# MARKET MECHANISMS AND MAXIMIZATION

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By

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# MARKET MECHANISMS AND MAXIMIZATION: PREFACE

Since at least the time of Adam Smith and Cournot economic theory has been concerned with maximum and minimum problems. Modern "neo—classical marginalism" represents the culmination of this interest.

In comparatively recent times, mathematicians concerned with complex problems of internal planning of the U.S. Air Force or other large units have developed a set of theories and procedures closely related to the maximum problems of economic theory. The name of this technical field is "linear programming". Within the U.S. Air Forces and outside, considerable progress has been made on the mathematical theory of linear programming and on its practical application to administrative problems. But to most economists it remains only a name, if indeed they have heard of it at all.

The following papers have one primary purpose — to provide an exposition of linear programming that students of economic theory can understand and can relate to the existing body of economic theory. I assume only a minimum of mathematical knowledge on the part of the reader.

I have benefitted from reading unpublished memoranda of Tjalling Koopmans of the Cowles Commission, George Dantzig of Project SCOOP. At RAND I am indebted to George W. Brown and others.

# Market Mechanisms and Maximization, I: The Theory of Comparative Advantage

## 1. Introduction

Professional economists are often impressed with the efficiency of a price system in realizing prescribed goals. A whole technical literature of "welfare economics" has grown up dealing with this subject; and there even exists a school of thought which would "play the game of competition" in order to solve the complex problems of a completely planned socialist state.

In the last year interest in linear programming has grown, within the Air Force and elsewhere. But just as many people have been speaking prose all their lives without knowing it; so for centuries have problems of linear programming been presenting themselves for solution.

The question therefore arises: In the new branch of "linear programming" will there be any computational methods suggested by pursuing the logic of a market mechanism? The present series of papers attempts to explore aspects of this problem, beginning with an analysis of the well-known "theory of comparative advantage".

## 2. International Comparative Advantage

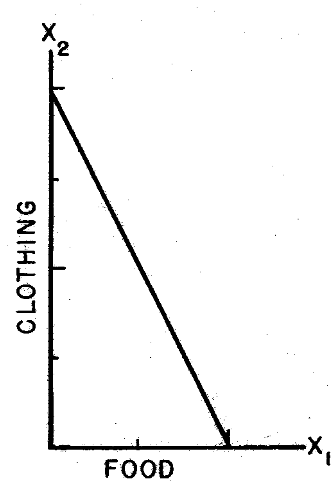
More than a century ago the English economist, David Ricardo, outlined a simple theory to explain the actual pattern of international trade and to proclaim its benefits to all participating countries. A traditional numerical example (somewhat simplified) is of the following form: Portugal can divert resources from food to clothing production and in effect convert one unit of food into one unit of clothing; England, on the other hand, can convert one unit of food into two units of clothing.

Almost certainly Portugal will specialize completely in food, England completely in clothing. England will export clothing in exchange for food imports, the "terms of trade" (or barter price ratio) being almost certainly somewhere between one-food-for-one-clothing and one-food-for-two-clothing. Both countries will be better off than if they do not specialize. World production will be "optimal".

These four economic conclusions are really nothing but mathematical truisms or theorems within the field of linear programming. Mathematically, we may consider any one country — say England — to be subject to a linear relation ("production possibility" curve) of the following type

Figure 1

$$2x_1 + x_2 \leq C$$



If there exists an international price ratio, $p_1/p_2$, somewhere between 1 and 2, at which it can convert food into clothing, the real value of England's National Product (expressed in clothing units) may be written as

$$(2) \qquad Z = (p_1/p_2)\, x_1 + x_2 \text{ or say } Z = 1.5\, x_1 + x_2$$

The problem is to maximize National Product, NP, subject to the technical production-possibility curve of England. Or mathematically, to maximize a linear sum of the form (2) subject to a linear inequality of the form (1). This is a typical problem of linear programming.

Graphical or numerical experimentation will soon convince one that the highest NP will be reached only when $x_1 = o$, $x_2 = C$, and $Z = NP = C$.*

* This may be seen by superimposing on Figure 1 contour lines of equal NP. These will be a family of parallel lines all with a slope of $(p_1/p_2)$; these parallel lines will be flatter than the single production-possibility line of Figure 1, because the price ratio is less than 2. We want to get up to the highest NP line, which we can only do by climbing northwest along the production-possibility line until we reach the $X_2$ intercept, where food production is zero. If negative numbers are allowed, we would want to continue moving northwest indefinitely. This is what happens in an arbitrage situation where two powerful and rich agencies try to maintain different relative price ratios between the same pair of commodities. In this case, little chaps named Gresham who are on their toes can make a lot of money, buying cheap and selling dear and repeating the process indefinitely - or rather until one of the agencies runs out of one of the goods or changes its mind about its price pegging. Buying and selling introduces negative as well as positive numbers. But in linear programming, all variables must usually be positive.

Any other solution will lead to lower NP. Thus, if unemployment is allowed to develop so that England is inside its production possibility curve, NP will suffer. Or if it should specialize on the wrong commodity, food, then NP will turn out to be only $.75C = (1.5/2.0)C$.

The reader may easily verify that for Portugal the best production pattern, when $p_1/p_2$ is greater than 1, involves zero clothing production and complete specialization on food. Only in that way will she maximize her NP

$$Z' = 1.5x'_1 + x'_2 = (p_1/p_2)x'_1 + x'_2 \quad \text{subject to}$$

$$x'_1 + x_2' \leq C'$$

## 3. The Mathematical Problem

Under atomistic competition, where there are numerous competing firms in each country, all this will come about automatically without the conscious deliberation or intervention of any planning body. But there is nothing very surprising about this: After all, we have been describing a commercial situation and it is very natural that we should have had to use commercial and economic language in doing so. There was nothing shadowy or contrived about the prices used, $p_1/p_2$; they were real and concrete.

However, fingers were made before forks and we can imagine this situation as it might appear to a naive scientist from Mars who had never heard about prices and competitive private enterprise. He might still ask the non-commercial question: What is the "optimal" pattern of world production of food and clothing between England and Portugal? If he were acute, the Martian scientist would be troubled by his own question, particularly by the word in quotation marks — "optimal". Optimal in what sense?

Certainly not — as we have already agreed — in the sense of money value, since this is a pre-commercial stage of world history. The scientist might be tempted to consider evaluating food and clothing by their "intrinsic worth"; but unless he had been contaminated by a course in heavy Germany philosophy, he would soon realize that this is an undefinable concept for food and clothing in a world where some people are more like peacocks and others more like gluttons.

An omniscient Martian would soon settle for a more modest definition of the optimum. He would say "I don't know how the final choice between food and clothing is to be made, whether English millionaires will have the greatest say, or the United Nations Commission on Living Standards. But it is my job as a production expert to give <u>the world the best menu from which to choose</u>. Or in other words, for each specified amount of any one good — say food — I must make sure that the production of the other good is as large as possible."

In saying this, the scientist has unwittingly defined a definite class of problems in linear programming. Prices as such have nothing to do with the problem, although — like Voltaire's God — it may be desirable to invent them if they do not exist! To the economist at least, it will seem natural to introduce a system of "shadow" or accounting prices or some sort of system of numerical points. If such shadow prices are really useful, then it follows that many problems of linear programming may benefit on the computational side from a process of imitating the market mechanism.

But first let us write down the mathematical problem that the scientist has posed for himself. He wants (a) to maximize the total of clothing production in Portugal and England, subject to (b) a prescribed total of food, and subject to (c) the two linear production-possibility constraints of the two countries, and where (d) all quantities must by their nature not be negative numbers. Mathematically,

$$Z = X_2 = x_2 + x_2' \text{ is to be a maximum subject to}$$

$$x_1 + x_1' = X_1$$

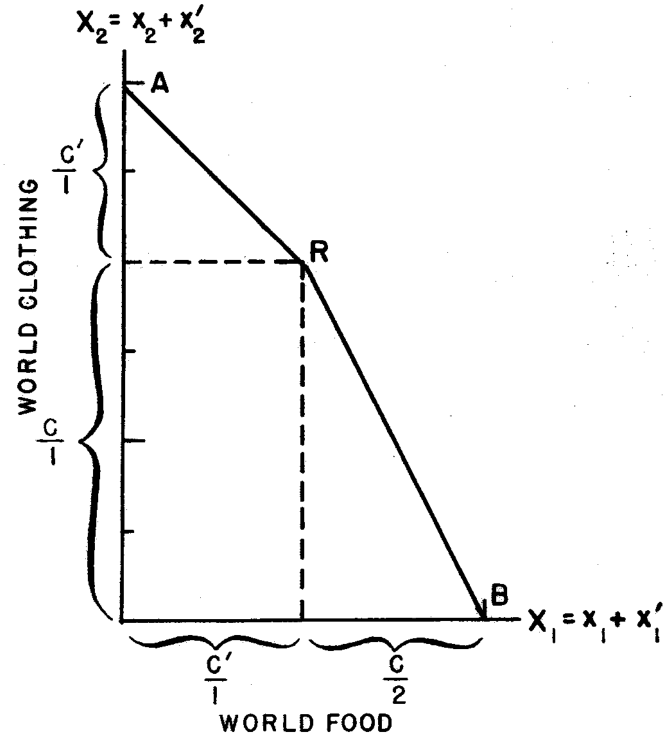
$$(4) \qquad 2x_1 + x_2 \leq C$$

$$x_1' + x_2' \leq C' \text{ and } x_1 \geq o,\ x_2 \geq o,\ x_1' \geq o,\ x_2' \geq o$$

This is a typical problem of linear programming, which is defined as the problem of maximizing one linear relation subject to a number of linear inequalities.

Note that with C and C' being given, there will be a different best Z (or $X_2$) for each prescribed $X_1$. Actually our economic intuition — if pushed far enough — tells us that the resulting "menu" or world production-possibility curve looks like

Figure 2




The flatter of the two line segments has a slope of −1, equal to Portugal's food—clothing technical ratio; the steeper segment has a slope of −2 corresponding to the similar technical ratio for England. The absolute maximum of $X_2$ (corresponding to zero $X_1$) occurs at A where all resources in both countries are used to produce $X_2$ alone, giving us (from the technical relations) C/1 + C'/1 of clothing. The maximum of food comes at B where all resources in both countries are going to food, and yielding in all C/2 + C'/1 of food.

The critical corner-point on the $X_2 - X_1$ curve occurs at R where England is specializing completely on clothing and Portugal on food. It might be called the "Ricardo point", since it is there that the classical theory of comparative advantage tells us we will almost certainly end up.*

* The classical economists recognize that the word "almost" is needed in order to take account of the possibility that the final market price ratio might be at the limit of the range of differing comparative costs instead of within the range. This case was considered especially likely if one country was big compared with the other. In such cases, where the final price ratio settles at the cost ratio of one of the countries, production within that country would be indeterminate. The actual result will have to be dictated by the final pattern of international demand.

There is another remarkable economic feature of our final so-called "world opportunity cost relation between food and clothing." The curve is a convex one: In economic terminology, the "marginal opportunity cost" of converting one good into the other is increasing as we want more of any one good. Or more accurately, it is non-decreasing, since along any one of the line-segments it is constant, neither increasing or decreasing. But between line segments it is strongly increasing, a result that may seem at first surprising in view of the so-called constant cost assumptions of the classical theory of comparative advantages.

Economic and mathematical importance attaches to the concept of marginal cost as defined by $(-dX_2/dX_1)$, the absolute slope of our curve. To the left of the Ricardo point this is exactly equal to one of the technical coefficients of the problem, that of Portugal. To the right, it is equal to the corresponding coefficient of England. At the critical point R it is, strictly speaking, undefined, since the right-hand and left-hand limits that define a mathematical derivative are different. We may adopt the convention whereby marginal cost at such a point is defined as any number between the limiting right and left hand slopes. A similar problem arises in defining marginal cost at the limiting intercept points where the curve hits the axes. It is natural and convenient to define MC at such a point as the upper intercept on the $X_2$ axis, A, as any number equal to or less than the absolute right-hand slope at that point; i.e., MC is one or any number less than one. Similarly, at the lower intercept the MC or slope is any number equal to or greater than 2.*

* We define MC at any point on the curve as the numerical slope of any straight line that "touches" but does not "cut" the curve at that point, i.e., the slope of a line that never lies inside the curve, but does touch it at the point in question.

## 4. Economic Considerations

An economic theorist who is used to thinking of problems in terms of market situations will immediately be struck by a rather remarkable fact: The purely technical concept of MC or slope, which could be arrived at from the pure logic of the maximum problem without reference to prices or markets, does behave remarkably like a market price ratio. His economic intuition tells him:

1. When $1 < p_1/p_2 < 2$, each country will specialize on its best product and the world will in fact be at the Ricardo point where $1 \leq MC \leq 2$.

2. When $p_1/p_2 < 1$ (or $> 2$), both countries will specialize completely on the same product and the world will actually be at one (or the other) intercept with $MC \leq 1$ or ($\geq 2$).

3. When $p_1/p_2 = 1$ (or $= 2$), we will actually be anywhere on one (or the other) of the two line segments with MC, if uniquely defined, being $= 1$ (or $= 2$).

Not only does a market price ratio have the properties of world MC, but by creating a shadow price ratio even where none existed and playing the game of competition, we could end up at any specific point of the final optimal locus. Moreover, in the special case of this problem where the production conditions in the two countries are independent, the problem of decision-making can be in a certain sense decentralized and partitioned into separate parts.

