Online appendix

Cross-temporal probabilistic forecast reconciliation: methodological and practical issues

4 September 2023

Contents

A Cross-sectional, temporal and cross-temporal covariance approximations

Table A.1 presents some approximations for the cross-sectional (Hyndman et al. 2011, 2016, Wickramasuriya et al. 2019) and the temporal (Athanasopoulos et al. 2017, Nystrup et al. 2020) covariance matrices. Di Fonzo & Girolimetto (2023*a*) consider the following approximations for the cross-temporal covariance matrix.

- oct(ols) - identity: $\mathbf{\Omega}_{ct} = \boldsymbol{I}_{n(k^*+m)}$.
- oct($struc$) - structural: $\mathbf{\Omega}_{ct} = \text{diag}(\boldsymbol{S}_{ct}\mathbf{1}_{mn_b})$.
- oct($wlsv$) - series variance scaling: $\mathbf{\Omega}_{ct} = \widehat{\mathbf{\Omega}}_{ct,wlsv}$, a straightforward extension of the series variance scaling matrix presented by Athanasopoulos et al. (2017).
- oct($bdshr$) - block-diagonal shrunk cross-covariance scaling: $\mathbf{\Omega}_{ct} = \boldsymbol{P}\widehat{\boldsymbol{W}}^{BD}_{ct,shr}\boldsymbol{P}'$, with $\widehat{\boldsymbol{W}}^{BD}_{ct,shr}$ a block diagonal matrix where each $k-$block ($k = m, k_{p-1}, \dots, 1$) is $\boldsymbol{I}_{M_k} \otimes \widehat{\boldsymbol{W}}^{[k]}_{shr}$, $\widehat{\boldsymbol{W}}^{[k]}_{shr}$ is the shrunk estimate of the cross-sectional covariance matrix proposed by Wickramasuriya et al. (2019), and $\boldsymbol{P}$ is the commutation matrix such that $\boldsymbol{P}\text{vec}(\boldsymbol{Y}_\tau) = \text{vec}(\boldsymbol{Y}'_\tau)$.
- oct(shr) - MinT-shr: $\mathbf{\Omega}_{ct} = \hat{\lambda}\widehat{\mathbf{\Omega}}_{ct,D} + (1-\hat{\lambda})\widehat{\mathbf{\Omega}}_{ct}$, where $\hat{\lambda}$ is an estimated shrinkage coefficient (Ledoit & Wolf 2004), $\widehat{\mathbf{\Omega}}_{ct,D} = \boldsymbol{I}_{n(k^*+m)} \odot \widehat{\mathbf{\Omega}}_{ct}$ with $\odot$ denoting the Hadamard product, and $\widehat{\mathbf{\Omega}}_{ct}$ is the covariance matrix of the cross-temporal one-step ahead in-sample forecast errors.
- oct(sam) - MinT-sam: $\mathbf{\Omega}_{ct} = \widehat{\mathbf{\Omega}}_{ct}$.

	Cross-sectional framework	**Temporal framework**
identity	cs(ols): $\boldsymbol{W} = \boldsymbol{I}_n$	te(ols): $\mathbf{\Omega} = \boldsymbol{I}_{k^*+m}$
structural	cs($struc$): $\boldsymbol{W} = \text{diag}(\boldsymbol{S}_{cs}\mathbf{1}_{nb})$	te($struc$): $\mathbf{\Omega} = \text{diag}(\boldsymbol{S}_{te}\mathbf{1}_m)$
series variance	cs(wls): $\boldsymbol{W} = \widehat{\boldsymbol{W}}_D = \boldsymbol{I}_n \odot \widehat{\boldsymbol{W}}$	te($wlsv$): $\mathbf{\Omega} = \widehat{\mathbf{\Omega}}_{wlsv}$
MinT-shr	cs(shr): $\boldsymbol{W} = \hat{\lambda}\widehat{\boldsymbol{W}}_D + (1-\hat{\lambda})\widehat{\boldsymbol{W}}$	te(shr): $\mathbf{\Omega} = \hat{\lambda}\widehat{\mathbf{\Omega}}_D + (1-\hat{\lambda})\widehat{\mathbf{\Omega}}$
MinT-sam	cs(sam): $\boldsymbol{W} = \widehat{\boldsymbol{W}}$	te(sam): $\mathbf{\Omega} = \widehat{\mathbf{\Omega}}$

Note: $\widehat{\boldsymbol{W}}$ ($\widehat{\mathbf{\Omega}}$) is the covariance matrix of the cross-sectional (temporal) one-step ahead in-sample forecast errors, $\widehat{\mathbf{\Omega}}_{wlsv}$ is a diagonal matrix presented by Athanasopoulos et al. (2017), and $\widehat{\mathbf{\Omega}}_D = \boldsymbol{I}_{k^*+m} \odot \widehat{\mathbf{\Omega}}$, where $\odot$ denotes the Hadamard product.

Table A.1: *Approximations for cross-sectional ($\boldsymbol{W}$) and temporal ($\mathbf{\Omega}$) covariance matrices.*

B Alternative forms of the cross-temporal covariance matrix

In this appendix, some derivations of the solutions proposed in Section 4 to obtain an estimator of the cross-temporal covariance matrix are reported. Starting from the the definition of cross-temporal covariance matrix we obtain the first equivalence in (10). Therefore, we have that

$$\begin{aligned}
&\lambda\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts,D} + (1-\lambda)\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts}\\
&\qquad\Downarrow\\
\widehat{\boldsymbol{\Omega}}_{HB} &= \boldsymbol{S}_{ct}\left[\lambda\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts,D} + (1-\lambda)\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts}\right]\boldsymbol{S}_{ct}'\\
&= \lambda\boldsymbol{S}_{ct}\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts,D}\boldsymbol{S}_{ct}' + (1-\lambda)\boldsymbol{S}_{ct}\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts}\boldsymbol{S}_{ct}'.
\end{aligned}$$

The high-frequency time series representation (the second equivalence) can be derived in the following manner:

$$\begin{aligned}
\widetilde{\boldsymbol{\Omega}} &= \boldsymbol{S}_{ct}\boldsymbol{\Omega}_{hf\text{-}bts}\boldsymbol{S}_{ct}'\\
&= (\boldsymbol{S}_{cs}\otimes\boldsymbol{S}_{te})\,\boldsymbol{\Omega}_{hf\text{-}bts}\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{S}_{te})'\\
&= (\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\,\boldsymbol{\Omega}_{hf\text{-}bts}\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})'\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})'\\
&= (\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\,\boldsymbol{\Omega}_{hf}\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})'
\end{aligned}$$

where $\boldsymbol{\Omega}_{hf} = (\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\,\boldsymbol{\Omega}_{hf\text{-}bts}\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})'$ and $\boldsymbol{S}_{ct} = \boldsymbol{S}_{cs}\otimes\boldsymbol{S}_{te} = (\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})$. We can apply the shrinkage estimator as

$$\begin{aligned}
&\lambda\widehat{\boldsymbol{\Omega}}_{hf,D} + (1-\lambda)\widehat{\boldsymbol{\Omega}}_{hf}\\
&\qquad\Downarrow\\
\widehat{\boldsymbol{\Omega}}_{H} &= (\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\left[\lambda\widehat{\boldsymbol{\Omega}}_{hf,D} + (1-\lambda)\widehat{\boldsymbol{\Omega}}_{hf}\right](\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})'\\
&= \lambda(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\widehat{\boldsymbol{\Omega}}_{hf,D}(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})' + (1-\lambda)(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\widehat{\boldsymbol{\Omega}}_{hf}(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})'.
\end{aligned}$$

The bottom time series representation (the third equivalence) follows by

$$\begin{aligned}
\widetilde{\boldsymbol{\Omega}} &= \boldsymbol{S}_{ct}\boldsymbol{\Omega}_{hf\text{-}bts}\boldsymbol{S}_{ct}'\\
&= (\boldsymbol{S}_{cs}\otimes\boldsymbol{S}_{te})\,\boldsymbol{\Omega}_{hf\text{-}bts}\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{S}_{te})'\\
&= (\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\,\boldsymbol{\Omega}_{hf\text{-}bts}\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})'\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})'\\
&= (\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\,\boldsymbol{\Omega}_{bts}\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})',
\end{aligned}$$

where $\boldsymbol{\Omega}_{bts} = (\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})\,\boldsymbol{\Omega}_{hf\text{-}bts}\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})'$ and $\boldsymbol{S}_{ct} = \boldsymbol{S}_{cs}\otimes\boldsymbol{S}_{te} = (\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\,(\boldsymbol{I}_n\otimes\boldsymbol{S}_{te})$. Finally we have that

$$\begin{aligned}
&\lambda\widehat{\boldsymbol{\Omega}}_{bts,D} + (1-\lambda)\widehat{\boldsymbol{\Omega}}_{bts}\\
&\qquad\Downarrow\\
\widehat{\boldsymbol{\Omega}}_{B} &= (\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\left[\lambda\widehat{\boldsymbol{\Omega}}_{bts,D} + (1-\lambda)\widehat{\boldsymbol{\Omega}}_{bts}\right](\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})'\\
&= \lambda\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})\,\widehat{\boldsymbol{\Omega}}_{bts,D}\,(\boldsymbol{S}_{cs}\otimes\boldsymbol{I}_{m+k^*})' +
\end{aligned}$$

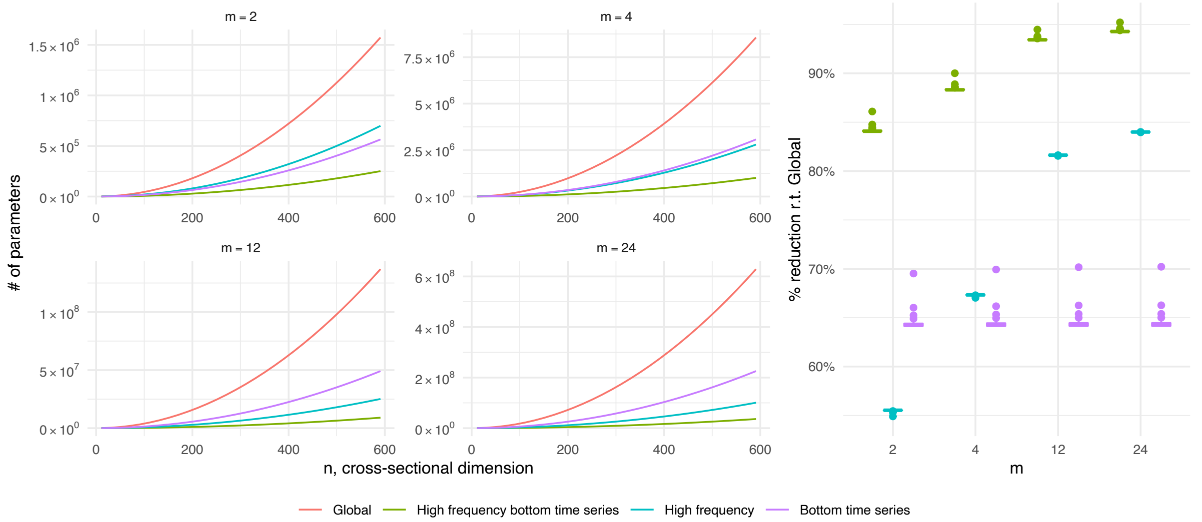

Figure B.1: *The four graphs on the left represent the number of different parameters in the covariance matrix for the various approaches presented for different values of m and n (n_b, the number of bottom time series, is about 60% of the total). On the right, we have the boxplot of the percentage reduction in the number of parameters compared to the global approach.*

$$(1-\lambda)\left(\boldsymbol{S}_{cs} \otimes \boldsymbol{I}_{m+k^*}\right) \widehat{\boldsymbol{\Omega}}_{bts} \left(\boldsymbol{S}_{cs} \otimes \boldsymbol{I}_{m+k^*}\right)'.$$

In general, the covariance matrix of the reconciled forecasts is equal to $\boldsymbol{M}\widehat{\boldsymbol{\Omega}}\boldsymbol{M}'$ where $\boldsymbol{M} = \boldsymbol{S}_{ct}\boldsymbol{G}$ is the projection matrix. When using the HB approach, the covariance matrix of the reconciliation and the base forecasts will be identical. Indeed, it can be shown (see Panagiotelis et al. 2021 for more details) that if $\boldsymbol{M}$ is a projection matrix (6) then $\boldsymbol{M}\boldsymbol{S}_{ct} = \boldsymbol{S}_{ct}\boldsymbol{G}\boldsymbol{S}_{ct} = \boldsymbol{S}_{ct}$, and we obtain that

$$\begin{aligned}
\boldsymbol{M}\widehat{\boldsymbol{\Omega}}_{HB}\boldsymbol{M}' &= \boldsymbol{M}\boldsymbol{S}_{ct}\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts,HB}\boldsymbol{S}_{ct}'\boldsymbol{M}' \\
&= \boldsymbol{S}_{ct}\boldsymbol{G}\boldsymbol{S}_{ct}\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts,HB}\boldsymbol{S}_{ct}'\boldsymbol{G}'\boldsymbol{S}_{ct}' \\
&= \boldsymbol{S}_{ct}\widehat{\boldsymbol{\Omega}}_{hf\text{-}bts,HB}\boldsymbol{S}_{ct}' = \widehat{\boldsymbol{\Omega}}_{HB}.
\end{aligned}$$

Figure B.1 shows the number of parameters for different values of m and n, with n_b fixed to approximately 60% of n. The right panel reports the boxplot of the percentage reductions in the number of parameters compared to the global approach. Figure B.2 gives some visual insights on the covariance matrices obtainable with $\lambda = 0$ and $\lambda = 1$, respectively, for a simple cross-temporal hierarchical structure with 3 time series and $\mathcal{K} = \{2, 1\}$ (e.g, cross-temporal semi-annual, see the Monte Carlo simulation in Appendix C).

