

Finite-volume method - Part II

Face interpolations: Recall the face interpolation schemes (from Part I) for

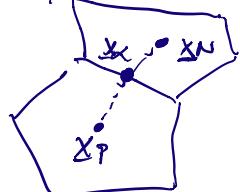
$$\int_{\Omega} \nabla \cdot \underline{u} \, dV = \boxed{\int_{\Omega} \underline{u} \cdot \underline{n} \, dS} \approx \sum_{\alpha} S_{\alpha} n_{\alpha}^{\alpha} \left[\begin{matrix} u_i \\ \dots \\ u_i \end{matrix} \right] \text{ value at face centroid}$$

- upwinding scheme: $u_{\alpha} = \begin{cases} u_p & \text{if flux} > 0 \\ u_N & \text{if flux} < 0 \end{cases}$

(convective flux involves transport velocities \underline{v})

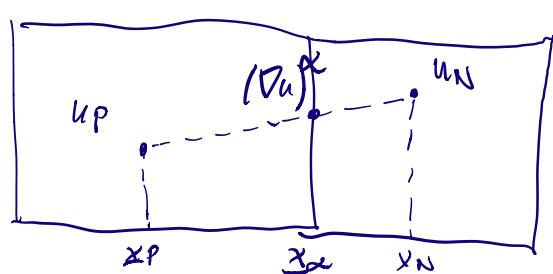
- linear upwinding scheme: $u_{\alpha} = \begin{cases} u_p + d_p \nabla u_p & \text{if flux} > 0 \\ u_N + d_N \nabla u_N & \text{if flux} < 0 \end{cases}$

- linear/central differencing: $u_{\alpha} = (1-\phi)u_p + \phi u_N$



$$\text{with } \phi = \frac{|x_{\alpha} - x_p|}{|x_N - x_p|}$$

Gradient calculation: central difference approach



$$(\nabla u)(x_{\alpha}) = \frac{u_N - u_p}{|x_N - x_p|}$$

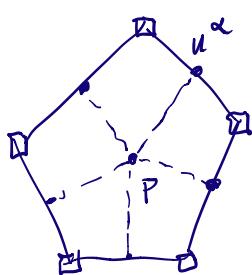
→ needs corrections for non-orthogonal / skewed meshes

$$\int_{\Omega} \underline{\nabla} \cdot (\underline{\nabla} u) dV = \int_{\partial\Omega} \underline{\nabla} u \cdot \underline{n} dS$$

$$\approx \sum_{\alpha} S_{\alpha} (\nabla u)^{\alpha} n_i^{\alpha}$$

what about finding $(\nabla u)^P = (\nabla u)(x_P)$?

(Green-)Gauss approach: In 3D, we find



$$(\nabla u)^P = \frac{1}{V} \sum_{\alpha} [u^{\alpha}]_n^{\alpha} S_{\alpha}$$

and how to calculate the face centroid values?

- linear interpolation
- nodal averaging

Discretization: Let's discretize the (physical) domain Ω into a mesh with grid cells Ω_e :

$$\Omega = \bigcup_e \Omega_e$$

The integral form of our equation becomes

$$\underbrace{\int_{\Omega} \partial_x^2 u \, dx}_{\text{diffusion}} + \underbrace{\int_{\Omega} f \, dx}_{\text{force}} = \sum_e \int_{\Omega_e} \partial_x^2 u \, dx + \sum_e \int_{\Omega_e} f \, dx = 0$$

Force term: For grid cell Ω_e and a piecewise-linear $f(x)$, we find

$$\begin{aligned} \underbrace{\int_{\Omega_e} f(x) \, dx}_{=} &= \int_{\Omega_e} f_p \, dx + (\nabla f)_p \underbrace{\int_{\Omega_e} (x - x_p) \, dx}_{=0 \text{ constant}} \\ &= f_p V_e \\ &\approx \bar{f} V_e \end{aligned}$$

