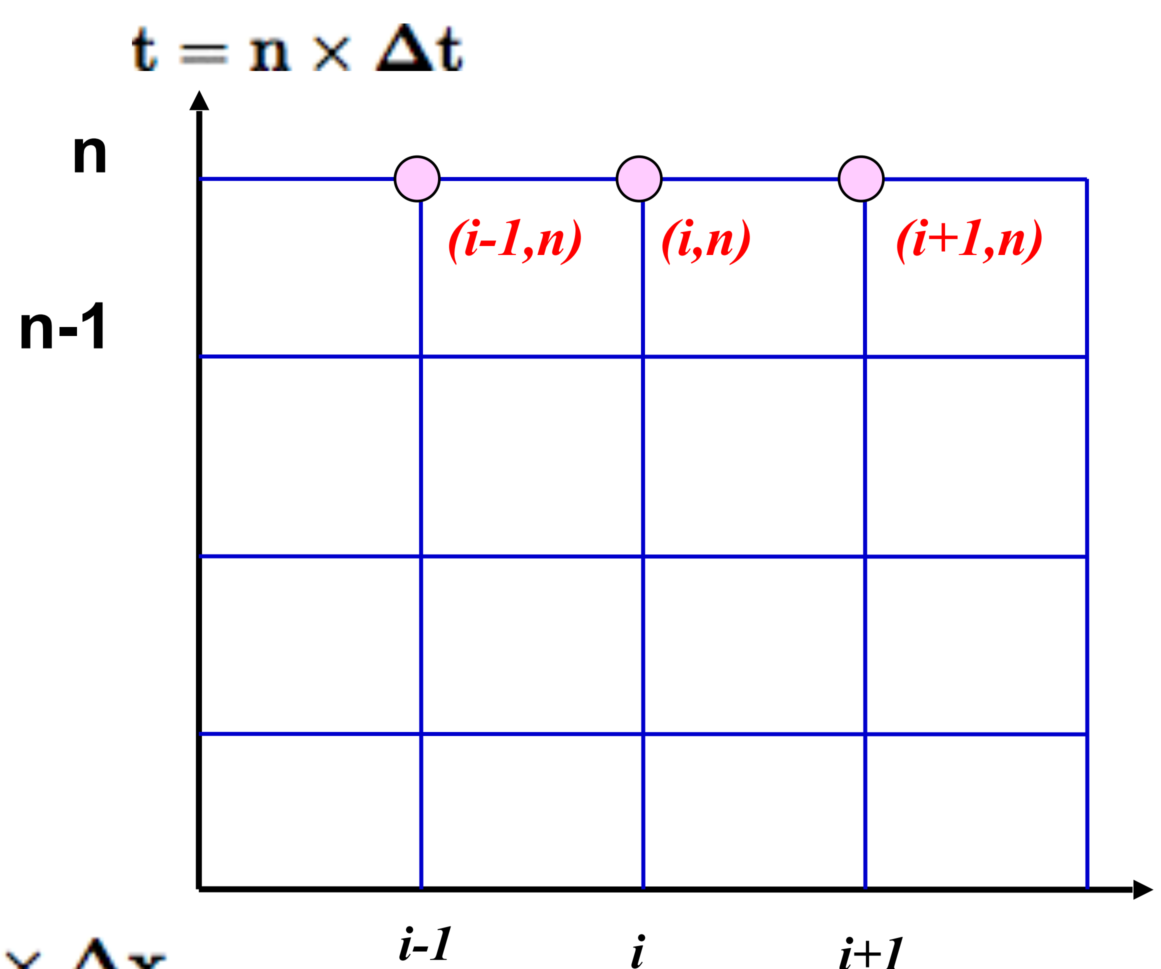
central difference in space

Backward scheme in time & central difference in space

$$\partial_t T(x,t) \approx \frac{T(x,t) - T(x,t-\Delta t)}{\Delta t}$$
$$\equiv \frac{T_i^n - T_i^{n-1}}{\Delta t}$$

$$\partial_x^2 T(x,t) \approx \frac{T(x+\Delta x,t) - 2T(x,t) + T(x-\Delta x,t)}{\Delta x^2}$$
$$\equiv \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}$$





