

Finite-volume method

1D wave equation: The 1D wave equation can be written as

$$\rho \partial_t^2 s = \partial_x (\mu \partial_x s)$$

for displacement $s(x, t)$, density $\rho(x)$ and shear moduli $\mu(x)$. We will set an initial condition $s(x, 0) = f(x)$ and boundary condition

$$\begin{cases} s(L, t) = 0 \\ s(0, t) = 0 \end{cases} \quad \text{or} \quad \begin{cases} \mu \partial_x s(L, t) = B_L(t) = 0 \\ \mu \partial_x s(0, t) = B_0(t) = 0 \end{cases}$$

"Dirichlet" "Neumann"

Integral form: For the finite-volume method, one way to solve the wave equation is to cast it as an advection problem of a first-order PDE:

$$\partial_t q(x, t) + a \partial_x q(x, t) = 0$$

q : quantity (conserved)

a : transport velocity

That is, the wave equation written in
a velocity-stress formulation

$$\begin{cases} \rho \partial_t v = \partial_x \sigma \\ \partial_t \sigma = \mu \partial_x v \end{cases} \xrightarrow{\text{advection form}} \begin{cases} \partial_t v - \frac{1}{\rho} \partial_x \sigma = 0 \quad (1) \\ \partial_t \sigma - \mu \partial_x v = 0 \quad (2) \end{cases}$$

Using $\underline{Q} = \begin{pmatrix} v \\ \sigma \end{pmatrix}$ and $\underline{A} = \begin{pmatrix} 0 & -\frac{1}{\rho} \\ -\mu & 0 \end{pmatrix}$ this becomes

$$\partial_t \underline{Q} + \underline{A} \partial_x \underline{Q} = 0$$

which is in "advection" form, but still
a coupled system. One would diagonalize \underline{A}
using its eigenvalues $\lambda_{1,2} = \pm \sqrt{\frac{\mu}{\rho}}$ and
eigenvectors to decouple the system
before applying the FVM discretization

→ see Heiner Igel's book

"Computational seismology", chapter 8

→ Lax-Wendroff scheme (for advection),
second-order accurate

→ discontinuous fields involved
solving a Riemann problem

Here, as an alternative (inside) approach, we use the second-order formulation directly to solve the integral form

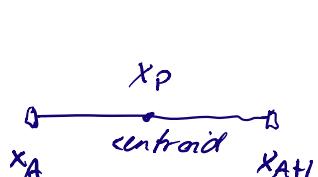
$$\int_{\Omega} \rho \partial_t^2 s \, dV = \int_{\Omega} \partial_x (\mu \partial_x s) \, dV$$

which we will put into a matrix form

$$\underline{\underline{M}} \underline{\partial_t^2 s} = \underline{\underline{K}} \underline{s}$$

Discretization: Discretizing the (physical) domain $\Omega = \bigcup_e \Omega_e$ into grid cells Ω_e and applying Gauss' theorem leads to

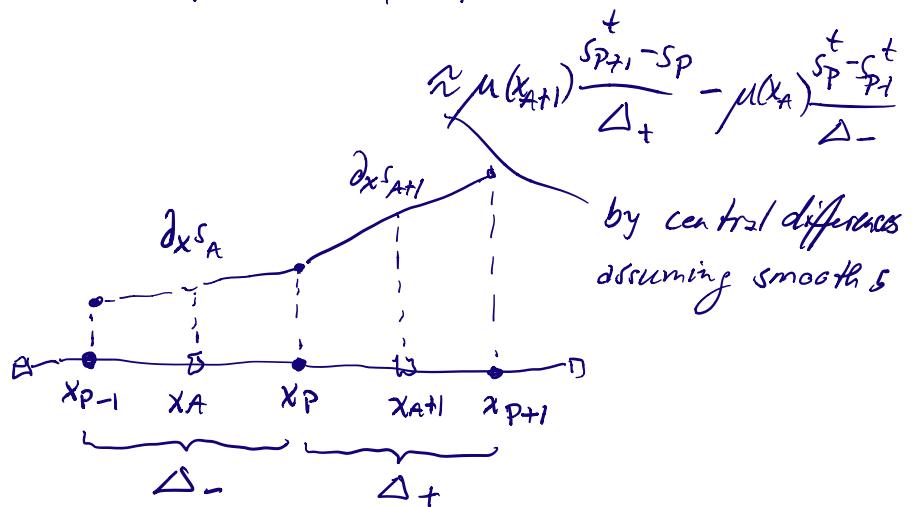
$$\int_{\Omega_e} \rho \partial_t^2 s \, dV = \int_{\partial \Omega_e} (\mu \partial_x s) n_x \, ds$$



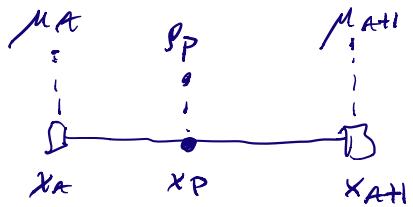
$$\int_{x_A}^{x_{A+1}} \rho \partial_t^2 s \, dx = \underbrace{\mu \partial_x s}_{\text{mass term}} \Big|_{x_A}^{x_{A+1}} = \underbrace{\mu \partial_x s(x_{A+1}, t)}_{\text{stiffness term}} - \underbrace{\mu \partial_x s(x_A, t)}_{\text{stiffness term}}$$

Mass term: $\int_{x_p}^{x_{p+1}} \rho \partial_t^2 s \, dx \approx s(x_p) \partial_t^2 s(x_p, t) V_e$

Stiffness term: $\mu \partial_x s / x_A = \mu \partial_x s(x_{p+1}, t) - \mu \partial_x s(x_p, t)$



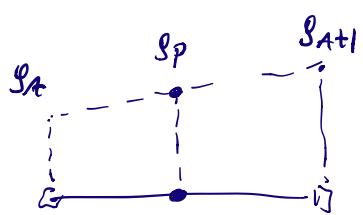
What about material properties $\rho(x_p)$, $\mu(x_A)$ & $\mu(x_{A+1})$?



