Constructive Algebra

Daniel Wessel, Università di Verona, Dip.to di Informatica

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github.com/danielwessel/pc18

Introduction

“Reunion of broken parts”

- Kronecker vs. Dedekind: advent of ideal theory
- Passing from potential to actual infinity
- Rise of modern algebra
- Topology enters, Grothendieck revolutionizes
- Development of point-free methods
- Call for a revised Hilbert programme in abstract algebra

Varieties of constructive algebra

- Operative mathematics (Lorenzen)
- Bishop-style constructive algebra (Mines, Richman, Ruitenburg)
- Topos-valid algebra (Wraith, Banaschewski, Johnstone, …)
- Dynamical algebra (Lombardi, Roy, Coste, Yengui, …)
- Integrated development in type theory (Coquand, Persson)
- Formal topology $\between$ comm. algebra (Coquand, Schuster, …)

Key instruments

"What would have happened if topologies without *points had been discovered before topologies* with *points, or if Grothendieck had known the theory of distributive lattices?"*

G.-C. Rota
Indiscrete Thoughts

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> *Indiscrete Thoughts*

Modern constructive algebra makes extensive use of **lattice theory** (sometimes in disguise).

Important example
Constructive Krull dimension (Joyal, Lombardi)

Leitmotiv

“When we say that we have a constructive version of an abstract algebraic theorem, this means that we have a theorem the proof of which is constructive, which has a clear computational content, and from which we can recover the usual version of the abstract theorem by an immediate application of a well classified non-constructive principle.”

T. Coquand and H. Lombardi
Hidden constructions in abstract algebra

Outline

- Preliminaries
- Examples and counterexamples
- Constructive proofs from ideal objects
- Entailment relations
- Around Hilbert's 17th problem
- Abstract ideal theory in commutative rings
- Injective modules and Baer's criterion (w.i.p.)

Textbooks

- Ray Mines, Fred Richman, and Wim Ruitenburg. *A Course in Constructive Algebra*. Universitext. New York: Springer-Verlag, 1988
- Harold M. Edwards. *Essays in Constructive Mathematics*. New York: Springer, 2005
- Henri Lombardi and Claude Quitté. *Commutative Algebra: Constructive Methods. Finite Projective Modules*. Dordrecht: Springer Netherlands, 2015
- Ihsen Yengui. *Constructive commutative algebra. Projective modules over polynomial rings and dynamical Gröbner bases*. Vol. 2138. Lecture Notes in Mathematics. Cham: Springer, 2015

Rudiments of ring theory

Basic notions

We work in constructive set theory **CZF**.

A set S is said to be **discrete** if

$$\forall x, y \in S\,(\,x = y \vee x \neq y\,)$$

A subset T of S is **detachable** if

$$\forall x \in S\,(\,x \in T \vee x \notin T\,)$$

$\mathrm{Fin}(S)$ consists of the **finitely enumerable** subsets of S, i.e., $U \in \mathrm{Fin}(S)$ iff

$$\exists n \in \mathbb{N}\, \exists f\,(\,f : \{\,1, \ldots, n\,\} \twoheadrightarrow U\,)$$

Caveat: subsets of f.e. sets need not be f.e.!

Basic notions

Throughout, let $\mathbf{R}$ be a commutative ring with 1.
We will incur only few and basic concepts.

$\mathbf{R}$ is **integral** if, for all $a \in \mathbf{R}$,

$$a = 0 \lor \forall b \in \mathbf{R} \, (\, ab = 0 \to \; b = 0 \,)$$

$\mathbf{R}$ is a **discrete field** if, for all $a \in \mathbf{R}$,

$$a = 0 \lor \exists b \in \mathbf{R} \, (\, ab = 1 \,)$$

E.g., the trivial ring is a discrete field!
A discrete field $\mathbf{R}$ is a discrete set iff $1 =_{\mathbf{R}} 0$ is decidable.

Basic notions

Recall that an **ideal** I of $\mathbf{R}$ is an additive subgroup s.t. $RI \subseteq I$.

If $U \in \mathrm{Pow}(\mathbf{R})$, then $\langle U \rangle$ denotes the ideal **generated** by U.

An ideal I is **prime** if, for all $a, b \in \mathbf{R}$,

$$ab \in I \;\rightarrow\; a \in I \vee b \in I$$

and **maximal** if, for all $a \in \mathbf{R}$,

$$a \in I \;\vee\; \exists b \in \mathbf{R}\,(\,1 - ab \in I\,)$$

Every maximal ideal is prime.
I is maximal iff the quotient ring R/I is a discrete field.

Basic notions

Let I be an ideal of $\mathbf{R}$.

The **radical** of I is

$$\sqrt{I} = \{\, a \in \mathbf{R} : \exists n \in \mathbb{N} \,(\, a^n \in I \,)\, \}$$

The **Jacobson radical** of I is

$$\begin{aligned} \mathrm{Jac}(I) &= \{\, a \in \mathbf{R} : \forall b \, \exists c \,(\, 1 - (1 - ab)c \in I \,)\, \} \\ &= \{\, a \in \mathbf{R} : \forall b \,(\, 1 \in \langle a, b \rangle \;\rightarrow\; 1 \in \langle I, b \rangle \,)\, \} \end{aligned}$$

Note that

$$\sqrt{I} \subseteq \mathrm{Jac}(I)$$

Basic notions

Ideals can be captured by **inductive definitions**:

$$\frac{\emptyset}{0} \qquad \frac{\{\, a, b \,\}}{a+b}\, a, b \in \mathbf{R} \qquad \frac{\{\, a \,\}}{ab}\, a, b \in \mathbf{R}$$

Prime ideals appear as **non-deterministic** inductive definitions:

$$\frac{\emptyset}{\{\, 0 \,\}} \qquad \frac{\{\, a, b \,\}}{\{\, a+b \,\}} \qquad \frac{\{\, a \,\}}{\{\, ab \,\}} \qquad \frac{\{\, ab \,\}}{\{\, a, b \,\}} \qquad \frac{\{\, 1 \,\}}{\emptyset}$$

N.B. Under SGA the class $\mathrm{Spec}(\mathbf{R})$ of primes is **set-generated**; the class of minimal primes is a set, cf. [Ber13; IN16].

