Cross-temporal Probabilistic Forecast Reconciliation: online appendix

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A Monte Carlo Simulation: one-step residuals and shrinkage covariance matrix

In Section 5.1, we discussed the use of one-step residuals in estimating the covariance matrix. In particular we point out that one-step residuals lead to a biased estimate of the covariance matrix where some correlation are zeros by definition. This can be seen clearly in Figure 1, where we have reported the covariance and correlation matrix corresponding to Figure 11 in the paper, but where we have used one-step residuals. In addition, Tables 1, 2 and 3 show the Frobenius norm, CRPS, and ES skill scores as explained in the paper to investigate the effectiveness of one-step residuals. Moreover, in Tables 4 and 5, we have utilized a shrinkage matrix rather than the sample covariance matrix to assess the performance of our approach.

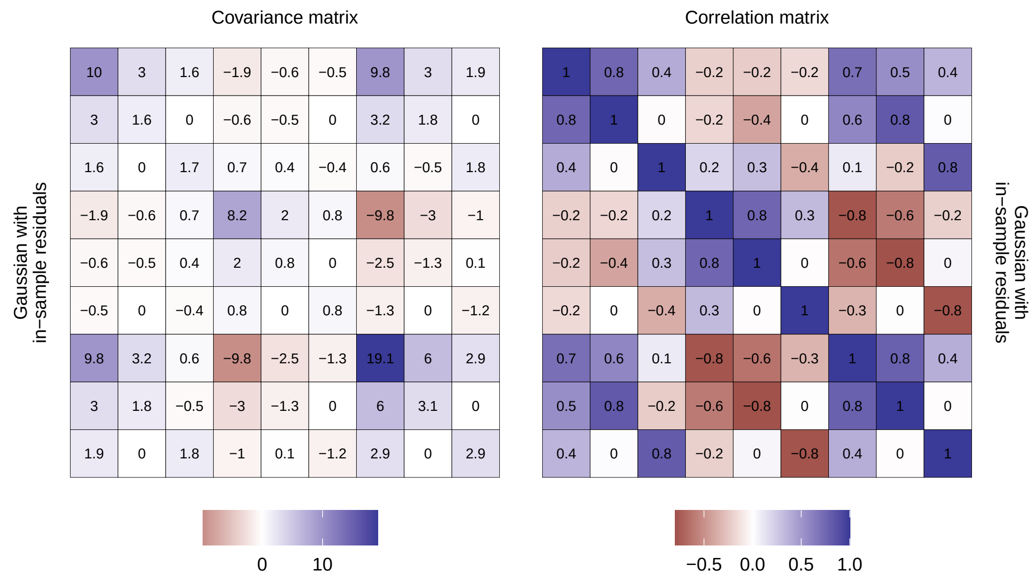

Figure 1: *Comparison of estimated covariance and correlation matrices (first simulation) for base forecasts using a parametric Gaussian (with one-step residuals) approach. The true covariance and correlation matrices are shown in Figure 10 in the paper.*

Reconciliation approach	**Base forecasts' sample approach**								
	Gaussian approach: sample covariance matrix								
	ctjb	In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
base	8.260	17.638	16.733	22.178	21.789	7.748	6.549	3.409	2.215
ct(bu)	3.195	21.789	21.789	**21.789**	21.789	2.215	2.215	**2.215**	2.215
ct(shr_{cs}, bu_{te})	3.202	21.942	21.789	21.942	21.789	2.224	2.215	2.224	2.215
ct($wlsv_{te}, bu_{cs}$)	**3.183**	18.237	18.237	21.789	21.789	**2.188**	2.188	**2.215**	2.215
oct($wlsv$)	3.766	19.174	18.611	22.304	21.789	3.082	2.191	2.910	2.215
oct($bdshr$)	3.203	18.559	18.416	21.937	21.789	2.195	**2.184**	2.224	**2.215**
oct(shr)	5.217	25.015	23.457	23.413	**21.789**	2.260	2.202	2.226	2.215
oct($bshr$)	5.282	23.772	23.997	22.146	21.789	2.720	2.220	2.756	2.215
oct($hshr$)	6.161	**11.336**	**10.940**	23.598	**21.789**	4.138	4.167	2.225	2.215
oct($hbshr$)	5.731	11.379	10.940	22.146	21.789	5.085	4.167	2.756	2.215
oct$_h$(shr)	3.251	20.965	19.992	22.079	**21.789**	2.260	2.202	2.226	2.215
oct$_h$($bshr$)	3.602	21.306	21.022	22.146	21.789	2.720	2.220	2.756	2.215
oct$_h$($hshr$)	4.869	11.405	10.940	22.037	21.789	4.138	4.167	2.225	2.215
oct$_h$($hbshr$)	5.731	11.379	10.940	22.146	21.789	5.085	4.167	2.756	2.215

Table 1: *Frobenius norm between the true (in Figure 10 in the paper) and the estimated covariance matrix for different reconciliation approaches and different techniques for simulating the base forecasts. In bold, it is reported the lowest value for each column, in blue the minimum.*

