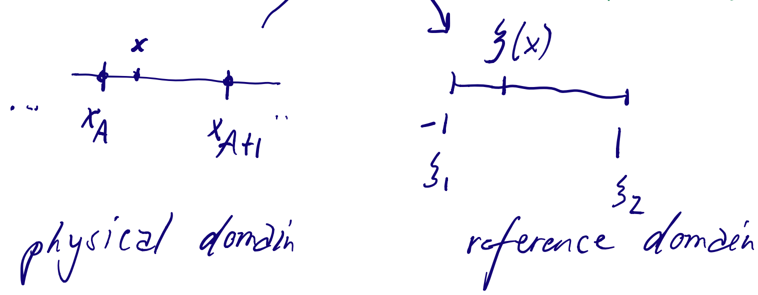
Finite-element method - part 3

Mapping between physical & reference domain:

$$\xi : [x_A, x_{A+1}] \longrightarrow [\xi_1, \xi_2]$$



We find a linear mapping

$$\xi(x) = \frac{2x - x_A - x_{A+1}}{h_A}$$

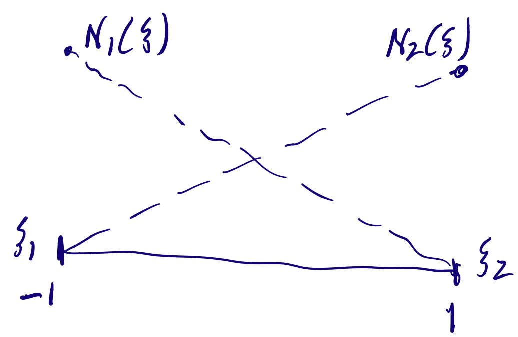
and

$$x(\xi) = \frac{1}{2}\left(h_A \xi + x_A + x_{A+1}\right)$$

Our linear shape functions become

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{when } a = 1, 2$$

$\xi_1 = -1$ and $\xi_2 = 1$



and

$$x(\xi) = \sum_{a=1}^{2} x_a N_a(\xi)$$

Let's go back to the stiffness matrix

$$a(w,u) = \int_0^L \partial_x w \; \partial_x u \; dx$$

$$= \sum_{e=1}^{E} \int_{\Omega_e} \partial_x w \; \partial_x u \; dx \quad \text{finite elements}$$

$$= \sum_{e=1}^{E} \int_{-1}^{+1} \partial_x w \; \partial_x u \left(\frac{\partial x}{\partial \xi}\right) d\xi \quad \text{reference element}$$

where $\left(\frac{\partial x}{\partial \xi}\right)$ is the Jacobian:

$$x(\xi) = \sum_{a=1}^{2} x_a N_a(\xi) \longrightarrow \frac{\partial x}{\partial \xi} = \sum_{a=1}^{2} x_a \, \partial_\xi N_a(\xi)$$

and

$$\partial_x u = \sum_{a=1}^{2} d_a \, \partial_\xi N_a(\xi) \left(\frac{\partial \xi}{\partial x}\right)$$

Thus, we get

$$a(w,u) = \sum_{e=1}^{E} \int_{-1}^{+1} \partial_\xi N_a \; \partial_\xi N_b \left(\frac{\partial \xi}{\partial x}\right) d\xi$$

$$= \sum_{e=1}^{E} (-1)^{a+b} \frac{1}{h_e}$$

with h_e element size

This gives

$$\underline{\underline{k}}^e = \frac{1}{h_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$ stiffness matrix at local level

Assembly: The assembling part allows us to move back from the local level to the global (physical) level

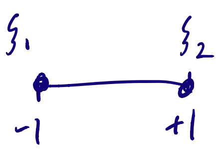
Let's define the Location Matrix:

$$LM(a, e) = \begin{cases} e & \text{if } a=1 \\ e+1 & \text{if } a=2 \end{cases}$$

a: local degree of freedom

e: element number

to provide the global matrix index



ξ_1	1	2	$\cdots$	e	$\cdots$	n	
ξ_2	2	3	$\cdots$	$e+1$	$\cdots$	$\boxed{0}$	← boundary condition

Then the assembly can be written as

$$\begin{pmatrix} K_{e,e} & K_{e,e+1} \\ K_{e+1,e} & K_{e+1,e+1} \end{pmatrix} \leftarrow \begin{pmatrix} K_{e,e} & K_{e,e+1} \\ K_{e+1,e} & K_{e+1,e+1} \end{pmatrix} + \underbrace{\begin{pmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{pmatrix}}_{\text{local element stiffness matrix}}$$

and

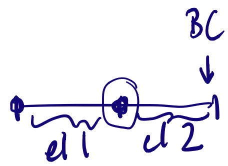
$$\begin{pmatrix} F_e \\ F_{e+1} \end{pmatrix} \leftarrow \begin{pmatrix} F_e \\ F_{e+1} \end{pmatrix} + \underbrace{\begin{pmatrix} f_1^e \\ f_2^e \end{pmatrix}}_{\text{local force contribution}}$$

at last element

$$K_{n,n} \leftarrow K_{n,n} + k_{11}^{nel}$$

$$F_n \leftarrow F_n + f_1^{nel}$$

Example: number of elements $n = 2$



global stiffness matrix $\underline{\underline{K}} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$

then

$$\begin{pmatrix} K_{11} & K_{12} & \cdot \\ K_{21} & K_{22} & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$K_{11} = K^1_{11} + K^2_{11} = k^1_{11} + 0 = \frac{1}{h_1}$$

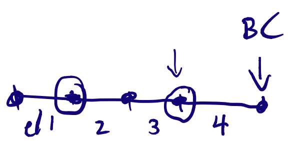
$$K_{12} = K^1_{12} + K^2_{12} = k^1_{12} + 0 = \frac{-1}{h_1}$$

$$K_{22} = K^1_{22} + K^2_{22} = \underline{k^1_{22}} + \underline{k^2_{11}} = \frac{1}{h_1} + \frac{1}{h_2}$$

Assuming $h_e = h_1 = h_2$ same element size

then $\underline{\underline{K}} = \frac{1}{h_e}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ global stiffness matrix

Illustration with 4 elements:



$$\begin{pmatrix} K_{11} & K_{12} & & 0 \\ K_{21} & K_{22} & K_{23} & \\ & K_{32} & K_{33} & K_{34} \\ 0 & & K_{43} & K_{44} \end{pmatrix}$$ global stiffness

with local contribution from element 4:

$$\begin{pmatrix} k^4_{11} & k^4_{12} \\ k^4_{21} & k^4_{22} \end{pmatrix} = \begin{pmatrix} K_{44} & K_{40}=0 \\ K_{04}=0 & K_{00}=0 \end{pmatrix}$$