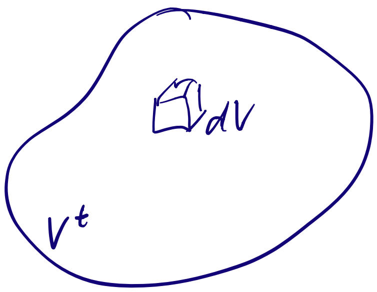
Conservation of linear momentum



momentum = mass × velocity

$$\text{total momentum} = \int_{V^t} \underbrace{\rho}_{\substack{\text{density} \\ \rho(x,t)}} \underbrace{\underline{v}}_{\text{velocity}} \, dV$$

rate of change $\frac{d}{dt}$ total momentum = applied forces

$$\frac{d}{dt}\int_{V^t} \rho \underline{v} \, dV = \int_{V^t} \partial_t (\rho \underline{v}) \, dV + \underbrace{\int_{\Sigma^t} (\rho \underline{v}) \underline{v} \cdot \hat{\underline{n}} \, d\Sigma}_{\text{how much momentum is carried over the surface}}$$

$$= \int_{V^t} \partial_t (\rho \underline{v}) + \underline{\nabla} \cdot (\rho \underbrace{\underline{v}\,\underline{v}}_{\text{tensor product}}) \, dV$$

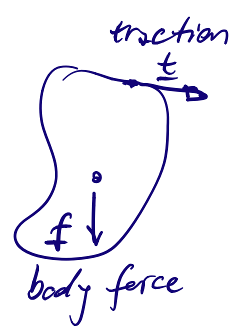
index notation: $\int \partial_t (\rho v^i) + \nabla_j (\rho \underbrace{v^i v^j}_{\text{tensor product}})$

tensor product

$$v \otimes v = \underline{v}\,\underline{v} = v^i v^j = \begin{pmatrix} v_1 v_1 & v_1 v_2 & v_1 v_3 \\ v_2 v_1 & v_2 v_2 & v_2 v_3 \\ v_3 v_1 & v_3 v_2 & v_3 v_3 \end{pmatrix}$$

$$\frac{d}{dt} \text{momentum} = \text{forces}$$

$$= \underbrace{\int_{V^t} \underline{f} \, dV}_{\text{body forces}} + \underbrace{\int_{\Sigma^t} \underline{t} \, d\Sigma}_{\text{traction}}$$



Traction $\underline{t} = \underline{\underline{T}} \cdot \underline{\hat{n}}$

$\underline{t}$: traction

$\underline{\underline{T}}$: stress tensor

$t^i = T^{ij} \hat{n}_j$

$$\int_{\Sigma^t} \underline{t} \, d\Sigma = \int_{\Sigma^t} \underline{\underline{T}} \cdot \underline{\hat{n}} \, d\Sigma \overset{\text{Gauss' theorem}}{=} \int_{V^t} \underline{\nabla} \cdot \underline{\underline{T}} \, dV$$

index notation : $\int_{V^t} \nabla_j T^{ij}$

We find

$$\int_{V^t} \partial_t (\rho \underline{v}) + \underline{\nabla} \cdot (\rho \underline{v}\,\underline{v}) \, dV = \int_{V^t} \underline{f} + \underline{\nabla} \cdot \underline{\underline{T}} \, dV$$

valid for any volume

Therefore

$$\boxed{\partial_t (\rho \underline{v}) + \underline{\nabla} \cdot (\rho \underline{v}\,\underline{v} - \underline{\underline{T}}) = \underline{f}}$$

conservation of momentum

Compare with mass conservation

quantity : density
momentum
⋮

$$\underbrace{\partial_t \rho}_{\text{rate of change of quantity}} + \underbrace{\nabla \cdot (\rho \underline{v} \qquad)}_{\text{divergence of quantity}} = 0$$

rate of change of quantity + divergence of quantity = something

typical form of a conservation law

Let's write out the conservation of momentum

$$\partial_t(\rho \underline{v}) + \underline{\nabla}\cdot(\rho \underline{v}\,\underline{v} - \underline{\underline{T}}) = \underline{f}$$

