The shallow-water equation

Recall the conservation laws for mass and linear momentum

$$\partial_t \rho + \underline{\nabla} \cdot (\rho \underline{v}) = 0 \qquad \text{mass}$$

$$\rho D_t \underline{v} - \underline{\nabla} \cdot \underline{\underline{T}} = \underline{f} \qquad \text{linear momentum}$$

Consider volume force due to gravity $\underline{f} = \rho \underline{g} h$

Assumptions: - fluid is incompressible:

$$\underline{\nabla} \cdot \underline{v} = 0 \longrightarrow D_t \rho = 0$$

- Newtonian fluid:

$$\underline{\underline{T}} = -p \underline{\underline{I}} + \underline{\underline{T}}^f$$

p: pressure hydrostatic

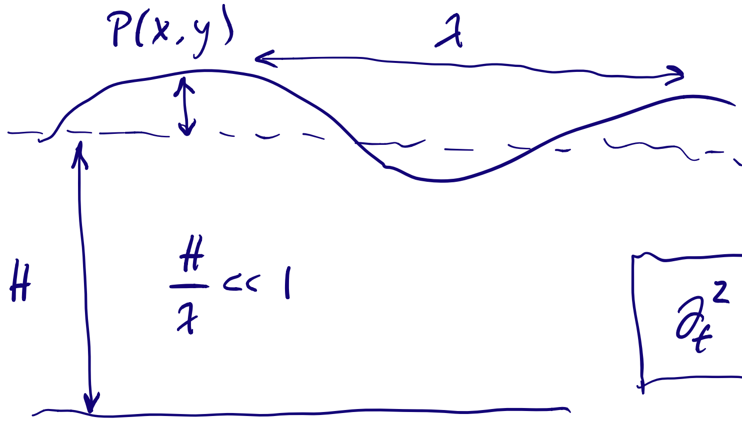
$\underline{\underline{T}}^f$: stress (fluid) due to viscosity

We find the Navier-Stokes equations

$$\underline{\nabla} \cdot \underline{v} = 0$$

$$\partial_t \rho \underline{v} + \underline{\nabla} \cdot (\rho \underline{v}\,\underline{v}) = -\underline{\nabla} p + \rho \underline{g} + \underline{\nabla} \cdot \underline{\underline{T}}^f$$

Tsunamis



Assuming long-wavelength and neglect the vertical accelerations, we find the shallow-water equation

$$\boxed{\partial_t^2 P = \underline{\nabla} \cdot \left(v^2 \underline{\nabla} P \right)}$$

with P : height of tsunami

and $v = \sqrt{gH}$ wave speed

H : ocean depth

Finite-difference method — Higher order schemes

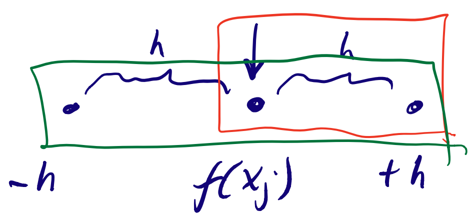
Recall the approximations

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \approx \frac{f(x+h)-f(x)}{h} + \mathcal{O}(h) \text{ forward}$$

$$\frac{f(x)-f(x-h)}{h} + \mathcal{O}(h) \text{ backward}$$

$$\frac{f(x+h)-f(x-h)}{2h} + \mathcal{O}(h^2) \text{ centered}$$

Finite-difference stencil



using small differences
$h = \Delta x$

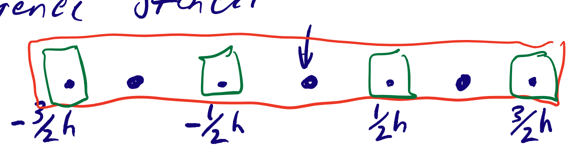
Can we ask for 4^{th}-order accuracy?

Let's see with intermediate steps

$$\frac{d}{dx}f(x) = \frac{1}{h}\left\{a\left[f\left(x+\tfrac{3}{2}h\right)-f\left(x-\tfrac{3}{2}h\right)\right]+b\left[f\left(x+\tfrac{1}{2}h\right)-f\left(x-\tfrac{1}{2}h\right)\right]\right\}$$

we find $a = -\frac{1}{24}$, $b = \frac{9}{8}$ (optimization problem)

Finite-difference stencil



for 4^{th}-order accuracy, the stencil becomes bigger

Recall, Taylor expansions

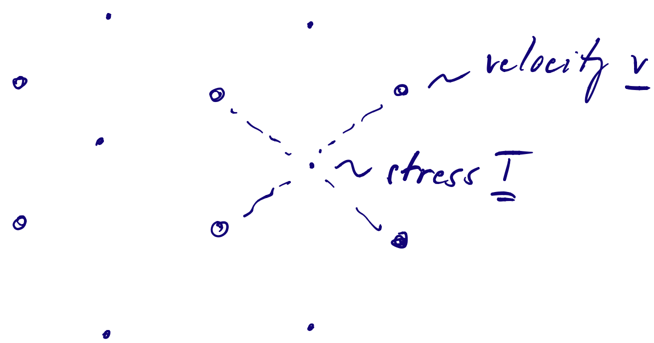
$$f(x+\Delta x) = f(x) + \Delta x\, f'(x) + \frac{1}{2}\Delta x^2 f''(x) + \frac{1}{6}\Delta x^3 f'''(x) + \frac{1}{24}\Delta x^4 f''''(x) + \frac{1}{120}\Delta x^5 f^{v}(x) + O(\Delta x^6)$$

$$f(x-\Delta x) = f(x) - \Delta x\, f'(x) + \frac{1}{2}\Delta x^2 f''(x) - \frac{1}{6}\Delta x^3 f'''(x) + \frac{1}{24}\Delta x^4 f''''(x) - \frac{1}{120}\Delta x^5 f^{v}(x) + O(\Delta x^6)$$

we can find the expression for $f''(x)$ with a truncation error $O(\Delta x^4)$

$\longrightarrow$ see your homework

Staggered scheme

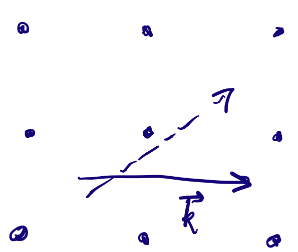


staggered scheme

$$\frac{f(x+\frac{1}{2})-f(x-\frac{1}{2})}{h}+O(h^2)$$

Luo & Schuster (1990)
"Parsimonious staggered grid finite-differencing of the wave equation", GRL

Resolution, grid dispersion & grid anisotropy



dispersion: wave speed as a function of wave vector $\vec{k}$