

1 Introduction

Previously, we introduced a tool for exploring the molecular parameter space of plastic and molecular crystals called a global phase diagram (GPD) [?, ?]. Global phase diagrams summarize the phase behavior of a class of substances as a function either of parameters in an empirical equation of state or of parameters in an intermolecular potential. The classic example of a GPD of the first type is the classification scheme for high pressure vapor liquid phase equilibria by van Konynenburg [?]. Their classification was based on the van der Waals equation of state with simple binary mixing rules. Despite the crude equation of state employed, it is still widely used to classify the phase behavior of real binary mixtures. Our GPD’s are of the second type. They use an intermolecular potential constructed from a complete set of basis functions for rotational space for all molecules of a particular point group and a set of molecular center-of-mass packings inferred from experimental crystal structures. The parameters of the intermolecular potential become axes on GPD’s for each packing and the crystal structures are phase regions in the diagrams.

An important benchmark of GPD’s utility is to show that experimentally observed structures can be found and their intermolecular potential parameters can be read from the axes. Our goal is to determine the number of translational packings, or reference lattices, required in GPD construction and the number of potential parameters needed to find all experimental phases. A small number of each will disprove previous hypotheses that many parameters are needed [?] and enhance the usability of GPD’s in materials design.

The outline of the balance of the paper is as follows. In Sec. 2 we discuss the derivation of our data set, its chemical and crystallographic characteristics, and group entries based on structural similarity. In Sec. 3 we deduce reference lattices for each structure by pseudosymmetry detection and describe the fit of the structures to these lattices. Section 4 closes with a discussion of these results and their application to GPDs, which is the subject of a subsequent publication. [?]

2 Experimental Data Set

Since our molecular crystal global phase diagrams are constructed for molecules of a given molecular point group symmetry, we have chosen to use the *CSDSymmetry* database as our primary source of crystal structures [?]. This database summarizes the point groups of molecules that form error-free, nonpolymeric, nonionic, coordinate-determined molecular crystals in the CSD. Duplicates were removed from the database and hydrogen atoms were not considered when assigning point groups. While the methods introduced in our previous work [?] are applicable to disordered structures, which are systematically absent from *CSDSymmetry*, it is a convenient source of crystal data for molecules of a particular point group. Our methods are restricted to single component crystals, however. Therefore we worked with the single component crystal subset of the *CSDSymmetry* database. This was accomplished by first querying the CSD for all single component crystals using CONQUEST, the interface to the CSD, and then using CONQUEST to take the intersection of the two data sets.

Continuing the example begun in our prior work [?], we chose to consider crystals composed of molecules with T_d molecular point group symmetry. We have augmented the data from *CSDSymmetry* with a recently determined structure of the low temperature ordered phase of heavy methane [?]. The data set contains 71 crystal structures of 70 different chemical substances. Only carbon tetrachloride (CCl_4) appeared twice in different polymorphs. [CSD structures CARBTC [?] and CARBTC07 [?]]. Names and chemical formulas for all the entries are given in Table ?? . The chemical structures include 15 hydrocarbons and their substituted derivatives and 56 organometallics. The organometallics contain 30 different metals: Al, As, Bi, Cd, Co, Cs, Cu, Ga, Ge, Hf, In, Mg, Mn, Na, Ni, Np, Pb, Pt, Re, Rh, Ru, Sb, Si, Se, Sn, Tc, Th, Ti, U, and Zn. Eight types of molecular frameworks are present in the data set: Cubane (29), Adamantane (17), MX_4 (16), Tetrahedrane (3), and others (6). There are entries from all seven crystal systems in the data set. Twenty crystals are cubic (or isometric), one is hexagonal, five are trigonal, ten are tetragonal, six are orthorhombic, 22 are monoclinic, and seven are triclinic as shown in Fig. ?? . This distribution is different than the CSD as a whole [?], but that is not

surprising given the larger-than-average number of symmetries in the T_d point group. The data set is sufficient however to test whether molecular crystal global phase diagrams apply to hydrocarbons and organometallics, a variety of molecular frameworks, and all seven crystal systems.

One entry [CSD structure XUWROW [?]] is a very unusual structure containing 70% voids as recorded by the CCDC staff in its cif file in the CSD, presumably using the standard van der Waals radii in the documentation of CCDC software [?]. It was grown as a thin epitaxial crystal under ultra-high vacuum. Apparently the crystal structure is strongly influenced by the substrate and we exclude it from further consideration.

