Cross-temporal Probabilistic Forecast Reconciliation: online appendix

Daniele Girolimetto
Department of Statistical Sciences, University of Padova
Email: daniele.girolimetto@phd.unipd.it

George Athanasopoulos
Department of Econometrics and Business Statistics, Monash University
Email: george.athanasopoulos@monash.edu

Tommaso Di Fonzo
Department of Statistical Sciences, University of Padova
Email: tommaso.difonzo@unipd.it

Rob J Hyndman
Department of Econometrics and Business Statistics, Monash University
Email: rob.hyndman@monash.edu

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A Alternative forms of the cross-temporal covariance matrix

In this appendix, some derivations of the solutions proposed in Section 4 to obtain an estimator of the cross-temporal covariance matrix are reported. Starting from the the definition of cross-temporal covariance matrix we obtain the first equivalence in (10). Therefore, we have that

$$\begin{aligned}
&\lambda\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts,D} + (1-\lambda)\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts} \\
&\qquad\Downarrow \\
\widehat{\boldsymbol{\Omega}}_{HB} &= \boldsymbol{S}_{ct}\left[\lambda\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts,D} + (1-\lambda)\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts}\right]\boldsymbol{S}_{ct}' \\
&= \lambda\boldsymbol{S}_{ct}\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts,D}\boldsymbol{S}_{ct}' + (1-\lambda)\boldsymbol{S}_{ct}\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts}\boldsymbol{S}_{ct}'.
\end{aligned}$$

The high-frequency time series representation (the second equivalence) can be derived in the following manner:

$$\begin{aligned}
\boldsymbol{\Omega} &= \boldsymbol{S}_{ct}\boldsymbol{\Omega}_{hf\text{-}bts}\boldsymbol{S}_{ct}' \\
&= (\boldsymbol{S}_{cs}\otimes\boldsymbol{S}_{te})\,\boldsymbol{\Omega}_{hf\text{-}bts}\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{S}_{te})' \\
&= (\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\,\boldsymbol{\Omega}_{hf\text{-}bts}\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})'\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})' \\
&= (\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\,\boldsymbol{\Omega}_{hf}\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})'
\end{aligned}$$

where $\boldsymbol{\Omega}_{hf} = (\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\,\boldsymbol{\Omega}_{hf\text{-}bts}\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})'$ and $\boldsymbol{S}_{ct} = \boldsymbol{S}_{cs}\otimes\boldsymbol{S}_{te} = (\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})$. We can apply the shrinkage estimator as

$$\begin{aligned}
&\lambda\widehat{\boldsymbol{\Omega}}_{hf,D} + (1-\lambda)\widehat{\boldsymbol{\Omega}}_{hf} \\
&\qquad\Downarrow \\
\widehat{\boldsymbol{\Omega}}_{H} &= (\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\left[\lambda\widehat{\boldsymbol{\Omega}}_{hf,D} + (1-\lambda)\widehat{\boldsymbol{\Omega}}_{hf}\right](\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})' \\
&= \lambda(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\widehat{\boldsymbol{\Omega}}_{hf,D}(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})' + (1-\lambda)(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\widehat{\boldsymbol{\Omega}}_{hf}(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})'.
\end{aligned}$$

The bottom time series representation (the third equivalence) follows by

$$\begin{aligned}
\boldsymbol{\Omega} &= \boldsymbol{S}_{ct}\boldsymbol{\Omega}_{hf\text{-}bts}\boldsymbol{S}_{ct}' \\
&= (\boldsymbol{S}_{cs}\otimes\boldsymbol{S}_{te})\,\boldsymbol{\Omega}_{hf\text{-}bts}\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{S}_{te})' \\
&= (\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\,\boldsymbol{\Omega}_{hf\text{-}bts}\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})'\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})' \\
&= (\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\,\boldsymbol{\Omega}_{bts}\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})',
\end{aligned}$$

where $\boldsymbol{\Omega}_{bts} = (\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\,\boldsymbol{\Omega}_{hf\text{-}bts}\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})'$ and $\boldsymbol{S}_{ct} = \boldsymbol{S}_{cs}\otimes\boldsymbol{S}_{te} = (\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})$. Finally we have that

$$\begin{aligned}
&\lambda\widehat{\boldsymbol{\Omega}}_{bts,D} + (1-\lambda)\widehat{\boldsymbol{\Omega}}_{bts} \\
&\qquad\Downarrow \\
\widehat{\boldsymbol{\Omega}}_{B} &= (\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\left[\lambda\widehat{\boldsymbol{\Omega}}_{bts,D} + (1-\lambda)\widehat{\boldsymbol{\Omega}}_{bts}\right](\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})' \\
&= \lambda\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\,\widehat{\boldsymbol{\Omega}}_{bts,D}\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})' +
\end{aligned}$$

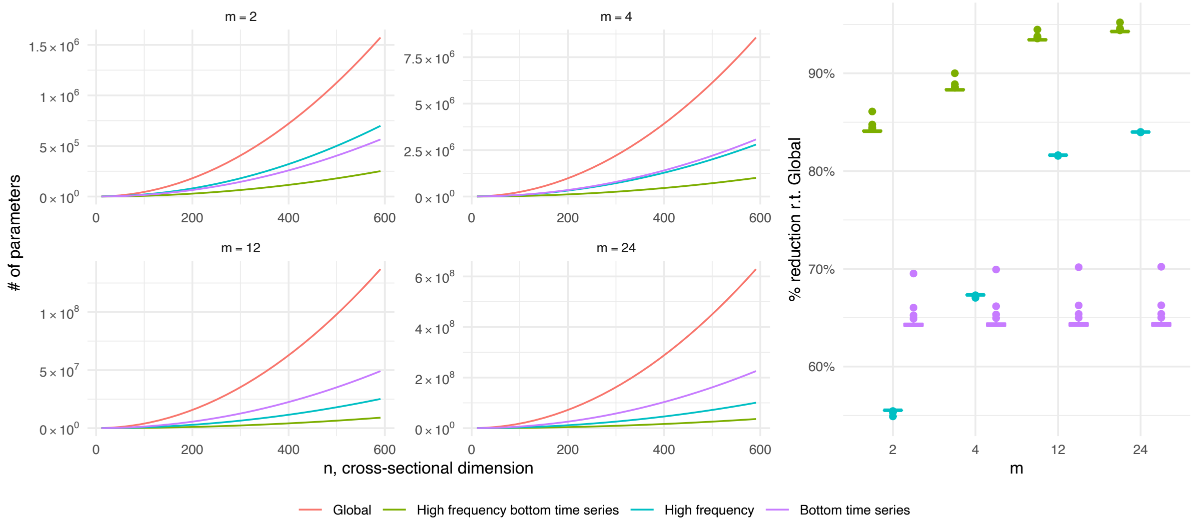

Figure A.1: *The four graphs on the left represent the number of different parameters in the covariance matrix for the various approaches presented for different values of m and n (n_b, the number of bottom time series, is about 60% of the total). On the right, we have the boxplot of the percentage reduction in the number of parameters compared to the global approach.*

$$(1-\lambda)\left(\boldsymbol{S}_{cs} \otimes \boldsymbol{I}_{m+k^*}\right) \widehat{\boldsymbol{\Omega}}_{bts} \left(\boldsymbol{S}_{cs} \otimes \boldsymbol{I}_{m+k^*}\right)'.$$

In general, the covariance matrix of the reconciled forecasts is equal to $\boldsymbol{M}\widehat{\boldsymbol{\Omega}}\boldsymbol{M}'$ where $\boldsymbol{M} = \boldsymbol{S}_{ct}\boldsymbol{G}$ is the projection matrix. When using the HB approach, the covariance matrix of the reconciliation and the base forecasts will be identical. Indeed, it can be shown (see Panagiotelis et al. 2021 for more details) that if $\boldsymbol{M}$ is a projection matrix (6) then $\boldsymbol{M}\boldsymbol{S}_{ct} = \boldsymbol{S}_{ct}\boldsymbol{G}\boldsymbol{S}_{ct} = \boldsymbol{S}_{ct}$, and we obtain that

$$\begin{aligned}
\boldsymbol{M}\widehat{\boldsymbol{\Omega}}_{HB}\boldsymbol{M}' &= \boldsymbol{M}\boldsymbol{S}_{ct}\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts,HB}\boldsymbol{S}_{ct}'\boldsymbol{M}' \\
&= \boldsymbol{S}_{ct}\boldsymbol{G}\boldsymbol{S}_{ct}\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts,HB}\boldsymbol{S}_{ct}'\boldsymbol{G}'\boldsymbol{S}_{ct}' \\
&= \boldsymbol{S}_{ct}\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts,HB}\boldsymbol{S}_{ct}' = \widehat{\boldsymbol{\Omega}}_{HB}.
\end{aligned}$$

Figure A.1 shows the number of parameters for different values of m and n, with n_b fixed to approximately 60% of n. The right panel reports the boxplot of the percentage reductions in the number of parameters compared to the global approach.

B Cross-temporal covariance matrix for the Monte Carlo simulation

We assume two AR(2) processes with correlated errors such that

$$y_{i,t} = \phi_{i,1} y_{i,t-1} + \phi_{i,2} y_{i,t-2} + \varepsilon_{i,t}$$

where $\boldsymbol{\varepsilon}_t \sim \mathcal{N}_2\left(\mathbf{0}_{(2\times 1)}, \boldsymbol{\Omega}_{cs}\right)$ and $i \in \{B, C\}$. Let $y_{i,T+h}$ be the true observation for the i^{th} series and $\widetilde{y}_{i,T+h}$ the corresponding forecasts such that

$$\begin{aligned} y_{i,T+1} &= \phi_{i,1} y_{i,T} + \phi_{i,2} y_{i,T-1} + \varepsilon_{i,T+1} \\ y_{i,T+2} &= \phi_{i,1} y_{i,T+1} + \phi_{i,2} y_{i,T} + \varepsilon_{i,T+2} \end{aligned} \quad \text{and} \quad \begin{aligned} \widetilde{y}_{i,T+1} &= \phi_{i,1} y_{i,T} + \phi_{i,2} y_{i,T-1} \\ \widetilde{y}_{i,T+2} &= \phi_{i,1} \widetilde{y}_{i,T+1} + \phi_{i,2} y_{i,T} \end{aligned},$$

then

$$\begin{aligned} y_{i,T+1} - \widetilde{y}_{i,T+1} &= \varepsilon_{i,T+1} \\ y_{i,T+2} - \widetilde{y}_{i,T+2} &= \varepsilon_{i,T+2} + \phi_{i,1}\varepsilon_{i,T+1}. \end{aligned}$$