Limitations

Omniscience

LPO

For every binary sequence $(a_i)_{i \in \mathbb{N}}$ either

(a) there exists n such that $a_n = 1$, or

(b) $a_n = 0$ for every n.

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Ideals of $\mathbb{Z}$ need not be principal

Let $(a_i)_{i\in\mathbb{N}}$ be a binary sequence.
Consider the generated ideal

$$I = \langle\{\, a_i : i \in \mathbb{N} \,\}\rangle \subseteq \mathbb{Z}$$

Suppose that I has a single generator …

Omniscience

Free modules need not be projective [MRR88, Ex. II.4.10]

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Let $(a_i)_{i\in\mathbb{N}}$ be a binary sequence.
Let $I = \{0, 1\}/R$, where $xRy \equiv (x = y) \vee \exists n\,(a_n = 1)$.
Let $\mathbf{R} = \mathbb{Z}/2\mathbb{Z}$.

$$\begin{tikzcd} & \mathbf{R}^2 \arrow[d, two heads, "e"] \\ \mathbf{R}^{(I)} \arrow[ur, dashed, "f"] \arrow[r, "\mathrm{id}"'] & \mathbf{R}^{(I)} \end{tikzcd}$$

Suppose that $\mathbf{R}^{(I)}$ is projective, and that $ef = \mathrm{id}$.
However, $\mathbf{R}^2$ is discrete ...

Omniscience

LLPO

For every binary sequence $(a_i)_{i \in \mathbb{N}}$ that contains at most one 1, either

(a) $a_n = 0$ for all odd n, or

(b) $a_n = 0$ for all even n.

Omniscience

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Counterexamples

The zero ideal is not prime in $\mathbb{R}$.
Splitting fields need not be unique [BR87, Thm. 4.6].
"Summands need not be summands" [MRR88, Ex. II.4.6].

The Upside Down

Invariance of rank

Theorem*

Let $\mathbf{R}$ be non-trivial. If $\varphi : \mathbf{R}^m \to \mathbf{R}^n$ is surjective, then $m \geqslant n$.

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Proof.

Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{R}$, and consider the field $k = \mathbf{R}/\mathfrak{m}$. Tensoring yields an epimorphism of k-vector spaces

$$\mathrm{id}_k \otimes \varphi : \underbrace{k \otimes_{\mathbf{R}} \mathbf{R}^m}_{\cong k^m} \to \underbrace{k \otimes_{\mathbf{R}} \mathbf{R}^n}_{\cong k^n}$$

which implies $m \geqslant n$ (Steinitz exchange). $\square$

Invariance of rank, constructively

Theorem (Richman [Ric88])

If $\varphi : \mathbf{R}^m \to \mathbf{R}^n$ is surjective, and $m < n$, then $\mathbf{R}$ is trivial.

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Proof.
There is $\psi : \mathbf{R}^n \to \mathbf{R}^m$ such that $\varphi \circ \psi = \mathrm{id}_{\mathbf{R}^n}$.
Extend φ to $\mathbf{R}^n = \mathbf{R}^m \oplus \mathbf{R}^{n-m}$ by setting $\varphi(\mathbf{R}^{n-m}) = 0$.
View ψ as a map into $\mathbf{R}^n$.
If $A, B \in \mathrm{Mat}_n(\mathbf{R})$ are the matrices of φ and ψ, resp., then

$$1 = \det I_n = \det AB = (\det A)(\det B) = (\det A) \cdot 0 = 0 \qquad \square$$

Nakayama's lemma

Theorem*

Let $J \subseteq \bigcap \mathrm{Max}(\mathbf{R})$ be an ideal, and let M be a finitely generated $\mathbf{R}$-module. If $JM = M$, then $M = 0$.

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Proof sketch [Eis04].

(i) Show first that there is $a \in J$ for which $(1 - a)M = 0$. (Corollary to Cayley-Hamilton)

(ii) Every (proper) maximal ideal avoids $1 - a$ which thus is a unit. It follows that $M = 0$. $\square$

Nakayama's lemma, constructively

Solution: observe that (classically)

$$\bigcap \mathrm{Max}(\mathbf{R}) = \mathrm{Jac}(\mathbf{R})$$

and substitute the former for the latter.

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Theorem

Let $J \subseteq \mathrm{Jac}(\mathbf{R})$ be an ideal, and let M be a finitely generated $\mathbf{R}$-module. If $JM = M$, then $M = 0$.

Proof.

Step (i) above holds constructively [LQ15].

Now we know that $1 - a$ is a unit since $a \in \mathrm{Jac}(\mathbf{R})$. $\square$

Computational content from ideal elements: Gauß' Lemma and nilpotent coefficients

Content of polynomials

Theorem (Gauß' Lemma)
Let $f, g \in \mathbf{R}[X]$.

$$\mathrm{c}(f)\mathrm{c}(g) \subseteq \sqrt{\mathrm{c}(fg)}$$

where $\mathrm{c}(f)$ is the **content** of f, i.e., the ideal generated by the coefficients of f.

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A classical proof [Eis04].

"It is enough to show that if a prime ideal $\mathfrak{p}$ contains $\mathrm{c}(fg)$, then it contains $\mathrm{c}(f)\mathrm{c}(g)$. Factoring out $\mathfrak{p}$, we may assume that $\mathbf{R}$ is a domain and $\mathfrak{p}$ is 0, and we must show that if $fg = 0$, then $f = 0$ or $g = 0$. Since $\mathbf{R}$ is a now domain, this is obvious." $\square$

Example

Let $f = a_0 + a_1 X$ and $g = b_0 + b_1 X$.
Suppose that $\mathfrak{p}$ contains $\mathrm{c}(fg) = \langle a_0 b_0, a_0 b_1 + a_1 b_0, a_1 b_1 \rangle$.

