Spectral-element method

The spectral-element method (SEM) is a weak-form method like the finite-element method, but differs in the choice of basis functions used to interpolate the function and the integration rule:

1.) SEM uses high-order polynomials, i.e., Lagrange polynomials

2.) together with a specific integration rule (quadrature) we will have a diagonal mass matrix

Weak form (or variational statement):

Let's consider the 1D heat equation

$$\rho c_v \partial_t T = \partial_x (K \partial_x T)$$

with temperature $T = T(x,t)$ in $[0, L]$, c_v is heat capacity, ρ density & K thermal conductivity.

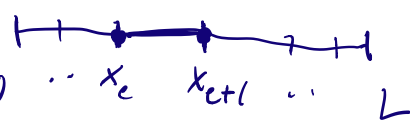
The weak form writes as

$$\int_0^L w\,\rho c_v\,\partial_t T\,dx = \int_0^L w\,\partial_x(K\,\partial_x T)\,dx$$

$$\underset{\text{integration by parts}}{=} -\int_0^L K\,\partial_x w\,\partial_x T\,dx + \underbrace{\int_0^L \partial_x(w\,K\,\partial_x T)\,dx}_{w\,K\,\partial_x T\big|_0^L}$$

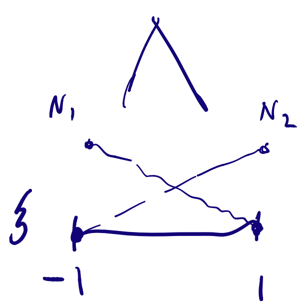
Choice of the shape functions:

Our discretization looks like:



Domain $\Omega = \bigcup_e \Omega_e$

with elements $\Omega_e = [x_e, x_{e+1}]$



The reference element will have $\xi \in [-1, 1]$ and linear mapping

$$x(\xi) = \sum_{a=1}^{2} N_a(\xi)\,x_a = N_1(\xi)\,x_e + N_2(\xi)\,x_{e+1}$$

linear mapping: $\xi : [x_e, x_{e+1}] \rightarrow [\xi_1, \xi_2]$

with degree-one Lagrange shape functions

$$\begin{cases} N_1(\xi) = \frac{1}{2}(1-\xi) \\ N_2(\xi) = \frac{1}{2}(1+\xi) \end{cases}$$

"anchor" shape functions

and the Jacobian

$$J = \frac{\partial x}{\partial \xi} = \frac{1}{2}(x_{e+1} - x_e)$$

Choice of the basis functions:

In the SEM, functions are expanded on high-order Lagrange polynomials:

$$f(x(\xi)) = \sum_{\alpha=0}^{N} f^{\alpha} \, l_{\alpha}^{N}(\xi)$$

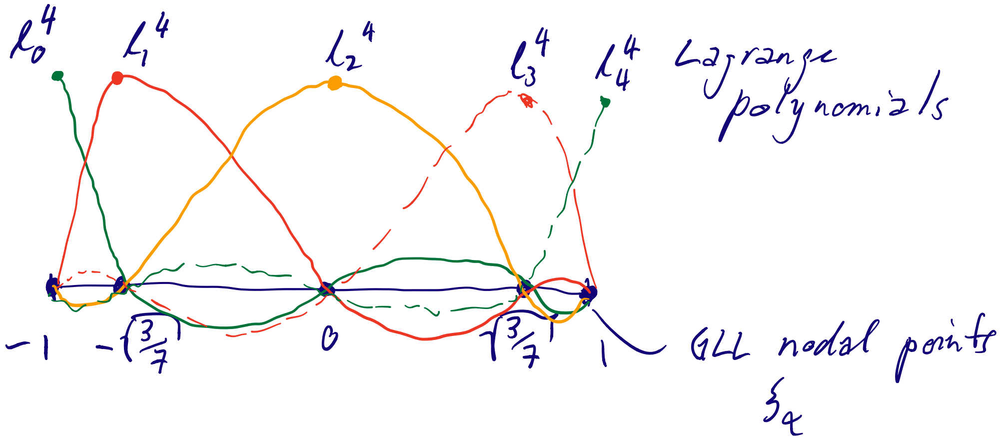
where l_α^N are Lagrange polynomials of degree N

$$l_\alpha^N(\xi) \equiv \frac{(\xi - \xi_0)\cdots(\xi - \xi_{\alpha-1})(\xi - \xi_{\alpha+1})\cdots(\xi - \xi_N)}{(\xi_\alpha - \xi_0)\cdots(\xi_\alpha - \xi_{\alpha-1})(\xi_\alpha - \xi_{\alpha+1})\cdots(\xi_\alpha - \xi_N)}$$

with nodal points ξ_α, $\alpha = 0, \ldots, N$ are the $N+1$ Gauss-Lobatto-Legendre (GLL) points, which are the roots of

$$(1 - \xi^2) P_N'(\xi) = 0$$

with P_N the degree N Legendre polynomial



Note the following properties:

- $\boxed{\ell_\alpha^N(\xi_\beta) = \delta_{\alpha\beta}}$ and $\delta_{\alpha\beta} = \begin{cases} 1, & \alpha = \beta \\ 0, & \text{else} \end{cases}$ Kronecker delta
- $f^\alpha = f(\xi_\alpha)$ evaluated at point ξ_α

Integration rule (quadrature): the SEM uses the same Gauss-Lobatto-Legendre (GLL) node and quadrature rule

$$\int_{-1}^{+1} f(\xi)\, d\xi \approx \sum_{\alpha=0}^{N} \hat{\omega}_\alpha\, f(\xi_\alpha)$$

with integration weights $\hat{\omega}_\alpha$

__Example:__ Integration weights $\hat{\omega}_\alpha$ for a 4th-order expansion

ξ_α	$\hat{\omega}_\alpha$
0	$32/45$
$\pm\sqrt{3/7}$	$49/90$
± 1	$1/10$

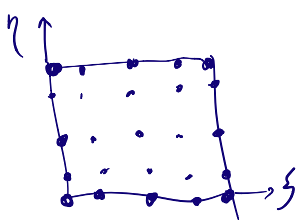
Thus, the integral starting in the physical domain can be written as

$$\int_0^L f(x)\,dx = \int_\Omega f(x)\,dx = \sum_e \int_{\Omega_e} f(x)\,dx$$

and (approximated by)

$$\int_{\Omega_e} f(x)\,dx = \int_{-1}^{+1} f(x(\xi))\,J(\xi)\,d\xi \approx \sum_{\alpha=0}^{N} \hat{\omega}_\alpha f^\alpha J^\alpha$$

__Example:__ In 2D, we have $x(\xi,\eta) = \sum_{a=1}^{n_a} N_a(\xi,\eta)\,x_a$ as anchor functions and



$$\int_\Omega f(x)\,d^2x = \sum_e \int_{\Omega_e} f(x)\,d^2x$$

with

$$\int_{\Omega_e} f(x)\,d^2x = \int_{-1}^{+1}\int_{-1}^{+1} f(x(\xi,\eta))\,J(\xi,\eta)\,d\xi\,d\eta \approx \sum_{\alpha=0}^{N}\sum_{\beta=0}^{N} \hat{\omega}_\alpha \hat{\omega}_\beta f^{\alpha\beta} J^{\alpha\beta}$$