Backward scheme in time & central difference in space

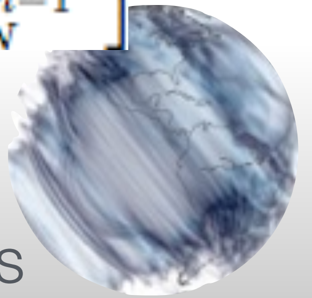
$$\partial_t T(x,t) \approx \frac{T(x,t)-T(x,t-\Delta t)}{\Delta t} \equiv \frac{T_i^n - T_i^{n-1}}{\Delta t}$$

$$\partial_x^2 T(x,t) \approx \frac{T(x+\Delta x,t)-2T(x,t)+T(x-\Delta x,t)}{\Delta x^2} \equiv \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}$$

$$\boxed{\partial_t T(x,t) = D\partial_x^2 T(x,t)}$$

$$-D\frac{\Delta t}{\Delta x^2}T_{i-1}^n + \left[1 + D\frac{2\Delta t}{\Delta x^2}\right]T_i^n - D\frac{\Delta t}{\Delta x^2}T_{i+1}^n = T_i^{n-1}$$

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & 0 & a_N & b_N \end{bmatrix} \begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \\ \dots \\ T_{N-1}^n \\ T_N^n \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{N-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_N \end{bmatrix} \begin{bmatrix} T_1^{n-1} \\ T_2^{n-1} \\ T_3^{n-1} \\ \dots \\ T_{N-1}^{n-1} \\ T_N^{n-1} \end{bmatrix}$$

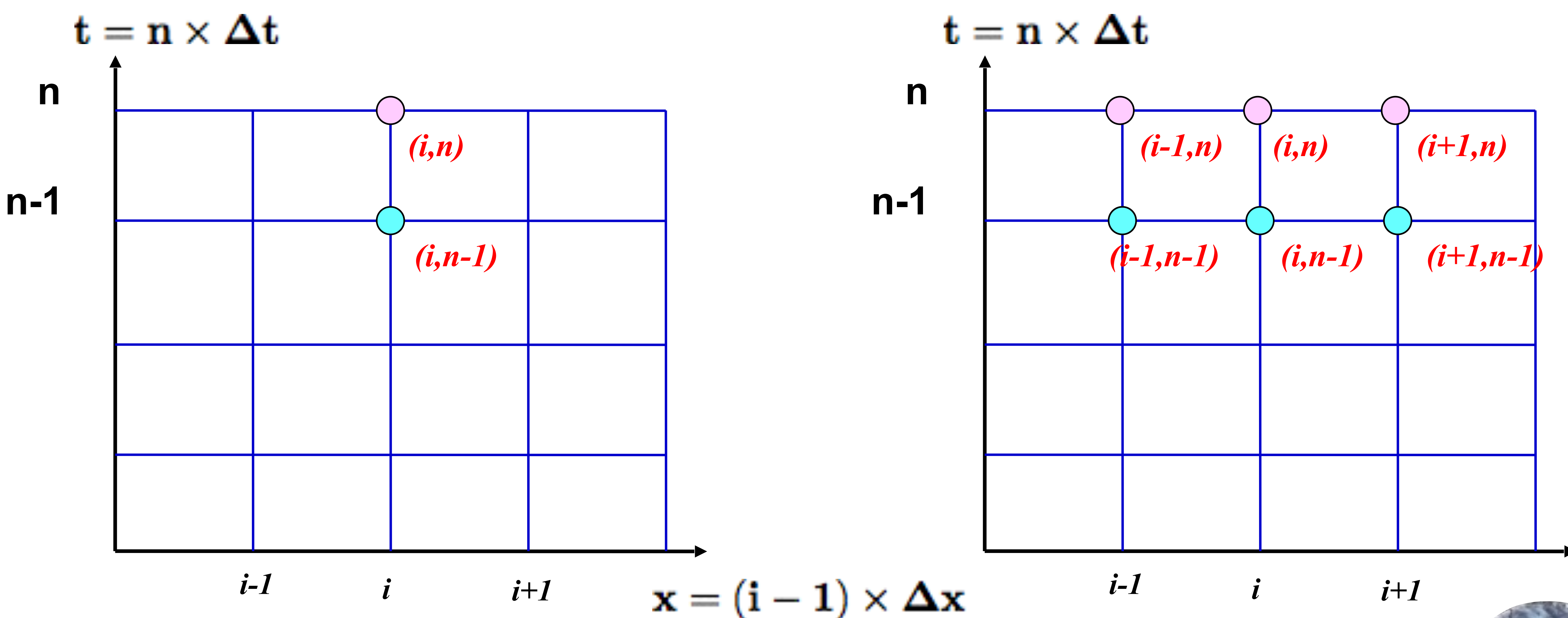


Crank-Nicolson scheme

Crank-Nicolson scheme

$$
\begin{aligned}
\partial_t T(x,t) &\approx \frac{T(x,t) - T(x,t-\Delta t)}{\Delta t} \\
&\equiv \frac{T_i^n - T_i^{n-1}}{\Delta t}
\end{aligned}
$$

