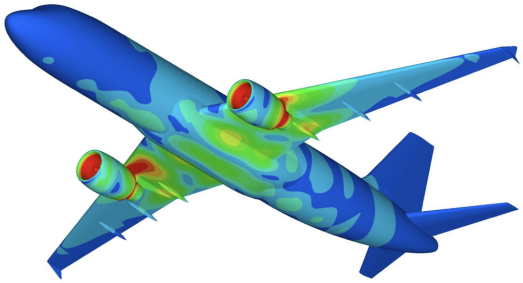

# Finite-element method

**Lecture notes**

The Finite-Element Method (FEM) differs from the Finite-Difference (FD) and the Pseudo-Spectral (PS) methods previously presented in class. The FD & PS methods are strong methods, that is, the partial differential equations (PDE's) are solved by approximating the operators (time & spatial derivatives), whereas the FEM is a weak method, where the operators are exact but the solution basis functions are approximated.

## Weak form or variational statement

Let us consider the following 1-D partial differential equation (PDE), with boundary conditions (the strong form): find $u \in [0,1]$ such that

$$\boxed{\partial_x^2\, u + f = 0}$$

$$\begin{cases} u(1) = q \\ \partial_x u(0) = -h \end{cases}$$

This is a two point boundary value problem.

Note that the exact solution to this problem is:

$$u(x) = q + (1-x)h + \int_x^1 \left( \int_0^y f(z)\mathrm{d}z \right) \mathrm{d}y$$

for any $f$. You can check that

$$\begin{cases} u(1) = q \\ \partial_x u(x) = -h - \int_0^x f(z)\mathrm{d}z \\ \partial_x u(0) = -h \\ \partial_x^2 u = -f \end{cases}$$

Now, we introduce a test function $\mathsf{w}(x)$, as an arbitrary function which satisfies $\mathsf{w}(1) = 0$. The weak form is derived as follows:

1) the PDE is dotted by the test function:

$$\mathsf{w}\,\partial_x^2 + \mathsf{w}\, f = 0$$

2) the dotted PDE is integrated over the problem domain:

$$\int_0^1 \mathsf{w}\,\partial_x^2 u \mathrm{d}x + \int_0^1 \mathsf{w}\, f \mathrm{d}x = 0$$

$$\begin{aligned}\text{(integration by parts)} &\iff -\int_0^1 \partial_x \mathsf{w}\,\partial_x u \mathrm{d}x + \int_0^1 \partial_x(\mathsf{w}\,\partial_x u)\mathrm{d}x + \int_0^1 \mathsf{w}\, f \mathrm{d}x = 0 \\ &\iff -\int_0^1 \partial_x \mathsf{w}\,\partial_x u \mathrm{d}x + [\mathsf{w}\,\partial_x u]_0^1 + \int_0^1 \mathsf{w}\, f \mathrm{d}x = 0\end{aligned}$$

Taking into account the boundary conditions and the nature of the test function, we finally get the weak form:

$$\boxed{-\int_0^1 \partial_x \mathsf{w}\,\partial_x u\,\mathrm{d}x + h\mathsf{w}(0) + \int_0^1 \mathsf{w}\, f \mathrm{d}x = 0}$$

$\forall \mathsf{w}$, such that $\mathsf{w}(1) = 0$.
To simplify the notations, we introduce the following definitions:

$$a(\mathsf{w}, u) \equiv \int_0^1 \partial_x \mathsf{w}\,\partial_x u \mathrm{d}x$$

a bilinear functional such that

$$\begin{cases} a(\alpha \mathsf{w}_1 + \beta \mathsf{w}_2, u) = \alpha a(\mathsf{w}_1, u) + \beta a(\mathsf{w}_2, u) \\ a(\mathsf{w}, u) = a(u, \mathsf{w}) \end{cases}$$

and

$$(\mathsf{w}, f) \equiv \int_0^1 \mathsf{w}\, f \mathrm{d}x$$

a linear functional.