The remaining crystal structures were organized into groups that bear a strong "structural relation" as discussed in the International Tables for Crystallography (ITC) [?]. For crystals to belong to the same group they must have the same space group symmetry, cell lengths in similar proportions, similar cell angles, and molecular centers at equivalent Wyckoff point(s) with similar structural parameter values where applicable. We refer to these groups as sharing a particular distinct structure. Note that we do not require that the atomic positions be similar to be classified in the same distinct structure. For example, two structures [CSD structures DILWIE01 [?] and ZEYHIU [?]] crystallize in space group 165 with molecules at Wyckoff point d. Their cell parameters are also in similar ratios. Therefore, we classify them in the same distinct structure, despite their different chemical structures with different numbers of atoms.

Two structures in *CSDSymmetry* [CSD structures HMGETP [?] and JUFWUC [?]] have molecules that are tetrahedral within the default error bounds for detecting molecular symmetry, but are sufficiently distorted to influence their space group classifications (195a and 197a, respectively). Either the molecules are very slightly distorted in the crystal or the crystal symmetries are underspecified and actually belong to space groups 215 and 217, respectively. Although slight distortions could result from crystal Jahn-Teller distortions, for example, we adopt the later interpretation. Therefore these crystals have been grouped with distinct structures 215a and 217a, respectively. This assignment is discussed from an energetic point of view in Ref. [?].

The 70 crystal structures fell into 46 distinct structures. Five structures are cubic (or isometric), one is hexagonal, four are trigonal, eight are tetragonal, six are orthorhombic, sixteen are monoclinic, and six are triclinic. These distinct structures are further characterized in the following section.

3 Reference Lattice Assignments

Our global phase diagrams are constructed starting with a fully disordered plastically crystalline reference state. At sufficiently high temperature the intermolecular potential, which hinders free rotation, becomes weaker than the thermal energy. In this limit molecules are free to disorder and ultimately to freely rotate. Since the molecules are disordered they may occupy high symmetry Wyckoff points in high symmetry space groups. These are space groups with special values of the cell parameters ($a, b, c, \alpha, \beta, \gamma$) and molecules located at special values of the unit cell fractional coordinates (x, y, z). Each molecule has four or more generally equivalent non-coplanar nearest neighbors. This is needed for structural stability in three dimensions. We call these reference lattices.

High symmetry reference lattices may be determined for an experimental crystal structure by determining the space group formed from the lattice of the molecular centers of mass. These calculations were done using FINDSYM in the ISOTROPY software suite [?] with coordinates scaled by the minimum unit cell length $\{a, b, c\}$ and a large error tolerance setting (0.1). Initially it may seem unexpected that molecular crystals, the majority of which belong to monoclinic or triclinic space groups [?], should be slight distortions of high symmetry lattices. However, prior work in this area has shown this is not only possible but quite common [?, ?]. Standard unit cell choices for low symmetry crystals systems often hide the similarity of a suitably-chosen supercell to a reference lattice in a higher symmetry crystal system. This high translational symmetry is a result of the close-packing tendency of molecules. Broken symmetries are due to orientational ordering and anisotropic intermolecular potentials.

Occasionally a structure arises with fewer than four nearest neighbors per molecule. Due to the inherent lack

of structural stability of this arrangement, this suggests the crystal is in fact a lattice of molecular clusters, typically *dimers*, acting as a single entity [?]. In such a case the center of mass of the dimers is used in FINDSYM again with a larger error tolerance.

Given a list of common reference lattices, we developed an automated technique to identify the reference lattices of experimental structures. From a list of possible reference lattices, we superimpose an experimental structure on each possible lattice so one molecule in each structure overlaps. Then the experimental structure is rotated and isotropically scaled in order to minimize the sum of the absolute values between the center of each molecule of the experimental structure and the center of the closest reference lattice molecule,

$$\min_{a, \omega} \sum_m |R(a, \omega) \cdot \mathbf{X}_m^{\text{exp}} - \mathbf{X}_{\text{closest}}^{\text{ref}}|, \quad (1)$$

where $R(a, \omega)$ is a Cartesian rotation matrix multiplied by an isotropic scaling factor a (Appendix A), $\mathbf{X}_m^{\text{exp}}$ is the center of mass of a molecule in the experimental structure, $\mathbf{X}_{\text{closest}}^{\text{ref}}$ is the center of mass of the closest molecule in one of the possible reference lattices, and m sums over all molecules in the experimental structure, which is typically limited to the first 14 neighbors of the superimposed center molecule. This cutoff is justified by CSD studies [?] showing 14 to be the most common molecular coordination number. This technique was used in a computer program to identify the majority of the fcc, bcc, hcp, sc, and diamond reference lattices.