$$
\begin{aligned}
\partial_x^2 T(x,t) \approx\ & \frac{1}{2}\frac{T(x+\Delta x,t) - 2T(x,t) + T(x-\Delta x,t)}{\Delta x^2} \\
&+ \frac{1}{2}\frac{T(x+\Delta x,t-\Delta t) - 2T(x,t-\Delta t) + T(x-\Delta x,t-\Delta t)}{\Delta x^2} \\
\equiv\ & \frac{1}{2}\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + \frac{1}{2}\frac{T_{i+1}^{n-1} - 2T_i^{n-1} + T_{i-1}^{n-1}}{\Delta x^2}
\end{aligned}
$$



Crank-Nicolson scheme

$$\partial_t T(x,t) \approx \frac{T(x,t) - T(x,t-\Delta t)}{\Delta t} \equiv \frac{T_i^n - T_i^{n-1}}{\Delta t}$$

$$\partial_x^2 T(x,t) \approx \frac{1}{2}\frac{T(x+\Delta x,t) - 2T(x,t) + T(x-\Delta x,t)}{\Delta x^2} + \frac{1}{2}\frac{T(x+\Delta x,t-\Delta t) - 2T(x,t-\Delta t) + T(x-\Delta x,t-\Delta t)}{\Delta x^2}$$

$$\equiv \frac{1}{2}\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + \frac{1}{2}\frac{T_{i+1}^{n-1} - 2T_i^{n-1} + T_{i-1}^{n-1}}{\Delta x^2}$$

$$\boxed{\partial_t T(x,t) = D\partial_x^2 T(x,t)}$$

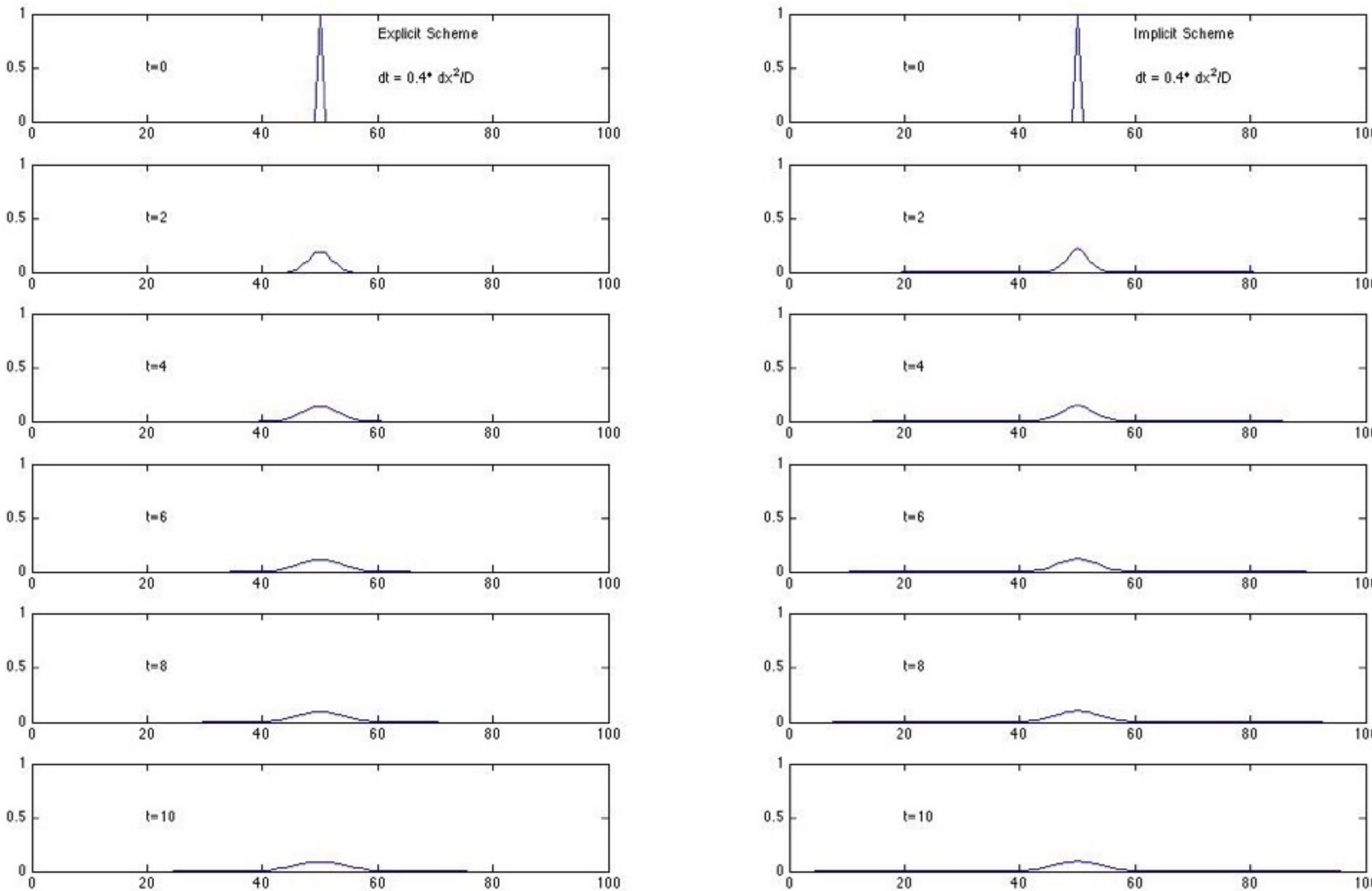
$$-D\frac{\Delta t}{2\Delta x^2}T_{i-1}^n + \left[1 + D\frac{\Delta t}{\Delta x^2}\right]T_i^n - D\frac{\Delta t}{2\Delta x^2}T_{i+1}^n = D\frac{\Delta t}{2\Delta x^2}T_{i-1}^{n-1} + \left[1 - D\frac{\Delta t}{\Delta x^2}\right]T_i^{n-1} + D\frac{\Delta t}{2\Delta x^2}T_{i+1}^{n-1}$$