Finally, we can compute each element of the high frequency bottom time series covariance matrix

$$\begin{aligned} Var\left(y_{i,T+1} - \widetilde{y}_{i,T+1}\right) &= \sigma_i^2, \quad \forall i \in \{B, C\} \\ Var\left(y_{i,T+2} - \widetilde{y}_{i,T+2}\right) &= \sigma_i^2\left(1 + \phi_{i,1}^2\right), \quad \forall i \in \{B, C\} \\ Cov\left[\left(y_{i,T+2} - \widetilde{y}_{i,T+2}\right), \left(y_{i,T+1} - \widetilde{y}_{i,T+1}\right)\right] &= Cov\left[\left(y_{i,T+1} - \widetilde{y}_{i,T+1}\right), \left(y_{i,T+2} - \widetilde{y}_{i,T+2}\right)\right] \\ &= \phi_{i,1}\sigma_i^2, \quad \forall i \in \{B, C\} \\ Cov\left[\left(y_{i,T+1} - \widetilde{y}_{i,T+1}\right), \left(y_{j,T+1} - \widetilde{y}_{j,T+1}\right)\right] &= Cov\left[\left(y_{j,T+1} - \widetilde{y}_{j,T+1}\right), \left(y_{i,T+1} - \widetilde{y}_{i,T+1}\right)\right] \\ &= \sigma_{i,j}, \quad \forall i,j \in \{B, C\}, \quad i \neq j \\ Cov\left[\left(y_{i,T+2} - \widetilde{y}_{i,T+2}\right), \left(y_{j,T+1} - \widetilde{y}_{j,T+1}\right)\right] &= Cov\left[\left(y_{j,T+1} - \widetilde{y}_{j,T+1}\right), \left(y_{i,T+2} - \widetilde{y}_{i,T+2}\right)\right] \\ &= \phi_{i,1}\sigma_{i,j}, \quad \forall i,j \in \{B, C\}, \quad i \neq j \\ Cov\left[\left(y_{i,T+2} - \widetilde{y}_{i,T+2}\right), \left(y_{j,T+2} - \widetilde{y}_{j,T+2}\right)\right] &= Cov\left[\left(y_{j,T+2} - \widetilde{y}_{j,T+2}\right), \left(y_{i,T+2} - \widetilde{y}_{i,T+2}\right)\right] \\ &= \sigma_{i,j}\left(1 + \phi_{i,1}\phi_{j,1}\right), \quad \forall i,j \in \{B, C\}, \quad i \neq j. \end{aligned}$$

In conclusion,

$$\boldsymbol{\Omega}_{hf\text{-}bts} = \begin{bmatrix} \sigma_B^2 & & & \\ \phi_{B,1}\sigma_B^2 & \sigma_B^2\left(1 + \phi_{B,1}^2\right) & & \\ \sigma_{BC} & \phi_{B,1}\sigma_{BC} & \sigma_C^2 & \\ \phi_{C,1}\sigma_{BC} & \sigma_{BC}\left(1 + \phi_{B,1}\phi_{C,1}\right) & \phi_{C,1}\sigma_C^2 & \sigma_C^2\left(1 + \phi_{C,1}^2\right) \end{bmatrix}$$

and

$$\boldsymbol{\Omega}_{ct} = \mathbf{S}_{ct}\boldsymbol{\Omega}_{hf\text{-}bts}\mathbf{S}_{ct}'.$$

C Monte Carlo Simulation: one-step residuals and shrinkage covariance matrix

In Section 4.1, we discussed the use of one-step residuals in estimating the covariance matrix. In particular we point out that one-step residuals lead to a biased estimate of the covariance matrix where some correlation are zeros by definition (see Figure C.2). In addition, Tables C.1, C.2 and C.3 show the Frobenius norm, CRPS, and ES skill scores as explained in the paper to investigate the effectiveness of one-step residuals. Moreover, in Tables C.4 and C.5, we have utilized a shrinkage matrix rather than the sample covariance matrix to assess the performance of our approach.

Reconciliation approach	**Generation of the base forecasts paths**								
	Gaussian approach: sample covariance matrix								
	ctjb	In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
base	8.260	17.638	16.733	22.178	21.789	7.748	6.549	3.409	2.215
ct(bu)	3.195	21.789	21.789	**21.789**	21.789	2.215	2.215	**2.215**	2.215
ct(shr_{cs}, bu_{te})	3.202	21.942	21.789	21.942	21.789	2.224	2.215	2.224	2.215
ct($wlsv_{te}, bu_{cs}$)	**3.183**	18.237	18.237	21.789	21.789	**2.188**	2.188	**2.215**	2.215
oct($wlsv$)	3.766	19.174	18.611	22.304	21.789	3.082	2.191	2.910	2.215
oct($bdshr$)	3.203	18.559	18.416	21.937	21.789	2.195	**2.184**	2.224	**2.215**
oct(shr)	5.217	25.015	23.457	23.413	**21.789**	2.260	2.202	2.226	2.215
oct($bshr$)	5.282	23.772	23.997	22.146	21.789	2.720	2.220	2.756	2.215
oct($hshr$)	6.161	**11.336**	**10.940**	23.598	**21.789**	4.138	4.167	2.225	2.215
oct($hbshr$)	5.731	11.379	10.940	22.146	21.789	5.085	4.167	2.756	2.215
oct$_h$(shr)	3.251	20.965	19.992	22.079	**21.789**	2.260	2.202	2.226	2.215
oct$_h$($bshr$)	3.602	21.306	21.022	22.146	21.789	2.720	2.220	2.756	2.215
oct$_h$($hshr$)	4.869	11.405	10.940	22.037	21.789	4.138	4.167	2.225	2.215
oct$_h$($hbshr$)	5.731	11.379	10.940	22.146	21.789	5.085	4.167	2.756	2.215

Table C.1: *Frobenius norm between the true and the estimated covariance matrix for different reconciliation approaches and different techniques for simulating the base forecasts. Entries in bold represent the lowest value for each column, while the blue entry represent the global minimum. The reconciliation approaches are described in Table 2.*

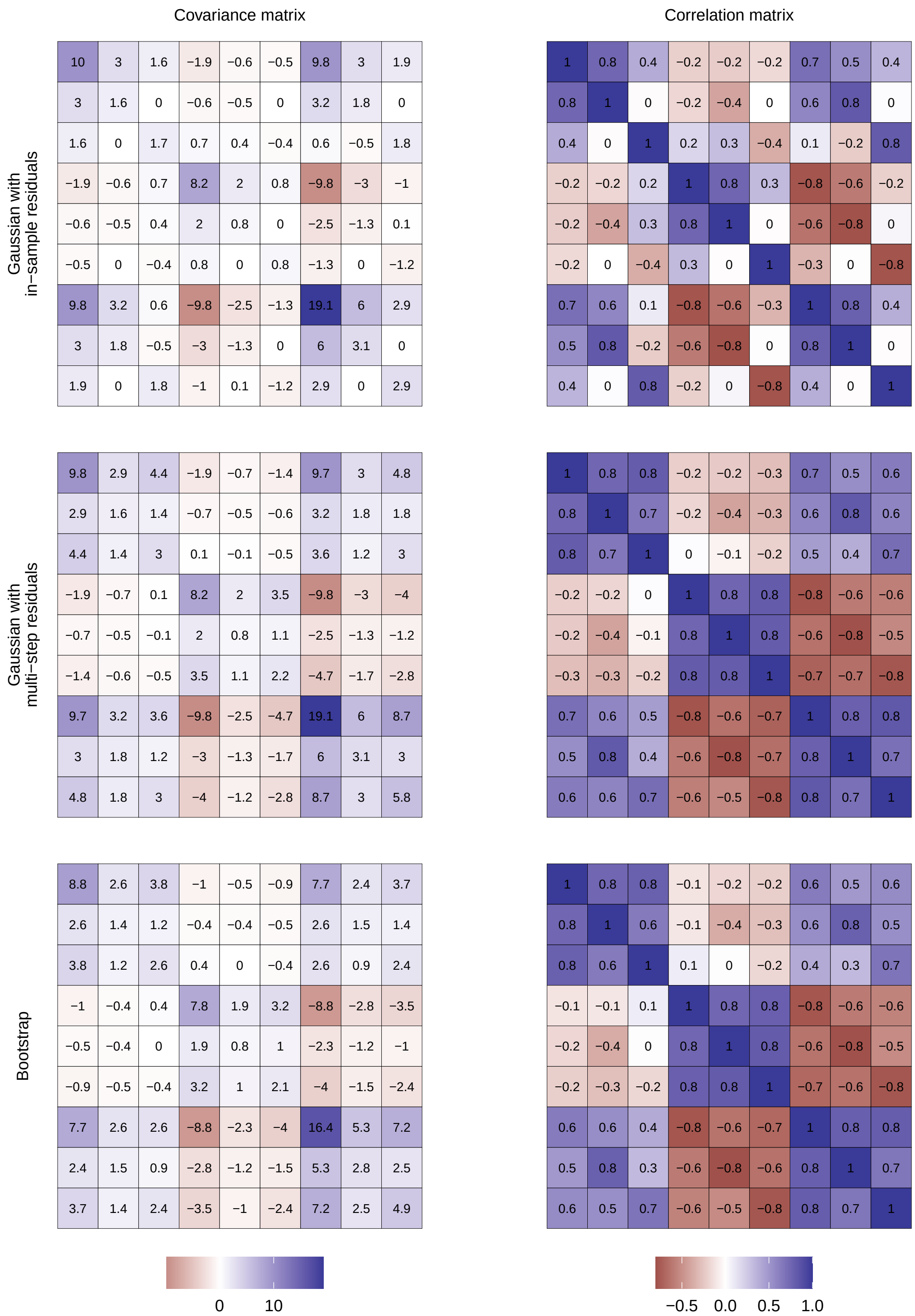

Figure C.2: *Comparison of estimated covariance and correlation matrices (first simulation) for base forecasts using a parametric Gaussian (with one-step residuals) approach. The true covariance and correlation matrices are shown in Figure 7.*

