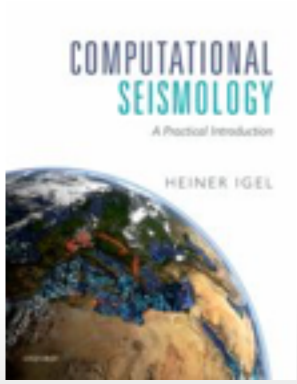
# Finite-element methods

# Finite-element literature

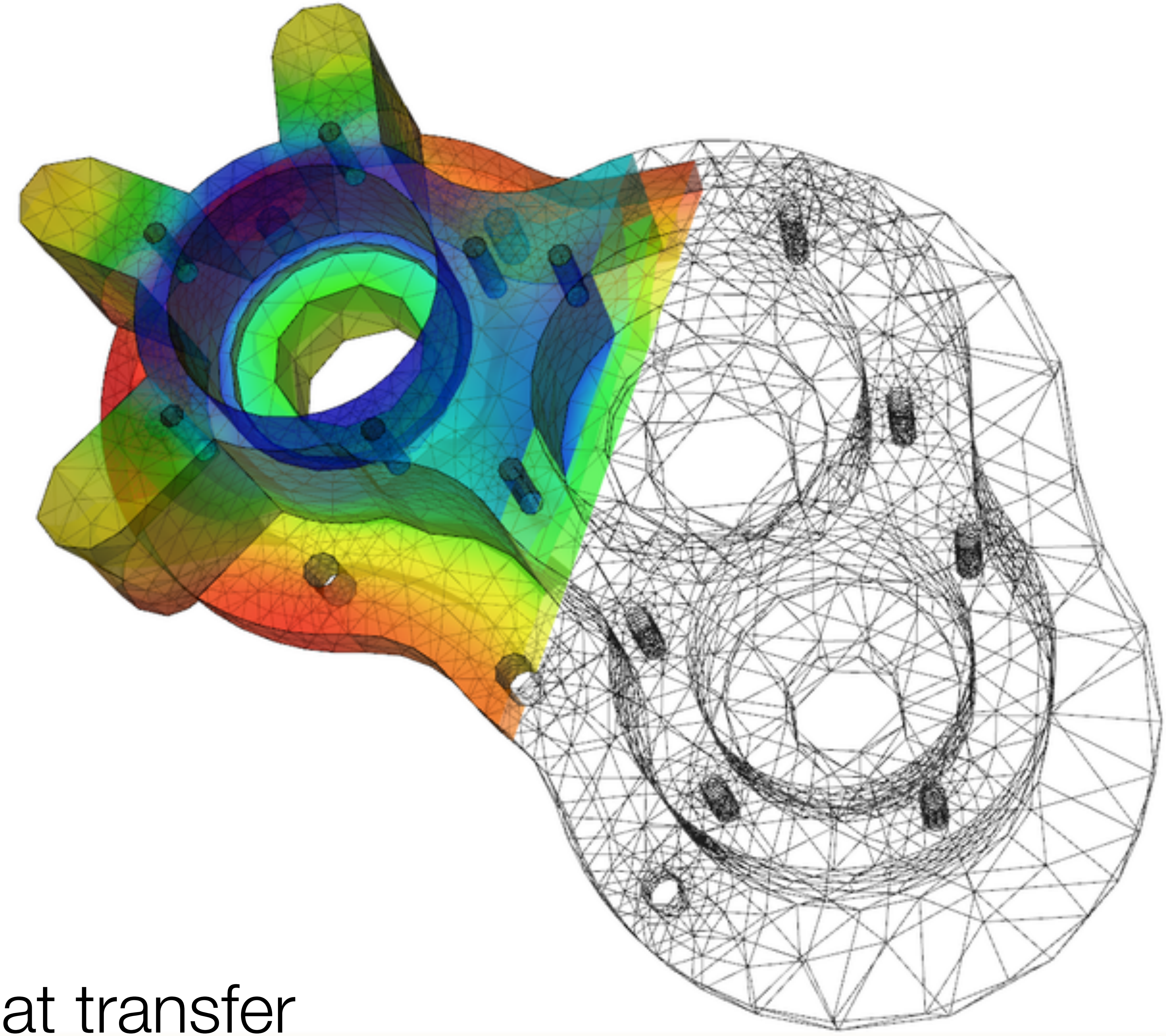
books


Thomas J.R. Hughes.
**The Finite Element Method :** linear static and dynamic finite element analysis,
Mineola, N.Y. : Dover, 2000.


O. C. Zienkiewicz, R. L. Taylor and J. Z. Zhu.
**The Finite Element Method :** its basis and fundamentals,
ISBN: 978-1-85617-633-0.



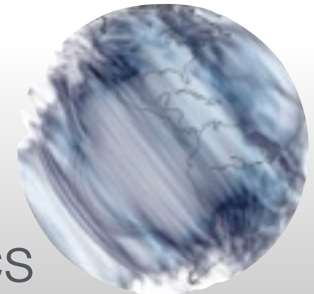

H. Igel
**Computational Seismology :** A practical introduction,
ISBN: 978-0-19871-740-9
http://www.oxfordscholarship.com/

