Finite-difference method

1D Wave equation

$$\boxed{\rho(x)\partial_t^2 u(x,t) = \partial_x[\kappa(x)\partial_x u(x,t)]}, \qquad (x \in [0,L], t \in [0,+\infty))$$

for a homogeneous material:

$$\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t)$$

with wave speed $c = \sqrt{\frac{\kappa}{\rho}}$

1D Wave equation

$$\boxed{\rho(x)\partial_t^2 u(x,t) = \partial_x[\kappa(x)\partial_x u(x,t)]}, \quad (x \in [0,L], t \in [0,+\infty))$$

as a 1st-order PDE system and a heterogeneous material:

$$\begin{aligned} \rho(x)\partial_t v(x,t) &= \partial_x T(x,t) \\ \partial_t T(x,t) &= \kappa(x)\partial_x v(x,t) \end{aligned} \qquad \begin{aligned} v(x,t) &= \partial_t u(x,t) \\ T(x,t) &= \kappa(x)\partial_x u(x,t) \end{aligned}$$

Finite-difference schemes

Central scheme in time & central difference in space

- Spatial derivatives:

$$\partial_x f(x,t) \approx \frac{f(x+\Delta x,t)-f(x-\Delta x,t)}{2\Delta x}$$

$$\partial_x^2 f(x,t) \approx \frac{f(x+\Delta x,t)-2f(x,t)+f(x-\Delta x,t)}{\Delta x^2}$$

- Temporal derivatives:

$$\partial_t^2 f(x,t) \approx \frac{f(x,t+\Delta t)-2f(x,t)+f(x,t-\Delta t)}{\Delta t^2}$$

Central scheme in time & central difference in space

for a homogeneous material: $$\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t)$$

$$\partial_t^2 u(x_i, t_n) = \frac{u(x_i, t_n + \Delta t) - 2u(x_i, t_n) + u(x_i, t_n - \Delta t)}{\Delta t^2} = \frac{1}{\Delta t^2}[u_i^{n+1} - 2u_i^n + u_i^{n-1}]$$

$$\partial_x^2 u(x_i, t_n) = \frac{u(x_i + \Delta x, t_n) - 2u(x_i, t_n) + u(x_i - \Delta x, t_n)}{\Delta x^2} = \frac{1}{\Delta x^2}[u_{i+1}^n - 2u_i^n + u_{i-1}^n]$$

$$\Downarrow$$

$$u_i^{n+1} = \frac{c^2 \Delta t^2}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + 2u_i^n - u_i^{n-1}$$

Problem set

Boundary conditions

- the Dirichlet boundary conditions correspond to a fix termination, e.g., $u(0,t) = 0$ that is no displacement at all time.
- the Neumann boundary conditions correspond to a termination free to move, e.g., $\partial_x u(0,t) = 0$.

Initial conditions

$$
\begin{aligned}
u(x,0) &= \exp^{-0.1(x-50)^2} \\
\partial_t u(x,0) &= 0
\end{aligned}
$$

Problem set

Two sets of equations:

$$\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t)$$

$$\begin{aligned} \rho(x)\partial_t v(x,t) &= \partial_x T(x,t) \\ \partial_t T(x,t) &= \kappa(x)\partial_x v(x,t) \end{aligned}$$

Two boundary conditions:

$$u(0,t) = 0$$

$$T(0,t) = 0$$

Two wavespeed models:

$$c = 1, \rho = 1, \kappa = 1 \quad (x \in [0,100])$$

$$c(x) = 1, \rho(x) = 1, \kappa(x) = 1 \quad (x \in [0,60])$$

$$c(x) = 2, \rho(x) = 1, \kappa(x) = 4 \quad (x \in (60,100])$$

Results

2nd-order vs. 1st-order PDE

Dirichlet boundary