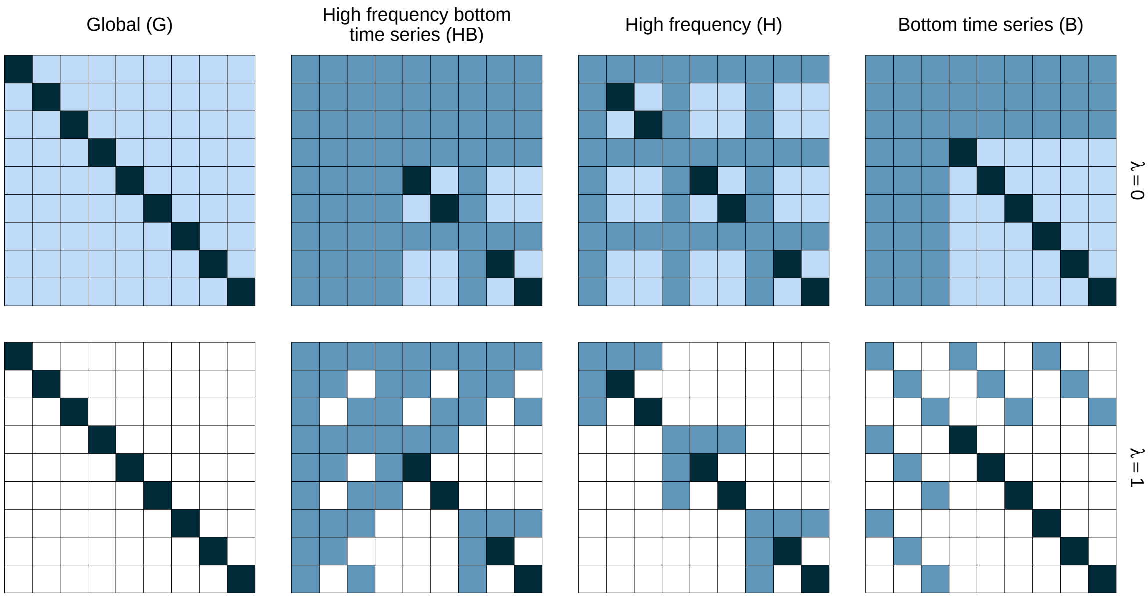

Figure B.2: *Representation of four types of covariance matrices that can be obtained from the cross-temporal hierarchical structure (3 time series and $m = 2$) for two different values of $\lambda \in \{0, 1\}$, the shrinkage parameter. The cells that are not modified by shrinkage are colored black, those actively involved in the shrinkage phase are colored light blue, and those derived from and not estimated by the base forecasts errors are colored blue. Additionally, for $\lambda = 1$, the cells corresponding to a zero value are colored white.*

B.1 Proof Theorem 4.1

Proof. Let $\mathbf{S}_{ct}^{+} = (\mathbf{S}_{ct}'\mathbf{S}_{ct})^{-1}\mathbf{S}_{ct}'$ be the generalized inverse of $\mathbf{S}_{ct}$ ($\mathbf{S}_{ct}$ has linearly independent columns by construction). Applying (6), we obtain

$$\widetilde{\boldsymbol{b}}_{ols}^{[1]} = (\mathbf{S}_{ct}'\mathbf{S}_{ct})^{-1}\mathbf{S}_{ct}'\widehat{\boldsymbol{x}}_h$$

and

$$\widetilde{\boldsymbol{b}}_{hb}^{[1]} = [\mathbf{S}_{ct}'(\mathbf{S}_{ct}\widehat{\Omega}_{hf-bts}\mathbf{S}_{ct}')^{+}\mathbf{S}_{ct}]^{-1}\mathbf{S}_{ct}'(\mathbf{S}_{ct}\widehat{\Omega}_{hf-bts}\mathbf{S}_{ct}')^{+}\widehat{\boldsymbol{x}}_h.$$

Since $(\boldsymbol{AB})^{+} = \boldsymbol{B}^{+}\boldsymbol{A}^{+}$ (Greville 1966), then

$$\begin{aligned} \mathbf{S}_{ct}'(\mathbf{S}_{ct}\widehat{\Omega}_{hf-bts}\mathbf{S}_{ct}')^{+}\mathbf{S}_{ct} &= \mathbf{S}_{ct}'(\widehat{\Omega}_{hf-bts}^{1/2}\mathbf{S}_{ct}')^{+}(\mathbf{S}_{ct}\widehat{\Omega}_{hf-bts}^{1/2})^{+}\mathbf{S}_{ct} \\ &= \mathbf{S}_{ct}'(\mathbf{S}_{ct}')^{+}\widehat{\Omega}_{hf-bts}^{+}\mathbf{S}_{ct}^{+}\mathbf{S}_{ct} = \widehat{\Omega}_{hf-bts}^{+} = \widehat{\Omega}_{hf-bts}^{-1} \end{aligned} \tag{B.1}$$

and

$$\begin{aligned} \mathbf{S}_{ct}'(\mathbf{S}_{ct}\widehat{\Omega}_{hf-bts}\mathbf{S}_{ct}')^{+} &= \mathbf{S}_{ct}'(\widehat{\Omega}_{hf-bts}^{1/2}\mathbf{S}_{ct}')^{+}(\mathbf{S}_{ct}\widehat{\Omega}_{hf-bts}^{1/2})^{+} \\ &= \widehat{\Omega}_{hf-bts}^{-1}\mathbf{S}_{ct}^{+} = \widehat{\Omega}_{hf-bts}^{-1}(\mathbf{S}_{ct}'\mathbf{S}_{ct})^{-1}\mathbf{S}_{ct}'. \end{aligned} \tag{B.2}$$

Therefore,

$$\begin{aligned} \widetilde{\boldsymbol{b}}_{hb}^{[1]} &= [\mathbf{S}_{ct}'(\mathbf{S}_{ct}\widehat{\Omega}_{hf-bts}\mathbf{S}_{ct}')^{+}\mathbf{S}_{ct}]^{-1}\mathbf{S}_{ct}'(\mathbf{S}_{ct}\widehat{\Omega}_{hf-bts}\mathbf{S}_{ct}')^{+}\widehat{\boldsymbol{x}}_h \\ &\overset{(B.1)}{=} \widehat{\Omega}_{hf-bts}\mathbf{S}_{ct}'(\mathbf{S}_{ct}\widehat{\Omega}_{hf-bts}\mathbf{S}_{ct}')^{+}\widehat{\boldsymbol{x}}_h \\ &\overset{(B.2)}{=} \widehat{\Omega}_{hf-bts}\widehat{\Omega}_{hf-bts}^{-1}(\mathbf{S}_{ct}'\mathbf{S}_{ct})^{-1}\mathbf{S}_{ct}'\widehat{\boldsymbol{x}}_h \\ &= (\mathbf{S}_{ct}'\mathbf{S}_{ct})^{-1}\mathbf{S}_{ct}'\widehat{\boldsymbol{x}}_h = \widetilde{\boldsymbol{b}}_{ols}^{[1]} \end{aligned}$$

□

C Monte Carlo simulation

We study the effect of combining cross-sectional and temporal aggregations, using a simple hierarchy that allows us to effectively visualize the quantities involved, such as the covariance matrix. Additionally, the small size and nature of the data generating process make it possible to exactly calculate the true cross-temporal covariance structure, thus providing insights into the nature of the time series data involved in the forecast reconciliation process.

Consider a 2-level hierarchical structure with three time series (one upper series, A, and two bottom series, B and C) such that the cross-sectional aggregation matrix is $\boldsymbol{A}_{cs} = [1 \quad 1]$ ($A = B + C$). The temporal structure we are considering is equivalent to using semi-annual data with $\mathcal{K} = \{2, 1\}$ and $m = 2$. The assumed Data-Generating Processes (DPG) for the semi-annual bottom level series are two AR(2) given by

$$y_{B,t} = \phi_{B,1} y_{B,t-1} + \phi_{B,2} y_{B,t-2} + \varepsilon_{B,t}$$
$$y_{C,t} = \phi_{C,1} y_{C,t-1} + \phi_{C,2} y_{C,t-2} + \varepsilon_{C,t}$$

with parameters[1] $\boldsymbol{\phi}_B = [\phi_{B,1} \ \phi_{B,2}]' = [1.34 \ \ -0.74]'$ and $\boldsymbol{\phi}_C = [\phi_{C,1} \ \phi_{C,2}]' = [0.95 \quad - \ 0.42]'$. The error $\boldsymbol{\varepsilon}_t = [\varepsilon_{B,t} \quad \varepsilon_{C,t}]'$ driving the process is drawn from a multivariate normal distribution with standard deviations simulated from a uniform distribution between 0.5 and 2 and a fixed correlation of -0.8. The cross-sectional error covariance matrix is thus given by

$$\boldsymbol{\Omega}_{cs} = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.8 \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} 0.9 & 0 \\ 0 & 1.8 \end{bmatrix} = \begin{bmatrix} \sigma_B^2 & \sigma_{BC} \\ \sigma_{BC} & \sigma_C^2 \end{bmatrix}.$$

To obtain the remaining series, the bottom series are then cross-temporally aggregated.

For the forecast experiment, the base forecasts are computing using AR models where the order is automatically determined by the algorithm proposed by Hyndman & Khandakar (2008) and implemented in the R package `forecast` (Hyndman et al. 2023), thus allowing for possible mis-specification in the models. The training window length is 500 years, consisting of 1000 high frequency observations. The experiment is replicated 500 times, with a forecast horizon of 1 year.

Since the AR(2) models used as DPG for the bottom series B and C at the most disaggregated temporal level are known, we may compute the true covariance matrix for one-step ahead forecasts at the annual level $\boldsymbol{\Omega}_{ct} = \boldsymbol{S}_{ct} \boldsymbol{\Omega}_{hf\text{-}bts} \boldsymbol{S}_{ct}'$, where

$$\boldsymbol{\Omega}_{hf\text{-}bts} = \begin{bmatrix} \sigma_B^2 & & & \\ \phi_{B,1}\sigma_B^2 & \sigma_B^2\left(1+\phi_{B,1}^2\right) & & \\ \sigma_{BC} & \phi_{B,1}\sigma_{BC} & \sigma_C^2 & \\ \phi_{C,1}\sigma_{BC} & \sigma_{BC}\left(1+\phi_{B,1}\phi_{C,1}\right) & \phi_{C,1}\sigma_C^2 & \sigma_C^2\left(1+\phi_{C,1}^2\right) \end{bmatrix}.$$

The detailed calculations can be found in Section C.2. Figure C.3 shows both the covariance matrix and the correlation matrix for fixed parameters.

[1] The $\boldsymbol{\phi}_B$ and $\boldsymbol{\phi}_C$ parameters are estimated from the "Lynx" and "Hare" time series contained in the `pelt` dataset of the `tsibbledata` package for R (O'Hara-Wild et al. 2022).

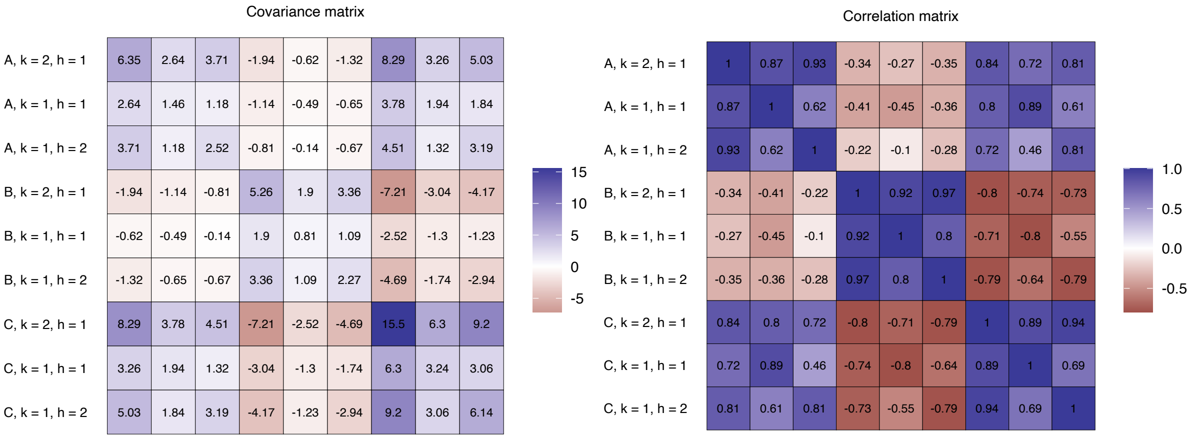

Figure C.3: *Simulation experiment. True cross-temporal covariance (left) and correlation (right) error matrix of the reconciled forecasts with* $\sigma_B = 0.9$, $\sigma_C = 1.8$, $\boldsymbol{\phi}_B = [1.34 \; -0.74]'$, $\boldsymbol{\phi}_C = [0.95 \; -0.42]'$ *and* $\rho = -0.8$.

To construct cross-temporal samples of the base forecasts, we use the Gaussian and bootstrap approaches discussed in Sections 3.1 and 3.2, respectively. For the parametric approach we use multi-step residuals with the different covariance matrix structures analyzed in 4.1, while for the non-parametric approach, we use regular one-step residuals. We do not use overlapping residuals in our analysis as we have the advantage of generating a large number of observation. Ten different reconciliation approaches have been adopted (see Table 2): ct(bu), ct(shr_{cs}, bu_{te}), ct$(wlsv_{te}, bu_{cs})$, oct$(wlsv)$, oct$(bdshr)$, oct$_h(shr)$, oct$_h(bshr)$, oct$_h(hshr)$ and oct$_h(hbshr)$.

C.1 Covariance matrix comparison and forecast accuracy scores

To compare the true covariance matrix $\boldsymbol{\Omega}_{ct}$ with the estimated covariance matrix $\widehat{\boldsymbol{\Omega}}$, we use the Frobenius norm to quantify the difference between two matrices:

$$\|\boldsymbol{D}\|_F = \sqrt{\sum_{i=1}^{n(k^*+m)} \sum_{j=1}^{n(k^*+m)} |d_{i,j}|^2}$$

where $\boldsymbol{D} = \widehat{\boldsymbol{\Omega}} - \boldsymbol{\Omega}_{ct}$. The true covariance matrix, shown in Figure C.3, was compared to the estimated covariance matrices obtained using various reconciliation approaches and techniques for generating sample paths of the base forecasts. Thus, we should be able to determine which reconciliation approach and simulation technique produce an accurate estimate of the covariance matrix. Other types of matrix norms were also considered with similar results.

From Table C.2, it appears that the reconciled covariance matrices are always closer to the true matrix than the base forecast matrix when using both the Gaussian and the bootstrap approach. Overall, there are no major differences in the findings when using either one-step or multi-step residuals in cross-temporal forecast reconciliation. In fact, using approaches like oct$(bdshr)$, we obtain results that are consistent with approaches such as oct$_h(shr)$, where no temporal and/or cross-sectional correlation assumptions are imposed. It is worth noting that the HB covariance matrix when used to calculate the base forecasts samples, is not changed by the reconciliation step (see Appendix B). In conclusion, our results suggest that using multi-step

Reconciliation approach	Generation of the base forecasts paths				
	ctjb	Gaussian approach*			
		G	B	H	HB
base	8.260	7.748	6.549	3.409	2.215
ct(bu)	3.195	2.215	2.215	**2.215**	2.215
ct(shr_{cs}, bu_{te})	3.202	2.224	2.215	2.224	2.215
ct($wlsv_{te}, bu_{cs}$)	**3.183**	**2.188**	2.188	**2.215**	2.215
oct($wlsv$)	3.766	3.082	2.191	2.910	2.215
oct($bdshr$)	3.203	2.195	**2.184**	2.224	**2.215**
oct$_h$(shr)	3.251	2.260	2.202	2.226	2.215
oct$_h$($bshr$)	3.602	2.720	2.220	2.756	2.215
oct$_h$($hshr$)	4.869	4.138	4.167	2.225	2.215
~~oct$_h$($hbshr$)~~	~~5.731~~	~~5.085~~	~~4.167~~	~~2.756~~	~~2.215~~

*The Gaussian method employs a sample covariance with multi-step residuals.

