Spectral-element method

1D wave equation: Let's recall the 1D wave equation

$$\rho \, \partial_t^2 s = \partial_x (\mu \, \partial_x s)$$

for displacement s and density ρ, shear modulus μ.

As example, we use one of the following initial & boundary conditions

$$\begin{cases} s(x,0) = f(x) \\ s(L,t) = 0 \\ s(0,t) = 0 \end{cases} \quad \text{or} \quad \begin{cases} s(x,0) = f(x) \\ \mu \, \partial_x s(L,t) = B_L(t) \\ \mu \, \partial_x s(0,t) = B_0(t) \end{cases}$$

"Dirichlet" "Neumann"

Weak form: The weak form of the wave equation is

$$\underbrace{\int_0^L w \rho \, \partial_t^2 s \, dx}_{\text{mass}} = \int_0^L w \, \partial_x (\mu \partial_x s) \, dx$$

$$\overset{\text{integration by part}}{=} - \underbrace{\int_0^L \mu \, \partial_x w \, \partial_x s \, dx}_{\text{stiffness}} + \underbrace{w \mu \partial_x s \Big|_0^L}_{\text{boundary}}$$

and similar to the heat equation problem, we have

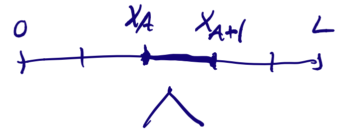
$$M_\alpha \, \partial_t^2 s^\alpha = \sum_{\gamma=0}^{N} K_{\alpha\gamma} s^\gamma$$

Note that the mass & stiffness look the same as for the heat equation. Here, we have a second-order time derivative.

<u>Meshing</u>: We subdivide the (physical) domain Ω into a number of non-overlapping elements Ω_e, $e=1,..,n$ such that $\Omega = \bigcup_{e=1}^{n} \Omega_e$

The mapping to the reference element

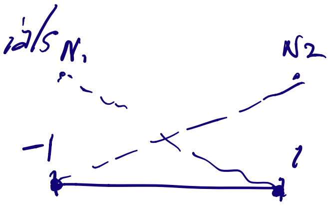
$$x(\xi) = \sum_{a=1}^{2} x_a \, N_a(\xi)$$



use low-degree Lagrange polynomials

$$N_1(\xi) = \frac{1}{2}(1-\xi)$$

$$N_2(\xi) = \frac{1}{2}(1+\xi)$$



The Jacobian of the mapping is

$$J = \frac{\partial x}{\partial \xi} = \frac{1}{2}(x_{A+1} - x_A) \quad \text{where} \quad \begin{aligned} x(-1) &= x_A \\ x(+1) &= x_{A+1} \end{aligned}$$

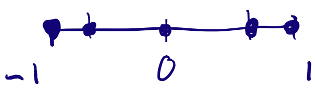
Interpolation on an element: We represent the displacement field by higher-degree Lagrange polynomials

$$f(x(\xi)) = \sum_{\alpha=0}^{N} f^{\alpha} \underbrace{l_{\alpha}^{N}(\xi)}_{\text{Lagrange polynomials of degree } N}$$

Integration over an element: For the spectral-element method, we use a Gauss-Lobatto-Legendre integration rule

$$\int_{\Omega_e} f(x)\,dx = \int_{-1}^{1} f(x(\xi))\,J(\xi)\,d\xi$$

$$\approx \sum_{\alpha=0}^{N} \underbrace{\hat{\omega}_{\alpha}}_{\text{integration weights}} f^{\alpha} J^{\alpha}$$



where the integration points are the GLL points, i.e., roots of $(1-\xi^2)\,P_N'(\xi) = 0$

Discretization of the weak form: To obtain explicit expressions for the weak form, we expand displacement and test functions as

$$s(x(\xi), t) = \sum_{\alpha=0}^{N} s^{\alpha}(t)\, l_{\alpha}^{N}(\xi)$$

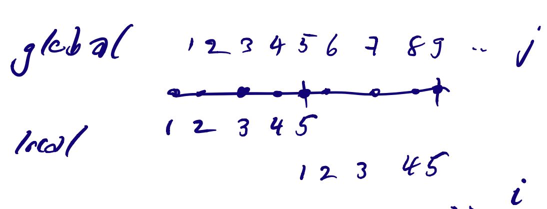
$$w(x(\xi)) = \sum_{\alpha=0}^{N} w^{\alpha}\, l_{\alpha}^{N}(\xi)$$

This choice makes the SEM a Galerkin method, where the basis functions are the same for the test (vector) function as for the displacement field.

At element level, we find the following local contributions

- mass matrix: $M_{\beta\beta'} = \hat{\omega}_{\beta'}\, \rho^{\beta'}\, J^{\beta'}$
- stiffness matrix: $K_{\beta',\alpha} = -\sum_{\gamma=0}^{N} \hat{\omega}_{\gamma}\, \mu^{\gamma}\, l'^{N}_{\beta'}(\xi_{\gamma})\, l'^{N}_{\alpha}(\xi_{\gamma})\, J^{\gamma}\, [\partial_x \xi(\xi_{\gamma})]^2$

Assembly: We use a local to global array indexing



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