Now that we have developed the whole set of Equations that describe E+M,

$$\nabla \cdot \vec{E} = \rho/\epsilon_0 \qquad \nabla \cdot \vec{B} = 0$$

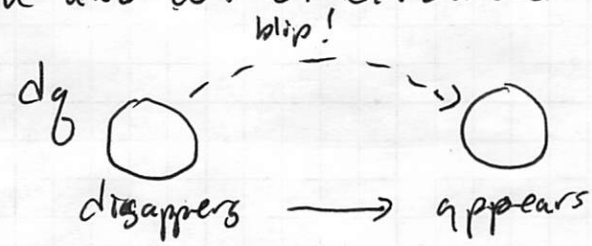
$$\nabla \times \vec{E} + \frac{d\vec{B}}{dt} = 0 \qquad \nabla \times \vec{B} - \epsilon_0 \mu_0 \frac{d\vec{E}}{dt} = \mu_0 \vec{J}$$

We will explore conservation laws in E+M.

You learned about one such conservation law → conservation of electric charge. (but there are others)

<u>Globally</u>: the total charge in the universe doesn't change.

It turns out this is a relatively weak statement because if it were all we know, nothing stops us from positing that charges can "blip" in and out of existence.




This would conserve charge but is not how the world works!

<u>Locally</u>: If a charge leaves a volume, it <u>must</u> flow past the boundary. (stronger statement)


We've expressed this at a point using the continuity equation $\frac{d\rho}{dt} = -\nabla \cdot \vec{J}$

increase in charge/volume = $-$ (outflow of current density)

For a volume:

$$\underbrace{\frac{dQ}{dt} = \frac{d}{dt}\iiint_V \rho \, d\tau}_{} = \underbrace{-\iiint_V \nabla\cdot\vec{J}\, d\tau = -\oiint_S \vec{J}\cdot d\vec{A}}_{} = -I_{out}$$

increase of $\frac{\text{charge}}{\text{time}}$ = −(outflow of current)

So we have both global & local statements of charge conservation. Are there other local conservation laws? I think that should expect:

- Energy (we will focus on this)
- Momentum (Discuss this)
- Angular Momentum (Touch on this)

In general, "conservation of $X$" means that

$$\frac{dX}{dt} = -\nabla\cdot\left(\begin{array}{c}\text{volume flow of a}\\ \text{current associated w/ } X\end{array}\right)$$

---

Reminders about Energy

① Stored Electrical Energy $\quad W_e = \frac{1}{2}\varepsilon_0 \iiint E^2 d\tau$

- work (energy) required to assemble charges to build this E field.

Electrical Energy Density $\quad w_e = \frac{1}{2}\varepsilon_0 E^2$ (energy/volume stored in E field at a point)

② Stored Magnetic Energy $\quad W_B = \frac{1}{2\mu_0}\iiint B^2 d\tau$

- work (energy) required to get currents flowing (against back EMFs) to build this B field.

Magnetic Energy Density $\quad w_B = \frac{1}{2\mu_0} B^2$ (energy/volume stored in a B field at a point)

So the total Energy is given by,

$$U_{tot,EM} = \iiint \left(\tfrac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2\right) d\tau = \text{total stored E\&M energy in fields}$$

or

$$u_{tot,EM} = \tfrac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 = \text{stored local E\&M energy/volume}$$
$$\equiv \text{"energy density"}$$

For a statement of conservation of Energy, we are looking for a relation that looks like,

$$\frac{d}{dt}(\text{energy density}) = -(\text{outflow/vol of some energy current})$$
$$= -\nabla\cdot(\text{"energy current density"})$$

So we are going to try to figure out what this ↑ is.

---

- Consider some general situation with charges and currents that produce "general" $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ fields throughout space. We will zoom in on a charge "$dq$" that is moving with a velocity $\vec{v}$ at a time $t$.
- the work done on this charge by the fields is,

$$dW_q = \vec{F}_{on\,q}\cdot d\vec{l} = \underbrace{dq\,(\vec{E}+\vec{v}\times\vec{B})}_{\text{Force on } dq}\cdot\underbrace{\vec{v}\,dt}_{d\vec{l}}$$

The magnetic field does no work so that,

$$dW_q = dq\,\vec{E}\cdot\vec{v}\,dt \quad \text{thus the work per unit time,}$$

$$\frac{dW_q}{dt} = dq\,\vec{E}\cdot\vec{v} = \underbrace{(\rho\,d\tau)}_{dq}\Big(\vec{E}\cdot\underbrace{\frac{\vec{J}}{\rho}}_{\rho\vec{v}=\vec{J}}\Big)$$

* Assume that we are Lorentz averaging here.