Reconciliation approach	Generation of the base forecasts paths								
	Gaussian approach: sample covariance matrix								
	ctjb	In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
				$\forall k \in \{2,1\}$					
base	1.000	1.008	1.009	1.044	1.047	0.998	0.999	1.002	1.004
ct(bu)	0.901	0.930	0.929	0.929	0.929	0.900	0.900	0.900	0.900
ct(shr_{cs}, bu_{te})	**0.901**	**0.929**	0.928	**0.929**	**0.928**	**0.900**	**0.899**	**0.900**	**0.900**
ct($wlsv_{te}, bu_{cs}$)	0.910	0.930	0.929	0.939	0.939	0.916	0.916	0.916	0.917
oct($wlsv$)	0.922	0.942	0.944	0.951	0.953	0.930	0.930	0.930	0.931
oct($bdshr$)	0.910	0.930	0.930	0.939	0.938	0.916	0.915	0.916	0.916
oct(shr)	0.941	0.999	0.985	0.983	0.973	0.903	0.902	0.902	0.903
oct($bshr$)	0.951	0.995	1.000	0.983	0.986	0.922	0.922	0.921	0.922
oct($hshr$)	0.987	0.995	0.993	1.039	1.026	0.972	0.972	0.974	0.975
oct($hbshr$)	0.987	0.995	0.996	1.024	1.028	0.985	0.985	0.987	0.989
oct$_h$(shr)	0.904	0.929	**0.928**	0.932	0.932	0.903	0.902	0.902	0.903
oct$_h$($bshr$)	0.923	0.948	0.952	0.951	0.954	0.922	0.922	0.921	0.922
oct$_h$($hshr$)	0.974	0.982	0.982	1.012	1.012	0.972	0.972	0.974	0.975
oct$_h$($hbshr$)	0.987	0.995	0.996	1.024	1.028	0.985	0.985	0.987	0.989
				$k = 1$					
base	1.000	1.017	1.019	1.017	1.019	0.998	0.999	0.999	1.000
ct(bu)	0.978	0.994	0.994	0.994	0.994	0.976	0.976	0.977	0.977
ct(shr_{cs}, bu_{te})	**0.977**	**0.993**	**0.993**	**0.994**	**0.993**	**0.976**	**0.976**	**0.976**	**0.976**
ct($wlsv_{te}, bu_{cs}$)	0.986	1.002	1.002	1.003	1.003	0.993	0.993	0.993	0.993
oct($wlsv$)	0.998	1.014	1.015	1.015	1.016	1.006	1.006	1.007	1.007
oct($bdshr$)	0.986	1.002	1.002	1.003	1.003	0.992	0.992	0.993	0.993
oct(shr)	1.037	1.082	1.067	1.064	1.056	0.979	0.978	0.979	0.979
oct($bshr$)	1.041	1.071	1.074	1.060	1.062	0.998	0.998	0.998	0.998
oct($hshr$)	1.080	1.090	1.091	1.119	1.105	1.050	1.050	1.053	1.053
oct($hbshr$)	1.065	1.080	1.081	1.088	1.090	1.063	1.064	1.066	1.068
oct$_h$(shr)	0.980	0.996	0.995	0.996	0.996	0.979	0.978	0.979	0.979
oct$_h$($bshr$)	0.999	1.016	1.018	1.016	1.018	0.998	0.998	0.998	0.998
oct$_h$($hshr$)	1.052	1.067	1.066	1.074	1.075	1.050	1.050	1.053	1.053
oct$_h$($hbshr$)	1.065	1.080	1.081	1.088	1.090	1.063	1.064	1.066	1.068
				$k = 2$					
base	1.000	0.998	0.999	1.071	1.075	0.998	0.999	1.005	1.008
ct(bu)	0.831	0.869	0.869	0.869	0.869	0.830	0.829	0.829	0.830
ct(shr_{cs}, bu_{te})	**0.830**	0.869	0.868	**0.868**	**0.868**	**0.830**	**0.829**	**0.829**	**0.830**
ct($wlsv_{te}, bu_{cs}$)	0.840	**0.863**	**0.862**	0.879	0.878	0.846	0.844	0.845	0.846
oct($wlsv$)	0.851	0.875	0.877	0.891	0.893	0.859	0.859	0.859	0.861
oct($bdshr$)	0.839	0.863	0.863	0.879	0.878	0.845	0.844	0.845	0.846
oct(shr)	0.854	0.922	0.909	0.908	0.897	0.833	0.831	0.832	0.832
oct($bshr$)	0.869	0.925	0.931	0.911	0.915	0.851	0.851	0.851	0.852
oct($hshr$)	0.901	0.908	0.904	0.966	0.952	0.900	0.899	0.901	0.902
oct($hbshr$)	0.915	0.917	0.919	0.964	0.969	0.913	0.913	0.914	0.917
oct$_h$(shr)	0.834	0.868	0.865	0.872	0.872	0.833	0.831	0.832	0.832
oct$_h$($bshr$)	0.852	0.886	0.890	0.890	0.894	0.851	0.851	0.851	0.852
oct$_h$($hshr$)	0.902	0.904	0.904	0.953	0.952	0.900	0.899	0.901	0.902
oct$_h$($hbshr$)	0.915	0.917	0.919	0.964	0.969	0.913	0.913	0.914	0.917

Table C.2: *Simulation experiment. AvgRelCRPS defined in Section 5.1. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths								
	Gaussian approach: sample covariance matrix								
	ctjb	In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
	$\forall k \in \{2,1\}$								
base	1.000	1.005	1.009	1.039	1.046	0.996	0.999	1.000	1.004
ct(bu)	0.897	0.924	0.923	0.924	0.923	0.895	0.896	0.897	**0.895**
ct(shr_{cs}, bu_{te})	**0.896**	0.924	0.923	**0.923**	**0.922**	**0.895**	**0.895**	**0.896**	0.896
ct($wlsv_{te}, bu_{cs}$)	0.906	0.924	0.923	0.933	0.932	0.912	0.911	0.910	0.912
oct($wlsv$)	0.916	0.935	0.937	0.944	0.945	0.923	0.923	0.923	0.924
oct($bdshr$)	0.906	**0.923**	0.923	0.932	0.932	0.910	0.910	0.911	0.912
oct(shr)	0.938	0.993	0.980	0.977	0.969	0.898	0.898	0.898	0.897
oct($bshr$)	0.947	0.990	0.995	0.979	0.981	0.915	0.915	0.915	0.915
oct($hshr$)	0.978	0.987	0.985	1.027	1.016	0.963	0.964	0.966	0.967
oct($hbshr$)	0.977	0.986	0.985	1.012	1.016	0.974	0.976	0.977	0.978
oct$_h$(shr)	0.900	0.923	**0.922**	0.926	0.925	0.898	0.898	0.897	0.898
oct$_h$($bshr$)	0.916	0.940	0.943	0.942	0.945	0.914	0.916	0.915	0.916
oct$_h$($hshr$)	0.967	0.974	0.974	1.002	1.002	0.964	0.964	0.966	0.967
oct$_h$($hbshr$)	0.978	0.984	0.986	1.012	1.015	0.975	0.976	0.977	0.980
	$k = 1$								
base	1.000	1.014	1.020	1.015	1.019	0.997	1.000	0.997	1.000
ct(bu)	0.969	0.985	0.983	0.985	0.984	**0.967**	**0.967**	0.968	**0.968**
ct(shr_{cs}, bu_{te})	**0.968**	**0.984**	**0.983**	**0.984**	**0.983**	0.968	0.967	**0.968**	0.968
ct($wlsv_{te}, bu_{cs}$)	0.977	0.991	0.991	0.992	0.992	0.984	0.983	0.981	0.984
oct($wlsv$)	0.989	1.002	1.004	1.003	1.004	0.994	0.995	0.995	0.997
oct($bdshr$)	0.977	0.989	0.991	0.992	0.992	0.981	0.982	0.983	0.985
oct(shr)	1.028	1.070	1.056	1.053	1.046	0.969	0.969	0.970	0.969
oct($bshr$)	1.034	1.061	1.065	1.051	1.053	0.985	0.987	0.986	0.987
oct($hshr$)	1.066	1.075	1.076	1.099	1.090	1.037	1.037	1.039	1.039
oct($hbshr$)	1.050	1.065	1.065	1.070	1.073	1.048	1.049	1.049	1.052
oct$_h$(shr)	0.971	0.985	0.985	0.986	0.986	0.969	0.969	0.969	0.969
oct$_h$($bshr$)	0.987	1.002	1.005	1.002	1.005	0.986	0.987	0.987	0.988
oct$_h$($hshr$)	1.040	1.053	1.053	1.059	1.058	1.036	1.036	1.040	1.040
oct$_h$($hbshr$)	1.051	1.064	1.063	1.071	1.073	1.047	1.049	1.051	1.052
	$k = 2$								
base	1.000	0.997	0.999	1.063	1.073	0.996	0.998	1.003	1.008
ct(bu)	0.831	0.867	0.867	0.867	0.867	0.829	0.829	0.830	**0.828**
ct(shr_{cs}, bu_{te})	**0.829**	0.867	0.866	**0.866**	**0.865**	**0.828**	**0.829**	**0.829**	0.829
ct($wlsv_{te}, bu_{cs}$)	0.839	**0.860**	**0.860**	0.877	0.876	0.844	0.844	0.844	0.845
oct($wlsv$)	0.849	0.872	0.875	0.887	0.890	0.858	0.856	0.856	0.857
oct($bdshr$)	0.839	0.861	0.861	0.876	0.875	0.845	0.843	0.845	0.844
oct(shr)	0.856	0.921	0.909	0.907	0.898	0.832	0.831	0.832	0.831
oct($bshr$)	0.868	0.924	0.930	0.911	0.915	0.849	0.848	0.849	0.848
oct($hshr$)	0.897	0.905	0.901	0.959	0.947	0.895	0.896	0.898	0.899
oct($hbshr$)	0.910	0.912	0.912	0.957	0.961	0.906	0.909	0.909	0.910
oct$_h$(shr)	0.835	0.865	0.862	0.870	0.868	0.833	0.833	0.831	0.832
oct$_h$($bshr$)	0.850	0.881	0.885	0.886	0.889	0.847	0.849	0.849	0.850
oct$_h$($hshr$)	0.900	0.902	0.901	0.947	0.948	0.897	0.896	0.897	0.899
oct$_h$($hbshr$)	0.910	0.910	0.914	0.957	0.961	0.907	0.908	0.909	0.912