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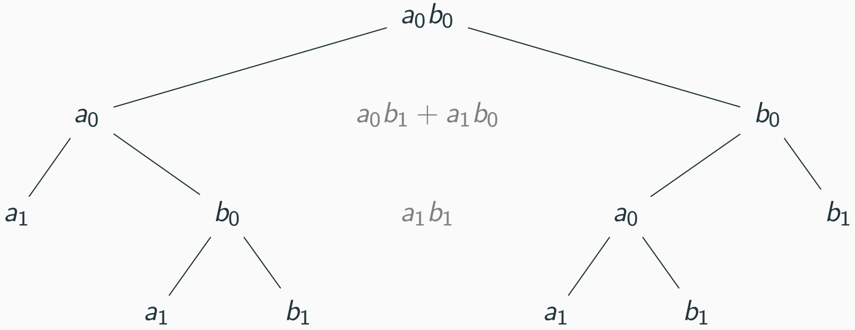

So $\mathfrak{p}$ contains $\mathrm{c}(f)\mathrm{c}(g) = \langle a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1 \rangle$.

Example

Now suppose that

$$c \in \mathrm{c}(f)\mathrm{c}(g)$$

It is immediate that

$$c \in \langle a_0, a_1 \rangle \cap \langle b_0, b_1 \rangle$$

We want to obtain a witness for

$$c \in \sqrt{\langle a_0 b_0, a_0 b_1 + a_1 b_0, a_1 b_1 \rangle}$$

$\boxed{\text{The tree is an instruction!}}$

Entailment

For $a \in \mathbf{R}$ and $U \in \mathrm{Fin}(\mathbf{R})$ write

$$U \rhd a \;\equiv\; a \in \sqrt{\langle U \rangle}$$

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This $\rhd$ is a (single-conclusion) **entailment relation**:

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \qquad \frac{U \rhd a}{U, V \rhd a}\ (\mathrm{M}) \qquad \frac{U \rhd b \quad V, b \rhd a}{U, V \rhd a}\ (\mathrm{T})$$

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N.B. (later)

1. This $\rhd$ is **inductively generated**.
2. The **models** of $\rhd$ precisely are the radical ideals of $\mathbf{R}$.

Entailment

In addition:

$$\frac{U, a \rhd c \quad V, b \rhd c}{U, V, ab \rhd c}\ (\pi)$$

Because if

$$c^n = u + ra \qquad \text{and} \qquad c^m = v + sb$$

for certain $n, m \in \mathbb{N}, u \in \langle U \rangle, v \in \langle V \rangle, r, s \in \mathbf{R}$, then

$$c^{n+m} = uv + sbu + rav + rsab \in \langle U, V, ab \rangle$$

Upside down

Following the left branch bottom-up:

$$\dfrac{a_0, a_0 b_1 + a_1 b_0 \rhd a_1 b_0 \qquad \dfrac{a_0, a_1 \rhd c \qquad \dfrac{a_0, a_1 \rhd c \quad b_0, b_1 \rhd c}{a_0, b_0, a_1 b_1 \rhd c}\,(\pi)}{a_0, a_1 b_0, a_1 b_1 \rhd c}\,(\pi)}{a_0, a_0 b_1 + a_1 b_0, a_1 b_1 \rhd c}\,(\mathrm{T})$$

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Similarly, we obtain

$$b_0, a_0 b_1 + a_1 b_0, a_1 b_1 \rhd c$$

Together this yields

$$a_0 b_0, a_0 b_1 + a_1 b_0, a_1 b_1 \rhd c$$

Cf. [BV96] and the course material.

Nilpotent coefficients

Theorem

If $f = \sum a_i X^i$ is a unit in $\mathbf{R}[X]$, then a_i is nilpotent for $i \geqslant 1$.

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A classical proof.

Suppose that $fg = 1$ in $\mathbf{R}[X]$. If $\mathbf{R}$ is a domain, then since

$$\deg(f) + \deg(g) = \deg(fg) = \deg(1) = 0$$

we see that $a_i = 0$ for $i \geqslant 1$. Thus, reducing modulo a generic prime $\mathfrak{p}$, we get

$$a_i \in \bigcap \mathrm{Spec}(\mathbf{R}) = \sqrt{0}$$

□

Example

Let $f = a_0 + a_1 X$ and $g = b_0 + b_1 X$.
Suppose that

$$a_0 b_0 = 1 \qquad a_0 b_1 + a_1 b_0 = 0 \qquad a_1 b_1 = 0$$

Consider a generic (proper) prime ideal $\mathfrak{p}$.

$$\begin{tikzcd}
 & a_1 b_1 \in \mathfrak{p} \arrow[dl, dash] \arrow[dr, dash] & & \\
{\color{blue} a_1 \in \mathfrak{p}} & & b_1 \in \mathfrak{p} \arrow[dl, dash] \arrow[dr, dash] & \\
 & {\color{blue} a_1 \in \mathfrak{p}} & & {\color{red} b_0 \in \mathfrak{p}}
\end{tikzcd}$$

Upside down

The tree tells us how to infer $a_1 \in \sqrt{0}$:

$$
\frac{\rhd a_1 b_1 \qquad \dfrac{a_1 \rhd a_1 \qquad \dfrac{\rhd a_0 b_1 + a_1 b_0 \qquad \dfrac{b_1, a_0 b_1 + a_1 b_0 \rhd a_1 b_0 \qquad \dfrac{a_1 \rhd a_1 \qquad b_0 \rhd a_1}{a_1 b_0 \rhd a_1}}{b_1, a_0 b_1 + a_1 b_0 \rhd a_1}}{b_1 \rhd a_1}}{a_1 b_1 \rhd a_1}}{\rhd a_1}
$$

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$$

Simple example:

$2X + 1 \in \mathbb{Z}/4\mathbb{Z}[X]$

See the course material for a proper discussion.

Yet another instance

Theorem (McAdam, Swan [MS04])
The following are equivalent.

1. **R** is reduced and **connected**, i.e., every idempotent is 0 or 1.
2. For all $f, g \in \mathbf{R}[X]$, if fg is monic, then the (formally) leading coefficients of f and g are units.

Again, this can be shown modulo a generic prime ideal.

A constructive argument is hidden in the classical proof [Yen03].

Theorem (Kronecker)

Let $a_0, \ldots, a_m, b_0, \ldots, b_n$ be indeterminates and let $\mathbf{R} = \mathbb{Z}[a_0, \ldots, a_m, b_1, \ldots, b_n]$. Put

$$c_k = \sum_{i+j=k} a_i b_j.$$

Then each element $a_i b_j$ is integral over the subring of $\mathbf{R}$ generated by $c_0, c_1, \ldots, c_{m+n}$.