Reconciliation approach	Base forecasts' sample approach								
		Gaussian approach: sample covariance matrix							
	ctjb	In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
				$\forall k \in \{2,1\}$					
base	1.000	1.008	1.009	1.044	1.047	0.998	0.999	1.002	1.004
ct(bu)	0.901	0.930	0.929	0.929	0.929	0.900	0.900	0.900	0.900
ct(shr_{cs}, bu_{te})	**0.901**	**0.929**	0.928	**0.929**	**0.928**	**0.900**	**0.899**	**0.900**	**0.900**
ct$(wlsv_{te}, bu_{cs})$	0.910	0.930	0.929	0.939	0.939	0.916	0.916	0.916	0.917
oct$(wlsv)$	0.922	0.942	0.944	0.951	0.953	0.930	0.930	0.930	0.931
oct$(bdshr)$	0.910	0.930	0.930	0.939	0.938	0.916	0.915	0.916	0.916
oct(shr)	0.941	0.999	0.985	0.983	0.973	0.903	0.902	0.902	0.903
oct$(bshr)$	0.951	0.995	1.000	0.983	0.986	0.922	0.922	0.921	0.922
oct$(hshr)$	0.987	0.995	0.993	1.039	1.026	0.972	0.972	0.974	0.975
oct$(hbshr)$	0.987	0.995	0.996	1.024	1.028	0.985	0.985	0.987	0.989
oct$_h(shr)$	0.904	0.929	**0.928**	0.932	0.932	0.903	0.902	0.902	0.903
oct$_h(bshr)$	0.923	0.948	0.952	0.951	0.954	0.922	0.922	0.921	0.922
oct$_h(hshr)$	0.974	0.982	0.982	1.012	1.012	0.972	0.972	0.974	0.975
oct$_h(hbshr)$	0.987	0.995	0.996	1.024	1.028	0.985	0.985	0.987	0.989
				$k = 1$					
base	1.000	1.017	1.019	1.017	1.019	0.998	0.999	0.999	1.000
ct(bu)	0.978	0.994	0.994	0.994	0.994	0.976	0.976	0.977	0.977
ct(shr_{cs}, bu_{te})	**0.977**	**0.993**	**0.993**	**0.994**	**0.993**	**0.976**	**0.976**	**0.976**	**0.976**
ct$(wlsv_{te}, bu_{cs})$	0.986	1.002	1.002	1.003	1.003	0.993	0.993	0.993	0.993
oct$(wlsv)$	0.998	1.014	1.015	1.015	1.016	1.006	1.006	1.007	1.007
oct$(bdshr)$	0.986	1.002	1.002	1.003	1.003	0.992	0.992	0.993	0.993
oct(shr)	1.037	1.082	1.067	1.064	1.056	0.979	0.978	0.979	0.979
oct$(bshr)$	1.041	1.071	1.074	1.060	1.062	0.998	0.998	0.998	0.998
oct$(hshr)$	1.080	1.090	1.091	1.119	1.105	1.050	1.050	1.053	1.053
oct$(hbshr)$	1.065	1.080	1.081	1.088	1.090	1.063	1.064	1.066	1.068
oct$_h(shr)$	0.980	0.996	0.995	0.996	0.996	0.979	0.978	0.979	0.979
oct$_h(bshr)$	0.999	1.016	1.018	1.016	1.018	0.998	0.998	0.998	0.998
oct$_h(hshr)$	1.052	1.067	1.066	1.074	1.075	1.050	1.050	1.053	1.053
oct$_h(hbshr)$	1.065	1.080	1.081	1.088	1.090	1.063	1.064	1.066	1.068
				$k = 2$					
base	1.000	0.998	0.999	1.071	1.075	0.998	0.999	1.005	1.008
ct(bu)	0.831	0.869	0.869	0.869	0.869	0.830	0.829	0.829	0.830
ct(shr_{cs}, bu_{te})	**0.830**	0.869	0.868	**0.868**	**0.868**	**0.830**	**0.829**	**0.829**	**0.830**
ct$(wlsv_{te}, bu_{cs})$	0.840	**0.863**	**0.862**	0.879	0.878	0.846	0.844	0.845	0.846
oct$(wlsv)$	0.851	0.875	0.877	0.891	0.893	0.859	0.859	0.859	0.861
oct$(bdshr)$	0.839	0.863	0.863	0.879	0.878	0.845	0.844	0.845	0.846
oct(shr)	0.854	0.922	0.909	0.908	0.897	0.833	0.831	0.832	0.832
oct$(bshr)$	0.869	0.925	0.931	0.911	0.915	0.851	0.851	0.851	0.852
oct$(hshr)$	0.901	0.908	0.904	0.966	0.952	0.900	0.899	0.901	0.902
oct$(hbshr)$	0.915	0.917	0.919	0.964	0.969	0.913	0.913	0.914	0.917
oct$_h(shr)$	0.834	0.868	0.865	0.872	0.872	0.833	0.831	0.832	0.832
oct$_h(bshr)$	0.852	0.886	0.890	0.890	0.894	0.851	0.851	0.851	0.852
oct$_h(hshr)$	0.902	0.904	0.904	0.953	0.952	0.900	0.899	0.901	0.902
oct$_h(hbshr)$	0.915	0.917	0.919	0.964	0.969	0.913	0.913	0.914	0.917

Table 2: *CRPS skill score defined in (17) and (18). The smaller this value, the more accurate the forecast. Approaches that performed worse than the benchmark model (Bootstrap base forecasts) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue.*

Reconciliation approach	Base forecasts' sample approach								
	ctjb	Gaussian approach: sample covariance matrix							
		In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
				$\forall k \in \{2,1\}$					
base	1.000	1.005	1.009	1.039	1.046	0.996	0.999	1.000	1.004
ct(bu)	0.897	0.924	0.923	0.924	0.923	0.895	0.896	0.897	**0.895**
ct(shr_{cs}, bu_{te})	**0.896**	0.924	0.923	**0.923**	**0.922**	**0.895**	**0.895**	**0.896**	0.896
ct($wlsv_{te}, bu_{cs}$)	0.906	0.924	0.923	0.933	0.932	0.912	0.911	0.910	0.912
oct($wlsv$)	0.916	0.935	0.937	0.944	0.945	0.923	0.923	0.923	0.924
oct($bdshr$)	0.906	**0.923**	0.923	0.932	0.932	0.910	0.910	0.911	0.912
oct(shr)	0.938	0.993	0.980	0.977	0.969	0.898	0.898	0.898	0.897
oct($bshr$)	0.947	0.990	0.995	0.979	0.981	0.915	0.915	0.915	0.915
oct($hshr$)	0.978	0.987	0.985	1.027	1.016	0.963	0.964	0.966	0.967
oct($hbshr$)	0.977	0.986	0.985	1.012	1.016	0.974	0.976	0.977	0.978
oct$_h$(shr)	0.900	0.923	**0.922**	0.926	0.925	0.898	0.898	0.897	0.898
oct$_h$($bshr$)	0.916	0.940	0.943	0.942	0.945	0.914	0.916	0.915	0.916
oct$_h$($hshr$)	0.967	0.974	0.974	1.002	1.002	0.964	0.964	0.966	0.967
oct$_h$($hbshr$)	0.978	0.984	0.986	1.012	1.015	0.975	0.976	0.977	0.980
				$k = 1$					
base	1.000	1.014	1.020	1.015	1.019	0.997	1.000	0.997	1.000
ct(bu)	0.969	0.985	0.983	0.985	0.984	**0.967**	**0.967**	0.968	**0.968**
ct(shr_{cs}, bu_{te})	**0.968**	**0.984**	**0.983**	**0.984**	**0.983**	0.968	0.967	**0.968**	0.968
ct($wlsv_{te}, bu_{cs}$)	0.977	0.991	0.991	0.992	0.992	0.984	0.983	0.981	0.984
oct($wlsv$)	0.989	1.002	1.004	1.003	1.004	0.994	0.995	0.995	0.997
oct($bdshr$)	0.977	0.989	0.991	0.992	0.992	0.981	0.982	0.983	0.985
oct(shr)	1.028	1.070	1.056	1.053	1.046	0.969	0.969	0.970	0.969
oct($bshr$)	1.034	1.061	1.065	1.051	1.053	0.985	0.987	0.986	0.987
oct($hshr$)	1.066	1.075	1.076	1.099	1.090	1.037	1.037	1.039	1.039
oct($hbshr$)	1.050	1.065	1.065	1.070	1.073	1.048	1.049	1.049	1.052
oct$_h$(shr)	0.971	0.985	0.985	0.986	0.986	0.969	0.969	0.969	0.969
oct$_h$($bshr$)	0.987	1.002	1.005	1.002	1.005	0.986	0.987	0.987	0.988
oct$_h$($hshr$)	1.040	1.053	1.053	1.059	1.058	1.036	1.036	1.040	1.040
oct$_h$($hbshr$)	1.051	1.064	1.063	1.071	1.073	1.047	1.049	1.051	1.052
				$k = 2$					
base	1.000	0.997	0.999	1.063	1.073	0.996	0.998	1.003	1.008
ct(bu)	0.831	0.867	0.867	0.867	0.867	0.829	0.829	0.830	**0.828**
ct(shr_{cs}, bu_{te})	**0.829**	0.867	0.866	**0.866**	**0.865**	**0.828**	**0.829**	**0.829**	0.829
ct($wlsv_{te}, bu_{cs}$)	0.839	**0.860**	**0.860**	0.877	0.876	0.844	0.844	0.844	0.845
oct($wlsv$)	0.849	0.872	0.875	0.887	0.890	0.858	0.856	0.856	0.857
oct($bdshr$)	0.839	0.861	0.861	0.876	0.875	0.845	0.843	0.845	0.844
oct(shr)	0.856	0.921	0.909	0.907	0.898	0.832	0.831	0.832	0.831
oct($bshr$)	0.868	0.924	0.930	0.911	0.915	0.849	0.848	0.849	0.848
oct($hshr$)	0.897	0.905	0.901	0.959	0.947	0.895	0.896	0.898	0.899
oct($hbshr$)	0.910	0.912	0.912	0.957	0.961	0.906	0.909	0.909	0.910
oct$_h$(shr)	0.835	0.865	0.862	0.870	0.868	0.833	0.833	0.831	0.832
oct$_h$($bshr$)	0.850	0.881	0.885	0.886	0.889	0.847	0.849	0.849	0.850
oct$_h$($hshr$)	0.900	0.902	0.901	0.947	0.948	0.897	0.896	0.897	0.899
oct$_h$($hbshr$)	0.910	0.910	0.914	0.957	0.961	0.907	0.908	0.909	0.912