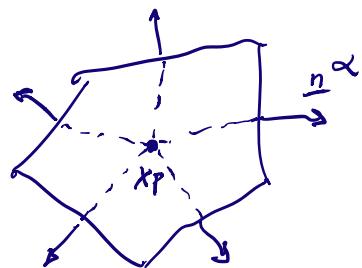
V_e : volume of Ω_e

↳ second-order accurate $f_p = f(\underline{x}_p)$

\bar{f} : cell average of $f(x)$

Diffusion term: In general, the diffusion term can be rewritten using Gauss' theorem

$$\int_{\Omega_e} \nabla \cdot (\underline{\nabla} u) dV = \int_{\partial \Omega_e} \underline{\nabla} u \cdot \underline{n} dS = \sum_{\alpha} \int_{\partial \Omega^{\alpha}} \underline{\nabla} u \cdot \underline{n}^{\alpha} dS$$



$$\approx \sum_{\alpha} S_{\alpha} \underline{n}^{\alpha} \cdot (\underline{\nabla} u)^{\alpha}$$

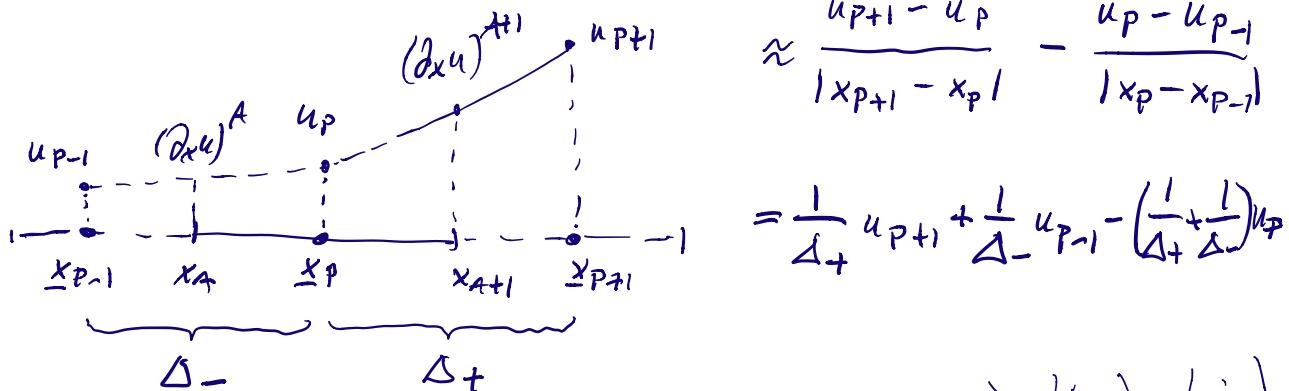
S_{α} : surface area of face α

$$(\underline{\nabla} u)^{\alpha} = (\underline{\nabla} u)(\underline{x}_{\alpha})$$

at face centroid \underline{x}_{α}

In 1D, this becomes

$$\int_{\Omega_e} \partial_x^2 u dx = \partial_x u \Big|_{x_A}^{x_{A+1}} = \partial_x u(x_{A+1}) - \partial_x u(x_A)$$

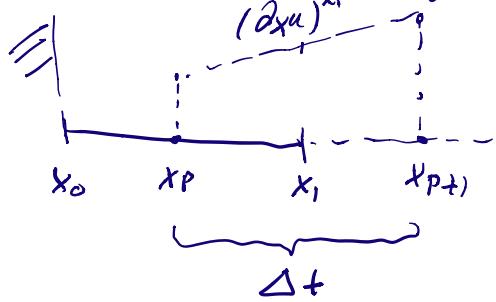


→ matrix form: $\underline{K} \underline{d} = \underline{F}$

$$\underline{K} = \begin{pmatrix} & & & \\ \cdots & \frac{1}{\Delta_-} & -\left(\frac{1}{\Delta_+} + \frac{1}{\Delta_-}\right) & \frac{1}{\Delta_+} & \cdots \end{pmatrix} \begin{pmatrix} u_{p-1} \\ u_p \\ u_{p+1} \\ \vdots \\ d \end{pmatrix} = \begin{pmatrix} & & & \\ \cdots & -f_p v_p & & \\ & & & \vdots \end{pmatrix} \begin{pmatrix} F \\ \vdots \end{pmatrix}$$

Boundary conditions:

Neumann boundary: $\partial_x u(0) = -h$ at left boundary



$$\begin{aligned} \int_{\Omega_e} \partial_x^2 u \, dx &= \partial_x u \Big|_{x_0=0}^{x_1} \\ &= \partial_x u(x_1) - \boxed{\partial_x u(0)} \\ &= \frac{1}{\Delta t} u_{p+1} - \frac{1}{\Delta t} u_p + h \end{aligned}$$

Thus, for cell element Ω_1 :