μ is needed at the cell faces rather than at the centroidal location as ρ is.

Assuming they piecewise linear, we can either define them at centroid positions together with the gradient information, or simply specify them at cell face positions directly

→ similar to staggered grid finite-differences approaches



For density, we need $\rho(x_p)$:

$$\begin{aligned}\rho(x_p) &= \bar{\rho} = \frac{1}{V_e} \int \rho(x) dV \\ &= \frac{\rho_A + \rho_{A+1}}{2}\end{aligned}$$

Note that μ, ρ can be defined at an element (local) level

→ similar to FEM and SEM approach

The discretized equation looks like

$$g(x_p) \partial_t^2 s(x_p, t) V_c = \mu(x_{A+1}) \frac{s_{P+1}^t - s_P^t}{\Delta_+} - \mu(x_A) \frac{s_P^t - s_{P-1}^t}{\Delta_-}$$

$$V_c \underbrace{\left(\frac{s_A + s_{A+1}}{2} \right)}_{\sim \underline{M}} \partial_t^2 s_P^t = \underbrace{\frac{\mu_{A+1}}{\Delta_+} s_{P+1}^t}_{\sim \underline{K}} - \underbrace{\left(\frac{\mu_{A+1}}{\Delta_+} - \frac{\mu_A}{\Delta_-} \right) s_P^t}_{\sim \underline{K}} + \underbrace{\frac{\mu_A}{\Delta_-} s_{P-1}^t}_{\sim \underline{K}}$$

Using the Newmark scheme for the time stepping

$$d_{n+1} = d_n + \Delta t v_n + \frac{1}{2} \Delta t^2 a_n$$

$$v_{n+1} = v_n + \frac{1}{2} \Delta t a_n$$

$$a_{n+1} = 0$$

solve

$$a_{n+1} = \underline{\underline{M}}^{-1} \underline{F}_{n+1} = \underline{\underline{M}}^{-1} \underline{\underline{K}} \underline{d}_{n+1}$$

correction

$$v_{n+1} = v_{n+1} + \frac{1}{2} \Delta t a_{n+1}$$

Boundary conditions:

Neumann : $\partial_x s(0) = \partial_x s(L) = 0$

left : $\partial_x s(0) = 0$

$$\Rightarrow \int_{\partial D_e} \mu \partial_x s \, ndS \approx \mu(x_{A+1}) \partial_x s(x_{A+1}, t) - 0$$

$$= \frac{\mu_{A+1}}{\Delta_+} s_{P+1}^t - \frac{\mu_{A+1}}{\Delta_+} s_p^t$$

right : $\partial_x s(L) = 0$

$$\Rightarrow \int_{\partial D_e} \mu \partial_x s \, ndS \approx 0 - \mu(x_A) \partial_x s(x_A, t)$$

$$= - \frac{\mu_A}{\Delta_-} s_p^t + \frac{\mu_A}{\Delta_-} s_{P-1}^t$$

Dirichlet : $s(0, t) = s(L, t) = 0$

left : $s(0, t) = 0$

$$\Rightarrow \int_{\partial D_e} \mu \partial_x s \, ndS \approx \mu_{A+1} \partial_x s_{A+1}^t - \underbrace{\mu_A \partial_x s_A^t}_{\mu(x_0) \partial_x s(0, t)}$$

$$\mu(x_0) \partial_x s(0, t) \approx \mu(x_0) \frac{s(x_P, t) - s(x_0, t)}{|x_P - x_0|}$$

$$= \frac{\mu_A}{\Delta x_0} s_p^t - 0$$

$$\rightarrow \int_{\partial D_e} \mu \partial_x s \, ndS \approx \frac{\mu_{A+1}}{\Delta_+} s_{P+1}^t - \frac{\mu_{A+1}}{\Delta_+} s_p^t - \frac{\mu_A}{\Delta x_0} s_p^t$$

$$= \left(\frac{\mu_{A+1}}{\Delta_+} s_{p+1}^t - \left(\frac{\mu_{A+1}}{\Delta_+} + \frac{\mu_A}{\Delta x_0} \right) s_p^t \right)$$

right: $s(L, t) = 0$

$$\Rightarrow \mu(x_{A+1}) \partial_x s(L, t) \approx \mu(x_N) \frac{s(L, t) - s(x_p, t)}{|x_N - x_p|}$$

$$= 0 - \frac{\mu_N}{\Delta x_N} s_p^t$$

$$\rightarrow \int_{\partial\Omega_e} \mu \partial_x s_n ds \approx - \frac{\mu_N}{\Delta x_N} s_p^t - \frac{\mu_A}{\Delta_-} s_p^t + \frac{\mu_A}{\Delta_-} s_{p-1}^t$$

$$= - \left(\frac{\mu_N}{\Delta x_N} + \frac{\mu_A}{\Delta_-} \right) s_p^t + \frac{\mu_A}{\Delta_-} s_{p-1}^t$$

Note that the Dirichlet boundary only uses a first-order approach to impose it here, whereas the Neumann boundary is exact.