Thus, the energy per unit time is given by,

$$\frac{dW_q}{dt} = \vec{E}\cdot\vec{J}\,d\tau$$

For many charges we can express a global form,

<u>Globally</u>: $$\frac{dW_q}{dt} = \iiint_V \underbrace{(\vec{E}\cdot\vec{J})}_{}d\tau$$

this is the E+M power density (Joules / sec · m³)

We can also make a local statement that uses $u$, the energy density,

<u>Locally</u>: $$\frac{du}{dt} = \vec{E}\cdot\vec{J}$$ E+M work done on charged particles per unit volume.

---

- This is not expressed in the way we intended (yet) so let's explore a few vector manipulations to see if we can get it there.
- We will make use of Maxwell's Equations to reexpress the statement above in terms of $\vec{E}$ + $\vec{B}$ fields.

Start with $\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{d\vec{E}}{dt}$

$$\vec{J} = \frac{1}{\mu_0}\left(\nabla\times\vec{B} - \mu_0\varepsilon_0\frac{d\vec{E}}{dt}\right)$$

so, $$\vec{E}\cdot\vec{J} = \frac{\vec{E}\cdot(\nabla\times\vec{B})}{\mu_0} - \varepsilon_0\vec{E}\cdot\frac{d\vec{E}}{dt}$$ (call this eqn.(a))

Here's an (inobvious) step,

$$\nabla\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{B})$$ (math from product rule #6 in Griffiths)

the first term in (a) is,

$$\vec{E}\cdot(\nabla\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{E}) - \nabla\cdot(\vec{E}\times\vec{B})$$

From Faraday's Law, we know $\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$,
so we have,

$$\vec{E}\cdot\vec{J} = \frac{\vec{E}\cdot(\nabla\times\vec{B})}{\mu_0} - \varepsilon_0 \vec{E}\cdot\frac{d\vec{E}}{dt}$$

$$= \frac{\vec{B}\cdot(\nabla\times\vec{E}) - \nabla\cdot(\vec{E}\times\vec{B})}{\mu_0} - \varepsilon_0 \vec{E}\cdot\frac{d\vec{E}}{dt}$$

$$\vec{E}\cdot\vec{J} = \frac{-\vec{B}\cdot\frac{d\vec{B}}{dt}}{\mu_0} - \varepsilon_0 \vec{E}\cdot\frac{d\vec{E}}{dt} + \frac{\nabla\cdot(\vec{E}\times\vec{B})}{\mu_0}$$

Now, here's a second (inobvious) step,

$$\frac{d}{dt}\vec{A}^2 = 2\vec{A}\cdot\frac{d\vec{A}}{dt}$$ → comes from the chain rule (you can prove this!)

so, $\vec{E}\cdot\frac{d\vec{E}}{dt} = \frac{1}{2}\frac{d}{dt}(E^2)$ and $\vec{B}\cdot\frac{d\vec{B}}{dt} = \frac{1}{2}\frac{d}{dt}(B^2)$

thus,

$$\vec{E}\cdot\vec{J} = -\frac{1}{2\mu_0}\frac{d}{dt}B^2 - \frac{1}{2}\varepsilon_0\frac{d}{dt}E^2 - \frac{1}{\mu_0}\nabla\cdot(\vec{E}\times\vec{B})$$

or,

$$\vec{E}\cdot\vec{J} = -\frac{d}{dt}\left(\frac{B^2}{2\mu_0} + \frac{\varepsilon_0}{2}E^2\right) - \nabla\cdot\underbrace{\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right)}$$

this term has a name $\vec{S} \equiv \frac{1}{\mu_0}\vec{E}\times\vec{B}$ (poynting vector)

Let's put this all together,

$$\frac{dW}{dt} \equiv \iiint \vec{E}\cdot\vec{J}\,d\tau = -\frac{d}{dt}\iiint\left(\frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right)d\tau - \iiint(\nabla\cdot\vec{S})\,d\tau$$