## 5. Shadow Prices

Thus, if we invent a parameter $\lambda$ which is to play the role of a shadow price, $p_1/p_2$, we can split our original maximum problem into two quasi-separate ones. Instead of maximizing

$$\text{I} \qquad Z = x_2 + x'_2 \qquad \text{subject to}$$

$$x_1 + x_1' = X_1$$

$$2x_1 + x_2 \leq C$$

$$x'_1 + x'_2 \leq C'$$

let us separately maximize

$$\text{II} \qquad z = \lambda x_1 + x_2 \qquad \text{subject to}$$

$$2x_1 + x_2 \leq C$$

and

$$\text{II}' \qquad z' = \lambda x'_1 + x'_2 \qquad \text{subject to}$$

$$x_1 + x_2 \leq C'$$

There is some choice of $\lambda$ for which separate optimal solutions to the latter problems, II and II', do add up to the optimal solution to I for any $X_1$. Also it is clear that solving both II and II' is equivalent to solving the world problem of maximizing

$$Z = z + z' = \lambda (x_1 + x'_1) + (x_2 + x'_2) \quad \text{subject to}$$

$$\text{I}' \qquad \begin{aligned} 2x_1 + x_2 &\leq C \\ x'_1 + x'_2 &\leq C' \end{aligned}$$

To arrive at an optimal point (on our $X_2 - X_1$ locus) of each of the following types:

(a) At the upper or $X_2$ intercept, A

(b) On the upper flat line segment, AR

(c) At the Ricardo Point, R

(d) On the lower steep line segment RB

(e) At the lower or $X_1$ intercept B,

we must have

(a) $\lambda \leq 1$

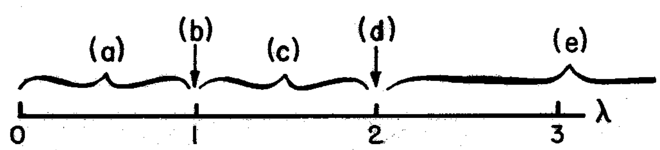
(b) $\lambda = 1$

(c) $1 \leq \lambda \leq 2$

(d) $\lambda = 2$

(e) $2 \leq \lambda$

Figure 3




As $\lambda$ grows from small to large, $X_1$ goes from nothing to the maximum, always at the expense of $X_2$.

## 6. Internal Shadow Prices

From an economist's viewpoint the problem can be decentralized even further. Consider the maximum problem within any one of the countries, such as England. A central planning board could issue shut-down orders so as to maximize

$$z = \lambda x_1 + x_2 \qquad \text{subject to}$$

$$2x_1 + x_2 \leq C$$

But alternatively we might abolish all planning agents and think of food and clothing industries each made up of a myriad of small independent firms. They incur costs and earn revenues from the sale of their products. This they do by converting resources into products. So far we have not spoken much about the character of the resources involved (labor as in Ricardo's theory, etc.). But it will be obvious on reflection that C is a sort of measure of total resources that can be parcelled out to the various firms in the two industries, provided that all the little c's provided to the firms must add up to not more than C. Also, the production function of, say the 999th food producer is of the form

$$x_{1,999} \leq \tfrac{1}{2} c_{1,999}$$

and the corresponding production function of the 77th clothing firm is

$$x_{2,77} \leq c_{2,77}$$

where total output of England's food is the sum of all food firms' output

$$X_1 = x_{1,1} + \cdots + x_{1,999} + \cdots$$

and total England's clothing is

$$X_2 = x_{2,1} + \cdots + x_{2,77} + \cdots$$

and where, so to speak, the total of all resources used by all firms cannot exceed the total of all resources available in the country; or where

$$C \geq c_{1,1} + \cdots + c_{1,999} + \cdots \text{ plus } c_{2,1} + \cdots + c_{2,77} + \cdots$$

Note too that England's grand production possibility curve as given earlier in Figure 1 or equation (1) is simply an aggregation of these individual firm production relations.

One minor point should be noted. Why is there an inequality in the production functions? This is because a firm may be inefficient and not be getting as much product as known technology permits. It must be shown that such inefficient behavior — which is clearly inconsistent with a final optimum — will in fact be heavily penalized by a competitive market and thus be eliminated. The similar inequality with respect to the sum of the c's, which would imply wasteful unemployment of available resources, will also turn out to be prohibited by a perfect market mechanism.

To the Martian with world vision, $\lambda$ is a shadow price constructed for a purpose; to the domestic English planners interested in maximizing the value of English NP, as in II, $\lambda$ is a real enough external price, or barter ratio at which goods can be converted into each other by international trade. The time has now arrived for the English planners to introduce some new shadow prices, or internal accounting point-prices, of the following form.

Establish the following three shadow prices:

$\lambda_1$ = price of food = $\lambda$ *

$\lambda_2$ = price of clothing = 1 by convention

$\lambda_3$ = price of resources, C

* The price of food has here been arbitrarily set equal to the given international price, $\lambda$. This step could be deduced as a theorem by creating a third class of firms who are arbitraging exporters or importers. But this is so obvious a result that I have short-circuited the exposition and simply assumed this step. It is clear in any case that only the ratios of the $\lambda$'s are important.

Then for any food producer, say the 999th, the profit statement

$$\text{Total Revenue} - \text{Total Cost} = \lambda_1\, x_{1,999} - \lambda_3\, c_{1,999}$$

and the "profitability per unit of output" will be

$$\pi_{1,999} = \lambda - \lambda_3 \frac{c_{1,999}}{x_{1,999}} \leq \lambda - \lambda_3\, 2$$

from the production function for food.

Similarly for the 77th clothing producer, unit profits are

$$\pi_{2,77} = 1 - \lambda_3 \frac{c_{2,77}}{x_{2,77}} \leq 1 - \lambda_3$$

In the above profit expressions, we may omit the inequality signs whenever we are speaking of most-efficient producers. How shall the prices of resources be determined? At first let us suppose that there is an all-powerful Office of Price Administration that will use high intelligence to solve this problem, but that once the best price has been established we shall try to rely on the quasi-automatic response of competing firms to determine appropriate output as quantities.

The planning authorities will probably set up some such rule of behavior for firms as follows:

(a) If you make losses, you must contract your scale of operations until finally you go out of business.

(b) If you make positive profits, you may (and hence will) expand your scale of operations at some positive rate.

(c) If you just break even with zero profits, let us for the moment say you stand pat at any existing level of activity.

In terms of this rule, there are certain obvious things that the OPA must do in setting the best price or wage for resources. At the least, $\lambda_3$ must be set so as <u>not to lead to positive profits anywhere in the system</u> — in other words, set so high that even the most efficient producers in clothing and food are unable to realize surplus profits. This means we must have

$$\pi_{1,999} = \lambda - 2\lambda_3 \leq 0 \; \ldots \text{ or } \lambda_3 \geq \frac{\lambda}{2}$$

$$\pi_{2,77} = 1 - \lambda_3 \leq 0 \qquad , \text{ etc. } \lambda_3 \geq 1$$

It will be noted that there is no longer an inequality sign accompanying the left-hand equality signs in the above profit expression. This is because we are considering the profitability of the most-efficient producers of food and clothing; not even they are permitted to have (excess) profits.

Our above conditions determine a minimum below which $\lambda_3$ must not go, but they do not rule out still higher shadow prices. However, it is reasonable to add the further condition that <u>profits are not to be everywhere negative</u> — that they are to be <u>somewhere</u> zero. Otherwise, no firms could stay permanently in business and the total of resources used would be zero, and all production and income would also be zero.

If the equality sign must hold for at least one of our profit expressions, it follows logically that

$$\lambda_3 = \frac{\lambda}{2} \text{ or } 1 \text{ whichever is greater}$$
$$= \text{Max } (\frac{\lambda}{2}, 1)$$

This means that if $\lambda = 1.5$ as in our earlier example, we must set our shadow price equal to Max. (3/4, 1) or to 1. This yields negative profits for all food firms and zero profits for all most-efficient clothing firms; profits for inefficient food firms are always negative.

If our extreme price ratio should be $\lambda = 4$, then $\lambda_3$ = Max (2, 1) = 2 and it is clothing that has negative profits. Only in the critical case where $\lambda = 2$ will both industries be capable of simultaneous operation.

Our OPA has solved its price problem satisfactorily. But there is one major difficulty about our set-up when it comes to determining the exact quantities of resources and output. This difficulty is a consequence of the extreme constant-cost assumption involved in all simple versions of linear programming, assumptions which put the word "linear" into the name of the subject. At our final best price, firms are not forced all to contract or all to expand indefinitely. Efficient firms in the proper industry are permitted to have a large or small output. But there is nothing driving them in total toward 100 percent use of society's resources, neither more nor less. The atomistic firms are suspended in kind of a <u>neutral</u> equilibrium: they have no incentive to do other than what they are doing. Profit considerations neither encourage nor discourage them from doing what society desires.

At the very last stage OPA must call upon WPB for a few mild direct quantity fiats. Once the proper prices have been promulgated by the pricing authorities, the production authority does not have to use much intelligence; but it does have to use a little. It must lead the neutral (efficient) producers – by their moustaches so to speak – to use up exactly 100 percent of the available resources.

If we call the actual amount of total resources currently in use at time t, C (t), then

$$C(t) = c_{1,1} + \dots + c_{1,999} + \dots$$

plus

$$c_{2,1} + \dots + c_{2,77} + \dots$$

The WPB must make sure that

$$C(t) = C$$

if a true optimum is to be reached, and it must have the intelligence to recognize this condition.

## 7. Automatic Determination of Price and Quantity

Actually, it seems reasonable to expect that a market mechanism can be set up which, more or less automatically and without the use of centralized intelligence or planning, will simultaneously determine both the best price and quantity.

Imagine an auctioneer (or ticker tape) that causes $\lambda_3$ to grow whenever total resources used, C(t), exceeds the available amount of C (thought of as being the average amount available over a longer period of time when no temporary stockpiles of resources can be drawn upon). Suppose the market price automatically falls at a rate proportional to any deficiency of C(t) below C. When there is exactly full employment and C(t) = C, then price is constant and $d\lambda_3/dt$ is zero.

Dynamically this means that we have the following law of growth or decay of price

$$\alpha \frac{d\lambda_3}{dt} = \left[C(t) - C\right] + \ldots$$

where $\alpha$ is a positive proportionality factor or time constant, and where any non-linear powers of $\left[C(t) - C\right]$ have been neglected.

To make our system completely automatic and self-generating we need to have some law of growth for C(t). Our earlier rule that firms have to follow provides us with such a law. Whenever $\lambda_3$ went above the level at which efficient firms could anywhere make profits, firms contracted their use of resources. When $\lambda_3$ fell below said critical level, the growth of resources in use was made to be proportional to this deficiency of price. In crude mathematical terms, we have in effect

$$-\beta \frac{d\,C(t)}{dt} = \left[\lambda_3 - \lambda_3^{\circ}\right] + \ldots$$

where $\lambda_3^{\circ}$ is the (as yet unknown) equilibrium value [which will turn out to be Max $(\frac{\lambda}{2}, 1)$], where higher powers of the term in brackets can be neglected, and where $\beta$ is a positive proportionality factor.*

*Strictly speaking, our constant cost assumptions leave the above relation undefined for $\lambda_3 - \lambda_3^{\circ} = 0$. It will not matter if we have the firms with zero profits stand pat, because in any case profits will only be exactly zero for an instant and not long enough for them to have changed their operations appreciably. Note that implicit "inertia" assumptions are involved in all of the above postulated dynamic laws.

Our combined dynamic equations form a determinate system. Any initial price and quantity situation, $\left[C(t), \lambda_3\ (t)\right]$, will generate its own laws of motion over time, according to the relations

$$\alpha \ \frac{d\lambda_3}{dt} = \left[C(t) - C\right]; \ \lambda_3\ (t) \geq 0.$$

$$-\beta \ \frac{dC(t)}{dt} = \left[\lambda_3 - \lambda_3^{o}\right]; \ C(t) \geq 0.$$

If price of resources or the wage stays too high, profits will be negative, and employment will be dropping; as employment drops below the full employment level, the wage will begin to fall; finally profits will be positive and employment will begin to grow rather than fall; it then may — and actually will — overshoot the full employment level, and we are off on another oscillatory cycle around the equilibrium position.

It is a fairly obvious that the system can never come to rest except at the equilibrium levels ($C$, $\lambda_3^{o}$). It is also fairly obvious from the arrows in Figure 4a that the system will oscillate around the equilibrium. [E.g. when we have both shadow wage and employment above their equilibrium levels, the wage will be still rising and employment will be falling — as shown by the arrows pointing Northwest in the Quadrant I above and to the right of the equilibrium point, E; and so forth for quadrants II, III, and IV.]

Figure 4a

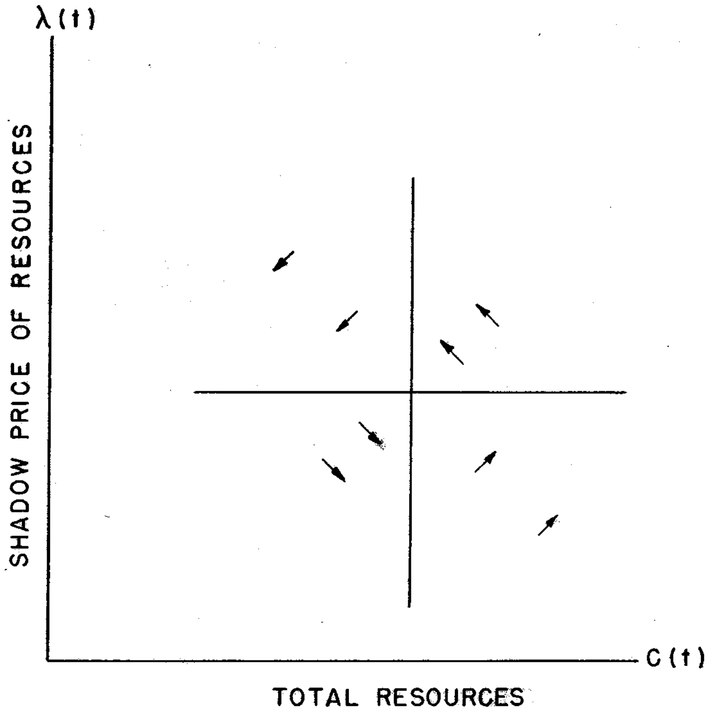
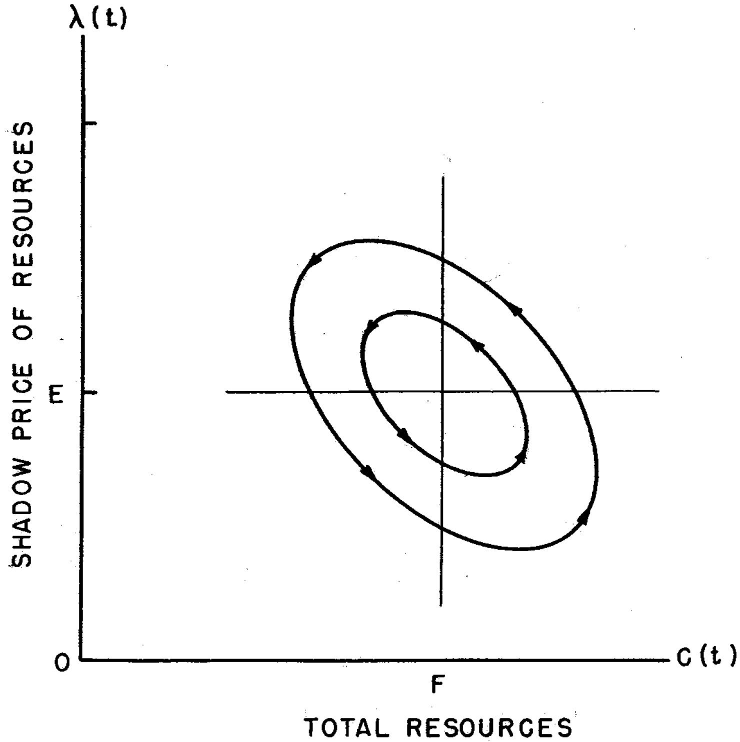


Figure 4b



It is not at all obvious as to whether the system will finally settle down to rest, or oscillate ever more explosively, or just wobble back and forth in an undamped and unexplosive "conservative" fashion. Actually, as Figure 4b shows, the system will behave like a frictionless pendulum and wobble indefinitely without growing or declining in amplitude. If we start out near enough to the equilibrium, the averages of our variables over time will be very near to the correct values; this is because the concentric ellipse that the system travels along will then be a very small one.