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & 0 & a_N & b_N \end{bmatrix} \begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \\ \dots \\ T_{N-1}^n \\ T_N^n \end{bmatrix} = \begin{bmatrix} e_1 & f_1 & 0 & 0 & 0 & 0 \\ d_2 & e_2 & f_2 & 0 & 0 & 0 \\ 0 & d_3 & e_3 & f_3 & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & d_{N-1} & e_{N-1} & f_{N-1} \\ 0 & 0 & 0 & 0 & d_N & e_N \end{bmatrix} \begin{bmatrix} T_1^{n-1} \\ T_2^{n-1} \\ T_3^{n-1} \\ \dots \\ T_{N-1}^{n-1} \\ T_N^{n-1} \end{bmatrix}$$

Explicit vs. Implicit schemes
Results

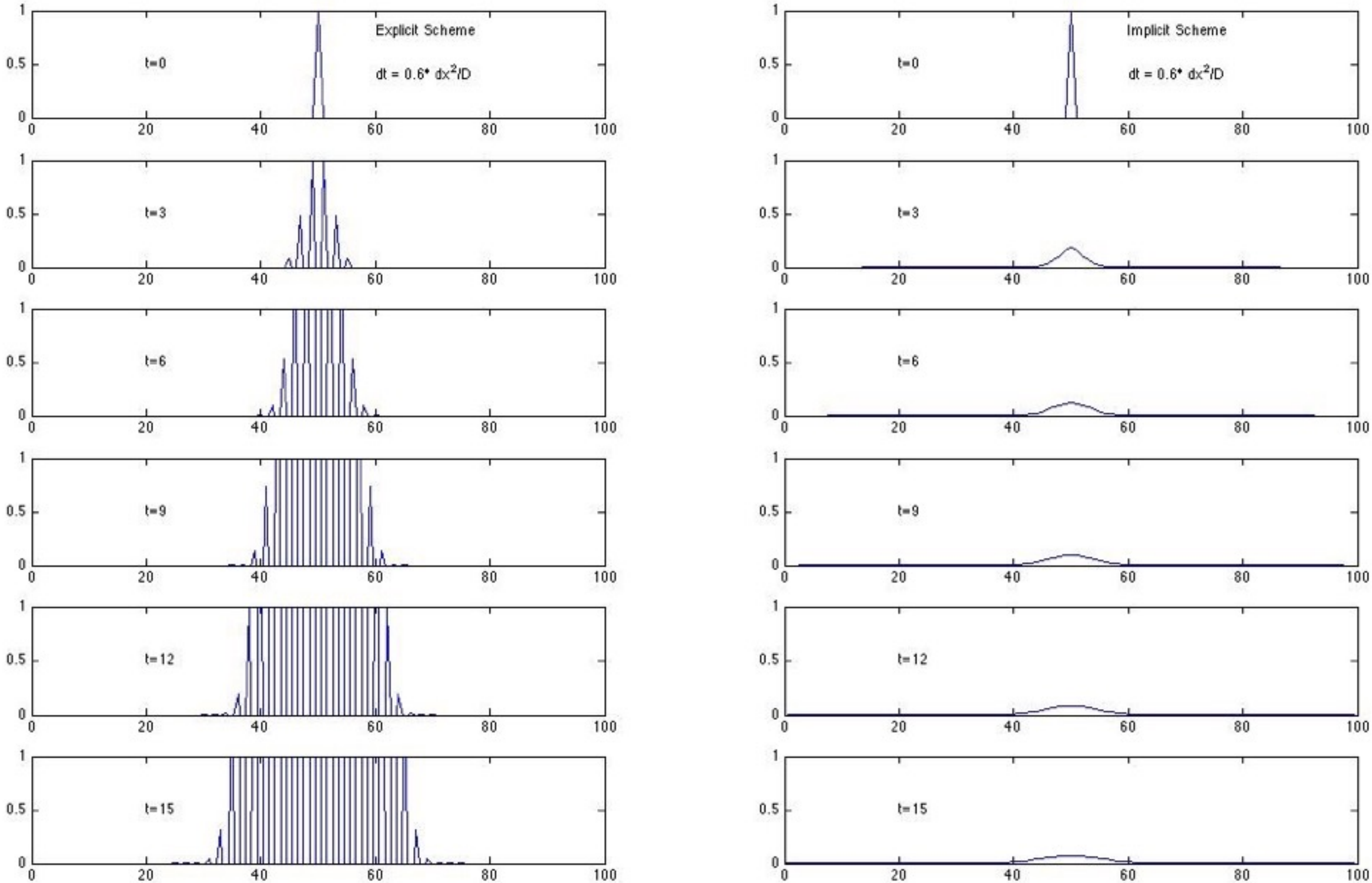
Explicit vs. Implicit schemes

