Finite-volume method – Part I

The finite-volume method is based on the integral form of partial differential equations. It discretizes the (physical) domain into grid cells and approximates the integral values over each grid cell.

Integral form:

Let's consider the following 1D partial differential equation for $u \in [0,1]$

$$\partial_x^2 u + f = 0$$

steady-state diffusion
Poisson's equation

with boundary conditions

$$u(1) = g \qquad \text{"Dirichlet"}$$

$$\partial_x u(0) = -h \qquad \text{"Neumann"}$$

and $f = f(x)$

The integral form is simply the PDE integrated over the domain $\Omega \in [0,1]$

$$\int_\Omega (\partial_x^2 u + f)\, dx = \int_0^1 \partial_x^2 u \, dx + \int_0^1 f \, dx = 0$$

Finite-volume approach:

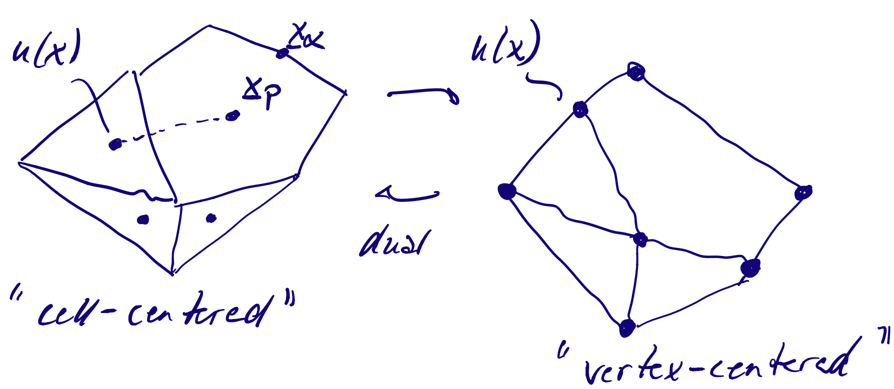
Motivated by conservation laws, where often the divergence of a quantity is involved, we look at Gauss' theorem which is fundamental for the finite-volume method

$$\int_{\Omega} \underline{\nabla} \cdot \underline{u} \, dV = \int_{\partial\Omega} \underline{u} \cdot \underline{n} \, dS$$

"divergence of u" "flux of u across surface $\partial\Omega$"

[continuity equation: $\int_{\Omega} \partial_t \rho + \underbrace{\int_{\Omega} \underline{\nabla} \cdot (\rho \underline{v})}_{= \int_{\partial\Omega} \rho \underline{v} \cdot \underline{n} \, dS} = 0$ conservation of mass]

Grid:



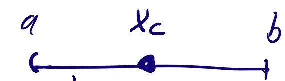
"cell-centred" "vertex-centred"

Here, we choose a cell-centred, i.e., centroid-based, approach. The centroid for grid cell Ω_e is defined as $\int_{\Omega_e} (\underline{x} - \underline{x}_p) \, dV = 0$

and similar the centroid of face α is given by

$$\int_{\partial\Omega_e^\alpha} (\underline{x} - \underline{x}_\alpha) \, dS = 0$$

1D line:



$$\int_a^b (x - x_c) \, dx = 0$$

To have at least second-order accuracy of our discretization, we assume that the field u varies linearly within each cell:

$$u(x) = u_P + (\underline{x} - \underline{x}_P) \cdot (\underline{\nabla} u)_P \quad \text{where } u_P = u(\underline{x}_P)$$

It follows that the mean value within the grid cell is represented by the variable value at centroid P:

$$\bar{u} = \frac{1}{V_e} \int_{\Omega_e} u(x) \, dV \qquad \text{mean / average}$$

$$= u(\underline{x}_P) = u_P$$

Note if u is non-linear, then this is at least second-order accurate

$$\bar{u} = \frac{1}{V_e} \int_{\Omega_e} u(x) \, dV = \frac{1}{V_e} \int_{\Omega_e} \left(u_P + (\underline{x} - \underline{x}_P) \cdot (\nabla u)_P + O(\nabla^2 u) \right) dV$$

$$= \frac{1}{V_e} \int_{\Omega_e} u_P \, dV + \frac{1}{V_e} \int_{\Omega_e} (\underline{x} - \underline{x}_P) \cdot (\nabla u)_P \, dV + \frac{1}{V_e} \int_{\Omega_e} O(\nabla^2 u) \, dV$$

$$= u_p \frac{1}{V_e} \underbrace{\int_{\Omega_e} dV}_{= V_e} + (\nabla u)_p \frac{1}{V_e} \underbrace{\int_{\Omega_e} (\underline{x} - \underline{x}_p)\, dV}_{= 0 \text{ centroid}} + O(\nabla^2 u)\, dV$$

