Constructive Algebra

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github.com/danielwessel/pc18

Introduction

“Reunion of broken parts”

- Kronecker vs. Dedekind: advent of ideal theory
- Passing from potential to actual infinity
- Rise of modern algebra
- Topology enters, Grothendieck revolutionizes
- Development of point-free methods
- Call for a revised Hilbert programme in abstract algebra

Varieties of constructive algebra

- Operative mathematics (Lorenzen)
- Bishop-style constructive algebra (Mines, Richman, Ruitenburg)
- Topos-valid algebra (Wraith, Banaschewski, Johnstone, ...)
- Dynamical algebra (Lombardi, Roy, Coste, Yengui, ...)
- Integrated development in type theory (Coquand, Persson)
- Formal topology $\Join$ comm. algebra (Coquand, Schuster, ...)

Key instruments

> *"What would have happened if topologies* without *points had been discovered before topologies* with *points, or if Grothendieck had known the theory of distributive lattices?"*
>
> G.-C. Rota
> *Indiscrete Thoughts*

Modern constructive algebra makes extensive use of **lattice theory** (sometimes in disguise).

Important example
Constructive Krull dimension (Joyal, Lombardi)

Leitmotiv

"When we say that we have a constructive version of an abstract algebraic theorem, this means that we have a theorem the proof of which is constructive, which has a clear computational content, and from which we can recover the usual version of the abstract theorem by an immediate application of a well classified non-constructive principle."

T. Coquand and H. Lombardi

Hidden constructions in abstract algebra

Outline

- Preliminaries
- Examples and counterexamples
- Constructive proofs from ideal objects
- Entailment relations
- Around Hilbert's 17th problem
- Abstract ideal theory in commutative rings
- Injective modules and Baer's criterion (w.i.p.)

Textbooks

- Ray Mines, Fred Richman, and Wim Ruitenburg. *A Course in Constructive Algebra*. Universitext. New York: Springer-Verlag, 1988
- Harold M. Edwards. *Essays in Constructive Mathematics*. New York: Springer, 2005
- Henri Lombardi and Claude Quitté. *Commutative Algebra: Constructive Methods. Finite Projective Modules*. Dordrecht: Springer Netherlands, 2015
- Ihsen Yengui. *Constructive commutative algebra. Projective modules over polynomial rings and dynamical Gröbner bases*. Vol. 2138. Lecture Notes in Mathematics. Cham: Springer, 2015

Rudiments of ring theory

Basic notions

We work in constructive set theory **CZF**.

A set S is said to be **discrete** if

$$\forall x, y \in S\,(\,x = y \vee x \neq y\,)$$

A subset T of S is **detachable** if

$$\forall x \in S\,(\,x \in T \vee x \notin T\,)$$

$\mathrm{Fin}(S)$ consists of the **finitely enumerable** subsets of S, i.e., $U \in \mathrm{Fin}(S)$ iff

$$\exists n \in \mathbb{N}\,\exists f\,(\,f : \{\,1, \ldots, n\,\} \twoheadrightarrow U\,)$$

Caveat: subsets of f.e. sets need not be f.e.!

Basic notions

Throughout, let $\mathbf{R}$ be a commutative ring with 1.
We will incur only few and basic concepts.

$\mathbf{R}$ is **integral** if, for all $a \in \mathbf{R}$,

$$a = 0 \ \vee \ \forall b \in \mathbf{R} \, (\, ab = 0 \ \rightarrow \ b = 0 \,)$$

$\mathbf{R}$ is a **discrete field** if, for all $a \in \mathbf{R}$,

$$a = 0 \ \vee \ \exists b \in \mathbf{R} \, (\, ab = 1 \,)$$

E.g., the trivial ring is a discrete field!
A discrete field $\mathbf{R}$ is a discrete set iff $1 =_{\mathbf{R}} 0$ is decidable.

Basic notions

Recall that an **ideal** I of $\mathbf{R}$ is an additive subgroup s.t. $RI \subseteq I$.

If $U \in \mathrm{Pow}(\mathbf{R})$, then $\langle U \rangle$ denotes the ideal **generated** by U.

An ideal I is **prime** if

$$ab \in I \;\rightarrow\; a \in I \vee b \in I$$

and **maximal** if, for all $a \in \mathbf{R}$,

$$a \in I \;\vee\; \exists b \in \mathbf{R}\,(\,1 - ab \in I\,)$$

Every maximal ideal is prime.
If I is maximal, then the quotient ring R/I is a discrete field.

Basic notions

Let I be an ideal of $\mathbf{R}$.

The **radical** of I is

$$\sqrt{I} = \{\, a \in \mathbf{R} : \exists n \in \mathbb{N}\,(\, a^n \in I \,)\,\}$$

The **Jacobson radical** of I is

$$\begin{aligned}\mathrm{Jac}(I) &= \{\, a \in \mathbf{R} : \forall b\, \exists c\,(\, 1 - (1 - ab)c \in I \,)\,\} \\ &= \{\, a \in \mathbf{R} : \forall b\,(\, 1 \in \langle a, b\rangle \;\to\; 1 \in \langle I, b\rangle \,)\,\}\end{aligned}$$

Note that

$$\sqrt{I} \subseteq \mathrm{Jac}(I)$$

Basic notions

Ideals can be captured by **inductive definitions**:

$$\frac{}{0} \qquad \frac{\{\, a, b \,\}}{a+b}\, a, b \in \mathbf{R} \qquad \frac{\{\, a \,\}}{ab}\, a, b \in \mathbf{R}$$

Prime ideals appear as **non-deterministic** inductive definitions:

$$\frac{}{\{\, 0 \,\}} \qquad \frac{\{\, a, b \,\}}{\{\, a+b \,\}} \qquad \frac{\{\, a \,\}}{\{\, ab \,\}} \qquad \frac{\{\, ab \,\}}{\{\, a, b \,\}} \qquad \frac{\{\, 1 \,\}}{\emptyset}$$

N.B. Under SGA the class $\mathrm{Spec}(\mathbf{R})$ of primes is **set-generated**; the class of minimal primes is a set, cf. [Ber13; IN16].

Limitations

Omniscience

LPO

For every binary sequence $(a_i)_{i\in\mathbb{N}}$ either

(a) there exists n such that $a_n = 1$, or

(b) $a_n = 0$ for every n.

Subgroups of $\mathbb{Z}$ need not be cyclic

Let $(a_i)_{i\in\mathbb{N}}$ be a binary sequence.
Consider the generated ideal

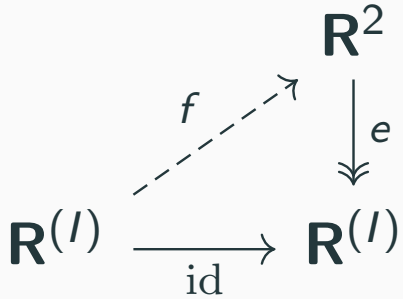
$$I = \langle \{\, a_i : i \in \mathbb{N} \,\} \rangle \subseteq \mathbb{Z}$$

Suppose that I has a single generator ...

Omniscience

Free modules need not be projective [MRR88, Ex. II.4.10]

Let $(a_i)_{i\in\mathbb{N}}$ be a binary sequence.
Let $I = \{0,1\}/R$, where $xRy \equiv (x = y) \vee \exists n\,(a_n = 1)$.
Let $\mathbf{R} = \mathbb{Z}/2\mathbb{Z}$.