using $\nabla\cdot(\rho \underline{v}\,\underline{v} - T) = \underline{v}\,\underline{\nabla}\cdot(\rho \underline{v}) + \rho \underline{v}\cdot\underline{\nabla}\underline{v} - \underline{\nabla}\cdot\underline{\underline{T}}$ we see

$$\rho\,\partial_t \underline{v} + \underline{v}\underbrace{\left[\partial_t \rho + \underline{\nabla}\cdot(\rho \underline{v})\right]}_{\text{continuity} \to = 0} + \rho \underline{v}\cdot\underline{\nabla}\underline{v} - \underline{\nabla}\cdot\underline{\underline{T}} = \underline{f}$$

$$\rho\,(\underbrace{\partial_t \underline{v} + \underline{v}\cdot\underline{\nabla}\underline{v}}) = \underline{\nabla}\cdot\underline{\underline{T}} + \underline{f}$$

$$\boxed{\rho\, D_t \underline{v} = \underline{\nabla}\cdot\underline{\underline{T}} + \underline{f}}$$

conservation of momentum

needs constitutive law to solve

Hooke's law $T_{ij} = c_{ijkl}\,\varepsilon_{kl}$

$$\underline{\underline{T}} = \underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}}$$

ε : strain tensor

c : elastic tensor

T : stress tensor

Using $\rho(\underline{x},t) = \rho_0(\underline{x}) + \underbrace{\rho_1(\underline{x},t)}_{\text{small perturbation}}$

$$\underline{v}(\underline{x},t) = \partial_t\, \underline{s}(\underline{x},t)$$

s : displacement

we find

$$\rho_0\,\partial_t^2 \underline{s} = \underline{\nabla}\cdot\underline{\underline{T}} + \underline{f}$$

$$\boxed{\rho_0\,\partial_t^2 \underline{s} = \underline{\nabla}\cdot(\underline{\underline{\underline{\underline{c}}}} : \underline{\underline{\varepsilon}}) + \underline{f}}$$

equations of motion (linearised in ρ_0)

index notation: $\rho \, \partial_t^2 s_i = \nabla_j (c_{ijkl} \, \varepsilon_{kl}) + f_i$

strain $\underline{\underline{\varepsilon}} := \frac{1}{2} (\underline{\nabla} \, \underline{s} + (\underline{\nabla} \, \underline{s})^T)$

<u>1-D equation</u>: wave equation

$$\rho \, \underline{\partial_t^2 s} = \underline{\underline{\partial_x} (\mu \, \underline{\partial_x s})} + f$$

s : displacement

μ : wave speed (shear modulus)

f : source

Finite-difference method

We approximate the derivative $\frac{\partial}{\partial x}$ in terms of small steps Δx of a function $f(x)$.

The Taylor-series for $f(x_0 + \Delta x)$ expands:

$$f(x+\Delta x) = f(x) + \Delta x\, f'(x) + \frac{1}{2}\Delta x^2\, f''(x) + \dots \quad (1)$$

$$f(x-\Delta x) = f(x) - \Delta x\, f'(x) + \frac{1}{2}\Delta x^2\, f''(x) - \dots \quad (2)$$

Forward scheme: $$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} \underbrace{- \frac{1}{2}\Delta x\, f''(x) - \dots}_{\substack{\text{truncation error} \\ O(\Delta x)}}$$

Backward scheme: $$f'(x) = \frac{f(x) - f(x-\Delta x)}{\Delta x} \underbrace{+ \frac{1}{2}\Delta x\, f''(x) - \dots}_{O(\Delta x)}$$

Central scheme: $$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} \underbrace{+ \frac{1}{3!}\Delta x^2 f'''(x) + \dots}_{\substack{\text{truncation error} \\ O(\Delta x^2)}}$$

central scheme: $$f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} \underbrace{+ \frac{1}{4!}\Delta x^2 f''''(x) + \dots}_{O(\Delta x^2)}$$