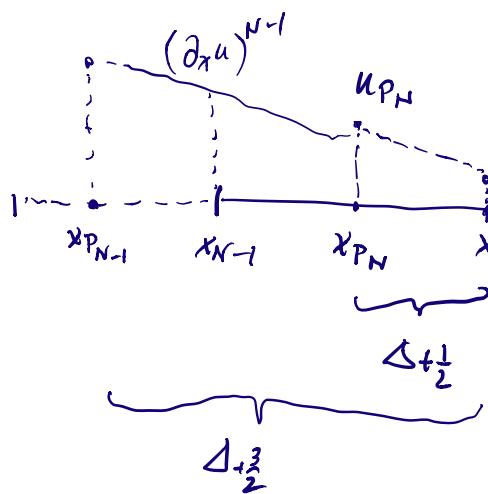
$$\underbrace{\int_{\Omega_1} \partial_x^2 u \, dx}_{\frac{1}{\Delta t} u_{p+1}} + \underbrace{\int_{\Omega_1} f \, dx}_{f_p V_i} = 0$$

$$\frac{1}{\Delta t} u_{p+1} - \frac{1}{\Delta t} u_p + h + f_p V_i = 0$$

→ matrix form

$$\begin{pmatrix} -\frac{1}{\Delta t} & \frac{1}{\Delta t} & 0 & \dots \end{pmatrix} \begin{pmatrix} u_p \\ u_{p+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} -f_p V_i - h \\ \vdots \end{pmatrix}$$

Dirichlet boundary: $u(1) = q$ at right boundary



For cell element Ω_N :

$$u(1) = q \int \partial_x^2 u \, dx = \frac{\partial_x u(x_N) - \partial_x u(x_{N-1})}{\Delta x}$$

what is $\partial_x u(x_N)$ when we have no information on $u_{P_{N+1}}$?

Using a backward difference:

$$\partial_x u(x_N) = \frac{u(x_N) - u(x_P)}{|x_N - x_P|}$$

$$= \frac{1}{\Delta_{+1/2}^{1/2}} q - \frac{1}{\Delta_{+1/2}^{1/2}} u_P$$

first-order accurate

How could we find higher order approximations?

Taylor series approach

$$u_{P_N} = u(x_N - \Delta_{+1/2}^{1/2}) = u(x_N) - \Delta_{+1/2}^{1/2} \partial_x u(x_N) + \frac{1}{2} \Delta_{+1/2}^{1/2} \partial_x^2 u(x_N) \dots$$

$$u_{P_{N-1}} = u(x_N - \Delta_{+1/2}^{1/2}) = u(x_N) - \Delta_{+1/2}^{1/2} \partial_x u(x_N) + \frac{1}{2} \Delta_{+1/2}^{1/2} \partial_x^2 u(x_N) \dots$$

$$\rightarrow \partial_x u(x_N) = \dots$$

cancel out second-order terms

$$\partial_x u(x_N) = a \underbrace{u(x_N)}_q + b u_{P_N} + c u_{P_{N-1}} + O(\partial_x^3 u)$$

a, b, c : factors for
second-order scheme

Thus, we find for cell element Ω_N

$$\begin{aligned}
 & \int_{\Omega_N} \partial_x^2 u \, dx + \int_{\Omega_N} f \, dx = 0 \\
 & \underbrace{\partial_x u(x_N) - \partial_x u(x_{N-1})}_{\Delta x} + \underbrace{f_N V_N}_{= 0} = 0 \\
 & \frac{1}{\Delta x^2} q - \frac{1}{\Delta x^2} u_{P_N} - \left(\frac{1}{\Delta_-} u_{P_N} - \frac{1}{\Delta_-} u_{P_{N-1}} \right) = -f_N V_N \\
 & + \frac{1}{\Delta_-} u_{P_{N-1}} - \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta_-} \right) u_P = -f_N V_N - \frac{1}{\Delta x^2} q
 \end{aligned}$$

→ matrix form

$$\left(\begin{array}{cccc} \dots & 0 & \Delta & -\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta}\right) \\ & & \Delta & -f_N V_N - \frac{1}{\Delta x^2} g \end{array} \right) = \left(\begin{array}{c} \vdots \\ u_{P_{N-1}} \\ u_{P_N} \\ \vdots \end{array} \right)$$

Use a linear solver for the unknowns \underline{d} ,
such that:

$$\underline{d} = \underline{\underline{K}}^{-1} \underline{F}$$