Spectral-element method

Homework - 1D heat equation:

We look at

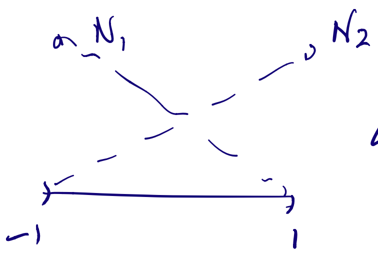
$$\rho c \, \partial_t T = \partial_x (\kappa \, \partial_x T)$$

(1.) which in the weak-form becomes

$$\int_0^L \underbrace{w \rho c \, \partial_t T \, dx}_{\text{mass/capacity}} = - \int_0^L \underbrace{\kappa \, \partial_x w \, \partial_x T \, dx}_{\text{stiffness}} + \underbrace{w \kappa \partial_x T \Big|_0^L}_{\text{boundary}}$$

(2.) shape functions:

$$\begin{cases} N_1(\xi) = \frac{1}{2}(1-\xi) \\ N_2(\xi) = \frac{1}{2}(1+\xi) \end{cases}$$



degree-1 Lagrange polynomials

$\longrightarrow$ Jacobian $J = \dfrac{\partial x}{\partial \xi} = \frac{1}{2}(\underset{e+1}{X_2} - \underset{e}{X_1})$ where $x(-1) = \underset{e}{X_1}$ and $x(+1) = \underset{e+1}{X_2}$

(3.) basis function:

$$f(x(\xi)) = \sum_{\alpha=0}^{N} f^{\alpha} \underbrace{l_{\alpha}^{N}(\xi)}_{\substack{\text{Lagrange polynomials} \\ \text{degree } N}}$$

(4.) local contributions:

- mass matrix: $M_{\beta_1} = \hat{\omega}_{\beta_1} \rho^{\beta_1} c^{\beta_1} J^{\beta_1}$

- stiffness matrix: $K_{\beta_1\gamma} = -\sum_{\alpha=0}^{N} \hat{\omega}_\alpha \kappa^\alpha \ell'^{N}_{\beta_1}(\xi_\alpha) \ell'^{N}_{\gamma}(\xi_\alpha) J^\alpha [\partial_x \xi(\xi_\alpha)]^2$

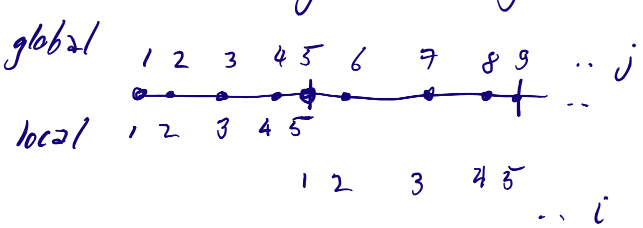
$$\longrightarrow M_{\beta_1} \partial_t T^{\beta_1}(t) = \sum_\gamma K_{\beta_1\gamma} T^\gamma(t)$$

system of equations

\+ boundary terms

(5.) Assembly:

- local to global array indexing



$\text{iglob}(\text{ielement}, i) = j$

$\longrightarrow$ global matrices $\underline{\underline{M}}$ (diagonal), $\underline{\underline{K}}$, $\underline{F}$

(6.) time marching:

- predictor - corrector scheme:

$$\left.\begin{aligned} T_{n+1} &= T_n + \tfrac{1}{2}\Delta t\, \dot{T}_n \\ \dot{T}_{n+1} &= 0 \end{aligned}\right\} \text{predictor step}$$

$$\dot{T}_{n+1} = \frac{1}{\underline{\underline{M}}} \, RHS$$

$$\left. T_{n+1} = T_{n+1} + \frac{1}{2} \Delta t \, \dot{T}_{n+1} \right\} \text{corrector step}$$