* * *

Actual market systems often behave something like the above, as modified by speculation and other factors.* But I am not so much concerned with the realism of this description of actual markets or with the feasibility of playing the game of shadow prices to solve actual administrative problems. Rather I am trying in this series of studies to elucidate the theory of linear programming and to see whether economic intuition suggests any computational methods for the solution of purely mathematical problems of linear programming. The paper following this one explores the matter further by means of an extended discussion of a problem involving the minimum adequate-diet problem, which involved many more variables.

* In the next paper, dealing with minimum cost of an adequate diet, I shall show how certain kinds of speculation may actually cause the system to damp down to the correct equilibrium. It should be noted that in many set-ups more than 100 percent of full employment is impossible so that the oscillations are asymmetrical.

# Market Mechanisms and Maximization, II: The Cheapest-Adequate-Diet Problem

## 1. Introduction

In the first paper of this series the simplest Ricardian theory of comparative advantage was shown to be a typical problem in the new field of "linear programming". Some tentative explorations into the role of real and shadow prices were indicated. The successive steps in the analysis were suggested in a natural way by a *post mortem* of familiar and well-understood market situations.

It would seem desirable now to apply the suggested mode of attack to a quite different problem of purely internal planning at the technical or biological level, where shadow prices play no obvious role. Finding the cheapest diet meeting prescribed nutritional standards provides such a problem. It has been studied by Stigler and others* and represents a good pedagogical

* G. J. Stigler, "The Cost of Subsistence", *Journal of Farm Economics*, Vol. XXVII (1945), pp. 303-14.
P. A. Samuelson, "Comparative Statics and the Logic of Economic Maximizing", *Review of Economic Studies*, Vol. XIV (1946-7), pp. 41-43; I understand that one lecture in the Washington course on linear programming, 1947, has been devoted to this topic.

case study in the field of linear programming.

## 2. Statement of the Problem

*Health Standards*: The National Research Council (NRC) has set forward a table purporting to show, on the basis of present scientific knowledge, the minimum (annual) amounts of different nutritional elements – calories, niacin, vitamin D, etc. – that a typical adult should have. Opinions change rapidly in this field and no claim can be made for great accuracy in such a specification. Moreover, the penalties for having less than these amounts are known only for extreme cases of unbalanced diets; and there is the further point that too much of some elements, such as calories, may be as harmful as too little. But for our purposes we may take the table as definitive and write it symbolically as

TABLE I

Table of Minimum Nutritional Elements

| Minimum Nutritional Elements | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | | i | | m |
| $C_1$ | $C_2$ | $C_3$ | | $C_i$ | | $C_m$ |

Nutritional Composition of Foods: Our second bit of information comes from biologists and chemists. It analyzes the nutritional content of a large number of common foods (cooked in some agreed-upon way). We may call these foods, measured in their appropriate units, $X_1$, $X_2$, ..., $X_n$. We shall make the (somewhat doubtful) assumption that there is a constant amount of each nutritional element in each unit of any given food; so that if 10 units of $X_1$ gives us 100 calories, 20 units will give us 200, and 100 units will give us 1,000 calories – all this independently of the other X's that are being simultaneously consumed. This "constant-return-to-scale" and "independence" assumption helps to keep the problem within the simpler realms of linear programming theory. It also permits us to summarize our second type of information in one rectangular table.

TABLE II

Nutritional Content of Various Commodities, Per Unit

| Nutritional Element | Commodities | | | | | | | Minimum Standards |
|---|---|---|---|---|---|---|---|---|
| | $X_1$ | $X_2$ | $X_3$ | ... | $X_k$ | ... | $X_n$ | |
| Element 1 | $a_{11}$ | $a_{12}$ | $a_{13}$ | ... | $a_{1k}$ | ... | $a_{1n}$ | $C_1$ |
| Element 2 | $a_{21}$ | $a_{22}$ | $a_{23}$ | ... | $a_{2k}$ | ... | $a_{2n}$ | $C_2$ |
| ... | .. | .. | .. | ... | .. | ... | .. | . |
| Element i | $a_{i1}$ | $a_{i2}$ | $a_{i3}$ | ... | $a_{ik}$ | ... | $a_{in}$ | $C_i$ |
| ... | .. | .. | .. | ... | .. | ... | .. | . |
| Element m | $a_{m1}$ | $a_{m2}$ | $a_{m3}$ | ... | $a_{mk}$ | ... | $a_{mn}$ | $C_m$ |

In words, the amount of the 3rd nutritional element contained in the 7th commodity is $a_{37}$. If we think of one slice of toast as having 50 calories we could say $a_{\text{calories, toast}} = 50$ (calories per slice), etc.

Usually the number of goods will be much greater than the known number of nutritional elements, so that $n>m$. But this need not be the case; indeed it would not be the case on a desert island or for a community subject to many taboos. So long as we can find one food that contains something of a given prescribed element, it is clear that the given standard of nutrition can somehow be reached. (This means that we must not have <u>all</u> the a's zero in any row). Ordinarily, the prescribed standard of nutrition ($C_1$, $C_2$,..., $C_i$, ..., $C_m$) can be reached and surpassed in a variety of different ways or diets; but the different diets will not all be equally tasty or cheap.

How do we test whether a given diet, say

$$(X_1, X_2, \ldots, X_k, \ldots, X_n) = (100, 550, \ldots, 3.5, \ldots, 25{,}000),$$

is adequate? We must test each nutritional element in turn. Since each unit of the first good contains $a_{11}$ units of the first element, we get altogether $a_{11}X_1$ of such an element from the first good. Similarly we get $a_{12}X_2$ of this first element from the second good. We must compare the sum of this element <u>from all goods in the diet</u> with the prescribed minimum, $C_1$, to make sure that

$$a_{11}X_1 + a_{12}X_2 + \ldots + a_{1k}X_k + \ldots + a_{1n}X_n \geq C_1$$

and similarly for the second element, we must have

$$a_{21}X_1 + a_{22}X_2 + \ldots + a_{2k}X_k + \ldots + a_{2n}X_n \geq C_2$$

and so forth for the $\underline{i}^{th}$ or $\underline{m}^{th}$ element.

We have not yet introduced the cost of food into the picture, but when we do it will become apparent that it is desirable not to have to pay for any excess consumption of food. In the above equations we should like — if possible — to have the equality signs hold rather than the inequalities. But this will not always be possible, as an ambitious dietician might discover after trying to find a diet which <u>exactly</u> reaches the prescribed standard in every respect. And even where it is in fact possible, she will discover that it is an exceedingly difficult arithmetical feat to find such an exact diet. Moreover, and this may surprise her still more, it may turn out to be most economical not to follow such an exact diet, since there will often turn out to be a cheaper diet that overshoots the mark in some respect.

To show that an exact diet may be impossible, consider the case where every food contains more than twice as much of the first element than of the second; and suppose that the NRC asks for equal amounts of the two elements. Obviously the guinea pig must end up consuming an excess of the first element if he is to have enough of the second element.*

* In this case the matrix $\underline{a}$ is such that the equation $ax = C$ does not have a non-negative solution in the variables, $x$. This may be because $\underline{a}$ is of rank less than $\underline{m}$, or because the solution yields negative $x$'s.

Where there are many "varied" foods a number of different exact diets may exist. But the dietician will feel frustrated in her attempt to find one by trial and error: If she adds more meat to make up a deficit of protein, she may create an excess of calories, and she finds herself constantly in the position of the old lady, who in picking up one bundle drops several others.

In some cases her search for an exact diet would be a relatively easy one. Suppose there exist "pure foods" that consist solely of each of the nutritional elements. Thus a concentrate of Vitamin D would be a commodity with the convenient property of containing nothing else; in its vertical column in Table II, there would be zeros everywhere except in the Vitamin D row. If such a pure food or concentrate existed for every element, an exact diet could be easily concocted from these pure elements alone; or such pure elements could be added in the easily-recognized necessary amounts to supplement any diet that fell anywhere within the prescribed standards. Such pure foods are commercially available in many cases, but as everyone knows, they are usually quite expensive, so that it is usually cheaper to sacrifice exactness and get some excess of cheaper foods.*

* If we could imagine that there existed a continuous range of meats, say, with all gradations of calorie-protein ratios, then some of the earlier mentioned difficulties in finding an exact diet would be obviated. This shows that where many "varied" foods are available, an "almost exact" diet can be fairly easily approximated.

The NRC standards refer to an average normal person. It is easy to imagine cases where, because of disease or other reasons, the prescribed standards must be altered and where the category of permitted foods might be greatly narrowed. This might even lead to an impossible situation in which all the equations could not be satisfied. Thus a patient suffering from certain digestive disturbances might require large amounts of fats; but if he is at the same time suffering from a circulatory disorder that cannot tolerate fats, there may still be found a starchy diet that will meet the situation. However, if a disorder of the pancreas is also present, there may be no known resolution of the incompatible biological demands, with the final outcome obvious and unpleasant.

Because I am interested in the deductive aspects of linear programming, I do not wish to go further into the biological relevance of the simplified linear relations assumed in this problem. But one of the big bottlenecks in the fruitful application of linear programming is the problem of setting up hypotheses which are in reasonable agreement with empirical reality and in filling in the postulated numerical constants. I suspect that a competent student of nutrition, who carefully examined the laboratory procedures for isolating vitamins and determining food-contents, could show that a person would be ill-advised to take Table I and II at their face value. In picking a best diet on the basis of such data, he might inadvertently aid science in isolating new vitamins, sacrificing his health in the attempt.

Economic Price Data: Thus far no mention has been made of the economic costs, in terms of dollars, of the various diets. In theory we can hope to get from the Bureau of Labor Statistics (BLS) data on the prices of the different goods, such as might be indicated in Table III.

TABLE III

Price (Per Unit) of Different Goods

| Number or Name of Commodity | $X_1$ | $X_2$ | ... | $X_k$ | ... | $X_n$ |
|---|---|---|---|---|---|---|
| Price (per unit) | $P_1$ | $P_2$ | ... | $P_k$ | ... | $P_n$ |

For any given diet, $(x_1, x_2, \ldots, x_n)$, the total cost would be easily calculated as the sum of the costs of each of the n goods (it being understood that in any one diet only a few of the possible array of foods would appear, the rest having zero weight). Mathematically, the total dollar cost of a diet would be

$$Z = P_1\ x_1 + P_2\ x_2 + \dots + P_k\ x_k + \dots + P_n\ x_n$$

We may state the full problem as that of minimizing this last sum subject to the m basic inequalities which guarantee that the minimum of each nutritional element is in fact secured. That is

(1) $Z = P_1\ x_1 + \dots + P_n x_n$ is to be a minimum subject to

$$\begin{array}{lcl} a_{11}\ x_1 + \dots + a_{1n}\ x_n \geq C_1 & & x_1 \geq 0 \\ & \text{and} & \\ a_{21}\ x_1 + \dots + a_{2n}\ x_n \geq C_2 & & x_2 \geq 0 \\ \cdot \quad \cdot \quad \cdot & & \cdot \quad \cdot \\ a_{m1}\ x_1 + \dots + a_{mn}\ x_n \geq C_m & & x_n \geq 0 \end{array}$$

It is clear that every such problem has a least-cost solution which will be realized by one or more optimal diets. It is also clear that information contained in Tables I, II and III uniquely define the problem and can be summarized by the following short-hand description of the data:

(2)
$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} & \dots & a_{1n} & C_1 \\ a_{21} & a_{22} & \dots & a_{2k} & \dots & a_{2n} & C_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mk} & \dots & a_{mn} & C_m \\ P_1 & P_2 & \dots & P_k & \dots & P_n & \end{bmatrix} \text{ or } \begin{bmatrix} a & C \\ P & \end{bmatrix}$$

## 3. A Numerical Example

A simple hypothetical example will illustrate the nature of the problem. Assume only two nutritional elements, 1 and 2, or "calories" and "vitamins", with $(C_1, C_2) = (700, 400)$. Assume there are 5 foods: let the first, $X_1$, contain only calories as given by a coefficient $a_{11} = 1$, with $a_{21}$ being zero; let the second, $X_2$, contain only vitamins as indicated by a given $a_{22} = 1$, with $a_{12}$ being zero; let the third good be like the first in that it contains only calories so that $a_{13} = 1$ and $a_{23} = 0$; let the fourth good contain something of both elements, and where for simplicity we can define our nutritional elements' units so that $a_{41}$ and $a_{42}$ are equal to each other and to unity; and finally let the fifth good possess relatively twice as much of calories as compared with vitamins as does the fourth good so that $a_{25} = 1$ and $a_{15} = 2$. Finally, we must assume some prices to make the problem complete: Let $(P_1, P_2, P_3, P_4, P_5) = (2, 20, 3, 11, 12)$, where all prices are in dollars per unit.

Our numerical data can be summarized in the table:

| | Goods | | | | | Standards | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Calories | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | $a_{15}$ | $C_1$ | = | 1 | 0 | 1 | 1 | 2 | 700 |
| Vitamins | $a_{21}$ | $a_{22}$ | $a_{23}$ | $a_{24}$ | $a_{25}$ | $C_2$ | | 0 | 1 | 0 | 1 | 1 | 400 |
| Prices | $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | Z | | 2 | 20 | 3 | 11 | 12 | ? |

and our problem is to find a best diet $(x_1, x_2, \ldots, x_5)$ and the least cost, Z, as indicated by the question mark.

If one tackled this problem by trial and error, by luck or good judgment, one would finally find that (1) the cheapest Z is 4700. It happens that (2) this can be reached in only one way: by the diet $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 100, 300)$, with nothing of the first three goods being bought. Note that (3) there are only as many goods bought as there are nutritional elements. The rest are zero. Finally, it happens that (4) this "best diet" is also an "exact" one, yielding no surplus of either element.

