Finite-element method — Part 2

Finite-element approach so far:

- weak form:

$$\boxed{\partial_x^2 u + f = 0} \longrightarrow -\int_0^1 \partial_x w\, \partial_x u\, dx + h\, w(0) + \int_0^1 w f\, dx = 0$$

strong form — weak form

with boundary conditions

$$\begin{cases} u(1) = q \\ \partial_x u(0) = -h \end{cases}$$

with test function $w(x)$ such that $w(1) = 0$

- function approximation:

$$w(x) \approx \sum_{A=1}^{n} c_A N_A(x)$$

$$u(x) \approx \sum_{B=1}^{n} \underline{d_B} N_B(x) + q N_{n+1}(x)$$

shape functions $N_A(x)$ with $\begin{cases} N_A(1) = 0 \\ N_{n+1}(1) = 1, \end{cases}$

since $u(1) = q$ & $w(1) = 0$

n: number of elements

- linear system of equations:

$$-\sum_B \underbrace{\int_0^1 \partial_x N_A\, \partial_x N_B\, dx}_{K_{AB}}\, d_B + \underbrace{\int_0^1 N_A f\, dx + h N_A(0) - q \int_0^1 \partial_x N_A\, \partial_x N_{n+1} dx}_{F_A} = 0$$

in matrix form

$$K_{AB} = \int_0^1 \partial_x N_A\, \partial_x N_B\, dx$$

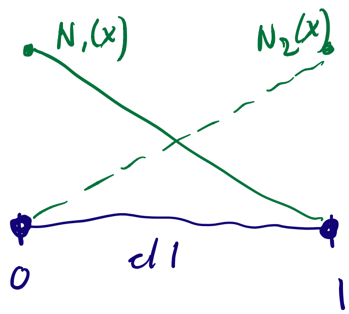
$$F_A = \int_0^1 N_A f\, dx + h N_A(0) - q \int_0^1 \partial_x N_A\, \partial_x N_{n+1}\, dx$$

$$\longrightarrow K_{AB}\, d_B = F_A \qquad A, B = 1, \ldots, n$$

Choice of shape functions:

Let's look at linear basis functions and discretize our domain $[0,1]$ with finite elements.

- discretization with number of elements $n=1$:



test function $w(x)$ has a single shape function $N_1(x)$:

$$\underline{w(x) = c_1 N_1(x)} \qquad \text{for } x \in [0,1]$$

$w(1) = 0 \rightarrow N_1(1) = 0$

solution $u(x)$ becomes

$$\underline{u(x) = d_1 N_1(x) + q N_2(x)} \qquad \text{for } x \in [0,1]$$

$u(1) = q$

$\rightarrow N_2(1) = 1$

We find linear shape functions

$$\begin{cases} N_1(x) = 1 - x \\ N_2(x) = x \end{cases}$$

Let's look at our problem

$$K_{11} = \int_0^1 \partial_x N_1 \, \partial_x N_1 \, dx = \int_0^1 \underbrace{\partial_x(1-x)}_{-1} \underbrace{\partial_x(1-x)}_{-1} dx$$

$$= 1$$

$$F_1 = \int_0^1 N_1 f \, dx + h N_1(0) - q \int_0^1 \underbrace{\partial_x N_1}_{-1} \underbrace{\partial_x N_2}_{1} dx$$

$$= \int_0^1 (1-x) f(x) dx + h + q$$

In matrix form

$$d_1 = \frac{1}{K_{11}} F_1$$

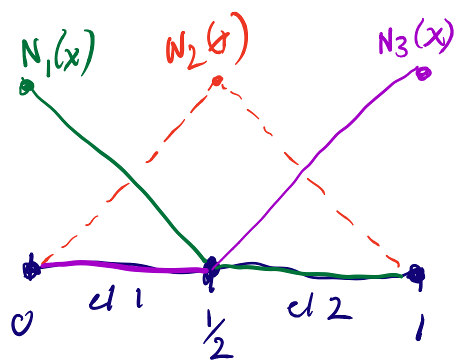
and

$$u(x) = d_1 N_1(x) + q N_2(x)$$

$$= \left[\int_0^1 (1-x) f(x)\,dx + h + q \right](1-x) + qx$$

Note that for $f(x)=0$ this is the exact solution.

- discretization with number of elements $n=2$:



For test function $w(x)$ we have two shape functions

$$w(x) = c_1 N_1(x) + c_2 N_2(x) \quad \text{for } x \in [0,1]$$

with $w(1)=0$

$\rightarrow \begin{cases} N_1(1)=0 \\ N_2(1)=0 \end{cases}$

For solution $u(x)$ we find

$$u(x) = d_1 N_1(x) + d_2 N_2(x) + q N_3(x)$$

with $u(1)=q$

$\rightarrow N_3(1)=1$

Choosing linear shape functions, we find

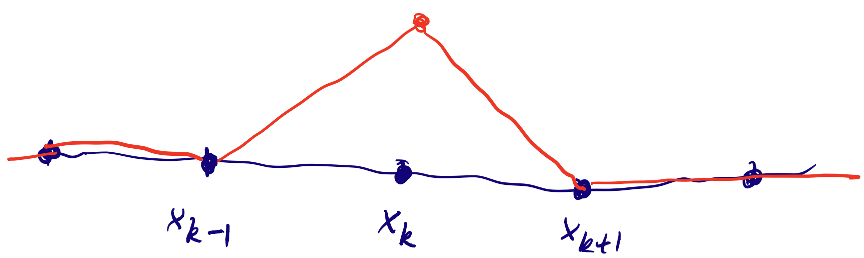
$$N_1(x) = \begin{cases} 1-2x, & 0 < x < \frac{1}{2} \\ 0, & \frac{1}{2} < x < 1 \end{cases}$$

$$N_2(x) = \begin{cases} 2x, & 0 < x < \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x < 1 \end{cases}$$

$$N_3(x) = \begin{cases} 0, & 0 < x < \frac{1}{2} \\ 2x-1, & \frac{1}{2} < x < 1 \end{cases}$$

piecewise linear functions "shifted & scaled" tent function in general

$$N_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{x_k - x_{k-1}} & , \; x \in [x_{k-1}, x_k] \\ \dfrac{x_{k+1} - x}{x_{k+1} - x_k} & , \; x \in [x_k, x_{k+1}] \\ 0 & , \; \text{otherwise} \end{cases}$$



