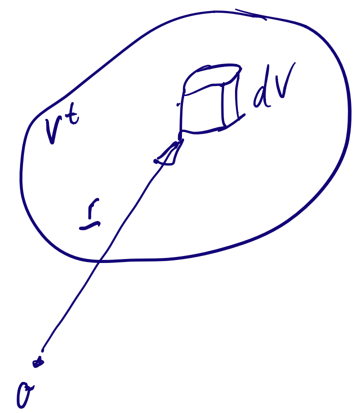
Conservation of angular momentum



total angular momentum $\displaystyle\int_{V^t} \rho\, \underbrace{\underline{r}}_{\text{position vector}} \times \underbrace{\underline{v}}_{\text{velocity}}\, dV$

volume torque $\displaystyle\int_{V^t} \underline{r} \times \underline{f}\, dV$ $\qquad \underline{f}$: body force

surface torque $\displaystyle\int_{\Sigma^t} \underline{r} \times \underline{t}\, d\Sigma$ $\qquad \underline{t}$: traction, $\underline{t} = \hat{\underline{n}} \cdot \underline{\underline{T}}$

rate of change of angular momentum

$$\frac{d}{dt}\int_{V^t} \rho\, \underline{r} \times \underline{v}\, dV = \int_{V^t} \underline{r} \times \underline{f}\, dV + \int_{\Sigma^t} \underline{r} \times \underline{t}\, d\Sigma$$

"rate of change of angular momentum is equal to the net torque"

$$\frac{d}{dt}\int_{V^t} \rho\, \underline{r} \times \underline{v}\, dV = \int_{V^t} \partial_t(\rho\, \underline{r} \times \underline{v}) + \nabla\cdot(\rho(\underline{r}\times\underline{v})\underline{v})\, dV$$

$$= \int_{V^t} (\underline{r}\times\underline{v})\underbrace{\left[\partial_t \rho + \underline{\nabla}\cdot(\rho\underline{v})\right]}_{\substack{\text{conservation of mass} \\ = 0}} + \rho\underbrace{\left[\partial_t(\underline{r}\times\underline{v}) + \underline{v}\cdot\underline{\nabla}(\underline{r}\times\underline{v})\right]}_{= \underline{r} \times \rho \underline{D_t} \underline{v}} dV$$

Let's consider surface torque

$$\int_{\Sigma^t} \underline{r} \times \underline{t}\, d\Sigma = \int_{\Sigma^t} \underline{r} \times (\hat{\underline{n}} \cdot \underline{\underline{T}})\, d\Sigma = \int_{V^t} \underline{r} \times (\underline{\nabla} \cdot \underline{\underline{T}}) + \underline{\underline{\underline{\varepsilon}}} : \underline{\underline{T}}\, dV$$

Kronecker delta

$$\delta_{ij} = \begin{cases} 0, & \text{else} \\ 1, & i=j \end{cases}$$

$\underline{\underline{\underline{\hat{\varepsilon}}}}$: Levi-Civita tensor

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{even permutation} \\ 0, & \text{if some are equal } i=j,\ j=k,\ i=k \\ -1, & \text{odd permutation} \end{cases}$$

even permutation $ijk = (1\ 2\ 3)$
$(2\ 3\ 1)$
$(3\ 1\ 2)$

odd permutation $ijk = (3\ 2\ 1)$
$(1\ 3\ 2)$
$(2\ 1\ 3)$

↳ cross product

$$(\underline{u} \times \underline{v})_i = \hat{\varepsilon}_{ijk} u^j v^k$$

It follows that we can rewrite

$$\underline{r} \times [\underbrace{\rho D_t \underline{v} - \underline{\nabla} \cdot \underline{\underline{T}} - \underline{f}}_{\text{conservation of linear momentum } = 0}] = \underline{\underline{\underline{\hat{\varepsilon}}}} : \underline{\underline{T}}$$

Thus, $\underline{\underline{\underline{\hat{\varepsilon}}}} : \underline{\underline{T}} = 0$ is satisfied if $T^{ij} = T^{ji}$, $\underline{\underline{T}} = \underline{\underline{T}}^T$, i.e. the stress tensor $\underline{\underline{T}}$ is symmetric.

For Hooke's law, $\underbrace{\underline{\underline{T}}}_{\text{stress}} = \underline{\underline{\underline{\underline{c}}}} : \underbrace{\underline{\underline{\varepsilon}}}_{\text{strain}}$ with $\left.\begin{array}{l} \underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^T \text{ strain} \\ \underline{\underline{T}} = \underline{\underline{T}}^T \text{ stress} \end{array}\right\}$ therefore $c_{ijkl} = c_{jikl} = c_{ijlk}$

In total, (all in) geophysics can be described by

$$\boxed{\begin{aligned} &\partial_t \rho + \underline{\nabla} \cdot (\rho \underline{v}) = 0 \\ &\rho D_t \underline{v} - \underline{\nabla} \cdot \underline{\underline{T}} = \underline{f} \\ &\underline{\underline{T}} = \underline{\underline{T}}^T \\ &\rho D_t U + \underline{\nabla} \cdot \underline{H} = \underline{\underline{T}} : \dot{\underline{\underline{\varepsilon}}} + h \end{aligned}}$$

conservation of mass

linear momentum

angular momentum

energy