Table 3: *ES skill score defined in equation (17) and (19). The smaller this value, the more accurate the forecast. Approaches that performed worse than the benchmark model (Bootstrap base forecasts) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue.*

Reconciliation approach	ctjb	In-sample residuals G	In-sample residuals B	In-sample residuals H	In-sample residuals HB	Multi-step residuals G	Multi-step residuals B	Multi-step residuals H	Multi-step residuals HB
	Base forecasts' sample approach								
		Gaussian approach: shrinkage covariance matrix							
				$\forall k \in \{2, 1\}$					
base	1.007	1.009	1.044	1.046	0.997	0.999	1.002	1.003	1.000
ct(bu)	0.929	0.929	0.929	0.929	0.899	0.900	0.900	0.900	0.901
ct(shr_{cs}, bu_{te})	**0.929**	0.928	**0.929**	**0.928**	**0.899**	**0.899**	**0.900**	**0.900**	**0.901**
ct($wlsv_{te}, bu_{cs}$)	0.930	0.930	0.939	0.938	0.915	0.916	0.917	0.916	0.910
oct($wlsv$)	0.943	0.944	0.951	0.952	0.929	0.930	0.931	0.930	0.922
oct($bdshr$)	0.930	0.930	0.938	0.938	0.915	0.916	0.916	0.916	0.910
oct(shr)	0.994	0.982	0.980	0.973	0.902	0.902	0.903	0.902	0.941
oct($bshr$)	0.995	0.998	0.983	0.986	0.921	0.922	0.922	0.922	0.951
oct($hshr$)	0.994	0.994	1.035	1.025	0.971	0.972	0.974	0.974	0.987
oct($hbshr$)	0.995	0.997	1.025	1.027	0.984	0.986	0.988	0.988	0.987
oct$_h$(shr)	0.929	**0.928**	0.932	0.932	0.902	0.902	0.903	0.902	0.904
oct$_h$($bshr$)	0.948	0.951	0.951	0.953	0.921	0.922	0.922	0.922	0.923
oct$_h$($hshr$)	0.982	0.982	1.011	1.011	0.971	0.972	0.974	0.974	0.974
oct$_h$($hbshr$)	0.995	0.997	1.025	1.027	0.984	0.986	0.988	0.988	0.987
				$k = 1$					
base	1.017	1.019	1.017	1.019	0.998	0.999	0.999	0.999	1.000
ct(bu)	0.994	0.994	0.994	0.994	0.976	0.976	0.977	0.976	0.978
ct(shr_{cs}, bu_{te})	**0.993**	**0.993**	**0.993**	**0.993**	**0.975**	**0.976**	**0.976**	**0.976**	**0.977**
ct($wlsv_{te}, bu_{cs}$)	1.002	1.002	1.003	1.003	0.992	0.993	0.993	0.993	0.986
oct($wlsv$)	1.015	1.015	1.015	1.016	1.005	1.007	1.007	1.007	0.998
oct($bdshr$)	1.002	1.002	1.003	1.002	0.992	0.992	0.993	0.992	0.986
oct(shr)	1.076	1.065	1.061	1.056	0.978	0.978	0.979	0.978	1.037
oct($bshr$)	1.070	1.072	1.060	1.062	0.997	0.998	0.998	0.998	1.041
oct($hshr$)	1.090	1.092	1.114	1.105	1.049	1.050	1.053	1.052	1.080
oct($hbshr$)	1.080	1.081	1.089	1.090	1.062	1.064	1.066	1.066	1.065
oct$_h$(shr)	0.996	0.995	0.996	0.996	0.978	0.978	0.979	0.978	0.980
oct$_h$($bshr$)	1.016	1.018	1.016	1.018	0.997	0.998	0.998	0.998	0.999
oct$_h$($hshr$)	1.066	1.067	1.075	1.075	1.049	1.050	1.053	1.052	1.052
oct$_h$($hbshr$)	1.080	1.081	1.089	1.090	1.062	1.064	1.066	1.066	1.065
				$k = 2$					
base	0.997	0.999	1.071	1.074	0.997	0.999	1.005	1.008	1.000
ct(bu)	0.869	0.868	0.868	0.868	0.829	0.829	0.830	0.830	0.831
ct(shr_{cs}, bu_{te})	0.868	0.867	**0.868**	**0.867**	**0.829**	**0.829**	**0.830**	**0.829**	**0.830**
ct($wlsv_{te}, bu_{cs}$)	**0.863**	**0.862**	0.878	0.878	0.845	0.845	0.846	0.846	0.840
oct($wlsv$)	0.876	0.877	0.891	0.892	0.859	0.860	0.860	0.860	0.851
oct($bdshr$)	0.863	0.863	0.878	0.877	0.844	0.845	0.846	0.845	0.839
oct(shr)	0.918	0.906	0.906	0.897	0.832	0.832	0.833	0.832	0.854
oct($bshr$)	0.924	0.928	0.911	0.915	0.850	0.851	0.852	0.851	0.869
oct($hshr$)	0.907	0.905	0.962	0.951	0.898	0.899	0.902	0.902	0.901
oct($hbshr$)	0.917	0.919	0.964	0.968	0.912	0.913	0.915	0.916	0.915
oct$_h$(shr)	0.867	0.864	0.872	0.871	0.832	0.832	0.833	0.832	0.834
oct$_h$($bshr$)	0.886	0.890	0.890	0.893	0.850	0.851	0.852	0.851	0.852
oct$_h$($hshr$)	0.904	0.905	0.952	0.952	0.898	0.899	0.902	0.902	0.902
oct$_h$($hbshr$)	0.917	0.919	0.964	0.968	0.912	0.913	0.915	0.916	0.915

Table 4: *CRPS skill score defined in (17) and (18). The smaller this value, the more accurate the forecast. Approaches that performed worse than the benchmark model (Bootstrap base forecasts) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue.*