Suppose that $\mathbf{R}^{(I)}$ is projective, and that $ef = \mathrm{id}$.
However, $\mathbf{R}^2$ is discrete ...

Omniscience

LLPO

For every binary sequence $(a_i)_{i\in\mathbb{N}}$ that contains at most one 1, either

(a) $a_n = 0$ for all odd n, or

(b) $a_n = 0$ for all even n.

Counterexamples

The zero ideal is not prime in $\mathbb{R}$.

Splitting fields need not be unique [BR87, Thm. 4.6].

"Summands need not be summands" [MRR88, Ex. II.4.6].

The Upside Down

Invariance of rank

Theorem*

Let $\mathbf{R}$ be non-trivial. If $\varphi : \mathbf{R}^m \to \mathbf{R}^n$ is surjective, then $m \geqslant n$.

Proof.

Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{R}$, and consider the field $k = \mathbf{R}/\mathfrak{m}$.

Tensoring yields an epimorphism of k-vector spaces

$$\mathrm{id}_k \otimes \varphi : \underbrace{k \otimes_{\mathbf{R}} \mathbf{R}^m}_{\cong k^m} \to \underbrace{k \otimes_{\mathbf{R}} \mathbf{R}^n}_{\cong k^n}$$

which implies $m \geqslant n$ (Steinitz exchange). □

Invariance of rank, constructively

Theorem (Richman [Ric88])

If $\varphi : \mathbf{R}^m \to \mathbf{R}^n$ is surjective, and $m < n$, then $\mathbf{R}$ is trivial.

Proof.

There is $\psi : \mathbf{R}^n \to \mathbf{R}^m$ such that $\varphi \circ \psi = \mathrm{id}_{\mathbf{R}^n}$.

Extend φ to $\mathbf{R}^n = \mathbf{R}^m \oplus \mathbf{R}^{n-m}$ by setting $\varphi(\mathbf{R}^{n-m}) = 0$.

View ψ as a map into $\mathbf{R}^n$.

If $A, B \in \mathrm{Mat}_n(\mathbf{R})$ are the matrices of φ and ψ, resp., then

$$1 = \det I_n = \det AB = (\det A)(\det B) = (\det A) \cdot 0 = 0 \qquad \square$$

Nakayama's lemma

Theorem*

Let $J \subseteq \bigcap \mathrm{Max}(\mathbf{R})$ be an ideal, and let M be a finitely generated $\mathbf{R}$-module. If $JM = M$, then $M = 0$.

Proof sketch [Eis04].

(i) Show first that there is $a \in J$ for which $(1 - a)M = 0$. (Corollary to Cayley-Hamilton)

(ii) Every (proper) maximal ideal avoids $1 - a$ which thus is a unit. It follows that $M = 0$. □

Nakayama's lemma, constructively

Solution: observe that (classically)

$$\bigcap \mathrm{Max}(\mathbf{R}) = \mathrm{Jac}(\mathbf{R})$$

and substitute the former for the latter.

Theorem

Let $J \subseteq \mathrm{Jac}(\mathbf{R})$ be an ideal, and let M be a finitely generated $\mathbf{R}$-module. If $JM = M$, then $M = 0$.

Proof.

Step (i) above holds constructively [LQ15].

Now we know that $1 - a$ is a unit since $a \in \mathrm{Jac}(\mathbf{R})$. □

Computational content from ideal elements: Gauß' Lemma and nilpotent coefficients

Content of polynomials

Theorem (Gauß' Lemma)

Let $f, g \in \mathbf{R}[X]$.

$$\mathrm{c}(f)\mathrm{c}(g) \subseteq \sqrt{\mathrm{c}(fg)}$$

where $\mathrm{c}(f)$ is the **content** of f, i.e., the ideal generated by the coefficients of f.

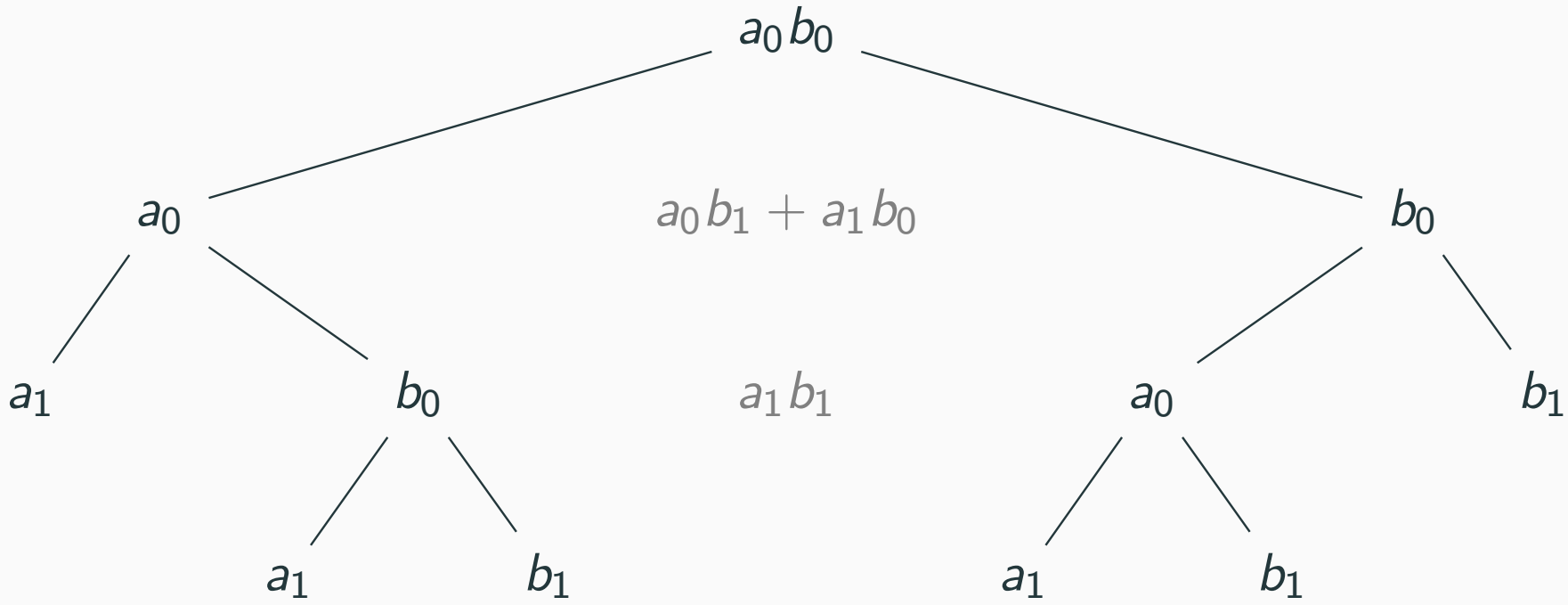
A classical proof [Eis04].

"It is enough to show that if a prime ideal $\mathfrak{p}$ contains $\mathrm{c}(fg)$, then it contains $\mathrm{c}(f)\mathrm{c}(g)$. Factoring out $\mathfrak{p}$, we may assume that $\mathbf{R}$ is a domain and $\mathfrak{p}$ is 0, and we must show that if $fg = 0$, then $f = 0$ or $g = 0$. Since R is a now domain, this is obvious." □

Example

Let $f = a_0 + a_1 X$ and $g = b_0 + b_1 X$.