- Direct consequence: **Dedekind's Prague theorem** [Edw90]
- Non-constructive proof is based on **valuations** (Bourbaki)
- Coquand and Persson have used (multi-conclusion) entailment relations [CP01; Coq09]

Summary

- Ideal objects like prime ideals and valuation rings can act as **useful fictions**.
- We can systematically keep track of certain identities by means of suitable **entailment relations**.
- The use of maximal ideals can be tackled with a similar **backtracking strategy**:

  Ihsen Yengui. "Making the use of maximal ideals constructive." In: *Theoret. Comput. Sci.* 392 (2008), pp. 174–178

Entailment relations

Entailment relations

Let S be a set, and let $\vdash \subseteq \mathrm{Fin}(S) \times \mathrm{Fin}(S)$.

$\vdash$ is an **entailment relation** if it is
reflexive, monotone, and transitive, i.e.,

$$\frac{U \between V}{U \vdash V}\,(\mathrm{R}) \qquad \frac{U \vdash V}{U, U' \vdash V, V'}\,(\mathrm{M}) \qquad \frac{U \vdash V, a \quad U, a \vdash V}{U \vdash V}\,(\mathrm{T})$$

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A **model** of $\vdash$ is a subset α of S which “splits entailments”, i.e.,

$$\frac{\alpha \supseteq U \quad U \vdash V}{\alpha \between V}$$

Semantics

Let $\mathrm{Spec}(\vdash)$ denote the class of models of $\vdash$.

Completeness theorem* (Scott)
The following are equivalent.

1. $U \vdash V$
2. $\forall \alpha \in \mathrm{Spec}(\vdash)\,(\,U \subseteq \alpha \;\to\; \alpha \between V\,)$

N.B.
Completeness implies excluded middle.
CT is classically equivalent to the prime ideal theorem.

The fundamental theorem: constructive semantics

Theorem (Cederquist, Coquand)

Every entailment relation $(S, \vdash)$ generates a distributive lattice L_S with a map $i : S \to L_S$ such that

$$U \vdash V \qquad \text{if and only if} \qquad \bigwedge_{a \in U} i(a) \leqslant \bigvee_{b \in V} i(b)$$

This i is *universal* among interpretations in distributive lattices:

$$\begin{tikzcd} (S, \vdash) \arrow[r, "i"] \arrow[dr, "\forall f"'] & L_S \arrow[d, dashed, "\exists ! g"] \\ & L \end{tikzcd}$$

Example: support of a ring

Consider the entailment relation of (proper) **prime ideal** of $\mathbf{R}$.

$$
\begin{aligned}
&\vdash 0 \\
a &\vdash ab \\
a, b &\vdash a + b \\
ab &\vdash a, b \\
1 &\vdash
\end{aligned}
$$

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\begin{aligned}
&\vdash 0 &\qquad\qquad& D(0) = 1 \\
a &\vdash ab && D(a) \leqslant D(ab) \\
a, b &\vdash a + b && D(a) \wedge D(b) \leqslant D(a+b) \\
ab &\vdash a, b && D(ab) \leqslant D(a) \vee D(b) \\
1 &\vdash && D(1) = 0
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1 \vdash & D(1) = 0
\end{array}
$$

We get Joyal's lattice (a "notion of zero") [Joy75].
The dual of $\vdash$ yields the **universal support** on **R**.

Vast applicability of entailment relations

- **Constructive algebra**
  e.g. Point-free spectra (Joyal, Coquand)
- **Proof theory**
  e.g. Szpilrajn's theorem (Negri, von Plato, Coquand)
- **Point-free topology**
  e.g. localic Hahn-Banach (Mulvey–Pelletier, Coquand)
- **Theoretical computer science**
  e.g. domain theory, resolution (Zhang–Rounds, Coquand)
- **Non-classical logic**
  e.g. many-valued logic (Scott)

Around Hilbert's 17th problem

Sums of squares

Example (Motzkin 1967)

$$\begin{aligned} M(x,y) &= x^4y^2 + x^2y^4 + 1 - 3x^2y^2 \\ &= \frac{x^2y^2(x^2+y^2+1)(x^2+y^2-2)^2 + (x^2-y^2)^2}{(x^2+y^2)^2} \end{aligned}$$

But $M(x,y)$ cannot be written as a sum of squares of polynomials.

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But $M(x,y)$ cannot be written as a sum of squares of polynomials.

Hilbert's 17th problem
Suppose that $f \in \mathbb{R}[x_1,\ldots,x_n]$ is nonnegative at all points of $\mathbb{R}^n$.
Is f a finite sum of squares of rational functions?

Artin gave an affirmative answer.

Sums of squares

> *"[Artin's] method was as remarkable as the result. It was perhaps the first triumph of what is sometimes called 'abstract' algebra."*

Richard Brauer
Emil Artin

Artin's key observation
The totally positive elements of a field
are precisely the sums of squares.

Partially ordered rings

A partial order $\leqslant$ of a ring $\mathbf{R}$ is **compatible** if, for all $a, b, c \in \mathbf{R}$,

$$a \leqslant b \;\rightarrow\; a + c \leqslant b + c$$

$$0 \leqslant a \wedge 0 \leqslant b \;\rightarrow\; 0 \leqslant ab$$

Classical problems

Determine a linear extension

Describe the totally positive elements

Total orders as ideal elements

Let **R** be an **integral ring**, i.e., such that, for all $a \in \mathbf{R}$,

$$a = 0 \lor \forall b \in \mathbf{R}\,(\, ab = 0 \;\to\; b = 0\,)$$

Let $\vdash$ be generated by all instances of

$$\begin{aligned} a, -a &\vdash \\ a, b &\vdash ab \\ a, b &\vdash a + b \\ &\vdash a, -a \qquad \text{for } a \neq 0 \end{aligned}$$

Total orders as ideal elements

Given $\alpha \in \mathrm{Spec}(\vdash)$, stipulate

$$a <_\alpha b \equiv b - a \in \alpha$$

This yields a compatible **strict** order which is total, i.e.,

$$\forall a, b \in \mathbf{R} \, (\, a \neq b \rightarrow a <_\alpha b \vee b <_\alpha a \,)$$

Conversely, every such $<$ has a **positive cone**

$$\alpha_< = \{ \, a \in \mathbf{R} : 0 < a \, \} \in \mathrm{Spec}(\vdash)$$

Total orders as ideal elements

Proposition

Let $U \in \mathrm{Fin}(\mathbf{R})$. The following are equivalent.