Reconciliation approach	Base forecasts' sample approach								
		Gaussian approach: shrinkage covariance matrix							
	ctjb	In-sample residuals				Multi-step residuals			
		G	B	H	HB	G	B	H	HB
				$\forall k \in \{2, 1\}$					
base	1.005	1.008	1.039	1.045	0.996	0.999	1.000	1.003	1.000
ct(bu)	**0.923**	0.923	0.923	0.923	**0.895**	0.896	0.897	0.897	0.897
ct(shr_{cs}, bu_{te})	0.923	0.922	**0.922**	**0.922**	0.896	**0.895**	**0.895**	**0.895**	**0.896**
ct$(wlsv_{te}, bu_{cs})$	0.924	0.924	0.932	0.932	0.910	0.911	0.911	0.911	0.906
oct$(wlsv)$	0.935	0.937	0.944	0.945	0.922	0.924	0.923	0.923	0.916
oct$(bdshr)$	0.924	0.924	0.932	0.931	0.909	0.911	0.911	0.910	0.906
oct(shr)	0.989	0.978	0.975	0.968	0.897	0.898	0.898	0.898	0.938
oct$(bshr)$	0.990	0.993	0.978	0.981	0.915	0.915	0.915	0.915	0.947
oct$(hshr)$	0.986	0.985	1.024	1.015	0.963	0.964	0.966	0.967	0.978
oct$(hbshr)$	0.985	0.986	1.012	1.015	0.973	0.976	0.977	0.978	0.977
oct$_h(shr)$	0.923	**0.922**	0.925	0.925	0.897	0.898	0.898	0.898	0.900
oct$_h(bshr)$	0.941	0.943	0.942	0.945	0.913	0.915	0.915	0.915	0.916
oct$_h(hshr)$	0.974	0.975	1.001	1.001	0.964	0.964	0.966	0.966	0.967
oct$_h(hbshr)$	0.985	0.986	1.013	1.016	0.973	0.976	0.977	0.978	0.978
				$k = 1$					
base	1.014	1.018	1.015	1.019	0.997	0.999	0.997	0.998	1.000
ct(bu)	0.983	0.984	0.984	0.984	0.967	0.967	0.969	0.969	0.969
ct(shr_{cs}, bu_{te})	**0.983**	**0.982**	**0.982**	**0.983**	**0.966**	**0.967**	**0.966**	**0.966**	**0.968**
ct$(wlsv_{te}, bu_{cs})$	0.991	0.992	0.993	0.992	0.983	0.983	0.983	0.983	0.977
oct$(wlsv)$	1.002	1.004	1.004	1.004	0.994	0.995	0.994	0.996	0.989
oct$(bdshr)$	0.990	0.991	0.992	0.991	0.981	0.983	0.984	0.982	0.977
oct(shr)	1.065	1.054	1.051	1.045	0.969	0.970	0.970	0.969	1.028
oct$(bshr)$	1.061	1.063	1.050	1.052	0.986	0.986	0.987	0.985	1.034
oct$(hshr)$	1.076	1.077	1.095	1.088	1.036	1.036	1.040	1.038	1.066
oct$(hbshr)$	1.064	1.065	1.071	1.073	1.047	1.048	1.050	1.050	1.050
oct$_h(shr)$	0.984	0.985	0.986	0.986	0.969	0.969	0.969	0.968	0.971
oct$_h(bshr)$	1.003	1.005	1.003	1.005	0.985	0.987	0.987	0.986	0.987
oct$_h(hshr)$	1.054	1.054	1.059	1.059	1.036	1.037	1.038	1.039	1.040
oct$_h(hbshr)$	1.063	1.065	1.071	1.074	1.046	1.048	1.049	1.051	1.051
				$k = 2$					
base	0.996	0.998	1.064	1.073	0.995	0.999	1.003	1.007	1.000
ct(bu)	0.867	0.866	0.867	0.866	**0.829**	0.829	0.830	0.830	0.831
ct(shr_{cs}, bu_{te})	0.867	0.866	**0.866**	**0.866**	0.830	**0.829**	**0.830**	**0.830**	**0.829**
ct$(wlsv_{te}, bu_{cs})$	**0.861**	**0.861**	0.875	0.875	0.843	0.845	0.845	0.845	0.839
oct$(wlsv)$	0.873	0.874	0.888	0.889	0.856	0.857	0.857	0.856	0.849
oct$(bdshr)$	0.862	0.861	0.876	0.874	0.843	0.844	0.844	0.844	0.839
oct(shr)	0.918	0.907	0.905	0.898	0.831	0.832	0.832	0.832	0.856
oct$(bshr)$	0.924	0.928	0.911	0.915	0.849	0.849	0.849	0.849	0.868
oct$(hshr)$	0.904	0.901	0.957	0.946	0.895	0.896	0.898	0.900	0.897
oct$(hbshr)$	0.912	0.913	0.956	0.961	0.905	0.909	0.909	0.911	0.910
oct$_h(shr)$	0.866	0.863	0.869	0.869	0.830	0.831	0.832	0.832	0.835
oct$_h(bshr)$	0.882	0.886	0.886	0.889	0.846	0.848	0.849	0.848	0.850
oct$_h(hshr)$	0.901	0.902	0.947	0.946	0.896	0.896	0.898	0.899	0.900
oct$_h(hbshr)$	0.912	0.914	0.958	0.961	0.905	0.908	0.910	0.909	0.910

Table 5: *ES skill score defined in equation (17) and (19). The smaller this value, the more accurate the forecast. Approaches that performed worse than the benchmark model (Bootstrap base forecasts) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue.*

B Australian GDP dataset: one-step residuals and shrinkage covariance matrix

Reconciliation approach	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G_h	H_h	G_{oh}	H_{oh}		G_h	H_h	G_{oh}	H_{oh}
	$\forall k \in \{4,2,1\}$					$k=1$				
base	1.000	0.979	0.995	0.968	0.976	1.000	0.988	0.988	0.971	0.971
$\text{ct}(shr_{cs}, bu_{te})$	0.937	0.956	0.956	0.976	0.976	0.992	1.008	1.008	1.029	1.029
$\text{ct}(wls_{cs}, bu_{te})$	0.930	0.917	0.917	0.898	0.898	0.986	0.974	0.975	0.956	0.956
$\text{oct}(wlsv)$	0.926	0.919	0.920	0.900	0.900	0.984	0.981	0.979	0.959	0.959
$\text{oct}(bdshr)$	0.940	0.965	0.945	0.992	0.957	0.997	1.019	1.003	1.044	1.018
$\text{oct}(shr)$	0.944	1.020	0.940	1.094	0.988	1.015	1.095	1.010	1.160	1.059
$\text{oct}(hshr)$	0.988	0.972	1.002	0.974	1.001	1.048	1.037	1.060	1.034	1.061
$\text{oct}_o(wlsv)$	**0.926**	**0.911**	**0.912**	**0.896**	**0.895**	**0.984**	**0.971**	**0.970**	**0.954**	**0.954**
$\text{oct}_o(bdshr)$	0.978	0.964	0.946	0.952	0.930	1.034	1.016	1.003	1.005	0.989
$\text{oct}_o(shr)$	0.950	0.946	0.922	0.925	0.903	1.014	1.003	0.985	0.987	0.968
$\text{oct}_o(hshr)$	0.989	0.966	0.984	0.954	0.965	1.047	1.028	1.038	1.012	1.023
$\text{oct}_{oh}(shr)$	1.102	1.059	1.001	1.094	0.988	1.172	1.109	1.066	1.160	1.059
$\text{oct}_{oh}(hshr)$	1.006	0.983	1.009	0.974	1.001	1.068	1.046	1.059	1.034	1.061
	$k=2$					$k=4$				
base	1.000	0.984	0.993	0.968	0.976	1.000	0.966	1.004	0.964	0.981
$\text{ct}(shr_{cs}, bu_{te})$	0.949	0.966	0.966	0.987	0.987	0.874	0.896	0.896	0.914	0.914
$\text{ct}(wls_{cs}, bu_{te})$	0.942	0.928	0.928	0.909	0.909	0.866	0.853	0.853	0.834	0.834
$\text{oct}(wlsv)$	0.938	0.929	0.931	0.911	0.911	0.860	0.853	0.855	0.835	0.834
$\text{oct}(bdshr)$	0.953	0.976	0.956	1.003	0.969	0.874	0.904	0.880	0.931	0.889
$\text{oct}(shr)$	0.955	1.031	0.951	1.113	1.002	0.866	0.940	0.864	1.015	0.909
$\text{oct}(hshr)$	1.001	0.985	1.014	0.987	1.016	0.919	0.900	0.935	0.904	0.931
$\text{oct}_o(wlsv)$	**0.938**	**0.921**	**0.923**	**0.907**	**0.906**	**0.860**	**0.847**	**0.848**	**0.832**	**0.830**
$\text{oct}_o(bdshr)$	0.991	0.974	0.957	0.964	0.942	0.914	0.905	0.883	0.892	0.865
$\text{oct}_o(shr)$	0.965	0.958	0.934	0.938	0.916	0.877	0.882	0.852	0.854	0.831
$\text{oct}_o(hshr)$	1.002	0.979	0.996	0.967	0.978	0.922	0.898	0.923	0.888	0.898
$\text{oct}_{oh}(shr)$	1.120	1.069	1.013	1.113	1.002	1.020	1.002	0.928	1.015	0.909
$\text{oct}_{oh}(hshr)$	1.021	0.996	1.021	0.987	1.016	0.934	0.912	0.951	0.904	0.931

