Heat equation

Internal energy $\mathcal{U}$ (per unit mass) is a thermodynamic quantity with an equation of state

$$\mathcal{U} = \mathcal{U}(T) \qquad T: \text{temperature}$$

The material derivative of $\mathcal{U}$ becomes

$$D_t\, \mathcal{U}(T) = \underbrace{\frac{\partial \mathcal{U}}{\partial T}}_{c_v} D_t T$$

c_v specific heat (at constant volume)

For solids, heat flow is by <u>conduction</u>, not by convection (or radiation). Assume that heat flux is proportional to the temperature gradient ∇T (i.e., Fourier's law), we write

$$\rho\, c_v\, D_t T + \underline{\nabla} \cdot (-K\, \underline{\nabla} T) = \tilde{h}$$

or

$$\rho\, c_v\, (\partial_t T + \underline{v} \cdot \underline{\nabla} T) = \underline{\nabla} \cdot (K\, \underline{\nabla} T) + \tilde{h}$$

"heat equation"

$\tilde{h}$: heat sources

K: conductivity

c_v: specific heat

ρ: density

T: temperature

1-D equation:

For a medium at rest, the heat equation in 1-D becomes

$$\rho \, c_v \, \partial_t T = \partial_x (K \, \partial_x T) + \hat{h}$$

c_v, K, ρ: material properties

$\hat{h}$: heat sources

T: temperature
$T = T(x,t)$
(unknown)

For a homogeneous medium with constant material properties, this simplifies to

$$\partial_t T = \underbrace{\frac{K}{\rho \, c_v}}_{:= D} \partial_x^2 T + \hat{h}$$

D: thermal diffusivity

This last equation is going to be the starting point for our first homework exercise.