Suppose that $\mathfrak{p}$ contains $\mathrm{c}(fg) = \langle a_0 b_0, a_0 b_1 + a_1 b_0, a_1 b_1 \rangle$.



So $\mathfrak{p}$ contains $\mathrm{c}(f)\mathrm{c}(g) = \langle a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1 \rangle$.

Example

Now suppose that

$$c \in \mathrm{c}(f)\mathrm{c}(g)$$

It is immediate that

$$c \in \langle a_0, a_1 \rangle \cap \langle b_0, b_1 \rangle$$

We want to obtain a witness for

$$c \in \sqrt{\langle a_0 b_0, a_0 b_1 + a_1 b_0, a_1 b_1 \rangle}$$

The tree is an instruction!

Entailment

For $a \in \mathbf{R}$ and $U \in \mathrm{Fin}(\mathbf{R})$ write

$$U \rhd a \;\equiv\; a \in \sqrt{\langle U \rangle}$$

This $\rhd$ is a (single-conclusion) **entailment relation**:

$$\frac{a \in U}{U \rhd a}\,(\mathrm{R}) \qquad \frac{V \supseteq U \rhd a}{V \rhd a}\,(\mathrm{M}) \qquad \frac{U \rhd b \quad V, b \rhd a}{U, V \rhd a}\,(\mathrm{T})$$

N.B. (later)

1. This $\rhd$ is **inductively generated**.
2. The **models** of $\rhd$ precisely are the radical ideals of $\mathbf{R}$.

Entailment

In addition:

$$\frac{U, a \rhd c \quad V, b \rhd c}{U, V, ab \rhd c}\ (\pi)$$

Because if

$$c^n = u + ra \qquad \text{and} \qquad c^m = v + sb$$

for certain $n, m \in \mathbb{N}, u \in \langle U \rangle, v \in \langle V \rangle, r, s \in \mathbf{R}$, then

$$c^{n+m} = uv + sbu + rav + rsab \in \langle U, V, ab \rangle$$

Upside down

Following the left branch bottom-up:

$$\dfrac{a_0, a_0 b_1 + a_1 b_0 \rhd a_1 b_0 \qquad \dfrac{a_0, a_1 \rhd c \qquad \dfrac{a_0, a_1 \rhd c \qquad b_0, b_1 \rhd c}{a_0, b_0, a_1 b_1 \rhd c}\,(\pi)}{a_0, a_1 b_0, a_1 b_1 \rhd c}\,(\pi)}{a_0, a_0 b_1 + a_1 b_0, a_1 b_1 \rhd c}\,(\mathrm{T})$$

Similarly, we obtain

$$b_0, a_0 b_1 + a_1 b_0, a_1 b_1 \rhd c$$

Together this yields

$$a_0 b_0, a_0 b_1 + a_1 b_0, a_1 b_1 \rhd c$$

Cf. [BV96] and the course material.

Nilpotent coefficients

Theorem

If $f = \sum a_i X^i$ is a unit in $\mathbf{R}[X]$, then a_i is nilpotent for $i \geqslant 1$.

A classical proof.

Suppose that $fg = 1$ in $\mathbf{R}[X]$. If $\mathbf{R}$ is a domain, then since

$$\deg(f) + \deg(g) = \deg(fg) = \deg(1) = 0$$

we see that $a_i = 0$ for $i \geqslant 1$. Thus, reducing modulo a generic prime $\mathfrak{p}$, we get

$$a_i \in \bigcap \operatorname{Spec}(\mathbf{R}) = \sqrt{0} \qquad \square$$

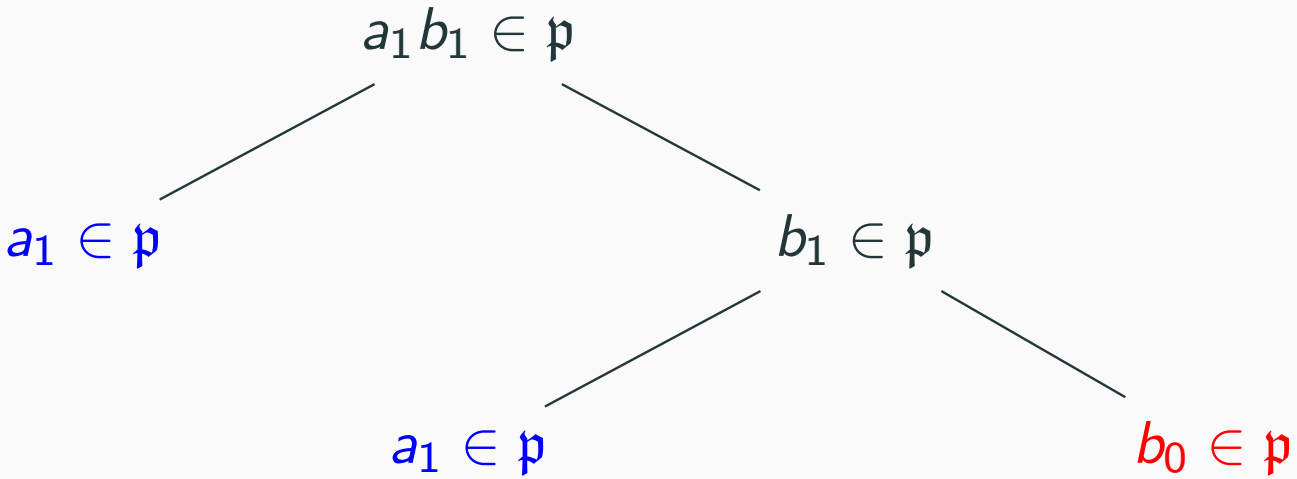
Example

Let $f = a_0 + a_1 X$ and $g = b_0 + b_1 X$.

Suppose that

$$a_0 b_0 = 1 \qquad a_0 b_1 + a_1 b_0 = 0 \qquad a_1 b_1 = 0$$

Consider a generic (proper) prime ideal $\mathfrak{p}$.



Upside down

The tree tells us how to infer $a_1 \in \sqrt{0}$:

$$\dfrac{\dfrac{}{\rhd a_1 b_1}\qquad \dfrac{\dfrac{}{a_1 \rhd a_1}\qquad \dfrac{\dfrac{\dfrac{}{a_1 \rhd a_1}\qquad \dfrac{}{b_0 \rhd a_1}}{a_1 b_0 \rhd a_1}}{b_1, a_0 b_1 + a_1 b_0 \rhd a_1 b_0}}{b_1, a_0 b_1 + a_1 b_0 \rhd a_1}}{\rhd a_0 b_1 + a_1 b_0}}{b_1 \rhd a_1}}{a_1 b_1 \rhd a_1}}{\rhd a_1}$$

Simple example:

$2X + 1 \in \mathbb{Z}/4\mathbb{Z}[X]$

See the course material for a proper discussion.

Yet another instance

Theorem (McAdam, Swan [MS04])
The following are equivalent.

1. $\mathbf{R}$ is reduced and **connected**, i.e., every idempotent is 0 or 1.
2. For all $f, g \in \mathbf{R}[X]$, if fg is monic, then the (formally) leading coefficients of f and g are units.

Again, this can be shown modulo a generic prime ideal.