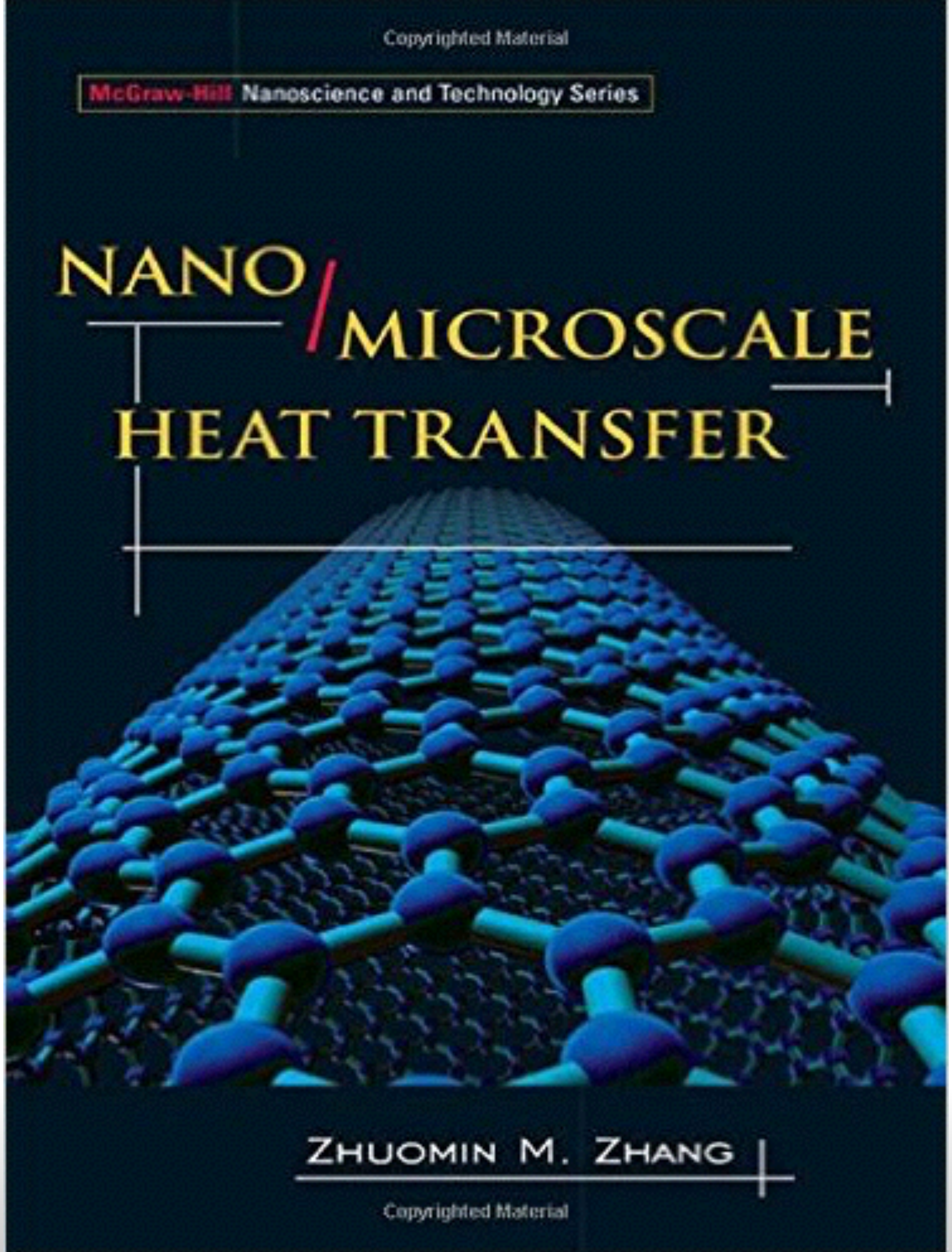


# Heat transfer

civil engineering

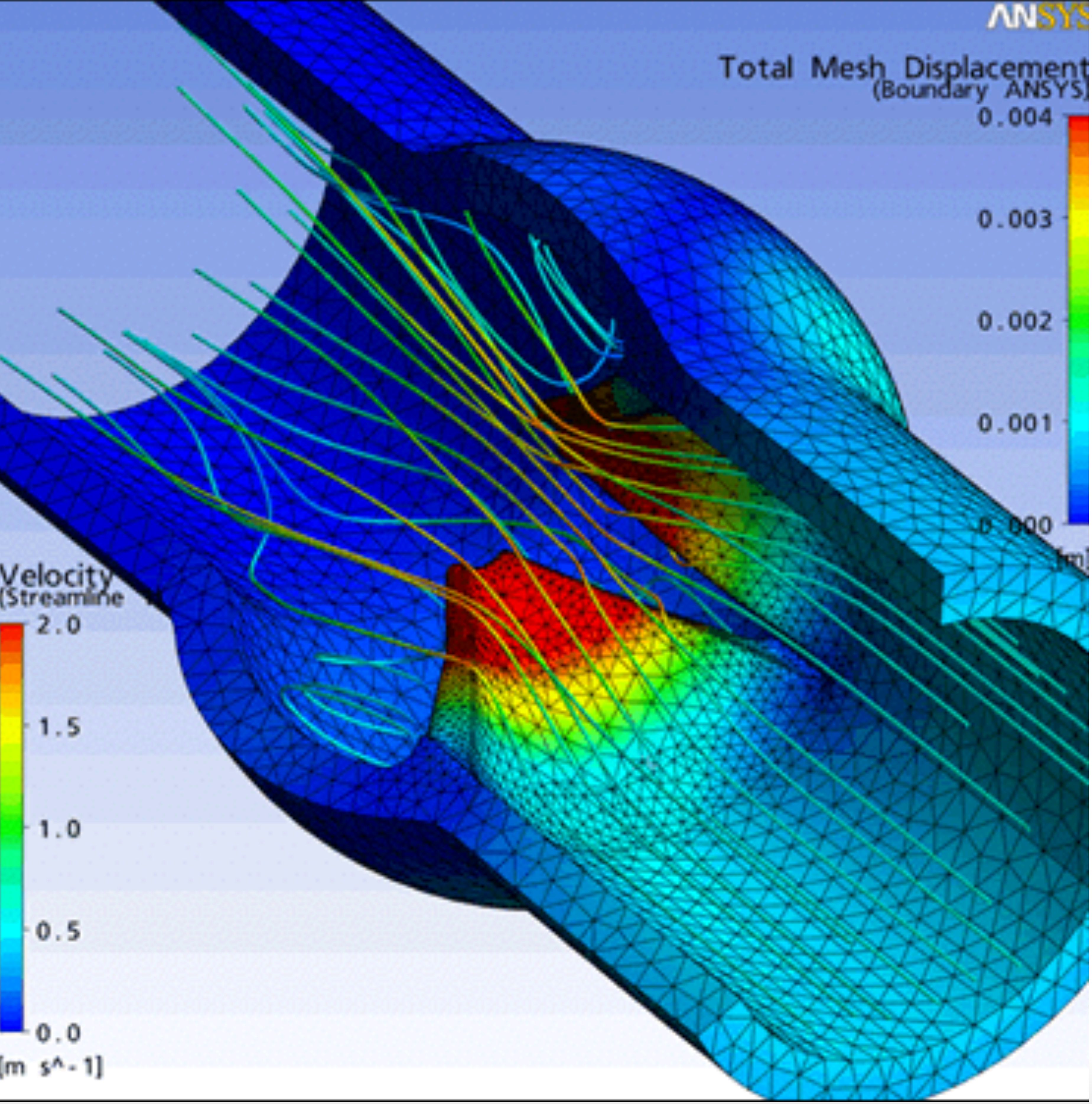






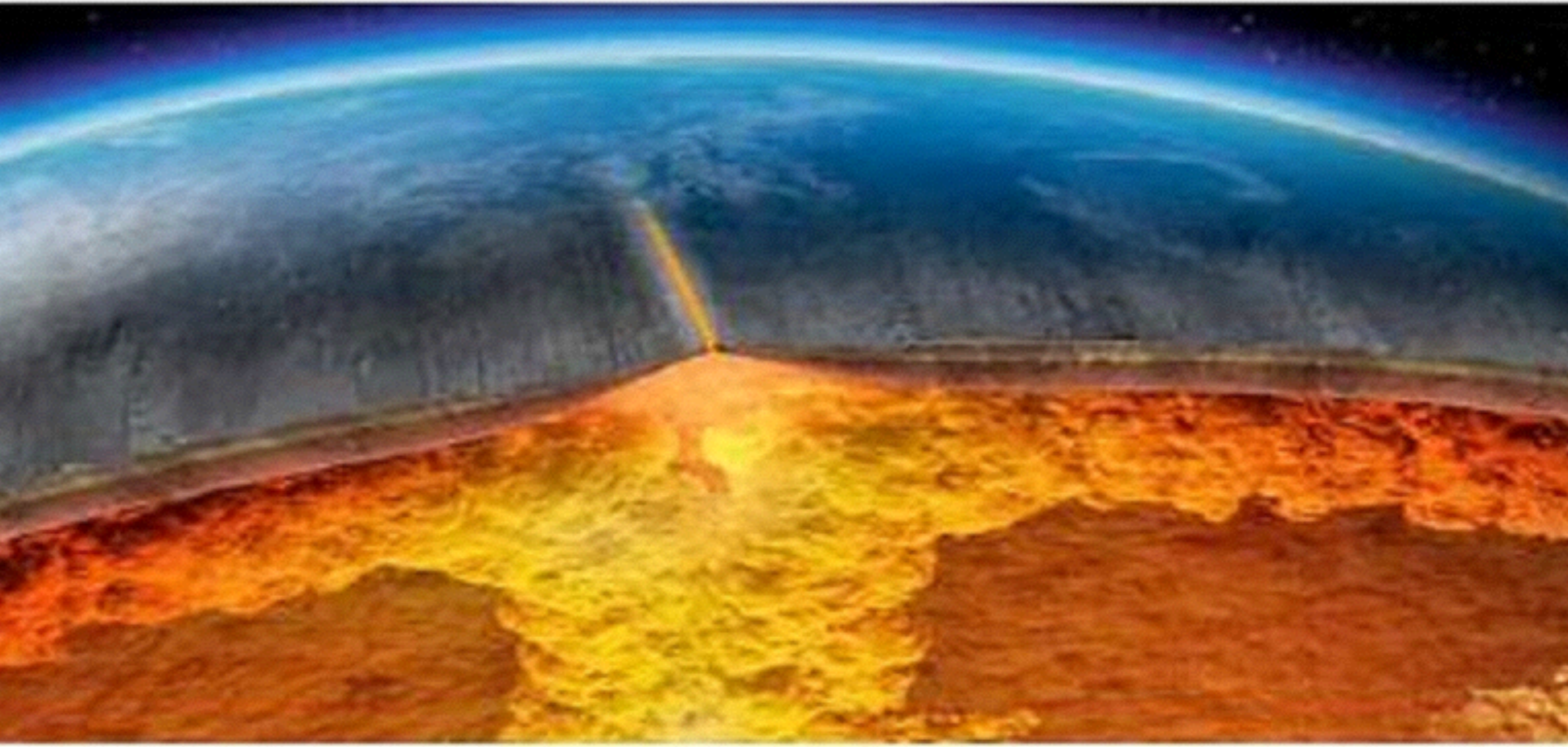

Heat transfer

nano-scales