factor = 0.4 -> dt = 0.4 * dt**2 / D



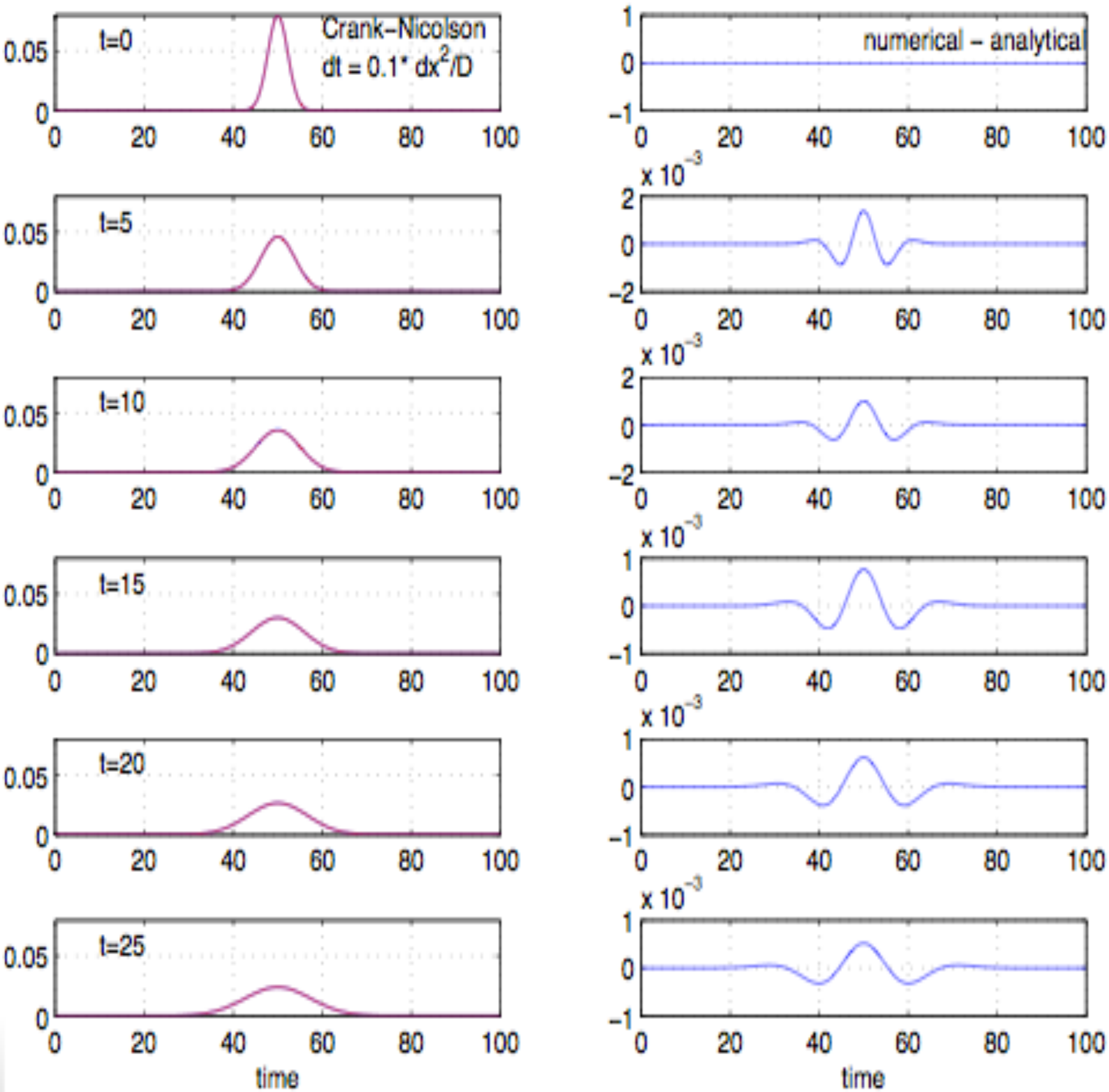
Explicit vs. Implicit schemes

factor = 0.6 -> dt = 0.6 * dt**2 / D



Crank-Nicolson scheme vs. analytic solution
Results

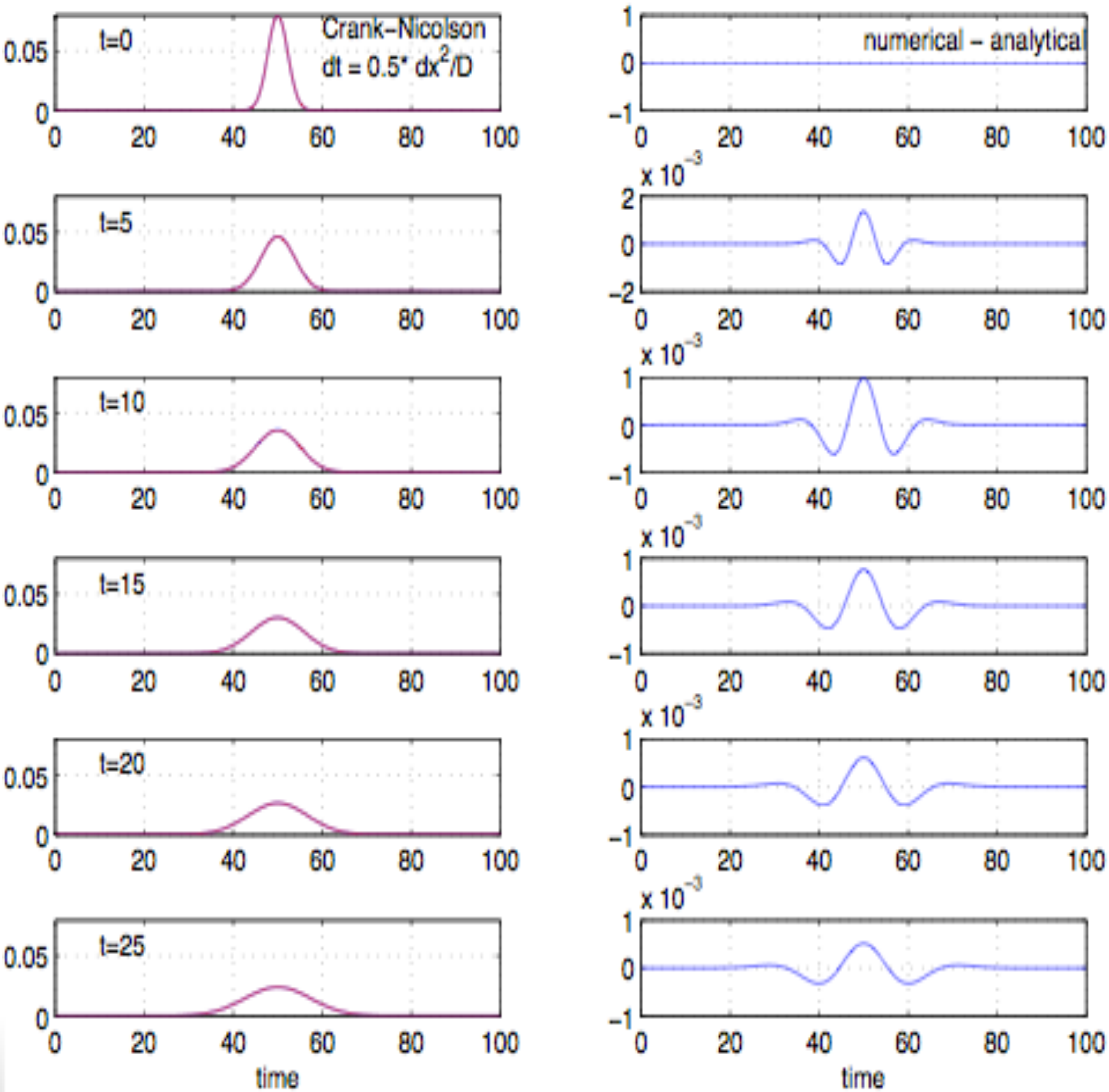
Crank-Nicolson scheme vs. analytic solution



