Pseudo-Spectral method

1D wave equation

2nd-order partial differential equation:

$$\rho(x)\partial_t^2 u(x,t) = \partial_x[\kappa(x)\partial_x u(x,t)], \quad (x \in [0,L], t \in [0,+\infty))$$

velocity-stress formulation

1st-order partial differential equations

$$\begin{aligned} \rho(x)\partial_t v(x,t) &= \partial_x T(x,t) \\ \partial_t T(x,t) &= \kappa(x)\partial_x v(x,t) \end{aligned}$$

$$\begin{aligned} v(x,t) &= \partial_t u(x,t) \\ T(x,t) &= \kappa(x)\partial_x u(x,t) \end{aligned}$$

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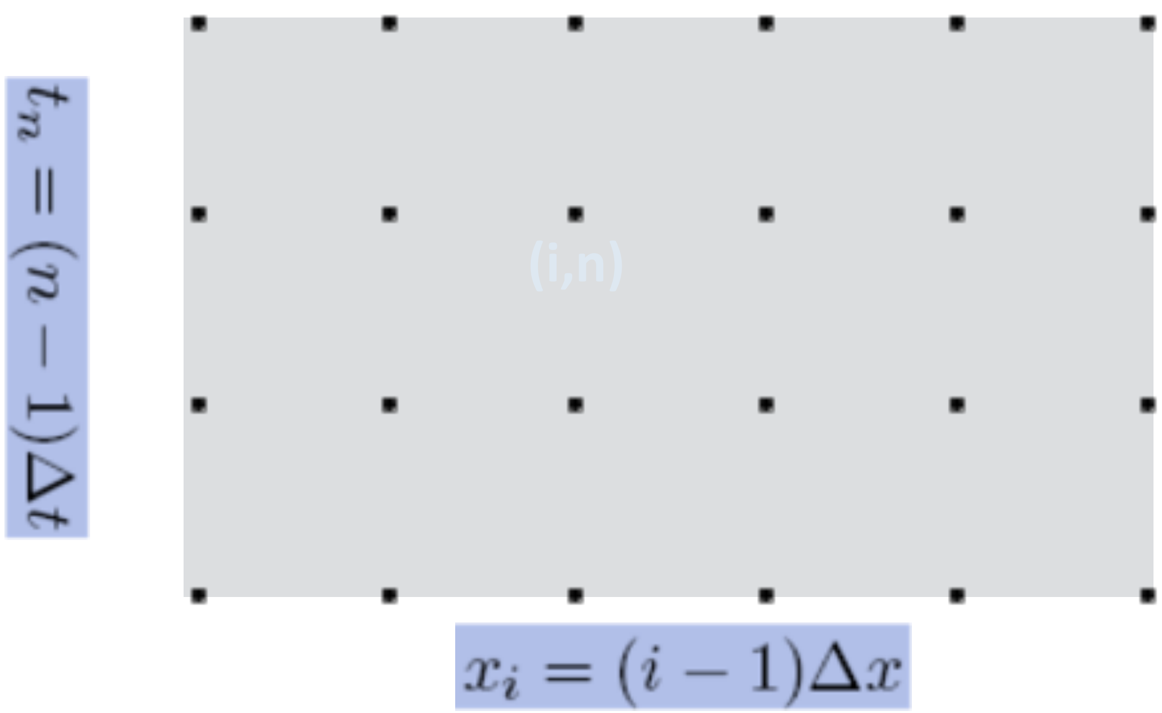
Discretization $u_i^n = u(x_i, t_n)$



continuous form:

$$\begin{aligned} \rho(x)\partial_t v(x,t) &= \partial_x T(x,t) \\ \partial_t T(x,t) &= \kappa(x)\partial_x v(x,t) \end{aligned}$$

discretized form:

$$\rho(x_i)\partial_t v(x_i,t_n) = \partial_x T(x_i,t_n)$$

$$\partial_t T(x_i,t_n) = \kappa(x_i)\partial_x v(x_i,t_n)$$

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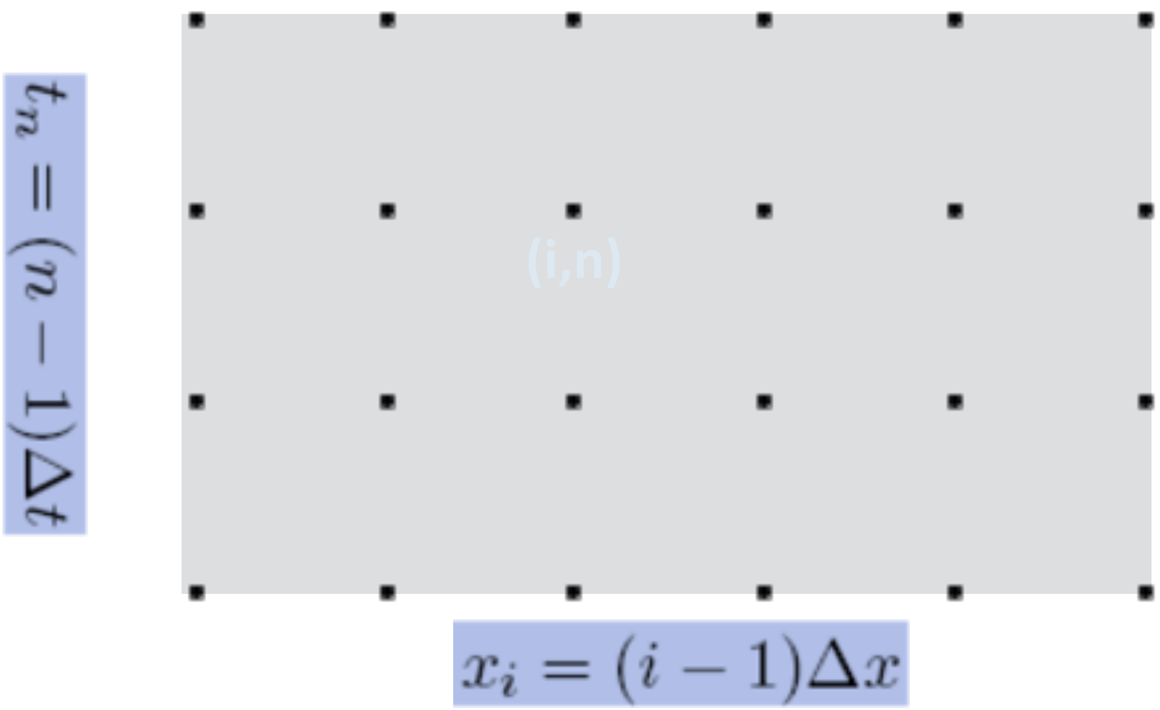
Derivatives



$$\tilde{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} \mathrm{d}x$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{F}(k) e^{ikx} \mathrm{d}k$$

$$\int_{-\infty}^{+\infty} |f(x)| \mathrm{d}x < \infty$$

temporal derivative: central differences

$$v_i^{n+1} = v_i^{n-1} + \frac{2\Delta t}{\rho_i} \times (\frac{\partial T}{\partial x})_i^n$$

$$T_i^{n+1} = T_i^{n-1} + 2\Delta t \times \kappa_i \times (\frac{\partial v}{\partial x})_i^n$$

spatial derivative: Fourier transforms

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{F}(k) e^{ikx} \mathrm{d}k] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} ik\tilde{F}(k) e^{ikx} \mathrm{d}k$$

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Recipe

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{F}(k) e^{ikx} \mathrm{d}k\right] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} ik\tilde{F}(k) e^{ikx} \mathrm{d}k
\end{aligned}
$$

- First, Fourier Transform whatever field $f(x)$ we need to differentiate.
- Second, multiply each Fourier coefficient $\tilde{F}(k)$ by ik.
- Finally, carry out inverse Fourier Transform to get desired derivatives.