Table C.2: *Simulation experiment. Frobenius norm between the true and the estimated covariance matrix for different reconciliation approaches and different techniques for simulating the base forecasts. Entries in bold represent the lowest value for each column, while the blue entry represent the global minimum. The reconciliation approaches are described in Table 2.*

residuals or bootstrap techniques may help find a "good" estimate of the covariance matrix, which can be further improved by the reconciliation.

A limitation of this simulation setting is that we are using a high number of residuals, which may result in undervaluing the benefit from using the parameterization form of the covariance matrix such as HB, H, or B (see Section 4). Additionally, shrinkage techniques often yield very similar results when we use the corresponding matrix with $\lambda = 0$ (full covariance matrix).

In Tables C.3 and C.4 are reported the $\overline{\text{RelCRPS}}$ and ES ratio indices introduced in Sections 5 where low values indicate better quality of the forecasts. The good performance of the ct(bu) approach can be explained by a good quality of the base forecasts at the bottom level for $k = 1$, and therefore it is difficult for the other approaches to correctly adjust them using the somewhat less good forecasts of the higher temporal and cross-sectional levels. This also explains the good performance of ct(shr_{cs}, bu_{te}), which by definition only takes into account the information provided by the most temporally disaggregated base forecasts. Looking at the optimal cross-temporal reconciliation approaches, it does not seem to be any advantage in using multi-step residuals to calculate the covariance matrix in the reconciliation step.

In conclusion, we found that simulating base forecasts from multi-step residuals allows us to estimate a covariance matrix close to the true one. Additionally, we observed that reconciliation could be used to further improve the accuracy of these estimates: accurate base forecasts for $k = 1$ assist the good performance for bottom-up and optimal cross-temporal reconciliation approaches, such as oct($wlsv$) and oct($bdshr$), which perform well in terms of both CRPS and ES.

C.2 Cross-temporal covariance matrix

We assume two AR(2) processes with correlated errors such that

$$y_{i,t} = \phi_{i,1} y_{i,t-1} + \phi_{i,2} y_{i,t-2} + \varepsilon_{i,t}$$

Reconciliation approach	Generation of the base forecasts sample paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G	B	H	HB		G	B	H	HB
	$\forall k \in \{2,1\}$					$k=1$				
base	1.000	0.998	0.999	1.002	1.004	1.000	0.998	0.999	0.999	1.000
ct(bu)	0.901	0.900	0.900	0.900	0.900	0.978	0.976	0.976	0.977	0.977
ct(shr_{cs}, bu_{te})	**0.901**	**0.900**	**0.899**	**0.900**	**0.900**	**0.977**	**0.976**	**0.976**	**0.976**	**0.976**
ct($wlsv_{te}, bu_{cs}$)	0.910	0.916	0.916	0.916	0.917	0.986	0.993	0.993	0.993	0.993
oct($wlsv$)	0.922	0.930	0.930	0.930	0.931	0.998	1.006	1.006	1.007	1.007
oct($bdshr$)	0.910	0.916	0.915	0.916	0.916	0.986	0.992	0.992	0.993	0.993
oct$_h$(shr)	0.904	0.903	0.902	0.902	0.903	0.980	0.979	0.978	0.979	0.979
oct$_h$($bshr$)	0.923	0.922	0.922	0.921	0.922	0.999	0.998	0.998	0.998	0.998
oct$_h$($hshr$)	0.974	0.972	0.972	0.974	0.975	1.052	1.050	1.050	1.053	1.053
oct$_h$($hbshr$)	0.987	0.985	0.985	0.987	0.989	1.065	1.063	1.064	1.066	1.068
	$k=2$									
base	1.000	0.998	0.999	1.005	1.008					
ct(bu)	0.831	0.830	0.829	0.829	0.830					
ct(shr_{cs}, bu_{te})	**0.830**	**0.830**	**0.829**	**0.829**	**0.830**					
ct($wlsv_{te}, bu_{cs}$)	0.840	0.846	0.844	0.845	0.846					
oct($wlsv$)	0.851	0.859	0.859	0.859	0.861					
oct($bdshr$)	0.839	0.845	0.844	0.845	0.846					
oct$_h$(shr)	0.834	0.833	0.831	0.832	0.832					
oct$_h$($bshr$)	0.852	0.851	0.851	0.851	0.852					
oct$_h$($hshr$)	0.902	0.900	0.899	0.901	0.902					
oct$_h$($hbshr$)	0.915	0.913	0.913	0.914	0.917					

*The Gaussian method employs a sample covariance matrix and includes four techniques (G, B, H, HB) with multi-step residuals.

Table C.3: *Simulation experiment. $\overline{RelCRPS}$ defined in Section 5. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

where $\boldsymbol{\varepsilon}_t \sim \mathcal{N}_2\left(\mathbf{0}_{(2\times 1)}, \boldsymbol{\Omega}_{cs}\right)$ and $i \in \{B, C\}$. Let $y_{i,T+h}$ be the true observation for the i^{th} series and $\widetilde{y}_{i,T+h}$ the corresponding forecasts such that

$$\begin{aligned} y_{i,T+1} &= \phi_{i,1} y_{i,T} + \phi_{i,2} y_{i,T-1} + \varepsilon_{i,T+1} \\ y_{i,T+2} &= \phi_{i,1} y_{i,T+1} + \phi_{i,2} y_{i,T} + \varepsilon_{i,T+2} \end{aligned} \quad \text{and} \quad \begin{aligned} \widetilde{y}_{i,T+1} &= \phi_{i,1} y_{i,T} + \phi_{i,2} y_{i,T-1} \\ \widetilde{y}_{i,T+2} &= \phi_{i,1} \widetilde{y}_{i,T+1} + \phi_{i,2} y_{i,T} \end{aligned},$$

then

$$\begin{aligned} y_{i,T+1} - \widetilde{y}_{i,T+1} &= \varepsilon_{i,T+1} \\ y_{i,T+2} - \widetilde{y}_{i,T+2} &= \varepsilon_{i,T+2} + \phi_{i,1}\varepsilon_{i,T+1}. \end{aligned}$$

Finally, we can compute each element of the high frequency bottom time series covariance matrix

$$\begin{aligned} Var\left(y_{i,T+1} - \widetilde{y}_{i,T+1}\right) &= \sigma_i^2, \quad \forall i \in \{B, C\} \\ Var\left(y_{i,T+2} - \widetilde{y}_{i,T+2}\right) &= \sigma_i^2\left(1 + \phi_{i,1}^2\right), \quad \forall i \in \{B, C\} \\ Cov\left[\left(y_{i,T+2} - \widetilde{y}_{i,T+2}\right), \left(y_{i,T+1} - \widetilde{y}_{i,T+1}\right)\right] &= Cov\left[\left(y_{i,T+1} - \widetilde{y}_{i,T+1}\right), \left(y_{i,T+2} - \widetilde{y}_{i,T+2}\right)\right] \\ &= \phi_{i,1}\sigma_i^2, \quad \forall i \in \{B, C\} \\ Cov\left[\left(y_{i,T+1} - \widetilde{y}_{i,T+1}\right), \left(y_{j,T+1} - \widetilde{y}_{j,T+1}\right)\right] &= Cov\left[\left(y_{j,T+1} - \widetilde{y}_{j,T+1}\right), \left(y_{i,T+1} - \widetilde{y}_{i,T+1}\right)\right] \\ &= \sigma_{i,j}, \quad \forall i, j \in \{B, C\}, \quad i \neq j \end{aligned}$$

Reconciliation approach	Generation of the base forecasts sample paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G	B	H	HB		G	B	H	HB
	$\forall k \in \{2,1\}$					$k=1$				
base	1.000	0.996	0.999	1.000	1.004	1.000	0.997	1.000	0.997	1.000
ct(bu)	0.897	0.895	0.896	0.897	**0.895**	0.969	**0.967**	**0.967**	0.968	**0.968**
ct(shr_{cs}, bu_{te})	**0.896**	**0.895**	**0.895**	**0.896**	0.896	**0.968**	0.968	0.967	**0.968**	0.968
ct($wlsv_{te}, bu_{cs}$)	0.906	0.912	0.911	0.910	0.912	0.977	0.984	0.983	0.981	0.984
oct($wlsv$)	0.916	0.923	0.923	0.923	0.924	0.989	0.994	0.995	0.995	0.997
oct($bdshr$)	0.906	0.910	0.910	0.911	0.912	0.977	0.981	0.982	0.983	0.985
oct$_h$(shr)	0.900	0.898	0.898	0.897	0.898	0.971	0.969	0.969	0.969	0.969
oct$_h$($bshr$)	0.916	0.914	0.916	0.915	0.916	0.987	0.986	0.987	0.987	0.988
oct$_h$($hshr$)	0.967	0.964	0.964	0.966	0.967	1.040	1.036	1.036	1.040	1.040
~~oct$_h$($hbshr$)~~	~~0.978~~	~~0.975~~	~~0.976~~	~~0.977~~	~~0.980~~	~~1.051~~	~~1.047~~	~~1.049~~	~~1.051~~	~~1.052~~
	$k=2$									
base	1.000	0.996	0.998	1.003	1.008					
ct(bu)	0.831	0.829	0.829	0.830	**0.828**					
ct(shr_{cs}, bu_{te})	**0.829**	**0.828**	**0.829**	**0.829**	0.829					
ct($wlsv_{te}, bu_{cs}$)	0.839	0.844	0.844	0.844	0.845					
oct($wlsv$)	0.849	0.858	0.856	0.856	0.857					
oct($bdshr$)	0.839	0.845	0.843	0.845	0.844					
oct$_h$(shr)	0.835	0.833	0.833	0.831	0.832					
oct$_h$($bshr$)	0.850	0.847	0.849	0.849	0.850					
oct$_h$($hshr$)	0.900	0.897	0.896	0.897	0.899					
~~oct$_h$($hbshr$)~~	~~0.910~~	~~0.907~~	~~0.908~~	~~0.909~~	~~0.912~~					

*The Gaussian method employs a sample covariance matrix and includes four techniques (G, B, H, HB) with multi-step residuals.

Table C.4: *Simulation experiment. ES ratio indices defined in Section 5. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

$$
\begin{aligned}
Cov\left[\left(y_{i,T+2}-\widetilde{y}_{i,T+2}\right),\left(y_{j,T+1}-\widetilde{y}_{j,T+1}\right)\right] &= Cov\left[\left(y_{j,T+1}-\widetilde{y}_{j,T+1}\right),\left(y_{i,T+2}-\widetilde{y}_{i,T+2}\right)\right] \\
&= \phi_{i,1}\sigma_{i,j}, \quad \forall i,j \in \{B,C\}, \quad i \neq j \\
Cov\left[\left(y_{i,T+2}-\widetilde{y}_{i,T+2}\right),\left(y_{j,T+2}-\widetilde{y}_{j,T+2}\right)\right] &= Cov\left[\left(y_{j,T+2}-\widetilde{y}_{j,T+2}\right),\left(y_{i,T+2}-\widetilde{y}_{i,T+2}\right)\right] \\
&= \sigma_{i,j}\left(1+\phi_{i,1}\phi_{j,1}\right), \quad \forall i,j \in \{B,C\}, \quad i \neq j.
\end{aligned}
$$

In conclusion,

$$
\mathbf{\Omega}_{hf\text{-}bts} = \begin{bmatrix}
\sigma_B^2 & & & \\
\phi_{B,1}\sigma_B^2 & \sigma_B^2\left(1+\phi_{B,1}^2\right) & & \\
\sigma_{BC} & \phi_{B,1}\sigma_{BC} & \sigma_C^2 & \\
\phi_{C,1}\sigma_{BC} & \sigma_{BC}\left(1+\phi_{B,1}\phi_{C,1}\right) & \phi_{C,1}\sigma_C^2 & \sigma_C^2\left(1+\phi_{C,1}^2\right)
\end{bmatrix}
$$

and

$$
\mathbf{\Omega}_{ct} = \mathbf{S}_{ct}\mathbf{\Omega}_{hf\text{-}bts}\mathbf{S}_{ct}'.
$$

C.3 One-step residuals and shrinkage covariance matrix

In Section 4.1, we discussed the use of one-step residuals in estimating the covariance matrix. In particular we point out that one-step residuals lead to a biased estimate of the covariance matrix where some correlation are zeros by definition (see Figure C.4). In addition, Tables C.5, C.6 and C.7 show the Frobenius norm, CRPS, and ES skill scores as explained in the paper to investigate the effectiveness of one-step residuals. Moreover, in Tables C.8 and C.9, we have utilized a shrinkage matrix rather than the sample covariance matrix to assess the performance of our approach.

Reconciliation approach	**Generation of the base forecasts paths**								
	Gaussian approach: sample covariance matrix								
	ctjb	In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
base	8.260	17.638	16.733	22.178	21.789	7.748	6.549	3.409	2.215
ct(bu)	3.195	21.789	21.789	**21.789**	21.789	2.215	2.215	**2.215**	2.215
ct(shr_{cs}, bu_{te})	3.202	21.942	21.789	21.942	21.789	2.224	2.215	2.224	2.215
ct($wlsv_{te}, bu_{cs}$)	**3.183**	18.237	18.237	21.789	21.789	**2.188**	2.188	**2.215**	2.215
oct($wlsv$)	3.766	19.174	18.611	22.304	21.789	3.082	2.191	2.910	2.215
oct($bdshr$)	3.203	18.559	18.416	21.937	21.789	2.195	**2.184**	2.224	**2.215**
oct(shr)	5.217	25.015	23.457	23.413	**21.789**	2.260	2.202	2.226	2.215
oct($bshr$)	5.282	23.772	23.997	22.146	21.789	2.720	2.220	2.756	2.215
oct($hshr$)	6.161	**11.336**	**10.940**	23.598	**21.789**	4.138	4.167	2.225	2.215
oct($hbshr$)	5.731	11.379	10.940	22.146	21.789	5.085	4.167	2.756	2.215
oct$_h$(shr)	3.251	20.965	19.992	22.079	**21.789**	2.260	2.202	2.226	2.215
oct$_h$($bshr$)	3.602	21.306	21.022	22.146	21.789	2.720	2.220	2.756	2.215
oct$_h$($hshr$)	4.869	11.405	10.940	22.037	21.789	4.138	4.167	2.225	2.215
~~oct$_h$($hbshr$)~~	~~5.731~~	~~11.379~~	~~10.940~~	~~22.146~~	~~21.789~~	~~5.085~~	~~4.167~~	~~2.756~~	~~2.215~~

Table C.5: *Frobenius norm between the true and the estimated covariance matrix for different reconciliation approaches and different techniques for simulating the base forecasts. Entries in bold represent the lowest value for each column, while the blue entry represent the global minimum. The reconciliation approaches are described in Table 2.*

