Proof for Concentration of Area in a Hypersphere

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Overview

Our main goal is to show that concentration occurs at *any* equator of an asymptotic hypersphere. This can be quite a deceptive statement as we are working in extremely high dimensions. Any description that relies on 3D intuition is unlikely to give us an accurate idea of what is really happening.

If we are working with an n-Sphere then its equator is its intersection with an n-dimensional hyperplane which contains the origin. In this proof, we will think of equators as the osculation of a northern hemisphere and its southern hemisphere.

The content of this article has been based on Chapter 14.1 of J. Matousek's *Lectures on Discrete Geometry* [1].

Groundwork

Definition. Unit n-Sphere $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$

Definition. n-Ball $\mathbb{B}^n(r) := \{x \in \mathbb{R}^n : \|x\| \leq r\}$

Definition. ε-neighbourhood For $A \subseteq \mathbb{S}^{n-1}$ we define

$$A_\varepsilon := \{x \in \mathbb{S}^{n-1} : \exists a \in A : \|x - a\| < \varepsilon\}$$

Definition. Union of Segments For $A \subseteq \mathbb{S}^{n-1}$ we define

$$\tilde{A} := \{\alpha x : x \in A, \alpha \in [0, 1]\} \subseteq \mathbb{B}^n(1)$$

Definition. $\mathbb{P}(.)$ denotes the surface measure on $\mathbb{S}^{n-1}$ scaled such that

$$\mathbb{P}(\mathbb{S}^{n-1}) = 1$$

This means $\mathbb{P}(.)$ is a probability measure. For $A \subset \mathbb{S}^{n-1}$ we can think of $\mathbb{P}(A)$ as the probability that a point in $\mathbb{S}^{n-1}$ lies in A. By geometry of spheres[1]

$$\mathbb{P}(A) = \mu(\tilde{A}) = \frac{\mathrm{Vol}(\tilde{A})}{\mathrm{Vol}(\mathbb{B}^n(1))}$$

Lemma. Brunn-Minkowski For any nonempty compact sets $A, B \subset \mathbb{R}^n$

$$\sqrt{\mathrm{Vol}(A)\mathrm{Vol}(B)} \leq \mathrm{Vol}\left(\frac{1}{2}(A+B)\right)$$

Lemma. If we define $B := \mathbb{S}^{n-1} \setminus A_\varepsilon$ then it follows by definition of A_ε

$$\forall a \in A, b \in B, \; \|a-b\| \geq \varepsilon$$

Lemma. Pythagorean Inequality

$$\forall \tilde{x} \in \tilde{A}, \tilde{y} \in \tilde{B}, \; \left\|\frac{\tilde{x}+\tilde{y}}{2}\right\| \leq 1 - \frac{\varepsilon^2}{8}$$

Proof of the lemma. **Let $\tilde{x} = \alpha x$, $\tilde{y} = \beta y$, $x \in A$, $y \in B$:**

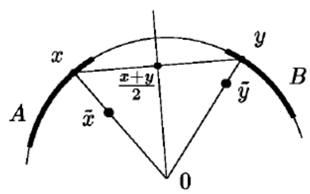

Figure 1: Diagram for inequality [1]

The parallelogram law states that in inner product spaces we have:

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + 2\|x-y\|^2$$

By construction we know that $\|x\| = 1, \|y\| = 1, \|x-y\| \geq \varepsilon$ hence

$$\|x+y\|^2 \leq 4 + \varepsilon^2 \Rightarrow \left\|\frac{x+y}{2}\right\| \leq \sqrt{1 - \frac{\varepsilon^2}{4}} \leq 1 - \frac{\varepsilon^2}{8}$$

Now we can control $\tilde{x}, \tilde{y}$, we may assume that $\beta = 1$. By symmetry WLOG $\alpha \geq \beta$. If $\beta = 0$ then $\alpha = 0$ so that is trivial. Otherwise

$$\|\alpha x + \beta y\| = \beta \left\|\frac{\alpha}{\beta} x + y\right\| \leq \|\alpha' x + y\|$$

By the triangle inequality and taking ε sufficiently small (say $[0,1]$)^2

$$\left\|\frac{\tilde{x}+\tilde{y}}{2}\right\| = \left\|\frac{\alpha x + y}{2}\right\| \leq \alpha \left\|\frac{x+y}{2}\right\| + (1-\alpha)\left\|\frac{y}{2}\right\|$$

$$\frac{\varepsilon^2}{8} \leq \frac{1}{2} \Rightarrow \alpha\left(1 - \frac{\varepsilon^2}{8}\right) + (1-\alpha)\left(1 - \frac{1}{2}\right) \leq 1 - \frac{\varepsilon^2}{8}$$

Proof

Theorem. Let $A \subseteq \mathbb{S}^{n-1}$ be a measurable set with $\mathbb{P}(A) \geq 1/2$ then[3]

$$\forall \varepsilon > 0,\, \mathbb{P}(A_\varepsilon) \geq 1 - 2e^{-\varepsilon^2 n/4}$$

By the previous lemma we deduce that

$$\frac{1}{2}(\tilde{A} + \tilde{B}) := \left\{ \frac{\tilde{x} + \tilde{y}}{2} : \tilde{x} \in \tilde{A}, \tilde{y} \in \tilde{B} \right\} \subseteq \mathbb{B}^n(1 - \varepsilon^2/8)$$