A constructive argument is hidden in the classical proof!
Cf. [Wes18; Yen03]

Dedekind's Prague theorem

Theorem (Kronecker)

Let $a_0, \ldots, a_m, b_0, \ldots, b_n$ be indeterminates and let $\mathbf{R} = \mathbb{Z}[a_i, b_j]$. Put

$$c_k = \sum_{i+j=k} a_i b_j.$$

Then each element $a_i b_j$ is integral over the subring of $\mathbf{R}$ generated by $c_0, c_1, \ldots, c_{m+n}$.

- Direct consequence: **Dedekind's Prague theorem** [Edw90]
- Non-constructive proof is based on **valuations** (Bourbaki)
- Coquand and Persson have used (multi-conclusion) entailment relations [CP01; Coq09]

Conclusion

- Ideal objects like prime ideals and valuation rings can act as **useful fictions**.
- We can systematically keep track of certain identities by means of suitable **entailment relations**.
- The use of maximal ideals can be tackled with a similar **backtracking strategy**:

  Ihsen Yengui. "Making the use of maximal ideals constructive." In: *Theoret. Comput. Sci.* 392 (2008), pp. 174–178

Entailment relations

Entailment relations

Let S be a set, and let $\vdash \subseteq \mathrm{Fin}(S) \times \mathrm{Fin}(S)$.

$\vdash$ is an **entailment relation** if it is
reflexive, monotone, and transitive, i.e.,

$$\frac{U \between V}{U \vdash V}\,(\mathrm{R}) \qquad \frac{U \vdash V}{U, U' \vdash V, V'}\,(\mathrm{M}) \qquad \frac{U \vdash V, a \quad U', a \vdash V'}{U, U' \vdash V, V'}\,(\mathrm{T})$$

A **model** of $\vdash$ is a subset α of S which "splits entailments", i.e.,

$$\frac{\alpha \supseteq U \quad U \vdash V}{\alpha \between V}$$

Semantics

Let $\mathrm{Spec}(\vdash)$ denote the class of models of $\vdash$.

Completeness theorem* (Scott)
The following are equivalent.

1. $U \vdash V$
2. $\forall \alpha \in \mathrm{Spec}(\vdash)\,(\, U \subseteq \alpha \;\rightarrow\; \alpha \between V \,)$

N.B. Completeness implies excluded middle.
CT is classically equivalent to the prime ideal theorem.

Completeness necessitates classical logic

Let ψ be a bounded formula, let $S = \{ * \}$ be a singleton set.

$$\{ (S, S), (S, \emptyset) \} \cup \{ (\emptyset, S) : \psi \} \cup \{ (\emptyset, \emptyset) : \psi \}$$

is an entailment relation. Notice that

$$\vdash * \qquad \text{iff} \qquad \psi$$

Completeness implies

$$\{ a \in S : \forall \alpha \in \mathrm{Spec}(\vdash)(a \in \alpha) \} = \{ a \in S : \psi \}$$

and therefore

$$\psi \qquad \text{iff} \qquad \mathrm{Spec}(\vdash) = \emptyset$$

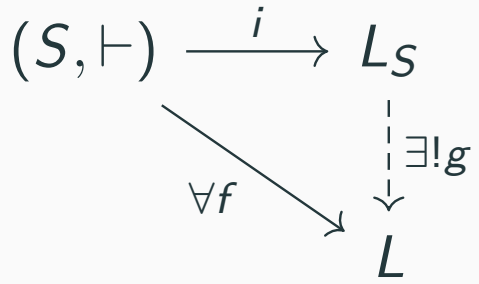
The fundamental theorem: constructive semantics

Theorem (Cederquist, Coquand)

Every entailment relation $(S, \vdash)$ generates a distributive lattice L_S with a map $i : S \to L_S$ such that

$$U \vdash V \qquad \text{if and only if} \qquad \bigwedge_{a \in U} i(a) \leqslant \bigvee_{b \in V} i(b)$$

This i is *universal* among interpretations in distributive lattices:



Example: Support of a ring

Consider the entailment relation of (proper) **prime ideal** of **R**.

$$\begin{aligned} \vdash 0 \qquad & D(0) = 1 \\ a \vdash ab \qquad & D(a) \leqslant D(ab) \\ a, b \vdash a + b \qquad & D(a) \wedge D(b) \leqslant D(a + b) \\ ab \vdash a, b \qquad & D(ab) \leqslant D(a) \vee D(b) \\ 1 \vdash \qquad & D(1) = 0 \end{aligned}$$

The fundamental theorem yields the **universal support** of **R**.
We get Joyal's lattice [Joy75]!

Vast applicability of entailment relations

- **Constructive algebra**
  e.g. dynamical methods (Coste, Lombardi, Roy, ...)
- **Proof theory**
  e.g. Szpilrajn's theorem (Negri, von Plato, Coquand)
- **Point-free topology**
  e.g. localic Hahn-Banach (Mulvey & Pelletier, Coquand)
- **Theoretical computer science**
  e.g. domain theory, resolution (Zhang & Rounds, Coquand)
- **Non-classical logic**
  e.g. many-valued logic (Scott)

Working with entailment relations

- Aim: capture and replace algebraic structures and their ideal objects syntactically
- Examples include ring spectra and several function spaces.
- We deal with inductively generated entailment relations.
- The task is to obtain a direct, non-inductive description.
- Often, the key is to understand the finite **inconsistent** subsets.

From inductive to non-inductive description: A case study around Hilbert's 17th problem

Sums of squares

Example (Motzkin 1967)

$$M(x,y) = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$$
$$= \frac{x^2y^2(x^2+y^2+1)(x^2+y^2-2)^2 + (x^2-y^2)^2}{(x^2+y^2)^2}$$

But $M(x,y)$ cannot be written as a sum of squares of polynomials.

Hilbert's 17th problem

Suppose that $f \in \mathbb{R}[x_1, \ldots, x_n]$ is nonnegative at all points of $\mathbb{R}^n$.

Is f a finite sum of squares of rational functions?

Artin gave an affirmative answer.

Sums of squares

"[Artin's] method was as remarkable as the result. It was perhaps the first triumph of what is sometimes called 'abstract' algebra."

Richard Brauer

Emil Artin

Key observation

The totally positive elements of a field are precisely the sums of squares.

Partially ordered rings

A partial order $\leqslant$ of a ring $\mathbf{R}$ is **compatible** if, for all $a, b, c \in \mathbf{R}$,

$$a \leqslant b \rightarrow a + c \leqslant b + c$$

$$0 \leqslant a \wedge 0 \leqslant b \rightarrow 0 \leqslant ab$$

Classical problems

Determine a linear extension

Describe the totally positive elements

Total orders as ideal elements

Let **R** be an **integral ring**, i.e., such that, for all $a \in R$,

$$a = 0 \ \vee \ \forall b \in \mathbf{R}\,(\, ab = 0 \ \rightarrow \ b = 0\,)$$

Let $\vdash$ be generated by all instances of

$$\begin{aligned} a, -a &\vdash \\ a, b &\vdash ab \\ a, b &\vdash a + b \\ &\vdash a, -a \qquad \text{for } a \neq 0 \end{aligned}$$

Total orders as ideal elements

Given $\alpha \in \mathrm{Spec}(\vdash)$, stipulate

$$a <_\alpha b \equiv b - a \in \alpha$$

This yields a compatible **strict** order which is total, i.e.,

$$\forall a, b \in \mathbf{R}\,(\, a \neq b \rightarrow a <_\alpha b \vee b <_\alpha a\,)$$

Conversely, every such $<$ has a **positive cone**

$$\alpha_< = \{\, a \in \mathbf{R} : 0 < a \,\} \in \mathrm{Spec}(\vdash)$$

Total orders as ideal elements

Proposition

Let $U \in \mathrm{Fin}(\mathbf{R})$. The following are equivalent.