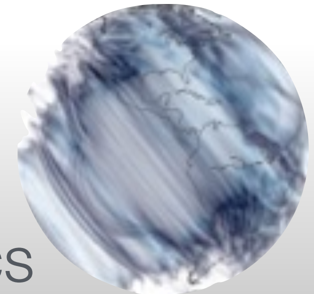

# Heat transfer

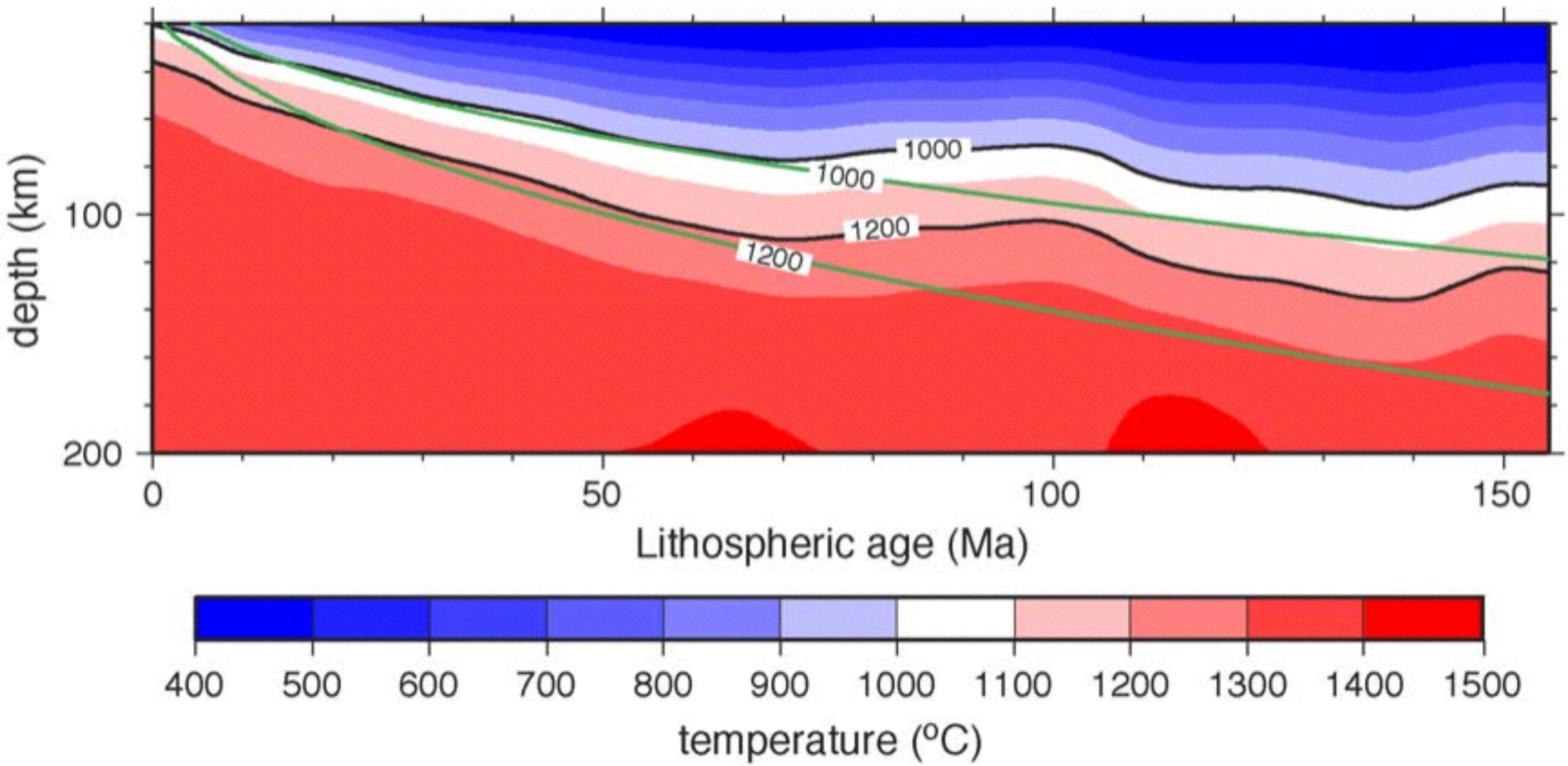
biological tissue



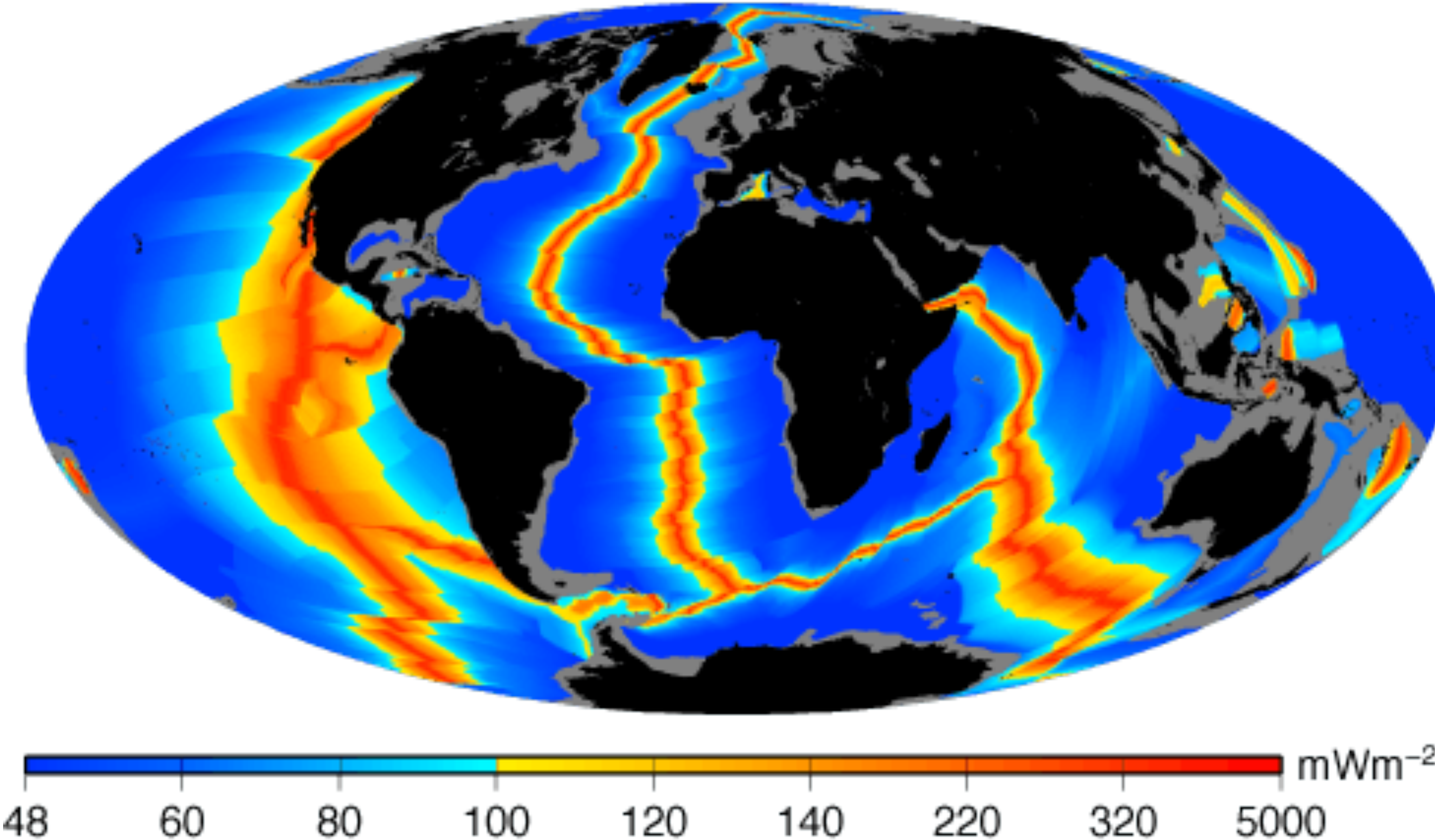
# Heat transfer

## half-space