*The Gaussian method employs a sample covariance matrix:
G_h and H_h use multi-step residuals and G_{oh} and H_{oh} use overlapping and multi-step residuals.

Table 6: *CRPS skill score defined in equation (17) and (18) for the Australian Quarterly National Accounts dataset (AusGDP). The smaller this value, the more accurate the forecast. Approaches that performed worse than the benchmark model (Bootstrap base forecasts) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue.*

Reconciliation approach	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G_h	H_h	G_{oh}	H_{oh}		G_h	H_h	G_{oh}	H_{oh}
			$\forall k \in \{4,2,1\}$					$k=1$		
base	1.000	0.970	0.988	0.960	0.970	1.000	0.977	0.977	0.965	0.965
$\mathrm{ct}(shr_{cs}, bu_{te})$	0.897	0.944	0.944	0.973	0.973	0.964	1.001	1.001	1.033	1.033
$\mathrm{ct}(wls_{cs}, bu_{te})$	**0.886**	0.880	0.880	**0.860**	0.860	**0.954**	**0.944**	0.945	**0.928**	**0.928**
$\mathrm{oct}(wlsv)$	0.890	0.890	0.894	0.872	0.872	0.958	0.957	0.957	0.938	0.939
$\mathrm{oct}(bdshr)$	0.905	0.956	0.934	0.992	0.954	0.972	1.014	0.994	1.048	1.018
$\mathrm{oct}(shr)$	0.895	0.979	0.895	1.053	0.944	0.973	1.060	0.969	1.121	1.015
$\mathrm{oct}(hshr)$	0.951	0.940	0.973	0.959	0.992	1.017	1.010	1.034	1.023	1.055
$\mathrm{oct}_o(wlsv)$	0.891	**0.879**	0.881	0.864	0.864	0.958	0.945	0.945	0.931	0.931
$\mathrm{oct}_o(bdshr)$	0.940	0.928	0.910	0.918	0.895	1.004	0.986	0.971	0.980	0.961
$\mathrm{oct}_o(shr)$	0.900	0.899	**0.876**	0.878	**0.858**	0.973	0.963	**0.944**	0.949	0.930
$\mathrm{oct}_o(hshr)$	0.956	0.936	0.955	0.922	0.936	1.021	1.004	1.012	0.987	1.000
$\mathrm{oct}_{oh}(shr)$	1.059	1.015	0.956	1.053	0.945	1.130	1.063	1.019	1.121	1.016
$\mathrm{oct}_{oh}(hshr)$	0.986	0.968	0.999	0.959	0.992	1.053	1.034	1.049	1.024	1.055
			$k=2$					$k=4$		
base	1.000	0.972	0.985	0.959	0.969	1.000	0.959	1.000	0.957	0.976
$\mathrm{ct}(shr_{cs}, bu_{te})$	0.915	0.961	0.960	0.991	0.991	0.818	0.874	0.874	0.899	0.900
$\mathrm{ct}(wls_{cs}, bu_{te})$	**0.904**	0.896	**0.896**	**0.877**	**0.877**	**0.807**	0.805	0.805	**0.782**	0.783
$\mathrm{oct}(wlsv)$	0.909	0.907	0.912	0.889	0.889	0.811	0.813	0.819	0.794	0.794
$\mathrm{oct}(bdshr)$	0.925	0.976	0.953	1.013	0.974	0.825	0.883	0.860	0.920	0.876
$\mathrm{oct}(shr)$	0.913	1.000	0.914	1.076	0.963	0.807	0.885	0.808	0.967	0.861
$\mathrm{oct}(hshr)$	0.973	0.960	0.993	0.978	1.014	0.871	0.856	0.897	0.881	0.913
$\mathrm{oct}_o(wlsv)$	0.908	**0.895**	0.898	0.881	0.882	0.812	**0.802**	0.806	0.786	0.786
$\mathrm{oct}_o(bdshr)$	0.960	0.947	0.929	0.938	0.915	0.860	0.856	0.836	0.841	0.816
$\mathrm{oct}_o(shr)$	0.921	0.919	0.896	0.898	0.878	0.814	0.821	**0.796**	0.794	**0.775**
$\mathrm{oct}_o(hshr)$	0.977	0.956	0.976	0.942	0.957	0.876	0.854	0.882	0.844	0.856
$\mathrm{oct}_{oh}(shr)$	1.082	1.029	0.973	1.076	0.963	0.971	0.954	0.882	0.967	0.861
$\mathrm{oct}_{oh}(hshr)$	1.007	0.988	1.017	0.979	1.014	0.904	0.888	0.934	0.881	0.913

Base forecasts' sample approach

*The Gaussian method employs a sample covariance matrix:
G_h and H_h use multi-step residuals and G_{oh} and H_{oh} use overlapping and multi-step residuals.