1. $U \vdash$
2. There are $a_0, \ldots, a_n \in (U)$ and $x_0, \ldots, x_n \in \mathbf{R} \setminus \{0\}$ s.t.

$$\sum_{i=0}^{n} a_i x_i^2 = 0$$

where (U) is the multiplicative monoid generated by U.

Total orders as ideal elements

Proof sketch.

Abbreviate the second item by $\mathsf{Inc}(U)$.

Show that

(i) $\mathsf{Inc}(U)$ implies $U \vdash$

(ii) Inc is monotone, i.e., if $\mathsf{Inc}(U)$ and $U \subseteq V$, then $\mathsf{Inc}(V)$

(iii)

$$\frac{U \vdash V \quad \forall b \in V\, \mathsf{Inc}(W, b)}{\mathsf{Inc}(U, W)}$$

□

Remarks

In (iii) it suffices to consider initial entailments.

This is a general strategy for describing the inconsistent subsets.

Sums of squares

Corollary

Let $\mathbf{K}$ be a (non-trivial) discrete field.

The following are equivalent.

1. $\vdash$ **collapses**, i.e., $\emptyset \vdash \emptyset$
2. -1 is a sum of squares

Corollary

Let $\mathbf{K}$ be a discrete formally real field and let $0 \neq a \in \mathbf{K}$.

The following are equivalent.

1. a is **totally positive**, i.e., $\vdash a$
2. a is a sum of squares.

Field extensions

Corollary

Let $\mathbf{K}$ be a **factorial field**, and let $f \in \mathbf{K}[X]$ be irreducible and of odd degree. Let $\vdash$ and $\vdash_f$ be the entailment relations of total order of $\mathbf{K}$ and $\mathbf{K}[X]/\langle f \rangle$, respectively.

Then $\vdash$ and $\vdash_f$ **collapse simultaneously**.

Classically, this means that every odd-degree extension of a formally real field is formally real.

Proof.

By induction on the degree of f, following the classical proof. $\square$

Perspectives

- Orderability criteria for groups.
  E.g., **Levi's theorem**: "An abelian group is orderable iff it is torsion-free" in terms of collapse.

- Ordered groups and topology.
  E.g., **Sikora's theorem**: "The space of compatible orders of $\mathbb{Z}^n$, where $n > 1$, is a Cantor space" by Stone duality.

- **Extendability** criteria for partial orders.
  E.g., Serre's theorem on extension of partial orders of fields.

- **Archimedean** property requires geometric sequents!

Generalized entailment relations

Generalized entailment relations

Let S be a set, and let $\vdash \subseteq \mathrm{Fin}(S) \times \mathrm{Pow}(S)$.

$\vdash$ is a **generalized entailment relation** if it is reflexive, monotone

$$\frac{U \between V}{U \vdash V}\,(\mathrm{R}) \qquad \frac{U' \supseteq_{\mathrm{f}} U \quad U \vdash V}{U' \vdash V}\,(\mathrm{M})$$

and transitive

$$\frac{U \vdash V \quad \forall b \in V\,(\,U', b \vdash W\,)}{U, U' \vdash W}\,(\mathrm{T})$$

Inductively generated entailment relations

An **axiom set** for $\vdash$ is given by a set-indexed family $(U_i, V_i)_{i \in I}$ of initial entailments.

Proposition (Cut elimination)
The relation $\vdash$ defined inductively by

$$\frac{U \between V}{U \vdash V}\,(\mathsf{R}) \qquad \frac{\forall b \in V_i\,(\,U, b \vdash W\,)}{U, U_i \vdash W}\,(\mathsf{T}_i)$$

is the least entailment relation to contain $(U_i, V_i)_{i \in I}$.

Example: discrete fields

Let $\mathbf{R}$ be a commutative ring with 1.

Consider on $\mathbf{R}$ the entailment relation generated by

$$\begin{aligned} &\vdash 0 \\ a &\vdash ab \\ a, b &\vdash a + b \\ &\vdash a, \{\, 1 - ab : b \in \mathbf{R} \,\} \end{aligned} \tag{f}$$

Axiom (f) captures the geometric sequent of **discrete field**:

$$\top \vdash x = 0 \;\; \mathbf{op} \;\; \exists y\, (xy = 1)$$

Example: discrete fields

Every discrete field is **reduced**, i.e.,

$$a^2 \vdash a$$

Indeed, for every $b \in \mathbf{R}$ notice that

$$a^2, 1 - ab \vdash a$$

is witnessed by $a^2 b + a(1 - ab) = a$. Now cut (f).

Geometric axioms can be used within **dynamical computations**. Cf. Michel Coste, Henri Lombardi, and Marie-Françoise Roy. "Dynamical method in algebra: Effective Nullstellensätze." In: *Ann. Pure Appl. Logic* 111.3 (2001), pp. 203–256.

Semantics, again

$\vdash$ is **complete** if, for all $(U, V) \in \mathrm{Fin}(S) \times \mathrm{Pow}(S)$,

$$\forall \alpha \in \mathrm{Spec}(\vdash)\,(\, U \subseteq \alpha \;\to\; \alpha \between V \,) \quad \text{implies} \quad U \vdash V.$$

Proposition[*]

1. Countably generated entailment relations are complete.
2. An entailment relation $\vdash$ is complete if

$$\frac{U \vdash V \quad \forall b \in V\, \mathrm{Fin}(W, b) \between \mathrm{Inc}}{\mathrm{Fin}(U, W) \between \mathrm{Inc}} \qquad \text{(P)}$$

   where $W \in \mathrm{Pow}(S)$ and $\mathrm{Inc} = \{\, U \in \mathrm{Fin}(S) : U \vdash \,\}$.
3. Every conventional entailment relation is complete.

Interpretation

An **interpretation**

$$i : (S, \vdash) \to (S', \vdash')$$

of entailment relations is given by a function $i : S \to S'$ such that

$$U \vdash V \qquad \text{implies} \qquad i(U) \vdash' i(V)$$

An interpretation is **conservative** if

$$i(U) \vdash' i(V) \qquad \text{implies} \qquad U \vdash V$$

and **weakly conservative** if

$$i(U) \vdash' \qquad \text{implies} \qquad U \vdash$$

Conservation

Every interpretation $i : (S, \vdash) \to (S', \vdash')$ induces a mapping of model classes:

$$i^{-1} : \mathrm{Spec}(\vdash') \to \mathrm{Spec}(\vdash), \quad \beta \to i^{-1}(\beta)$$

Proposition* ("Lying over")

Suppose that $\vdash$ is complete and $\vdash'$ satisfies (P).

The following are equivalent.

1. i is weakly conservative.
2. $\forall \alpha \in \mathrm{Spec}(\vdash)\, \exists \beta \in \mathrm{Spec}(\vdash')\, (\, \alpha \subseteq i^{-1}(\beta)\,)$.