Thus far, all we have done is dotting by an arbitrary test function and integrating over the problem domain, which has allowed us to restate the problem in a weak formulation. To move into the actual FEM realm, we are going to solve the problem locally on an element, by introducing a discretization scheme and a finite-dimensional approximation. This corresponds to a Galerkin's method. To that end, we need to expand our functions on a finite basis.

## Galerkin's method

We define the shape functions, $N_A(x)$, as basis functions on which we expand our functions. The test function writes then as

$$\mathsf{w}(x) = \sum_{A=1}^{N} \mathsf{c}_A\, N_A(x)$$

Remember that the test function is such that $\mathsf{w}(1) = 0$, therefore $N_A(1) = 0$. Since the boundary conditions give $u(1) = q$, let's introduce one more shape function $N_{n+1}$, such that $N_{n+1}(1) = 1$, then

$$u(x) = \sum_{A=1}^{N} \mathsf{d}_A\, N_A(x) + q\, N_{n+1}(x)$$

where the coefficients $\mathsf{d}_A$ are the unknowns we are solving for.

Replacing these expressions into the weak form yields to

$$-\int_0^1 \sum_A \mathsf{c}_A\, \partial_x N_A \sum_B \mathsf{d}_B\, \partial_x N_B \mathrm{d}x - \int_0^1 \sum_A \mathsf{c}_A\, \partial_x N_A q \partial_x N_{n+1} \mathrm{d}x + \int_0^1 \sum_A \mathsf{c}_A\, N_A f \mathrm{d}x + h \sum_A \mathsf{c}_A\, N_A(0) = 0$$

which is valid for any coefficient $\mathsf{c}_A$. We can thus simplify this expression zeroing most coefficients except one put to 1, such that

$$\boxed{-\sum_B \int_0^1 \partial_x N_A \mathsf{d}_B\, \partial_x N_B \mathrm{d}x - \int_0^1 \partial_x N_A q \partial_x N_{n+1} \mathrm{d}x + \int_0^1 N_A f \mathrm{d}x + h N_A(0) = 0}$$

We can introduce the following matricial forms:

$$\begin{cases} K_{AB} \equiv a(N_A, N_B) = \int_0^1 \partial_x N_A \partial_x N_B \mathrm{d}x \\ F_A \equiv \int_0^1 N_A f \mathrm{d}x + N_A(0) h - a(N_A, N_{n+1}) q \end{cases}$$

such that we can rewrite the weak form as

$$K_{AB}\, \mathsf{d}_B = F_A, \text{ with } A\text{=}1,\ldots,\mathsf{n}$$

leading to a system of n equations for $\mathsf{d}_B$, $B$=1,...,n, the n unknowns. The solution is simply

$$\mathsf{d} = \mathbf{K}^{-1}\mathbf{F}$$

$\mathbf{K}$ is referred to as the stiffness matrix and $\mathbf{F}$ as the force vector.

## Choice of the shape functions

Let us have a closer look at the basis functions.

### Discretization $n = 1$

First we consider the case where $n = 1$, that is, we only have one shape function. Then we know that

$$\begin{aligned} &\mathsf{w}(x) = \mathsf{c}_1 \, N_1(x), \; x \in [0, 1] \\ &\Rightarrow N_1(1) = 0 \end{aligned}$$

and

$$\begin{aligned} &u(x) = \mathsf{d}_1 \, N_1(x) + q N_2(x) \\ &\Rightarrow N_2(1) = 1 \end{aligned}$$