Table 7: *ES skill score defined in equation (17) and (19) for the Australian Quarterly National Accounts dataset (AusGDP). The smaller this value, the more accurate the forecast. Approaches that performed worse than the benchmark model (Bootstrap base forecasts) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue.*

Reconciliation approach	Base forecasts' sample approach									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G_h	H_h	G_{oh}	H_{oh}		G_h	H_h	G_{oh}	H_{oh}
	$\forall k \in \{4,2,1\}$					$k=1$				
base	1.000	0.979	1.011	0.968	0.987	1.000	0.988	0.988	0.971	0.971
ct(shr_{cs}, bu_{te})	0.937	0.960	0.961	0.962	0.960	0.992	1.001	1.001	1.004	1.000
ct(wls_{cs}, bu_{te})	0.930	**0.951**	0.953	**0.911**	0.915	0.986	0.997	0.998	**0.964**	0.967
oct$(wlsv)$	0.926	0.972	0.957	0.918	0.917	0.984	1.010	1.003	0.971	0.970
oct$(bdshr)$	0.940	0.986	0.966	0.981	0.956	0.997	1.015	1.006	1.016	1.000
oct(shr)	0.944	0.999	0.962	1.051	0.995	1.015	1.047	1.021	1.105	1.058
oct$(hshr)$	0.988	1.000	1.021	0.979	1.002	1.048	1.045	1.066	1.034	1.053
oct$_o(wlsv)$	**0.926**	0.961	0.948	0.914	**0.912**	**0.984**	1.000	0.993	0.966	**0.965**
oct$_o(bdshr)$	0.978	0.956	0.949	0.949	0.934	1.034	**0.984**	**0.983**	0.988	0.977
oct$_o(shr)$	0.950	0.957	**0.946**	0.933	0.917	1.014	0.998	0.995	0.986	0.974
oct$_o(hshr)$	0.989	0.997	1.013	0.967	0.982	1.047	1.039	1.054	1.019	1.032
oct$_{oh}(shr)$	1.102	1.010	1.006	1.051	0.995	1.172	1.059	1.063	1.105	1.058
oct$_{oh}(hshr)$	1.006	0.989	1.004	0.979	1.002	1.068	1.037	1.050	1.034	1.053
	$k=2$					$k=4$				
base	1.000	0.984	1.009	0.968	0.987	1.000	0.966	1.037	0.964	1.002
ct(shr_{cs}, bu_{te})	0.949	0.972	0.972	0.974	0.971	0.874	0.910	0.911	0.910	0.910
ct(wls_{cs}, bu_{te})	0.942	**0.962**	0.964	**0.923**	0.927	0.866	**0.897**	0.900	**0.851**	0.855
oct$(wlsv)$	0.938	0.988	0.968	0.931	0.929	0.860	0.921	0.903	0.856	0.856
oct$(bdshr)$	0.953	1.004	0.979	0.996	0.970	0.874	0.942	0.914	0.932	0.900
oct(shr)	0.955	1.016	0.973	1.070	1.010	0.866	0.937	0.895	0.981	0.922
oct$(hshr)$	1.001	1.015	1.034	0.993	1.017	0.919	0.942	0.965	0.913	0.937
oct$_o(wlsv)$	**0.938**	0.976	**0.959**	0.927	**0.925**	**0.860**	0.910	0.894	0.853	0.852
oct$_o(bdshr)$	0.991	0.970	0.963	0.963	0.948	0.914	0.917	0.905	0.899	0.880
oct$_o(shr)$	0.965	0.973	0.959	0.948	0.931	0.877	0.903	**0.886**	0.868	**0.850**
oct$_o(hshr)$	1.002	1.013	1.026	0.980	0.996	0.922	0.943	0.962	0.905	0.921
oct$_{oh}(shr)$	1.120	1.026	1.019	1.070	1.010	1.020	0.947	0.939	0.981	0.922
oct$_{oh}(hshr)$	1.021	1.005	1.017	0.993	1.017	0.934	0.929	0.946	0.913	0.937

*The Gaussian method employs a shrinkage covariance matrix:
G_h and H_h use multi-step residuals and G_{oh} and H_{oh} use overlapping and multi-step residuals.

Table 8: *CRPS skill score defined in equation (17) and (18) for the Australian Quarterly National Accounts dataset (AusGDP). The smaller this value, the more accurate the forecast. Approaches that performed worse than the benchmark model (Bootstrap base forecasts) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue.*

Reconciliation approach	Base forecasts' sample approach									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G_h	H_h	G_{oh}	H_{oh}		G_h	H_h	G_{oh}	H_{oh}
	$\forall k \in \{4, 2, 1\}$					$k = 1$				
base	1.000	0.967	1.002	0.957	0.980	1.000	0.973	0.973	0.961	0.962
$\mathrm{ct}(shr_{cs}, bu_{te})$	0.897	0.968	0.969	0.963	0.962	0.964	1.012	1.012	1.009	1.004
$\mathrm{ct}(wls_{cs}, bu_{te})$	**0.886**	0.939	0.944	**0.882**	0.888	**0.954**	0.994	0.998	**0.947**	0.952
$\mathrm{oct}(wlsv)$	0.890	0.966	0.959	0.897	0.901	0.958	1.017	1.012	0.960	0.965
$\mathrm{oct}(bdshr)$	0.905	0.997	0.981	0.986	0.960	0.972	1.031	1.021	1.024	1.005
$\mathrm{oct}(shr)$	0.895	0.979	0.945	1.021	0.962	0.973	1.041	1.011	1.083	1.028
$\mathrm{oct}(hshr)$	0.951	0.997	1.023	0.973	1.005	1.017	1.051	1.073	1.034	1.063
$\mathrm{oct}_o(wlsv)$	0.891	0.950	0.945	0.889	0.892	0.958	1.002	0.997	0.953	0.956
$\mathrm{oct}_o(bdshr)$	0.940	0.935	0.933	0.922	0.909	1.004	**0.965**	**0.964**	0.969	0.959
$\mathrm{oct}_o(shr)$	0.900	**0.935**	**0.928**	0.895	**0.884**	0.973	0.984	0.982	0.960	**0.950**
$\mathrm{oct}_o(hshr)$	0.956	0.997	1.015	0.945	0.965	1.021	1.049	1.062	1.007	1.024
$\mathrm{oct}_{oh}(shr)$	1.059	0.981	0.983	1.021	0.962	1.130	1.034	1.041	1.083	1.029
$\mathrm{oct}_{oh}(hshr)$	0.986	0.996	1.014	0.973	1.005	1.053	1.050	1.064	1.034	1.063
	$k = 2$					$k = 4$				
base	1.000	0.970	0.999	0.955	0.980	1.000	0.958	1.033	0.953	1.000
$\mathrm{ct}(shr_{cs}, bu_{te})$	0.915	0.987	0.988	0.983	0.982	0.818	0.909	0.910	0.902	0.902
$\mathrm{ct}(wls_{cs}, bu_{te})$	**0.904**	**0.958**	0.962	**0.900**	0.906	**0.807**	0.871	0.876	**0.805**	0.812
$\mathrm{oct}(wlsv)$	0.909	0.988	0.979	0.916	0.920	0.811	0.896	0.891	0.820	0.825
$\mathrm{oct}(bdshr)$	0.925	1.024	1.005	1.010	0.984	0.825	0.938	0.919	0.926	0.895
$\mathrm{oct}(shr)$	0.913	1.006	0.967	1.045	0.982	0.807	0.898	0.864	0.940	0.881
$\mathrm{oct}(hshr)$	0.973	1.020	1.046	0.994	1.028	0.871	0.924	0.954	0.897	0.929
$\mathrm{oct}_o(wlsv)$	0.908	0.972	0.964	0.908	0.911	0.812	0.882	0.876	0.812	0.816
$\mathrm{oct}_o(bdshr)$	0.960	0.959	0.957	0.945	0.932	0.860	0.884	0.879	0.857	0.841
$\mathrm{oct}_o(shr)$	0.921	0.958	**0.950**	0.917	**0.905**	0.814	**0.867**	**0.857**	0.815	**0.803**
$\mathrm{oct}_o(hshr)$	0.977	1.021	1.038	0.966	0.987	0.876	0.926	0.949	0.868	0.889
$\mathrm{oct}_{oh}(shr)$	1.082	1.002	1.003	1.045	0.982	0.971	0.910	0.911	0.941	0.882
$\mathrm{oct}_{oh}(hshr)$	1.007	1.017	1.036	0.994	1.028	0.904	0.924	0.947	0.896	0.929

*The Gaussian method employs a shrinkage covariance matrix:
G_h and H_h use multi-step residuals and G_{oh} and H_{oh} use overlapping and multi-step residuals.