How are these answers arrived at? For the moment such questions may be deferred. Let us first ascertain how general our results are. Is our first conclusion, of a single best Z universally true in linear programming? The answer is "yes" for all well-behaved problems. If there are two different Z's, one would be better than the other. Moreover, the set of admissible X's in linear programming is often — but not always — made a closed one, so that Z can never be made indefinitely better but will remain within a definite bound, which will in fact be reached for some set of X's. It may be remarked that in the problem of this paper Z is to be at a minimum, whereas in the previous discussion of comparative advantage Z was to be at a maximum. These are essentially the same mathematical problem.

But our second conclusion — that the X's are unique — is not universally true. Often the best Z will be reached by a number of alternative X's, quite possibly by an infinite number of such. For example, suppose that the first three goods had been given extremely cheap prices compared with the last two. Obviously, the best diet would be found among the first three goods. But suppose good $X_1$ and good $X_3$, which are exactly alike, were given equally low prices. Then the best way of getting our calories could involve any one of an infinite number of combinations of $X_1$ and $X_3$, providing only that their sum adds up to 700.

In this last case our final diet might involve three goods instead of only two. But even in this case there could be no harm in setting either $X_1$ or $X_3$ to zero and achieving the best Z with <u>as few goods as there are nutritional elements</u>. This suggests a general proposition in the field of linear programming:

> <u>THEOREM</u>: In a linear minimum or maximum problem involving $\underline{n}$ variables (i.e. x's) and $\underline{m}$ inequalities (i.e. C's), the number of non-zero x's will never have to be greater than $\underline{m}$.

This general statement of our above third conclusion will have to be proved later. Note that the theorem would not help much if $\underline{m}$ were greater than $\underline{n}$, as can happen in many problems. Note too that we may sometimes have more than this number of zeros. A simple example will show this possibility: suppose the price of $X_4$ were extremely low compared with all other prices; then it stands to reason that all of our required calories and vitamins can be bought most cheaply by purchasing 700 units of $X_4$, and buying a single good would be the best way to get two elements.*

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* The reader may be tempted to stretch the theorem to cover this case. He may argue that for this example we can forget about vitamins, since $C_2$ is no bottleneck. In effect then we have only 1 rather than 2 effective constraints and a new $\underline{m} = 1$ can be substituted for $\underline{m} = 2$. But there are many problems involved in defining such a new $\underline{m}$ that are glossed over in this discussion. It should be emphasized that $\underline{m}$ is the number of inequalities or "possible equalities" not the actual number of equalities that turn out to be satisfied.

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The last example shows that our best diets will not always be exact, so that our fourth conclusion is not generally true. Often some of the $\underline{m}$ side conditions or constraints will turn out

<u>not</u> to be binding; however, they cannot be thrown away because for other prices or standards, they may become binding. It is intuitively clear that changing any $C_i$ — e.g. increasing the calorie requirements — will cause a definite change in the best Z; but it is also clear that changing a non-binding C will have zero effect on Z, until it begins to be binding. The rates of change of the form $(dZ/dC_i)$ are in the nature of <u>marginal costs</u> and will be found to have an important economic and mathematical significance, related to "shadow-prices" and so-called "Lagrangean multipliers".

## 4. Possible Methods of Attack

<u>Random Trial and Error</u>: Let us now examine various possible ways of trying to solve our minimum problem. The first and simplest method would consist of aimlessly trying different diets in the hope that the best one will soon be found. This is clearly a hopeless method, even if only an approximate solution is desired. The number of possibilities is endless, and in any case if we should happen to stumble on the best solution we might never be aware of its optimal properties.

<u>Use of Elementary Calculus</u>: An economist of any sophistication is accustomed to solving maximum or minimum problems by moving his variables until "marginal something" is just balanced by "marginal something else", or until "marginal net something" is zero. This corresponds in the language of calculus to "differentiating the quantity to be minimized, Z, with respect to the independent variables, which are X's, and then setting the

resulting expression equal to zero." It is hoped that this gives enough equations to determine the optimal point uniquely.

Obviously this method cannot be used in problems of linear programming. In the first place if we differentiate a linear sum

$$Z = P_1 x_1 + P_2 x_2 + \ldots + P_n x_n$$

with respect to any variable such as $x_1$, we get

$$\frac{\partial Z}{\partial x_1} = P_1$$

which is either never zero or always zero and does not give us determining equations for the x's. If it made any sense to regard $x_1$ as an independent variable, the obvious information given by the above derivative would be the following: You will always lower Z by decreasing $x_1$ indefinitely, $P_1$ being positive; or if $P_1$ is negative, by increasing $x_1$ indefinitely; or if $P_1 = 0$, the value of $x_1$ is indifferent.

Determining Free Variables: But there is a more serious objection to the procedure of simple differentiation. A variable like $x_1$ is far from being independent; it is constrained in a number of different ways. In the first place, no quantity of food consumed can be negative so that $x_1 \geq 0$. This tells us that even in the strange case where $x_1$ is not subject to constraint, we would not wish to decrease $x_1$ indefinitely, but only to zero; and our equilibrium condition would be stated, not by a derivative set equal to zero, but by an inequality.

Even this is a gross over-simplification. $x_1$ is not free to move over the range of positive numbers independently of the other x's. Our constraints – minimum calories and vitamins in this problem, maximum domestic production possibilities in the theory of comparative advantage – link up the x's. This is not peculiar to linear programming. It also arises in well-behaved problems of economics, such as the problem of maximizing a consumer utility

(3) $$U = U(x_1, x_2, \ldots, x_n) \text{ subject to the "budget equation"}$$
$$P_1 x_1 + P_2 x_2 + \ldots + P_n x_n = I$$

where I is the prescribed maximum total expenditure. There should really be a $<$ symbol before the I, but this is usually omitted, since it is obvious that short of the point of satiation the consumer will not let purchasing power go to waste.

The x's are clearly not all independent. If all but the last one, $x_n$, are prescribed, then the latter is determined by the budget equation. It is rational – and in this simple example easy – to "eliminate" $x_n$ as an independent variable by substituting into U the value of

$$x_n = \frac{I}{P_n} - \frac{P_1}{P_n} x_1 - \frac{P_2}{P_n} x_2 - \ldots - \frac{P_{n-1}}{P_n} x_{n-1}$$

as determined by the budget equation.

This gives us U as a function of only (n – 1) x's, and all of them are free and independent variables; or

$$U = U(x_1, x_2, \ldots, x_{n-1}, \frac{I}{P_n} - \frac{P_1}{P_2} x_1 - \ldots - \frac{P_{n-1}}{P_n} x_{n-1})$$

Now we may differentiate U with respect to each of the independent x's, holding the other x's constant but implicity letting $x_n$ vary in accordance with the constraints. This gives us for any x, such as $x_1$ the equation

$$\left(\frac{\partial U}{\partial x_1}\right)_{\substack{\text{other x's except} \\ x_n \text{ constant}}} = \frac{\partial U}{\partial x_1} - \frac{\partial U}{\partial x_n} \frac{P_1}{P_n}$$

Now if we set all these partial derivatives simultaneously equal to zero and if U has the well-behaved concavity properties usually assumed - law of diminishing (relative, ordinal) marginal utility, etc. - this defines a unique optimum. In words, the optimum is characterized by "marginal utilities proportional to respective prices."

In linear programming the side-conditions or constraints are all linear. This would seem to be a great advantage, since it is often difficult or impossible to solve non-linear relations for "dependent variables" that we wish to "eliminate" from our Z function. The advantage of linearity in the constraints is partly illusory: It is rendered nugatory by the appearance of the innocent looking inequality sign, >, in these side conditions. We cannot be sure in advance that we shall be able to find, or shall want to find, an exact diet. This means that some of the side conditions will <u>not</u> have the equality sign holding, and in such cases we will not be able to use those linear side conditions to eliminate variables. Unfortunately, we are not told in advance which side-conditions will be "binding" (i.e. which ones will have equality signs) and which will be "non-binding". We must find this out the hard way.

Occasionally economic intuition will tell us which we can disregard. Thus, imagine a rich man who takes no thought of the future. Suppose he has a certain sum of money he may spend during a war-year. But suppose the government forces him to pay ration points for everything he buys and sets the ration points equal in value to the dollar prices of the goods he buys. Further suppose that the government gives him less ration points than he has dollars. He must now maximize his utility subject to two budget constraints: a dollar and a ration-point limitation. But it is obvious in this case that the dollar constraint is redundant. It is physically impossible for him to spend more dollars than ration points; and since we have assumed no future use for the dollars, it is clear that they may be disregarded. The interested reader may verify that in this case the consumer's equilibrium involves "marginal utilities proportional to ration-points", etc.*

* See P. A. Samuelson, Foundations of Economic Analysis, pp. 163-171 for further treatment and references. In recent years, A. Henderson and H. Makower have discussed similar issues in the Review of Economic Studies. The particular example here discussed is related to the proposal for war-time "expenditure control".

Even in the simplest problem of rationing, economic intuition cannot usually be relied upon to give information as to which equalities will or will not be effective. Thus, suppose that ration points set for each good are not proportional to quoted prices. Then it is quite possible that a given consumer will be able - if he wants to - to spend the total of both his dollars and his ration points. But will he always want to? The case of a rich diabetic during sugar rationing shows that the answer is no:

dollars and not points may be binding. The case of a rich glutton provides an opposite example where points and not dollars are binding.*

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* Imagine a consumer confronted by dollar prices, $(P_1, P_2) = (1, 1)$ and with dollar income, $I = 95$; let him be faced by point prices, $(P_1', P_2') = (4, 1)$ and with total points to spend, $I' = 200$. He can buy $(X_1, X_2) = (35, 60)$ if he spends all his points and dollars. In the first of the accompanying figure, he would choose to do so; but in the other two, he would let either dollars or ration points go to waste. (The above mentioned figures of this footnote on the following page.)

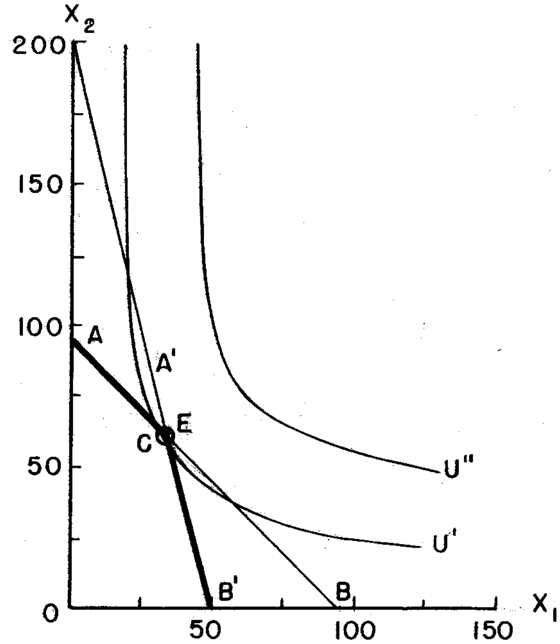


FIG. 1a

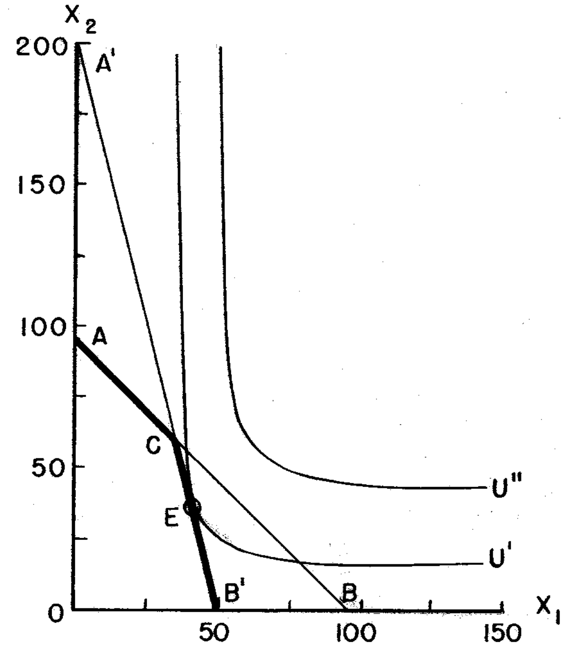


FIG. 1b

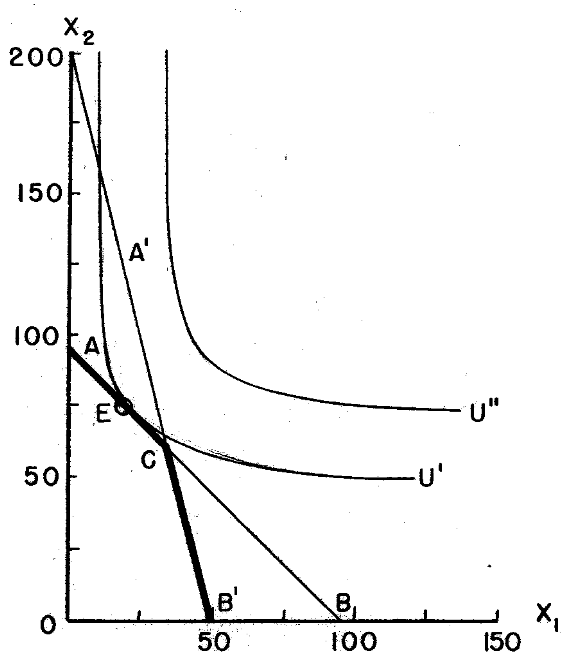


FIG. 1c

The lines AB and A'B' represent the dollar and ration point budget equations respectively. The heavy locus ACB' represents the locus available to the consumer since the "scarcest currency" is always the bottleneck. In Fig. 1a, this locus touches but doesn't cross the highest indifference curve at C. In Fig. 1b, this phenonemon occurs along CB', where dollars are redundant; in Fig. 1c, the ration points are redundant. When there are only two x's and when both constraints are known to be binding, there is no room left for maximizing behavior; only when there are more goods does the problem become interesting.

As we shall see presently, there is a formal artifice by which we can get rid of all the inequality signs: i.e. by defining new variables such as "unused" dollars, "unused" ration points, "excess" calories, etc. Of course, this does not avoid the question as to which of these new variables will turn out to be zero.

But for the moment let us suppose that we know which of the constraints will be binding.* Let them be $r \geq m$ in number.

