Nilpotent coefficients

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Let $\mathbf{R}$ be a commutative ring with 1. The following describes the units of $\mathbf{R}[X]$ (see, e.g., [1, Ex. 1.2]) and is readily proved by reduction modulo a generic prime ideal. Richman has given a very short constructive proof [3]. Here we use the constructive prime spectrum.

Proposition. *If $f = a_0 + a_1X + \cdots + a_kX^k \in \mathbf{R}[X]^\times$, then $a_0 \in \mathbf{R}^\times$ and $a_1, \ldots, a_k \in \sqrt{0}$.*

Proof. We show that the formally leading coefficient a_k is nilpotent. This will suffice, for then

$$a_0 + a_1X + \cdots + a_{k-1}X^{k-1} = f - a_kX^k$$

is a unit (being the the sum of a unit and a nilpotent element), and so we may argue by induction on the formal degree.

Suppose now that $k > 0$ and let $g = b_0 + b_1X + \cdots + b_\ell X^\ell \in \mathbf{R}[X]$ such that $fg = 1$. Put

$$c_s = \sum_{\substack{i,j \\ i+j=s}} a_ib_j \qquad \text{where } k \leqslant s \leqslant k+\ell.$$

These are the coefficients of fg in which a_k occurs. Keep in mind that each c_s vanishes. Next we use the entailment relation $\vdash$ of (proper) prime ideal of $\mathbf{R}$ [2]. Recall the *formal Nullstellensatz*, which asserts that

$$U \vdash b_1, \ldots, b_n \qquad \text{if and only if} \qquad b_1 \cdots b_n \in \sqrt{\langle U \rangle}.$$

Each of the following entailments is witnessed by the corresponding identity in the right column:

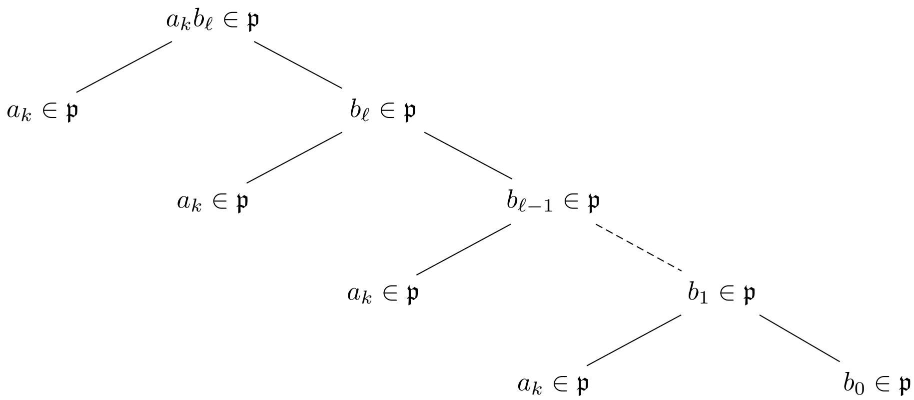
$$\begin{array}{rlr}
 & \vdash c_k & c_k = 0 \\
 & \vdash c_{k+1} & c_{k+1} = 0 \\
 & \vdots & \vdots \\
 & \vdash c_{k+\ell-1} & c_{k+\ell-1} = 0 \\
 & \vdash c_{k+\ell} & c_{k+\ell} = 0 \\
c_{k+\ell} & \vdash a_k, b_\ell & a_kb_\ell = c_{k+\ell} \\
b_\ell, c_{k+\ell-1} & \vdash a_k, b_{\ell-1} & a_kb_{\ell-1} = c_{k+\ell-1} - a_{k-1}b_\ell \\
b_{\ell-1}, b_\ell, c_{k+\ell-2} & \vdash a_k, b_{\ell-2} & a_kb_{\ell-2} = c_{k+\ell-2} - a_{k-1}b_{\ell-1} - a_{k-2}b_\ell \\
 & \vdots & \vdots \\
b_2, \ldots, b_\nu, \ldots, b_\ell, c_{k+1} & \vdash a_k, b_1 & a_kb_1 = c_{k+1} - a_{k-1}b_2 - \cdots - a_{k-\nu+1}b_\nu \\
b_1, \ldots, b_\nu, \ldots, b_\ell, c_k & \vdash a_k, b_0 & a_kb_0 = c_k - a_{k-1}b_1 - \cdots - a_{k-\nu}b_\nu \\
b_0 & \vdash & 1 = a_0b_0
\end{array}$$

where $\nu = \min\{k, \ell\}$. A series of cuts yields $\vdash a_k$, which is to say that a_k is nilpotent. □

Remark 1. Consider a generic prime ideal $\mathfrak{p}$ of $\mathbf{R}$. As $a_k b_\ell = 0 \in \mathfrak{p}$, one has $a_k \in \mathfrak{p}$ or $b_\ell \in \mathfrak{p}$. In the latter case, since $c_{k+\ell-1} = 0 \in \mathfrak{p}$, it follows that

$$c_{k+\ell-1} - a_{k-1} b_\ell = a_k b_{\ell-1} \in \mathfrak{p},$$

which leads to another branching, $a_k \in \mathfrak{p}$ or $b_{\ell-1} \in \mathfrak{p}$. And so on, travelling down the coefficients:



The rightmost branch would assert that $\mathfrak{p}$ is improper (since $b_0 \in \mathbf{R}^\times$), so we conclude that $a_k \in \mathfrak{p}$. By a variant of Krull's lemma,

$$a_k \in \bigcap \mathsf{Spec}(\mathbf{R}) = \sqrt{0}.$$

This is the underlying heuristic for our constructive proof.

Remark 2. The entailment relation of prime ideal is a *conservative extension* of the (single-conclusion) entailment relation of radical ideal [4]. At the heart of this conservation lies that

$$\sqrt{\langle U, a\rangle} \cap \sqrt{\langle U, b\rangle} \subseteq \sqrt{\langle U, ab\rangle}. \tag{1}$$

The above proof boils down to applications of (1). In deduction terms this means to fold up branchings of proof trees [4].

References

[1] M.F. Atiyah and I.G. MacDonald. *Introduction to Commutative Algebra.* Reading, MA: Addison-Wesley Publishing Company, Inc., 1969.

[2] Jan Cederquist and Thierry Coquand. "Entailment relations and distributive lattices". In: *Logic Colloquium '98: Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic.* Ed. by Samuel R. Buss, Petr Hájek, and Pavel Pudlák. Vol. 13. Lecture Notes in Logic. AK Peters/Springer, 2000, pp. 110–123.

[3] Fred Richman. "Nontrivial uses of trivial rings". In: *Proc. Amer. Math. Soc.* 103.4 (1988), pp. 1012–1014.

[4] Davide Rinaldi, Peter Schuster, and Daniel Wessel. "Eliminating disjunctions by disjunction elimination". In: *Indagationes Mathematicae* 29.1 (2018), pp. 226–259.