1. $U \vdash$
2. There are $a_0, \ldots, a_n \in (U)$ and $x_0, \ldots, x_n \in \mathbf{R} \setminus \{ 0 \}$ s.t.

$$\sum_{i=0}^{n} a_i x_i^2 = 0$$

where (U) is the multiplicative monoid generated by U.

Total orders as ideal elements

Proof sketch.

Abbreviate the second item by $\mathrm{Inc}(U)$.

Show that

(i) $\mathrm{Inc}(U)$ implies $U \vdash$

(ii) Inc is monotone, i.e., if $\mathrm{Inc}(U)$ and $U \subseteq V$, then $\mathrm{Inc}(V)$

(iii)

$$\frac{U \vdash V \quad \forall b \in V \, \mathrm{Inc}(W, b)}{\mathrm{Inc}(U, W)}$$

$\square$

<u>Remarks</u>

In (iii) it suffices to consider initial entailments.

This is a general strategy for describing the inconsistent subsets.

Sums of squares

Corollary

Let $\mathbf{K}$ be a (non-trivial) discrete field.
The following are equivalent.

1. $\vdash$ **collapses**, i.e., $\emptyset \vdash \emptyset$
2. -1 is a sum of squares

Corollary

Let $\mathbf{K}$ be a (non-trivial) discrete field.
The following are equivalent.

1. $\vdash$ **collapses**, i.e., $\emptyset \vdash \emptyset$
2. -1 is a sum of squares

Corollary

Let $\mathbf{K}$ be a discrete formally real field and let $0 \neq a \in \mathbf{K}$.
The following are equivalent.

1. a is **totally positive**, i.e., $\vdash a$
2. a is a sum of squares.

Corollary

Let $\mathbf{K}$ be a **factorial field**, and let $f \in \mathbf{K}[X]$ be irreducible and of odd degree. Let $\vdash$ and $\vdash_f$ be the entailment relations of total order of $\mathbf{K}$ and $\mathbf{K}[X]/\langle f \rangle$, respectively.

Then $\vdash$ and $\vdash_f$ **collapse simultaneously**.

Classically, this means that every odd-degree extension of a formally real field is formally real.

Proof.

By induction on the degree of f, following the classical proof. $\square$

Perspectives

- Orderability criteria for groups.
  E.g., **Levi's theorem**: "An abelian group is orderable iff it is torsion-free" in terms of collapse.

- Ordered groups and topology.
  E.g., **Sikora's theorem**: "The space of compatible orders of $\mathbb{Z}^n$, where $n > 1$, is a Cantor space" by Stone duality.

- **Extendability** criteria for partial orders.
  E.g., Serre's theorem on extension of partial orders of fields.

- **Archimedean** property requires geometric sequents!

Generalized entailment relations

Generalized entailment relations

Let S be a set, and let $\vdash \subseteq \mathrm{Fin}(S) \times \mathrm{Pow}(S)$.

$\vdash$ is a **generalized entailment relation** if it is reflexive

$$\frac{U \between V}{U \vdash V}\,(\mathrm{R})$$

and transitive

$$\frac{U \vdash V \qquad \forall b \in V\,(\,U', b \vdash W\,)}{U, U' \vdash W}\,(\mathrm{T})$$

Note that monotonicity (M) is a consequence of (R) and (T).

Inductively generated entailment relations

An **axiom set** for $\vdash$ is given by a set-indexed family $(U_i, V_i)_{i \in I}$ of initial entailments.

Proposition (Cut elimination)

The relation $\vdash$ defined inductively by

$$\frac{U \between V}{U \vdash V}\,(\mathsf{R}) \qquad \frac{\forall b \in V_i\,(\,U, b \vdash W\,)}{U, U_i \vdash W}\,(\mathsf{T}_i)$$

is the least entailment relation to contain $(U_i, V_i)_{i \in I}$.

Example: discrete fields

Let $\mathbf{R}$ be a commutative ring with 1.
Consider on $\mathbf{R}$ the entailment relation generated by

$$
\begin{aligned}
&\vdash 0 \\
a &\vdash ab \\
a, b &\vdash a + b \\
&\vdash a, \{\, 1 - ab : b \in \mathbf{R} \,\} \qquad\qquad \text{(f)}
\end{aligned}
$$

Axiom (f) captures the geometric sequent of **discrete field**:

$$\top \;\vdash\; x = 0 \;\; \textbf{op} \;\; \exists y \, (xy = 1)$$

Geometric axioms can be used within **dynamical proofs** [CLR01].

Example: discrete fields

A discrete field is **without zerodivisors**, i.e.,

$$ab \vdash a, b$$

Indeed, for every $x \in \mathbf{R}$ notice that

$$ab, 1 - ax \vdash b$$

is witnessed by $xab + b(1 - ax) = b$. Now apply (T) with (f).

Notice the special case of **reducedness**, i.e.,

$$a^2 \vdash a$$

Semantics, again

$\vdash$ is **complete** if, for all $(U, V) \in \mathrm{Fin}(S) \times \mathrm{Pow}(S)$,

$$\forall \alpha \in \mathrm{Spec}(\vdash)\,(\, U \subseteq \alpha \;\to\; \alpha \between V\,) \quad \text{implies} \quad U \vdash V.$$

Semantics, again

$\vdash$ is **complete** if, for all $(U, V) \in \mathsf{Fin}(S) \times \mathsf{Pow}(S)$,

$$\forall \alpha \in \mathrm{Spec}(\vdash)\,(\,U \subseteq \alpha \,\rightarrow\, \alpha \between V\,) \quad \text{implies} \quad U \vdash V.$$

Proposition*

1. Countably generated entailment relations are complete.
2. An entailment relation $\vdash$ is complete if

$$\frac{U \vdash V \quad \forall b \in V\, \mathsf{Fin}(W, b) \between \mathsf{Inc}}{\mathsf{Fin}(U, W) \between \mathsf{Inc}} \tag{P}$$

   where $W \in \mathsf{Pow}(S)$ and $\mathsf{Inc} = \{\, U \in \mathsf{Fin}(S) : U \vdash \,\}$.

