Spectral-element method

1D wave equation

1D wave equation

Strong form: $\rho \partial_t^2 s = \partial_x(\mu \partial_x s)$

IC & BC:
$$\left\{\begin{array}{lcl} s(x,0) & = & f(x) \\ s(L,t) & = & 0 \\ s(0,t) & = & 0 \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} s(x,0) & = & f(x) \\ \partial_x s(L,t) & = & 0 \\ \partial_x s(0,t) & = & 0 \end{array}\right.$$

Dirichlet boundary   Neumann boundary

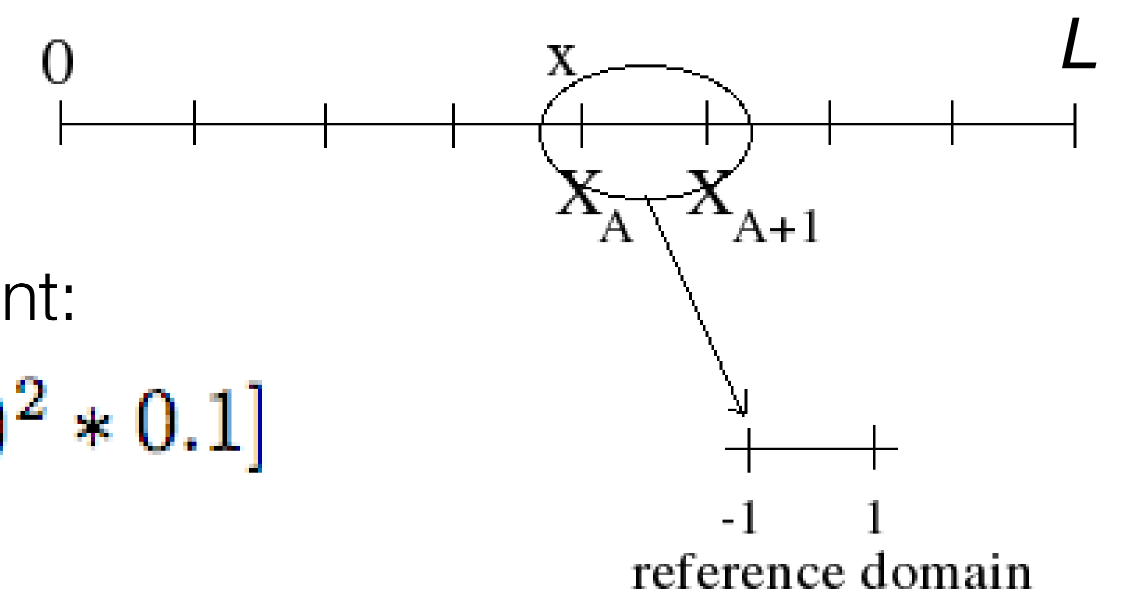
1D wave equation

Strong form: $\rho \partial_t^2 s = \partial_x (\mu \partial_x s)$

IC & BC: $\begin{cases} s(x,0) & = & f(x) \\ s(L,t) & = & 0 \\ s(0,t) & = & 0 \end{cases}$ and $\begin{cases} s(x,0) & = & f(x) \\ \partial_x s(L,t) & = & 0 \\ \partial_x s(0,t) & = & 0 \end{cases}$

Dirichlet boundary

Neumann boundary



string with initial displacement:

$f(x) = \exp[-(x-50)^2 * 0.1]$

string properties:

$\rho = 1$ & $\mu = 1$

Weak form

Weak form: $$\int_0^L w\,\rho\,\partial_t^2 s\,\mathrm{d}x = -\int_0^L \mu\,\partial_x w\,\partial_x s\,\mathrm{d}x + w\,\mu\,\partial_x s\Big|_0^L$$

displacement field (and test function) expanded on basis functions:

$$s(x(\xi),t) = \sum_{\alpha}^{N} s^{\alpha}(t)\, l_{\alpha}^{N}(\xi)$$

unknowns

Weak form

Weak form: $$\int_0^L w\,\rho\,\partial_t^2 s\,\mathrm{d}x = -\int_0^L \mu\,\partial_x w\,\partial_x s\,\mathrm{d}x + w\,\mu\,\partial_x s\Big|_0^L$$

