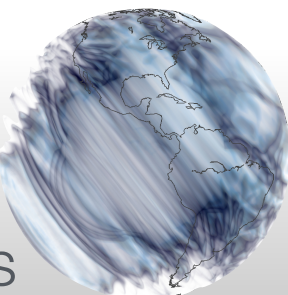


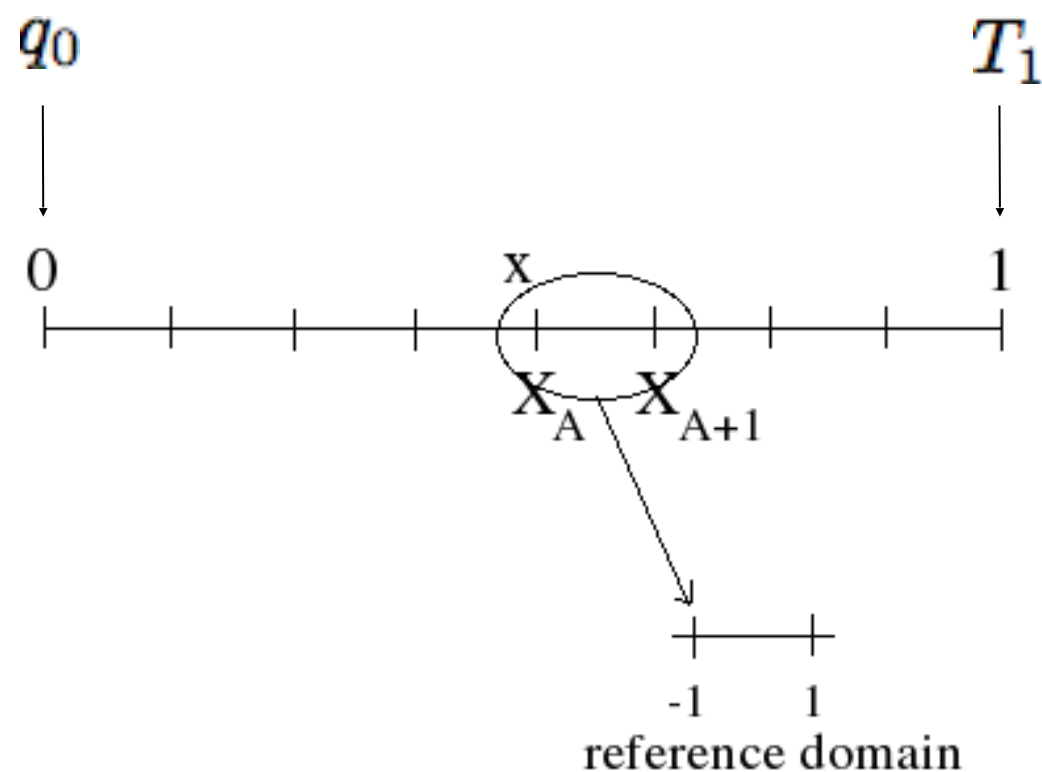
Finite-element methods



1D steady-state diffusion equation

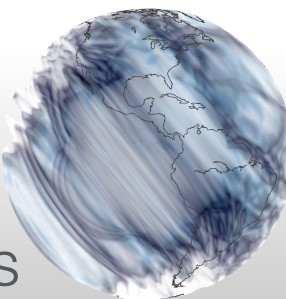
Strong form: $\partial_x^2 T + f = 0$

Boundary conditions: $\begin{cases} T(1) = T_1 \\ -\partial_x T(0) = q_0 \end{cases}$



Explore two sets of boundary conditions:

- $T_1 = 1$, $q_0 = 1$, and $f = 0$
- $T_1 = 1$, $q_0 = 1$, and $f = 1$



FEM - 1D steady-state diffusion equation

Weak form:

$$-\int_0^1 \partial_x w \partial_x T dx + q_0 w(0) + \int_0^1 w f dx = 0$$

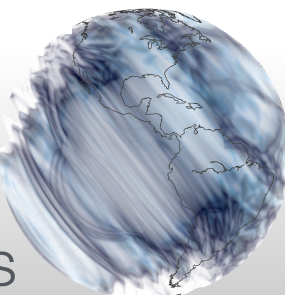
Test function and temperature field expanded on some basis functions

$$w(x) = \sum_{A=1}^N c_A N_A(x)$$

$$T(x) = \sum_{A=1}^{N_{el}} d_A N_A(x) + T_1 N_{n+1}(x)$$

unknown

$$-\sum_B \int_0^1 \partial_x N_A d_B \partial_x N_B dx - \int_0^1 \partial_x N_A q \partial_x N_{n+1} dx + \int_0^1 N_A f dx + h N_A(0) = 0$$



