Spectral-element method

Wave equation - time marching:

For the wave equation, we employ a Newmark scheme which belongs to the family of predictor-corrector schemes.

The Newmark scheme keeps track of displacement d_n, velocity v_n and acceleration a_n

$$M a_{n+1} = F_{n+1} \quad , \text{thus} \quad a_{n+1} = \frac{1}{M} F_{n+1}$$

the stepping uses

$$d_{n+1} = d_n + \Delta t \, v_n + \Delta t^2 \left[\left(\frac{1}{2} - \beta\right) a_n + \beta a_{n+1} \right]$$

$$v_{n+1} = v_n + \Delta t \left[(1-\gamma) a_n + \gamma a_{n+1} \right]$$

involving parameters β and γ:

- if $\beta \geq \gamma \geq \frac{1}{2}$: unconditionally stable
- if $\gamma \geq \frac{1}{2}, \beta < \frac{1}{2}$: conditionally stable

It is second-order accurate only if $\gamma = \frac{1}{2}$

The angular momentum is conserved only if $\beta = 0$ & $\gamma = \frac{1}{2}$

This leads to

* predictor: $d_{n+1} = d_n + \Delta t\, v_n + \frac{1}{2}\Delta t^2 a_n$

$$v_{n+1} = v_n + \frac{1}{2}\Delta t\, a_n$$

$$a_{n+1} = 0$$

* solve: $M\,\Delta a = F_{n+1} \longrightarrow \Delta a = \frac{1}{M} F_{n+1}$

* corrector: $a_{n+1} = a_{n+1} + \Delta a$

$$v_{n+1} = v_{n+1} + \frac{1}{2}\Delta t\, a_{n+1}$$

$$d_{n+1} = d_{n+1}$$

The stability criterion is

- about 5 grid points per wavelength to accurately resolve a given wavelength (at least for 4th-order polynomials → number of GLL points per element is equal to 5)

- Courant condition for numerical stability:

$$\frac{c\,\Delta t}{\Delta x} \leq 1$$

Δt: time step size
c: wave speed ($\sqrt{\frac{\mu}{\rho}}$)
Δx: grid node spacing

$$\Delta t \leq \frac{\Delta x^{\text{minimum}}}{c^{\text{maximum}}}$$

estimate for maximum time step size