Gaussian heat source
dx = 0.5
dt = 0.1 dx^2 /D

Red: analytical
Blue: numerical

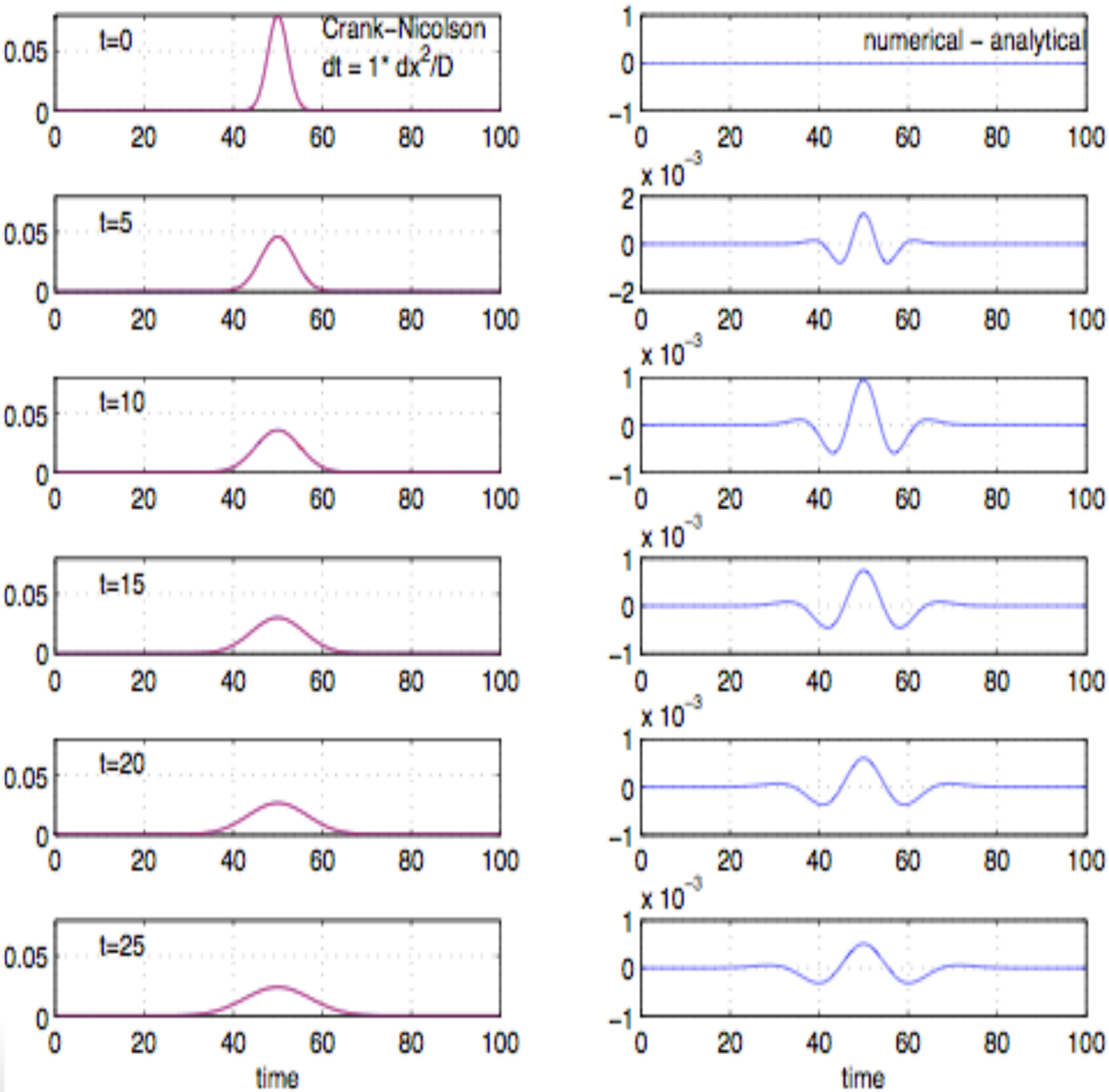
Crank-Nicolson scheme vs. analytic solution