- the first term is $U_{em}$; the second can make use of the divergence theorem.

$$\frac{dW}{dt} = -\frac{d}{dt}U_{em} - \oint\!\!\oint \vec{S}\cdot d\vec{A}$$

So our statement of energy conservation globally is,

$$\frac{dW}{dt} = -\frac{d}{dt}U_{em} - \oint\!\!\oint \vec{S}\cdot d\vec{A}$$

① ② ③

In words,

① = work done on charges by EM fields

= ② decrease in energy stored in the fields minus

③ whatever energy flowed across the boundary

<u>Does this make sense?</u>

If no energy flows across the boundary (if ③ = 0),

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt}$$ increase in particle energy = loss of stored field energy

seems ok. just energy conservation.

If ③ ≠ 0, there's another mechanism to feed energy to the particles, through $\vec{S}$.

$\vec{S}$ is the outflow of energy so <u>negative</u> outflow (inflow) yields <u>positive</u> work on charges.

$\vec{S} = \dfrac{\text{energy flow}}{\text{per unit time (+ area)}}$ transported by $\vec{E}$ & $\vec{B}$ $= \dfrac{\vec{E}\times\vec{B}}{\mu_0}$

A local statement of energy conservation looks at densities,

<u>Locally:</u> $\dfrac{du_q}{dt} = \vec{E}\cdot\vec{J} = -\dfrac{\partial}{\partial t} u_{em} - \nabla\cdot\vec{S}$

this is poynting's theorem (derived in 1884)

We can reorganize this statement,

$$\frac{\partial}{\partial t}\left(u_q + u_{em}\right) = -\nabla\cdot\vec{S}$$

this is Griffith's $u_{mech}$, particle's energy density (could be complicated KE obviously & thermal and other forms of PE)

this is the energy density of the $\vec{E}$ & $\vec{B}$ fields

this is the $\dfrac{\text{outflow}}{\text{volume}}$ of energy current.
$\vec{S} = \vec{E}\times\vec{B}/\mu_0$

the statement,

$\dfrac{\partial}{\partial t}\left(u_q + u_{em}\right) = -\nabla\cdot\vec{S}$ is our classic conservation law structure

$$\frac{\partial}{\partial t}(\text{something}) = -\nabla\cdot\left(\text{that something's associated current density}\right)$$

$\vec{S}$ energy current ~~den~~ density $= \dfrac{\text{flow of energy}}{\text{sec}\cdot\text{m}^2}$

Compare this to,

$\dfrac{\partial}{\partial t}(\rho) = -\nabla\cdot\vec{J}$ $\quad \vec{J} = \dfrac{\text{flow of charge}}{\text{sec}\cdot\text{m}^2}$

<u>Globally</u>: (integrating over a volume) we get back to

$$\frac{d}{dt}\iiint (u_q + u_{em})\, d\tau = -\iiint \nabla\cdot\vec{S}\, d\tau = -\oiint \vec{S}\cdot d\vec{A}$$

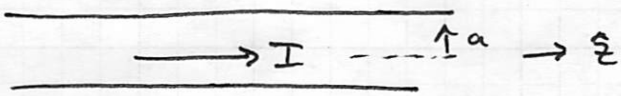
rate of increase of <u>all</u> energy = − (outflow of energy/second)

<u>Side note</u>:

In materials $\vec{S} = \vec{E}\times\vec{H}$ and

$$u_{em} = \frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{H}\cdot\vec{B}$$

Example: Steady current in a wire



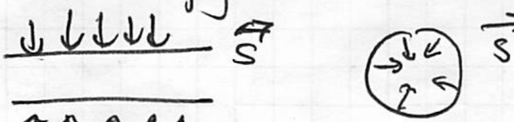

Consider a long wire with a steady current.

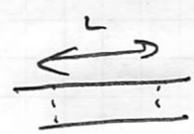
We know that $\vec{E} = E_0 \hat{z}$ and $\vec{J} = \sigma \vec{E} = \sigma E_0 \hat{z}$

As we have done in the past $\vec{B}_{inside} = \frac{\mu_0 J \pi r^2}{2\pi r}\hat{\varphi} = \frac{\mu_0 \sigma E_0}{2} r \hat{\varphi}$

At the edge $\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} = \frac{\sigma E_0^2}{2} a \underbrace{(\hat{z} \times \hat{\varphi})}_{-\hat{s}}$

So the energy flows inwards!