Table 9: *ES skill score defined in equation (17) and (19) and* (**??**) *for the Australian Quarterly National Accounts dataset (AusGDP). The smaller this value, the more accurate the forecast. Approaches that performed worse than the benchmark model (Bootstrap base forecasts) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue.*

C Australian Tourism Demand dataset: shrinkage covariance matrix

Reconciliation approach	Base forecasts' sample approach									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G	B	H	HB		G	B	H	HB
	$\forall k \in \{12, 6, 4, 3, 2, 1\}$					$k = 1$				
base	1.000	**0.971**	0.972	**0.971**	0.972	1.000	**0.972**	0.971	0.972	0.971
ct(bu)	1.321	1.017	1.018	1.017	1.017	1.077	0.983	0.983	0.983	0.983
ct(shr_{cs}, bu_{te})	1.057	1.013	**0.971**	1.013	0.971	0.976	0.987	**0.961**	0.988	0.961
ct($wlsv_{te}, bu_{cs}$)	1.062	1.069	1.070	0.974	0.974	0.976	0.986	0.986	**0.965**	0.965
oct(ols)	0.989	1.163	1.052	1.139	0.987	0.982	1.038	0.992	1.047	0.987
oct($struc$)	0.982	1.099	1.039	1.037	0.960	0.970	1.007	0.971	0.999	0.962
oct($wlsv$)	0.987	1.080	1.041	0.992	0.958	0.952	1.004	0.969	0.978	0.956
oct($bdshr$)	0.975	1.072	1.032	0.985	**0.950**	**0.949**	0.999	0.965	0.975	**0.952**
oct$_h$($hbshr$)	0.989	1.189	1.076	1.171	1.021	0.982	1.045	1.000	1.063	1.009
oct$_h$($bshr$)	0.994	1.202	1.073	1.168	1.021	0.988	1.046	1.012	1.063	1.012
oct$_h$($hshr$)	**0.969**	1.066	1.052	1.008	0.994	0.953	0.994	0.972	0.991	0.979
oct$_h$(shr)	1.007	1.090	1.046	1.000	0.970	1.000	1.035	0.992	0.998	0.973
	$k = 2$					$k = 3$				
base	1.000	**0.969**	0.969	**0.968**	0.968	1.000	**0.971**	**0.970**	**0.969**	0.970
ct(bu)	1.189	1.000	1.000	1.000	1.000	1.273	1.013	1.013	1.013	1.013
ct(shr_{cs}, bu_{te})	1.015	1.004	**0.968**	1.004	0.968	1.041	1.013	0.973	1.014	0.973
ct($wlsv_{te}, bu_{cs}$)	1.016	1.043	1.044	0.969	0.969	1.046	1.067	1.068	0.974	0.974
oct(ols)	0.992	1.118	1.037	1.092	0.989	0.994	1.153	1.053	1.124	0.990
oct($struc$)	0.982	1.075	1.022	1.020	0.963	0.986	1.099	1.041	1.033	0.964
oct($wlsv$)	0.972	1.064	1.021	0.987	0.958	0.983	1.083	1.041	0.993	0.960
oct($bdshr$)	**0.964**	1.057	1.015	0.983	**0.953**	0.972	1.075	1.033	0.988	**0.955**
oct$_h$($hbshr$)	0.992	1.136	1.055	1.116	1.014	0.994	1.178	1.075	1.153	1.020
oct$_h$($bshr$)	0.997	1.145	1.059	1.114	1.016	0.999	1.190	1.075	1.151	1.021
oct$_h$($hshr$)	0.965	1.050	1.029	1.001	0.986	**0.971**	1.067	1.051	1.009	0.994
oct$_h$(shr)	1.005	1.083	1.035	1.001	0.973	1.009	1.097	1.050	1.004	0.974
	$k = 4$					$k = 6$				
base	1.000	**0.973**	**0.973**	**0.971**	0.973	1.000	**0.976**	0.977	**0.975**	0.977
ct(bu)	1.340	1.021	1.021	1.021	1.021	1.450	1.032	1.033	1.032	1.033
ct(shr_{cs}, bu_{te})	1.061	1.018	0.974	1.018	0.974	1.094	1.023	**0.974**	1.024	0.974
ct($wlsv_{te}, bu_{cs}$)	1.068	1.087	1.089	0.976	0.976	1.103	1.108	1.110	0.978	0.978
oct(ols)	0.993	1.186	1.068	1.148	0.989	0.989	1.223	1.080	1.184	0.987
oct($struc$)	0.986	1.120	1.057	1.042	0.962	0.986	1.141	1.071	1.054	0.959
oct($wlsv$)	0.990	1.100	1.059	0.996	0.959	1.001	1.115	1.076	0.998	0.958
oct($bdshr$)	0.977	1.091	1.049	0.989	**0.952**	0.985	1.103	1.064	0.989	**0.949**
oct$_h$($hbshr$)	0.993	1.215	1.095	1.182	1.022	0.989	1.258	1.112	1.225	1.026
oct$_h$($bshr$)	0.997	1.230	1.089	1.178	1.023	0.994	1.278	1.101	1.219	1.025
oct$_h$($hshr$)	**0.973**	1.084	1.071	1.012	0.996	**0.976**	1.097	1.091	1.017	1.002
oct$_h$(shr)	1.009	1.108	1.062	1.003	0.972	1.010	1.113	1.070	1.000	0.968
	$k = 12$									
base	1.000	**0.968**	**0.969**	**0.969**	0.971					
ct(bu)	1.675	1.056	1.057	1.057	1.057					
ct(shr_{cs}, bu_{te})	1.163	1.032	0.974	1.033	0.974					
ct($wlsv_{te}, bu_{cs}$)	1.174	1.128	1.130	0.982	0.982					
oct(ols)	0.982	1.277	1.085	1.252	0.982					
oct($struc$)	0.982	1.158	1.074	1.075	0.950					
oct($wlsv$)	1.025	1.122	1.085	1.001	0.954					
oct($bdshr$)	1.002	1.110	1.071	0.989	**0.941**					
oct$_h$($hbshr$)	0.982	1.322	1.125	1.305	1.033					
oct$_h$($bshr$)	0.987	1.347	1.107	1.297	1.031					
oct$_h$($hshr$)	**0.978**	1.106	1.107	1.021	1.010					
oct$_h$(shr)	1.010	1.107	1.067	0.991	0.959					

*The Gaussian method employs a shrikage covariance matrix and includes four techniques (G, B, H, HB) with multi-step residuals..

