Constructive Algebra – Exercises

Autumn School Proof & Computation
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Throughout, let $\mathbf{R}$ be a commutative ring with 1. The subset of invertible elements (*units*) of $\mathbf{R}$ will be denoted $\mathbf{R}^\times$.

Exercise 1. (Constructive prime spectrum) The (single-conclusion) entailment relation $\rhd$ of *radical ideal* of $\mathbf{R}$ and the (multi-conclusion) entailment relation $\vdash$ of *proper prime ideal* [1] are inductively generated by all instances of the following axioms, respectively:

$$\begin{array}{cc} \rhd\, 0 & \vdash 0 \\ a \rhd ra & a \vdash ra \\ a, b \rhd a+b & a, b \vdash a+b \\ a^2 \rhd a & ab \vdash a, b \\ & 1 \vdash \end{array}$$

Find direct, non-inductive descriptions for $\rhd$ and $\vdash$. Show that $\vdash$ is a *conservative extension* of $\rhd$, i.e., for all $a \in R$ and $U \in \mathsf{Fin}(R)$,

$$U \rhd a \qquad \text{if and only if} \qquad U \vdash a.$$

What does this say from a classical point of view? (There is a versatile criterion on the interplay of single- and multi-conclusion entailment relations due to Dana Scott [8], which has turned out widely applicable in algebra and order theory [7].)

Exercise 2. (Nil- and Jacobson radicals) Recall that the *nilradical* of $\mathbf{R}$ is

$$\mathsf{Nil}(\mathbf{R}) = \{\, a \in \mathbf{R} : \exists n \in \mathbb{N}\, (a^n = 0) \,\}.$$

The *Jacobson radical* of $\mathbf{R}$ is defined as

$$\mathsf{Jac}(\mathbf{R}) = \left\{\, a \in \mathbf{R} : \forall b \in \mathbf{R}\, (1 - ab \in \mathbf{R}^\times) \,\right\}.$$

(a) Use the entailment relation $\rhd$ of radical ideal (cf. Exercise 1) to show that the sum of a unit and a nilpotent element is a unit.

(b) Show that the polynomial ring $\mathbf{R}[X]$ is *rad-nil*, i.e., $\mathsf{Jac}(\mathbf{R}[X]) = \mathsf{Nil}(\mathbf{R}[X])$. (Hint: what do we know about the coefficients of an invertible polynomial?)

Exercise 3. (Local rings, constructively [2, 4, 6, 9]) A ring $\mathbf{R}$ is *local* if, for all $a \in \mathbf{R}$, either a is a unit or $1 - a$ is a unit.

(a) Show that $\mathbf{R}$ is local if and only if, for all $a, b \in \mathbf{R}$, if $a + b$ is a unit, then so is a or b.

(b) Let $\mathbf{R}$ be local. Show that $\mathsf{Jac}(\mathbf{R}) = \{\, a \in \mathbf{R} : a \in \mathbf{R}^\times \to \mathbf{R} = 0 \,\}$ [6, Theorem III.6.5].

Exercise 4. (Minimal prime ideals) Suppose that $\mathbf{R}$ is *reduced*, i.e., for all $a \in R$, if $a^2 = 0$, then $a = 0$. Matlis [5, Proposition 1.2(2)] has shown that a finitely generated ideal $\mathfrak{a}$ of $\mathbf{R}$ is contained in a minimal prime ideal of $\mathbf{R}$ if and only if $\mathsf{Ann}(\mathfrak{a}) \neq 0$, where

$$\mathsf{Ann}(\mathfrak{a}) = \{\, x \in \mathbf{R} : x\mathfrak{a} = 0 \,\}.$$

Use the entailment relation $\vdash_{\mathfrak{m}}$ of maximal filter of $\mathbf{R}$ [3] to give a constructive interpretation of Matlis' result. (Hint: write $\mathfrak{a} = \langle a_1, \ldots, a_k \rangle$ and describe $\vdash_{\mathfrak{m}} a_1, \ldots, a_k$.)

References

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