Table C.3: *Simulation experiment. ES ratio indices defined in Section 5.1. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths								
	Gaussian approach: shrinkage covariance matrix								
	ctjb	In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
	$\forall k \in \{2,1\}$								
base	1.007	1.009	1.044	1.046	0.997	0.999	1.002	1.003	1.000
ct(bu)	0.929	0.929	0.929	0.929	0.899	0.900	0.900	0.900	0.901
ct(shr_{cs}, bu_{te})	**0.929**	0.928	**0.929**	**0.928**	**0.899**	**0.899**	**0.900**	**0.900**	**0.901**
ct($wlsv_{te}, bu_{cs}$)	0.930	0.930	0.939	0.938	0.915	0.916	0.917	0.916	0.910
oct($wlsv$)	0.943	0.944	0.951	0.952	0.929	0.930	0.931	0.930	0.922
oct($bdshr$)	0.930	0.930	0.938	0.938	0.915	0.916	0.916	0.916	0.910
oct(shr)	0.994	0.982	0.980	0.973	0.902	0.902	0.903	0.902	0.941
oct($bshr$)	0.995	0.998	0.983	0.986	0.921	0.922	0.922	0.922	0.951
oct($hshr$)	0.994	0.994	1.035	1.025	0.971	0.972	0.974	0.974	0.987
oct($hbshr$)	0.995	0.997	1.025	1.027	0.984	0.986	0.988	0.988	0.987
oct$_h$(shr)	0.929	**0.928**	0.932	0.932	0.902	0.902	0.903	0.902	0.904
oct$_h$($bshr$)	0.948	0.951	0.951	0.953	0.921	0.922	0.922	0.922	0.923
oct$_h$($hshr$)	0.982	0.982	1.011	1.011	0.971	0.972	0.974	0.974	0.974
oct$_h$($hbshr$)	0.995	0.997	1.025	1.027	0.984	0.986	0.988	0.988	0.987
	$k = 1$								
base	1.017	1.019	1.017	1.019	0.998	0.999	0.999	0.999	1.000
ct(bu)	0.994	0.994	0.994	0.994	0.976	0.976	0.977	0.976	0.978
ct(shr_{cs}, bu_{te})	**0.993**	**0.993**	**0.993**	**0.993**	**0.975**	**0.976**	**0.976**	**0.976**	**0.977**
ct($wlsv_{te}, bu_{cs}$)	1.002	1.002	1.003	1.003	0.992	0.993	0.993	0.993	0.986
oct($wlsv$)	1.015	1.015	1.015	1.016	1.005	1.007	1.007	1.007	0.998
oct($bdshr$)	1.002	1.002	1.003	1.002	0.992	0.992	0.993	0.992	0.986
oct(shr)	1.076	1.065	1.061	1.056	0.978	0.978	0.979	0.978	1.037
oct($bshr$)	1.070	1.072	1.060	1.062	0.997	0.998	0.998	0.998	1.041
oct($hshr$)	1.090	1.092	1.114	1.105	1.049	1.050	1.053	1.052	1.080
oct($hbshr$)	1.080	1.081	1.089	1.090	1.062	1.064	1.066	1.066	1.065
oct$_h$(shr)	0.996	0.995	0.996	0.996	0.978	0.978	0.979	0.978	0.980
oct$_h$($bshr$)	1.016	1.018	1.016	1.018	0.997	0.998	0.998	0.998	0.999
oct$_h$($hshr$)	1.066	1.067	1.075	1.075	1.049	1.050	1.053	1.052	1.052
oct$_h$($hbshr$)	1.080	1.081	1.089	1.090	1.062	1.064	1.066	1.066	1.065
	$k = 2$								
base	0.997	0.999	1.071	1.074	0.997	0.999	1.005	1.008	1.000
ct(bu)	0.869	0.868	0.868	0.868	0.829	0.829	0.830	0.830	0.831
ct(shr_{cs}, bu_{te})	0.868	0.867	**0.868**	**0.867**	**0.829**	**0.829**	**0.830**	**0.829**	**0.830**
ct($wlsv_{te}, bu_{cs}$)	**0.863**	**0.862**	0.878	0.878	0.845	0.845	0.846	0.846	0.840
oct($wlsv$)	0.876	0.877	0.891	0.892	0.859	0.860	0.860	0.860	0.851
oct($bdshr$)	0.863	0.863	0.878	0.877	0.844	0.845	0.846	0.845	0.839
oct(shr)	0.918	0.906	0.906	0.897	0.832	0.832	0.833	0.832	0.854
oct($bshr$)	0.924	0.928	0.911	0.915	0.850	0.851	0.852	0.851	0.869
oct($hshr$)	0.907	0.905	0.962	0.951	0.898	0.899	0.902	0.902	0.901
oct($hbshr$)	0.917	0.919	0.964	0.968	0.912	0.913	0.915	0.916	0.915
oct$_h$(shr)	0.867	0.864	0.872	0.871	0.832	0.832	0.833	0.832	0.834
oct$_h$($bshr$)	0.886	0.890	0.890	0.893	0.850	0.851	0.852	0.851	0.852
oct$_h$($hshr$)	0.904	0.905	0.952	0.952	0.898	0.899	0.902	0.902	0.902
oct$_h$($hbshr$)	0.917	0.919	0.964	0.968	0.912	0.913	0.915	0.916	0.915

Table C.4: *Simulation experiment. AvgRelCRPS defined in Section 5.1. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths								
		Gaussian approach: shrinkage covariance matrix							
	ctjb	In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
	$\forall k \in \{2, 1\}$								
base	1.005	1.008	1.039	1.045	0.996	0.999	1.000	1.003	1.000
ct(bu)	**0.923**	0.923	0.923	0.923	**0.895**	0.896	0.897	0.897	0.897
ct(shr_{cs}, bu_{te})	0.923	0.922	**0.922**	**0.922**	0.896	**0.895**	**0.895**	**0.895**	**0.896**
ct($wlsv_{te}, bu_{cs}$)	0.924	0.924	0.932	0.932	0.910	0.911	0.911	0.911	0.906
oct($wlsv$)	0.935	0.937	0.944	0.945	0.922	0.924	0.923	0.923	0.916
oct($bdshr$)	0.924	0.924	0.932	0.931	0.909	0.911	0.911	0.910	0.906
oct(shr)	0.989	0.978	0.975	0.968	0.897	0.898	0.898	0.898	0.938
oct($bshr$)	0.990	0.993	0.978	0.981	0.915	0.915	0.915	0.915	0.947
oct($hshr$)	0.986	0.985	1.024	1.015	0.963	0.964	0.966	0.967	0.978
oct($hbshr$)	0.985	0.986	1.012	1.015	0.973	0.976	0.977	0.978	0.977
oct$_h$(shr)	0.923	**0.922**	0.925	0.925	0.897	0.898	0.898	0.898	0.900
oct$_h$($bshr$)	0.941	0.943	0.942	0.945	0.913	0.915	0.915	0.915	0.916
oct$_h$($hshr$)	0.974	0.975	1.001	1.001	0.964	0.964	0.966	0.966	0.967
oct$_h$($hbshr$)	0.985	0.986	1.013	1.016	0.973	0.976	0.977	0.978	0.978
	$k = 1$								
base	1.014	1.018	1.015	1.019	0.997	0.999	0.997	0.998	1.000
ct(bu)	0.983	0.984	0.984	0.984	0.967	0.967	0.969	0.969	0.969
ct(shr_{cs}, bu_{te})	**0.983**	**0.982**	**0.982**	**0.983**	**0.966**	**0.967**	**0.966**	**0.966**	**0.968**
ct($wlsv_{te}, bu_{cs}$)	0.991	0.992	0.993	0.992	0.983	0.983	0.983	0.983	0.977
oct($wlsv$)	1.002	1.004	1.004	1.004	0.994	0.995	0.994	0.996	0.989
oct($bdshr$)	0.990	0.991	0.992	0.991	0.981	0.983	0.984	0.982	0.977
oct(shr)	1.065	1.054	1.051	1.045	0.969	0.970	0.970	0.969	1.028
oct($bshr$)	1.061	1.063	1.050	1.052	0.986	0.986	0.987	0.985	1.034
oct($hshr$)	1.076	1.077	1.095	1.088	1.036	1.036	1.040	1.038	1.066
oct($hbshr$)	1.064	1.065	1.071	1.073	1.047	1.048	1.050	1.050	1.050
oct$_h$(shr)	0.984	0.985	0.986	0.986	0.969	0.969	0.969	0.968	0.971
oct$_h$($bshr$)	1.003	1.005	1.003	1.005	0.985	0.987	0.987	0.986	0.987
oct$_h$($hshr$)	1.054	1.054	1.059	1.059	1.036	1.037	1.038	1.039	1.040
oct$_h$($hbshr$)	1.063	1.065	1.071	1.074	1.046	1.048	1.049	1.051	1.051
	$k = 2$								
base	0.996	0.998	1.064	1.073	0.995	0.999	1.003	1.007	1.000
ct(bu)	0.867	0.866	0.867	0.866	**0.829**	0.829	0.830	0.830	0.831
ct(shr_{cs}, bu_{te})	0.867	0.866	**0.866**	**0.866**	0.830	**0.829**	**0.830**	**0.830**	**0.829**
ct($wlsv_{te}, bu_{cs}$)	**0.861**	**0.861**	0.875	0.875	0.843	0.845	0.845	0.845	0.839
oct($wlsv$)	0.873	0.874	0.888	0.889	0.856	0.857	0.857	0.856	0.849
oct($bdshr$)	0.862	0.861	0.876	0.874	0.843	0.844	0.844	0.844	0.839
oct(shr)	0.918	0.907	0.905	0.898	0.831	0.832	0.832	0.832	0.856
oct($bshr$)	0.924	0.928	0.911	0.915	0.849	0.849	0.849	0.849	0.868
oct($hshr$)	0.904	0.901	0.957	0.946	0.895	0.896	0.898	0.900	0.897
oct($hbshr$)	0.912	0.913	0.956	0.961	0.905	0.909	0.909	0.911	0.910
oct$_h$(shr)	0.866	0.863	0.869	0.869	0.830	0.831	0.832	0.832	0.835
oct$_h$($bshr$)	0.882	0.886	0.886	0.889	0.846	0.848	0.849	0.848	0.850
oct$_h$($hshr$)	0.901	0.902	0.947	0.946	0.896	0.896	0.898	0.899	0.900
oct$_h$($hbshr$)	0.912	0.914	0.958	0.961	0.905	0.908	0.910	0.909	0.910

Table C.5: *Simulation experiment. ES ratio indices defined in Section 5.1. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

D Forecast reconciliation of the Australian GDP dataset

Athanasopoulos et al. (2020) proposed using state-of-the-art forecast reconciliation methods to improve the accuracy of macroeconomic forecasts and facilitate aligned decision-making. In their empirical analysis, they applied cross-sectional forecast reconciliation to 95 Australian QNA time series that represent the Gross Domestic Product (GDP) calculated using both the income and expenditure approaches. These two approaches correspond to two distinct hierarchical structures, with GDP at the top and 15 lower-level aggregates in the income approach, and GDP as the top-level aggregate in a hierarchy of 79 time series in the expenditure approach (for more information, see Athanasopoulos et al. 2020, pp. 702–705 and figures 21.4–21.7). Bisaglia et al. (2020) showed how to obtain a "one-number" forecast where the GDP reconciled forecasts are coherent for both the expenditure and income sides. Di Fonzo & Girolimetto (2022*c*,*d*) extended the one number forecasts idea to obtain fully reconciled probabilistic forecasts, and Di Fonzo & Girolimetto (2023*a*) computed cross-temporally reconciled point forecasts.