* In the calorie-vitamin numerical example, I have already indicated that the best solution happens to involve both constraints and we may imagine a problem in which an umpire had provided us with this information. In other problems we might imagine setting up in turn the hypothesis that all possible combinations of sets of inequalities will hold. This would involve considering $2^m$ possible cases, a fierce number when $m$ is large.

How do we proceed to use these equations in order to eliminate redundant variables and arrive at a set of independent variables? We may first illustrate by the rather trivial case where $r = 1$ and there is only one constraint. Suppose that we can forget vitamins and have only the calorie side condition.

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + a_{14} x_4 + a_{15} x_5 =$$
$$1x_1 + 0 x_2 + 1x_3 + 1x_4 + 2x_5 = 700$$

We can use this equation to solve for any variable, except $x_2$ which has a zero coefficient. Suppose we decide to eliminate $x_5$ as redundant; then we solve for it and substitute into our cost data to get

$$Z = P_1x_1 + P_2x_2 + \dots + P_5x_5 = 2x_1 + 20x_2 + 3x_3 + 11x_4 + 12x_5$$

$$= P_1x_1 + P_2x_2 + \dots + P_5 \left(\frac{C_1 - a_{11}x_1 - a_{12}x_2 - \dots - a_{14}x_4}{a_{15}}\right)$$

$$= (P_1 - \frac{a_{11}}{a_{15}} P_5)\ x_1 + (P_2 - \frac{a_{12}}{a_{15}} P_5)\ x_2 + \dots + (P_4 - \frac{a_{14}}{a_{15}} P_5)\ x_4 + \text{constant}$$

$$= (2 - 6)\ x_1 + (20 - 0)\ x_2 + (2 - 6)\ x_3 + (11 - 6)\ x_4 + \text{constant}$$

$$= -4x_1 + 20\ x_2 - 3\ x_3 + 5\ x_4 + \text{constant}$$

Our cost function has now been expressed in terms of one less variable than we had originally. But presumably these four variables are now free to move independently over all positive values.

How should we optimally adjust $x_2$? Because it has a positive coefficient, it is clear that $(\partial Z/\partial x_2) = 20>0$ and every increase in $x_2$ sends up costs. Therefore, we go into reverse and reduce $x_2$ in order to effect savings. This we continue until we reach the limit $x_2 = 0$. The exact same can be said for $x_4$ which has a positive coefficient.

So far so good. But applying the same reasoning to the other x's leads to a perplexing situation: $x_1$ and $x_3$ have negative coefficients and it would seem that increasing them indefinitely would be in order. This is surely absurd. Or is it? Will increasing $x_1$ and/or $x_3$ cause the total of calories to become excessive, and therefore be a foolish procedure? No it will not; it will not because $x_5$ is always being reduced so as to keep total calories = 700 = $C_1$. That is the meaning of our earlier substitution.

In disposing of one objection we encounter another. If $x_1$ and $x_3$ are increased enough, $x_5$ will ultimately become a negative number, which it is not permitted to do. If we could regard foods as bundles of calories and could "convert" $x_1$ and $x_3$ into $x_5$, and then could <u>sell</u> (as well as buy) $x_5$ in unlimited amounts at $P_5 = 12$ — then it would indeed be optimal to increase $x_1$ and $x_3$ indefinitely at the expense of $x_5$. But all this is not possible; in our problem no x can ever become negative. This means that $x_1$ and $x_3$ can only be increased until $x_5$ is zero; from that point on, if we increase $x_1$, we must decrease $x_3$ and vice versa. This means that, with $x_2$ and $x_4$ being already set equal to zero, and with $x_1$ and $x_3$ being made so large as to set $x_5 = 0$, we are finally left with the following choices for $x_1$ and $x_3$.

$$\frac{1}{2} x_1 + \frac{1}{2} x_3 - \frac{700}{2} = x_5 = 0 \quad \text{or} \quad x_1 = -x_3 + 700$$

We substitute this into Z to get

$$\begin{aligned} Z &= P_1 x_1 + P_3 x_3 = 2x_1 + 3x_3 \\ &= P_1 (-x_3 + 700) + P_3 x_3 = (P_3 - P_1) x_3 + \text{constant} \\ &= 1 x_3 + \text{constant} \end{aligned}$$

Since $x_3$ is now our remaining free variable and since it has a positive coefficient, we will realize economies by making it as small as possible: When $x_3$ is set equal to zero, then obviously — from the above relation between $x_1$ and $x_3$, or from the original calorie constraint — we must have $x_1 = 700$.

At long last we have our optimal diet: $(x_1, x_2, x_3, x_4, x_5) = (700, 0, 0, 0, 0)$. As we had reason to expect earlier, where there is only one effective constraint, there must be only one non-zero variable.

An intuitive economist might have arrived at this result almost immediately. He is used to working with the concept "marginal utility of the (last) dollar spent on each commodity". In this problem, he would replace utility by calories and look for the most calories per dollar; or for the maximum of

$$(4) \qquad \frac{a_{11}}{P_1} = \frac{1}{2}, \ \frac{a_{12}}{P_2} = 0, \ \frac{a_{13}}{P_3} = \frac{1}{3}, \ \frac{a_{14}}{P_4} = \frac{1}{11}, \ \frac{a_{15}}{P_5} = \frac{2}{12}$$

Clearly $x_1$ is the cheapest way of getting calories. It is too bad that this simple device will not get the solution to more complicated problems.

As a matter of fact the more tedious method of substitution outlined above can follow many paths. With good luck we might have picked a path which would have gotten us our solution in almost a single step. Suppose we had used our calorie relations to solve for $x_1$ rather than $x_5$. Then our cost would have turned out to be

$$Z = (P_2 - \frac{a_{12}}{a_{11}}P_1)x_2 + (P_3 - \frac{a_{13}}{a_{11}}P_1)x_3 + (P_4 - \frac{a_{14}}{a_{11}}P_1)x_4 + (P_5 - \frac{a_{15}}{a_{11}}P_1)x_5 + \text{constant}$$

$$= 20\ x_2 + 1x_3 + 10\ x_4 + 8\ x_5 + \text{constant}$$

All the coefficients of the variables are positive; each variable is best set to zero; from our constraint we find that $x_1 = 700$. Hindsight always helps.

* * *

We have labored hard to get the best solution. The only trouble with our solution is that <u>it is wrong</u>. We have already been informed that the best diet for our original problem is $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 100, 300)$. What is lacking about (700, 0, 0, 0, 0)? Clearly it yields the correct calories, but it fails to yield the specified amount of vitamines. It is definitely the cheapest diet under the assumption made that only the calorie constraint would be binding. But this was a gratuitous assumption that we had no right to make, as can be checked-up-on by seeing whether the full conditions of the problem are satisfied.

Our work has not been entirely in vain. We have not answered our original problem but we have given the correct answer to some other problems. We have the best answer to the problem where vitamins are of no importance. (Or alternatively if the first food, $x_1$, contained an awful lot of vitamins so that $a_{21}$ were very large instead of being zero and so that we could be sure that the vitamin requirements would be more than satisfied, then our solution would be a correct one.)

There is one further virtue to our solution to the problem where only calories count. It gives us a <u>lower bound</u> to the best obtainable cost: If calories alone cost at least the amount $700 \times 2 = 1400$, then a diet adequate in every respect must cost that much or more.*

* Thus, our optimal Z must be at least as great as the cheapest way of buying calories alone: or $Z \geq \underset{i}{\text{Min}} \frac{P_i C_1}{a_{1i}}$ This must equally be true with respect to vitamins or any other element: or $Z \geq \underset{i}{\text{Min}} \frac{P_i C_k}{a_{ki}}$ where k = 1, 2, ..., m. The best of these lower bounds is given then by $Z \geq \underset{k}{\text{Max}} \underset{i}{\text{Min}} \frac{P_i C_k}{a_{ki}}$. In these expressions $\underset{k}{\text{Max}}$ means "maximum with respect to k" and similarly for $\underset{i}{\text{Min}}$. Note finally that if some of the a's in our problem could be negative, this line of reasoning would fail.

The main purpose of this discussion was, however, expository. When only one constraint is binding, the problem of elimination by substitution is at its simplest and the logic of the process is revealed most clearly.

We may recapitulate just what was done in this process:

1. We found an expression for one of the dependent variables by using our constraint.

2. We substituted this expression into our Z sum wherever the dependent variable appeared, thus eliminating the dependent variable from our Z sum.

3. The remaining variables were not perfectly free to move as they pleased. When one became zero, it hit an inflexible stop. Worse than that when a movement of the independent variables caused the eliminated dependent variables to hit zero, we again ran into an inflexible wall and could at best move along that wall.

4. But our minimizing procedure, within these constraints, was logically simple. We kept repeating firmly to ourselves:

"Every day in every way, we must be getting better and better. We just keep moving, so long as we are moving <u>down</u> the cost trail." (Specifically, when we had chosen to eliminate $x_5$, we then moved to $x_2 = 0$ because the positive coefficient of $x_2$ meant that this would be a downward direction; then we moved further downward by setting $x_4 = 0$. Since $x_1$ and $x_3$ had negative coefficients, our next downward move involved increasing one or both of them; this went on until we hit the "geometrical plane or wall" represented by

$$x_2 = x_4 = x_5 = 0 = \frac{C_1}{a_{15}} - \frac{a_{11}}{a_{15}} x_1 - \frac{a_{13}}{a_{15}} x_3$$

We proceeded to edge our way along this wall in a downward direction by decreasing $x_3$ which had a positive coefficient in the expression for Z defined in term of $x_3$ along this final wall. If $x_3$'s coefficient had been negative instead of positive, we would have increased it at the expense of $x_1$, up to the $C_1 = 700$ limit. If its coefficient had been zero, any point on the wall would have been indifferently good.)

So much for the process of elimination of dependent variables when there is only one constraint. If there are two or more constraints, the logic of the process is unchanged; but the numerical steps are considerably more tedious. Let us illustrate by examining briefly our simple calorie-vitamin problem, where we have been told that both constraints are in fact to be binding. Here we have two effective constraints and so we can eliminate two variables. Actually, in this case, except for $x_1$ and $x_3$ we can eliminate <u>any</u> two variables from the numerical relations numerical relations

$$1x_1 + 0x_2 + 1x_3 + 1x_4 + 2x_5 = 700$$
$$0x_1 + 1x_2 + 0x_3 + 1x_4 + 1x_5 = 400$$

Applying the methods of high school or more advanced algebra, we will soon find that it is much easier to eliminate the "pure" variables, $x_1$ and $x_2$ or $x_3$ and $x_2$, than any other pair. In fact, to express $x_4$ and $x_5$ in terms of the remaining variables involves solving two simultaneous equations (or as a mathematician would say, involves "inverting a matrix"). This is logically easy to do but tedious in practice.

Let us therefore agree to eliminate $x_1$ and $x_2$, to get

$$x_1 = 700 - 1x_3 - 1x_4 - 2x_5$$
$$x_2 = 400 - 0x_3 - 1x_4 - 1x_5$$

We now substitute these into our cost expression

$$Z = \sum_1^5 P_i\, x_i = 2x_1 + 20x_2 + 3x_3 + 11x_4 + 12x_5$$
$$= 2(700 - 1x_3 - 1x_4 - 2x_5)+20\,(400 - 1x_4 - 1x_5)+3x_3 + 11x_4 + 12x_5$$
$$= (-2 + 3)x_3 + (-2 -20 +11)x_4 + (-4 -20 +12)\,x_5$$
$$= 1x_3 - 11x_4 - 12x_5$$

Because $x_3$ has a positive coefficient (or "<u>net</u> cost"), we must obviously reduce it to zero. Just as clearly the negative coefficients of $x_4$ and $x_5$ mean we must increase them <u>at the expense of $x_1$ and $x_2$</u>. But $x_1$ and $x_2$ can never be reduced below zero. When they both reach zero, $x_4$ and $x_5$ take on the values

(100, 300), which we earlier said were the best values.*

* To be rigorous, we must verify that <u>both</u> $x_1$ and $x_2$ should be forced to zero levels. If we had eliminated $x_4$ and $x_5$, the resulting coefficients of $x_1$, $x_2$, $x_3$ would all be positive, providing round-about verification of what can be directly shown.

In the general case where we know there are $\underline{r}$ ($\leq \underline{m}$ and $\leq \underline{n}$) independent and consistent binding constraints, we can always eliminate $\underline{r}$ variables and substitute for them in the Z expression. The resulting expression for Z will be defined in terms of the remaining $n - r$ quasi-free variables; and depending on their coefficients, we can proceed to find some $n-r$ variables that can be set equal to zero. The final values of the non-zero variables can be found by solving our $\underline{r}$ effective equations. If, and only if, we have selected the right set of effective constraints, the whole process will be consistent.

The picturization of this process in terms of higher geometry is conceptually very helpful, but will be reserved for a later section.

<u>Use of Lagrangean Multipliers</u>. In ordinary well-behaved maximum or minimum problems, there is a well-known artifice, due to Lagrange, for dealing with side conditions or constraints. If we have

$$(5) \qquad \begin{aligned} Z &= F_o(x_1, x_2, \ldots, x_n) \text{ to be a minimum subject to} \\ &F_1(x_1, x_2, \ldots, x_n) = 0 \\ &F_2(x_1, x_2, \ldots, x_n) = 0 \\ &\quad . \quad . \quad . \quad . \quad . \qquad\qquad m \leq n \\ &F_m(x_1, x_2, \ldots, x_n) = 0 \end{aligned}$$

the trick is to form the Lagrangean expression

$$G = \lambda_0 F_0 + \lambda_1 F_1 + \ldots + \lambda_m F_m = G(x_1, x_2, \ldots, x_n; \lambda_0, \lambda_1, \ldots, \lambda_m)$$

where $\lambda_0$ may be set equal to one, and where the other $\lambda$'s are "undetermined multipliers". Usually it is said: "We have added nothing to the $F_0$". But we next are told to treat the x's "as if they were independent variables" and we end up with the minimum conditions:

$$(6) \qquad \frac{\partial G}{\partial x_i} = \lambda_0 \frac{\partial F_0}{\partial x_i} + \lambda_1 \frac{\partial F_1}{\partial x_i} + \ldots + \lambda_m \frac{\partial F_m}{\partial x_i} = 0 \qquad (i = 1, 2, \ldots, n)$$

We can eliminate the $\lambda$'s and end up with $(n - m)$ minimizing equations involving the partial derivatives of the F's.

This is purely a formal trick, whose sole justification lies in the fact that it rapidly gives us what other rigorous methods assure us are the true first-order minimum conditions. But it is <u>not</u> true that $G(x_1, \ldots, x_n; \lambda_0, \lambda_1, \ldots, \lambda_m)$ is necessarily at a relative minimum with respect to freely variable x's, even when the correct $\lambda$'s are prescribed. And the correct secondary conditions on the higher partial derivatives are much more complicated than the usual textbook treatment would ever suggest.

