Problem Set 4

Pseudo-Spectral Method

The Pseudo-Spectral Method, like the Finite-Difference Method, is a strong form solution of Partial Differential Equations. Instead of using finite differencing, the Pseudo-Spectral Method may utilize Fourier Transforms to calculate the approximation of spatial derivatives. The Pseudo-Spectral Method has very high accuracy, when the field to be differentiated is smooth. However, when singularities exist, this method suffers from the so-called Gibbs Phenomenon.

The Continuous Fourier Transform $\tilde{F}(k)$ of any function $f(x)$ ($\int_{-\infty}^{+\infty} |f(x)| \mathrm{d}x < \infty$) and its inverse transform are defined as:

$$\begin{aligned}
\tilde{F}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} \mathrm{d}x \\
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{F}(k) e^{ikx} \mathrm{d}k
\end{aligned}$$

As a consequence, the derivatives of $f(x)$ can be expressed as:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x} [\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \tilde{F}(k) e^{ikx} \mathrm{d}k] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} ik\tilde{F}(k) e^{ikx} \mathrm{d}k
\end{aligned}$$

Therefore, we can use Fourier Transforms to calculate the derivatives of a function $f(x)$, given the Fourier Transform for that function exists.

The steps are clear:

- First, Fourier Transform whatever field $f(x)$ we need to differentiate.
- Second, multiply each Fourier coefficient $\tilde{F}(k)$ by ik.
- Finally, carry out an inverse Fourier Transform to get desired derivatives.

For simplicity, we are going to solve the 1-D wave equation again, using the Pseudo-Spectral Method.

Wave Equation

The 1-D expression of the wave equation is:

$$\rho(x)\ \partial_t^2 u(x,t) = \partial_x[\,\kappa(x)\ \partial_x u(x,t)], \quad (x \in [0,L], t \in [0,+\infty))$$

where $u(x,t)$ is the displacement field at the position x at instant t, $\rho(x)$ is the material density and $\kappa(x)$ is the material bulk modulus.
The initial conditions are:

$$\begin{aligned} u(x,0) &= \exp^{-0.1*(x-50)^2}, \\ \partial_t u(x,0) &= 0 \end{aligned}$$

To solve the wave equation, you can recognize that it is equivalent to

$$\begin{aligned} \rho(x)\partial_t v(x,t) &= \partial_x T(x,t) \\ \partial_t T(x,t) &= \kappa(x)\partial_x v(x,t) \end{aligned}$$

where

$$\begin{aligned} v(x,t) &= \partial_t u(x,t) \quad \text{is a velocity field}\,, \\ T(x,t) &= \kappa(x)\partial_x u(x,t) \quad \text{is a stress field}\,. \end{aligned}$$

Problem:

Write the discretized form of the system to solve for (v,T). The grid size Δx is chosen to be 0.1. The string length is $L = 100$. Plot your numerical results at several time steps for a homogeneous material case:

- $\rho = 1, \kappa = 1 \quad (x \in [0, 100])$

Heterogeneous material

Use the same code you just wrote and investigate the evolution of the displacement field time series, when

- $\rho(x) = 1, \kappa(x) = 1 \quad (x \in [0, 60])$

  and

- $\rho(x) = 1, \kappa(x) = 4 \quad (x \in (60, 100])$

using the same initial conditions.

Extra Question – Boundary conditions

You may have already noticed that we didn't talk about boundary conditions in previous sections. That's because boundary conditions in the Pseudo-Spectral Method are quite difficult to deal with. Think about how you may implement both first kind and second kind boundary conditions in the Pseudo-Spectral Method. Implement them in your code and run long simulations to see what you get.