3. Every conventional entailment relation is complete.

Interpretation

An **interpretation**

$$i : (S, \vdash) \to (S', \vdash')$$

of entailment relations is given by a function $i : S \to S'$ such that

$$U \vdash V \qquad \text{implies} \qquad i(U) \vdash' i(V)$$

An interpretation is **conservative** if

$$i(U) \vdash' i(V) \qquad \text{implies} \qquad U \vdash V$$

and **weakly conservative** if

$$i(U) \vdash' \qquad \text{implies} \qquad U \vdash$$

Conservation

Every interpretation $i : (S, \vdash) \to (S', \vdash')$ induces a mapping of model classes:

$$i^{-1} : \mathrm{Spec}(\vdash') \to \mathrm{Spec}(\vdash), \quad \beta \to i^{-1}(\beta)$$

Conservation

Every interpretation $i : (S, \vdash) \to (S', \vdash')$ induces a mapping of model classes:

$$i^{-1} : \mathrm{Spec}(\vdash') \to \mathrm{Spec}(\vdash), \quad \beta \to i^{-1}(\beta)$$

Proposition* ("Lying over")
Suppose that $\vdash$ is complete and $\vdash'$ satisfies (P).
The following are equivalent.

1. i is weakly conservative.
2. $\forall \alpha \in \mathrm{Spec}(\vdash)\, \exists \beta \in \mathrm{Spec}(\vdash')\, (\, \alpha \subseteq i^{-1}(\beta)\,)$.

Digression: formal spaces

Formal spaces from entailment relations

Let $\vdash$ be inductively generated by $(U_i, V_i)_{i \in I}$.
Consider

$$\begin{aligned} C : \mathrm{Fin}(S) &\to \mathrm{Pow}(\mathrm{Pow}(\mathrm{Fin}(S))) \\ U &\mapsto \bigcup_{i \in I} \{\, \{\, U \cup \{\, b \,\} : b \in V_i \,\} : U_i \subseteq U \,\} \end{aligned}$$

Let $\vdash$ be inductively generated by $(U_i, V_i)_{i \in I}$.
Consider

$$\begin{aligned} C : \mathrm{Fin}(S) &\to \mathrm{Pow}(\mathrm{Pow}(\mathrm{Fin}(S))) \\ U &\mapsto \bigcup_{i \in I} \{\, \{\, U \cup \{\, b \,\} : b \in V_i \,\} : U_i \subseteq U \,\} \end{aligned}$$

$(\mathrm{Fin}(S), \supseteq, C)$ is a **covering system**, i.e.,
for every $U \in \mathrm{Fin}(S)$ and $X \in C(U)$,

1. $X \subseteq {\downarrow}\{\, U \,\}$
2. if $U' \supseteq U$, then there is $Y \in C(U')$ such that $Y \subseteq {\downarrow}X$.

Formal spaces from entailment relations

C gives way to an inductive definition:

$$\Phi = \{\, (X, U) : U \in \mathrm{Fin}(S) \text{ and } X \in C(U) \,\},$$

and, for $\mathcal{U} \in \mathrm{Pow}(\mathrm{Fin}(S))$, we put

$$\mathcal{A}\mathcal{U} = I(\Phi, {\downarrow}\mathcal{U}).$$

Formal spaces from entailment relations

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and, for $\mathcal{U} \in \mathrm{Pow}(\mathrm{Fin}(S))$, we put

$$\mathcal{A}\mathcal{U} = I(\Phi, {\downarrow}\mathcal{U}).$$

The operator $\mathcal{A}$ has the following properties:

1. ${\downarrow}\mathcal{A}\mathcal{U} \subseteq \mathcal{A}\mathcal{U}$
2. $\mathcal{U} \subseteq \mathcal{A}\mathcal{V}$ implies $\mathcal{A}\mathcal{U} \subseteq \mathcal{A}\mathcal{V}$
3. $\mathcal{A}\mathcal{U} \cap \mathcal{A}\mathcal{V} \subseteq \mathcal{A}(\mathcal{U}{\downarrow}\mathcal{V})$

Formal spaces from entailment relations

A subset $\mathcal{U} \subseteq \mathsf{Fin}(S)$ is $\mathcal{A}$-**saturated** if $\mathcal{U} = \mathcal{A}\mathcal{U}$.

The class $\mathsf{Sat}(\mathcal{A})$ of all $\mathcal{A}$-saturated subsets of $\mathsf{Fin}(S)$ is a **set-generated frame**, where

1. $\mathcal{A}\mathcal{U} \wedge \mathcal{A}\mathcal{U} = \mathcal{A}(\mathcal{U}{\downarrow}\mathcal{V})$
2. $\bigvee_{i \in I} \mathcal{A}\mathcal{U}_i = \mathcal{A}(\bigcup_{i \in I} \mathcal{U}_i)$

A set of generators is given by

$$\{ \mathcal{A}\{ U \} : U \in \mathsf{Fin}(S) \}$$

Formal spaces from entailment relations

Theorem (CZF$^+$)
Let $\vdash$ be an inductively generated generalized entailment relation. There is a set-generated frame F together with a map $i : S \to F$ such that

$$U \vdash V \qquad \text{if and only if} \qquad \bigwedge_{a \in U} i(a) \leqslant \bigvee_{b \in V} i(b)$$

This i is universal among interpretations in frames:

$$\begin{tikzcd} (S, \vdash) \arrow[r, "i"] \arrow[dr, "\forall f"'] & F \arrow[d, dashed, "\exists! f'"] \\ & F' \end{tikzcd}$$

Completely prime filters

Let $i : (S, \vdash) \to F$ be the universal interpretation.

1. If $\mathfrak{p}$ is a completely prime filter of F, then

$$i^{-1}(\mathfrak{p}) \in \mathrm{Spec}(\vdash).$$

2. If $\alpha \in \mathrm{Spec}(\vdash)$, then

$$\mathfrak{p}_\alpha = \left\{ x \in F : \exists U \in \mathsf{Fin}(\alpha) \bigwedge i(U) \leqslant x \right\}$$

is a completely prime filter such that $\alpha = i^{-1}(\mathfrak{p}_\alpha)$.