Table 10: *CRPS skill score defined in equation (17) and (18) for VN525 dataset. The smaller this value, the more accurate the forecast. Approaches that performed worse than the benchmark model (Bootstrap base forecasts) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue.*

Reconciliation approach	Base forecasts' sample approach									
	ctjb	Gaussian approach*				ctjb	Gaussian approach*			
		G	B	H	HB		G	B	H	HB
	$\forall k \in \{12, 6, 4, 3, 2, 1\}$					$k = 1$				
base	1.000	**0.958**	0.984	**0.972**	0.992	1.000	**0.954**	0.958	**0.954**	0.958
$\mathrm{ct}(bu)$	2.427	1.040	1.042	1.040	1.041	1.759	1.001	1.002	1.002	1.002
$\mathrm{ct}(shr_{cs}, bu_{te})$	1.243	0.988	**0.913**	0.990	0.913	1.098	1.011	**0.938**	1.013	0.938
$\mathrm{ct}(wlsv_{te}, bu_{cs})$	1.499	1.117	1.120	1.025	1.025	1.241	1.019	1.020	0.990	0.990
$\mathrm{oct}(ols)$	0.955	1.000	0.984	0.985	0.922	0.975	0.983	0.961	0.987	0.945
$\mathrm{oct}(struc)$	1.085	1.094	1.047	1.018	0.952	1.027	1.054	0.981	1.022	0.953
$\mathrm{oct}(wlsv)$	1.132	1.137	1.065	1.059	0.969	1.050	1.078	0.989	1.043	0.960
$\mathrm{oct}(bdshr)$	1.047	1.085	1.013	1.011	0.927	1.009	1.050	0.966	1.019	0.942
$\mathrm{oct}_h(hbshr)$	0.956	1.018	0.981	1.016	0.919	0.975	0.991	0.961	1.002	0.947
$\mathrm{oct}_h(bshr)$	**0.931**	1.002	1.001	0.982	**0.889**	**0.965**	0.980	0.975	0.985	0.933
$\mathrm{oct}_h(hshr)$	1.081	1.109	1.039	1.076	0.973	1.028	1.061	0.978	1.052	0.963
$\mathrm{oct}_h(shr)$	1.068	1.088	1.008	0.995	0.896	1.023	1.061	0.966	1.011	**0.924**
	$k = 2$					$k = 3$				
base	1.000	**0.960**	0.971	**0.958**	0.972	1.000	**0.963**	0.981	**0.966**	0.986
$\mathrm{ct}(bu)$	2.176	1.035	1.036	1.035	1.035	2.428	1.042	1.044	1.042	1.043
$\mathrm{ct}(shr_{cs}, bu_{te})$	1.192	1.020	**0.942**	1.021	0.942	1.245	1.009	**0.931**	1.011	0.931
$\mathrm{ct}(wlsv_{te}, bu_{cs})$	1.400	1.104	1.106	1.018	1.019	1.500	1.127	1.130	1.029	1.029
$\mathrm{oct}(ols)$	0.985	1.028	1.008	1.002	0.950	0.976	1.020	1.004	0.994	0.938
$\mathrm{oct}(struc)$	1.075	1.115	1.051	1.039	0.967	1.096	1.117	1.064	1.033	0.965
$\mathrm{oct}(wlsv)$	1.110	1.149	1.065	1.070	0.979	1.142	1.160	1.082	1.073	0.981
$\mathrm{oct}(bdshr)$	1.045	1.105	1.024	1.033	0.949	1.060	1.109	1.032	1.029	0.943
$\mathrm{oct}_h(hbshr)$	0.984	1.041	1.007	1.024	0.951	0.975	1.036	1.002	1.023	0.937
$\mathrm{oct}_h(bshr)$	**0.967**	1.029	1.025	0.998	0.928	**0.954**	1.024	1.025	0.993	**0.911**
$\mathrm{oct}_h(hshr)$	1.073	1.122	1.042	1.083	0.983	1.093	1.129	1.054	1.090	0.984
$\mathrm{oct}_h(shr)$	1.064	1.110	1.019	1.018	**0.922**	1.082	1.116	1.030	1.015	0.915
	$k = 4$					$k = 6$				
base	1.000	**0.962**	0.987	**0.973**	0.996	1.000	**0.963**	0.998	**0.984**	1.011
$\mathrm{ct}(bu)$	2.585	1.052	1.054	1.053	1.053	2.849	1.083	1.085	1.083	1.084
$\mathrm{ct}(shr_{cs}, bu_{te})$	1.277	1.000	**0.923**	1.002	0.923	1.339	0.999	**0.921**	1.000	0.920
$\mathrm{ct}(wlsv_{te}, bu_{cs})$	1.559	1.150	1.153	1.037	1.037	1.662	1.189	1.193	1.066	1.066
$\mathrm{oct}(ols)$	0.966	1.022	1.008	0.994	0.931	0.962	1.023	1.014	1.003	0.930
$\mathrm{oct}(struc)$	1.106	1.120	1.076	1.031	0.963	1.132	1.132	1.100	1.039	0.972
$\mathrm{oct}(wlsv)$	1.157	1.167	1.097	1.075	0.982	1.192	1.187	1.124	1.090	0.995
$\mathrm{oct}(bdshr)$	1.065	1.112	1.041	1.025	0.939	1.084	1.121	1.058	1.029	0.940
$\mathrm{oct}_h(hbshr)$	0.967	1.041	1.005	1.027	0.929	0.964	1.046	1.008	1.042	0.924
$\mathrm{oct}_h(bshr)$	**0.943**	1.028	1.028	0.994	**0.900**	**0.932**	1.029	1.032	1.000	**0.887**
$\mathrm{oct}_h(hshr)$	1.101	1.137	1.068	1.093	0.986	1.126	1.153	1.089	1.110	0.999
$\mathrm{oct}_h(shr)$	1.089	1.118	1.039	1.012	0.910	1.107	1.118	1.045	1.006	0.902
	$k = 12$									
base	1.000	0.948	1.010	1.002	1.033					
$\mathrm{ct}(bu)$	2.990	1.028	1.031	1.029	1.029					
$\mathrm{ct}(shr_{cs}, bu_{te})$	1.326	**0.897**	**0.830**	**0.899**	0.830					
$\mathrm{ct}(wlsv_{te}, bu_{cs})$	1.679	1.119	1.123	1.009	1.009					
$\mathrm{oct}(ols)$	0.872	0.927	0.914	0.930	0.840					
$\mathrm{oct}(struc)$	1.077	1.028	1.012	0.950	0.894					
$\mathrm{oct}(wlsv)$	1.149	1.089	1.041	1.006	0.922					
$\mathrm{oct}(bdshr)$	1.021	1.015	0.964	0.935	0.855					
$\mathrm{oct}_h(hbshr)$	0.872	0.955	0.906	0.978	0.833					
$\mathrm{oct}_h(bshr)$	**0.833**	0.927	0.927	0.927	**0.784**					
$\mathrm{oct}_h(hshr)$	1.066	1.056	1.005	1.026	0.926					
$\mathrm{oct}_h(shr)$	1.043	1.011	0.952	0.909	0.809					

*The Gaussian method employs a shrikage covariance matrix and includes four techniques (G, B, H, HB) with multi-step residuals.

Table 11: *ES skill score defined in equation (17) and (19) for VN525 dataset. The smaller this value, the more accurate the forecast. Approaches that performed worse than the benchmark model (Bootstrap base forecasts) are highlighted in red, the best for each column is marked in bold, and the overall lowest value is highlighted in blue.*