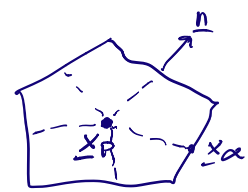
$$= u_p + O(\nabla^2 u) \qquad \text{second-order}$$

and $\bar{u} = u_p$ is exact for linear fields u

For smooth enough u:

$$\int_{\Omega_e} \underline{\nabla} \cdot \underline{u}\, dV \approx \underline{\nabla} \cdot \underline{u} \int_{\Omega_e} dV = \underline{\nabla} \cdot \underline{u}\, V_e$$

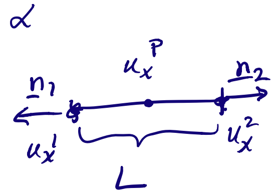
$$= \int_{\partial \Omega_e} \underline{u} \cdot \underline{n}\, dS$$



Thus, in 3D: $$(\partial_i u_i)_p \approx \underbrace{\frac{1}{V_e}}_{\text{Volume } \Omega_e} \sum_\alpha \underbrace{S_\alpha}_{\text{surface area } \partial\Omega^\alpha} n_i^\alpha \underbrace{\boxed{u_i^\alpha}}_{= u_i(\underline{x}_\alpha)}$$

2D: $$\partial_i u_i \approx \frac{1}{S} \sum_\alpha \underbrace{L_\alpha}_{\text{edge length } \alpha} n_i^\alpha u_i^\alpha$$

1D: $$\partial_x u_x \approx \frac{1}{L} \sum_\alpha n_x^\alpha u_x^\alpha$$



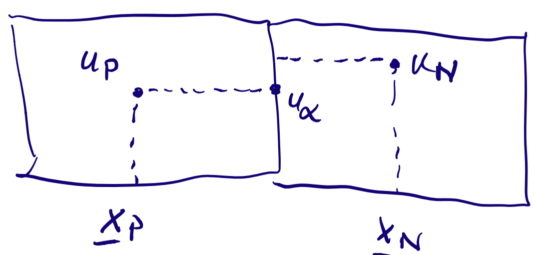
$$\approx \frac{u_x^2 - u_x^1}{L}$$

→ Dormy & Tarantola (1995)

Face interpolations: How do we find the surface integral values for the grid cell face? Which translates to how do we find the values at the surface centroid locations?

$$\int_\Omega \underline{\nabla} \cdot \underline{u} \, dV = \int_{\partial\Omega} \underline{u} \cdot \underline{n} \, dS \approx \sum_\alpha S_\alpha n_i^\alpha \underbrace{u_i^\alpha}_{\text{at face centroid}}$$

upwinding scheme: $u_\alpha = \begin{cases} u_P & \text{, if flux} > 0 \\ u_N & \text{, if flux} < 0 \end{cases}$



convective flux

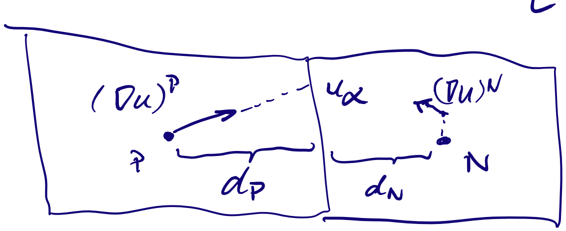
$F = (\rho \underline{v} u)^\alpha n S_\alpha$

$\underline{v}$: transport velocity

+ stable

− "smears out" solution

linear upwinding: $u_\alpha = \begin{cases} u_P + d_P \nabla u_P & \text{, if flux} > 0 \\ u_N + d_N \nabla u_N & \text{, if flux} < 0 \end{cases}$

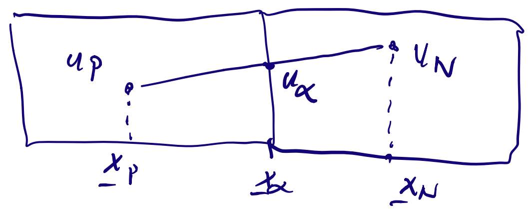


+ second-order accurate

− artificial oscillations

→ limited schemes

linear / central differencing: $u_\alpha = (1-\phi) u_P + \phi\, u_N$



with

$$\phi = \frac{|\underline{x}_\alpha - \underline{x}_P|}{|\underline{x}_N - \underline{x}_P|}$$

+ second-order accurate

− artificial oscillations