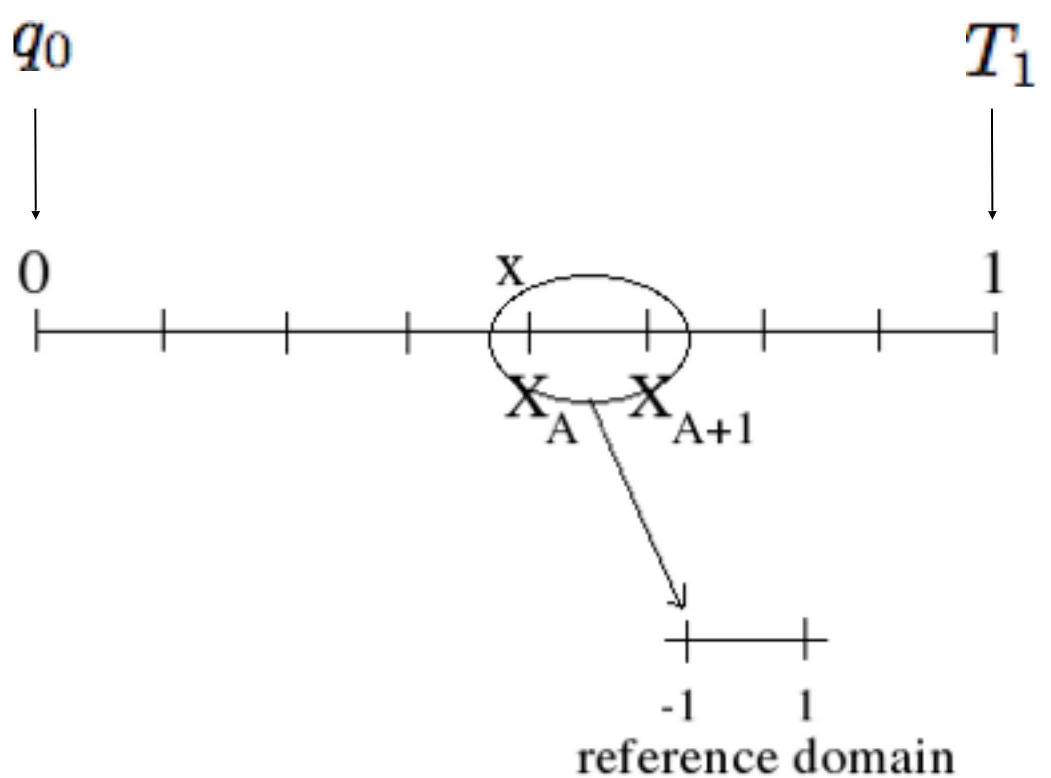
Finite-element methods

1D steady-state diffusion equation

Strong form: $$\partial_x^2 T + f = 0$$

Boundary conditions: $$\begin{cases} T(1) & = & T_1 \\ -\partial_x T(0) & = & q_0 \end{cases}$$



Explore two sets of boundary conditions:

- $T_1 = 1$, $q_0 = 1$, and $f = 0$
- $T_1 = 1$, $q_0 = 1$, and $f = 1$

FEM - 1D steady-state diffusion equation

Weak form: $$-\int_0^1 \partial_x w \partial_x T \mathrm{d}x + q_0 w(0) + \int_0^1 w f \mathrm{d}x = 0$$

Test function and temperature field expanded on some basis functions

$$w(x) = \sum_{A=1}^{N} c_A N_A(x)$$

$$T(x) = \sum_{A=1}^{N_{el}} d_A N_A(x) + T_1 N_{n+1}(x)$$

unknown

$$\boxed{-\sum_B \int_0^1 \partial_x N_A d_B \partial_x N_B \mathrm{d}x - \int_0^1 \partial_x N_A q \partial_x N_{n+1} \mathrm{d}x + \int_0^1 N_A f \mathrm{d}x + h N_A(0) = 0}$$

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$$\boxed{-\sum_B \int_0^1 \partial_x N_A d_B \partial_x N_B \mathrm{d}x - \int_0^1 \partial_x N_A q \partial_x N_{n+1} \mathrm{d}x + \int_0^1 N_A f \mathrm{d}x + h N_A(0) = 0}$$

$$\begin{cases} K_{AB} \equiv a(N_A, N_B) = \int_0^1 \partial_x N_A \partial_x N_B \mathrm{d}x \\ F_A \equiv \int_0^1 N_A f \mathrm{d}x + N_A(0)h - a(N_A, N_{n+1})q \end{cases}$$

$$a(w,u) \equiv \int_0^1 \partial_x w \partial_x u \mathrm{d}x$$

We are solving:

$$\boxed{\mathbf{Kd} = \mathbf{F}}$$

stiffness matrix

unknown vector

right-hand-side (force) vector

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Global level:

$$\begin{cases} K_{AB} \equiv a(N_A, N_B) = \int_0^1 \partial_x N_A \partial_x N_B \mathrm{d}x \\ F_A \equiv \int_0^1 N_A f \mathrm{d}x + N_A(0)h - a(N_A, N_{n+1})q \end{cases}$$