cooling






debate: half-space cooling of the Pacific plate
Ritzwoller et al. (EPSL, 2004)

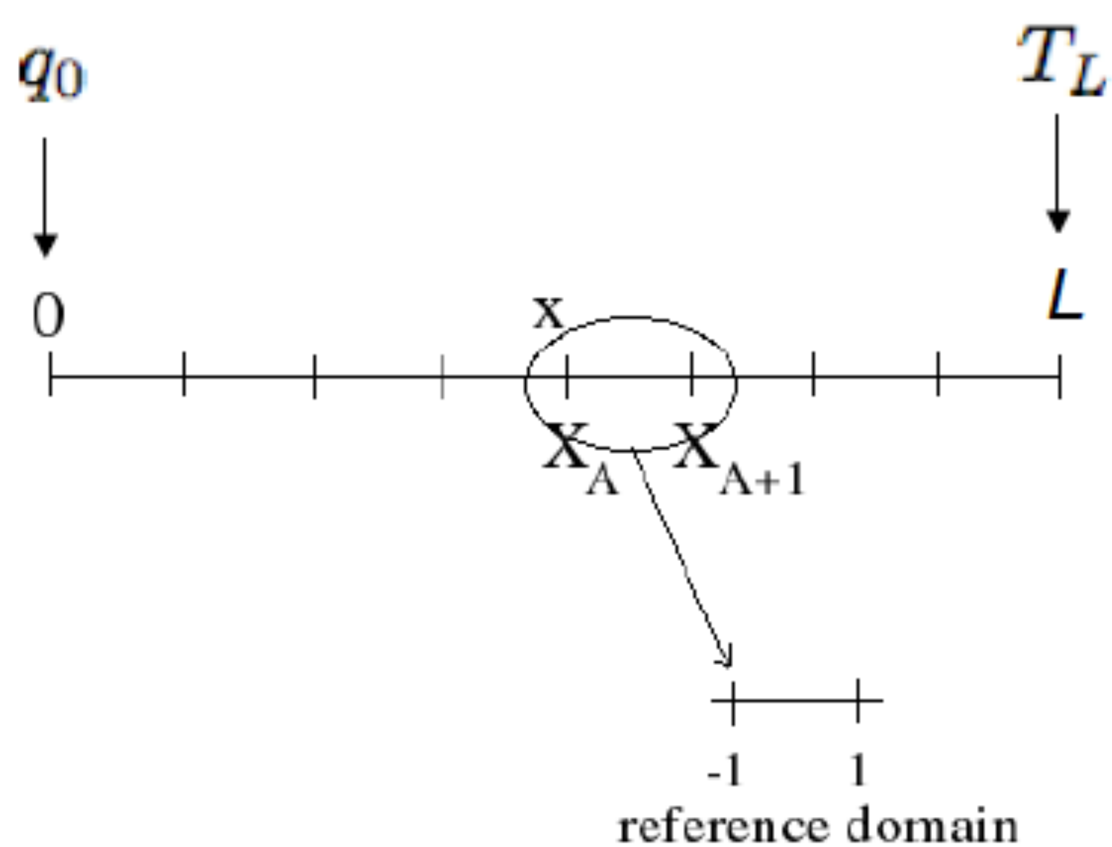



debate: geothermal heat transfer into the oceans
Labrosse (2009)

