**Aim**: Bound the following quantity

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} f(X_i) - \int_0^1 f(y)dy\right| > \varepsilon\right)$$

Defining the new random variable

$$Y_i \coloneqq f(X_i) - \int_0^1 f(y)dy \Rightarrow \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\sum_{i=1}^{n} f(X_i) - \int_0^1 f(y)dy$$

Therefore, we deduce that

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} f(X_i) - \int_0^1 f(y)dy\right| > \varepsilon\right) = P\left(\left|\frac{1}{n}\sum_{i=1}^{n} Y_i\right| > \varepsilon\right)$$

We use Markov's inequality

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} Y_i\right| > \varepsilon\right) = P\left(\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2 > \varepsilon^2\right) < \frac{1}{n^2\varepsilon^2} E\left(\left(\sum_{i=1}^{n} Y_i\right)^2\right) = \frac{\operatorname{Var}(Y_i)}{n\varepsilon^2}$$

$$\operatorname{Var}(Y_i) = \operatorname{Var}\left(f(X_i) - \int_0^1 f(y)dy\right) = \operatorname{Var}(X_i)$$

As we have a continuous function on a closed interval we know that it must bounded so let's define

$$|f|_\infty \coloneqq \sup\{|f(x): x \in [0,1]|\}$$

As $f(X_i)$ is bounded by $|f|_\infty$ we know from Term 2 MATH40005 that

$$\operatorname{Var}\big(f(X_i)\big) \leq \frac{\big(|f|_\infty - (-|f|_\infty)\big)^2}{4} = |f|_\infty^2$$

Therefore, we deduce that

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} f(X_i) - \int_0^1 f(y)dy\right| > \varepsilon\right) \leq \frac{|f|_\infty^2}{n\varepsilon^2}$$