D.1 One-step residuals and shrinkage covariance matrix

Reconciliation approach	**Generation of the base forecasts paths**									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G_h	H_h	G_{oh}	H_{oh}		G_h	H_h	G_{oh}	H_{oh}
	$\forall k \in \{4,2,1\}$					$k=1$				
base	1.000	0.979	0.995	0.968	0.976	1.000	0.988	0.988	0.971	0.971
ct(shr_{cs}, bu_{te})	0.937	0.956	0.956	0.976	0.976	0.992	1.008	1.008	1.029	1.029
ct(wls_{cs}, bu_{te})	0.930	0.917	0.917	0.898	0.898	0.986	0.974	0.975	0.956	0.956
oct($wlsv$)	0.926	0.919	0.920	0.900	0.900	0.984	0.981	0.979	0.959	0.959
oct($bdshr$)	0.940	0.965	0.945	0.992	0.957	0.997	1.019	1.003	1.044	1.018
oct(shr)	0.944	1.020	0.940	1.094	0.988	1.015	1.095	1.010	1.160	1.059
oct($hshr$)	0.988	0.972	1.002	0.974	1.001	1.048	1.037	1.060	1.034	1.061
oct$_o$($wlsv$)	**0.926**	**0.911**	**0.912**	**0.896**	**0.895**	**0.984**	**0.971**	**0.970**	**0.954**	**0.954**
oct$_o$($bdshr$)	0.978	0.964	0.946	0.952	0.930	1.034	1.016	1.003	1.005	0.989
oct$_o$(shr)	0.950	0.946	0.922	0.925	0.903	1.014	1.003	0.985	0.987	0.968
oct$_o$($hshr$)	0.989	0.966	0.984	0.954	0.965	1.047	1.028	1.038	1.012	1.023
oct$_{oh}$(shr)	1.102	1.059	1.001	1.094	0.988	1.172	1.109	1.066	1.160	1.059
oct$_{oh}$($hshr$)	1.006	0.983	1.009	0.974	1.001	1.068	1.046	1.059	1.034	1.061
	$k=2$					$k=4$				
base	1.000	0.984	0.993	0.968	0.976	1.000	0.966	1.004	0.964	0.981
ct(shr_{cs}, bu_{te})	0.949	0.966	0.966	0.987	0.987	0.874	0.896	0.896	0.914	0.914
ct(wls_{cs}, bu_{te})	0.942	0.928	0.928	0.909	0.909	0.866	0.853	0.853	0.834	0.834
oct($wlsv$)	0.938	0.929	0.931	0.911	0.911	0.860	0.853	0.855	0.835	0.834
oct($bdshr$)	0.953	0.976	0.956	1.003	0.969	0.874	0.904	0.880	0.931	0.889
oct(shr)	0.955	1.031	0.951	1.113	1.002	0.866	0.940	0.864	1.015	0.909
oct($hshr$)	1.001	0.985	1.014	0.987	1.016	0.919	0.900	0.935	0.904	0.931
oct$_o$($wlsv$)	**0.938**	**0.921**	**0.923**	**0.907**	**0.906**	**0.860**	**0.847**	**0.848**	**0.832**	**0.830**
oct$_o$($bdshr$)	0.991	0.974	0.957	0.964	0.942	0.914	0.905	0.883	0.892	0.865
oct$_o$(shr)	0.965	0.958	0.934	0.938	0.916	0.877	0.882	0.852	0.854	0.831
oct$_o$($hshr$)	1.002	0.979	0.996	0.967	0.978	0.922	0.898	0.923	0.888	0.898
oct$_{oh}$(shr)	1.120	1.069	1.013	1.113	1.002	1.020	1.002	0.928	1.015	0.909
oct$_{oh}$($hshr$)	1.021	0.996	1.021	0.987	1.016	0.934	0.912	0.951	0.904	0.931

*The Gaussian method employs a sample covariance matrix:
G_h and H_h use multi-step residuals and G_{oh} and H_{oh} use overlapping and multi-step residuals.

Table D.6: *AvgRelCRPS indices defined in Section 5.1 for the Australian QNA dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G_h	H_h	G_{oh}	H_{oh}		G_h	H_h	G_{oh}	H_{oh}
	$\forall k \in \{4,2,1\}$					$k=1$				
base	1.000	0.970	0.988	0.960	0.970	1.000	0.977	0.977	0.965	0.965
ct(shr_{cs}, bu_{te})	0.897	0.944	0.944	0.973	0.973	0.964	1.001	1.001	1.033	1.033
ct(wls_{cs}, bu_{te})	**0.886**	0.880	0.880	**0.860**	0.860	**0.954**	**0.944**	0.945	**0.928**	**0.928**
oct($wlsv$)	0.890	0.890	0.894	0.872	0.872	0.958	0.957	0.957	0.938	0.939
oct($bdshr$)	0.905	0.956	0.934	0.992	0.954	0.972	1.014	0.994	1.048	1.018
oct(shr)	0.895	0.979	0.895	1.053	0.944	0.973	1.060	0.969	1.121	1.015
oct($hshr$)	0.951	0.940	0.973	0.959	0.992	1.017	1.010	1.034	1.023	1.055
oct$_o$($wlsv$)	0.891	**0.879**	0.881	0.864	0.864	0.958	0.945	0.945	0.931	0.931
oct$_o$($bdshr$)	0.940	0.928	0.910	0.918	0.895	1.004	0.986	0.971	0.980	0.961
oct$_o$(shr)	0.900	0.899	**0.876**	0.878	**0.858**	0.973	0.963	**0.944**	0.949	0.930
oct$_o$($hshr$)	0.956	0.936	0.955	0.922	0.936	1.021	1.004	1.012	0.987	1.000
oct$_{oh}$(shr)	1.059	1.015	0.956	1.053	0.945	1.130	1.063	1.019	1.121	1.016
oct$_{oh}$($hshr$)	0.986	0.968	0.999	0.959	0.992	1.053	1.034	1.049	1.024	1.055
	$k=2$					$k=4$				
base	1.000	0.972	0.985	0.959	0.969	1.000	0.959	1.000	0.957	0.976
ct(shr_{cs}, bu_{te})	0.915	0.961	0.960	0.991	0.991	0.818	0.874	0.874	0.899	0.900
ct(wls_{cs}, bu_{te})	**0.904**	0.896	**0.896**	**0.877**	**0.877**	**0.807**	0.805	0.805	**0.782**	0.783
oct($wlsv$)	0.909	0.907	0.912	0.889	0.889	0.811	0.813	0.819	0.794	0.794
oct($bdshr$)	0.925	0.976	0.953	1.013	0.974	0.825	0.883	0.860	0.920	0.876
oct(shr)	0.913	1.000	0.914	1.076	0.963	0.807	0.885	0.808	0.967	0.861
oct($hshr$)	0.973	0.960	0.993	0.978	1.014	0.871	0.856	0.897	0.881	0.913
oct$_o$($wlsv$)	0.908	**0.895**	0.898	0.881	0.882	0.812	**0.802**	0.806	0.786	0.786
oct$_o$($bdshr$)	0.960	0.947	0.929	0.938	0.915	0.860	0.856	0.836	0.841	0.816
oct$_o$(shr)	0.921	0.919	0.896	0.898	0.878	0.814	0.821	**0.796**	0.794	**0.775**
oct$_o$($hshr$)	0.977	0.956	0.976	0.942	0.957	0.876	0.854	0.882	0.844	0.856
oct$_{oh}$(shr)	1.082	1.029	0.973	1.076	0.963	0.971	0.954	0.882	0.967	0.861
oct$_{oh}$($hshr$)	1.007	0.988	1.017	0.979	1.014	0.904	0.888	0.934	0.881	0.913

*The Gaussian method employs a sample covariance matrix:
G_h and H_h use multi-step residuals and G_{oh} and H_{oh} use overlapping and multi-step residuals.

Table D.7: *ES ratio indices defined in Section 5.1 for the Australian QNA dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G_h	H_h	G_{oh}	H_{oh}		G_h	H_h	G_{oh}	H_{oh}
	$\forall k \in \{4,2,1\}$					$k=1$				
base	1.000	0.979	1.011	0.968	0.987	1.000	0.988	0.988	0.971	0.971
ct(shr_{cs}, bu_{te})	0.937	0.960	0.961	0.962	0.960	0.992	1.001	1.001	1.004	1.000
ct(wls_{cs}, bu_{te})	0.930	**0.951**	0.953	**0.911**	0.915	0.986	0.997	0.998	**0.964**	0.967
oct($wlsv$)	0.926	0.972	0.957	0.918	0.917	0.984	1.010	1.003	0.971	0.970
oct($bdshr$)	0.940	0.986	0.966	0.981	0.956	0.997	1.015	1.006	1.016	1.000
oct(shr)	0.944	0.999	0.962	1.051	0.995	1.015	1.047	1.021	1.105	1.058
oct($hshr$)	0.988	1.000	1.021	0.979	1.002	1.048	1.045	1.066	1.034	1.053
oct_o($wlsv$)	**0.926**	0.961	0.948	0.914	**0.912**	**0.984**	1.000	0.993	0.966	**0.965**
oct_o($bdshr$)	0.978	0.956	0.949	0.949	0.934	1.034	**0.984**	**0.983**	0.988	0.977
oct_o(shr)	0.950	0.957	**0.946**	0.933	0.917	1.014	0.998	0.995	0.986	0.974
oct_o($hshr$)	0.989	0.997	1.013	0.967	0.982	1.047	1.039	1.054	1.019	1.032
oct_{oh}(shr)	1.102	1.010	1.006	1.051	0.995	1.172	1.059	1.063	1.105	1.058
oct_{oh}($hshr$)	1.006	0.989	1.004	0.979	1.002	1.068	1.037	1.050	1.034	1.053
	$k=2$					$k=4$				
base	1.000	0.984	1.009	0.968	0.987	1.000	0.966	1.037	0.964	1.002
ct(shr_{cs}, bu_{te})	0.949	0.972	0.972	0.974	0.971	0.874	0.910	0.911	0.910	0.910
ct(wls_{cs}, bu_{te})	0.942	**0.962**	0.964	**0.923**	0.927	0.866	**0.897**	0.900	**0.851**	0.855
oct($wlsv$)	0.938	0.988	0.968	0.931	0.929	0.860	0.921	0.903	0.856	0.856
oct($bdshr$)	0.953	1.004	0.979	0.996	0.970	0.874	0.942	0.914	0.932	0.900
oct(shr)	0.955	1.016	0.973	1.070	1.010	0.866	0.937	0.895	0.981	0.922
oct($hshr$)	1.001	1.015	1.034	0.993	1.017	0.919	0.942	0.965	0.913	0.937
oct_o($wlsv$)	**0.938**	0.976	**0.959**	0.927	**0.925**	**0.860**	0.910	0.894	0.853	0.852
oct_o($bdshr$)	0.991	0.970	0.963	0.963	0.948	0.914	0.917	0.905	0.899	0.880
oct_o(shr)	0.965	0.973	0.959	0.948	0.931	0.877	0.903	**0.886**	0.868	**0.850**
oct_o($hshr$)	1.002	1.013	1.026	0.980	0.996	0.922	0.943	0.962	0.905	0.921
oct_{oh}(shr)	1.120	1.026	1.019	1.070	1.010	1.020	0.947	0.939	0.981	0.922
oct_{oh}($hshr$)	1.021	1.005	1.017	0.993	1.017	0.934	0.929	0.946	0.913	0.937

*The Gaussian method employs a shrinkage covariance matrix:
G_h and H_h use multi-step residuals and G_{oh} and H_{oh} use overlapping and multi-step residuals.