# 1D unsteady-state diffusion equation

Strong form: $\rho c_p \partial_t T - \partial_x(\kappa \partial_x T) = f$

Boundary conditions: $\begin{cases} T(L,t) & = & T_L \\ -\kappa\partial_x T(0,t) & = & q_0 \\ T(x,0) & = & T_0(x) \end{cases}$



# FEM - 1D unsteady-state diffusion equation

**Weak form:** $$\int_0^L \rho c_p w \partial_t T \mathrm{d}x = -\int_0^L \kappa \partial_x w \partial_x T \mathrm{d}x + q_0 w(0) + \int_0^L w f \mathrm{d}x$$

Test function and temperature field expanded on some basis functions:

$$w(x) = \sum_{A=1}^{N} c_A N_A(x)$$

$$T(x) = \sum_{A=1}^{N_{el}} d_A N_A(x) + T_1 N_{n+1}(x)$$

unknown

# FEM - 1D unsteady-state diffusion equation

Weak form: $$\int_0^L \rho c_p w \partial_t T \mathrm{d}x = -\int_0^L \kappa \partial_x w \partial_x T \mathrm{d}x + q_0 w(0) + \int_0^L w f \mathrm{d}x$$

We are solving: $\boxed{\mathbf{M}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F}}$

capacity matrix

stiffness matrix

force vector

# FEM - 1D unsteady-state diffusion equation

**Global level:**

$$
\begin{aligned}
M_{AB} &= (N_A, \rho c_p N_B) \\
&= \int_0^L \rho c_p N_A N_B \mathrm{d}x
\end{aligned}
$$

$$K_{AB} = a(N_A, N_B) = \int_0^L \kappa \partial_x N_A \partial_x N_B \mathrm{d}x$$

and

$$
\begin{aligned}
F_A &= (N_A, f) + N_A(0) q_0 - a(N_A, N_{n+1}) T_L \\
&= \int_0^L N_A f \mathrm{d}x + N_A(0) q_0 - \int_0^L \partial_x N_A \partial_x N_{n+1} \mathrm{d}x
\end{aligned}
$$

# FEM - 1D unsteady-state diffusion equation

**Global level:**

$$M_{AB} = \int_0^L \rho c_p N_A N_B \mathrm{d}x$$

$$K_{AB} = \int_0^L \kappa \partial_x N_A \partial_x N_B \mathrm{d}x$$

$$F_A = \int_0^L N_A f \mathrm{d}x + N_A(0) q_0 - \int_0^L \partial_x N_A \partial_x N_{n+1} \mathrm{d}x$$

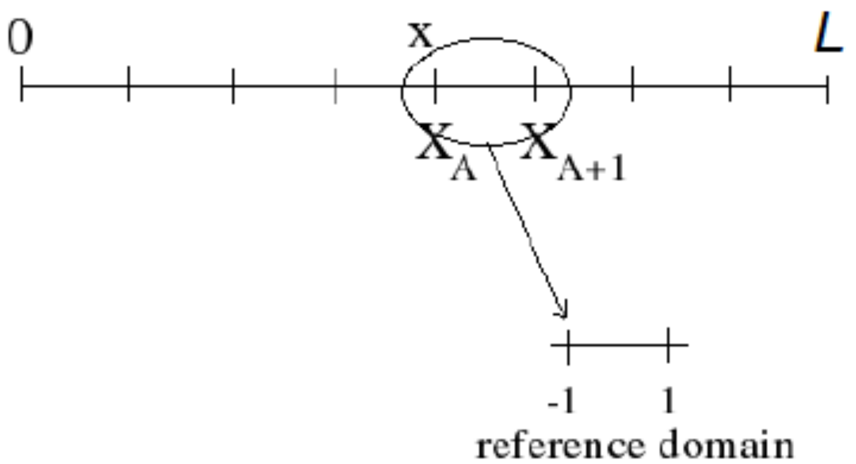
➔ $\mathbf{M}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F}$ where $\dot{\mathbf{d}} = \partial_\mathbf{t} \mathbf{d}$

mapping using Jacobian:

$$\int_0^1 g(x) dx = \sum_{\Omega_e} \int_{\Omega_e} g(x) dx = \sum_{\Omega_e} \int_{-1}^1 g(x(\xi)) J d\xi$$

# FEM - 1D unsteady-state diffusion equation

**Local level:** Consider the mapping $\xi : [X_A, X_{A+1}] \rightarrow [\xi_1, \xi_2]$, such that




$$\begin{cases} \xi(X_A) & = \xi_1 = -1 \\ \xi(X_{A+1}) & = \xi_2 = 1 \end{cases}$$

=> Linear shape functions:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{a=1,2}$$

# FEM - 1D unsteady-state diffusion equation

**Local (element) resolution:**

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{a=1,2}$$

Capacity matrix:

$$m^e_{ab} = (N_a, \rho c_p N_b) = \int_{\Omega_e} \rho c_p N_a N_b \mathrm{d}x$$

$\Downarrow$ *Change of variables (reference domain)*

$$m^e_{ab} = \frac{h_e}{2} \int_{-1}^{1} \rho c_p N_a N_b \mathrm{d}\xi$$

$\Downarrow$ *Matricial form*

$$m^e = \frac{\rho c_p h_e}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

# FEM - 1D unsteady-state diffusion equation

**Local (element) resolution:**

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{a=1,2}$$

Stiffness matrix: $$k^e_{ab} = a(N_a, \kappa N_b) = \int_{\Omega_e} \kappa \partial_x N_a \partial_x N_b \mathrm{d}x$$

$\Downarrow$ *Change of variables (reference domain)*

$$k^e_{ab} = \frac{2}{h_e} \int_{-1}^{1} \kappa \partial_\xi N_a \partial_\xi N_b \mathrm{d}\xi$$

$\Downarrow$ *Matricial form*

$$k^e = \frac{\kappa}{h_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

# FEM - 1D unsteady-state diffusion equation

**Local (element) resolution:**

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{a=1,2}$$

Force vector: $$f_a^e = \int_{\Omega_e} N_A f \mathrm{d}x + \begin{cases} \delta_{a1} q_0 & \text{for } e = 1 \\ -k_{a2}^e T_L & \text{for } e = N_{el} \\ 0 & \text{else} \end{cases}$$