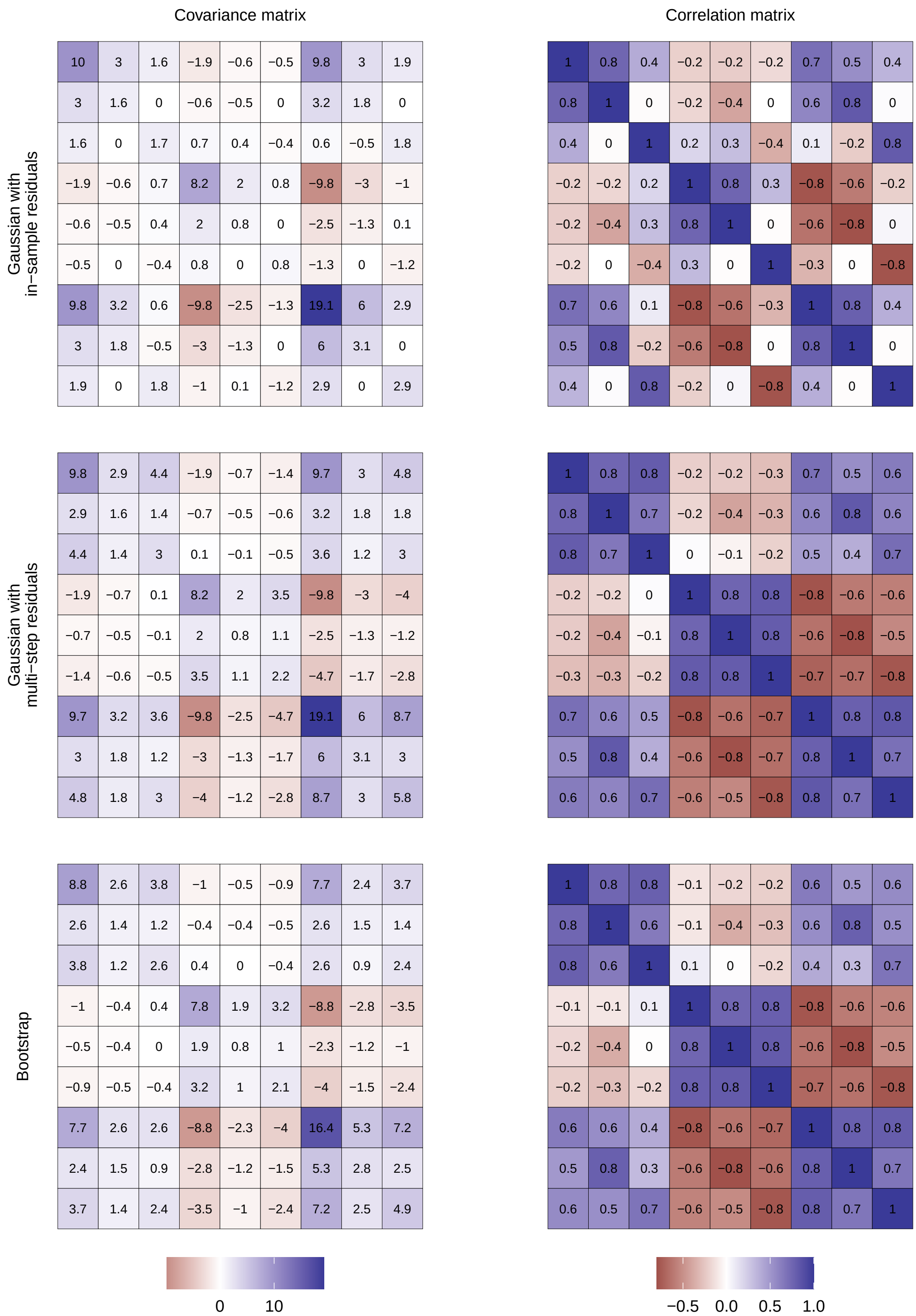

Figure C.4: *Comparison of estimated covariance and correlation matrices (first simulation) for base forecasts using a parametric Gaussian (with one-step residuals) approach. The true covariance and correlation matrices are shown in Figure 7.*

Reconciliation approach	Generation of the base forecasts paths								
	Gaussian approach: sample covariance matrix								
	ctjb	In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
				$\forall k \in \{2,1\}$					
base	1.000	1.008	1.009	1.044	1.047	0.998	0.999	1.002	1.004
ct(bu)	0.901	0.930	0.929	0.929	0.929	0.900	0.900	0.900	0.900
ct(shr_{cs}, bu_{te})	**0.901**	**0.929**	0.928	**0.929**	**0.928**	**0.900**	**0.899**	**0.900**	**0.900**
ct$(wlsv_{te}, bu_{cs})$	0.910	0.930	0.929	0.939	0.939	0.916	0.916	0.916	0.917
oct$(wlsv)$	0.922	0.942	0.944	0.951	0.953	0.930	0.930	0.930	0.931
oct$(bdshr)$	0.910	0.930	0.930	0.939	0.938	0.916	0.915	0.916	0.916
oct(shr)	0.941	0.999	0.985	0.983	0.973	0.903	0.902	0.902	0.903
oct$(bshr)$	0.951	0.995	1.000	0.983	0.986	0.922	0.922	0.921	0.922
oct$(hshr)$	0.987	0.995	0.993	1.039	1.026	0.972	0.972	0.974	0.975
oct$(hbshr)$	0.987	0.995	0.996	1.024	1.028	0.985	0.985	0.987	0.989
oct$_h(shr)$	0.904	0.929	**0.928**	0.932	0.932	0.903	0.902	0.902	0.903
oct$_h(bshr)$	0.923	0.948	0.952	0.951	0.954	0.922	0.922	0.921	0.922
oct$_h(hshr)$	0.974	0.982	0.982	1.012	1.012	0.972	0.972	0.974	0.975
~~oct$_h(hbshr)$~~	~~0.987~~	~~0.995~~	~~0.996~~	~~1.024~~	~~1.028~~	~~0.985~~	~~0.985~~	~~0.987~~	~~0.989~~
				$k = 1$					
base	1.000	1.017	1.019	1.017	1.019	0.998	0.999	0.999	1.000
ct(bu)	0.978	0.994	0.994	0.994	0.994	0.976	0.976	0.977	0.977
ct(shr_{cs}, bu_{te})	**0.977**	**0.993**	**0.993**	**0.994**	**0.993**	**0.976**	**0.976**	**0.976**	**0.976**
ct$(wlsv_{te}, bu_{cs})$	0.986	1.002	1.002	1.003	1.003	0.993	0.993	0.993	0.993
oct$(wlsv)$	0.998	1.014	1.015	1.015	1.016	1.006	1.006	1.007	1.007
oct$(bdshr)$	0.986	1.002	1.002	1.003	1.003	0.992	0.992	0.993	0.993
oct(shr)	1.037	1.082	1.067	1.064	1.056	0.979	0.978	0.979	0.979
oct$(bshr)$	1.041	1.071	1.074	1.060	1.062	0.998	0.998	0.998	0.998
oct$(hshr)$	1.080	1.090	1.091	1.119	1.105	1.050	1.050	1.053	1.053
oct$(hbshr)$	1.065	1.080	1.081	1.088	1.090	1.063	1.064	1.066	1.068
oct$_h(shr)$	0.980	0.996	0.995	0.996	0.996	0.979	0.978	0.979	0.979
oct$_h(bshr)$	0.999	1.016	1.018	1.016	1.018	0.998	0.998	0.998	0.998
oct$_h(hshr)$	1.052	1.067	1.066	1.074	1.075	1.050	1.050	1.053	1.053
~~oct$_h(hbshr)$~~	~~1.065~~	~~1.080~~	~~1.081~~	~~1.088~~	~~1.090~~	~~1.063~~	~~1.064~~	~~1.066~~	~~1.068~~
				$k = 2$					
base	1.000	0.998	0.999	1.071	1.075	0.998	0.999	1.005	1.008
ct(bu)	0.831	0.869	0.869	0.869	0.869	0.830	0.829	0.829	0.830
ct(shr_{cs}, bu_{te})	**0.830**	0.869	0.868	**0.868**	**0.868**	**0.830**	**0.829**	**0.829**	**0.830**
ct$(wlsv_{te}, bu_{cs})$	0.840	**0.863**	**0.862**	0.879	0.878	0.846	0.844	0.845	0.846
oct$(wlsv)$	0.851	0.875	0.877	0.891	0.893	0.859	0.859	0.859	0.861
oct$(bdshr)$	0.839	0.863	0.863	0.879	0.878	0.845	0.844	0.845	0.846
oct(shr)	0.854	0.922	0.909	0.908	0.897	0.833	0.831	0.832	0.832
oct$(bshr)$	0.869	0.925	0.931	0.911	0.915	0.851	0.851	0.851	0.852
oct$(hshr)$	0.901	0.908	0.904	0.966	0.952	0.900	0.899	0.901	0.902
oct$(hbshr)$	0.915	0.917	0.919	0.964	0.969	0.913	0.913	0.914	0.917
oct$_h(shr)$	0.834	0.868	0.865	0.872	0.872	0.833	0.831	0.832	0.832
oct$_h(bshr)$	0.852	0.886	0.890	0.890	0.894	0.851	0.851	0.851	0.852
oct$_h(hshr)$	0.902	0.904	0.904	0.953	0.952	0.900	0.899	0.901	0.902
~~oct$_h(hbshr)$~~	~~0.915~~	~~0.917~~	~~0.919~~	~~0.964~~	~~0.969~~	~~0.913~~	~~0.913~~	~~0.914~~	~~0.917~~

Table C.6: *Simulation experiment.* $\overline{\text{RelCRPS}}$ *defined in Section 5. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths								
	Gaussian approach: sample covariance matrix								
	ctjb	In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
	$\forall k \in \{2,1\}$								
base	1.000	1.005	1.009	1.039	1.046	0.996	0.999	1.000	1.004
ct(bu)	0.897	0.924	0.923	0.924	0.923	0.895	0.896	0.897	**0.895**
ct(shr_{cs}, bu_{te})	**0.896**	0.924	0.923	**0.923**	**0.922**	**0.895**	**0.895**	**0.896**	0.896
ct($wlsv_{te}, bu_{cs}$)	0.906	0.924	0.923	0.933	0.932	0.912	0.911	0.910	0.912
oct($wlsv$)	0.916	0.935	0.937	0.944	0.945	0.923	0.923	0.923	0.924
oct($bdshr$)	0.906	**0.923**	0.923	0.932	0.932	0.910	0.910	0.911	0.912
oct(shr)	0.938	0.993	0.980	0.977	0.969	0.898	0.898	0.898	0.897
oct($bshr$)	0.947	0.990	0.995	0.979	0.981	0.915	0.915	0.915	0.915
oct($hshr$)	0.978	0.987	0.985	1.027	1.016	0.963	0.964	0.966	0.967
oct($hbshr$)	0.977	0.986	0.985	1.012	1.016	0.974	0.976	0.977	0.978
oct$_h$(shr)	0.900	0.923	**0.922**	0.926	0.925	0.898	0.898	0.897	0.898
oct$_h$($bshr$)	0.916	0.940	0.943	0.942	0.945	0.914	0.916	0.915	0.916
oct$_h$($hshr$)	0.967	0.974	0.974	1.002	1.002	0.964	0.964	0.966	0.967
oct$_h$($hbshr$)	0.978	0.984	0.986	1.012	1.015	0.975	0.976	0.977	0.980
	$k = 1$								
base	1.000	1.014	1.020	1.015	1.019	0.997	1.000	0.997	1.000
ct(bu)	0.969	0.985	0.983	0.985	0.984	**0.967**	**0.967**	0.968	**0.968**
ct(shr_{cs}, bu_{te})	**0.968**	**0.984**	**0.983**	**0.984**	**0.983**	0.968	0.967	**0.968**	0.968
ct($wlsv_{te}, bu_{cs}$)	0.977	0.991	0.991	0.992	0.992	0.984	0.983	0.981	0.984
oct($wlsv$)	0.989	1.002	1.004	1.003	1.004	0.994	0.995	0.995	0.997
oct($bdshr$)	0.977	0.989	0.991	0.992	0.992	0.981	0.982	0.983	0.985
oct(shr)	1.028	1.070	1.056	1.053	1.046	0.969	0.969	0.970	0.969
oct($bshr$)	1.034	1.061	1.065	1.051	1.053	0.985	0.987	0.986	0.987
oct($hshr$)	1.066	1.075	1.076	1.099	1.090	1.037	1.037	1.039	1.039
oct($hbshr$)	1.050	1.065	1.065	1.070	1.073	1.048	1.049	1.049	1.052
oct$_h$(shr)	0.971	0.985	0.985	0.986	0.986	0.969	0.969	0.969	0.969
oct$_h$($bshr$)	0.987	1.002	1.005	1.002	1.005	0.986	0.987	0.987	0.988
oct$_h$($hshr$)	1.040	1.053	1.053	1.059	1.058	1.036	1.036	1.040	1.040
oct$_h$($hbshr$)	1.051	1.064	1.063	1.071	1.073	1.047	1.049	1.051	1.052
	$k = 2$								
base	1.000	0.997	0.999	1.063	1.073	0.996	0.998	1.003	1.008
ct(bu)	0.831	0.867	0.867	0.867	0.867	0.829	0.829	0.830	**0.828**
ct(shr_{cs}, bu_{te})	**0.829**	0.867	0.866	**0.866**	**0.865**	**0.828**	**0.829**	**0.829**	0.829
ct($wlsv_{te}, bu_{cs}$)	0.839	**0.860**	**0.860**	0.877	0.876	0.844	0.844	0.844	0.845
oct($wlsv$)	0.849	0.872	0.875	0.887	0.890	0.858	0.856	0.856	0.857
oct($bdshr$)	0.839	0.861	0.861	0.876	0.875	0.845	0.843	0.845	0.844
oct(shr)	0.856	0.921	0.909	0.907	0.898	0.832	0.831	0.832	0.831
oct($bshr$)	0.868	0.924	0.930	0.911	0.915	0.849	0.848	0.849	0.848
oct($hshr$)	0.897	0.905	0.901	0.959	0.947	0.895	0.896	0.898	0.899
oct($hbshr$)	0.910	0.912	0.912	0.957	0.961	0.906	0.909	0.909	0.910
oct$_h$(shr)	0.835	0.865	0.862	0.870	0.868	0.833	0.833	0.831	0.832
oct$_h$($bshr$)	0.850	0.881	0.885	0.886	0.889	0.847	0.849	0.849	0.850
oct$_h$($hshr$)	0.900	0.902	0.901	0.947	0.948	0.897	0.896	0.897	0.899
oct$_h$($hbshr$)	0.910	0.910	0.914	0.957	0.961	0.907	0.908	0.909	0.912