Table D.8: *AvgRelCRPS indices defined in Section 5.1 for the Australian QNA dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G_h	H_h	G_{oh}	H_{oh}		G_h	H_h	G_{oh}	H_{oh}
	$\forall k \in \{4,2,1\}$					$k=1$				
base	1.000	0.967	1.002	0.957	0.980	1.000	0.973	0.973	0.961	0.962
ct(shr_{cs}, bu_{te})	0.897	0.968	0.969	0.963	0.962	0.964	1.012	1.012	1.009	1.004
ct(wls_{cs}, bu_{te})	**0.886**	0.939	0.944	**0.882**	0.888	**0.954**	0.994	0.998	**0.947**	0.952
oct($wlsv$)	0.890	0.966	0.959	0.897	0.901	0.958	1.017	1.012	0.960	0.965
oct($bdshr$)	0.905	0.997	0.981	0.986	0.960	0.972	1.031	1.021	1.024	1.005
oct(shr)	0.895	0.979	0.945	1.021	0.962	0.973	1.041	1.011	1.083	1.028
oct($hshr$)	0.951	0.997	1.023	0.973	1.005	1.017	1.051	1.073	1.034	1.063
oct$_o$($wlsv$)	0.891	0.950	0.945	0.889	0.892	0.958	1.002	0.997	0.953	0.956
oct$_o$($bdshr$)	0.940	0.935	0.933	0.922	0.909	1.004	**0.965**	**0.964**	0.969	0.959
oct$_o$(shr)	0.900	**0.935**	**0.928**	0.895	**0.884**	0.973	0.984	0.982	0.960	**0.950**
oct$_o$($hshr$)	0.956	0.997	1.015	0.945	0.965	1.021	1.049	1.062	1.007	1.024
oct$_{oh}$(shr)	1.059	0.981	0.983	1.021	0.962	1.130	1.034	1.041	1.083	1.029
oct$_{oh}$($hshr$)	0.986	0.996	1.014	0.973	1.005	1.053	1.050	1.064	1.034	1.063
	$k=2$					$k=4$				
base	1.000	0.970	0.999	0.955	0.980	1.000	0.958	1.033	0.953	1.000
ct(shr_{cs}, bu_{te})	0.915	0.987	0.988	0.983	0.982	0.818	0.909	0.910	0.902	0.902
ct(wls_{cs}, bu_{te})	**0.904**	**0.958**	0.962	**0.900**	0.906	**0.807**	0.871	0.876	**0.805**	0.812
oct($wlsv$)	0.909	0.988	0.979	0.916	0.920	0.811	0.896	0.891	0.820	0.825
oct($bdshr$)	0.925	1.024	1.005	1.010	0.984	0.825	0.938	0.919	0.926	0.895
oct(shr)	0.913	1.006	0.967	1.045	0.982	0.807	0.898	0.864	0.940	0.881
oct($hshr$)	0.973	1.020	1.046	0.994	1.028	0.871	0.924	0.954	0.897	0.929
oct$_o$($wlsv$)	0.908	0.972	0.964	0.908	0.911	0.812	0.882	0.876	0.812	0.816
oct$_o$($bdshr$)	0.960	0.959	0.957	0.945	0.932	0.860	0.884	0.879	0.857	0.841
oct$_o$(shr)	0.921	0.958	**0.950**	0.917	**0.905**	0.814	**0.867**	**0.857**	0.815	**0.803**
oct$_o$($hshr$)	0.977	1.021	1.038	0.966	0.987	0.876	0.926	0.949	0.868	0.889
oct$_{oh}$(shr)	1.082	1.002	1.003	1.045	0.982	0.971	0.910	0.911	0.941	0.882
oct$_{oh}$($hshr$)	1.007	1.017	1.036	0.994	1.028	0.904	0.924	0.947	0.896	0.929

*The Gaussian method employs a shrinkage covariance matrix:
G_h and H_h use multi-step residuals and G_{oh} and H_{oh} use overlapping and multi-step residuals.

Table D.9: *ES ratio indices defined in Section 5.1 for the Australian QNA dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

E Australian Tourism Demand dataset

Table E.10: *Geographic divisions of Australia in States, Zones e Regions. Zones formed by a single region are highlighted in italics and not numbered.*

Series	Name	Label
Total		
1	Australia	Total
States		
2	New South Wales (NSW)	A
3	Victoria (VIC)	B
4	Queensland (QLD)	C
5	South Australia (SA)	D
6	Western Australia (WA)	E
7	Tasmania (TAS)	F
8	Northern Territory (NT)	G
Zones		
9	Metro NSW	AA
10	Nth Coast NSW	AB
	Sth Coast NSW	*AC*
11	Sth NSW	AD
12	Nth NSW	AE
	ACT	*AF*
13	Metro VIC	BA
	West Coast VIC	*BB*
14	East Coast VIC	BC
15	Nth East VIC	BD
16	Nth West VIC	BE
17	Metro QLD	CA
18	Central Coast QLD	CB
19	Nth Coast QLD	CC
20	Inland QLD	CD
21	Metro SA	DA
22	Sth Coast SA	DB
23	Inland SA	DC
24	West Coast SA	DD
25	West CoastWA	EA
	Nth WA	*EB*
	SthWA	*EC*
	Sth TAS	*FA*
26	Nth East TAS	FB
27	Nth West TAS	FC
28	Nth Coast NT	GA
29	Central NT	GB
Regions		
30	Sydney	AAA
31	Central Coast	AAB
32	Hunter	ABA
33	North Coast NSW	ABB
34	South Coast	ACA
35	Snowy Mountains	ADA
36	Capital Country	ADB
37	The Murray	ADC
38	Riverina	ADD
39	Central NSW	AEA
40	New England North West	AEB
41	Outback NSW	AEC
42	Blue Mountains	AED
43	Canberra	AFA
44	Melbourne	BAA
45	Peninsula	BAB
46	Geelong	BAC
47	Western	BBA
48	Lakes	BCA
continues Regions		
49	Gippsland	BCB
50	Phillip Island	BCC
51	Central Murray	BDA
52	Goulburn	BDB
53	High Country	BDC
54	Melbourne East	BDD
55	Upper Yarra	BDE
56	MurrayEast	BDF
57	Mallee	BEA
58	Wimmera	BEB
59	Western Grampians	BEC
60	Bendigo Loddon	BED
61	Macedon	BEE
62	Spa Country	BEF
63	Ballarat	BEG
64	Central Highlands	BEG
65	Gold Coast	CAA
66	Brisbane	CAB
67	Sunshine Coast	CAC
68	Central Queensland	CBA
69	Bundaberg	CBB
70	Fraser Coast	CBC
71	Mackay	CBD
72	Whitsundays	CCA
73	Northern	CCB
74	Tropical North Queensland	CCC
75	Darling Downs	CDA
76	Outback	CDB
77	Adelaide	DAA
78	Barossa	DAB
79	Adelaide Hills	DAC
80	Limestone Coast	DBA
81	Fleurieu Peninsula	DBB
82	Kangaroo Island	DBC
83	Murraylands	DCA
84	Riverland	DCB
85	Clare Valley	DCC
86	Flinders Range and Outback	DCD
87	Eyre Peninsula	DDA
88	Yorke Peninsula	DDB
89	Australia's Coral Coast	EAA
90	Experience Perth	EAB
91	Australia's SouthWest	EAC
92	Australia's North West	EBA
93	Australia's Golden Outback	ECA
94	Hobart and the South	FAA
95	East Coast	FBA
96	Launceston, Tamar and the North	FBB
97	North West	FCA
98	WildernessWest	FCB
99	Darwin	GAA
100	Kakadu Arnhem	GAB
101	Katherine Daly	GAC
102	Barkly	GBA
103	Lasseter	GBB
104	Alice Springs	GBC
105	MacDonnell	GBD

Source: Wickramasuriya et al. (2019), Di Fonzo & Girolimetto (2022*b*)

E.1 Dealing with negative reconciled forecasts

One issue in working with time series data is the presence of negative values, which can cause difficulties for certain types of models or analyses. For the base forecasts, using the bootstrap approach produces forecasts naturally non negative (ETS model with the log-transformation), while this is not true for the Gaussian approach. In this case, any negative forecast is set equal to zero. For the cross-temporal reconciliation, Di Fonzo & Girolimetto (2022*a*, 2023*b*) propose two solutions: either a state-of-the-art numerical optimization procedure (`osqp`, Stellato et al. 2020, 2022), or a simple heuristic strategy called set-negative-to-zero (sntz). With sntz, any negative high frequency bottom time series reconciled forecasts are set to zero, and then a cross-temporal reconciliation bottom-up is used to obtain the complete set of fully coherent forecasts. Di Fonzo & Girolimetto (2023*b*) found that both methods produce similar quality forecasts, but the optimization method required much more time and computational effort compared to the sntz heuristic. To reduce computational demands, we used the less time-intensive heuristic approach for reconciliation.