$$\rightarrow \quad \mathbf{M}\partial_t^2 \mathbf{s} = \mathbf{K}\mathbf{s}$$

mass matrix

stiffness matrix

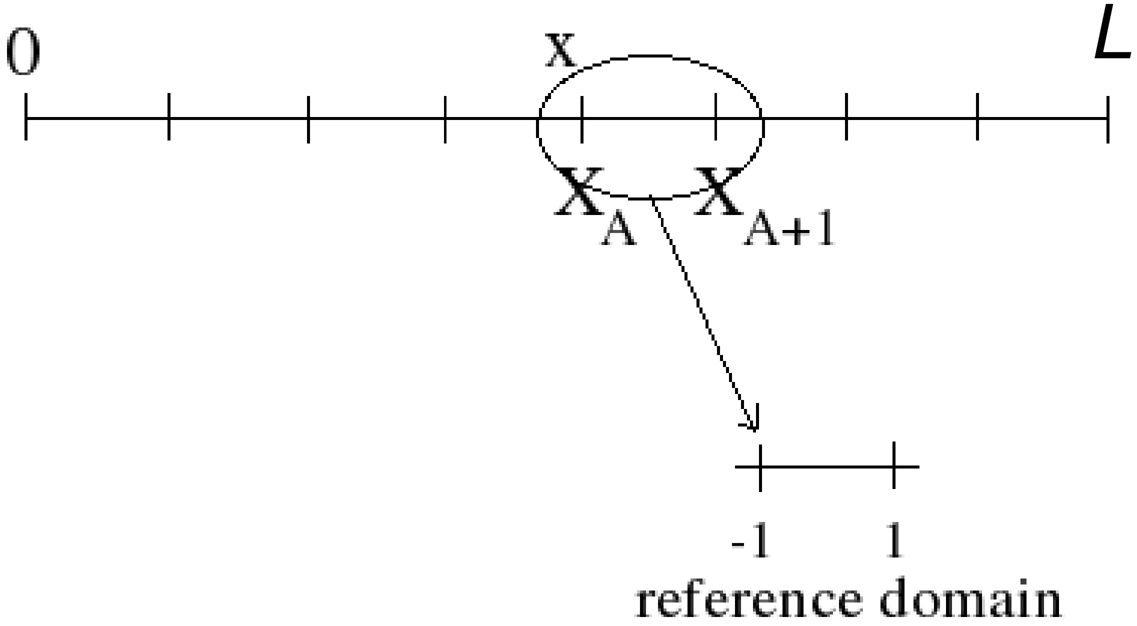
zero

force vector

Reference domain

Definition of the reference domain:

Consider the mapping $\xi : [X_A, X_{A+1}] \rightarrow [\xi_1, \xi_2]$, such that



$$\begin{cases} \xi(X_A) & = \xi_1 = -1 \\ \xi(X_{A+1}) & = \xi_2 = 1 \end{cases}$$

$$x(\xi) = \sum_{a=1}^{2} X_a N_a(\xi)$$

with shape functions being degree-1 Lagrange polynomials

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \quad \text{a=1,2}$$

Jacobian: $J = \dfrac{\partial x}{\partial \xi}$

Reference domain

Interpolation:

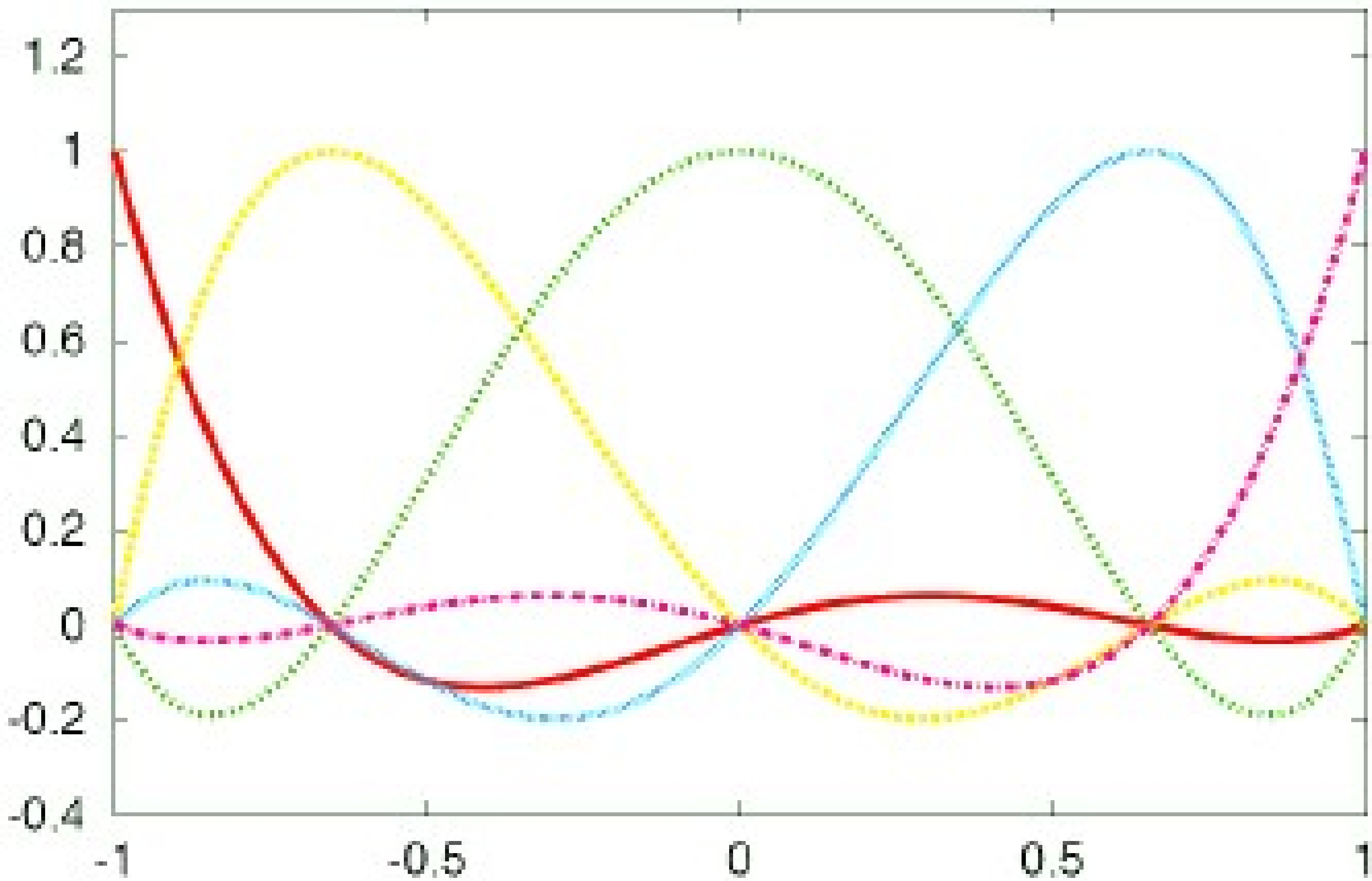
$$s(x(\xi), t) = \sum_{\alpha}^{N} s^{\alpha}(t) l_{\alpha}^{N}(\xi)$$

Gauss-Lobatto-Legendre quadrature integration rule:

$$\begin{aligned}\int_{\Omega_e} s(x,t)\mathrm{d}x &= \int_{-1}^{1} s(x(\xi),t)J(\xi)\mathrm{d}\xi \\ &\sim \sum_{\alpha=0}^{N} \hat{\omega}_{\alpha}\, s^{\alpha}(t) J^{\alpha}\end{aligned}$$

Basis functions

Lagrange polynomials:

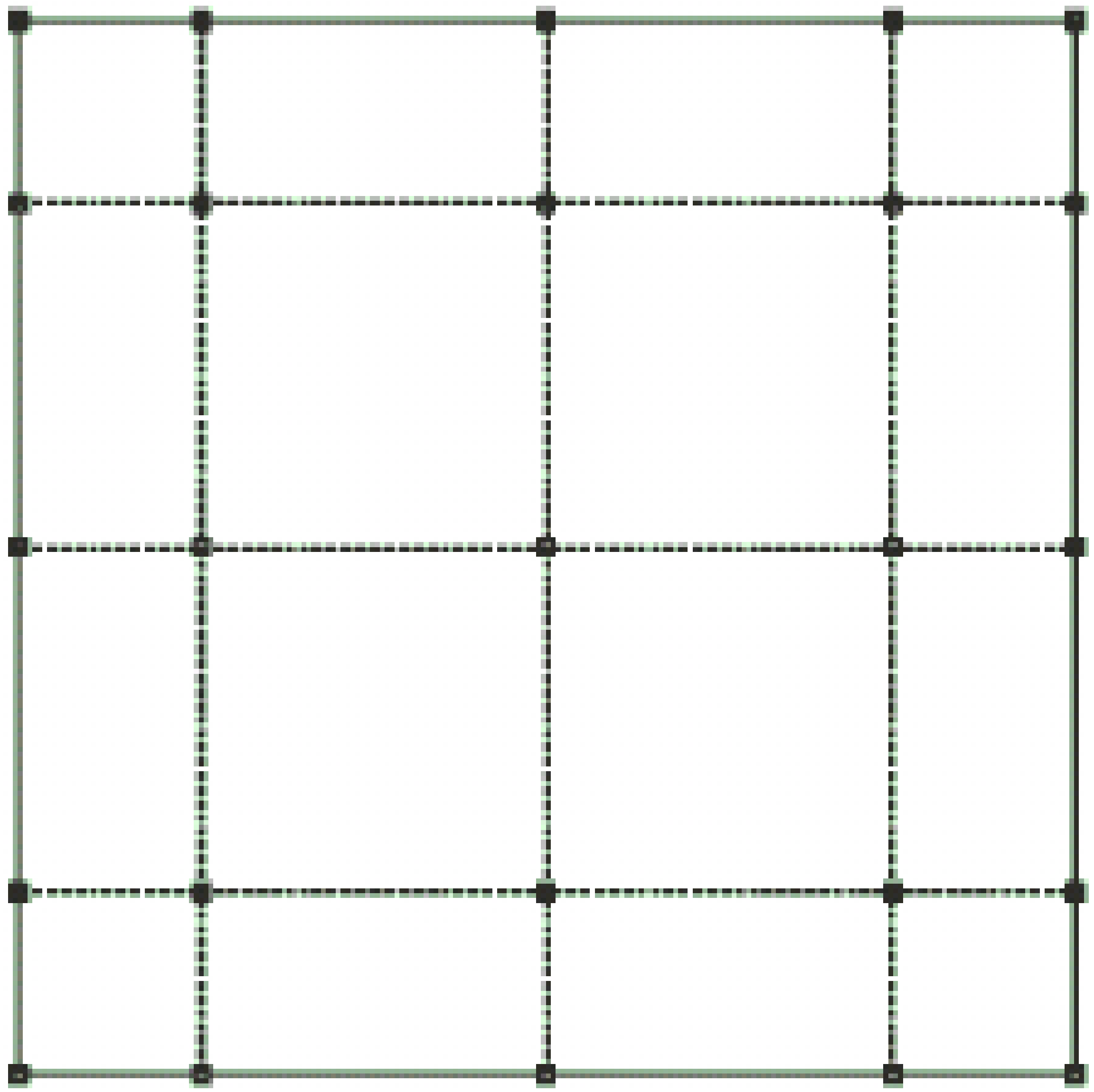


degree-4 polynomials

Lagrange polynomials property: $l_\alpha^N(\xi_\beta) = \delta_{\alpha\beta}$

Basis functions

Gauss-Lobatto-Legendre points:



degree-4 GLL points
(2D quad example)