Digression: formal spaces

Formal spaces from entailment relations

Let $\vdash$ be inductively generated by $(U_i, V_i)_{i \in I}$.
Consider

$$C : \mathrm{Fin}(S) \to \mathrm{Pow}(\mathrm{Pow}(\mathrm{Fin}(S)))$$
$$U \mapsto \bigcup_{i \in I} \{ \{ U \cup \{ b \} : b \in V_i \} : U_i \subseteq U \}$$

$(\mathrm{Fin}(S), \supseteq, C)$ is a **covering system**, i.e.,
for every $U \in \mathrm{Fin}(S)$ and $X \in C(U)$,

1. $X \subseteq \downarrow \{ U \}$
2. if $U' \supseteq U$, then there is $Y \in C(U')$ such that $Y \subseteq \downarrow X$.

Formal spaces from entailment relations

C gives way to an inductive definition:

$$\Phi = \{\, (X, U) : U \in \mathrm{Fin}(S) \text{ and } X \in C(U) \,\},$$

and, for $\mathcal{U} \in \mathrm{Pow}(\mathrm{Fin}(S))$, we put

$$\mathcal{A}\mathcal{U} = I(\Phi, \downarrow \mathcal{U}).$$

The operator $\mathcal{A}$ has the following properties:

1. $\downarrow (\mathcal{A}\mathcal{U}) \subseteq \mathcal{A}\mathcal{U}$
2. $\mathcal{U} \subseteq \mathcal{A}\mathcal{V}$ implies $\mathcal{A}\mathcal{U} \subseteq \mathcal{A}\mathcal{V}$
3. $\mathcal{A}\mathcal{U} \cap \mathcal{A}\mathcal{V} \subseteq \mathcal{A}(\mathcal{U} \downarrow \mathcal{V})$

Formal spaces from entailment relations

A subset $\mathcal{U} \subseteq \mathrm{Fin}(S)$ is $\mathcal{A}$-**saturated** if $\mathcal{U} = \mathcal{A}\mathcal{U}$.

The class $\mathrm{Sat}(\mathcal{A})$ of all $\mathcal{A}$-saturated subsets of $\mathrm{Fin}(S)$ is a **set-generated frame**, where

1. $\mathcal{A}\mathcal{U} \wedge \mathcal{A}\mathcal{U} = \mathcal{A}(\mathcal{U} \downarrow \mathcal{V})$
2. $\bigvee_{i \in I} \mathcal{A}\mathcal{U}_i = \mathcal{A}(\bigcup_{i \in I} \mathcal{U}_i)$

A set of generators is given by

$$\{\, \mathcal{A}\{\, U \,\} : U \in \mathrm{Fin}(S) \,\}$$

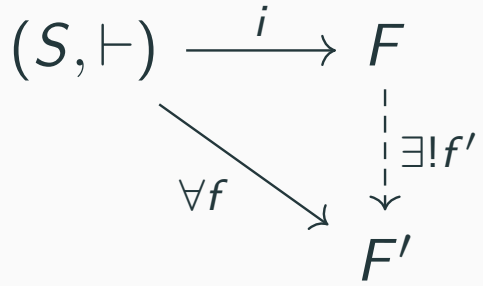
Formal spaces from entailment relations

Theorem

Let $\vdash$ be an inductively generated generalized entailment relation. There is a set-generated frame F together with a map $i : S \to F$ such that

$$U \vdash V \qquad \text{if and only if} \qquad \bigwedge_{a \in U} i(a) \leqslant \bigvee_{b \in V} i(b)$$

This i is universal among interpretations in frames:



Completely prime filters

Let $i : (S, \vdash) \to F$ be the universal interpretation.

1. If $\mathfrak{p}$ is a completely prime filter of F, then

$$i^{-1}(\mathfrak{p}) \in \mathrm{Spec}(\vdash).$$

2. If $\alpha \in \mathrm{Spec}(\vdash)$, then

$$\mathfrak{p}_\alpha = \left\{ x \in F : \exists U \in \mathsf{Fin}(\alpha) \bigwedge i(U) \leqslant x \right\}$$

is a completely prime filter such that $\alpha = i^{-1}(\mathfrak{p}_\alpha)$.

Cf. Thierry Coquand and Guo-Qiang Zhang. "Sequents, Frames, and Completeness". In: *Computer Science Logic. 14th International Workshop, CSL 2000 Annual Conference of the EACSL*. ed. by Helmut Schwichtenberg and Peter G. Clote.

Proper, prime, and maximal ideals

Spectra

Let **R** be a commutative ring with 1.

The entailment relation $\vdash$ of **proper ideal** of **R** is generated by:

$$1 \vdash$$

$$a \vdash ra$$

$$a, b \vdash a + b$$

$$\vdash 0$$

On top of $\vdash$ we can put axioms for **primality**

$$ab \vdash_{\mathfrak{p}} a, b$$

and **maximality**

$$\vdash_{\mathfrak{m}} a, \{\, 1 - ab : b \in \mathbf{R} \,\}$$

Proposition

The following are equivalent.

1. $U \vdash_{\mathfrak{m}} a_1, \ldots, a_k$
2. $a_1 \cdots a_k \in \mathrm{Jac}(\langle U \rangle)$

Proposition (Constructive PIT and Krull)

The inclusions

$$(\mathbf{R}, \vdash) \hookrightarrow (\mathbf{R}, \vdash_{\mathfrak{p}}) \hookrightarrow (\mathbf{R}, \vdash_{\mathfrak{m}})$$

are weakly conservative.

Spectra

Proposition

If $\mathbf{R}$ is discrete and non-trivial, then the following are equivalent.

1. $(\mathbf{R}, \vdash) \hookrightarrow (\mathbf{R}, \vdash_{\mathfrak{p}})$ is conservative.
2. $\mathbf{R}$ is a field.

Proposition

The following are equivalent.

1. $(\mathbf{R}, \vdash_{\mathfrak{p}}) \hookrightarrow (\mathbf{R}, \vdash_{\mathfrak{m}})$ is conservative.
2. $\operatorname{Kdim} \mathbf{R} \leqslant 0$, i.e., $\forall x \in \mathbf{R}\, \exists n \in \mathbb{N}\, \exists a \in \mathbf{R}\, (\, x^n = a x^{n+1}\,)$.

Primary ideals

The entailment relation $\vdash_{\mathfrak{p}'}$ of **primary ideal** of **R** is generated by the axioms of proper ideal together with

$$ab \vdash_{\mathfrak{p}'} a, \{\, b^n : n > 0 \,\}$$

Proposition

The following are equivalent.

1. **R** is reduced and $(\mathbf{R}, \vdash_{\mathfrak{p}'}) \hookrightarrow (\mathbf{R}, \vdash_{\mathfrak{m}})$ is conservative.
2. **R** is **von Neumann regular**, i.e., $\forall a \, \exists x \, (\, a = xa^2 \,)$.

("If **R** is absolutely flat, every primary ideal is maximal" [AM69, Ex. 4.3])

Minimal prime ideals

The entailment relation $\vdash$ of **proper prime filter** of **R** is dual to the entailment relation of proper prime ideal:

$$0 \vdash$$

$$ab \vdash a$$

$$a, b \vdash ab$$

$$a + b \vdash a, b$$

$$\vdash 1$$

On top of $\vdash$ we can put the axiom for **maximal** filter:

$$\vdash_{\mathfrak{m}} a, \mathrm{Ann}(a)$$

where $\mathrm{Ann}(a) = \{\, x \in \mathbf{R} : xa = 0 \,\}$.