E.2 Tables for all the temporal aggregation orders

Reconciliation approach	Generation of the base forecasts paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G	B	H	HB		G	B	H	HB
			$\forall k \in \{12, 6, 4, 3, 2, 1\}$					$k = 1$		
base	1.000	0.971	0.971	0.973	0.973	1.000	0.972	0.972	0.972	0.972
ct(bu)	1.321	1.011	1.011	1.011	1.011	1.077	0.983	0.982	0.982	0.982
ct(shr_{cs}, bu_{te})	1.057	0.974	0.969	0.974	0.969	0.976	0.963	0.962	0.963	0.962
ct($wlsv_{te}, bu_{cs}$)	1.062	0.974	0.974	0.972	0.972	0.976	0.965	0.965	0.966	0.966
oct(ols)	0.989	0.989	0.989	0.987	0.987	0.982	0.986	0.988	0.986	0.989
oct($struc$)	0.982	0.962	0.961	0.961	0.959	0.970	0.963	0.963	0.963	0.963
oct($wlsv$)	0.987	0.959	0.959	0.958	0.957	0.952	0.957	0.957	0.957	0.957
oct($bdshr$)	0.975	**0.956**	**0.953**	**0.952**	**0.951**	**0.949**	**0.955**	**0.953**	**0.954**	**0.954**
oct$_h$($hbshr$)	0.989	1.018	1.020	1.016	1.018	0.982	1.004	1.007	1.004	1.009
oct$_h$($bshr$)	0.994	1.018	1.020	1.016	1.019	0.988	1.007	1.013	1.006	1.012
oct$_h$($hshr$)	**0.969**	0.993	0.993	0.990	0.991	0.953	0.977	0.977	0.979	0.979
oct$_h$(shr)	1.007	0.980	0.972	0.970	0.970	1.000	0.986	0.977	0.976	0.974
			$k = 2$					$k = 3$		
base	1.000	0.970	0.969	0.970	0.971	1.000	0.971	0.971	0.972	0.973
ct(bu)	1.189	0.999	0.999	0.999	0.999	1.273	1.010	1.010	1.010	1.010
ct(shr_{cs}, bu_{te})	1.015	0.972	0.970	0.972	0.970	1.041	0.977	0.974	0.977	0.974
ct($wlsv_{te}, bu_{cs}$)	1.016	0.971	0.971	0.970	0.970	1.046	0.976	0.976	0.974	0.974
oct(ols)	0.992	0.991	0.991	0.990	0.991	0.994	0.992	0.993	0.991	0.992
oct($struc$)	0.982	0.966	0.965	0.965	0.965	0.986	0.967	0.966	0.966	0.965
oct($wlsv$)	0.972	0.961	0.960	0.960	0.960	0.983	0.963	0.962	0.962	0.962
oct($bdshr$)	**0.964**	**0.958**	**0.957**	**0.956**	**0.956**	0.972	**0.960**	**0.958**	**0.957**	**0.957**
oct$_h$($hbshr$)	0.992	1.013	1.015	1.012	1.015	0.994	1.019	1.021	1.018	1.020
oct$_h$($bshr$)	0.997	1.015	1.018	1.013	1.017	0.999	1.021	1.022	1.018	1.022
oct$_h$($hshr$)	0.965	0.987	0.987	0.986	0.987	**0.971**	0.994	0.994	0.992	0.993
oct$_h$(shr)	1.005	0.986	0.978	0.976	0.975	1.009	0.986	0.978	0.976	0.976
			$k = 4$					$k = 6$		
base	1.000	0.973	0.973	0.974	0.975	1.000	0.976	0.976	0.978	0.978
ct(bu)	1.340	1.016	1.015	1.015	1.015	1.450	1.023	1.023	1.023	1.023
ct(shr_{cs}, bu_{te})	1.061	0.978	0.973	0.978	0.973	1.094	0.978	0.972	0.978	0.972
ct($wlsv_{te}, bu_{cs}$)	1.068	0.977	0.977	0.974	0.974	1.103	0.977	0.977	0.974	0.974
oct(ols)	0.993	0.991	0.992	0.990	0.990	0.989	0.989	0.989	0.987	0.986
oct($struc$)	0.986	0.965	0.964	0.964	0.963	0.986	0.961	0.960	0.959	0.957
oct($wlsv$)	0.990	0.962	0.961	0.961	0.960	1.001	0.960	0.959	0.958	0.957
oct($bdshr$)	0.977	**0.959**	**0.956**	**0.955**	**0.954**	0.985	**0.956**	**0.953**	**0.950**	**0.948**
oct$_h$($hbshr$)	0.993	1.021	1.023	1.019	1.021	0.989	1.024	1.026	1.022	1.022
oct$_h$($bshr$)	0.997	1.022	1.022	1.019	1.022	0.994	1.022	1.022	1.020	1.022
oct$_h$($hshr$)	**0.973**	0.996	0.997	0.994	0.995	**0.976**	1.000	1.001	0.996	0.997
oct$_h$(shr)	1.009	0.984	0.976	0.973	0.973	1.010	0.978	0.970	0.967	0.967
			$k = 12$							
base	1.000	0.968	0.967	0.969	0.969					
ct(bu)	1.675	1.038	1.037	1.037	1.038					
ct(shr_{cs}, bu_{te})	1.163	0.977	0.965	0.977	0.965					
ct($wlsv_{te}, bu_{cs}$)	1.174	0.978	0.978	0.971	0.971					
oct(ols)	0.982	0.982	0.983	0.980	0.975					
oct($struc$)	0.982	0.951	0.949	0.947	0.943					
oct($wlsv$)	1.025	0.954	0.953	0.949	0.947					
oct($bdshr$)	1.002	**0.950**	**0.944**	**0.939**	**0.935**					
oct$_h$($hbshr$)	0.982	1.027	1.029	1.024	1.021					
oct$_h$($bshr$)	0.987	1.024	1.021	1.021	1.019					
oct$_h$($hshr$)	**0.978**	1.003	1.005	0.996	0.997					
oct$_h$(shr)	1.010	0.963	0.956	0.952	0.952					

*The Gaussian method employs a sample covariance matrix and includes four techniques (G, B, H, HB) with multi-step residuals.

Table E.11: *AvgRelCRPS defined in Section 5.1 for the Australian Tourism Demand dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G	B	H	HB		G	B	H	HB
	$\forall k \in \{12,6,4,3,2,1\}$					$k=1$				
base	1.000	0.956	0.955	0.958	0.951	1.000	0.952	0.950	0.952	0.950
$\text{ct}(bu)$	2.427	0.983	0.983	0.983	0.983	1.759	0.982	0.982	0.982	0.982
$\text{ct}(shr_{cs}, bu_{te})$	1.243	0.886	0.879	0.886	0.879	1.098	0.929	0.928	0.930	0.927
$\text{ct}(wlsv_{te}, bu_{cs})$	1.499	0.977	0.977	0.971	0.972	1.241	0.975	0.975	0.973	0.974
$\text{oct}(ols)$	0.955	0.893	0.891	0.893	0.888	0.975	0.937	0.936	0.936	0.935
$\text{oct}(struc)$	1.085	0.917	0.915	0.916	0.912	1.027	0.943	0.942	0.943	0.942
$\text{oct}(wlsv)$	1.132	0.933	0.929	0.931	0.927	1.050	0.951	0.949	0.950	0.949
$\text{oct}(bdshr)$	1.047	0.904	0.897	0.897	0.891	1.009	0.936	0.933	0.934	0.931
$\text{oct}_h(hbshr)$	0.956	0.889	0.886	0.888	0.884	0.975	0.937	0.936	0.937	0.935
$\text{oct}_h(bshr)$	**0.931**	**0.867**	**0.866**	**0.863**	**0.860**	**0.965**	**0.927**	0.927	0.925	0.923
$\text{oct}_h(hshr)$	1.081	0.935	0.931	0.935	0.927	1.028	0.952	0.951	0.952	0.950
$\text{oct}_h(shr)$	1.068	0.899	0.878	0.875	0.864	1.023	0.935	**0.923**	**0.921**	**0.916**
	$k=2$					$k=3$				
base	1.000	0.958	0.954	0.956	0.953	1.000	0.961	0.958	0.960	0.955
$\text{ct}(bu)$	2.176	1.001	1.001	1.001	1.001	2.428	0.998	0.997	0.997	0.997
$\text{ct}(shr_{cs}, bu_{te})$	1.192	0.927	0.921	0.927	0.921	1.245	0.911	0.904	0.911	0.904
$\text{ct}(wlsv_{te}, bu_{cs})$	1.400	0.992	0.992	0.988	0.988	1.500	0.991	0.991	0.986	0.987
$\text{oct}(ols)$	0.985	0.935	0.932	0.934	0.930	0.976	0.918	0.915	0.917	0.912
$\text{oct}(struc)$	1.075	0.949	0.947	0.948	0.944	1.096	0.939	0.936	0.938	0.933
$\text{oct}(wlsv)$	1.110	0.960	0.958	0.958	0.955	1.142	0.953	0.949	0.951	0.946
$\text{oct}(bdshr)$	1.045	0.938	0.933	0.933	0.929	1.060	0.926	0.920	0.921	0.915
$\text{oct}_h(hbshr)$	0.984	0.933	0.931	0.933	0.928	0.975	0.915	0.912	0.915	0.909
$\text{oct}_h(bshr)$	**0.967**	**0.917**	**0.916**	0.913	0.908	**0.954**	**0.895**	**0.895**	**0.892**	**0.887**
$\text{oct}_h(hshr)$	1.073	0.962	0.959	0.963	0.956	1.093	0.955	0.951	0.956	0.949
$\text{oct}_h(shr)$	1.064	0.933	0.916	**0.913**	**0.904**	1.082	0.923	0.903	0.900	0.890
	$k=4$					$k=6$				
base	1.000	0.960	0.960	0.962	0.956	1.000	0.961	0.959	0.964	0.956
$\text{ct}(bu)$	2.585	0.996	0.996	0.995	0.996	2.849	1.004	1.003	1.003	1.004
$\text{ct}(shr_{cs}, bu_{te})$	1.277	0.898	0.890	0.899	0.891	1.339	0.882	0.873	0.883	0.874
$\text{ct}(wlsv_{te}, bu_{cs})$	1.559	0.990	0.990	0.984	0.985	1.662	0.997	0.997	0.991	0.992
$\text{oct}(ols)$	0.966	0.905	0.902	0.904	0.899	0.962	0.889	0.887	0.890	0.885
$\text{oct}(struc)$	1.106	0.930	0.927	0.928	0.924	1.132	0.923	0.919	0.922	0.916
$\text{oct}(wlsv)$	1.157	0.947	0.943	0.945	0.939	1.192	0.942	0.937	0.941	0.934
$\text{oct}(bdshr)$	1.065	0.917	0.909	0.910	0.903	1.084	0.907	0.897	0.898	0.890
$\text{oct}_h(hbshr)$	0.967	0.901	0.898	0.900	0.895	0.964	0.882	0.880	0.883	0.877
$\text{oct}_h(bshr)$	**0.943**	**0.879**	**0.878**	**0.876**	**0.871**	**0.932**	**0.856**	**0.855**	**0.851**	**0.848**
$\text{oct}_h(hshr)$	1.101	0.949	0.944	0.949	0.941	1.126	0.945	0.939	0.945	0.936
$\text{oct}_h(shr)$	1.089	0.915	0.893	0.890	0.878	1.107	0.899	0.875	0.871	0.858
	$k=12$									
base	1.000	0.942	0.947	0.951	0.937					
$\text{ct}(bu)$	2.990	0.922	0.921	0.923	0.923					
$\text{ct}(shr_{cs}, bu_{te})$	1.326	0.779	0.767	0.777	0.766					
$\text{ct}(wlsv_{te}, bu_{cs})$	1.679	0.917	0.917	0.906	0.908					
$\text{oct}(ols)$	0.872	0.783	0.784	0.783	0.779					
$\text{oct}(struc)$	1.077	0.826	0.822	0.823	0.818					
$\text{oct}(wlsv)$	1.149	0.851	0.845	0.847	0.840					
$\text{oct}(bdshr)$	1.021	0.808	0.796	0.796	0.787					
$\text{oct}_h(hbshr)$	0.872	0.775	0.772	0.772	0.770					
$\text{oct}_h(bshr)$	**0.833**	**0.741**	**0.741**	**0.737**	**0.735**					
$\text{oct}_h(hshr)$	1.066	0.851	0.846	0.848	0.838					
$\text{oct}_h(shr)$	1.043	0.797	0.768	0.764	0.750					

*The Gaussian method employs a sample covariance matrix and includes four techniques (G, B, H, HB) with multi-step residuals.