Gaussian heat source
dx = 0.5
dt = 0.5 dx^2 /D

Red: analytical
Blue: numerical

Crank-Nicolson scheme vs. analytic solution

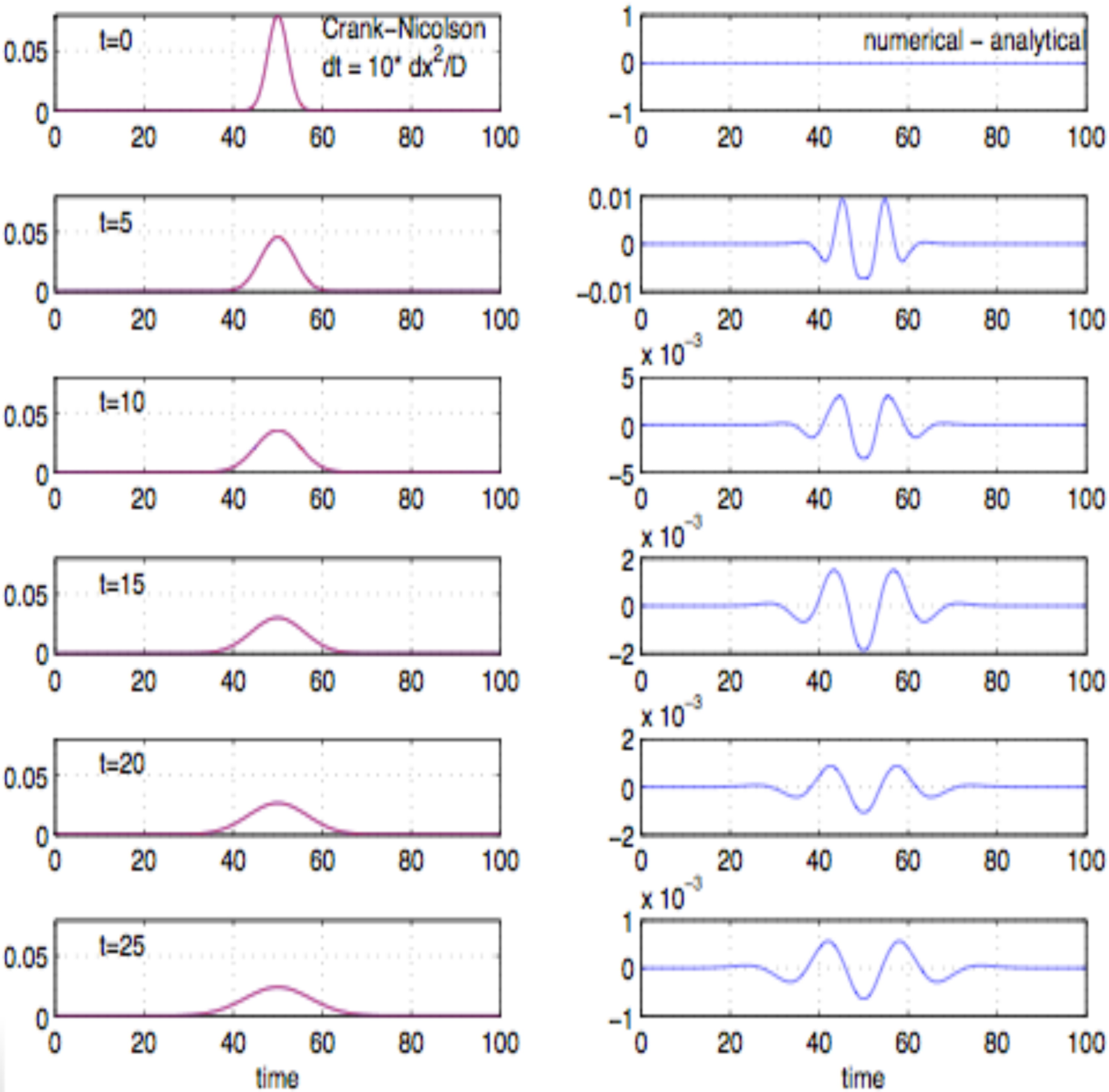


Gaussian heat source
dx = 0.5
dt = 1 dx^2/D

where
explicit scheme
becomes
unstable

Red: analytical
Blue: numerical

Crank-Nicolson scheme vs. analytic solution



Gaussian heat source
$dx = 0.5$
$dt = 10\ dx^2 / D$

where
explicit scheme
becomes
unstable

Red: analytical
Blue: numerical