Table C.7: *Simulation experiment. ES ratio indices defined in Section 5. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths								
		Gaussian approach: shrinkage covariance matrix							
	ctjb	In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
	$\forall k \in \{2,1\}$								
base	1.007	1.009	1.044	1.046	0.997	0.999	1.002	1.003	1.000
ct(bu)	0.929	0.929	0.929	0.929	0.899	0.900	0.900	0.900	0.901
ct(shr_{cs}, bu_{te})	**0.929**	0.928	**0.929**	**0.928**	**0.899**	**0.899**	**0.900**	**0.900**	**0.901**
ct($wlsv_{te}, bu_{cs}$)	0.930	0.930	0.939	0.938	0.915	0.916	0.917	0.916	0.910
oct($wlsv$)	0.943	0.944	0.951	0.952	0.929	0.930	0.931	0.930	0.922
oct($bdshr$)	0.930	0.930	0.938	0.938	0.915	0.916	0.916	0.916	0.910
oct(shr)	0.994	0.982	0.980	0.973	0.902	0.902	0.903	0.902	0.941
oct($bshr$)	0.995	0.998	0.983	0.986	0.921	0.922	0.922	0.922	0.951
oct($hshr$)	0.994	0.994	1.035	1.025	0.971	0.972	0.974	0.974	0.987
oct($hbshr$)	0.995	0.997	1.025	1.027	0.984	0.986	0.988	0.988	0.987
oct$_h$(shr)	0.929	**0.928**	0.932	0.932	0.902	0.902	0.903	0.902	0.904
oct$_h$($bshr$)	0.948	0.951	0.951	0.953	0.921	0.922	0.922	0.922	0.923
oct$_h$($hshr$)	0.982	0.982	1.011	1.011	0.971	0.972	0.974	0.974	0.974
oct$_h$($hbshr$)	0.995	0.997	1.025	1.027	0.984	0.986	0.988	0.988	0.987
	$k = 1$								
base	1.017	1.019	1.017	1.019	0.998	0.999	0.999	0.999	1.000
ct(bu)	0.994	0.994	0.994	0.994	0.976	0.976	0.977	0.976	0.978
ct(shr_{cs}, bu_{te})	**0.993**	**0.993**	**0.993**	**0.993**	**0.975**	**0.976**	**0.976**	**0.976**	**0.977**
ct($wlsv_{te}, bu_{cs}$)	1.002	1.002	1.003	1.003	0.992	0.993	0.993	0.993	0.986
oct($wlsv$)	1.015	1.015	1.015	1.016	1.005	1.007	1.007	1.007	0.998
oct($bdshr$)	1.002	1.002	1.003	1.002	0.992	0.992	0.993	0.992	0.986
oct(shr)	1.076	1.065	1.061	1.056	0.978	0.978	0.979	0.978	1.037
oct($bshr$)	1.070	1.072	1.060	1.062	0.997	0.998	0.998	0.998	1.041
oct($hshr$)	1.090	1.092	1.114	1.105	1.049	1.050	1.053	1.052	1.080
oct($hbshr$)	1.080	1.081	1.089	1.090	1.062	1.064	1.066	1.066	1.065
oct$_h$(shr)	0.996	0.995	0.996	0.996	0.978	0.978	0.979	0.978	0.980
oct$_h$($bshr$)	1.016	1.018	1.016	1.018	0.997	0.998	0.998	0.998	0.999
oct$_h$($hshr$)	1.066	1.067	1.075	1.075	1.049	1.050	1.053	1.052	1.052
oct$_h$($hbshr$)	1.080	1.081	1.089	1.090	1.062	1.064	1.066	1.066	1.065
	$k = 2$								
base	0.997	0.999	1.071	1.074	0.997	0.999	1.005	1.008	1.000
ct(bu)	0.869	0.868	0.868	0.868	0.829	0.829	0.830	0.830	0.831
ct(shr_{cs}, bu_{te})	0.868	0.867	**0.868**	**0.867**	**0.829**	**0.829**	**0.830**	**0.829**	**0.830**
ct($wlsv_{te}, bu_{cs}$)	**0.863**	**0.862**	0.878	0.878	0.845	0.845	0.846	0.846	0.840
oct($wlsv$)	0.876	0.877	0.891	0.892	0.859	0.860	0.860	0.860	0.851
oct($bdshr$)	0.863	0.863	0.878	0.877	0.844	0.845	0.846	0.845	0.839
oct(shr)	0.918	0.906	0.906	0.897	0.832	0.832	0.833	0.832	0.854
oct($bshr$)	0.924	0.928	0.911	0.915	0.850	0.851	0.852	0.851	0.869
oct($hshr$)	0.907	0.905	0.962	0.951	0.898	0.899	0.902	0.902	0.901
oct($hbshr$)	0.917	0.919	0.964	0.968	0.912	0.913	0.915	0.916	0.915
oct$_h$(shr)	0.867	0.864	0.872	0.871	0.832	0.832	0.833	0.832	0.834
oct$_h$($bshr$)	0.886	0.890	0.890	0.893	0.850	0.851	0.852	0.851	0.852
oct$_h$($hshr$)	0.904	0.905	0.952	0.952	0.898	0.899	0.902	0.902	0.902
oct$_h$($hbshr$)	0.917	0.919	0.964	0.968	0.912	0.913	0.915	0.916	0.915

Table C.8: *Simulation experiment. $\overline{RelCRPS}$ defined in Section 5. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	**Generation of the base forecasts paths**								
		Gaussian approach: shrinkage covariance matrix							
	ctjb	In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
					$\forall k \in \{2,1\}$				
base	1.005	1.008	1.039	1.045	0.996	0.999	1.000	1.003	1.000
ct(bu)	**0.923**	0.923	0.923	0.923	**0.895**	0.896	0.897	0.897	0.897
ct(shr_{cs}, bu_{te})	0.923	0.922	**0.922**	**0.922**	0.896	**0.895**	**0.895**	**0.895**	**0.896**
ct($wlsv_{te}, bu_{cs}$)	0.924	0.924	0.932	0.932	0.910	0.911	0.911	0.911	0.906
oct($wlsv$)	0.935	0.937	0.944	0.945	0.922	0.924	0.923	0.923	0.916
oct($bdshr$)	0.924	0.924	0.932	0.931	0.909	0.911	0.911	0.910	0.906
oct(shr)	0.989	0.978	0.975	0.968	0.897	0.898	0.898	0.898	0.938
oct($bshr$)	0.990	0.993	0.978	0.981	0.915	0.915	0.915	0.915	0.947
oct($hshr$)	0.986	0.985	1.024	1.015	0.963	0.964	0.966	0.967	0.978
oct($hbshr$)	0.985	0.986	1.012	1.015	0.973	0.976	0.977	0.978	0.977
oct$_h$(shr)	0.923	**0.922**	0.925	0.925	0.897	0.898	0.898	0.898	0.900
oct$_h$($bshr$)	0.941	0.943	0.942	0.945	0.913	0.915	0.915	0.915	0.916
oct$_h$($hshr$)	0.974	0.975	1.001	1.001	0.964	0.964	0.966	0.966	0.967
oct$_h$($hbshr$)	0.985	0.986	1.013	1.016	0.973	0.976	0.977	0.978	0.978
					$k = 1$				
base	1.014	1.018	1.015	1.019	0.997	0.999	0.997	0.998	1.000
ct(bu)	0.983	0.984	0.984	0.984	0.967	0.967	0.969	0.969	0.969
ct(shr_{cs}, bu_{te})	**0.983**	**0.982**	**0.982**	**0.983**	**0.966**	**0.967**	**0.966**	**0.966**	**0.968**
ct($wlsv_{te}, bu_{cs}$)	0.991	0.992	0.993	0.992	0.983	0.983	0.983	0.983	0.977
oct($wlsv$)	1.002	1.004	1.004	1.004	0.994	0.995	0.994	0.996	0.989
oct($bdshr$)	0.990	0.991	0.992	0.991	0.981	0.983	0.984	0.982	0.977
oct(shr)	1.065	1.054	1.051	1.045	0.969	0.970	0.970	0.969	1.028
oct($bshr$)	1.061	1.063	1.050	1.052	0.986	0.986	0.987	0.985	1.034
oct($hshr$)	1.076	1.077	1.095	1.088	1.036	1.036	1.040	1.038	1.066
oct($hbshr$)	1.064	1.065	1.071	1.073	1.047	1.048	1.050	1.050	1.050
oct$_h$(shr)	0.984	0.985	0.986	0.986	0.969	0.969	0.969	0.968	0.971
oct$_h$($bshr$)	1.003	1.005	1.003	1.005	0.985	0.987	0.987	0.986	0.987
oct$_h$($hshr$)	1.054	1.054	1.059	1.059	1.036	1.037	1.038	1.039	1.040
oct$_h$($hbshr$)	1.063	1.065	1.071	1.074	1.046	1.048	1.049	1.051	1.051
					$k = 2$				
base	0.996	0.998	1.064	1.073	0.995	0.999	1.003	1.007	1.000
ct(bu)	0.867	0.866	0.867	0.866	**0.829**	0.829	0.830	0.830	0.831
ct(shr_{cs}, bu_{te})	0.867	0.866	**0.866**	**0.866**	0.830	**0.829**	**0.830**	**0.830**	**0.829**
ct($wlsv_{te}, bu_{cs}$)	**0.861**	**0.861**	0.875	0.875	0.843	0.845	0.845	0.845	0.839
oct($wlsv$)	0.873	0.874	0.888	0.889	0.856	0.857	0.857	0.856	0.849
oct($bdshr$)	0.862	0.861	0.876	0.874	0.843	0.844	0.844	0.844	0.839
oct(shr)	0.918	0.907	0.905	0.898	0.831	0.832	0.832	0.832	0.856
oct($bshr$)	0.924	0.928	0.911	0.915	0.849	0.849	0.849	0.849	0.868
oct($hshr$)	0.904	0.901	0.957	0.946	0.895	0.896	0.898	0.900	0.897
oct($hbshr$)	0.912	0.913	0.956	0.961	0.905	0.909	0.909	0.911	0.910
oct$_h$(shr)	0.866	0.863	0.869	0.869	0.830	0.831	0.832	0.832	0.835
oct$_h$($bshr$)	0.882	0.886	0.886	0.889	0.846	0.848	0.849	0.848	0.850
oct$_h$($hshr$)	0.901	0.902	0.947	0.946	0.896	0.896	0.898	0.899	0.900
oct$_h$($hbshr$)	0.912	0.914	0.958	0.961	0.905	0.908	0.910	0.909	0.910

Table C.9: *Simulation experiment. ES ratio indices defined in Section 5. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

D Forecast reconciliation of the Australian GDP dataset

D.1 The dataset

Athanasopoulos et al. (2020) proposed using state-of-the-art forecast reconciliation methods to improve the accuracy of macroeconomic forecasts and facilitate aligned decision-making. In their empirical analysis, they applied cross-sectional forecast reconciliation to 95 Australian QNA time series that represent the Gross Domestic Product (GDP) calculated using both the income and expenditure approaches. These two approaches correspond to two distinct hierarchical structures, with GDP at the top and 15 lower-level aggregates in the income approach, and GDP as the top-level aggregate in a hierarchy of 79 time series in the expenditure approach (for more information, see Athanasopoulos et al. 2020, pp. 702–705 and figures 21.4–21.7). Bisaglia et al. (2020) showed how to obtain a "one-number" forecast where the GDP reconciled forecasts are coherent for both the expenditure and income sides. Di Fonzo & Girolimetto (2022*c*,*d*) extended the one number forecasts idea to obtain fully reconciled probabilistic forecasts, and Di Fonzo & Girolimetto (2023*a*) computed cross-temporally reconciled point forecasts.

D.2 One-step residuals and shrinkage covariance matrix

Reconciliation approach	**Generation of the base forecasts paths**									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G_h	H_h	G_{oh}	H_{oh}		G_h	H_h	G_{oh}	H_{oh}
	$\forall k \in \{4,2,1\}$					$k=1$				
base	1.000	0.979	0.995	0.968	0.976	1.000	0.988	0.988	0.971	0.971
ct(shr_{cs}, bu_{te})	0.937	0.956	0.956	0.976	0.976	0.992	1.008	1.008	1.029	1.029
ct(wls_{cs}, bu_{te})	0.930	0.917	0.917	0.898	0.898	0.986	0.974	0.975	0.956	0.956
oct($wlsv$)	0.926	0.919	0.920	0.900	0.900	0.984	0.981	0.979	0.959	0.959
oct($bdshr$)	0.940	0.965	0.945	0.992	0.957	0.997	1.019	1.003	1.044	1.018
oct(shr)	0.944	1.020	0.940	1.094	0.988	1.015	1.095	1.010	1.160	1.059
oct($hshr$)	0.988	0.972	1.002	0.974	1.001	1.048	1.037	1.060	1.034	1.061
oct$_o$($wlsv$)	**0.926**	**0.911**	**0.912**	**0.896**	**0.895**	**0.984**	**0.971**	**0.970**	**0.954**	**0.954**
oct$_o$($bdshr$)	0.978	0.964	0.946	0.952	0.930	1.034	1.016	1.003	1.005	0.989
oct$_o$(shr)	0.950	0.946	0.922	0.925	0.903	1.014	1.003	0.985	0.987	0.968
oct$_o$($hshr$)	0.989	0.966	0.984	0.954	0.965	1.047	1.028	1.038	1.012	1.023
oct$_{oh}$(shr)	1.102	1.059	1.001	1.094	0.988	1.172	1.109	1.066	1.160	1.059
oct$_{oh}$($hshr$)	1.006	0.983	1.009	0.974	1.001	1.068	1.046	1.059	1.034	1.061
	$k=2$					$k=4$				
base	1.000	0.984	0.993	0.968	0.976	1.000	0.966	1.004	0.964	0.981
ct(shr_{cs}, bu_{te})	0.949	0.966	0.966	0.987	0.987	0.874	0.896	0.896	0.914	0.914
ct(wls_{cs}, bu_{te})	0.942	0.928	0.928	0.909	0.909	0.866	0.853	0.853	0.834	0.834
oct($wlsv$)	0.938	0.929	0.931	0.911	0.911	0.860	0.853	0.855	0.835	0.834
oct($bdshr$)	0.953	0.976	0.956	1.003	0.969	0.874	0.904	0.880	0.931	0.889
oct(shr)	0.955	1.031	0.951	1.113	1.002	0.866	0.940	0.864	1.015	0.909
oct($hshr$)	1.001	0.985	1.014	0.987	1.016	0.919	0.900	0.935	0.904	0.931
oct$_o$($wlsv$)	**0.938**	**0.921**	**0.923**	**0.907**	**0.906**	**0.860**	**0.847**	**0.848**	**0.832**	**0.830**
oct$_o$($bdshr$)	0.991	0.974	0.957	0.964	0.942	0.914	0.905	0.883	0.892	0.865
oct$_o$(shr)	0.965	0.958	0.934	0.938	0.916	0.877	0.882	0.852	0.854	0.831
oct$_o$($hshr$)	1.002	0.979	0.996	0.967	0.978	0.922	0.898	0.923	0.888	0.898
oct$_{oh}$(shr)	1.120	1.069	1.013	1.113	1.002	1.020	1.002	0.928	1.015	0.909
oct$_{oh}$($hshr$)	1.021	0.996	1.021	0.987	1.016	0.934	0.912	0.951	0.904	0.931

*The Gaussian method employs a sample covariance matrix:
G_h and H_h use multi-step residuals and G_{oh} and H_{oh} use overlapping and multi-step residuals.