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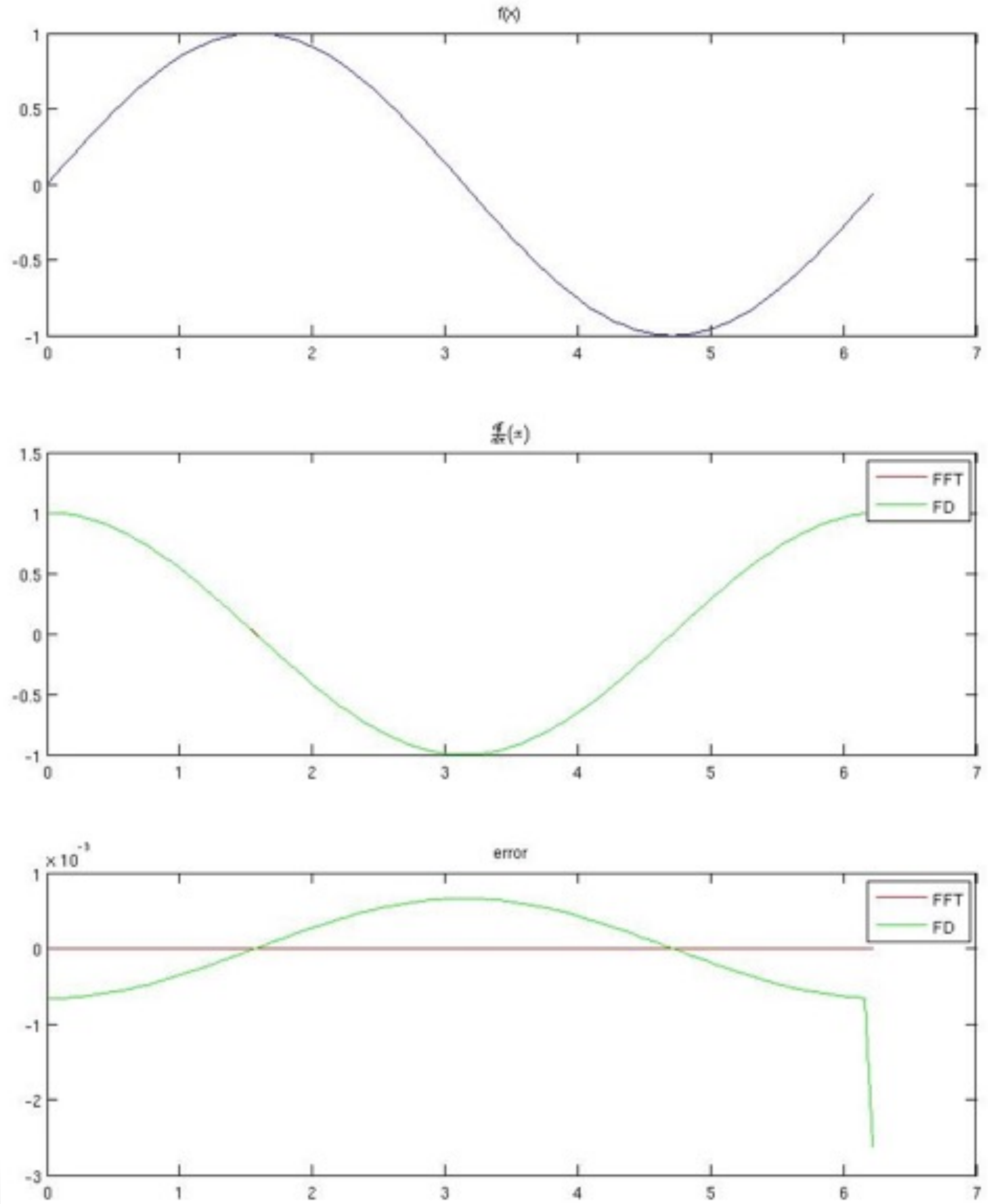