For linear programming, we already know that most minimum conditions involve boundary inequalities rather than vanishing derivatives. Therefore, the Lagrangean device is still more in need of justification. Actually, it is closely related to "shadow prices" and can be a suggestive method. But here only a brief outline of its possible application can be indicated.

Given $Z = \sum_{k=1}^{n} B_k x_k$ to be a minimum subject to

(7)

$$\sum_{k=1}^{n} a_{ik} x_k - C_k \geq 0 \qquad (i = 1, 2, \ldots, m \gtreqless n)$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n)$$

We form the Lagrangean expression

$$(8) \quad G = Z + \lambda_1 \left(\sum_{k=1}^{n} a_{1k} x_k - C_1\right) + \lambda_2 \left(\sum_{k=1}^{n} a_{2k} x_k - C_2\right) + \ldots$$

$$= \sum_{k=1}^{n} B_k x_k + \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik} \lambda_i x_k - \sum_{i=1}^{m} C_i \lambda_i$$

$$= \sum_{i=o}^{m} \sum_{k=o}^{n} a_{ik} \lambda_i x_k = G(x_o, x_1, \ldots, x_n; \lambda_o, \lambda_1, \ldots, \lambda_m)$$

where $\lambda_o = 1$, $x_o = -1$, $a_{ok} = B_k$, $a_{io} = C_i$. G is a so-called bilinear form. If it is to be at a minimum with respect to the x's, for suitably constant $\lambda$'s, we must have certain equalities or inequalities holding for

$$\frac{\partial G}{\partial x_j} = B_j + \sum_{i=1}^{m} a_{ij} \lambda_i$$

As yet this tells us not very much. We suspect that wherever a constraint is not binding, the corresponding $\lambda$ will be zero. This being the case, we can determine unique $\lambda$'s by the equations

$$(9) \qquad \frac{\partial G}{\partial x_j} = 0 = B_j + \sum_{i=1}^{m} a_{ij} \lambda_i$$

where j represents the subscript of any of the _r_ non-zero x's; and where i is the subscript of any of the _r_ effective constraints. For all other x's, we must have negative profitability, or

$$(9)' \qquad \frac{\partial G}{\partial x_j} = B_j + \sum_{i=1}^{m} a_{ij} \lambda_i \geq 0$$

But all this is as yet conjecture, based upon an analogy with shadow prices (these being the $\lambda$'s, except for sign) and "unit profitability" (being $\frac{\partial G}{\partial x_j}$ except for sign). It may be said that our G function is closely related to similar bilinear forms introduced by von Neumann in his article on economics and book on the theory of games.*

* J. v. Neumann, "A Model of General Equilibrium", _Review of Economic Studies_ (1945046) Vol. XIII, pp. 1-9, a translation of an earlier German work dating back to the 1930's; J. v. Neumann and O. Morgenstern, _The Theory of Games and Economic Behavior_, 2nd edition, 1947.

## 5. Intuitive Economic Considerations

For the moment let us abandon a direct mathematical attack and investigate whether any common sense hunches can suggest a new line of action. If only one constraint, such as calories, were involved, it would seem natural to concentrate on finding a food with the "lowest cost per calorie" or "greatest calories per dollar". Let us try to define the concept of "profit" a little more formally so that it will conveniently handle more complicated cases. This will involve the concept of "shadow-prices", or accounting prices created for the purpose of solving our minimum problem.

When a consumers' advisory committee says that "spinach is a good buy this week", it probably means that the housewife can get a lot of iron for her cash outlay, and other elements as well. But how can we evaluate and compare the vitamin content of the spinach with its iron content? One rule of thumb would be to give a certain number of points to each nutritional element, so that a dollar spent on spinach is thought of as buying us some number of points in the form of calories plus some number of points derived from its iron content plus the points derived from its other elements. It would seem reasonable only to buy foods which give us the maximum number of points per dollar, avoiding all other foods.

This rule would give us a definite answer to our problem. But we cannot apply it until we know how to score our different elements (calories, vitamins, etc.) in terms of points. If calories are "dirt cheap" or so plentiful that we are sure to get enough of them, we obviously will have to stay away from buying expensive starchy foods. This is because they will have a low total "point-score" or "profitability" if we prefer to use that term. On the other hand, it may turn out that calories are the expensive bottleneck and each calorie should be awarded a high number of points relative to what is awarded to each vitamin. But how can we know what relative point evaluation to give the different elements?

An economist well versed in utility theory might hope to be able to reduce all the different elements down to comparable "nutrition utiles". He might try to draw up a curve showing the different combinations of calories and vitamins that give the human guinea pig the same "level of nutritional well being". In Figure 2a, such as "indifference contour", is drawn up for vitamins and calories. Our economist would infer from the steep slope of the curve that many calorie units have the same point value as one vitamin unit.

All this is illegitimate. The NRC has defined basic nutritional requirements and <u>each</u> of these elements must attain its minimum level. In economic terms our correct indifference-contour looks like Figure 2b. We must get to the point C and anything above or to the right of that point is irrelevant.

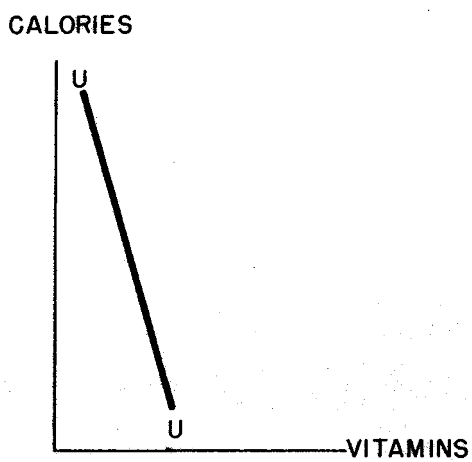


Figure 2a: Economist's attempt to define relative evaluation of calories and vitamins by nutritional indifference contour.

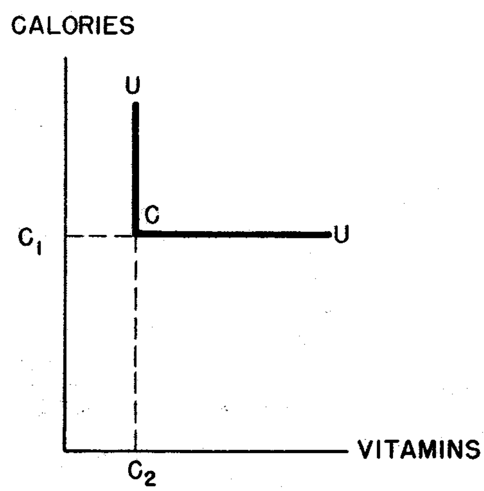


Figure 2b: Biologist's contour of minimum needs.

The slope at C is anything between 0 and infinity, so once again we are at sea as to how to award points, or how to set shadow prices for calories and vitamins. It is no good to tell us what to do on the assumption that so much $C_1$ can be substituted for so much $C_2$. This is not the problem set to us.*

* If we could consider some linear relation between the C's as equally satisfactory, our problem would become a different (and much simpler) problem in linear programming. We would minimize $\sum_{i=1}^{n} P_i x_i$ subject to the <u>single</u> constraint

$$(10)\quad K_1 \sum_{j=1}^{n} a_{1j}x_j + K_2 \sum_{j=1}^{n} a_{2j}x_j + \ldots = A_1x_1 + \ldots + A_nx_n \geq C = K_1C_1 + \ldots + K_mC_m$$

where the K's are the substitution ratios (or relative points) of the respective elements. We would only have to compare $A_i/P_i$, the "final net utiles per dollar" yielded by the <u>ith</u> commodity, and pick the highest value. Presumably one food would be enough if the K's were really constants (as they are usually assumed not to be in economics).

For our original problem it would seem only reasonable to expect the relative point values of calories and vitamins to vary with commodity market prices rather than to be natural physiological (or psychological) constants. To find the appropriate shadow price or points in any problem will not be easy. This should not discourage us so much as to lead us to abandon the present line of attack.

## 6. The Simple Case of "Pure" Foods

Suppose we tackle a rather simple problem first — that in which all foods are "pure" foods, each containing something of one nutritional element and nothing of all others. Thus, $x_1$ may have only calories, $x_2$ only vitamins, $x_3$ perhaps with calories only (but not necessarily with the same number as $x_1$ or with the same market price), etc. In this pure foods case our problem obviously breaks up into m different, independent, simple problems. Among all the calorie foods we select that one which most cheaply gives us our calorie requirements. Similarly we select among the pure foods containing only iron for the cheapest way of buying iron. Our final cost of an optimal diet is the sum of the cost of getting calories <u>and</u> <u>each</u> of the other elements.

Let us consider calories alone. For this purpose we might as well number all the pure calorie foods $x_1$, $x_2$, $x_3$, ... with market prices $P_1$, $P_2$, $P_3$, ... and with unit calorie contents $a_{11}$, $a_{12}$, $a_{13}$, ... We must minimize

$$(11) \qquad Z = P_1x_1 + P_2x_2 + \dots \qquad \text{subject to}$$
$$a_{11}x_1 + a_{12}x_2 + \dots \geq C_1$$
$$x_1 \geq 0,\ x_2 \geq 0,\ \dots.$$

We could get our answer by simply picking the greatest of $a_{11}/P_1$, $a_{12}/P_2$, etc. and concentrating on the corresponding food.

It will turn out to be a little more convenient to define a shadow price for calories. Let this be called $y_1$. As yet we don't know its value (in dollars per calories). The "unit profitability" of any food can be thought of as dollars of calorie revenue it brings in minus its cash cost per unit; or as

$$\pi_1 = a_{11}\, y_1 - P_1$$
$$\pi_2 = a_{12}\, y_1 - P_2$$
$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$
$$\pi_i = a_{1i}\, y_1 - P_i$$
$$\cdot \qquad \cdot$$

These are definite numbers once $y_1$ is given a definite value. At first glance one might suppose that we should pick the largest of these profitability numbers and get our calories from the corresponding food. But a second thought will convince us that this is not a valid procedure. If one food cost twice as much as another and had twice as many calories, its profitability would be twice as great; but there would be no advantage whatever in choosing one good over the other.

Our profitabilities have to be put on a per-dollar basis if they are to be comparable. We could work with

$$\frac{\pi_1}{P_1} = \frac{a_{11}}{P_1} y_1 - 1$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\frac{\pi_i}{P_i} = \frac{a_{1i}}{P_i} y_1 - 1$$
$$\vdots \qquad \vdots \qquad \vdots$$

The greatest of these would give us our cheapest food. For any positive $y_1$, or calorie shadow price, this would be the same thing as picking the greatest $a_{1i}/P_i$, or number of calories per dollar, which was our earlier approach.

Let us suppose that the <u>k</u>th food is the best one, so that

$$\frac{a_{1k}}{P_k} \geq \frac{a_{1i}}{P_i} \qquad i \neq k$$

Our best calorie diet is $(X_1, X_2, \ldots, X_k, \ldots X_i, \ldots) = (0, 0, \ldots, \frac{C_1}{a_{1k}}, \ldots, 0, \ldots)$ and its cost is

$$Z = 0 + 0 + \ldots + P_k \frac{C_1}{a_{1k}} + \ldots + 0 + \ldots$$

Obviously the (cheapest) extra or marginal cost of increasing our calorie requirements from $C_1$ to $C_1 + 1$ units would be

$$MC_1 = \frac{P_k}{a_{1k}}(C_1 + 1) - \frac{P_k}{a_{1k}} C_1 = \frac{P_k}{a_{1k}} = \frac{\partial Z}{\partial C_1}$$

The <u>cost</u> of calories - and note I do <u>not</u> quite say the <u>worth</u> of calories - would seem to be given by $P_k/a_{1k}$. It seems natural therefore to say that this is the proper (shadow) price of calories; or

$$y_1 = \underset{i}{\text{Min}} \frac{P_i}{a_{1i}} = \frac{\partial Z}{\partial C_1} = MC_1$$

With our shadow price now determined, we can go back and look at our original profit figures $\pi_1$, $\pi_2$, ... etc. Our objection to them - that they are not a per dollar basis - now disappears. In every case the profits are negative except in the case of our very cheapest calorie source, $x_k$. Thus

$$\pi_1 = a_{11}y_1 - P_1 \leq 0$$
$$\pi_2 = a_{12}y_1 - P_2 \leq 0$$
$$\cdot \quad \cdot \quad \cdot$$
$$\pi_k = a_{1k}y_1 - P_k = 0$$
$$\cdot \quad \cdot \quad \cdot$$
$$\pi_i = a_{1i}y_1 - P_i \leq 0$$

If the unusual should happen and some other good also had exactly zero profit, then it would be a matter of indifference as to how we divided our calorie expenditure between the cheap good and $x_k$.

In effect we have determined the highest possible price for calories that is possible. Or more accurately, the highest possible calorie price compatible with some good in the system "breaking even". All other non-optimal goods will show a loss. In order to solve a minimum problem (lowest cost from selecting best x's) we have in effect chosen to solve a maximum problem (highest price for calories). This is just like what turned out to be true in my earlier examination of comparative advantage as a problem in linear programming. Then in order to get the x's that would give maximum national income or product, we had to find the price of shadows that would yield minimum income (or cost outlay).

Is there any advantage in trying to solve a problem of best quantities by replacing it by a hardly less difficult price problem in linear programming? Thus our correct $y_1$ is the solution of

(11) $$Z' = C_1 y_1 \qquad \text{to be a maximum subject to}$$

$$\begin{aligned} a_{11}y_1 &\leq P_1 \\ a_{12}y_1 &\leq P_2 \qquad , \; y_1 \geq 0 \\ &\cdots \\ a_{1i}y_1 &\leq P_i \end{aligned}$$

Whether there is any advantage to this indirect approach or not, it well illustrates an important theoretical point: the "principle of duality".

To each minimum problem in the x's of the form

(12) $$Z = \sum_{j=1}^{n} P_j x_j \qquad \text{to be a minimum subject to}$$

$$\sum_{j=1}^{n} a_{kj}x_j \geq C_k \qquad k = 1, 2, \ldots, m$$

$$x_j \geq 0 \qquad j = 1, 2, \ldots, n$$

there corresponds a dual maximum problem of the form

$$(12)' \qquad Z' = \sum_{k=1}^{m} C_k y_k \qquad \text{to be a maximum subject to}$$

$$\sum_{k=1}^{m} a_{kj} y_k \leq P_j \qquad j = 1, 2, \ldots, n$$

$$y_k \geq 0 \qquad k = 1, 2, \ldots, m$$

Note that everything has been transposed or "turned on its side". This duality is remarkable. Even more remarkable is the fact that the best (maximum) Z' turns out to be equal to the best (minimum) Z. And any non-optimal Z' will be algebraically less than any Z (optimal or not).