Table D.10: *$\overline{RelCRPS}$ indices defined in Section 5 for the Australian QNA dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G$_h$	H$_h$	G$_{oh}$	H$_{oh}$		G$_h$	H$_h$	G$_{oh}$	H$_{oh}$
	$\forall k \in \{4,2,1\}$					$k=1$				
base	1.000	0.970	0.988	0.960	0.970	1.000	0.977	0.977	0.965	0.965
ct(shr_{cs}, bu_{te})	0.897	0.944	0.944	0.973	0.973	0.964	1.001	1.001	1.033	1.033
ct(wls_{cs}, bu_{te})	**0.886**	0.880	0.880	**0.860**	0.860	**0.954**	**0.944**	0.945	**0.928**	**0.928**
oct($wlsv$)	0.890	0.890	0.894	0.872	0.872	0.958	0.957	0.957	0.938	0.939
oct($bdshr$)	0.905	0.956	0.934	0.992	0.954	0.972	1.014	0.994	1.048	1.018
oct(shr)	0.895	0.979	0.895	1.053	0.944	0.973	1.060	0.969	1.121	1.015
oct($hshr$)	0.951	0.940	0.973	0.959	0.992	1.017	1.010	1.034	1.023	1.055
oct$_o$($wlsv$)	0.891	**0.879**	0.881	0.864	0.864	0.958	0.945	0.945	0.931	0.931
oct$_o$($bdshr$)	0.940	0.928	0.910	0.918	0.895	1.004	0.986	0.971	0.980	0.961
oct$_o$(shr)	0.900	0.899	**0.876**	0.878	**0.858**	0.973	0.963	**0.944**	0.949	0.930
oct$_o$($hshr$)	0.956	0.936	0.955	0.922	0.936	1.021	1.004	1.012	0.987	1.000
oct$_{oh}$(shr)	1.059	1.015	0.956	1.053	0.945	1.130	1.063	1.019	1.121	1.016
oct$_{oh}$($hshr$)	0.986	0.968	0.999	0.959	0.992	1.053	1.034	1.049	1.024	1.055
	$k=2$					$k=4$				
base	1.000	0.972	0.985	0.959	0.969	1.000	0.959	1.000	0.957	0.976
ct(shr_{cs}, bu_{te})	0.915	0.961	0.960	0.991	0.991	0.818	0.874	0.874	0.899	0.900
ct(wls_{cs}, bu_{te})	**0.904**	0.896	**0.896**	**0.877**	**0.877**	**0.807**	0.805	0.805	**0.782**	0.783
oct($wlsv$)	0.909	0.907	0.912	0.889	0.889	0.811	0.813	0.819	0.794	0.794
oct($bdshr$)	0.925	0.976	0.953	1.013	0.974	0.825	0.883	0.860	0.920	0.876
oct(shr)	0.913	1.000	0.914	1.076	0.963	0.807	0.885	0.808	0.967	0.861
oct($hshr$)	0.973	0.960	0.993	0.978	1.014	0.871	0.856	0.897	0.881	0.913
oct$_o$($wlsv$)	0.908	**0.895**	0.898	0.881	0.882	0.812	**0.802**	0.806	0.786	0.786
oct$_o$($bdshr$)	0.960	0.947	0.929	0.938	0.915	0.860	0.856	0.836	0.841	0.816
oct$_o$(shr)	0.921	0.919	0.896	0.898	0.878	0.814	0.821	**0.796**	0.794	**0.775**
oct$_o$($hshr$)	0.977	0.956	0.976	0.942	0.957	0.876	0.854	0.882	0.844	0.856
oct$_{oh}$(shr)	1.082	1.029	0.973	1.076	0.963	0.971	0.954	0.882	0.967	0.861
oct$_{oh}$($hshr$)	1.007	0.988	1.017	0.979	1.014	0.904	0.888	0.934	0.881	0.913

*The Gaussian method employs a sample covariance matrix:
G$_h$ and H$_h$ use multi-step residuals and G$_{oh}$ and H$_{oh}$ use overlapping and multi-step residuals.

Table D.11: *ES ratio indices defined in Section 5 for the Australian QNA dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G_h	H_h	G_{oh}	H_{oh}		G_h	H_h	G_{oh}	H_{oh}
			$\forall k \in \{4,2,1\}$					$k=1$		
base	1.000	0.979	1.011	0.968	0.987	1.000	0.988	0.988	0.971	0.971
$\text{ct}(shr_{cs}, bu_{te})$	0.937	0.960	0.961	0.962	0.960	0.992	1.001	1.001	1.004	1.000
$\text{ct}(wls_{cs}, bu_{te})$	0.930	**0.951**	0.953	**0.911**	0.915	0.986	0.997	0.998	**0.964**	0.967
$\text{oct}(wlsv)$	0.926	0.972	0.957	0.918	0.917	0.984	1.010	1.003	0.971	0.970
$\text{oct}(bdshr)$	0.940	0.986	0.966	0.981	0.956	0.997	1.015	1.006	1.016	1.000
$\text{oct}(shr)$	0.944	0.999	0.962	1.051	0.995	1.015	1.047	1.021	1.105	1.058
$\text{oct}(hshr)$	0.988	1.000	1.021	0.979	1.002	1.048	1.045	1.066	1.034	1.053
$\text{oct}_o(wlsv)$	**0.926**	0.961	0.948	0.914	**0.912**	**0.984**	1.000	0.993	0.966	**0.965**
$\text{oct}_o(bdshr)$	0.978	0.956	0.949	0.949	0.934	1.034	**0.984**	**0.983**	0.988	0.977
$\text{oct}_o(shr)$	0.950	0.957	**0.946**	0.933	0.917	1.014	0.998	0.995	0.986	0.974
$\text{oct}_o(hshr)$	0.989	0.997	1.013	0.967	0.982	1.047	1.039	1.054	1.019	1.032
$\text{oct}_{oh}(shr)$	1.102	1.010	1.006	1.051	0.995	1.172	1.059	1.063	1.105	1.058
$\text{oct}_{oh}(hshr)$	1.006	0.989	1.004	0.979	1.002	1.068	1.037	1.050	1.034	1.053
			$k=2$					$k=4$		
base	1.000	0.984	1.009	0.968	0.987	1.000	0.966	1.037	0.964	1.002
$\text{ct}(shr_{cs}, bu_{te})$	0.949	0.972	0.972	0.974	0.971	0.874	0.910	0.911	0.910	0.910
$\text{ct}(wls_{cs}, bu_{te})$	0.942	**0.962**	0.964	**0.923**	0.927	0.866	**0.897**	0.900	**0.851**	0.855
$\text{oct}(wlsv)$	0.938	0.988	0.968	0.931	0.929	0.860	0.921	0.903	0.856	0.856
$\text{oct}(bdshr)$	0.953	1.004	0.979	0.996	0.970	0.874	0.942	0.914	0.932	0.900
$\text{oct}(shr)$	0.955	1.016	0.973	1.070	1.010	0.866	0.937	0.895	0.981	0.922
$\text{oct}(hshr)$	1.001	1.015	1.034	0.993	1.017	0.919	0.942	0.965	0.913	0.937
$\text{oct}_o(wlsv)$	**0.938**	0.976	**0.959**	0.927	**0.925**	**0.860**	0.910	0.894	0.853	0.852
$\text{oct}_o(bdshr)$	0.991	0.970	0.963	0.963	0.948	0.914	0.917	0.905	0.899	0.880
$\text{oct}_o(shr)$	0.965	0.973	0.959	0.948	0.931	0.877	0.903	**0.886**	0.868	**0.850**
$\text{oct}_o(hshr)$	1.002	1.013	1.026	0.980	0.996	0.922	0.943	0.962	0.905	0.921
$\text{oct}_{oh}(shr)$	1.120	1.026	1.019	1.070	1.010	1.020	0.947	0.939	0.981	0.922
$\text{oct}_{oh}(hshr)$	1.021	1.005	1.017	0.993	1.017	0.934	0.929	0.946	0.913	0.937

*The Gaussian method employs a shrinkage covariance matrix:
G_h and H_h use multi-step residuals and G_{oh} and H_{oh} use overlapping and multi-step residuals.

Table D.12: *$\overline{RelCRPS}$ indices defined in Section 5 for the Australian QNA dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G_h	H_h	G_{oh}	H_{oh}		G_h	H_h	G_{oh}	H_{oh}
	$\forall k \in \{4,2,1\}$					$k=1$				
base	1.000	0.967	1.002	0.957	0.980	1.000	0.973	0.973	0.961	0.962
ct(shr_{cs}, bu_{te})	0.897	0.968	0.969	0.963	0.962	0.964	1.012	1.012	1.009	1.004
ct(wls_{cs}, bu_{te})	**0.886**	0.939	0.944	**0.882**	0.888	**0.954**	0.994	0.998	**0.947**	0.952
oct($wlsv$)	0.890	0.966	0.959	0.897	0.901	0.958	1.017	1.012	0.960	0.965
oct($bdshr$)	0.905	0.997	0.981	0.986	0.960	0.972	1.031	1.021	1.024	1.005
oct(shr)	0.895	0.979	0.945	1.021	0.962	0.973	1.041	1.011	1.083	1.028
oct($hshr$)	0.951	0.997	1.023	0.973	1.005	1.017	1.051	1.073	1.034	1.063
oct$_o$($wlsv$)	0.891	0.950	0.945	0.889	0.892	0.958	1.002	0.997	0.953	0.956
oct$_o$($bdshr$)	0.940	0.935	0.933	0.922	0.909	1.004	**0.965**	**0.964**	0.969	0.959
oct$_o$(shr)	0.900	**0.935**	**0.928**	0.895	**0.884**	0.973	0.984	0.982	0.960	**0.950**
oct$_o$($hshr$)	0.956	0.997	1.015	0.945	0.965	1.021	1.049	1.062	1.007	1.024
oct$_{oh}$(shr)	1.059	0.981	0.983	1.021	0.962	1.130	1.034	1.041	1.083	1.029
oct$_{oh}$($hshr$)	0.986	0.996	1.014	0.973	1.005	1.053	1.050	1.064	1.034	1.063
	$k=2$					$k=4$				
base	1.000	0.970	0.999	0.955	0.980	1.000	0.958	1.033	0.953	1.000
ct(shr_{cs}, bu_{te})	0.915	0.987	0.988	0.983	0.982	0.818	0.909	0.910	0.902	0.902
ct(wls_{cs}, bu_{te})	**0.904**	**0.958**	0.962	**0.900**	0.906	**0.807**	0.871	0.876	**0.805**	0.812
oct($wlsv$)	0.909	0.988	0.979	0.916	0.920	0.811	0.896	0.891	0.820	0.825
oct($bdshr$)	0.925	1.024	1.005	1.010	0.984	0.825	0.938	0.919	0.926	0.895
oct(shr)	0.913	1.006	0.967	1.045	0.982	0.807	0.898	0.864	0.940	0.881
oct($hshr$)	0.973	1.020	1.046	0.994	1.028	0.871	0.924	0.954	0.897	0.929
oct$_o$($wlsv$)	0.908	0.972	0.964	0.908	0.911	0.812	0.882	0.876	0.812	0.816
oct$_o$($bdshr$)	0.960	0.959	0.957	0.945	0.932	0.860	0.884	0.879	0.857	0.841
oct$_o$(shr)	0.921	0.958	**0.950**	0.917	**0.905**	0.814	**0.867**	**0.857**	0.815	**0.803**
oct$_o$($hshr$)	0.977	1.021	1.038	0.966	0.987	0.876	0.926	0.949	0.868	0.889
oct$_{oh}$(shr)	1.082	1.002	1.003	1.045	0.982	0.971	0.910	0.911	0.941	0.882
oct$_{oh}$($hshr$)	1.007	1.017	1.036	0.994	1.028	0.904	0.924	0.947	0.896	0.929

*The Gaussian method employs a shrinkage covariance matrix:
G_h and H_h use multi-step residuals and G_{oh} and H_{oh} use overlapping and multi-step residuals.

Table D.13: *ES ratio indices defined in Section 5 for the Australian QNA dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

E Australian Tourism Demand dataset

Table E.14: *Geographic divisions of Australia in States, Zones e Regions. Zones formed by a single region are highlighted in italics and not numbered.*

Series	Name	Label	Series	Name	Label
Total			*continues Regions*		
1	Australia	Total	49	Gippsland	BCB
States			50	Phillip Island	BCC
2	New South Wales (NSW)	A	51	Central Murray	BDA
3	Victoria (VIC)	B	52	Goulburn	BDB
4	Queensland (QLD)	C	53	High Country	BDC
5	South Australia (SA)	D	54	Melbourne East	BDD
6	Western Australia (WA)	E	55	Upper Yarra	BDE
7	Tasmania (TAS)	F	56	MurrayEast	BDF
8	Northern Territory (NT)	G	57	Mallee	BEA
Zones			58	Wimmera	BEB
9	Metro NSW	AA	59	Western Grampians	BEC
10	Nth Coast NSW	AB	60	Bendigo Loddon	BED
	Sth Coast NSW	*AC*	61	Macedon	BEE
11	Sth NSW	AD	62	Spa Country	BEF
12	Nth NSW	AE	63	Ballarat	BEG
	ACT	*AF*	64	Central Highlands	BEG
13	Metro VIC	BA	65	Gold Coast	CAA
	West Coast VIC	*BB*	66	Brisbane	CAB
14	East Coast VIC	BC	67	Sunshine Coast	CAC
15	Nth East VIC	BD	68	Central Queensland	CBA
16	Nth West VIC	BE	69	Bundaberg	CBB
17	Metro QLD	CA	70	Fraser Coast	CBC
18	Central Coast QLD	CB	71	Mackay	CBD
19	Nth Coast QLD	CC	72	Whitsundays	CCA
20	Inland QLD	CD	73	Northern	CCB
21	Metro SA	DA	74	Tropical North Queensland	CCC
22	Sth Coast SA	DB	75	Darling Downs	CDA
23	Inland SA	DC	76	Outback	CDB
24	West Coast SA	DD	77	Adelaide	DAA
25	West CoastWA	EA	78	Barossa	DAB
	Nth WA	*EB*	79	Adelaide Hills	DAC
	SthWA	*EC*	80	Limestone Coast	DBA
	Sth TAS	*FA*	81	Fleurieu Peninsula	DBB
26	Nth East TAS	FB	82	Kangaroo Island	DBC
27	Nth West TAS	FC	83	Murraylands	DCA
28	Nth Coast NT	GA	84	Riverland	DCB
29	Central NT	GB	85	Clare Valley	DCC
Regions			86	Flinders Range and Outback	DCD
30	Sydney	AAA	87	Eyre Peninsula	DDA
31	Central Coast	AAB	88	Yorke Peninsula	DDB
32	Hunter	ABA	89	Australia's Coral Coast	EAA
33	North Coast NSW	ABB	90	Experience Perth	EAB
34	South Coast	ACA	91	Australia's SouthWest	EAC
35	Snowy Mountains	ADA	92	Australia's North West	EBA
36	Capital Country	ADB	93	Australia's Golden Outback	ECA
37	The Murray	ADC	94	Hobart and the South	FAA
38	Riverina	ADD	95	East Coast	FBA
39	Central NSW	AEA	96	Launceston, Tamar and the North	FBB
40	New England North West	AEB	97	North West	FCA
41	Outback NSW	AEC	98	WildernessWest	FCB
42	Blue Mountains	AED	99	Darwin	GAA
43	Canberra	AFA	100	Kakadu Arnhem	GAB
44	Melbourne	BAA	101	Katherine Daly	GAC
45	Peninsula	BAB	102	Barkly	GBA
46	Geelong	BAC	103	Lasseter	GBB
47	Western	BBA	104	Alice Springs	GBC
48	Lakes	BCA	105	MacDonnell	GBD

Source: Wickramasuriya et al. (2019), Di Fonzo & Girolimetto (2022*b*)

E.1 Dealing with negative reconciled forecasts

One issue in working with time series data is the presence of negative values, which can cause difficulties for certain types of models or analyses. For the base forecasts, using the bootstrap approach produces forecasts naturally non negative (ETS model with the log-transformation), while this is not true for the Gaussian approach. In this case, any negative forecast is set equal to zero. For the cross-temporal reconciliation, Di Fonzo & Girolimetto (2022*a*, 2023*b*) propose two solutions: either a state-of-the-art numerical optimization procedure (`osqp`, Stellato et al. 2020, 2022), or a simple heuristic strategy called set-negative-to-zero (sntz). With sntz, any negative high frequency bottom time series reconciled forecasts are set to zero, and then a cross-temporal reconciliation bottom-up is used to obtain the complete set of fully coherent forecasts. Di Fonzo & Girolimetto (2023*b*) found that both methods produce similar quality forecasts, but the optimization method required much more time and computational effort compared to the sntz heuristic. To reduce computational demands, we used the less time-intensive heuristic approach for reconciliation.