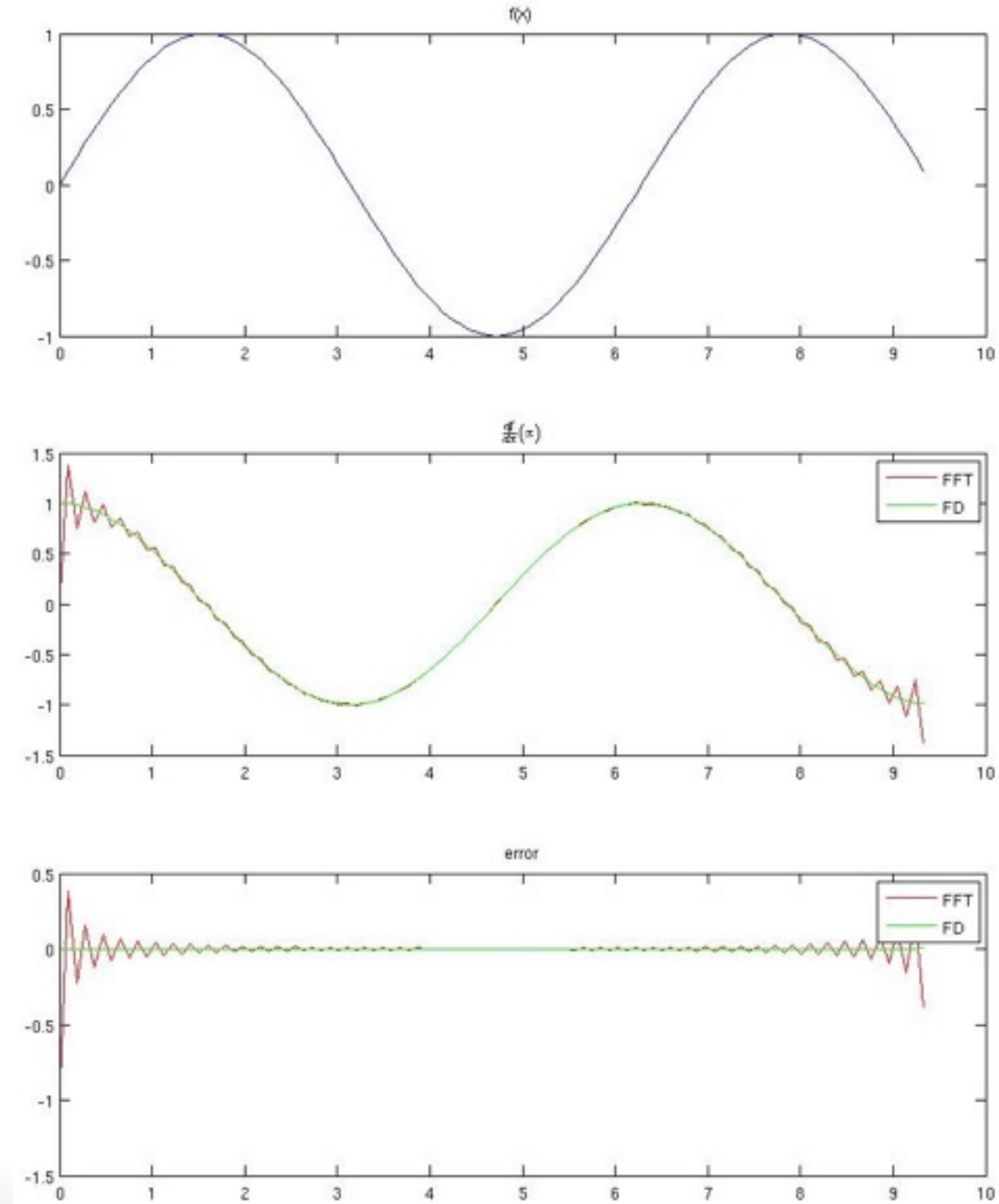
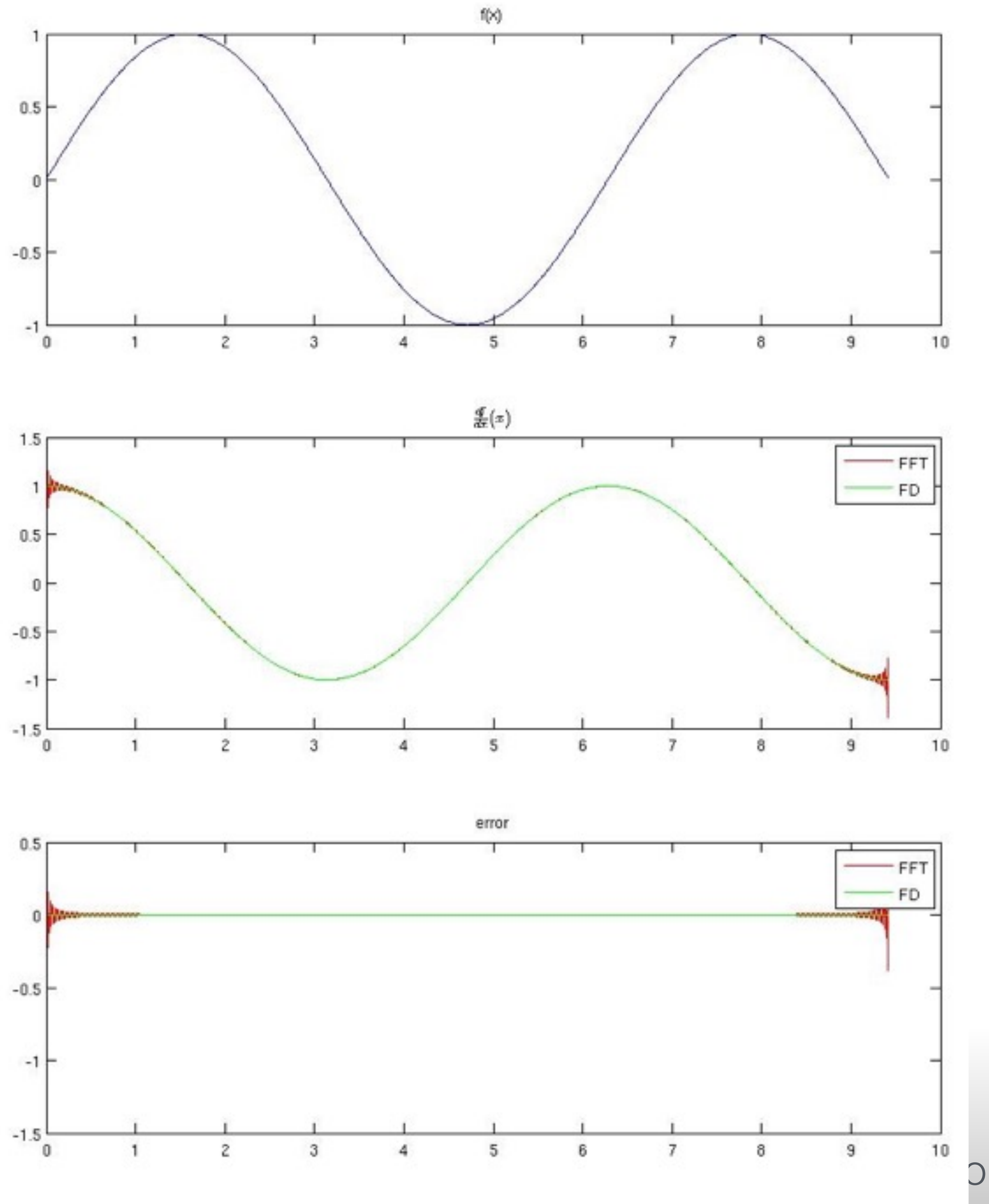
Results





Pseudo-spectral method

Results



Pseudo-spectral method

Results