We must reserve judgment as to whether shadow prices are at all helpful. But at the moment it looks rather discouraging, since to find the right shadow prices appears to be just as difficult a problem in linear programming as to find the right quantities. In fact the thoughtful reader should be more discouraged still by the proposed method. Let us give the names problem 1 to the task of finding the best quantities and problem 2 to the task of finding the best prices. The penetrating reader will say, "You tell me to solve problem 1 by first solving an equally difficult problem 2. By the same reasoning, I should try to solve problem 2 by a method of — so to speak — shadow prices to shadow prices, or by an equally difficult problem 3."

All this looks like an infinite regress. And if our critic is still more acute, he will claim that it involves a vicious circle. For after you have "turned a problem on its side" to get a second problem, when you then turn that problem on its side, you end up not with a new third problem but rather with your first problem back again. If problem 2 is "dual" to problem 1, then 1 is also dual to 2. We seem almost to be perpetrating the swindle: "Solve a problem by solving it, and waste a little time on the way."

There is some merit in this contention. Unless we can reverse the doctrine "divide and conquer" to make it read "expand your problem and conquer", we shall have gained nothing by using prices. Our hope must be that solving the two dual problems simultaneously will have some advantages over solving each separately.

* * *

All that we have established in the pure calorie foods case also holds for the vitamins pure food case and for any other pure foods. Let me recapitulate what we have established:

1. We can define the profitability of any pure food in terms of dollars per unit; i.e. the $\pi_1$, $\pi_2$, ...
2. To do this we must first know the (shadow) price of the nutritional elements; i.e. the $y_1$ per calories, the $y_2$ per vitamins, ... $y_m$, etc.
3. The shadow prices have a number of properties:
   a. They are in dollars per unit of each nutritional element.
   b. They represent marginal costs; i.e. the least cost of an extra prescribed unit of the element in question.
   c. They must be such as to make profits negative for a category of goods — namely those goods that are not bought at all.

d. They must be such as to make profits zero for any good that is bought in positive quantities.

e. Their values are in economic jargon "derived demands" and depend on the prices of the goods, $(P_1, \ldots, P_n)$, on the food contents $(a_{ij})$, and (possibly) on the specified requirements, $(C_1, \ldots, C_m)$.

f. The shadow prices $(y_1, \ldots y_m)$ are the solution of the "dual" or "transpose" problem to our original problem. They give us the maximum amount of what might be called "economic rent" that can be imputed to the nutritional elements; and the sum total of optimal rents must add up to exactly the same value as the total optimal cost of foods.*

---

* The interested reader can verify all this by working out our earlier arithmetical example of calories and vitamins on the assumption that only the goods $x_1$, $x_2$, $x_3$ can be bought. The data and solution are respectively

$$\left[\begin{array}{c|c} a & C \\ \hline P & Z \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 700 \\ 0 & 1 & 0 & 400 \\ \hline 2 & 20 & 3 & ? \end{array}\right], \qquad \begin{array}{r} (x_1, x_2, x_3) = (700, 400, 0) \\ (y_1, y_2) = (2, 20) \end{array}$$

$$2(700) + 20(400) = 9400 = Z = Z' = 700(2) + 400(20)$$

---

## 7. General Case of Mixed Commodities

The above conclusions were developed for the almost trivial case of pure goods, where no food had more than one nutritional element. It so happens that they remain substantially unchanged if we consider the more realistic and complicated case where foods are mixtures of many elements. A slightly more complicated argument is necessary to demonstrate this; at the same time, the actual task of finding a best diet is also slightly more complex.

To handle the case of foods which contain more than one element, let us think of any such commodity as a package containing these different components. Thus a package of mixed nuts contains so many almonds, filberts, peanuts and so forth. Likewise in our earlier 5-variable arithmetic example, $x_4$ was a package of one calorie and one vitamin unit; $x_5$ was a package of two calories and one vitamin units.

Let us suppose that there were some way of costlessly decomposing any commodity into its different elements. Then the price of any food, say $P_5$, must be just equal to the value of its components, where each is evaluated at its shadow price, $y_1$, $y_2$,... etc. Were this not so, an excess profit would be possible and eventually in a competitive market the price of the elements would have to change so as to bring this about, in much the same way as the price of pork comes into relation to the price of corn over the long run.

Thus, for any good that is actually ever bought and sold, we must end up with

$$P_k = a_{1k}\, y_1 + a_{2k}\, y_2 + \dots + a_{mk}\, y_m \quad ,$$

or what is the same thing, with

$$\pi_k = a_{1k}\, y_1 + a_{2k}\, y_2 + \dots + a_{mk}\, y_m - P_k = 0 \;.$$

Profitability must end up equal to zero. More precisely, profitability must be zero if it is rational to buy the food at all. If a food is non-optimal, then its profitability will have to be negative. Thus if $x_i$ is <u>not</u> a good buy, we will have

$$\pi_i = a_{1i}\, y_1 + a_{2i}\, y_2 + \dots + a_{mi}\, y_m - P_i < 0$$

Let us revert back to our five variable example

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | NRC Requirements | Shadow Price |
|---|---|---|---|---|---|---|---|
| Calorie | 1 | 0 | 1 | 1 | 2 | 700 | ? |
| Vitamins | 0 | 1 | 0 | 1 | 1 | 400 | ? |
| Prices | 2 | 20 | 3 | 11 | 12 | Z = ? | |
| Best diet | ? | ? | ? | ? | ? | | |

Our unknowns are the best diet and its cost; also the correct shadow prices for vitamins and calories. We happen to have been told that the best diet is (0, 0, 0, 100, 300). It can also be revealed that the correct shadow prices for calories and vitamins are $(y_1, y_2) = (1, 10)$. How can we find this out for ourselves? And what good is this last information once we have acquired it?

To find it out for ouselves, we might set down the rule:

Find shadow prices which permit of no positive profits anywhere in system and which permit zero profits somewhere in the system.

This rule certainly will help us to rule out a number of price configurations. Thus we could never have the vitamin price greater than 20 because that would make

$$\pi_2 = 1y_2 - 20 > 0$$

Likewise, we couldn't ever have the calories price greater than 2 since that would make $\pi_1 > 0$. We might simply examine all of our profit figures, and try all combination of the y's until we end up with a pattern of profits

$$\begin{aligned} \pi_1 &= a_{11}y_1 + \dots + a_{m1}\, y_m - P_1 \le 0 \\ &\vdots \\ \pi_n &= a_{n1}y_1 + \dots + a_{mn}\, y_m - P_n \le 0 \end{aligned} \tag{13}$$

and where the equality sign holds at least once.

After much lucky experimentation, we might be lucky enough to hit upon the combination $(y_1, y_2) = (1, 10)$ which results in

$$\pi_1 = 1(1) + 0\ (10) - 2 = -1 < 0$$

$$\pi_2 = 0(1) + 1\ (10) - 20 = -10 < 0$$

$$\pi_3 = 1(1) + 0\ (10) - 3 = -2 < 0$$

$$\pi_4 = 1(1) + 1\ (10) - 11 = 0$$

$$\pi_5 = 2(1) + 1\ (10) - 12 = 0$$

We might conclude that our rule had led us to the true equilibrium prices. But this is too hasty a conclusion. Perhaps there are other lucky guesses for the y's that would also give us a pattern of profits compatible with the above rule. Even worse, our result seems to be independent of the C's prescribed by the NRC; we certainly have not used the C's in applying the rule or calculating profits.

Our darkest suspicions are confirmed when we happen to try any of the following price combinations:

| | | |
|---|---|---|
| $(y_1, y_2) =$ | $(0, 11)$ | A |
| or | $(1, 10)$ | B |
| or | $(2, 8)$ | C |
| or | $(2, 0)$ | D |

We have already seen that B satisfies our profit rule. The reader may verify that any of these other points also gives profits which are nowhere positive. Thus, he will find that for D,

$$(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = (0, -20, -1, -9, -8)$$

Our rule leads not to one solution but to four solutions. Actually, the multiplicity of price patterns is much greater. It is infite. Any weighted average of the prices in A and B, or in B and C, or in C and D, also satisfy our rule. For example, the point $(y_1, y_2) = (1.3, 9.4)$, which is 3/10 of the way between B and C, gives a profit pattern whose algebraic signs are (-, -, -, -, 0), etc.

All this is summarized in Figure 3. The lines $\pi_1 - \pi_1, \ldots, \pi_5 - \pi_5$, represent the combinations of the shadow prices that will make each good show zero profitability. The "pure goods", $x_1$, $x_2$ and $x_3$, all yield east-west or north-south lines. The mixed goods have profit boundaries that slope downward depending upon the mixture of calories and vitamins.

Our rule of no positive profits means we must always be below or to the left of every line. Also shadow prices cannot be negative; so that in all our profitability rule constrains us to the five-sided area ABCDE.

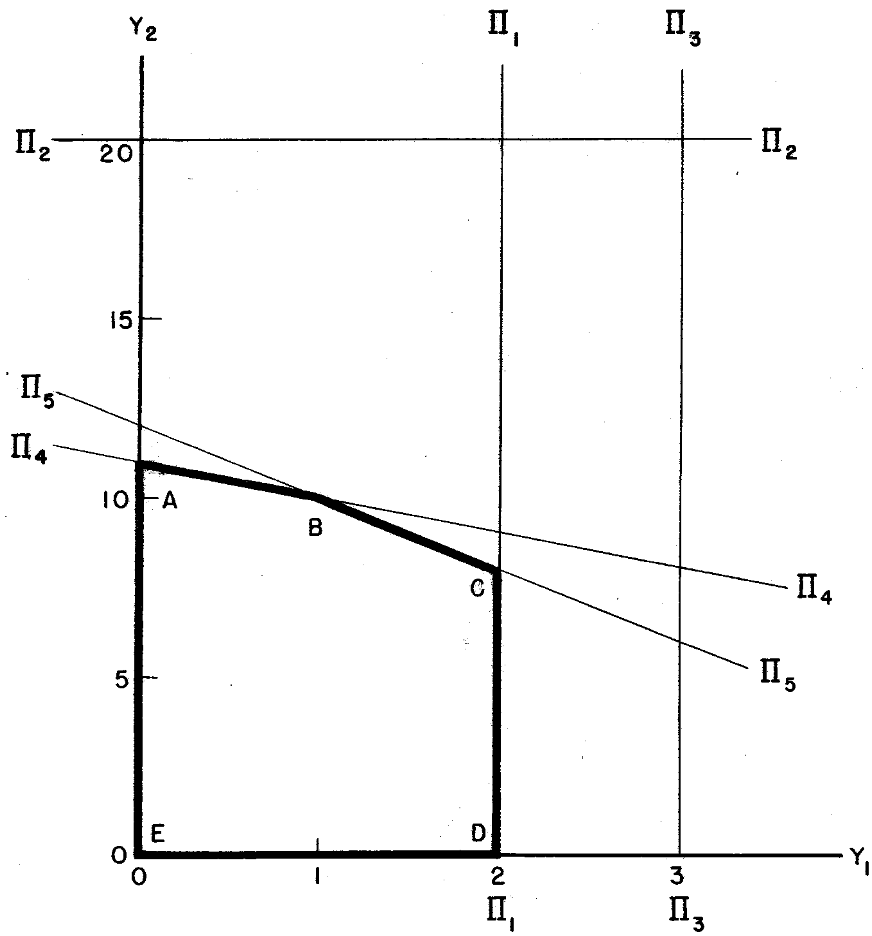


FIG.3

Our rule of no positive profits certainly needs amplification. So long as we had only one constraint — as in the simple theory of comparative advantage — the rule worked out satisfactorily. But now that we have two (or more) constraints, the rule gives us too many possible sets of equilibrium prices. What we seem to need in Figure 3 is a "best" direction to aim toward. And intuitively one feels that this best direction can only be supplied by a knowledge of the C's, or minimum nutritional requirements.

How can we generalize our rule so as to introduce the C's and lead to the relevant equilibrium price pattern? A clue is provided by a market analogy. Suppose that there are middlemen who buy the mixed foods and decompose them into their elements, selling the resulting pure calories, pure vitamins, etc. The (shadow) prices offered to middlemen for the elements must be such as to leave them with no (excess) profits even when they handle only the optimal foods, and with negative profits if they are rash enough to handle uneconomical goods. So much, so good, but this carries us no farther than our previous rule, which we saw needed amplification.

Our more complete rule for setting shadow prices can now be formulated as the following:

> Consistently with there being no positive profits anywhere in the system, set the y prices so that a maximum of total dollar revenue is realized from all the total required elements. (In economic jargon, the equilibrium prices must maximize the sum total of imputed or "derived" rents.)

The total revenue from all elements is the sum of calorie revenue, $C_1y_1$, plus vitamine revenue, $C_2y_2$, ... and so forth. Hence, our procedure for setting shadow prices may be stated mathematically as:

(14) $Z' = C_1y_1 + \ldots + C_my_m$ is to be a maximum subject to

$$a_{11}y_1 + a_{21}\ y_2 + \ldots + a_{m1}y_m \leq P_1$$

$$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \leq P_2$$

$$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \leq P_n$$

$$y_1 \geq 0,\ y_2 \geq 0,\ \ldots\ y_m \geq 0.$$

A minimum cost problem has been turned into a maximum rent problem! A quantities problem has been turned into a prices problem! This is the remarkable duality feature, discovered by von Neumann and other mathematicians but quite consistent with economic reasoning. Thus we should expect the national income ("rents") paid to factors to equal in a simple economic system the value of national output sold; or (maximum) Z' = (minimum) Z.

Let us apply our generalized rule to see whether it does select out the correct one of the four consistent price patterns: (0, 11), (1, 10), (2, 8), (2, 0). Our four different total of rents is

$$
\begin{aligned}
Z' &= C_1\, y_1 + C_2\, y_2 = 700\, y_1 + 400\, y_2 \\
\text{or} \quad &= 700\ (0) + 400\ (11) = 400, && \text{A} \\
\text{or} \quad &= 700\ (1) + 400\ (10) = 4700, && \text{B} \\
\text{or} \quad &= 700\ (2) + 400\ (8) = 4600, && \text{C} \\
&= 700\ (2) + 400\ (0) = 1400, && \text{D}
\end{aligned}
$$

Clearly the second case, B, represents the true optimum or maximum Z', which does equal the minimum Z = 4700 that we have earlier seen.*

* ABCDE is a convex contour. It follows, as common sense will confirm, that raising calorie requirements will — if anything — raise the shadow price of calories.

