

Finite-difference method



1D Wave equation

$$\rho(x)\partial_t^2 u(x, t) = \partial_x[\kappa(x)\partial_x u(x, t)], \quad (x \in [0, L], t \in [0, +\infty))$$

for a homogeneous material:

$$\partial_t^2 u(x, t) = c^2 \partial_x^2 u(x, t)$$

with wave speed $c = \sqrt{\frac{\kappa}{\rho}}$



1D Wave equation

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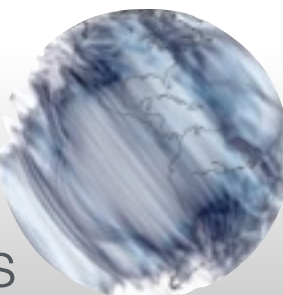
as a 1st-order PDE system and a heterogeneous material:

$$\rho(x)\partial_t v(x, t) = \partial_x T(x, t)$$

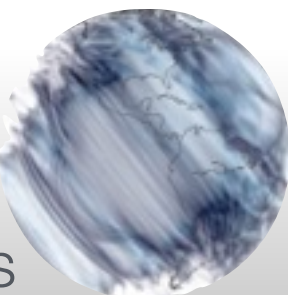
$$v(x, t) = \partial_t u(x, t)$$

$$\partial_t T(x, t) = \kappa(x)\partial_x v(x, t)$$

$$T(x, t) = \kappa(x)\partial_x u(x, t)$$



Finite-difference schemes



Central scheme in time & central difference in space

- Spatial derivatives:

$$\partial_x f(x, t) \approx \frac{f(x + \Delta x, t) - f(x - \Delta x, t)}{2\Delta x}$$

$$\partial_x^2 f(x, t) \approx \frac{f(x + \Delta x, t) - 2f(x, t) + f(x - \Delta x, t)}{\Delta x^2}$$

- Temporal derivatives:

$$\partial_t^2 f(x, t) \approx \frac{f(x, t + \Delta t) - 2f(x, t) + f(x, t - \Delta t)}{\Delta t^2}$$



Central scheme in time & central difference in space

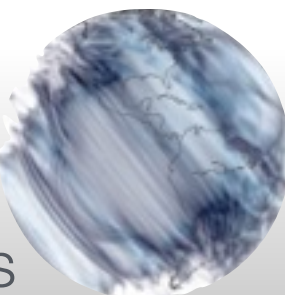
for a homogeneous material:

$$\partial_t^2 u(x, t) = c^2 \partial_x^2 u(x, t)$$

$$\partial_t^2 u(x_i, t_n) = \frac{u(x_i, t_n + \Delta t) - 2u(x_i, t_n) + u(x_i, t_n - \Delta t))}{\Delta t^2} = \frac{1}{\Delta t^2} [u_i^{n+1} - 2u_i^n + u_i^{n-1}]$$
$$\partial_x^2 u(x_i, t_n) = \frac{u(x_i + \Delta x, t_n) - 2u(x_i, t_n) + u(x_i - \Delta x, t_n))}{\Delta x^2} = \frac{1}{\Delta x^2} [u_{i+1}^n - 2u_i^n + u_{i-1}^n]$$



$$u_i^{n+1} = \frac{c^2 \Delta t^2}{\Delta x^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n) + 2u_i^n - u_i^{n-1}$$



Problem set



Boundary conditions

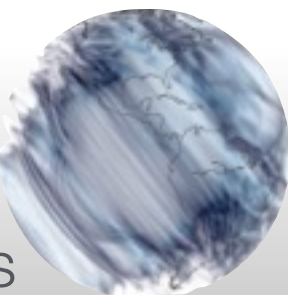
- the Dirichlet boundary conditions correspond to a fix termination, e.g., $u(0, t) = 0$ that is no displacement at all time.
- the Neumann boundary conditions correspond to a termination free to move, e.g., $\partial_x u(0, t) = 0$.



Initial conditions

$$u(x, 0) = \exp^{-0.1(x-50)^2}$$

$$\partial_t u(x, 0) = 0$$



Problem set

Two sets of equations:

$$\partial_t^2 u(x, t) = c^2 \partial_x^2 u(x, t)$$

$$\rho(x) \partial_t v(x, t) = \partial_x T(x, t)$$

$$\partial_t T(x, t) = \kappa(x) \partial_x v(x, t)$$

Two boundary conditions:

$$u(0, t) = 0$$

$$T(0, t) = 0$$

Two wavespeed models:

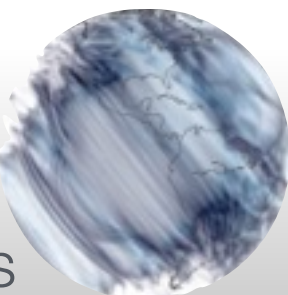
$$c = 1, \rho = 1, \kappa = 1 \quad (x \in [0, 100])$$

$$c(x) = 1, \rho(x) = 1, \kappa(x) = 1 \quad (x \in [0, 60])$$

$$c(x) = 2, \rho(x) = 1, \kappa(x) = 4 \quad (x \in (60, 100])$$



Results



2nd-order vs. 1st-order PDE

Dirichlet boundary