Cf. Thierry Coquand and Guo-Qiang Zhang. "Sequents, Frames, and Completeness". In: *Computer Science Logic. 14th International Workshop, CSL 2000 Annual Conference of the EACSL*. ed. by Helmut Schwichtenberg and Peter G. Clote.

Proper, prime, and maximal ideals

Spectra

Let **R** be a commutative ring with 1.

The entailment relation $\vdash$ of **proper ideal** of **R** is generated by:

$$\vdash 0$$

$$a \vdash ab$$

$$a, b \vdash a + b$$

$$1 \vdash$$

On top of $\vdash$ we can put axioms for **primality**

$$ab \vdash_{\mathfrak{p}} a, b$$

and **maximality**

$$\vdash_{\mathfrak{m}} a, \{\, 1 - ab : b \in \mathbf{R} \,\}$$

Proposition

The following are equivalent.

1. $U \vdash_{\mathfrak{m}} a_1, \ldots, a_k$
2. $a_1 \cdots a_k \in \mathrm{Jac}(\langle U \rangle)$

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2. $a_1 \cdots a_k \in \mathrm{Jac}(\langle U \rangle)$

Proposition (Constructive PIT and Krull)

The inclusions

$$(\mathbf{R}, \vdash) \hookrightarrow (\mathbf{R}, \vdash_{\mathfrak{p}}) \hookrightarrow (\mathbf{R}, \vdash_{\mathfrak{m}})$$

are weakly conservative.

Proposition

If $\mathbf{R}$ is discrete and non-trivial, then the following are equivalent.

1. $(\mathbf{R}, \vdash) \hookrightarrow (\mathbf{R}, \vdash_{\mathfrak{p}})$ is conservative.
2. $\mathbf{R}$ is a field.

Spectra

Proposition

If $\mathbf{R}$ is discrete and non-trivial, then the following are equivalent.

1. $(\mathbf{R}, \vdash) \hookrightarrow (\mathbf{R}, \vdash_{\mathfrak{p}})$ is conservative.
2. $\mathbf{R}$ is a field.

Proposition

The following are equivalent.

1. $(\mathbf{R}, \vdash_{\mathfrak{p}}) \hookrightarrow (\mathbf{R}, \vdash_{\mathfrak{m}})$ is conservative.
2. $\mathrm{Kdim}\, \mathbf{R} \leqslant 0$, i.e., $\forall x \in \mathbf{R}\, \exists n \in \mathbb{N}\, \exists a \in \mathbf{R}\, (\, x^n = a x^{n+1}\,)$.

Primary ideals

The entailment relation $\vdash_{\mathfrak{p}'}$ of **primary ideal** of $\mathbf{R}$ is generated by the axioms of proper ideal together with

$$ab \vdash_{\mathfrak{p}'} a, \{\, b^n : n > 0 \,\}$$

Primary ideals

The entailment relation $\vdash_{\mathfrak{p}'}$ of **primary ideal** of $\mathbf{R}$ is generated by the axioms of proper ideal together with

$$ab \vdash_{\mathfrak{p}'} a, \{\, b^n : n > 0 \,\}$$

Proposition

The following are equivalent.

1. $\mathbf{R}$ is reduced and $(\mathbf{R}, \vdash_{\mathfrak{p}'}) \hookrightarrow (\mathbf{R}, \vdash_{\mathfrak{m}})$ is conservative.
2. $\mathbf{R}$ is **von Neumann regular**, i.e., $\forall a\, \exists x\, (\, a = xa^2 \,)$.

("If **R** is absolutely flat, every primary ideal is maximal" [AM69, Ex. 4.3])

Minimal prime ideals

The entailment relation $\vdash$ of **proper prime filter** of **R** is dual to the entailment relation of proper prime ideal:

$$\begin{aligned} &\vdash 1 \\ ab &\vdash a \\ a, b &\vdash ab \\ a + b &\vdash a, b \\ 0 &\vdash \end{aligned}$$

Minimal prime ideals

The entailment relation $\vdash$ of **proper prime filter** of $\mathbf{R}$ is dual to the entailment relation of proper prime ideal:

$$\begin{aligned} &\vdash 1 \\ ab &\vdash a \\ a, b &\vdash ab \\ a + b &\vdash a, b \\ 0 &\vdash \end{aligned}$$

On top of $\vdash$ we can put the axiom for **maximal** filter:

$$\vdash_{\mathfrak{m}} a, \mathrm{Ann}(a)$$

where $\mathrm{Ann}(a) = \{\, x \in \mathbf{R} : xa = 0 \,\}$.

Minimal prime ideals

Proposition (Coquand, Lombardi [CL06])

Suppose that $\mathbf{R}$ is reduced. The following are equivalent.

1. $a_1, \ldots, a_k \vdash_{\mathfrak{m}} b_1, \ldots, b_\ell$
2. $\mathrm{Ann}(b_1, \ldots, b_\ell) \subseteq \mathrm{Ann}(a_1 \cdots a_k)$.

Minimal prime ideals

Proposition (Coquand, Lombardi [CL06])

Suppose that **R** is reduced. The following are equivalent.

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2. $\mathrm{Ann}(b_1, \ldots, b_\ell) \subseteq \mathrm{Ann}(a_1 \cdots a_k)$.

Proposition

The following are equivalent.

1. **R** is von Neumann regular.
2. **R** is reduced and $(\mathbf{R}, \vdash) \hookrightarrow (\mathbf{R}, \vdash_{\mathfrak{m}})$ is conservative.
3. **R** is reduced and every prime ideal of **R** is minimal.* [Mat83]

Ideals in lattices

Let L be a distributive lattice.
Let $\vdash$ be the entailment relation of (proper) prime ideal of L,
let $\vdash_{\mathfrak{m}}$ extend $\vdash$ with the axiom of maximality.

Proposition
The following are equivalent.

1. $(L, \vdash) \hookrightarrow (L, \vdash_{\mathfrak{m}})$ is conservative.
2. L is Boolean.

