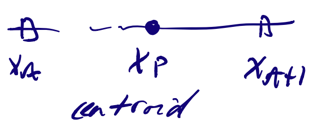
Finite-volume method

Riemann problem: In our homework, solving the 1D wave equation with a finite-volume approach leads to approximating the cell Ω_e integrals

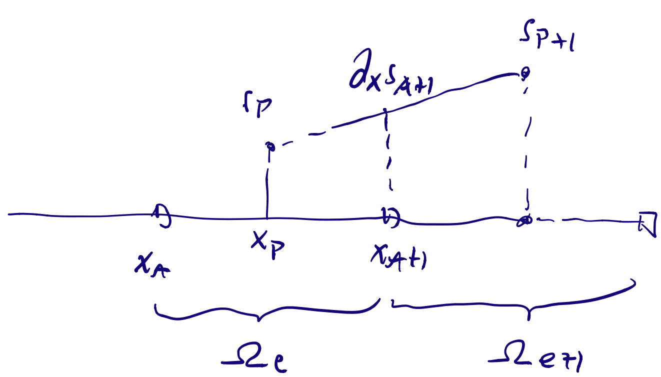
$$\int_{\Omega_e} \rho\, \partial_t^2 s \, dV = \int_{\partial\Omega_e} (\mu\, \partial_x s)\, n_x \, dS$$



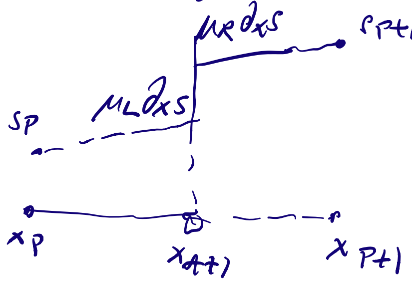
$$\underbrace{\int_{x_A}^{x_{A+1}} \rho\, \partial_t^2 s \, dx}_{\text{mass term}} = \underbrace{\mu\, \partial_x s \Big|_{x_A}^{x_{A+1}}}_{\text{stiffness term}}$$

where we have to evaluate the "fluxes" $\mu\, \partial_x s(x)$ at the cell face centroid (in 1D the cell corner)

$$\mu\, \partial_x s(x_{A+1}, t) \approx \mu_{A+1} \frac{s_{P+1}^t - s_P^t}{\Delta x}$$



What happens when we have a material discontinuity at x_{A+1}, i.e., a jump of $\mu(x_{A+1})$ between grid cell Ω_c and its neighbor cell Ω_{c+1}?



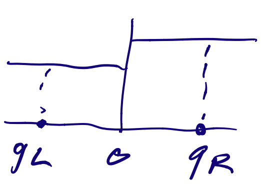
$$\mu(x) = \begin{cases} \mu_L, & x < x_{A+1} \\ \mu_R, & x > x_{A+1} \end{cases}$$

This leads to a jump in the stress (or "flux")

$$T_{A+1} = \mu(x_{A+1})\,\partial_x s(x_{A+1}, t) = \begin{cases} \mu_L \partial_x s(x_{A+1}, t), & x < x_{A+1} \\ \mu_R \partial_x s(x_{A+1}, t), & x > x_{A+1} \end{cases}$$

In a FVM perspective, this is a Riemann problem where we have a jump/discontinuity of a quantity at the cell boundary. This needs to be addressed by a Riemann solver, which determines how much of the discontinuity propagates into the connected cells.

Riemann problem: "initial value problem for a conservation equation given a discontinuity"



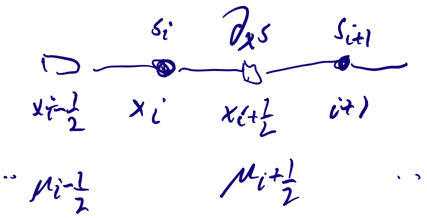
$$\begin{cases} \partial_t q + u\,\partial_x q = 0 \\ q(x) = \begin{cases} q_L, & \text{if } x < 0 \\ q_R, & \text{if } x > 0 \end{cases} \end{cases}$$

For our wave problem and physics perspective, we want the (normal) stress at the (solid-solid) interface of a material discontinuity to be continuous.

Thus, for the 1D wave equation we can take the average stress at the cell boundary

$$\text{average} \quad \hat{T}_{A+1} = \frac{1}{2}\left(\mu_L \partial_x s_{A+1} + \mu_R \partial_x s_{A+1}\right)$$

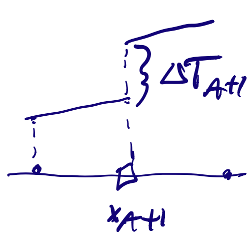
$$= \mu_L \partial_x s_{A+1} + \frac{1}{2}\left(\mu_R \partial_x s_{A+1} - \mu_L \partial_x s_{A+1}\right)$$



$$= \mu_L \partial_x s_{A+1} + \underbrace{\frac{\mu_R - \mu_L}{2} \partial_x s_{A+1}}_{\text{“correction”}}$$

$\partial_x s_{A+1}$ is continuous

This means that if we encounter a material jump $\left|\frac{\mu_R - \mu_L}{2}\right| > 0$, then we have to add a "correction" term



$$\frac{\Delta T_{A+1}}{2} = \frac{\mu_R - \mu_L}{2} \partial_x s_{A+1}$$

$$\longrightarrow \quad \hat{T}_{A+1} = \underbrace{\mu_L \partial_x s_{A+1}}_{T^L_{A+1}} + \frac{1}{2}\Delta T_{A+1}$$

to impose the continuity of stress at the jump interface. This will lead to proper reflections of the waves.

In terms of Riemann solvers, this is related to a Lax-Friedrich approach solution.