Consider some length of wire, L

Across this length, $\Delta V = E_0 L$ and $I = J\pi a^2 = \sigma E_0 \pi a^2$

So, with $\frac{d}{dt}(W + U_{em}) = -\oint\!\!\oint \vec{S} \cdot d\vec{A}$

$U_{em}$ is steady b/c neither $\vec{E}$ nor $\vec{B}$ change with time,

$\frac{dU_{em}}{dt} = 0$ so,

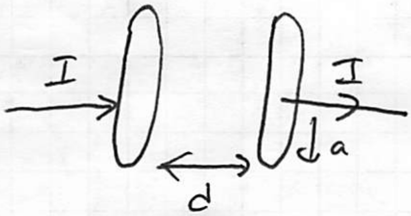
$$\frac{dW}{dt} = -\oint\!\!\oint \vec{S} \cdot d\vec{A} = -\frac{\sigma E_0^2}{2} a (\hat{s}) \cdot (\underbrace{2\pi a L}_{\text{outer area}} \hat{s})$$

(note: end caps don't contribute)

$$= +\underbrace{(\sigma E_0 \pi a^2)}_{\text{current, } I}\underbrace{(E_0 L)}_{\text{potential diff, } \Delta V}$$

The total power entering the wire is $P = I\Delta V$! as we've always said. It enters via the fields! It's converted to $W (U_{mech}) \rightarrow$ thermal energy.

Example: A slowly (quasi-static) charging capacitor




We are going to investigate the energy as the capacitor charges up.

with $d \ll a$,
By Gauss' Law $\vec{E} = \frac{Q}{A\varepsilon_0}\hat{z}$ (and zero outside, right?)

By the Maxwell-Ampere Law, the magnetic field due to the wire is,

$$\oint \vec{B}(s)\cdot d\vec{l} = \mu_0 I \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\varphi}$$

At the edge of the capacitor we showed that

$$\oint \vec{B}(s)\cdot d\vec{l} = \mu_0 \varepsilon_0 \iint \vec{J}_D \cdot d\vec{A} \quad \text{gave us}$$

$$\vec{B}(a) = \frac{\mu_0 I}{2\pi a}\hat{\varphi}$$ so the fields match there! remember?

At the edge of the capacitor ($s=a$),

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\frac{Q}{A\varepsilon_0}\frac{\mu_0 I}{2\pi a}\underbrace{[\hat{z}\times\hat{\varphi}]}_{-\hat{s}}$$

so,

$$\vec{S} = \frac{-Q}{A\varepsilon_0}\frac{I}{2\pi a}\hat{s}$$

energy flows in as we charge!

The total energy out / time is,

$$\oiint \vec{S}\cdot d\vec{A}$$

this integral is taken in cylindrical coordinates just outside the capacitor.

$d\vec{A} = s\,d\phi\,dz\,\hat{s}$ (area points outward)

at surface of capacitor edge $s=a$

$$\oiint \vec{S}\cdot d\vec{A} = \frac{QI}{2\pi\varepsilon_0 aA}(-\hat{s})\cdot \underbrace{(2\pi a d)}_{\text{just the outer area}}\hat{s}$$

$$= -\frac{QI}{\varepsilon_0}\frac{d}{A}$$

So the energy flows <u>into</u> the capacitor from external fields.

The stored energy between the plates is

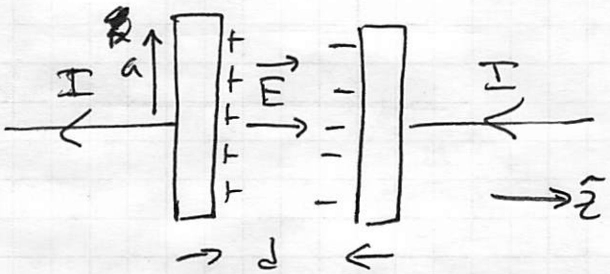
$$U_{em} = \left(\tfrac{1}{2}\varepsilon_0 E^2\right)\text{Volume} = \tfrac{1}{2}\varepsilon_0\left(\frac{Q}{A\varepsilon_0}\right)^2 (Ad)$$

↑ constant field.