Applying these techniques to our data set gives the reference lattice assignments in Tables ??- ???. Generally we name reference lattices first by their common name (*i.e.* fcc) if available, then by their Strukturbericht designation (*i.e.* A15) if available, and then by their space group/filled Wyckoff positions if necessary (*i.e.* 70a). The raw atomic positions, symmetry operators, etc. contained in the cif and fdat data files from the CSD were processed using the cctbx collection of algorithms [?]. Algorithms from the ISOTROPY software suite [?] and the Bilbao Crystallographic Server [?] were also used to identify group-subgroup relations, track Wyckoff point evolution, etc. In the following subsections we review the reference lattices for the structures in the experimental data set.

3.1 Cubic (Isometric) Space Groups

The cubic crystal structures are summarized in Table ?? . Each structure is listed as a space group number followed by its occupied Wyckoff points. A representative of each structure is named in the next column. Under each representative structure is the reference lattice. In the third column is shown the ratio $|G|/Z$. This is the number of symmetry operations $|G|$ per molecules in the unit cell Z . We call this ratio the symmetry density of a crystal structure. Defined in this way, the symmetry density is independent of the choice of unit cells. For instance the primitive and conventional fcc cells both have $|G|/Z = 48$. The symmetry density is a maximum of 48 for fcc, bcc, and sc sphere packings. Symmetry breaking, by loss of a class of symmetry operations and/or loss of translations, reduces the symmetry density. The index $[i]$ of a group-subgroup relation is the ratio of $|G|/Z$ of the group and its subgroup. Thus, the geometric interpretation of the index is the amount by which the symmetry of the crystal is diluted by symmetry breaking (ITC, p726) [?]. The number of entries bearing a strong structural similarity to the representative is shown in the fourth column. For example, structure 217a has space group 217 with molecules situated at Wyckoff point a. Its representative is DEQPAQ and in a disordered state the crystal has the bcc symmetry or reference lattice. Its symmetry density is 24 and that of its reference lattice 48. Therefore, its index in bcc is 2. There are 11 tetrahedral molecular crystals in *CSDSymmetry* with this structure.

The 20 crystal structures in Table ?? fall into five distinct structural types listed in order of decreasing symmetry density. Structure 227a is unique in that the Wyckoff point symmetry in the reference lattice is consistent with T_d molecular point group symmetry. Therefore the (ordered) experimental structure has the full symmetry of the (disordered) reference lattice. Structures 217a and 215a are subgroups of bcc and sc, both of index two. The molecular symmetry (T_d) is a subgroup of the Wyckoff site symmetry in the reference lattice (O_h). Thus the crystal symmetry is reduced by molecular ordering. Structure 218a,c is similarly a subgroup of index two of A15, but the molecules on the two occupied Wyckoff sites are inequivalent, even in the reference lattice. For each of the preceding structures,

there are no arbitrary cell parameters or structural parameters. Thus the molecular centers of mass coincide exactly with the reference lattice.

Structure 205c has one arbitrary parameter, x , for its occupied Wyckoff point. This is called a Wyckoff structural parameter. Figure ?? shows the neighbor distances as a function of the structural parameter. The representative experimental structure [tetracarbonyl-nickel CSD structure FOJBUB02 [?]] has molecules at Wyckoff point c with a structural parameter $x = 0.122$ which is approximately midway between the sc limit ($x = 1/4$) and a doubly occupied fcc limit ($x = 0$ or $1/2$). It is evident from the figure that the crystal structure is equivalent for structural parameters x and $1/2 - x$. Since the neighbor distances at the experimental values of the structural parameter are qualitatively different than in the sc lattice, the simple cubic reference lattice is not acceptable for our purposes. The crystal is best viewed as a dimer packing with dimer centers of mass on an fcc lattice as shown in Fig. ?? for a dimer situated on the $(1/2, 1/2, 0)$ fcc coordinate [?]. Its three-fold axis is parallel to the body diagonal direction. The dimer has point group symmetry D_{3d} so we would expect to identify this structure on a D_{3d} global phase diagram using an fcc reference lattice. This possibility is not pursued further here.