This yields **Nachbin's theorem**:
A distributive lattice is Boolean if and only if
all of its prime ideals are maximal.*Cf. [Bel99]

Towards formal Baer criteria

Injective modules

Let M be an $\mathbf{R}$-module.
The following are classically equivalent:

1. Given any exact sequence

   $$0 \to A \to B \to C \to 0$$

   of $\mathbf{R}$-modules, the sequence

   $$0 \to \mathrm{Hom}_{\mathbf{R}}(C, M) \to \mathrm{Hom}_{\mathbf{R}}(B, M) \to \mathrm{Hom}_{\mathbf{R}}(A, M) \to 0$$

   is exact.
2. M is a direct summand of every extension of itself.

Detecting injective modules

Baer's criterion*

A module is injective iff it is injective w.r.t. inclusions of ideals

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & R \arrow[dl, dashed] \\ & M & \end{tikzcd}$$

Detecting injective modules

Baer's criterion*

A module is injective iff it is injective w.r.t. inclusions of ideals

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & R \arrow[dl, dashed] \\ & M & \end{tikzcd}$$

Consequences*

Abelian groups are injective (as $\mathbb{Z}$-modules).

The category of **R**-modules has **enough injectives**.

Stepwise extension

Suppose that M is "ideal-injective" and let $A \subseteq B$.
Given $\varphi : A \to M$ and $b \in B$, consider the conductor ideal

$$I = \{\, r \in \mathbf{R} : rb \in A \,\}$$

and put

$$\mu : I \to M \quad r \mapsto \varphi(rb)$$

By assumption, there is

$$\nu : \mathbf{R} \to M \quad \text{s.t.} \quad \nu\big|_I = \mu$$

Now define

$$\varphi' : A + \mathbf{R}b \to M, \quad a + rb \mapsto \varphi(a) + \nu(r)$$

Stepwise extension

To “exhaust” B we need transfinite methods, e.g., Zorn’s lemma.

$$\begin{tikzcd}
A \arrow[r] \arrow[dr, "\varphi"'] & A + \mathbf{R}b \arrow[r, dashed] \arrow[d, "\varphi'"] & B \arrow[dl, dashed] \\
& M &
\end{tikzcd}$$

A finite and de-Zornified version lives in the Upside Down!

Extension as conservation

"Regarding the theorem instead from a logical viewpoint, it is clear that the import of the assertion is that the extension of the theory of functionals on the subspace A to that on the seminormed space B is actually conservative [...]. In other words, no more may be proved about the subspace A in terms of functionals on the seminormed space B than may already be proved by considering only functionals on the subspace A."

C.J. Mulvey and J. Wick Pelletier
A globalization of the Hahn-Banach theorem

Hom-sets as spectra

Let $\vdash$ on $A \times M$ be generated by all instances of

$$\begin{aligned} (a, m), (a, m') &\vdash && (m \neq m') \\ (a, m), (b, n) &\vdash (ra + sb, rm + sn) \\ &\vdash (0_A, 0_M) \\ &\vdash \{\, (a, m) : m \in M \,\} \end{aligned}$$

Hom-sets as spectra

Let $\vdash$ on $A \times M$ be generated by all instances of

$$\begin{aligned} (a, m), (a, m') &\vdash & \qquad\qquad\qquad\qquad\qquad (m \neq m') \\ (a, m), (b, n) &\vdash (ra + sb, rm + sn) \\ &\vdash (0_A, 0_M) \\ &\vdash \{ (a, m) : m \in M \} \end{aligned}$$

Notice that

$$\mathrm{Spec}(\vdash) = \mathrm{Hom}_{\mathbf{R}}(A, M)$$

Every $\mu \in \mathrm{Hom}_{\mathbf{R}}(A, B)$ induces an interpretation

$$i_\varphi : (A \times M, \vdash) \to (B \times M, \vdash'), \quad (a, m) \to (\mu(a), m)$$

Proposition

Suppose that $\mathbf{K}$ is a non-trivial discrete field.

Let A, B be $\mathbf{K}$-vector spaces, and $\mu \in \mathrm{Hom}_{\mathbf{K}}(A, B)$ be injective.

Then i_μ is weakly conservative.

Towards formal Baer criteria: Dual spaces

Proposition

Suppose that **K** is a non-trivial discrete field.
Let A, B be **K**-vector spaces, and $\mu \in \mathrm{Hom}_{\mathbf{K}}(A, B)$ be injective.
Then i_μ is weakly conservative.

Proof sketch.

Show that the following are equivalent:

1. $(a_1, m_1), \ldots, (a_k, m_k) \vdash$
2. There are $\lambda_1, \ldots, \lambda_k \in \mathbf{K}$ such that

$$\sum_{i=1}^{k} \lambda_i (a_i, m_i) = (0, 1). \qquad \square$$

Towards formal Baer criteria: Divisible modules

Proposition*

Let $A, B, M \in \mathbb{Z}\mathrm{-Mod}$, and $\mu \in \mathrm{Hom}_{\mathbb{Z}}(A, B)$ be injective.
If M is divisible, then i_μ is weakly conservative.

Towards formal Baer criteria: Divisible modules

Proposition*

Let $A, B, M \in \mathbb{Z}\text{−}\mathrm{Mod}$, and $\mu \in \mathrm{Hom}_{\mathbb{Z}}(A, B)$ be injective.
If M is divisible, then i_μ is weakly conservative.

Proof sketch.

Show that the following are equivalent:

1. $(a_1, m_1), \ldots, (a_k, m_k) \vdash$
2. There are $n_1, \ldots, n_k \in \mathbb{Z}$ and $0 \neq c \in M$ such that

$$\sum_{i=1}^{k} n_i(a_i, m_i) = (0, c).$$

□

More instances

Proposition*

Suppose that **R** is an integral ring.
Let $A, B, M \in \mathbf{R}{-}\mathrm{Mod}$ and $\mu \in \mathrm{Hom}_{\mathbf{R}}(A, B)$ be injective.

1. If M is torsion-free and divisible, then i_μ is weakly conservative.
2. If **R** is a Dedekind ring, and M is divisible, then i_μ is weakly conservative.

In both cases, the classical arguments reapply.

Summary

Summary

- We have made combined use of ideas and principles from proof theory and formal topology.
- Hilbert's programme, i.e., the constructive explanation of ideal objects, works for large parts of abstract algebra.
- Statements involving ideal objects are cases of make-believe—however, often we can do "as if", and gather computationally relevant information.

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