E.2 Tables for all the temporal aggregation orders

Reconciliation approach	Generation of the base forecasts paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G	B	H	HB		G	B	H	HB
	$\forall k \in \{12,6,4,3,2,1\}$					$k=1$				
base	1.000	0.971	0.971	0.973	0.973	1.000	0.972	0.972	0.972	0.972
$\text{ct}(bu)$	1.321	1.011	1.011	1.011	1.011	1.077	0.983	0.982	0.982	0.982
$\text{ct}(shr_{cs}, bu_{te})$	1.057	0.974	0.969	0.974	0.969	0.976	0.963	0.962	0.963	0.962
$\text{ct}(wlsv_{te}, bu_{cs})$	1.062	0.974	0.974	0.972	0.972	0.976	0.965	0.965	0.966	0.966
$\text{oct}(ols)$	0.989	0.989	0.989	0.987	0.987	0.982	0.986	0.988	0.986	0.989
$\text{oct}(struc)$	0.982	0.962	0.961	0.961	0.959	0.970	0.963	0.963	0.963	0.963
$\text{oct}(wlsv)$	0.987	0.959	0.959	0.958	0.957	0.952	0.957	0.957	0.957	0.957
$\text{oct}(bdshr)$	0.975	**0.956**	**0.953**	**0.952**	**0.951**	**0.949**	**0.955**	**0.953**	**0.954**	**0.954**
~~$\text{oct}_h(hbshr)$~~	~~0.989~~	~~1.018~~	~~1.020~~	~~1.016~~	~~1.018~~	~~0.982~~	~~1.004~~	~~1.007~~	~~1.004~~	~~1.009~~
$\text{oct}_h(bshr)$	0.994	1.018	1.020	1.016	1.019	0.988	1.007	1.013	1.006	1.012
$\text{oct}_h(hshr)$	**0.969**	0.993	0.993	0.990	0.991	0.953	0.977	0.977	0.979	0.979
$\text{oct}_h(shr)$	1.007	0.980	0.972	0.970	0.970	1.000	0.986	0.977	0.976	0.974
	$k=2$					$k=3$				
base	1.000	0.970	0.969	0.970	0.971	1.000	0.971	0.971	0.972	0.973
$\text{ct}(bu)$	1.189	0.999	0.999	0.999	0.999	1.273	1.010	1.010	1.010	1.010
$\text{ct}(shr_{cs}, bu_{te})$	1.015	0.972	0.970	0.972	0.970	1.041	0.977	0.974	0.977	0.974
$\text{ct}(wlsv_{te}, bu_{cs})$	1.016	0.971	0.971	0.970	0.970	1.046	0.976	0.976	0.974	0.974
$\text{oct}(ols)$	0.992	0.991	0.991	0.990	0.991	0.994	0.992	0.993	0.991	0.992
$\text{oct}(struc)$	0.982	0.966	0.965	0.965	0.965	0.986	0.967	0.966	0.966	0.965
$\text{oct}(wlsv)$	0.972	0.961	0.960	0.960	0.960	0.983	0.963	0.962	0.962	0.962
$\text{oct}(bdshr)$	**0.964**	**0.958**	**0.957**	**0.956**	**0.956**	0.972	**0.960**	**0.958**	**0.957**	**0.957**
~~$\text{oct}_h(hbshr)$~~	~~0.992~~	~~1.013~~	~~1.015~~	~~1.012~~	~~1.015~~	~~0.994~~	~~1.019~~	~~1.021~~	~~1.018~~	~~1.020~~
$\text{oct}_h(bshr)$	0.997	1.015	1.018	1.013	1.017	0.999	1.021	1.022	1.018	1.022
$\text{oct}_h(hshr)$	0.965	0.987	0.987	0.986	0.987	**0.971**	0.994	0.994	0.992	0.993
$\text{oct}_h(shr)$	1.005	0.986	0.978	0.976	0.975	1.009	0.986	0.978	0.976	0.976
	$k=4$					$k=6$				
base	1.000	0.973	0.973	0.974	0.975	1.000	0.976	0.976	0.978	0.978
$\text{ct}(bu)$	1.340	1.016	1.015	1.015	1.015	1.450	1.023	1.023	1.023	1.023
$\text{ct}(shr_{cs}, bu_{te})$	1.061	0.978	0.973	0.978	0.973	1.094	0.978	0.972	0.978	0.972
$\text{ct}(wlsv_{te}, bu_{cs})$	1.068	0.977	0.977	0.974	0.974	1.103	0.977	0.977	0.974	0.974
$\text{oct}(ols)$	0.993	0.991	0.992	0.990	0.990	0.989	0.989	0.989	0.987	0.986
$\text{oct}(struc)$	0.986	0.965	0.964	0.964	0.963	0.986	0.961	0.960	0.959	0.957
$\text{oct}(wlsv)$	0.990	0.962	0.961	0.961	0.960	1.001	0.960	0.959	0.958	0.957
$\text{oct}(bdshr)$	0.977	**0.959**	**0.956**	**0.955**	**0.954**	0.985	**0.956**	**0.953**	**0.950**	**0.948**
~~$\text{oct}_h(hbshr)$~~	~~0.993~~	~~1.021~~	~~1.023~~	~~1.019~~	~~1.021~~	~~0.989~~	~~1.024~~	~~1.026~~	~~1.022~~	~~1.022~~
$\text{oct}_h(bshr)$	0.997	1.022	1.022	1.019	1.022	0.994	1.022	1.022	1.020	1.022
$\text{oct}_h(hshr)$	**0.973**	0.996	0.997	0.994	0.995	**0.976**	1.000	1.001	0.996	0.997
$\text{oct}_h(shr)$	1.009	0.984	0.976	0.973	0.973	1.010	0.978	0.970	0.967	0.967
	$k=12$									
base	1.000	0.968	0.967	0.969	0.969					
$\text{ct}(bu)$	1.675	1.038	1.037	1.037	1.038					
$\text{ct}(shr_{cs}, bu_{te})$	1.163	0.977	0.965	0.977	0.965					
$\text{ct}(wlsv_{te}, bu_{cs})$	1.174	0.978	0.978	0.971	0.971					
$\text{oct}(ols)$	0.982	0.982	0.983	0.980	0.975					
$\text{oct}(struc)$	0.982	0.951	0.949	0.947	0.943					
$\text{oct}(wlsv)$	1.025	0.954	0.953	0.949	0.947					
$\text{oct}(bdshr)$	1.002	**0.950**	**0.944**	**0.939**	**0.935**					
~~$\text{oct}_h(hbshr)$~~	~~0.982~~	~~1.027~~	~~1.029~~	~~1.024~~	~~1.021~~					
$\text{oct}_h(bshr)$	0.987	1.024	1.021	1.021	1.019					
$\text{oct}_h(hshr)$	**0.978**	1.003	1.005	0.996	0.997					
$\text{oct}_h(shr)$	1.010	0.963	0.956	0.952	0.952					

*The Gaussian method employs a sample covariance matrix and includes four techniques (G, B, H, HB) with multi-step residuals.

Table E.15: *$\overline{RelCRPS}$ defined in Section 5 for the Australian Tourism Demand dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G	B	H	HB		G	B	H	HB
		$\forall k \in \{12, 6, 4, 3, 2, 1\}$						$k = 1$		
base	1.000	0.956	0.955	0.958	0.951	1.000	0.952	0.950	0.952	0.950
$\text{ct}(bu)$	2.427	0.983	0.983	0.983	0.983	1.759	0.982	0.982	0.982	0.982
$\text{ct}(shr_{cs}, bu_{te})$	1.243	0.886	0.879	0.886	0.879	1.098	0.929	0.928	0.930	0.927
$\text{ct}(wlsv_{te}, bu_{cs})$	1.499	0.977	0.977	0.971	0.972	1.241	0.975	0.975	0.973	0.974
$\text{oct}(ols)$	0.955	0.893	0.891	0.893	0.888	0.975	0.937	0.936	0.936	0.935
$\text{oct}(struc)$	1.085	0.917	0.915	0.916	0.912	1.027	0.943	0.942	0.943	0.942
$\text{oct}(wlsv)$	1.132	0.933	0.929	0.931	0.927	1.050	0.951	0.949	0.950	0.949
$\text{oct}(bdshr)$	1.047	0.904	0.897	0.897	0.891	1.009	0.936	0.933	0.934	0.931
~~$\text{oct}_h(hbshr)$~~	~~0.956~~	~~0.889~~	~~0.886~~	~~0.888~~	~~0.884~~	~~0.975~~	~~0.937~~	~~0.936~~	~~0.937~~	~~0.935~~
$\text{oct}_h(bshr)$	**0.931**	**0.867**	**0.866**	**0.863**	**0.860**	**0.965**	**0.927**	0.927	0.925	0.923
$\text{oct}_h(hshr)$	1.081	0.935	0.931	0.935	0.927	1.028	0.952	0.951	0.952	0.950
$\text{oct}_h(shr)$	1.068	0.899	0.878	0.875	0.864	1.023	0.935	**0.923**	**0.921**	**0.916**
			$k = 2$					$k = 3$		
base	1.000	0.958	0.954	0.956	0.953	1.000	0.961	0.958	0.960	0.955
$\text{ct}(bu)$	2.176	1.001	1.001	1.001	1.001	2.428	0.998	0.997	0.997	0.997
$\text{ct}(shr_{cs}, bu_{te})$	1.192	0.927	0.921	0.927	0.921	1.245	0.911	0.904	0.911	0.904
$\text{ct}(wlsv_{te}, bu_{cs})$	1.400	0.992	0.992	0.988	0.988	1.500	0.991	0.991	0.986	0.987
$\text{oct}(ols)$	0.985	0.935	0.932	0.934	0.930	0.976	0.918	0.915	0.917	0.912
$\text{oct}(struc)$	1.075	0.949	0.947	0.948	0.944	1.096	0.939	0.936	0.938	0.933
$\text{oct}(wlsv)$	1.110	0.960	0.958	0.958	0.955	1.142	0.953	0.949	0.951	0.946
$\text{oct}(bdshr)$	1.045	0.938	0.933	0.933	0.929	1.060	0.926	0.920	0.921	0.915
~~$\text{oct}_h(hbshr)$~~	~~0.984~~	~~0.933~~	~~0.931~~	~~0.933~~	~~0.928~~	~~0.975~~	~~0.915~~	~~0.912~~	~~0.915~~	~~0.909~~
$\text{oct}_h(bshr)$	**0.967**	**0.917**	**0.916**	0.913	0.908	**0.954**	**0.895**	**0.895**	**0.892**	**0.887**
$\text{oct}_h(hshr)$	1.073	0.962	0.959	0.963	0.956	1.093	0.955	0.951	0.956	0.949
$\text{oct}_h(shr)$	1.064	0.933	0.916	**0.913**	**0.904**	1.082	0.923	0.903	0.900	0.890
			$k = 4$					$k = 6$		
base	1.000	0.960	0.960	0.962	0.956	1.000	0.961	0.959	0.964	0.956
$\text{ct}(bu)$	2.585	0.996	0.996	0.995	0.996	2.849	1.004	1.003	1.003	1.004
$\text{ct}(shr_{cs}, bu_{te})$	1.277	0.898	0.890	0.899	0.891	1.339	0.882	0.873	0.883	0.874
$\text{ct}(wlsv_{te}, bu_{cs})$	1.559	0.990	0.990	0.984	0.985	1.662	0.997	0.997	0.991	0.992
$\text{oct}(ols)$	0.966	0.905	0.902	0.904	0.899	0.962	0.889	0.887	0.890	0.885
$\text{oct}(struc)$	1.106	0.930	0.927	0.928	0.924	1.132	0.923	0.919	0.922	0.916
$\text{oct}(wlsv)$	1.157	0.947	0.943	0.945	0.939	1.192	0.942	0.937	0.941	0.934
$\text{oct}(bdshr)$	1.065	0.917	0.909	0.910	0.903	1.084	0.907	0.897	0.898	0.890
~~$\text{oct}_h(hbshr)$~~	~~0.967~~	~~0.901~~	~~0.898~~	~~0.900~~	~~0.895~~	~~0.964~~	~~0.882~~	~~0.880~~	~~0.883~~	~~0.877~~
$\text{oct}_h(bshr)$	**0.943**	**0.879**	**0.878**	**0.876**	**0.871**	**0.932**	**0.856**	**0.855**	**0.851**	**0.848**
$\text{oct}_h(hshr)$	1.101	0.949	0.944	0.949	0.941	1.126	0.945	0.939	0.945	0.936
$\text{oct}_h(shr)$	1.089	0.915	0.893	0.890	0.878	1.107	0.899	0.875	0.871	0.858
			$k = 12$							
base	1.000	0.942	0.947	0.951	0.937					
$\text{ct}(bu)$	2.990	0.922	0.921	0.923	0.923					
$\text{ct}(shr_{cs}, bu_{te})$	1.326	0.779	0.767	0.777	0.766					
$\text{ct}(wlsv_{te}, bu_{cs})$	1.679	0.917	0.917	0.906	0.908					
$\text{oct}(ols)$	0.872	0.783	0.784	0.783	0.779					
$\text{oct}(struc)$	1.077	0.826	0.822	0.823	0.818					
$\text{oct}(wlsv)$	1.149	0.851	0.845	0.847	0.840					
$\text{oct}(bdshr)$	1.021	0.808	0.796	0.796	0.787					
~~$\text{oct}_h(hbshr)$~~	~~0.872~~	~~0.775~~	~~0.772~~	~~0.772~~	~~0.770~~					
$\text{oct}_h(bshr)$	**0.833**	**0.741**	**0.741**	**0.737**	**0.735**					
$\text{oct}_h(hshr)$	1.066	0.851	0.846	0.848	0.838					
$\text{oct}_h(shr)$	1.043	0.797	0.768	0.764	0.750					

*The Gaussian method employs a sample covariance matrix and includes four techniques (G, B, H, HB) with multi-step residuals.