FEM - 1D steady-state diffusion equation

$$-\sum_B \int_0^1 \partial_x N_A d_B \partial_x N_B dx - \int_0^1 \partial_x N_A q \partial_x N_{n+1} dx + \int_0^1 N_A f dx + h N_A(0) = 0$$

$$\begin{cases} K_{AB} \equiv a(N_A, N_B) = \int_0^1 \partial_x N_A \partial_x N_B dx \\ F_A \equiv \int_0^1 N_A f dx + N_A(0)h - a(N_A, N_{n+1})q \end{cases}$$

$$a(w, u) \equiv \int_0^1 \partial_x w \partial_x u dx$$

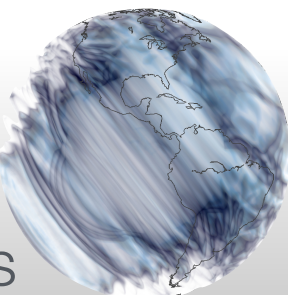
We are solving:

$$\mathbf{Kd} = \mathbf{F}$$

stiffness matrix

right-hand-side (force) vector

unknown vector



FEM - 1D steady-state diffusion equation

Global level:

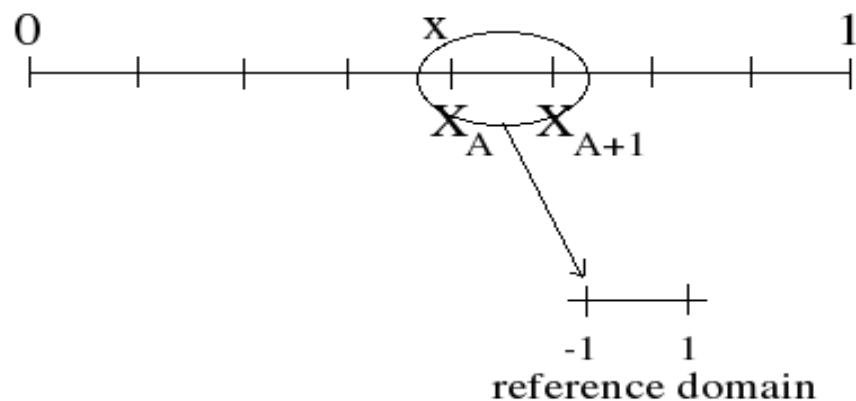
$$\begin{cases} K_{AB} \equiv a(N_A, N_B) = \int_0^1 \partial_x N_A \partial_x N_B dx \\ F_A \equiv \int_0^1 N_A f dx + N_A(0)h - a(N_A, N_{n+1})q \end{cases}$$

$$\rightarrow \mathbf{Kd} = \mathbf{F}$$

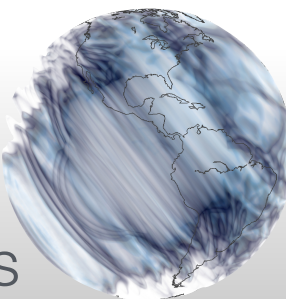
$$\int_0^1 g(x) dx = \sum_{\Omega_e} \int_{\Omega_e} g(x) dx = \sum_{\Omega_e} \int_{-1}^1 g(x(\xi)) J d\xi \quad J = \frac{dx}{d\xi}$$

Local level:

Consider the mapping $\xi : [X_A, X_{A+1}] \rightarrow [\xi_1, \xi_2]$, such that



$$\begin{cases} \xi(X_A) = \xi_1 = -1 \\ \xi(X_{A+1}) = \xi_2 = 1 \end{cases}$$



FEM - 1D steady-state diffusion equation

Local (element) resolution:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad a=1,2$$

Stiffness matrix:

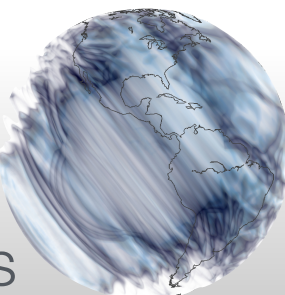
$$k_{ab}^e = a(N_a, N_b) = \int_{\Omega_e} \partial_x N_a \partial_x N_b dx$$

↓ Change of variables (reference domain)

$$k_{ab}^e = \frac{2}{h_e} \int_{-1}^1 \partial_\xi N_a \partial_\xi N_b d\xi$$

↓ Matrix form

$$k^e = \frac{1}{h_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$



FEM - 1D steady-state diffusion equation

Local (element) resolution:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad a=1,2$$

Force vector:

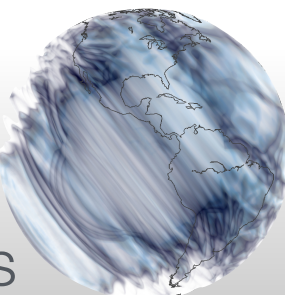
$$f_a^e = \int_{\Omega_e} N_A f dx + \begin{cases} \delta_{a1} q_0 & \text{for } e = 1 \\ -k_{a2}^e T_1 & \text{for } e = N_{el} \\ 0 & \text{else} \end{cases}$$

↓ Change of variables (reference domain)

$$f_a^e = \frac{h_e}{2} \int_{-1}^1 N_a f d\xi + \text{boundary terms}$$

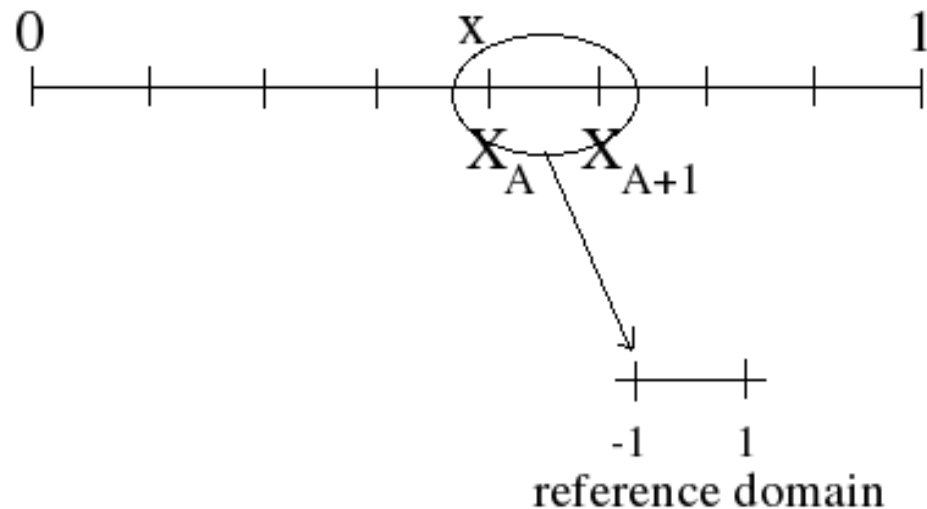
↓ Matrix form

$$f^e = \frac{h_e}{6} \begin{pmatrix} 2f_1 + f_2 \\ f_1 + 2f_2 \end{pmatrix} + \text{boundary terms}$$



FEM - 1D steady-state diffusion equation

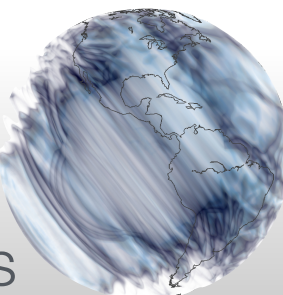
Assembling: back to global level



$$LM(i,ielem) = \begin{cases} ielem & \text{if } i=1 \\ ielem+1 & \text{if } i=2 \end{cases}$$

code example:

```
do ielem = 1, nelem
  do i = 1,2
    do j = 1,2
      K(LM(i,ielem),LM(j,ielem)) = K(LM(i,ielem),LM(j,ielem))
      + k_local_ielem(i,j)
    end
  end
end
```



FEM - 1D steady-state diffusion equation

FEM solution