Figure 4 indicates this same solution for optimal prices. We are free to move in ABCDE so as to maximize Z'. Contours of equal Z' are given by parallel lines with absolute slopes $C_1/C_2$ = 700/400. The arrows (perpendicular to the contour)

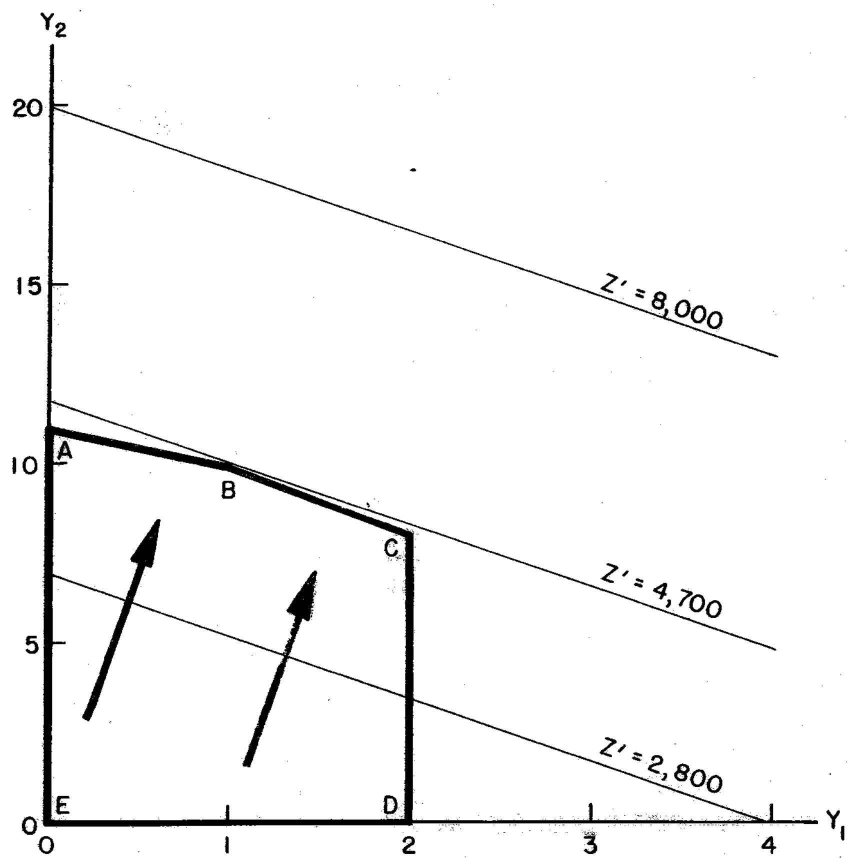


FIG. 4

indicate the direction in which our optimum lies. Clearly the best place to end up is at B where Z' = 4700; anywhere else will give a lower Z'; only at B will the broken line ABC be touching (but not crossing) the highest Z' contour.

If we imagine an Office of Price Administration (OPA) that tries to find the best prices by the deliberate use of intelligence, its task is now finished. It has posed for itself a prices—maximum problem in linear programming, rather than the quantities—minimum problem that a War Production Board (WPB) might formulate.

If the OPA mathematicians are omniscient, they will have found the best solution to this dual problem and set prices accordingly.

## 8. Determining Quantities by Use of Prices

Let us imagine that we have somehow been given the solution to our dual problem and know the correct set of (shadow) prices. We can use them immediately in two ways. First, we may directly compute the minimum cost of the best diet even before we know the composition of that diet! The minimum cost, Z, has been shown to be equal to the maximum rents, $Z' = \sum C_i y_i$ (= 4700 in our arithmetic example).

Second, we now know what the marginal costs of specifying one extra calorie unit or other nutritional element will be. These marginal costs are nothing but our shadow prices.*

$$(15) \qquad MC_1 = y_1 = \left(\frac{dZ}{dC_1}\right); \; \ldots \; ; MC_i = y_i = \left(\frac{dZ}{dC_i}\right), \ldots$$

* As mentioned earlier, the Langrangean multipliers, $\lambda_i$, and the prices, $y_i$, are essentially the same thing. In ordinary well-behaved non-linear maximum problems a general theorem assures us that the derivatives of the form $\partial Z/\partial C_i$ are exactly equal to the Lagrangean multiplier, $\lambda_i$. See P.A. Samuelson, Foundations, p. 132. It is remarkable that in the boundary maxima of linear programming the $\lambda$'s have the same property. Also, when a $C_k$ is not binding, $\lambda_k = 0$, just as $MC_k$ obviously should be too. By direct differentiation of $Z' = \sum^k C_k y_k$, we find $(\partial Z'/\partial C_k) = y_k \mp$ (strangely enough) $(dZ/dC_k)$.

But most important of all, once the OPA or anyone else has given us best shadow prices, we can use this information to select out deliberately the best quantities in the diet. Or we can set up quasi-automatic little market units and mechanisms which will end up near the best solution.

Given the best prices — e.g. $(y_1, y_2) = (1, 10)$ in our problem — we know that at least all but $\underline{m}$ of our goods can be regarded as unprofitable and their quantities set equal to zero. Thus, for $(y_1, y_2) = (1, 10)$, we have $(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = (-, -, -, 0, 0)$, so that our best diet will be $(X_1, X_2, X_3, X_4, X_5) = (0, 0, 0, X_4?, X_5?)$. Because there are only two elements, calories and vitamins, we can concentrate on only two foods, in this case $X_4$ and $X_5$. We must select the precise amount of these two foods so as to achieve our nutritional constraints. These constraints are also 2( or m) in number, so that we have enough equations to determine our 2 unknowns.

Our general procedure is as follows:

Suppose we know our best prices. They are m in number at most; but some prices may be zero, so that we may have only $r \leq m$ effective constraints. This means there will be some r economical goods which will exactly satisfy the effective constraints. We solve these r equations for exact values of the r economical X's.

The process of solving simultaneous equations is straight forward, but very tedious to do even in a mathematics laboratory. How do people ordinarily solve the problem of not starving to death or becoming malnourished? In part they are taught good habits: Those taught very bad habits may have died out long ago. In part there is some physiological evidence that Nature's evolutionary development has resulted in built-in "thermostats" of hunger and appetite so that when an animal has eaten too little salt over a period of time, it "hankers for" and seeks out salt.

In a crude way, it is as if there is a dynamic mechanism tending to make the animal seek foods containing much of the missing element. If we call $C_i$ (t) the amount of the ith nutritional element that the animal is currently getting, and if $C_i$ is the amount it needs for good health, then specific hunger causes $dC_i(t)/dt$ to be negative whenever $C_i(t) - C_i$ is positive, and vice versa. In this way, over time, the animal averages out a tolerable if not optimal diet.

* * *

The exact solution to our arithmetic problem is to determine the amounts of $X_4$ and $X_5$ from our calorie and our vitamin equations; or from

$$1\ X_4 + 2\ X_5 = 700$$

$$1\ X_4 + 1\ X_5 = 400$$

or $$X_4 = 100, \quad X_5 = 300$$

as substitution or elementary algebra will verify.*

* A sophisticated economist might notice in this problem that it would be possible to define certain composite commodities, or market baskets of $X_4$ and $X_5$, which if properly weighted, would be found to consist entirely of calories and entirely of vitamins. He would have to be sophisticated because the relative weights could not be positive numbers. I cannot buy $X_4$ and $X_5$ and put them into a basket and hope that there will be only vitamins in the basket. But if I buy 2 units of $X_4$ and sell 1 unit of $X_5$, I will end up with one unit of vitamins and nothing else! My market basket or composite commodity has weights (+2, −1). Similarly to get pure calories, I must buy 1 of $X_5$ and sell 1 of $X_4$, leaving me with one calorie unit and nothing else.

What is the market cost or price of each of these composite commodities or constructed "pure" foods? For the vitamin basket it is twice the price of $X_4$ minus once the price of $X_5$, or 2(11) −1(12) = 10. For the calories, it is the net algebraic cost of the foods in the second basket or +1(12) −1(11) = 1. It is not surprising that we have ended up with (1, 10), already seen to be the shadow prices $(y_1, y_2)$ and $(MC_1, MC_2)$.

The reader may be referred to Foundations, pp. 135–146, for a discussion of such composite commodities and the laws of their price formation.

The reader may verify that if the NRC set the vitamin requirement $C_2 = 0$, then a best set of prices would be $(y_1, y_2) = (2, 0)$, and we would have only $X_1$ a non-zero quantity.* We would solve for the exact $X_1$, by the r = 1 equation- system

$$1 X_1 = 700 \qquad \text{or}$$

$$X_1 = 700$$

* At $C_2 = 0$, $MC_2$ is ambiguous and ill-defined. This is because $(y_1, y_2) = (2, h)$ are solutions to our dual problem for all $0 \leq h \leq 8$. This means that $0 \leq MC_2 \leq 8$. Our best direction in Figure 4 is eastward and the line-segment CD represents the set of optimal prices for our dual problem. The point C, itself, is not truly a correct point for our original problem since it falsely tells us that $X_4$ shows a zero profit and can be bought. In the most general case of m constraints, even when we have best prices, care is necessary in selecting the proper sets of variables (x's) and equations (constraints).

## 9. Simultaneous, Automatic Solutions of the Dual Problems

We have explored a great number of aspects of the general problem of linear programming. We have seen that to find equilibrium prices is just as hard as to find the equilibrium quantities directly. The task of a central price board is no less difficult than that of a central production board. The intelligence and data needed are very great.

This is somewhat disappointing. But to economists like A. P. Lerner or O. Lange,* who have advocated the use of pricing

* See A. P. Lerner, Economics of Control; O. Lange, "On the Economic Theory of Socialism", in S. Lippincott's collection of the same name.

systems even in a socialist state, there is still one important saving feature. They believe that an automatic market mechanism can be set up which will minimize the need for central authority and intelligence and will simultaneously solve both the price and quantity problem. The executives who run the different factories will take prices as given and according to profitability will and we would have only $X_1$ a non-zero quantity.* We would solve for the exact $X_1$, by the r = 1 equation- system

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At the end of my earlier paper on linear programming and international theory, we saw that something like this could be envisaged in the case of comparative advantage. In the diet problem, can we also set up mechanisms which will automatically give us both prices and quantities?

The answer seems to be in the affirmative. It is in a sense easier to define dynamic processes which give us quantities and prices at the same time than it is to get either alone.

A procedure that will seem reasonable to an economist is pretty much as follows:

1. Let the value of any $x_i$ grow at a rate which is proportional to its profitability, $\pi_i$, where this is computed as described earlier in terms of any initial set of shadow prices.

   Let $x_i$ decrease if profits are negative, at a rate proportional to these losses — but with the proviso that when any $x_i = 0$, it can fall no further.

2. Our rule for prices is similar. Let the shadow price of any nutritional element, say $y_j$, diminish whenever the total of that element forthcoming is greater than the minimum prescribed $C_j$. Similarly, when there is a deficiency of any element, then let its price grow at a

rate proportional to the deficiency, subject to the proviso that no price can be negative.

These rules define our system's movement over time starting with any non-negative x's and y's. Let us call $C_j(t)$ the total of the jth element being yielded by the system's current X values, so the $\left[C_j(t) - C_j\right] = c_j(t)$ is the "algebraic excess" of the jth element. Then mathematically our two rules state that

$$\alpha_1 \frac{dx_1}{dt} = \pi_1, \ \ldots, \ \alpha_n \frac{dx_n}{dt} = \pi_n \tag{16}$$

$$\text{with } x_1 \geq 0, \ \ldots \ , \ x_n \geq 0$$

and

$$-\beta_1 \frac{dy_1}{dt} = c_1, \ \ldots \ , -\beta_m \frac{dy_m}{dt} = c_m \tag{16'}$$

$$\text{with } y_1 \geq 0, \ \ldots \ , \ y_m \geq 0.$$

and where the $\alpha$'s and $\beta$'s are positive proportionality time constants.

The system can only settle down and come to rest if profits are everywhere zero for all goods that are not themselves zero; and if "excesses" are everywhere zero for all elements whose prices are not zero. For all zero y's and x's the corresponding excesses are positive and the profits are negative.

The economic common sense of all this is something like the following: Too low shadow prices will yield high profits and great expansion of quantities; but expansion of quantities will flood the market with excessive supplies and cause prices to fall; this in turn will choke off the excess by diminishing output; and the system will oscillate around its equilibrium values.

Note that I do not say the system will necessarily settle down. Actually for this system it will tend to oscillate in non-damped but also non-explosive oscillations, rather like a pendulum in a "conservative" physical system without friction.*

* If we start the system out with a very good approximate guess to the true solution $(y_1^o, \ldots, y_m^o, x_1^o, \ldots, x_m^o)$, then the time averages of $[y_j(t), x_i(t)]$ will closely approximate the values $[y_j^o, x_i^o]$. Any bias will vanish with our initial error.

A little intelligent speculation or foresight can be expected to cause the system actually to settle down. I would conjecture that if we substract something from profits whenever there are great general surpluses of elements and make the growth rate, $dx_i/dt$, proportional to profits so adjusted rather than to $\pi_i$, then the system will finally damp down. Since some excesses are optimal — those for which the y's are zero — this will tend to bias our solution. But this bias can be avoided if we make sure that our correction to profits are only for "unwanted" excesses and are of the form

$$\alpha_i \frac{dx_i}{dt} = \pi_i - \sum_{j=1}^{m} K_{ij}\, c_j\, y_j$$

where the K's are non-negative proportionality time constants.

Likewise it makes economic sense to add still more dampening, if we wish to do so, by causing the growth of prices to be reduced whenever profits are "generally" too great.

Our final dynamic system might be something like

$$\alpha_i \frac{dx_i}{dt} = \pi_i - \sum_{j=1}^{m} K_{ij}\, c_j\, y_j \qquad (i = 1, 2, ..n)$$

$$x_i \geq 0$$

$$-\beta_j \frac{dy_j}{dt} = c_j + \sum_{i=1}^{n} M_{ji}\, \pi_i\, x_i \qquad (j = 1, 2, .., m)$$

$$y_j \geq 0$$

where the M's like the K's are non-negative time constants.

Later I hope to return to a mathematical consideration of the problems dealt with here.

Paul A. Samuelson