↓ *Change of variables (reference domain)*

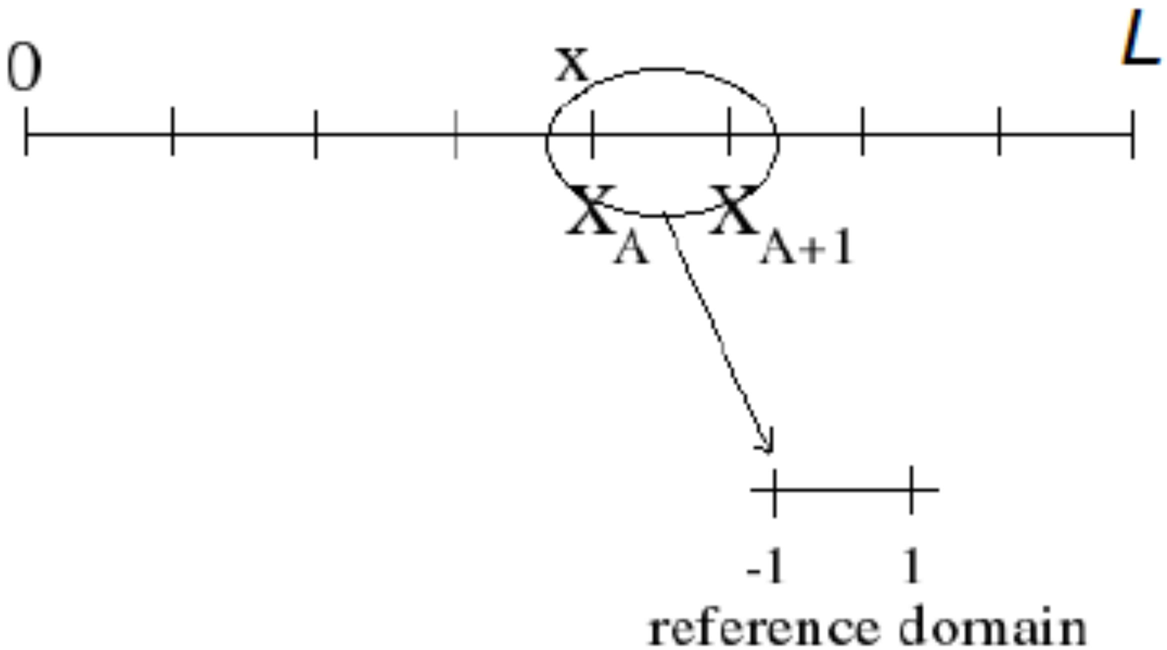
$$f_a^e = \frac{h_e}{2} \int_{-1}^{1} N_a f \mathrm{d}\xi + \text{boundary terms}$$

↓ *Matricial form*

$$f^e = \frac{h_e}{6} \begin{pmatrix} 2f_1 + f_2 \\ f_1 + 2f_2 \end{pmatrix} + \text{boundary terms}$$

# FEM - 1D unsteady-state diffusion equation

**Assembling:** back to global level




$$\text{iglob(i,ielem)} = \begin{cases} \text{ielem} & \text{if i=1} \\ \text{ielem+1} & \text{if i=2} \end{cases}$$

**code example:**

```
do ielem = 1, nelem
    do i = 1,2
        u(iglobe(i,ielem)) = u(iglobe(i,ielem)) + u_local(i,ielem)
    end
end
```

# FEM - 1D unsteady-state diffusion equation

**Time scheme:** Predictor-Corrector algorithm

- Predictor:

$$\begin{aligned} d_{n+1} &= d_n + (1-\alpha)\Delta t \dot{d}_n \\ \dot{d}_{n+1} &= 0 \quad \text{(initialization at the beginning of each time step)} \end{aligned}$$

- Solve:

$$\begin{aligned} rhs &= F - M\dot{d}_{n+1} - Kd_{n+1} \\ \delta\dot{d}_{n+1} &= M^{-1} rhs \end{aligned}$$

- Corrector:

$$\begin{aligned} d_{n+1} &= d_{n+1} + \alpha\Delta t \dot{d}_{n+1} \\ \dot{d}_{n+1} &= \dot{d}_{n+1} + \delta\dot{d}_{n+1} \end{aligned}$$

where $\Delta t$ is the time step.

$$\begin{cases} \alpha = 0 & \text{forward differences} \\ \boxed{\alpha = 1/2 \quad \text{midpoint rule}} \\ \alpha = 1 & \text{backward differences} \end{cases}$$

# FEM - 1D unsteady-state diffusion equation

**FEM solution:** *initial, simple harmonic function*

$$\rho c_p \partial_t T - \partial_x(\kappa \partial_x T) = f = 0$$

Boundary conditions:

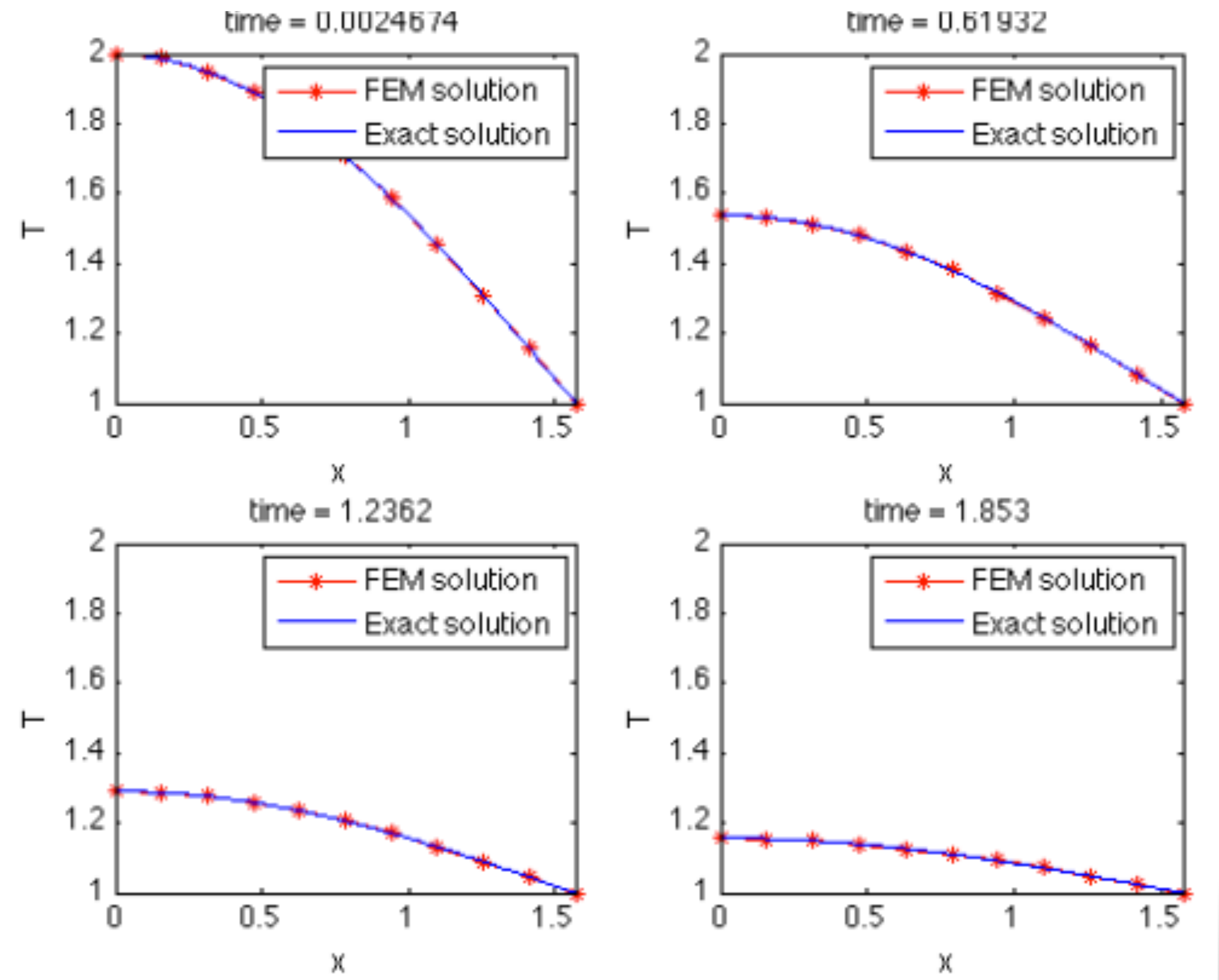
$$\begin{cases} T(L,t) & = & T_L = 1 \\ -\kappa\partial_x T(0,t) & = & q_0 = 0 \\ T(x,0) & = & T_0(x) = 1 + cos(x) \quad \text{in [0,L] and } L = \dfrac{\pi}{2} \end{cases}$$