GLL points are the n+1 roots of $(1-\xi^2)P_n'(\xi) = 0$

P_n : Legendre polynomial of degree n

Mass matrix

Local (element) resolution: Mass matrix

$$\begin{aligned}
\int_{\Omega_e} w\,\rho\,\partial_t^2 s \mathrm{d}x &= \int_{-1}^{1} \rho(x(\xi))\,w(x(\xi))\,\partial_t^2 s(x(\xi),t)\,J(\xi)\mathrm{d}\xi \\
&\sim \sum_{\alpha=0}^{N} \hat{\omega}_\alpha \rho^\alpha J^\alpha \sum_{\beta}^{N} w^\beta l_\beta^N(\xi_\alpha) \sum_{\gamma}^{N} \partial_t^2 s^\gamma l_\gamma^N(\xi_\alpha) \\
&= \sum_{\alpha=0}^{N} \boxed{\hat{\omega}_\alpha \rho^\alpha J^\alpha\, w^\alpha}\, \partial_t^2 s^\alpha
\end{aligned}$$

diagonal matrix

Stiffness matrix

Local (element) resolution: Stiffness matrix

$$
\begin{aligned}
\int_{\Omega_e} \mu\, \partial_x w\, \partial_x s \mathrm{d}x &= \int_{-1}^{1} \mu(x(\xi))\, [\partial_x w(x(\xi))][\partial_x s(x(\xi),t)] J(\xi) \mathrm{d}\xi \\
&\sim \sum_{\alpha=0}^{N} \hat{\omega}_\alpha \mu^\alpha [\sum_{\beta}^{N} w^\beta l_\beta'^N(\xi_\alpha) \partial_x \xi(\xi_\alpha)][\sum_{\gamma}^{N} s^\gamma l_\gamma'^N(\xi_\alpha) \partial_x \xi(\xi_\alpha)] J^\alpha
\end{aligned}
$$

=> $$M_{\alpha_1} \partial_t^2 s^{\alpha_1}(t) = \sum_{\gamma=0}^{N} K_{\alpha_1 \gamma} s^\gamma(t)$$

Matricial form:

$$\mathbf{M} \partial_t^2 \mathbf{s} = \mathbf{K} \mathbf{s}$$

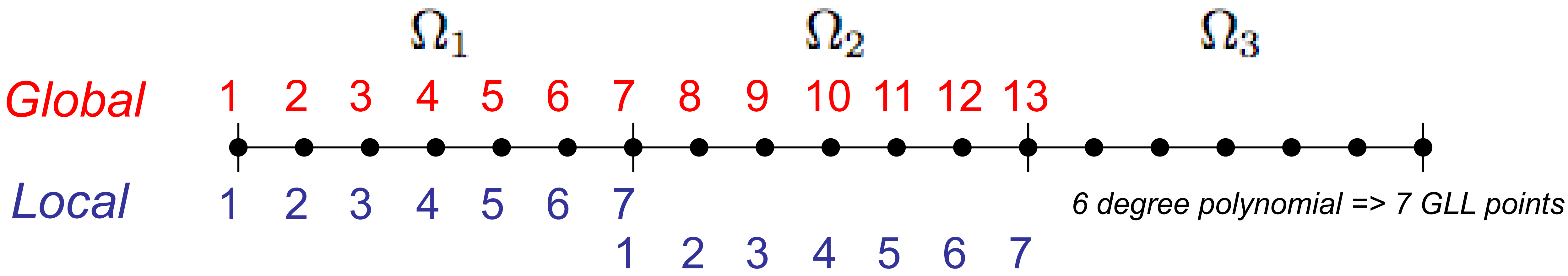
Boundaries

Boundary conditions:

$$
\begin{aligned}
w\mu\,\partial_x s &= w^N \mu^N \sum_{\alpha=0}^{N} s^\alpha(t)\, l_\alpha'^N(\xi_N)\,\partial_x\xi(\xi_N) \quad \text{at } x = L \\
w\mu\,\partial_x s &= w^0 \mu^0 \sum_{\alpha=0}^{N} s^\alpha(t)\, l_\alpha'^N(\xi_0)\,\partial_x\xi(\xi_0) \quad \text{at } x = 0
\end{aligned}
$$

Assembly

Local to Global:



You need an array which links Local *(where the calculation is done)* to Global *(where you want to know the results which are marched in time):*

$$\text{iglob}(\text{ielmn}, i) = j$$

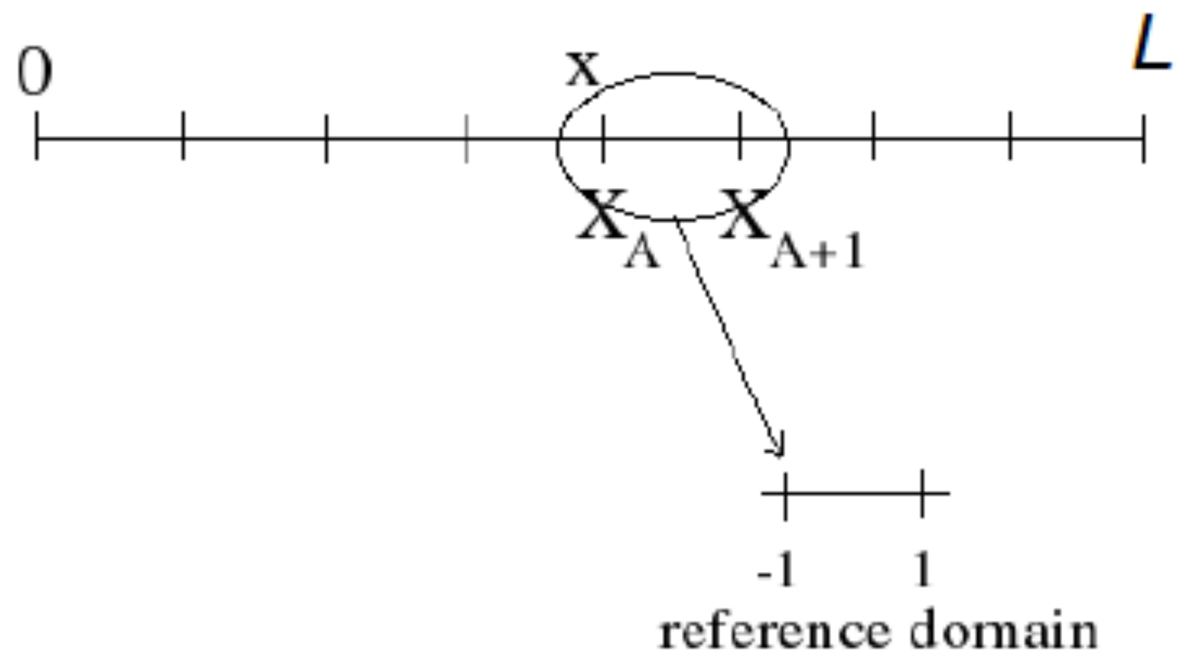
The element number (2)

The local number (4)

The global number (10)

Assembly

Assembling: back to global level



$M_{local}(\text{ielmn}, i) = \omega_\alpha \rho^\alpha J^\alpha$

$M_{global}(:) = 0$

!loop over the elements

do ielmn=1,Nel

!loop over the GLL points

  do i=1,NGLL

!get the global index

  j=iglob(ielmn,i) $M_{global}(j) = M_{global}(j) + M_{local}(\text{ielmn}, i)$

  enddo

enddo

Time stepping

Time scheme: Newmark algorithm

- Predictor:

$$\begin{aligned} d_{n+1} &= d_n + \Delta t v_n + \frac{1}{2}\Delta t^2 a_n \\ v_{n+1} &= v_n + \frac{1}{2}\Delta t a_n \\ a_{n+1} &= 0 \quad \text{(initialization at the beginning of each time step)} \end{aligned}$$

- Solve:

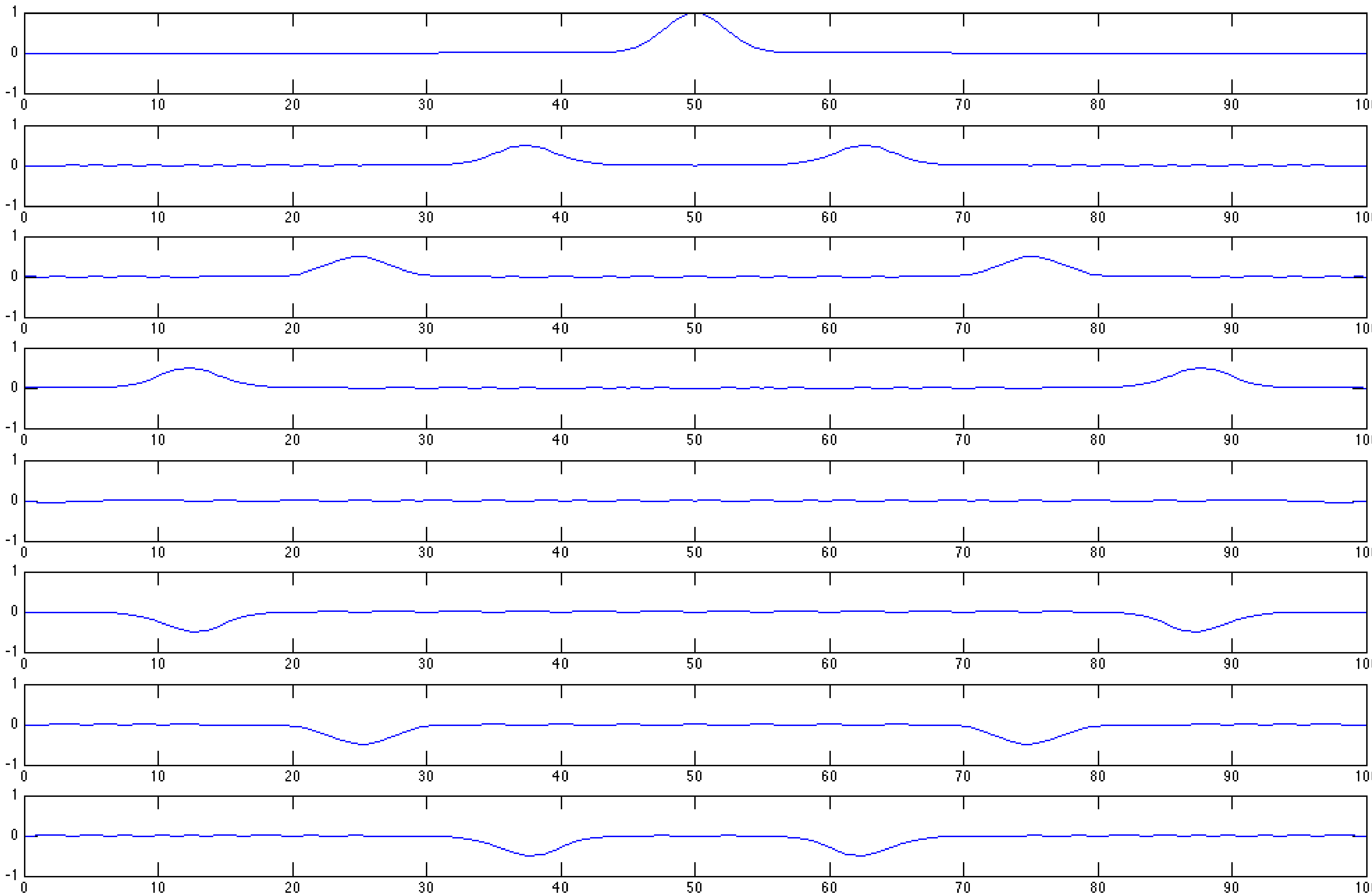
$$\begin{aligned} F_{n+1} &= K d_{n+1} \\ M\Delta a &= F_{n+1} \end{aligned}$$

- Corrector:

$$\begin{aligned} a_{n+1} &= a_{n+1} + \Delta a \\ v_{n+1} &= v_{n+1} + \frac{1}{2} a_{n+1} \\ d_{n+1} &= d_{n+1} \end{aligned}$$

SEM - 1D unsteady-state diffusion equation

Results: *(a) Dirichlet boundary*



SEM - 1D unsteady-state diffusion equation

Results: *(b) Neumann boundary*