So $$\frac{dU_{em}}{dt} = \frac{2Q}{2\varepsilon_0}\frac{dQ}{dt}\frac{d}{A} = \frac{QI}{\varepsilon_0}d/A$$ which is $\oiint \vec{S}\cdot d\vec{A}$!

increase of stored energy / sec $=$ $\frac{\text{flow of energy in}}{\text{sec.}}$

## Example: A discharging Capacitor




A capacitor is connected to very long leads. It has a circular cross section, radius, a, and a separation, d. with $d \ll a$.

We intend to find $\vec{S}$ to see how the energy is transported.

Between the plates $\vec{E} = \dfrac{Q}{\pi a^2 \varepsilon_0}\hat{z}$ like usual for a capacitor.

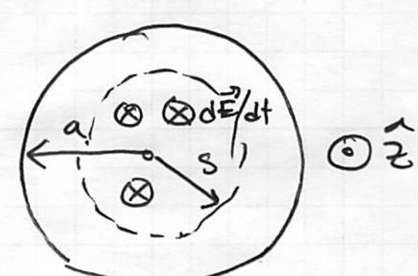
But now, $\dfrac{d\vec{E}}{dt}$ points in $-\hat{z}$! see why?

This also makes sense from a conservation of charge situation, $\dfrac{dQ}{dt} = -I$

Ok so we can compute $\vec{J}_D$,

$$\vec{J}_D = \varepsilon_0 \frac{d\vec{E}}{dt} = \frac{dQ/dt}{\pi a^2}\hat{z} = -\frac{I}{\pi a^2}\hat{z}$$

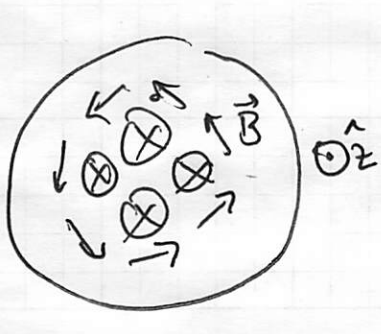
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \iint \vec{J}_D \cdot d\vec{A}$$

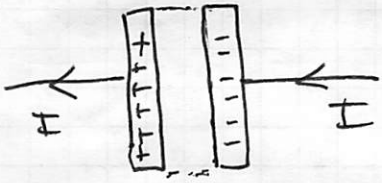



$$B 2\pi s = -\frac{\mu_0 I}{\pi a^2}\pi s^2$$

so, $\vec{B} = -\dfrac{\mu_0 I s}{2\pi a^2}\hat{\phi}$ circulates opposite our other example.

Makes sense b/c $d\vec{E}/dt$ points the other way.






$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

We will evaluate this at the surface of the dashed cylinder to see what the energy density current is doing.

$$\vec{S} = \frac{1}{\mu_0}\left(\frac{Q}{\pi a^2 \varepsilon_0}\hat{z} \times -\frac{\mu_0 I s}{2\pi a^2}\hat{\phi}\right)\Bigg|_{s=a}$$

$$= \frac{QIa}{2(\pi a^2)^2\varepsilon_0}\big(\underbrace{\hat{z}\times -\hat{\phi}}_{+\hat{s}}\big) = \frac{QIa}{2(\pi a^2)^2\varepsilon_0}\hat{s}$$

energy flows out of the region!

this is not Quasistatic! If RC circuit,

then $I(t) = \frac{V_0}{R}e^{-t/RC}$ and

$$Q(t) = CV_0\, e^{-t/RC} \qquad \text{note: } C = \frac{A\varepsilon_0}{d}$$

$$\vec{S} = \frac{\left(\frac{V_0}{R}\right)\left(e^{-t/RC}\right)\left(\frac{A\varepsilon_0}{d}V_0\right)\left(e^{-t/RC}\right)a}{2A^2\varepsilon_0}\hat{s}$$

$$\vec{S} = \frac{V_0^2}{R}\frac{a}{2Ad}e^{-2t/RC}\hat{s} \qquad \tau = \frac{RC}{2}$$

energy dissapation has time constant that is 1/2 that of I or Q.

B/c not Quasistatic

$$U_{em}(t) = \iiint \frac{\varepsilon_0}{2}E^2 d\tau + \iiint \frac{1}{2\mu_0}B^2 d\tau$$

is needed to find $dU_{em}/dt$!