Proof for Weak Law of Large Numbers

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Weak Law of Large Numbers - Given a sequence of iid r.vs $\{X_i\}$ which have finite expectation μ and variance σ then:

$$\forall \varepsilon > 0, \lim_{n\to\infty} \mathbb{P}\left(\left|\sum_{i=1}^{n} \frac{X_i}{n} - \mu\right| < \varepsilon\right) = 1$$

Note that there are multiple versions of the Weak Law. Generally speaking we don't even require finite variance. Asserting finite variance is all we need for the project and it makes the proof remarkably easy. There are even cases where we just require variances to bounded but not necessarily the same. Setting that aside we will focus on proving the version stated above.

By Chebyshev's Inequality we deduce that:

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} \frac{X_i}{n} - \mu\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2}\operatorname{Var}\left(\sum_{i=1}^{n} \frac{X_i}{n}\right)$$

Now we use the fact that $\{X_i\}$ are i.i.d which yields

$$\operatorname{Var}\left(\sum_{i=1}^{n} \frac{X_i}{n}\right) = \frac{n}{n^2}\operatorname{Var}\left(X_i\right) = \frac{\sigma^2}{n}$$

Therefore we deduce that

$$\forall n \in \mathbb{N}, \quad 1 - \frac{\sigma^2}{n\varepsilon^2} \leq \mathbb{P}\left(\left|\sum_{i=1}^{n} \frac{X_i}{n} - \mu\right| < \varepsilon\right) \leq 1$$

As $\varepsilon > 0$ is fixed and $\sigma^2 < \infty$, we deduce that the LHS converges to 1 for $n \to \infty$. Applying the Squeeze Theorem we prove the Weak Law. $\square$