Hence we deduce that

$$\mu\left(\frac{1}{2}(\tilde{A} + \tilde{B})\right) = \frac{\mathrm{Vol}(\mathbb{B}^n(1 - \varepsilon^2/8))}{\mathrm{Vol}(\mathbb{B}^n(1))} = \left(1 - \frac{\varepsilon^2}{8}\right)^n$$

Applying Brunn-Minkowski we deduce that

$$\sqrt{\mu(\tilde{A})\mu(\tilde{B})} = \frac{\sqrt{\mathrm{Vol}(\tilde{A})\mathrm{Vol}(\tilde{B})}}{\mathrm{Vol}(\mathbb{B}^n(1))} \leq \frac{\mathrm{Vol}(\frac{1}{2}(\tilde{A} + \tilde{B}))}{\mathrm{Vol}(\mathbb{B}^n(1))} = \mu\left(\frac{1}{2}(\tilde{A} + \tilde{B})\right)$$

By definition of $\mathbb{P}(.)$ and the requirement that $\mathbb{P}(A) \geq 1/2$ we deduce

$$\sqrt{\frac{1}{2}\mathbb{P}(B)} \leq \sqrt{\mathbb{P}(A)\mathbb{P}(B)} = \sqrt{\mu(\tilde{A})\mu(\tilde{B})} \leq \mu\left(\frac{1}{2}(\tilde{A} + \tilde{B})\right) \leq \left(1 - \frac{\varepsilon^2}{8}\right)^n$$

By transitivity of inequality we deduce that

$$1 - \mathbb{P}(A_\varepsilon) = \mathbb{P}(\mathbb{S}^{n-1} \setminus A_\varepsilon) = \mathbb{P}(B) \leq 2\left(1 - \frac{\varepsilon^2}{8}\right)^{2n} \leq 2e^{-\varepsilon^2 n/4}$$

$$\forall \varepsilon > 0,\, \mathbb{P}(A_\varepsilon) \geq 1 - 2e^{-n^2\varepsilon^2/4} \quad \square$$

Corollary. Asymptotic Concentration on regions of significant area
As $\mathbb{P}(.)$ is a probability measure, then by the squeeze theorem we deduce that

$$\forall \varepsilon > 0,\, 1 - 2e^{-n^2\varepsilon^2/4} \leq \mathbb{P}(A_\varepsilon) \leq 1 \Rightarrow \mathbb{P}(A_\varepsilon) \to 1$$

Corollary. Concentration at equators of hyperspheres We can define the northern hemisphere as (and the southern hemisphere S as its compliment):

$$N := \{x \in \mathbb{B}^n : x_n \geq 0\}$$

Clearly both the northern hemisphere and the southern hemisphere occupy at least half the sphere. By **Theorem** we know that concentration must occur at each hemisphere. It follows that it also occurs at their osculation: the equator.

We can prove this formally by considering neighborhoods around the equator to include the concentrated areas on each 'side'.

Further Remarks

1. **Motivation for the definition of** $\mathbb{P}(.)$ **-** For $n = 2$ when we have $\mathbb{S}^1$ (a circle), A is an arc length. We can see that $\mathrm{Vol}(\tilde{A})$ gives us the area of a sector. $\mu(\tilde{A})$ gives us the probability that we fall inside the sector if we sample from the ball, which intuitively is the same as the probability of falling in the arc length if you sample from the arc sphere.

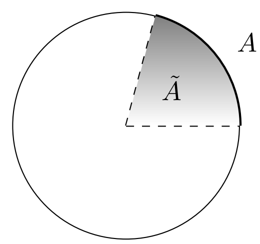

Figure 2: Relation between sector and arc length probabilities

2. If $\varepsilon > 1$ then as the radius of the sphere is 1 then $A_\varepsilon = \mathbb{S}^{n-1}$, $\mathbb{P}(A_\varepsilon) = 1$.
3. **Stronger statements of Theorem -** With more advanced methods we can show that

$$\mathbb{P}(A_\varepsilon) \geq 1 - 2e^{-\varepsilon^2 n/2}$$

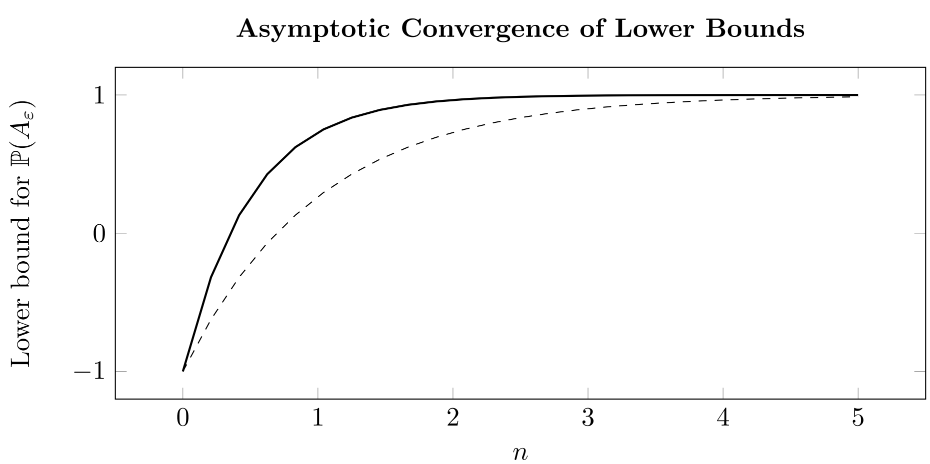

Figure 3: Both bounds clearly converge to 1, but the new one is faster.

References

[1] Jiri Matousek. *Lectures on discrete geometry*, volume 212. Springer Science & Business Media, 2013.