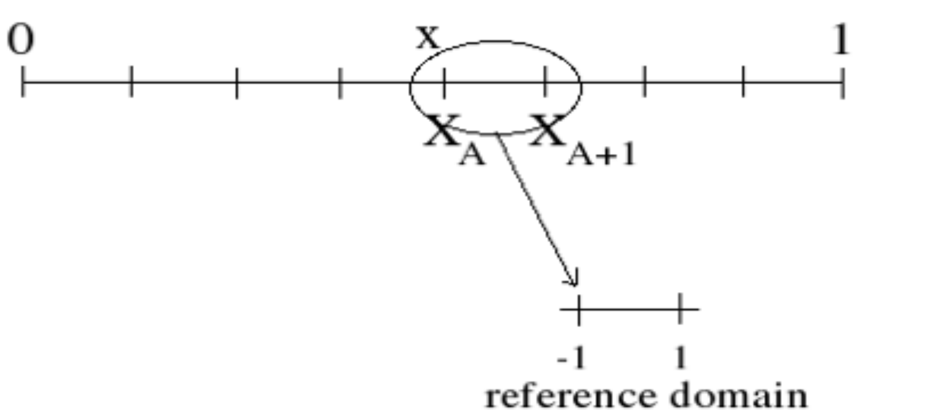
$$\rightarrow \mathbf{Kd} = \mathbf{F}$$

$$\int_0^1 g(x)dx = \sum_{\Omega_e} \int_{\Omega_e} g(x)dx = \sum_{\Omega_e} \int_{-1}^{1} g(x(\xi))Jd\xi \qquad J = \frac{dx}{d\xi}$$

Local level:

Consider the mapping $\xi : [X_A, X_{A+1}] \rightarrow [\xi_1, \xi_2]$, such that

$$\begin{cases} \xi(X_A) & = \xi_1 = -1 \\ \xi(X_{A+1}) & = \xi_2 = 1 \end{cases}$$



FEM - 1D steady-state diffusion equation

Local (element) resolution:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{a=1,2}$$

Stiffness matrix:

$$k^e_{ab} = a(N_a, N_b) = \int_{\Omega_e} \partial_x N_a \partial_x N_b \mathrm{d}x$$

$\Downarrow$ Change of variables (reference domain)

$$k^e_{ab} = \frac{2}{h_e} \int_{-1}^{1} \partial_\xi N_a \partial_\xi N_b \mathrm{d}\xi$$

$\Downarrow$ Matrix form

$$k^e = \frac{1}{h_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

FEM - 1D steady-state diffusion equation

Local (element) resolution:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{a=1,2}$$

Force vector:

$$f_a^e = \int_{\Omega_e} N_A f \mathrm{d}x + \begin{cases} \delta_{a1} q_0 & \text{for } e = 1 \\ -k_{a2}^e T_1 & \text{for } e = N_{el} \\ 0 & \text{else} \end{cases}$$

$\Downarrow$ Change of variables (reference domain)

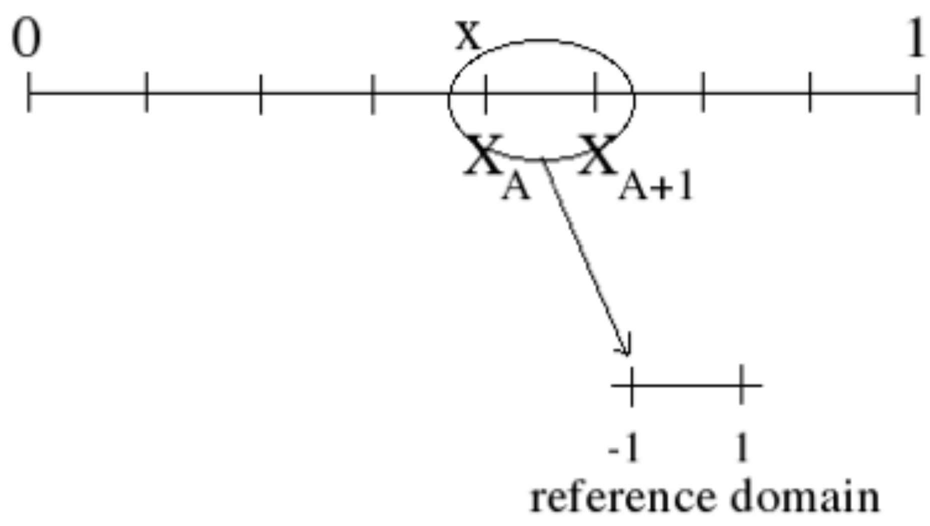
$$f_a^e = \frac{h_e}{2} \int_{-1}^{1} N_a f \mathrm{d}\xi + \text{boundary terms}$$

$\Downarrow$ Matrix form

$$f^e = \frac{h_e}{6} \begin{pmatrix} 2f_1 + f_2 \\ f_1 + 2f_2 \end{pmatrix} + \text{boundary terms}$$

FEM - 1D steady-state diffusion equation

Assembling: back to global level



$$\text{iglob(i,ielem)} = \begin{cases} \text{ielem} & \text{if i=1} \\ \text{ielem+1} & \text{if i=2} \end{cases}$$

code example:

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do ielem = 1, nelem
    do i = 1,2
        u(iglobe(i,ielem)) = u(iglobe(i,ielem)) + u_local(i,ielem)
    end
end
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FEM - 1D steady-state diffusion equation

FEM solution