has exact solution:

$$T(x,t) = 1 + e^{-t}\cos(x)$$

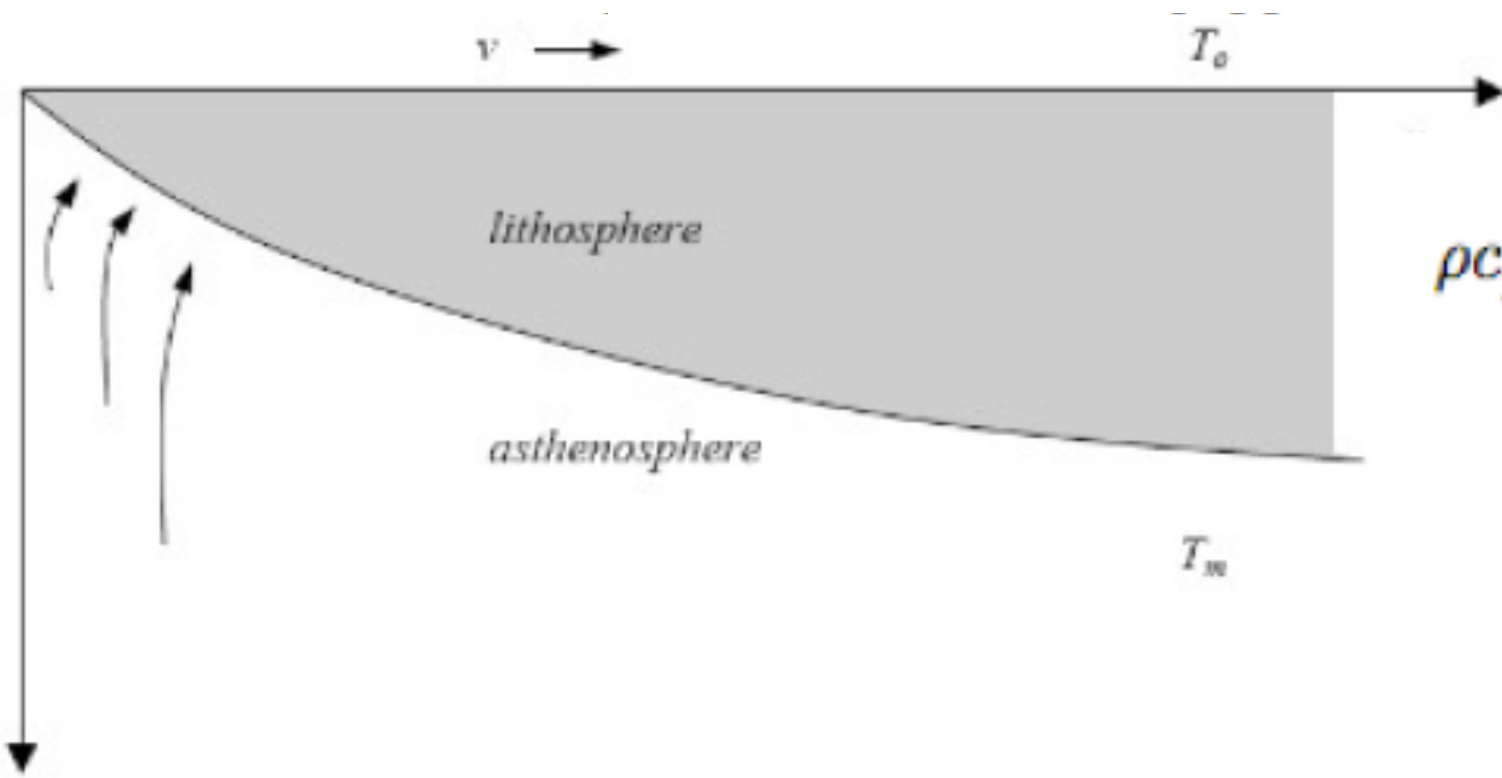
# FEM - 1D unsteady-state diffusion equation

**FEM solution:** *simple harmonic function*



# FEM - 1D unsteady-state diffusion equation

**FEM solution:** *half-space cooling*




$$\rho c_p \partial_t \theta = \partial_x (\kappa \partial_x \theta) \quad \text{in } 0 < x < \infty,$$

$$\text{where } \theta = \frac{T - T_m}{T_0 - T_m}$$

non-dimensional variable

Boundary conditions:

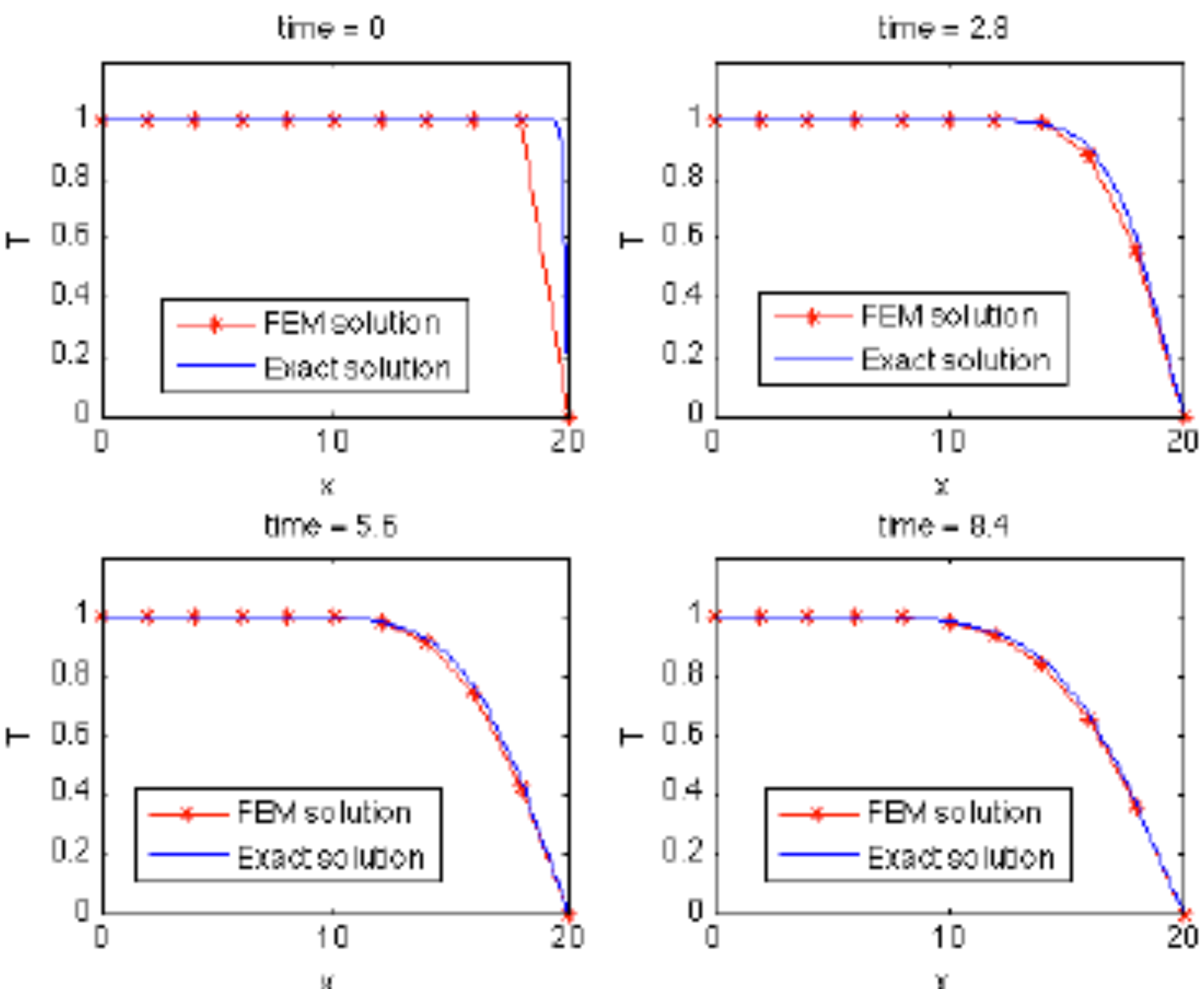
$$\begin{aligned} T(0,t) &= T_0 \quad \text{(surface temperature)} \\ T(x \to \infty, t) &\to T_m \\ T(x,0) &= T_m \quad \text{(initial temperature)} \end{aligned}$$

has exact solution:

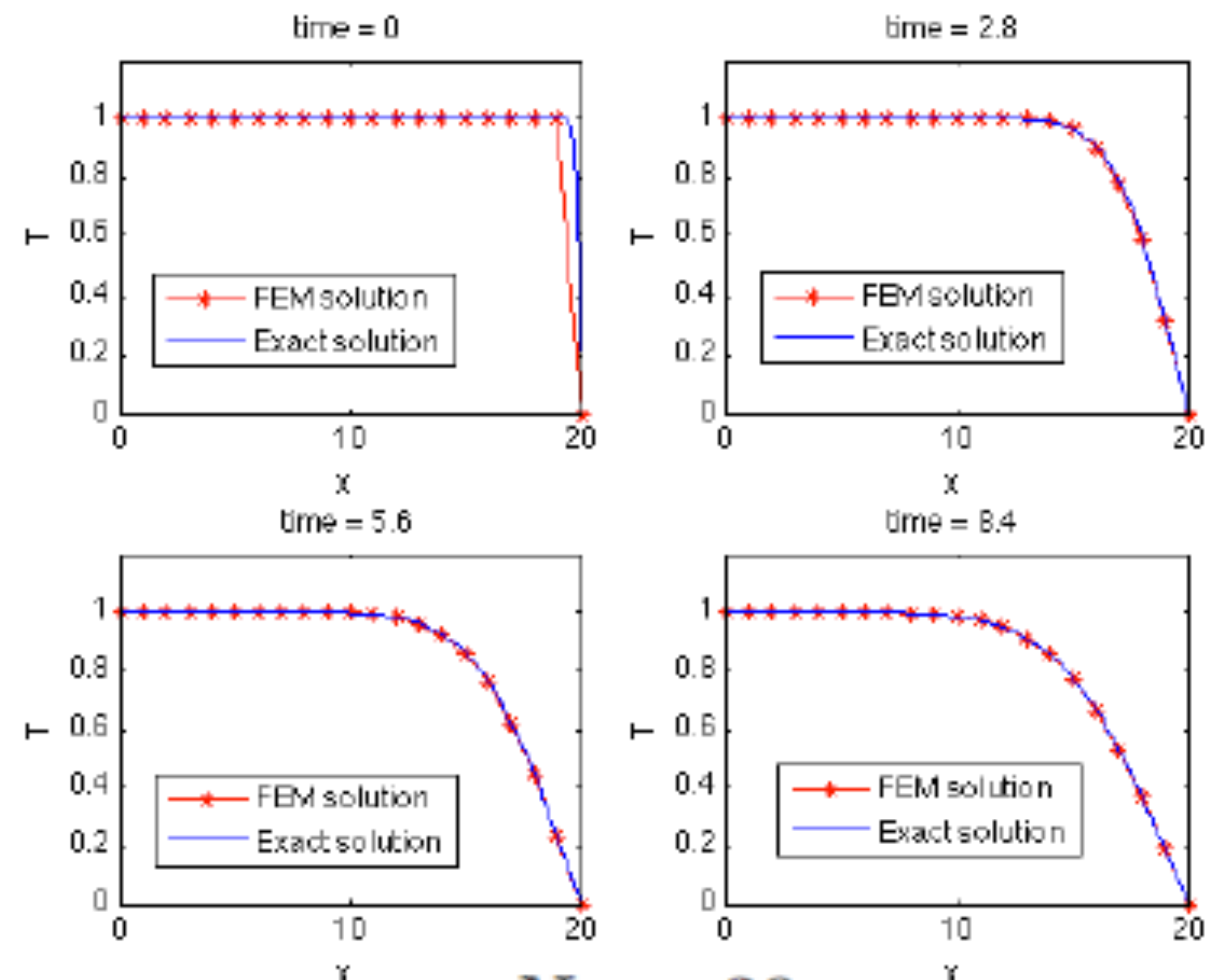
$$\theta = \operatorname{erfc} \frac{x}{2\sqrt{\frac{\kappa}{\rho c_p} t}}$$

# FEM - 1D unsteady-state diffusion equation

**FEM solution:** *half-space cooling*




$N_{el} = 10$




$N_{el} = 20$