Problem Set 2

Finite Difference Method

The following is what you may need in order to solve this problem set:

$$\begin{aligned}
\partial_t^2 f(x,t) &\approx \frac{f(x,t+\Delta t) - 2f(x,t) + f(x,t-\Delta t)}{\Delta t^2} \\
\partial_x^2 f(x,t) &\approx \frac{f(x+\Delta x,t) - 2f(x,t) + f(x-\Delta x,t)}{\Delta x^2} \\
\partial_x f(x,t) &\approx \frac{f(x+\Delta x,t) - f(x-\Delta x,t)}{2\Delta x}
\end{aligned}$$

where $f(x,t)$ is any given field.

Wave Equation

The 1-D expression of the wave equation is:

$$\rho(x)\partial_t^2 u(x,t) = \partial_x[\kappa(x)\partial_x u(x,t)], \quad (x \in [0,L], t \in [0,+\infty))$$

where $u(x,t)$ is the displacement field at the position x at instant t, $\rho(x)$ is the material density and $\kappa(x)$ is the material (bulk) modulus. If the material properties are constant, we can write:

$$\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t)$$

where we have introduced the wavespeed c

$$c = \sqrt{\frac{\kappa}{\rho}}.$$

Initially, let us consider constant material properties, and use the finite difference method to solve the 1-D 'homogeneous' wave equation:

$$\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t)$$

The initial conditions are:

$$
\begin{aligned}
u(x,0) &= \exp^{-0.1(x-50)^2} \\
\partial_t u(x,0) &= 0
\end{aligned}
$$

We are going to first investigate 2 types of boundary conditions:

- the **Dirichlet** boundary conditions: corresponds to a fix termination, e.g., displacement $u(0,t) = 0$ that is no displacement at all time.
- the **Neumann** boundary conditions: corresponds to a termination free to move, e.g., stress $T(0,t) = 0$.

First-Order System

To solve the wave equation, you can recognize that it is equivalent to

$$
\begin{aligned}
\rho(x)\partial_t v(x,t) &= \partial_x T(x,t) \\
\partial_t T(x,t) &= \kappa(x)\partial_x v(x,t)
\end{aligned}
$$

where

$$
\begin{aligned}
v(x,t) &= \partial_t u(x,t) \quad \text{is} \quad \text{a} \quad \text{velocity}\,, \\
T(x,t) &= \kappa(x)\partial_x u(x,t) \quad \text{is} \quad \text{a} \quad \text{stress}\,.
\end{aligned}
$$

Problem:

Write the discretized form of the system to solve for

- (v, T) using the first-order system, and

- directly for u using the second-order system.

The grid size Δx is chosen to be 0.1. The string length is $L = 100$. Plot your numerical results at several time steps for the following cases:

- $c = 1, \rho = 1, \kappa = 1 \quad (x \in [0, 100])$, Dirichlet boundary conditions on both ends of the string.
- $c = 1, \rho = 1, \kappa = 1 \quad (x \in [0, 100])$, Neumann boundary conditions on both ends of the string.

Extra Question - Heterogeneous materials

Use the same code you just wrote and investigate the evolution of the displacement field time series, when the material properties change:

- $c(x) = 1, \rho(x) = 1, \kappa(x) = 1 \quad (x \in [0, 60])$ and
- $c(x) = 2, \rho(x) = 1, \kappa(x) = 4 \quad (x \in (60, 100])$

Problem:

Use the same initial conditions as previously, together with the Dirichlet boundary conditions in $x = 0$ and the Neumann boundary conditions in $x = 100$ and plot the numerical results.