Table E.12: *ES ratio indices defined in Section 5.1 for the Australian Tourism Demand dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G	B	H	HB		G	B	H	HB
	$\forall k \in \{12,6,4,3,2,1\}$					$k=1$				
base	1.000	**0.971**	0.972	**0.971**	0.972	1.000	**0.972**	0.971	0.972	0.971
ct(bu)	1.321	1.017	1.018	1.017	1.017	1.077	0.983	0.983	0.983	0.983
ct(shr_{cs}, bu_{te})	1.057	1.013	**0.971**	1.013	0.971	0.976	0.987	**0.961**	0.988	0.961
ct$(wlsv_{te}, bu_{cs})$	1.062	1.069	1.070	0.974	0.974	0.976	0.986	0.986	**0.965**	0.965
oct(ols)	0.989	1.163	1.052	1.139	0.987	0.982	1.038	0.992	1.047	0.987
oct$(struc)$	0.982	1.099	1.039	1.037	0.960	0.970	1.007	0.971	0.999	0.962
oct$(wlsv)$	0.987	1.080	1.041	0.992	0.958	0.952	1.004	0.969	0.978	0.956
oct$(bdshr)$	0.975	1.072	1.032	0.985	**0.950**	**0.949**	0.999	0.965	0.975	**0.952**
oct$_h(hbshr)$	0.989	1.189	1.076	1.171	1.021	0.982	1.045	1.000	1.063	1.009
oct$_h(bshr)$	0.994	1.202	1.073	1.168	1.021	0.988	1.046	1.012	1.063	1.012
oct$_h(hshr)$	**0.969**	1.066	1.052	1.008	0.994	0.953	0.994	0.972	0.991	0.979
oct$_h(shr)$	1.007	1.090	1.046	1.000	0.970	1.000	1.035	0.992	0.998	0.973
	$k=2$					$k=3$				
base	1.000	**0.969**	0.969	**0.968**	0.968	1.000	**0.971**	**0.970**	**0.969**	0.970
ct(bu)	1.189	1.000	1.000	1.000	1.000	1.273	1.013	1.013	1.013	1.013
ct(shr_{cs}, bu_{te})	1.015	1.004	**0.968**	1.004	0.968	1.041	1.013	0.973	1.014	0.973
ct$(wlsv_{te}, bu_{cs})$	1.016	1.043	1.044	0.969	0.969	1.046	1.067	1.068	0.974	0.974
oct(ols)	0.992	1.118	1.037	1.092	0.989	0.994	1.153	1.053	1.124	0.990
oct$(struc)$	0.982	1.075	1.022	1.020	0.963	0.986	1.099	1.041	1.033	0.964
oct$(wlsv)$	0.972	1.064	1.021	0.987	0.958	0.983	1.083	1.041	0.993	0.960
oct$(bdshr)$	**0.964**	1.057	1.015	0.983	**0.953**	0.972	1.075	1.033	0.988	**0.955**
oct$_h(hbshr)$	0.992	1.136	1.055	1.116	1.014	0.994	1.178	1.075	1.153	1.020
oct$_h(bshr)$	0.997	1.145	1.059	1.114	1.016	0.999	1.190	1.075	1.151	1.021
oct$_h(hshr)$	0.965	1.050	1.029	1.001	0.986	**0.971**	1.067	1.051	1.009	0.994
oct$_h(shr)$	1.005	1.083	1.035	1.001	0.973	1.009	1.097	1.050	1.004	0.974
	$k=4$					$k=6$				
base	1.000	**0.973**	**0.973**	**0.971**	0.973	1.000	**0.976**	0.977	**0.975**	0.977
ct(bu)	1.340	1.021	1.021	1.021	1.021	1.450	1.032	1.033	1.032	1.033
ct(shr_{cs}, bu_{te})	1.061	1.018	0.974	1.018	0.974	1.094	1.023	**0.974**	1.024	0.974
ct$(wlsv_{te}, bu_{cs})$	1.068	1.087	1.089	0.976	0.976	1.103	1.108	1.110	0.978	0.978
oct(ols)	0.993	1.186	1.068	1.148	0.989	0.989	1.223	1.080	1.184	0.987
oct$(struc)$	0.986	1.120	1.057	1.042	0.962	0.986	1.141	1.071	1.054	0.959
oct$(wlsv)$	0.990	1.100	1.059	0.996	0.959	1.001	1.115	1.076	0.998	0.958
oct$(bdshr)$	0.977	1.091	1.049	0.989	**0.952**	0.985	1.103	1.064	0.989	**0.949**
oct$_h(hbshr)$	0.993	1.215	1.095	1.182	1.022	0.989	1.258	1.112	1.225	1.026
oct$_h(bshr)$	0.997	1.230	1.089	1.178	1.023	0.994	1.278	1.101	1.219	1.025
oct$_h(hshr)$	**0.973**	1.084	1.071	1.012	0.996	**0.976**	1.097	1.091	1.017	1.002
oct$_h(shr)$	1.009	1.108	1.062	1.003	0.972	1.010	1.113	1.070	1.000	0.968
	$k=12$									
base	1.000	**0.968**	**0.969**	**0.969**	0.971					
ct(bu)	1.675	1.056	1.057	1.057	1.057					
ct(shr_{cs}, bu_{te})	1.163	1.032	0.974	1.033	0.974					
ct$(wlsv_{te}, bu_{cs})$	1.174	1.128	1.130	0.982	0.982					
oct(ols)	0.982	1.277	1.085	1.252	0.982					
oct$(struc)$	0.982	1.158	1.074	1.075	0.950					
oct$(wlsv)$	1.025	1.122	1.085	1.001	0.954					
oct$(bdshr)$	1.002	1.110	1.071	0.989	**0.941**					
oct$_h(hbshr)$	0.982	1.322	1.125	1.305	1.033					
oct$_h(bshr)$	0.987	1.347	1.107	1.297	1.031					
oct$_h(hshr)$	**0.978**	1.106	1.107	1.021	1.010					
oct$_h(shr)$	1.010	1.107	1.067	0.991	0.959					

*The Gaussian method employs a shrikage covariance matrix and includes four techniques (G, B, H, HB) with multi-step residuals..

Table E.13: *AvgRelCRPS defined in Section 5.1 for the Australian Tourism Demand dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G	B	H	HB		G	B	H	HB
	$\forall k \in \{12, 6, 4, 3, 2, 1\}$					$k = 1$				
base	1.000	**0.958**	0.984	**0.972**	0.992	1.000	**0.954**	0.958	**0.954**	0.958
ct(bu)	2.427	1.040	1.042	1.040	1.041	1.759	1.001	1.002	1.002	1.002
ct(shr_{cs}, bu_{te})	1.243	0.988	**0.913**	0.990	0.913	1.098	1.011	**0.938**	1.013	0.938
ct$(wlsv_{te}, bu_{cs})$	1.499	1.117	1.120	1.025	1.025	1.241	1.019	1.020	0.990	0.990
oct(ols)	0.955	1.000	0.984	0.985	0.922	0.975	0.983	0.961	0.987	0.945
oct$(struc)$	1.085	1.094	1.047	1.018	0.952	1.027	1.054	0.981	1.022	0.953
oct$(wlsv)$	1.132	1.137	1.065	1.059	0.969	1.050	1.078	0.989	1.043	0.960
oct$(bdshr)$	1.047	1.085	1.013	1.011	0.927	1.009	1.050	0.966	1.019	0.942
oct$_h(hbshr)$	0.956	1.018	0.981	1.016	0.919	0.975	0.991	0.961	1.002	0.947
oct$_h(bshr)$	**0.931**	1.002	1.001	0.982	**0.889**	**0.965**	0.980	0.975	0.985	0.933
oct$_h(hshr)$	1.081	1.109	1.039	1.076	0.973	1.028	1.061	0.978	1.052	0.963
oct$_h(shr)$	1.068	1.088	1.008	0.995	0.896	1.023	1.061	0.966	1.011	**0.924**
	$k = 2$					$k = 3$				
base	1.000	**0.960**	0.971	**0.958**	0.972	1.000	**0.963**	0.981	**0.966**	0.986
ct(bu)	2.176	1.035	1.036	1.035	1.035	2.428	1.042	1.044	1.042	1.043
ct(shr_{cs}, bu_{te})	1.192	1.020	**0.942**	1.021	0.942	1.245	1.009	**0.931**	1.011	0.931
ct$(wlsv_{te}, bu_{cs})$	1.400	1.104	1.106	1.018	1.019	1.500	1.127	1.130	1.029	1.029
oct(ols)	0.985	1.028	1.008	1.002	0.950	0.976	1.020	1.004	0.994	0.938
oct$(struc)$	1.075	1.115	1.051	1.039	0.967	1.096	1.117	1.064	1.033	0.965
oct$(wlsv)$	1.110	1.149	1.065	1.070	0.979	1.142	1.160	1.082	1.073	0.981
oct$(bdshr)$	1.045	1.105	1.024	1.033	0.949	1.060	1.109	1.032	1.029	0.943
oct$_h(hbshr)$	0.984	1.041	1.007	1.024	0.951	0.975	1.036	1.002	1.023	0.937
oct$_h(bshr)$	**0.967**	1.029	1.025	0.998	0.928	**0.954**	1.024	1.025	0.993	**0.911**
oct$_h(hshr)$	1.073	1.122	1.042	1.083	0.983	1.093	1.129	1.054	1.090	0.984
oct$_h(shr)$	1.064	1.110	1.019	1.018	**0.922**	1.082	1.116	1.030	1.015	0.915
	$k = 4$					$k = 6$				
base	1.000	**0.962**	0.987	**0.973**	0.996	1.000	**0.963**	0.998	**0.984**	1.011
ct(bu)	2.585	1.052	1.054	1.053	1.053	2.849	1.083	1.085	1.083	1.084
ct(shr_{cs}, bu_{te})	1.277	1.000	**0.923**	1.002	0.923	1.339	0.999	**0.921**	1.000	0.920
ct$(wlsv_{te}, bu_{cs})$	1.559	1.150	1.153	1.037	1.037	1.662	1.189	1.193	1.066	1.066
oct(ols)	0.966	1.022	1.008	0.994	0.931	0.962	1.023	1.014	1.003	0.930
oct$(struc)$	1.106	1.120	1.076	1.031	0.963	1.132	1.132	1.100	1.039	0.972
oct$(wlsv)$	1.157	1.167	1.097	1.075	0.982	1.192	1.187	1.124	1.090	0.995
oct$(bdshr)$	1.065	1.112	1.041	1.025	0.939	1.084	1.121	1.058	1.029	0.940
oct$_h(hbshr)$	0.967	1.041	1.005	1.027	0.929	0.964	1.046	1.008	1.042	0.924
oct$_h(bshr)$	**0.943**	1.028	1.028	0.994	**0.900**	**0.932**	1.029	1.032	1.000	**0.887**
oct$_h(hshr)$	1.101	1.137	1.068	1.093	0.986	1.126	1.153	1.089	1.110	0.999
oct$_h(shr)$	1.089	1.118	1.039	1.012	0.910	1.107	1.118	1.045	1.006	0.902
	$k = 12$									
base	1.000	0.948	1.010	1.002	1.033					
ct(bu)	2.990	1.028	1.031	1.029	1.029					
ct(shr_{cs}, bu_{te})	1.326	**0.897**	**0.830**	**0.899**	0.830					
ct$(wlsv_{te}, bu_{cs})$	1.679	1.119	1.123	1.009	1.009					
oct(ols)	0.872	0.927	0.914	0.930	0.840					
oct$(struc)$	1.077	1.028	1.012	0.950	0.894					
oct$(wlsv)$	1.149	1.089	1.041	1.006	0.922					
oct$(bdshr)$	1.021	1.015	0.964	0.935	0.855					
oct$_h(hbshr)$	0.872	0.955	0.906	0.978	0.833					
oct$_h(bshr)$	**0.833**	0.927	0.927	0.927	**0.784**					
oct$_h(hshr)$	1.066	1.056	1.005	1.026	0.926					
oct$_h(shr)$	1.043	1.011	0.952	0.909	0.809					

*The Gaussian method employs a shrikage covariance matrix and includes four techniques (G, B, H, HB) with multi-step residuals.

Table E.14: *ES ratio indices defined in Section 5.1 for the Australian Tourism Demand dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

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