Minimal prime ideals

Proposition (Coquand, Lombardi [CL06])

Suppose that $\mathbf{R}$ is reduced. The following are equivalent.

1. $a_1, \ldots, a_k \vdash_{\mathfrak{m}} b_1, \ldots, b_\ell$
2. $\mathrm{Ann}(b_1, \ldots, b_\ell) \subseteq \mathrm{Ann}(a_1 \cdots a_k)$.

Proposition

The following are equivalent.

1. $\mathbf{R}$ is von Neumann regular.
2. $\mathbf{R}$ is reduced and $(\mathbf{R}, \vdash) \hookrightarrow (\mathbf{R}, \vdash_{\mathfrak{m}})$ is conservative.
3. $\mathbf{R}$ is reduced and every prime ideal of $\mathbf{R}$ is minimal.* [Mat83]

Ideals in lattices

Let L be a distributive lattice.
Let $\vdash$ be the entailment relation of (proper) prime ideal of L,
let $\vdash_{\mathfrak{m}}$ extend $\vdash$ with the axiom of maximality.

Proposition
The following are equivalent.

1. $(L, \vdash) \hookrightarrow (L, \vdash_{\mathfrak{m}})$ is conservative.
2. L is Boolean.

This yields **Nachbin's theorem**:
A distributive lattice is Boolean if and only if
all of its prime ideals are maximal.*Cf. [Bel99]

Towards formal Baer criteria

Injective modules

Let M be an $\mathbf{R}$-module.

The following are classically equivalent:

1. Given any exact sequence

$$0 \to A \to B \to C \to 0$$

of $\mathbf{R}$-modules, the sequence

$$0 \to \mathrm{Hom}_{\mathbf{R}}(C, M) \to \mathrm{Hom}_{\mathbf{R}}(B, M) \to \mathrm{Hom}_{\mathbf{R}}(A, M) \to 0$$

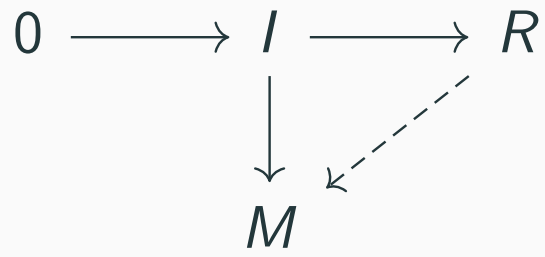
is exact.

2. M is a direct summand of every extension of itself.

Detecting injective modules

Baer's criterion*

A module is injective iff it is injective w.r.t. inclusions of ideals



Consequences*

Abelian groups are injective (as $\mathbb{Z}$-modules).

The category of **R**-modules has **enough injectives**.

Stepwise extension

Suppose that M is "ideal-injective" and let $A \subseteq B$.
Given $\varphi : A \to M$ and $b \in B$, consider the conductor ideal

$$I = \{\, r \in \mathbf{R} : rb \in A \,\}$$

and put

$$\mu : I \to M \quad r \mapsto \varphi(rb)$$

By assumption, there is

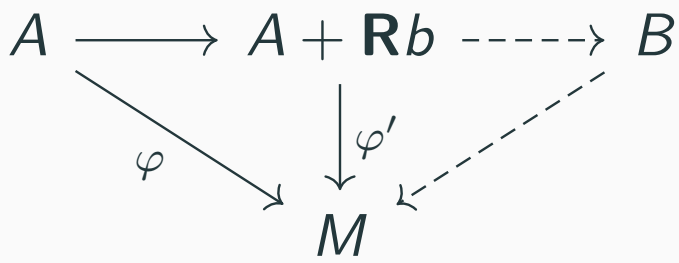
$$\nu : \mathbf{R} \to M \quad \text{s.t.} \quad \nu\big|_I = \mu$$

Now define

$$\varphi' : A + \mathbf{R}b \to M, \quad a + rb \mapsto \varphi(a) + \nu(r)$$

Stepwise extension

To "exhaust" B we need transfinite methods, e.g., Zorn's lemma.



A finite and de-Zornified version lives in the Upside Down!

Extension as conservation

"Regarding the theorem instead from a logical viewpoint, it is clear that the import of the assertion is that the extension of the theory of functionals on the subspace A to that on the seminormed space B is actually conservative [...]. In other words, no more may be proved about the subspace A in terms of functionals on the seminormed space B than may already be proved by considering only functionals on the subspace A."

C.J. Mulvey and J. Wick Pelletier

A globalization of the Hahn-Banach theorem

Hom-sets as spectra

Let $\vdash$ on $A \times M$ be generated by all instances of

$$\begin{aligned} (a, m), (a, m') &\vdash && (m \neq m') \\ (a, m), (b, n) &\vdash (ra + sb, rm + sn) \\ &\vdash (0_A, 0_M) \\ &\vdash \{\, (a, m) : m \in M \,\} \end{aligned}$$

Notice that

$$\mathrm{Spec}(\vdash) = \mathrm{Hom}_{\mathbf{R}}(A, M)$$

Every $\mu \in \mathrm{Hom}_{\mathbf{R}}(A, B)$ induces an interpretation

$$i_\varphi : (A \times M, \vdash) \to (B \times M, \vdash'), \quad (a, m) \to (\mu(a), m)$$

Towards formal Baer criteria: Dual spaces

Proposition

Suppose that $\mathbf{K}$ is a non-trivial discrete field.

Let A, B be $\mathbf{K}$-vector spaces, and $\mu \in \mathrm{Hom}_{\mathbf{K}}(A, B)$ be injective.

Then i_μ is weakly conservative.

Proof sketch.

Show that the following are equivalent:

1. $(a_1, m_1), \ldots, (a_k, m_k) \vdash$
2. There are $\lambda_1, \ldots, \lambda_k \in \mathbf{K}$ such that

$$\sum_{i=1}^{k} \lambda_i (a_i, m_i) = (0, 1). \qquad \square$$

Towards formal Baer criteria: Divisible modules

Proposition*

Let $A, B, M \in \mathbb{Z}\text{–Mod}$, and $\mu \in \mathrm{Hom}_{\mathbb{Z}}(A, B)$ be injective.
If M is divisible, then i_μ is weakly conservative.

Proof sketch.

Show that the following are equivalent:

1. $(a_1, m_1), \ldots, (a_k, m_k) \vdash$
2. There are $n_1, \ldots, n_k \in \mathbb{Z}$ and $0 \neq c \in M$ such that

$$\sum_{i=1}^{k} n_i(a_i, m_i) = (0, c).$$

□

More instances

Proposition*

Suppose that $\mathbf{R}$ is an integral ring.

Let $A, B, M \in \mathbf{R}\text{–}\mathrm{Mod}$ and $\mu \in \mathrm{Hom}_{\mathbf{R}}(A, B)$ be injective.

1. If M is torsion-free and divisible, then i_μ is weakly conservative.
2. If $\mathbf{R}$ is a Dedekind ring, and M is divisible, then i_μ is weakly conservative.

In both cases, the classical arguments reapply.

Conclusion

Conclusion

- We have made combined use of ideas and principles from proof theory and formal topology.

- Hilbert's programme, i.e., the constructive explanation of ideal objects, works for large parts of abstract algebra.

- Statements involving ideal objects are cases of make-believe—however, often we can do "as if", and gather computationally relevant information.

