Spectral-element method - part 2

Recall: 1D heat equation

$$\rho c_v \partial_t T = \partial_x (K \partial_x T)$$

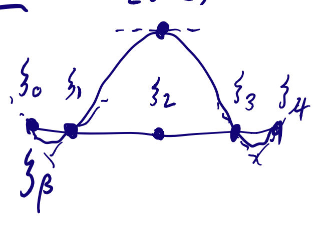
with density ρ, heat capacity c_v, conductivity K and temperature $T = T(x,t)$

The weak-form becomes

$$\underbrace{\int_0^L w \rho c \partial_t T \, dx}_{\text{mass / capacity term}} = - \underbrace{\int_0^L K \partial_x w \, \partial_x T \, dx}_{\text{stiffness}} + \underbrace{w K \partial_x T \Big|_0^L}_{\text{boundary}}$$

Matrix form: We expand our functions

$w^2 = 1 : \; l_2^4(\xi_2) = 1$



$w(x(\xi)) = 0$
except
$w(x(\xi_2)) = 1$

$$w(x(\xi)) = \sum_{\beta=0}^{N} w^\beta \underbrace{l_\beta^N(\xi)}_{\text{high-order degree } N \text{ Lagrange polynomial}}$$

$$T(x(\xi),t) = \sum_{\alpha=0}^{N} T^\alpha(t) \, l_\alpha^N(\xi)$$

Note that we can choose the test function coefficients w^β to be all equal to zero except one (which is set to 1) such that we can treat of the Lagrange polynomials one at a time (which means one GLL point at a time)

<u>mass matrix</u>: Let's look at the left-hand side term

$$\int_0^L w \rho c \partial_t T \, dx = \sum_e \int_{\Omega_e} w \rho c \partial_t T \, dx$$

The contribution from element Ω_e

$$\int_{\Omega_e} w \rho c \partial_t T \, dx = \int_{-1}^{+1} \rho(x(\xi)) c(x(\xi)) \underline{w(x(\xi))} \, \underline{\partial_t T(x(\xi),t)} J d\xi$$

$$\approx \sum_{\gamma=0}^{N} \hat{\omega}_\gamma \rho^\gamma c^\gamma J^\gamma \underbrace{\sum_{\beta=0}^{N} w^\beta l_\beta^N(\xi_\gamma)}_{l_\beta(\xi_\gamma) = \delta_{\beta\gamma}} \underbrace{\sum_{\alpha=0}^{N} \partial_t T^\alpha l_\alpha^N(\xi_\gamma)}_{\delta_{\alpha\gamma}}$$

$$= \sum_{\gamma=0}^{N} \hat{\omega}_\gamma \rho^\gamma c^\gamma J^\gamma w^\gamma \partial_t T^\gamma$$

local contribution in element Ω_e

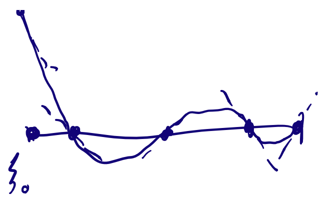
<u>Stiffness matrix</u>: For the contribution of element Ω_e

$$\int_{\Omega_e} k \, \partial_x w \, \partial_x T \, dx = \int_{-1}^{+1} k(x(\xi)) \left[\underline{\partial_x w(x(\xi))}\right] \left[\partial_x T(x(\xi),t)\right] J \, d\xi$$

$$\approx \sum_{\gamma=0}^{N} \hat{\omega}_\gamma k^\gamma J^\gamma \left[\sum_{\beta=0}^{N} w^\beta l_\beta^{\prime N}(\xi_\gamma) \underline{\partial_x \xi(\xi_\gamma)}\right] \left[\sum_{\alpha=0}^{N} T^\alpha l_\alpha^{\prime N}(\xi_\gamma) \partial_x \xi(\xi_\gamma)\right]$$

note: $\dfrac{\partial l_\alpha}{\partial x} = \dfrac{\partial l_\alpha}{\partial \xi} \dfrac{\partial \xi}{\partial x}$

$l_0^4(\xi)$



and $l_\beta^{\prime N}(\xi_\gamma) \neq \delta_{\beta\gamma}$

Let's choose as test function

$$\begin{cases} w^{\beta_1} = 1 \\ w^{\beta} = 0 \quad \text{for all } \beta \neq \beta_1 \end{cases}$$

then we find

$$\underbrace{\hat{\omega}_{\beta_1} \, \rho^{\beta_1} c^{\beta_1} J^{\beta_1} \underbrace{w^{\beta_1}}_{=1}}_{M_{\beta_1}} \partial_t T^{\beta_1} = - \sum_{\gamma=0}^{N} \hat{\omega}_\gamma \, k^\gamma J^\gamma \left[l_{\beta_1}^{\prime N}(\xi_\gamma) \, \partial_x \xi(\xi_\gamma) \right] \cdot \left[\sum_{\alpha=0}^{N} T^\alpha \, l_\alpha^{\prime}(\xi_\gamma) \partial_x \xi(\xi_\gamma) \right]$$

or

$$M_{\beta_1} \; \partial_t T^{\beta_1}(t) = \sum_{\alpha=0}^{N} K_{\beta_1 \alpha} \; T^\alpha(t)$$

$$\underline{\underline{M}} \; \partial_t \underline{T} = \underline{\underline{K}} \; \underline{T} \quad + \text{boundary}$$

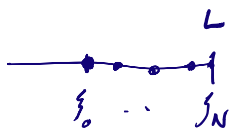
where $M_{\beta_1} = \hat{\omega}_{\beta_1} \, \underline{\rho^{\beta_1}} \, \underline{c^{\beta_1}} \, J^{\beta_1}$

$$K_{\beta_1 \alpha} = - \sum_{\gamma=0}^{N} \hat{\omega}_\gamma \, \underline{k^\gamma} \, J^\gamma \, l_{\beta_1}^{\prime N}(\xi_\gamma) \, l_\alpha^{\prime N}(\xi_\gamma) \left[\partial_x \xi(\xi_\gamma) \right]^2$$

material properties

Notice that mass & stiffness matrices are constructed at (reference) element level, which allows for material heterogeneities within an element.

Boundary conditions: In 1D, the conditions write as



$$\text{at } x = L: \quad w\, k\, \partial_x T = w^N K^N \sum_{\alpha=0}^{N} T^\alpha(t)\, \underline{\ell'^N_\alpha(\xi_N)}\, \partial_x \xi(\xi_N)$$

$$x = 0: \quad w\, k\, \partial_x T = w^0 K^0 \sum_{\alpha=0}^{N} T^\alpha(t)\, \ell'^N_\alpha(\xi_0)\, \partial_x \xi(\xi_0)$$