

Problem Set 6

FEM solution of 1D unsteady-state diffusion equation

Write a finite-element program, using linear shape functions, to find the temperature $T(x, t)$ in $[0, L]$ such that (strong form)

$$\rho c_p \partial_t T - \partial_x (\kappa \partial_x T) = f$$

where ρ is the medium density, c_p is the specific heat, κ is the conductivity, and f is a source or sink, with the following initial & boundary conditions:

$$\begin{aligned} T(L, t) &= T_L \\ -\kappa \partial_x T(0, t) &= q_0 \\ T(x, 0) &= T_0(x) \end{aligned}$$

Recipe:

Address this problem as follows:

1. Write the weak form of the equation, which translates into solving

$$\mathbf{M}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F}$$

where \mathbf{M} is the capacity matrix, defined as

$$\begin{aligned} M_{AB} &= (N_A, \rho c_p N_B) \\ &= \int_0^L \rho c_p N_A N_B \, dx \end{aligned}$$

and \mathbf{K} is the conductivity (or stiffness) matrix, defined as

$$\begin{aligned} K_{AB} &= a(N_A, N_B) \\ &= \int_0^L \kappa \partial_x N_A \partial_x N_B \, dx \end{aligned}$$

and \mathbf{F} is the right-hand-side (force) vector , defined as

$$F_A = (N_A, f) + N_A(0)q_0 - a(N_A, N_{n+1})T_L - (N_A, \rho c_p N_{n+1})\dot{T}_L$$

finally, \mathbf{d} are the unknown temperatures and $\dot{\mathbf{d}} = \partial_t \mathbf{d}$.

2. Define the local (element level) capacity & stiffness matrices and the right-hand-side vector.
3. Assemble these local matrices into global matrices (global level) using the Location Matrix (LM).
4. March in time, using the predictor-corrector algorithm:

- Predictor:

$$d_{n+1} = d_n + (1 - \alpha)\Delta t \dot{d}_n$$

$$\dot{d}_{n+1} = 0 \quad (\text{initialization at the beginning of each time step})$$

- Solve:

$$rhs = F - M\dot{d}_{n+1} - Kd_{n+1}$$

$$\delta\dot{d}_{n+1} = M^{-1}rhs$$

- Corrector:

$$d_{n+1} = d_{n+1} + \alpha\Delta t \dot{d}_{n+1}$$

$$\dot{d}_{n+1} = \dot{d}_{n+1} + \delta\dot{d}_{n+1}$$

where Δt is the time step.

The parameter α can take several values, leading to different methods with different level of stability (e.g., see Hughes, "The Finite Element Method - Linear Static and Dynamic Finite Element Analysis", page 459, for more details) such that

- $\alpha = 0$: Forward differences,
- $\alpha = 1/2$: Midpoint rule, Crank-Nicolson,
- $\alpha = 1$: Backward differences.

For $\alpha \geq 1/2$ the scheme is unconditionally stable.

Use $\alpha = 1/2$.

Compare the FEM solution to the exact solution of the strong form, plotting the temperature T versus x at different time steps.

Solve the following two cases:

Problem A: Simple harmonic function

Single harmonic $T_0(x)$ as

$$T_0(x) = 1 + \cos(x) \quad \text{in } [0, L]$$

where $L = \frac{\pi}{2}$ and the following conditions & parameters:

$$f = 0$$

$$q_0 = 0$$

$$T_L = 1$$

$$\kappa = 1$$

$$\rho c_p = 1$$

Try $N_{el} = 10$.

The exact solution to this problem is $T(x, t) = 1 + e^{-t} \cos(x)$.

Problem B: Half-space cooling

Here we consider the uniformly hot half-space cooling (e.g., oceanic lithospheric cooling). The differential equation for this problem is

$$\rho c_p \partial_t \theta = \partial_x (\kappa \partial_x \theta) \quad \text{in } 0 < x < \infty,$$

where $\theta = \frac{T - T_m}{T_0 - T_m}$ is a non-dimensional variable, using the following boundary conditions:

$$T(0, t) = T_0 \quad (\text{surface temperature})$$

$$T(x \rightarrow \infty, t) \rightarrow T_m$$

$$T(x, 0) = T_m \quad (\text{initial temperature})$$

The exact solution to this problem is

$$\begin{aligned} \theta &= \frac{T - T_m}{T_0 - T_m} \\ &= \operatorname{erfc} \frac{x}{2\sqrt{\frac{\kappa}{\rho c_p} t}} \end{aligned}$$

where $\operatorname{erfc}(x)$ is the complementary error function defined as $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$.

Notice that you can use the same algorithm developed for (a), by changing variable $y = L - x$, such that you solve

$$\rho c_p \partial_t T - \partial_y (\kappa \partial_y T) = f$$

with the boundary conditions

$$T(L, t) = T_0$$

$$-\kappa \partial_y T(0, t) = q_0$$

$$T(y, 0) = T_m$$

Let $L = 20$, $\kappa = 1$, $q_0 = 0$, $\rho c_p = 1$, $f = 0$, $T_0 = 0$, and $T_m = 1$.

Please be aware that:

- The initial condition of this problem is not compatible with the boundary condition at

$y = L$. To handle that you can set the temperature value at $y = L$ to be T_0 , and all the other nodes to be T_m . You can also try and use mode elements, which will make the transition from T_0 to T_m smoother.

- We are solving the problem for large L , that is, such that $L \gg 2\sqrt{\frac{\kappa}{\rho c_p}}t$, say $L = 20$ and $t = (0, 9)$, therefore, we can approximately use the exact solution for the half-space problem as the analytical solution for our problem.