So if we choose linear shape functions, we find

$$\begin{cases} N_1(x) = 1 - x \\ N_2(x) = x \end{cases}$$

We can now resolve $u(x)$ as follows, now that we have the shape functions defined, knowing that since $n = 1$ we are only dealing with $K_{11}$ and $F_1$:

$$\begin{aligned} &K_{11} = \int_0^1 \partial_x N_1 \partial_x N_1 \mathrm{d}x = 1 \\ &F_1 = \int_0^1 (1 - x) f(x) \mathrm{d}x + h + q \end{aligned}$$

thus

$$\begin{aligned} &\mathsf{d}_1 = \tfrac{F_1}{K_{11}} \\ &\Rightarrow u(x) = \left[ \int_0^1 (1 - x) f(x) \mathrm{d}x + h + q \right] (1 - x) + qx \end{aligned}$$

### Discretization $n = 2$

Let us consider now the case where $n = 2$, leading to two shape functions, such that

$$\begin{aligned} &\mathsf{w}(x) = \mathsf{c}_1 \, N_1(x) + \mathsf{c}_2 \, N_2(x), \; x \in [0, 1] \\ &\Rightarrow \begin{cases} N_1(1) = 0 \\ N_2(1) = 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} &u(x) = \mathsf{d}_1 \, N_1(x) + \mathsf{d}_2 \, N_2(x) + q N_3(x) \\ &\Rightarrow N_3(1) = 1 \end{aligned}$$

So if again we choose linear shape functions, we find

$$N_1(x) = \begin{cases} 1 - 2x & 0 < x < 1/2 \\ 0 & 1/2 < x < 1 \end{cases}$$

$$N_2(x) = \begin{cases} 2x & 0 < x < 1/2 \\ 2(1 - x) & 1/2 < x < 1 \end{cases}$$

$$N_3(x) = \begin{cases} 0 & 0 < x < 1/2 \\ 2x - 1 & 1/2 < x < 1 \end{cases}$$

Notice that we could choose higher order shape function, leading to higher order Finite-Element (FE) and to Spectral-Element (SE) methods.
Notice too how in the case $n = 2$, introducing a second shape functions increases the degree of freedom compared to the case $n = 1$, capturing more information leading to more accurate results.

As a homework:
Show that

$$\begin{cases} K_{11} = 2 \\ K_{12} = K_{21} = -2 \\ K_{22} = 4 \\ F_1 = \int_0^{1/2} (1-2x) f(x) \mathrm{d}x + h \\ F_2 = 2 \int_0^2 1/2x f(x) \mathrm{d}x + 2 \int_{1/2}^1 (1-x) f(x) \mathrm{d}x + 2q \end{cases}$$

leading to

$$\begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} \mathsf{d}_1 \\ \mathsf{d}_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

$\Rightarrow$ two linear equations for two unknowns.
Note that if $f(x) = 0 \Rightarrow u(x) = q + h(1-x)$, which is the exact solution, that is, the FE solution is exact in that case.

## Global vs. local view

Now that we have discussed the interpolation of the functions on the basis functions defined by the shape functions, let us introduce the linear finite elements on which the problem is solved. Because the physical problem is transposed and solved on elements, we need to define a global (corresponding to the physical domain) and a local (corresponding to a reference domain) view (see Figure 1).

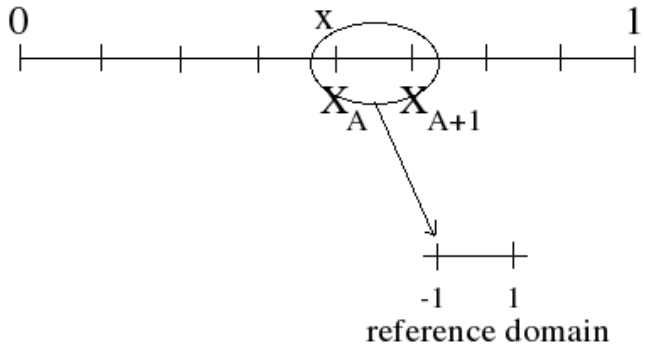


Figure 1: Global vs. Local view

### Global view

In this view we define:

- The domain $[X_A, X_{A+1}]$
- The nodes $\{X_A, X_{A+1}\}$
- The degrees of freedom $\{\mathrm{d}_A, \mathrm{d}_{A+1}\}$
- The shape functions $\{N_A(x), N_{A+1}(x)\}$

where

$$N_A(x) = \begin{cases} \frac{X - X_{A-1}}{h_{A-1}} & X_{A-1} < x < X_A \\ \frac{X_{A+1} - X}{h_A} & X_A < x < X_{A+1} \\ 0 & \text{elsewhere} \end{cases}$$

where $h_A = X_{A+1} - X_A$ is the element size, which can be irregular from one element to the other.
The shape functions for the first and last nodes are solely

$$N_1(x) = \begin{cases} \frac{X_1 - X}{h_1} & X_1 < x < X_2 \\ 0 & \text{elsewhere} \end{cases}$$

and

$$N_{n+1}(x) = \begin{cases} \frac{X - X_n}{h_n} & X_n < x < X_{n+1} \\ 0 & \text{elsewhere} \end{cases}$$

The stiffness matrix $\mathbf{K}$ is symmetric and $n \times n$, with the particular characteristics that it is banded due to the form of the shape functions, which cancel everywhere except within one element:

$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} & & & \\ K_{12} & K_{22} & K_{23} & & \\ & K_{32} & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & K_{nn} \end{pmatrix}$$

**Local view**

This is definitively the important view, especially on a parallel computational point of view.

- The domain is defined as $[\xi_1, \xi_2]$, this is the reference or standard domain. By convention we use in practice
$$\begin{cases} \xi_1 = -1 \\ \xi_2 = 1 \end{cases}$$
as it appears in Figure 1.
- The nodes are $\{\xi_1, \xi_2\}$
- Degrees of freedom $\{\mathsf{d}_1, \mathsf{d}_2\}$
- Shape functions $\{N_1, N_2\}$

Then the interpolation is $u(\xi) = \mathsf{d}_1 N_1(\xi) + \mathsf{d}_2 N_2(\xi)$.

**The mapping**

Now we need a mapping between both views, such that $\xi : [X_A, X_{A+1}] \to [\xi_1, \xi_2]$ and

$$\begin{cases} \xi(X_A) = \xi_1 \\ \xi(X_{A+1}) = \xi_2 \end{cases}$$

So if we suppose a linear mapping, we have

$$\xi(x) = c_1 \, + \, c_2 \, x$$

Due to the definition of the mapping function, we can show that it yields to

$$\xi(x) = \frac{2x - X_A - X_{A+1}}{h_A}$$

and

$$x(\xi) = \frac{1}{2}(h_A \xi + X_A + X_{A+1})$$

The linear shape functions then take the following standard form

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \text{ where } a\text{=1, 2}$$

Now we can write $\boxed{x(\xi) = \sum_{a=1}^{2} X_a N_a(\xi)}$. Doing so, you can see that the geometry is totally prescribed by this equation.

Let's go back to the stiffness matrix derivation, and summarize:

$$\begin{aligned} a(\mathsf{w}, u) = &\quad \int_0^1 \partial_x \mathsf{w}\, \partial_x u \mathrm{d}x \\ \text{(brake into finite elements)} \Rightarrow &\quad \sum_{e=1}^{E} \int_{\Omega_e} \partial_x \mathsf{w}\, \partial_x u \mathrm{d}x \\ \text{(move to the reference domain)} \Rightarrow &\quad \sum_{e=1}^{E} \int_{-1}^{1} \partial_x \mathsf{w}\, \partial_x u \left(\frac{\partial x}{\partial \xi}\right) \mathrm{d}\xi \end{aligned}$$

where $\left(\frac{\partial x}{\partial \xi}\right)$ is the Jacobian, which is calculated using

$$\begin{aligned} &x(\xi) = \textstyle\sum_{a=1}^{2} X_a N_a(\xi) \\ &\Rightarrow \tfrac{\partial x}{\partial \xi} = \textstyle\sum_{a=1}^{2} X_a \partial_\xi N_a(\xi) \end{aligned}$$

