Stability analysis

Diffusion, heat equation : $\rho c_V \partial_t \hat{T} = \partial_x (K \partial_x \hat{T})$

where $q = -K \partial_x \hat{T}$ heat flux

simplifies to

$$\partial_t \hat{T} = D \partial_x^2 \hat{T}$$

when $\underbrace{K}_{\text{conductivity}}$ is constant & $\underbrace{D}_{\text{diffusivity}} = \frac{K}{\rho c_V}$

Consider the discretization

$x_j = x_0 + j\Delta x$ $\quad$ Δx grid spacing

$t_n = t_0 + n\Delta t$ $\quad$ Δt time step

temperature $\hat{T}(x_j, t_n) \equiv \hat{T}_j^n$ $\quad$ notation for discrete temperature field

Then,

forward difference : $\partial_t \hat{T}\Big|_{j,n} \approx \dfrac{\hat{T}_j^{n+1} - \hat{T}_j^n}{\Delta t} + O(\Delta t)$

center difference : $\partial_x^2 \hat{T}\Big|_{j,n} \approx \dfrac{\hat{T}_{j+1}^n - 2\hat{T}_j^n + \hat{T}_{j-1}^n}{\Delta x^2} + O(\Delta x^2)$

The heat equation (simplified) becomes

$$\boxed{\frac{\hat{T}_j^{n+1} - \hat{T}_j^n}{\Delta t} = D \frac{\hat{T}_{j+1}^n - 2\hat{T}_j^n + \hat{T}_{j-1}^n}{\Delta x^2}}$$

Look for solutions of the form $\hat{T}_j^n = \xi^n e^{ikx_j} = \underbrace{\xi^n e^{ikj\Delta x}}_{\text{plane-wave assumption}}$

k: real spatial wavenumber

ξ: amplitude (complex)

The scheme is stable if $|\xi| < 1$, that is $\xi^n \to 0$ as $n \to \infty$.

Finding ξ

$$\frac{\xi^{n+1} e^{ikj\Delta x} - \xi^n e^{ikj\Delta x}}{\Delta t} = D \frac{\xi^n e^{ik(j+1)\Delta x} - 2\xi^n e^{ikj\Delta x} + \xi^n e^{ik(j-1)\Delta x}}{\Delta x^2}$$

dividing by $\xi^n e^{ikj\Delta x}$ and $\xi = \dfrac{\xi^{n+1}}{\xi^n}$ leads

$$\xi = 1 - \frac{4D\Delta t}{\Delta x^2} \sin^2\left(\frac{k\Delta x}{2}\right) = 1 - 2\alpha \sin^2\left(\frac{k\Delta x}{2}\right) \text{ and } \alpha = \frac{2D\Delta t}{\Delta x^2}$$

For $|\xi| < 1$, it follows $\dfrac{2D\Delta t}{\Delta x^2} < 1$

$$\longrightarrow \boxed{\Delta t < \frac{\Delta x^2}{2D}}$$ conditionally stable

Stability analysis – Part II

Consider this variant

$$\underbrace{\frac{\hat{T}_j^{n+1} - \hat{T}_j^n}{\Delta t}}_{\text{forward difference in time (FT)}} = D \underbrace{\frac{\hat{T}_{j+1}^{n+1} - 2\hat{T}_j^{n+1} + \hat{T}_{j-1}^{n+1}}{\Delta x^2}}_{\text{central difference at time (n+1) in space (CS)}}$$

This scheme is fully implicit; it may be rewritten as

$$-\alpha \hat{T}_{j-1}^{n+1} + (1-2\alpha)\hat{T}_j^{n+1} - \alpha \hat{T}_{j+1}^{n+1} = \hat{T}_j^n$$

with $\alpha = \dfrac{D\Delta t}{\Delta x^2}$

Stability: $\hat{T}_j^n = \xi^n e^{ikj\Delta x}$ plane-wave assumption $\xi = \dfrac{\xi^{n+1}}{\xi^n}$

$$\longrightarrow \xi = \left[1 + 4\alpha \sin^2\left(\frac{k\Delta x}{2}\right)\right]^{-1}$$

no matter how you choose Δx, it is always < 1

Thus, $|\xi| < 1$ for all Δt

<u>unconditionally</u> stable

Combine the stability of the implicit method with accuracy of a second-order grid in space & time:

$$\frac{\hat{T}_j^{n+1} - \hat{T}_j^n}{\Delta t} = \frac{D}{2\Delta x^2}\left(\hat{T}_{j+1}^{n+1} - 2\hat{T}_j^{n+1} + \hat{T}_{j-1}^{n+1} + \hat{T}_{j+1}^n - 2\hat{T}_j^n + \hat{T}_{j-1}^n\right)$$

center at $t_n = t_0 + (n + \frac{1}{2})\Delta t$

$\longrightarrow$ Crank - Nicholson scheme

leads to $\xi(k) = \dfrac{1 - 2\alpha \sin^2(\frac{k\Delta x}{2})}{1 + 2\alpha \sin^2(\frac{k\Delta x}{2})}$ stable for all Δt and second order!

In a nutshell:

"Let's suppose $\hat{T}_j^n = \xi^n e^{ikj\Delta x}$ is a solution to our differential equation. $\xi(k)$ is giving now the dispersion relation. If the norm of $\xi(k)$ is less than 1, then the solution is stable."

Wave equation : $\rho\, \partial_t^2 u = \partial_x (k\, \partial_x u)$

u : displacement

ρ : mass density

k : bulk modulus

$k(x) = \text{const.}$

using bulk sound speed $c = \sqrt{\frac{k}{\rho}}$

and constant k leads to

$$\boxed{\partial_t^2 u = c^2 \partial_x^2 u}$$

Discretize with central difference in time & space

$$\frac{\boxed{u_i^{n+1}} - 2u_i^n - u_i^{n-1}}{\Delta t^2} = c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

stability : $u_i^n = \xi^n e^{ikj\Delta x}$ plane-wave assumption

$|\xi| < 1 \longrightarrow \frac{c\,\Delta t}{\Delta x} \leq 1$ Courant stability condition

$$\boxed{\Delta t \leq \frac{\Delta x}{c}}$$

conditionally stable