Table E.16: *ES ratio indices defined in Section 5 for the Australian Tourism Demand dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G	B	H	HB		G	B	H	HB
	$\forall k \in \{12, 6, 4, 3, 2, 1\}$					$k = 1$				
base	1.000	**0.971**	0.972	**0.971**	0.972	1.000	**0.972**	0.971	0.972	0.971
$\mathrm{ct}(bu)$	1.321	1.017	1.018	1.017	1.017	1.077	0.983	0.983	0.983	0.983
$\mathrm{ct}(shr_{cs}, bu_{te})$	1.057	1.013	**0.971**	1.013	0.971	0.976	0.987	**0.961**	0.988	0.961
$\mathrm{ct}(wlsv_{te}, bu_{cs})$	1.062	1.069	1.070	0.974	0.974	0.976	0.986	0.986	**0.965**	0.965
$\mathrm{oct}(ols)$	0.989	1.163	1.052	1.139	0.987	0.982	1.038	0.992	1.047	0.987
$\mathrm{oct}(struc)$	0.982	1.099	1.039	1.037	0.960	0.970	1.007	0.971	0.999	0.962
$\mathrm{oct}(wlsv)$	0.987	1.080	1.041	0.992	0.958	0.952	1.004	0.969	0.978	0.956
$\mathrm{oct}(bdshr)$	0.975	1.072	1.032	0.985	**0.950**	**0.949**	0.999	0.965	0.975	**0.952**
$\mathrm{oct}_h(hbshr)$	0.989	1.189	1.076	1.171	1.021	0.982	1.045	1.000	1.063	1.009
$\mathrm{oct}_h(bshr)$	0.994	1.202	1.073	1.168	1.021	0.988	1.046	1.012	1.063	1.012
$\mathrm{oct}_h(hshr)$	**0.969**	1.066	1.052	1.008	0.994	0.953	0.994	0.972	0.991	0.979
$\mathrm{oct}_h(shr)$	1.007	1.090	1.046	1.000	0.970	1.000	1.035	0.992	0.998	0.973
	$k = 2$					$k = 3$				
base	1.000	**0.969**	0.969	**0.968**	0.968	1.000	**0.971**	**0.970**	**0.969**	0.970
$\mathrm{ct}(bu)$	1.189	1.000	1.000	1.000	1.000	1.273	1.013	1.013	1.013	1.013
$\mathrm{ct}(shr_{cs}, bu_{te})$	1.015	1.004	**0.968**	1.004	0.968	1.041	1.013	0.973	1.014	0.973
$\mathrm{ct}(wlsv_{te}, bu_{cs})$	1.016	1.043	1.044	0.969	0.969	1.046	1.067	1.068	0.974	0.974
$\mathrm{oct}(ols)$	0.992	1.118	1.037	1.092	0.989	0.994	1.153	1.053	1.124	0.990
$\mathrm{oct}(struc)$	0.982	1.075	1.022	1.020	0.963	0.986	1.099	1.041	1.033	0.964
$\mathrm{oct}(wlsv)$	0.972	1.064	1.021	0.987	0.958	0.983	1.083	1.041	0.993	0.960
$\mathrm{oct}(bdshr)$	**0.964**	1.057	1.015	0.983	**0.953**	0.972	1.075	1.033	0.988	**0.955**
$\mathrm{oct}_h(hbshr)$	0.992	1.136	1.055	1.116	1.014	0.994	1.178	1.075	1.153	1.020
$\mathrm{oct}_h(bshr)$	0.997	1.145	1.059	1.114	1.016	0.999	1.190	1.075	1.151	1.021
$\mathrm{oct}_h(hshr)$	0.965	1.050	1.029	1.001	0.986	**0.971**	1.067	1.051	1.009	0.994
$\mathrm{oct}_h(shr)$	1.005	1.083	1.035	1.001	0.973	1.009	1.097	1.050	1.004	0.974
	$k = 4$					$k = 6$				
base	1.000	**0.973**	**0.973**	**0.971**	0.973	1.000	**0.976**	0.977	**0.975**	0.977
$\mathrm{ct}(bu)$	1.340	1.021	1.021	1.021	1.021	1.450	1.032	1.033	1.032	1.033
$\mathrm{ct}(shr_{cs}, bu_{te})$	1.061	1.018	0.974	1.018	0.974	1.094	1.023	**0.974**	1.024	0.974
$\mathrm{ct}(wlsv_{te}, bu_{cs})$	1.068	1.087	1.089	0.976	0.976	1.103	1.108	1.110	0.978	0.978
$\mathrm{oct}(ols)$	0.993	1.186	1.068	1.148	0.989	0.989	1.223	1.080	1.184	0.987
$\mathrm{oct}(struc)$	0.986	1.120	1.057	1.042	0.962	0.986	1.141	1.071	1.054	0.959
$\mathrm{oct}(wlsv)$	0.990	1.100	1.059	0.996	0.959	1.001	1.115	1.076	0.998	0.958
$\mathrm{oct}(bdshr)$	0.977	1.091	1.049	0.989	**0.952**	0.985	1.103	1.064	0.989	**0.949**
$\mathrm{oct}_h(hbshr)$	0.993	1.215	1.095	1.182	1.022	0.989	1.258	1.112	1.225	1.026
$\mathrm{oct}_h(bshr)$	0.997	1.230	1.089	1.178	1.023	0.994	1.278	1.101	1.219	1.025
$\mathrm{oct}_h(hshr)$	**0.973**	1.084	1.071	1.012	0.996	**0.976**	1.097	1.091	1.017	1.002
$\mathrm{oct}_h(shr)$	1.009	1.108	1.062	1.003	0.972	1.010	1.113	1.070	1.000	0.968
	$k = 12$									
base	1.000	**0.968**	**0.969**	**0.969**	0.971					
$\mathrm{ct}(bu)$	1.675	1.056	1.057	1.057	1.057					
$\mathrm{ct}(shr_{cs}, bu_{te})$	1.163	1.032	0.974	1.033	0.974					
$\mathrm{ct}(wlsv_{te}, bu_{cs})$	1.174	1.128	1.130	0.982	0.982					
$\mathrm{oct}(ols)$	0.982	1.277	1.085	1.252	0.982					
$\mathrm{oct}(struc)$	0.982	1.158	1.074	1.075	0.950					
$\mathrm{oct}(wlsv)$	1.025	1.122	1.085	1.001	0.954					
$\mathrm{oct}(bdshr)$	1.002	1.110	1.071	0.989	**0.941**					
$\mathrm{oct}_h(hbshr)$	0.982	1.322	1.125	1.305	1.033					
$\mathrm{oct}_h(bshr)$	0.987	1.347	1.107	1.297	1.031					
$\mathrm{oct}_h(hshr)$	**0.978**	1.106	1.107	1.021	1.010					
$\mathrm{oct}_h(shr)$	1.010	1.107	1.067	0.991	0.959					

*The Gaussian method employs a shrikage covariance matrix and includes four techniques (G, B, H, HB) with multi-step residuals..

Table E.17: *$\overline{RelCRPS}$ defined in Section 5 for the Australian Tourism Demand dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

Reconciliation approach	Generation of the base forecasts paths									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G	B	H	HB		G	B	H	HB
		$\forall k \in \{12, 6, 4, 3, 2, 1\}$						$k = 1$		
base	1.000	**0.958**	0.984	**0.972**	0.992	1.000	**0.954**	0.958	**0.954**	0.958
$\text{ct}(bu)$	2.427	1.040	1.042	1.040	1.041	1.759	1.001	1.002	1.002	1.002
$\text{ct}(shr_{cs}, bu_{te})$	1.243	0.988	**0.913**	0.990	0.913	1.098	1.011	**0.938**	1.013	0.938
$\text{ct}(wlsv_{te}, bu_{cs})$	1.499	1.117	1.120	1.025	1.025	1.241	1.019	1.020	0.990	0.990
$\text{oct}(ols)$	0.955	1.000	0.984	0.985	0.922	0.975	0.983	0.961	0.987	0.945
$\text{oct}(struc)$	1.085	1.094	1.047	1.018	0.952	1.027	1.054	0.981	1.022	0.953
$\text{oct}(wlsv)$	1.132	1.137	1.065	1.059	0.969	1.050	1.078	0.989	1.043	0.960
$\text{oct}(bdshr)$	1.047	1.085	1.013	1.011	0.927	1.009	1.050	0.966	1.019	0.942
~~$\text{oct}_h(hbshr)$~~	~~0.956~~	~~1.018~~	~~0.981~~	~~1.016~~	~~0.919~~	~~0.975~~	~~0.991~~	~~0.961~~	~~1.002~~	~~0.947~~
$\text{oct}_h(bshr)$	**0.931**	1.002	1.001	0.982	**0.889**	**0.965**	0.980	0.975	0.985	0.933
$\text{oct}_h(hshr)$	1.081	1.109	1.039	1.076	0.973	1.028	1.061	0.978	1.052	0.963
$\text{oct}_h(shr)$	1.068	1.088	1.008	0.995	0.896	1.023	1.061	0.966	1.011	**0.924**
			$k = 2$					$k = 3$		
base	1.000	**0.960**	0.971	**0.958**	0.972	1.000	**0.963**	0.981	**0.966**	0.986
$\text{ct}(bu)$	2.176	1.035	1.036	1.035	1.035	2.428	1.042	1.044	1.042	1.043
$\text{ct}(shr_{cs}, bu_{te})$	1.192	1.020	**0.942**	1.021	0.942	1.245	1.009	**0.931**	1.011	0.931
$\text{ct}(wlsv_{te}, bu_{cs})$	1.400	1.104	1.106	1.018	1.019	1.500	1.127	1.130	1.029	1.029
$\text{oct}(ols)$	0.985	1.028	1.008	1.002	0.950	0.976	1.020	1.004	0.994	0.938
$\text{oct}(struc)$	1.075	1.115	1.051	1.039	0.967	1.096	1.117	1.064	1.033	0.965
$\text{oct}(wlsv)$	1.110	1.149	1.065	1.070	0.979	1.142	1.160	1.082	1.073	0.981
$\text{oct}(bdshr)$	1.045	1.105	1.024	1.033	0.949	1.060	1.109	1.032	1.029	0.943
~~$\text{oct}_h(hbshr)$~~	~~0.984~~	~~1.041~~	~~1.007~~	~~1.024~~	~~0.951~~	~~0.975~~	~~1.036~~	~~1.002~~	~~1.023~~	~~0.937~~
$\text{oct}_h(bshr)$	**0.967**	1.029	1.025	0.998	0.928	**0.954**	1.024	1.025	0.993	**0.911**
$\text{oct}_h(hshr)$	1.073	1.122	1.042	1.083	0.983	1.093	1.129	1.054	1.090	0.984
$\text{oct}_h(shr)$	1.064	1.110	1.019	1.018	**0.922**	1.082	1.116	1.030	1.015	0.915
			$k = 4$					$k = 6$		
base	1.000	**0.962**	0.987	**0.973**	0.996	1.000	**0.963**	0.998	**0.984**	1.011
$\text{ct}(bu)$	2.585	1.052	1.054	1.053	1.053	2.849	1.083	1.085	1.083	1.084
$\text{ct}(shr_{cs}, bu_{te})$	1.277	1.000	**0.923**	1.002	0.923	1.339	0.999	**0.921**	1.000	0.920
$\text{ct}(wlsv_{te}, bu_{cs})$	1.559	1.150	1.153	1.037	1.037	1.662	1.189	1.193	1.066	1.066
$\text{oct}(ols)$	0.966	1.022	1.008	0.994	0.931	0.962	1.023	1.014	1.003	0.930
$\text{oct}(struc)$	1.106	1.120	1.076	1.031	0.963	1.132	1.132	1.100	1.039	0.972
$\text{oct}(wlsv)$	1.157	1.167	1.097	1.075	0.982	1.192	1.187	1.124	1.090	0.995
$\text{oct}(bdshr)$	1.065	1.112	1.041	1.025	0.939	1.084	1.121	1.058	1.029	0.940
~~$\text{oct}_h(hbshr)$~~	~~0.967~~	~~1.041~~	~~1.005~~	~~1.027~~	~~0.929~~	~~0.964~~	~~1.046~~	~~1.008~~	~~1.042~~	~~0.924~~
$\text{oct}_h(bshr)$	**0.943**	1.028	1.028	0.994	**0.900**	**0.932**	1.029	1.032	1.000	**0.887**
$\text{oct}_h(hshr)$	1.101	1.137	1.068	1.093	0.986	1.126	1.153	1.089	1.110	0.999
$\text{oct}_h(shr)$	1.089	1.118	1.039	1.012	0.910	1.107	1.118	1.045	1.006	0.902
			$k = 12$							
base	1.000	0.948	1.010	1.002	1.033					
$\text{ct}(bu)$	2.990	1.028	1.031	1.029	1.029					
$\text{ct}(shr_{cs}, bu_{te})$	1.326	**0.897**	**0.830**	**0.899**	0.830					
$\text{ct}(wlsv_{te}, bu_{cs})$	1.679	1.119	1.123	1.009	1.009					
$\text{oct}(ols)$	0.872	0.927	0.914	0.930	0.840					
$\text{oct}(struc)$	1.077	1.028	1.012	0.950	0.894					
$\text{oct}(wlsv)$	1.149	1.089	1.041	1.006	0.922					
$\text{oct}(bdshr)$	1.021	1.015	0.964	0.935	0.855					
~~$\text{oct}_h(hbshr)$~~	~~0.872~~	~~0.955~~	~~0.906~~	~~0.978~~	~~0.833~~					
$\text{oct}_h(bshr)$	**0.833**	0.927	0.927	0.927	**0.784**					
$\text{oct}_h(hshr)$	1.066	1.056	1.005	1.026	0.926					
$\text{oct}_h(shr)$	1.043	1.011	0.952	0.909	0.809					

*The Gaussian method employs a shrikage covariance matrix and includes four techniques (G, B, H, HB) with multi-step residuals.

Table E.18: *ES ratio indices defined in Section 5 for the Australian Tourism Demand dataset. Approaches performing worse than the benchmark (bootstrap base forecasts, ctjb) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue. The reconciliation approaches are described in Table 2.*

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