As for the spatial derivations:

$$\partial_x u = \sum_{a=1}^{2} \mathsf{d}_a\, \partial_\xi N_a(\xi) \left(\frac{\partial \xi}{\partial x}\right)$$

So we finally get

$$\begin{aligned} a(\mathsf{w}, u) = &\quad \textstyle\sum_{e=1}^{E} \int_{-1}^{1} \partial_\xi N_a \partial_\xi N_b \left(\frac{\partial \xi}{\partial x}\right) \mathrm{d}\xi \\ = &\quad \textstyle\sum_{e=1}^{E} (-1)^{a+b} \frac{1}{h_e} \end{aligned}$$

where $h_e$ is the element size. This gives

$$\mathbf{k}^{\mathbf{e}} = \frac{1}{h_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

where $\mathbf{k}^{\mathbf{e}}$ is the stiffness matrix at the local level.

### Assembly

The assembling allows you to move back from the local level to the global (physical) level. To do so, it is convenient to define an array named $LM$, the Location Matrix, which stores the information corresponding to the location of $\mathbf{k^e}$ (local level) within $\mathbf{K}$ (global level). $LM$ is such that given a particular degree of freedom, $a$, and an element number, $e$, the value returned by the $LM$ array is the corresponding global position, $A$:

$$A = LM(a,e) = \begin{cases} e & \text{if } a{=}1 \\ e+1 & \text{if } a{=}2 \end{cases}$$

such that

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | ... | $e$ | ... | $n$ |
| 2 | 2 | 3 | ... | $e+1$ | ... | 0 |

where the first column refers to the local node number $(\xi_1, \xi_2)$, and the following columns give the corresponding node numbers within the elements. Notice that the last column has only one entry. Indeed the degree of freedom of that node is prescribed by the boundary conditions, and is thus not an unknown.
Then the assembly can be written as

$$\begin{pmatrix} K_{e,e} & K_{e,e+1} \\ K_{e+1,e} & K_{e+1,e+1} \end{pmatrix} \leftarrow \begin{pmatrix} K_{e,e} & K_{e,e+1} \\ K_{e+1,e} & K_{e+1,e+1} \end{pmatrix} + \begin{pmatrix} k^e_{11} & k^e_{12} \\ k^e_{21} & k^e_{22} \end{pmatrix}$$

and

$$\begin{pmatrix} F_e \\ F_{e+1} \end{pmatrix} \leftarrow \begin{pmatrix} F_e \\ F_{e+1} \end{pmatrix} + \begin{pmatrix} f^e_1 \\ f^e_2 \end{pmatrix}$$

but only

$$\begin{aligned} K_{n,n} &\leftarrow K_{n,n} + k^{n_{el}}_{11} \\ F_n &\leftarrow F_n + f^{n_{el}}_1 \end{aligned}$$

at the last element $n_{el}$.

If we go back to the stiffness matrix, then we showed in the previous section that

$$\mathbf{k^e} = \frac{1}{h_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Let us assume that we have $n_{el} = 2$, that is, two elements. Then we can assemble $\mathbf{K}$ which is then a $2 \times 2$ matrix:

$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

with (where the upper index refers to the element number)

$$\begin{aligned} K_{11} &= K^1_{11} + K^2_{11} \\ &= k^e_{11} + 0 \\ &= k^e_{11} \end{aligned}$$

$$\begin{aligned} K_{12} &= K_{12}^1 + K_{12}^2 \\ &= k_{12}^e + 0 \\ &= k_{12}^e \\ &= K_{21} \end{aligned}$$

and

$$\begin{aligned} K_{22} &= K_{22}^1 + K_{22}^2 \\ &= k_{22}^e + k_{11}^e \end{aligned}$$

Thus, finally we obtain the fully assembled global stiffness matrix

$$\mathbf{K} = \frac{1}{h_e} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

to solve for the degrees of freedom d in the matricial form.