References

References I

[Acz06] Peter Aczel. "Aspects of general topology in constructive set theory". In: *Annals of Pure and Applied Logic* 137.1–3 (2006), pp. 3–29.

[AM69] M.F. Atiyah and I.G. MacDonald. *Introduction to Commutative Algebra*. Reading, MA: Addison-Wesley Publishing Company, Inc., 1969.

[Bel99] John L. Bell. "Boolean algebras and distributive lattices treated constructively". In: *Mathematical Logic Quarterly* 45.1 (1999), pp. 135–143.

References II

[Ber13] Benno van den Berg. "Non-deterministic inductive definitions". In: *Arch. Math. Logic* 52.1-2 (2013), pp. 113–135. ISSN: 0933-5846.

[BR87] Douglas Bridges and Fred Richman. *Varieties of Constructive Mathematics*. Vol. 97. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 1987.

[BV96] B. Banaschewski and J.J.C Vermeulen. "Polynomials and radical ideals". In: *Journal of Pure and Applied Algebra* 113.3 (1996), pp. 219–227.

References III

[CC00] Jan Cederquist and Thierry Coquand. “Entailment relations and distributive lattices”. In: *Logic Colloquium ’98. Proceedings of the Annual European Summer Meeting of the Association for Symbolic Logic, Prague, Czech Republic, August 9–15, 1998*. Ed. by Samuel R. Buss, Petr Hájek, and Pavel Pudlák. Vol. 13. Lect. Notes Logic. Natick, MA: A. K. Peters, 2000, pp. 127–139.

References IV

[CCN98] Jan Cederquist, Thierry Coquand, and Sara Negri. "The Hahn-Banach theorem in type theory". In: *Twenty-Five Years of Constructive Type Theory (Venice, 1995)*. Ed. by G. Sambin and J.M. Smith. Vol. 36. Oxford Logic Guides. Oxford University Press, New York, 1998, pp. 57–72.

[CL06] Thierry Coquand and Henri Lombardi. "A logical approach to abstract algebra". In: *Mathematical Structures in Computer Science* 16.5 (2006), pp. 885–900.

References V

[CLR01] Michel Coste, Henri Lombardi, and Marie-Françoise Roy. "Dynamical method in algebra: Effective Nullstellensätze." In: *Ann. Pure Appl. Logic* 111.3 (2001), pp. 203–256.

[Coq+03] Thierry Coquand et al. "Inductively generated formal topologies". In: *Ann. Pure Appl. Logic* 124 (2003), pp. 71–106.

[Coq06] Thierry Coquand. "Geometric Hahn-Banach theorem". In: *Math. Proc. Cambridge Philos. Soc.* 140 (2006), pp. 313–315.

[Coq09] Thierry Coquand. "Space of valuations". In: *Annals of Pure and Applied Logic* 157.2–3 (2009), pp. 97–109.

References VI

[CP01] Thierry Coquand and Henrik Persson. "Valuations and Dedekind's Prague theorem". In: *Journal of Pure and Applied Algebra* 155 (2001), pp. 121–129.

[CZ] Thierry Coquand and Guo-Qiang Zhang. "Sequents, Frames, and Completeness". In: *Computer Science Logic. 14th International Workshop, CSL 2000 Annual Conference of the EACSL*. Ed. by Helmut Schwichtenberg and Peter G. Clote.

[CZ00] Thierry Coquand and Guo-Qiang Zhang. "Sequents, frames, and completeness". In: *Computer Science Logic (Fischbachau, 2000)*. Ed. by Peter G. Clote and Helmut Schwichtenberg. Vol. 1862. Lecture Notes in Comput. Sci. Springer, Berlin, 2000, pp. 277–291.

References VII

[Edw90] Harold M. Edwards. *Divisor Theory*. Birkhäuser Boston, 1990.

[Eis04] David Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*. Vol. 150. Graduate Texts in Mathematics. Springer, 2004.

[FG82] Michael Fourman and Robin Grayson. "Formal spaces". In: *The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*. Vol. 110. Stud. Logic Found. Math. Amsterdam: North-Holland, 1982, pp. 107–122.

References VIII

[IN16] Hajime Ishihara and Takako Nemoto. "Non-Deterministic Inductive Definitions and Fullness". In: *Concepts of Proof in Mathematics, Philosophy, and Computer Science*. Ed. by D. Probst and P. Schuster. Vol. 6. Ontos Mathematical Logic. Berlin: Walter de Gruyter, 2016, pp. 163–170.

[Joy75] André Joyal. "Les théorèmes de Chevalley-Tarski". In: *Cahiers de topologie et géométrie différentielle catégoriques* 16.3 (1975), pp. 256–258.

[LQ15] Henri Lombardi and Claude Quitté. *Commutative Algebra: Constructive Methods. Finite Projective Modules*. Dordrecht: Springer Netherlands, 2015.

References IX

[M E05] Harold M. Edwards. *Essays in Constructive Mathematics*. New York: Springer, 2005.

[Mat83] Eben Matlis. "The minimal prime spectrum of a reduced ring". In: *Illinois Journal of Mathematics* 27.3 (1983), pp. 353–391.

[MRR88] Ray Mines, Fred Richman, and Wim Ruitenburg. *A Course in Constructive Algebra*. Universitext. New York: Springer-Verlag, 1988.

References X

[MS04] Stephen McAdam and Richard G. Swan. "Factorizations of Monic Polynomials". In: *Rings, Modules, Algebras, and Abelian Groups*. Ed. by Alberto Facchini, Evan Houston, and Luigi Salce. Vol. 236. Lecture Notes in Pure and Applied Mathematics. Marcel Dekker, 2004.

[MW91] Christopher J. Mulvey and Joan Wick-Pelletier. "A globalization of the Hahn-Banach theorem". In: *Advances in Mathematics* 89 (1991), pp. 1–59.

[NPC04] Sara Negri, Jan von Plato, and Thierry Coquand. "Proof-theoretical analysis of order relations". In: *Arch. Math. Logic* 43 (2004), pp. 297–309.

References XI

[Ric88] Fred Richman. "Nontrivial uses of trivial rings". In: *Proceedings of the American Mathematical Society* 103.4 (1988), pp. 1012–1014.

[Sam03] Giovanni Sambin. "Some points in formal topology". In: *Theoretical Computer Science* 305.1-3 (2003), pp. 347–408.

[Sco74] Dana Scott. "Completeness and axiomatizability in many-valued logic". In: *Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971)*. Ed. by Leon Henkin et al. Providence, RI: Amer. Math. Soc., 1974, pp. 411–435.

References XII

[VS72] P. Vámos and D. W. Sharpe. *Injective modules*. Cambridge Tracts in Mathematics and Mathematical Physics, 62. Cambridge University Press, 1972.

[Wes18] Daniel Wessel. *A note on connected reduced rings*. 2018.

[Yen03] Ihsen Yengui. "An algorithm for the divisors of monic polynomials over a commutative ring". In: *Mathematische Nachrichten* 260.1 (2003), pp. 93–99.

[Yen08] Ihsen Yengui. "Making the use of maximal ideals constructive." In: *Theoret. Comput. Sci.* 392 (2008), pp. 174–178.

References XIII

[Yen15] Ihsen Yengui. *Constructive commutative algebra. Projective modules over polynomial rings and dynamical Gröbner bases*. Vol. 2138. Lecture Notes in Mathematics. Cham: Springer, 2015.