Note that $n=2$ increases the degree of freedom (d_1, d_2). Basis functions $N_k(x)$ have only local support, thus this becomes a local method (compared to pseudo-spectral method). We could choose higher-order shape functions (polynomials, ..), which would lead to a higher-order finite-element scheme.

<u>Homework:</u> Derive the matrix form for $n=2$:

$$K_{AB}\, d_B = F_A$$

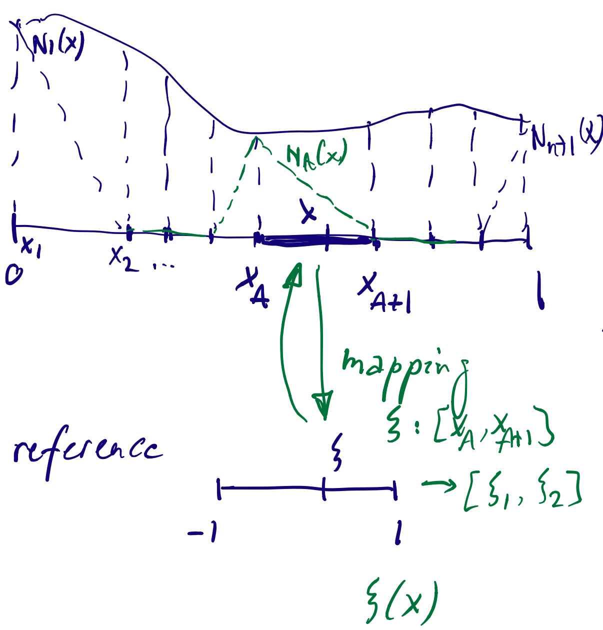
Show that
$$\begin{cases} K_{11} = 2 \\ K_{12} = K_{21} = -2 \\ K_{22} = 4 \\ F_1 = \int_0^{1/2} (1-2x)\, f(x)\, dx + h \\ F_2 = 2\int_0^{1/2} x f(x)\, dx + 2\int_{1/2}^{1} (1-x) f(x)\, dx + 2g \end{cases}$$

$$\rightarrow \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

How about flexibility of meshing and have arbitrary element sizes?

Global versus local view:

Let us introduce the linear finite elements on which the problem is solved. We separate between a global view (corresponds to the physical domain) and a local view (corresponds to a reference domain).



global view:

$$\left\{\begin{array}{ll} \text{domains} & [x_A, x_{A+1}] \\ \text{nodes} & \{x_A, x_{A+1}\} \\ \text{degrees of freedom} & \{d_A, d_{A+1}\} \\ \text{shape functions} & \{N_A(x), N_{A+1}(x)\} \end{array}\right.$$

where
$$N_A(x) = \begin{cases} \dfrac{x - x_{A-1}}{h_{A-1}}, & x_{A-1} < x < x_A \\ \dfrac{x_{A+1} - x}{h_A}, & x_A < x < x_{A+1} \\ 0, & \text{elsewhere} \end{cases}$$

with $h_A = x_{A+1} - x_A$ element size

the shape function for first & last nodes are

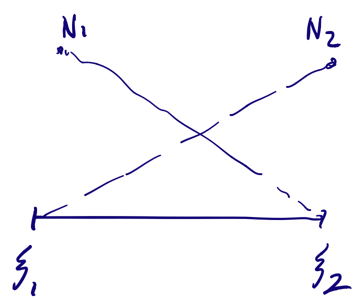
$$N_1(x) = \begin{cases} \frac{x_1 - x}{h_1}, & x_1 < x < x_2 \\ 0, & \text{else} \end{cases}$$

$$N_{n+1}(x) = \begin{cases} \frac{x - x_n}{h_n}, & x_n < x < x_{n+1} \\ 0, & \text{else} \end{cases}$$

Notice: the stiffness matrix K_{AB} becomes symmetric & banded for these shape functions

$$K = \begin{pmatrix} K_{11} & K_{12} & & 0 \\ K_{12} & K_{22} & K_{23} & \\ & K_{23} & K_{33} & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}$$

Local view:



$$\begin{cases} \text{domain } [\xi_1, \xi_2] & \text{in practice } [-1, 1] \\ \text{nodes } \{\xi_1, \xi_2\} \\ \text{degrees of freedom } \{d_1, d_2\} \\ \text{shape function } \{N_1, N_2\} \end{cases}$$

Then the interpolation is

$$u(\xi) = d_1 N_1(\xi) + d_2 N_2(\xi)$$

Mapping: We need a mapping between both views, such that $\xi : [x_A, x_{A+1}] \longrightarrow [\xi_1, \xi_2]$

and $\begin{cases} \xi(x_A) = \xi_1 \\ \xi(x_{A+1}) = \xi_2 \end{cases}$

Assuming a linear mapping $\xi(x) = c_1 + c_2 x$, we find

$$\xi(x) = \frac{2x - x_A - x_{A+1}}{h_A}$$

and $x(\xi) = \frac{1}{2}\left(h_A \xi + x_A + x_{A+1}\right)$