3.2 Hexagonal and Trigonal Space Groups

The six structures with hexagonal or trigonal space groups fall into five structural types summarized in Table ?. The c/a ratio is given for the experimental lattice in the third column. The c/a ratio for the reference lattice is also given. These are the ratios of the experimental structure embedded in the reference lattice. Thus $c/a=1.22$ for the bcc reference lattice row of the TCYMET structure entry is not c/a of the bcc crystal but c/a of TCYMET embedded in the bcc reference lattice. This embedding is shown in Fig. ?? in which the lattice constant of the bcc crystal, a' , is compared to the lattice constants a and c of TCYMET. In each case in Table ? the arbitrary parameters differ by only a few percent from the ideal reference lattices. All five reference lattices are sphere packings (hcp, bcc, or fcc).

In contrast to the cubic lattices, all of the hexagonal and trigonal structures in the data set have molecules at

Wyckoff points with variable Wyckoff structural parameters, so their value is given in the fourth column. Below these are shown the structural parameters of the reference lattices. Again the agreement between the translational arrangements of molecules in a crystal with a high-symmetry reference lattice is quite good.

3.3 Tetragonal Space Groups

Tetragonal structures are summarized in Table ???. We adopt the unique c-axis orientation for tetragonal space groups. The ten structures fall into eight structural types [?]. Structure 142a contains eight molecular centers in the unit cell. Their molecular centers differ by only 1% from two conventional fcc cells stacked in the z-direction. The other structures are less trivially related to their reference lattices and are tetragonally distorted to various degrees.

With the exception of structure 142a, discussed above, each of the tetragonal structures is a sub-group of structures 141a, 139a, or 123a. Figure ?? gives the neighbor distances for tetragonal distortions of these lattices. Note that there is a tetragonal distortion that transforms fcc and bcc into one another. As a result, structures 137b, 121a, 142a, 114a, and 88f could not be uniquely assigned to a reference lattice solely based on their group/sub-group relationships. For these structures, the c/a ratio provides the requisite additional information. The experimental c/a ratios are marked with circles in Fig. ?. There appear to be fewer than nine circles in the figure because several of them overlap. Note that the experimental structures cluster around the sphere packings (marked with dashed lines) or distorted lattices with equidistant neighbors (marked with dotted lines). For instance, the diamond lattice can be tetragonally distorted to yield six equidistant second-nearest neighbors (A5' in Fig. ?? for $c/a = 2/\sqrt{7}$) or eight equidistant nearest neighbors (A5'' in Fig. ?? for $c/a = 2\sqrt{3}$). The Aa reference lattice is a distorted bcc structure and has ten equidistant nearest neighbors for $c/a = \sqrt{2/3}$. The A6 reference lattice at $c/a = \sqrt{6}$ is a reference lattice for other molecules discussed in later sections. Also there is an additional dimer structure, 88f, among them.

3.4 Orthorhombic Space Groups

Orthorhombic structures are summarized in Table ??.

We have adopted the C-centered setting for end centered unit cells. Each of the six structures constitutes its own distinct structural type. Two structures, 64d,f and 60c,d very nearly match the fcc and bcc reference lattices respectively. Structure 19a is identified with the 63c reference lattice which is an orthorhombic structure with 10 nearest neighbors and equal c and a lengths. The $b/a=b/c$ ratio is $\sqrt{5} + 2\sqrt{6}$ and the structural parameter for Wyckoff point c ($0,y,1/4$) is $y = (\sqrt{6} - 1)/4$ [?]. The small discrepancies in the 19a x and y structural parameters in this case indicate that the molecules have further relaxed from their reference lattice locations. The three structures in space group 62 provide additional challenges, discussed below.

Whereas the structures in the data set from the higher crystal classes are uniquely identified by their space group and Wyckoff point(s), there are three distinct orthorhombic structures in space group 62 with molecules at Wyckoff point c . Therefore, 62c is insufficient as an identifier for these structures. Since there are two arbitrary cell length parameters and two structural parameters for 62c, it is impractical to plot the neighbor distances as a function of arbitrary parameters as in Fig. ?. Instead we note that Wyckoff point c in space group 62 has four equivalent positions, two at $y = 1/4$ and two at $y = 3/4$. Therefore we view the 3-dimensional structure as a set of layers in the xz -plane uniformly stacked in the y -direction. Then the c/a ratio and the structural parameters determine the symmetry of the layers. CSD structure GUTCED [?] is composed of alternating layers that are nearly on a square lattice. Based on the stacking of these layers this is a slightly distorted fcc lattice. CSD structure RIMMOP [?] has alternating rectangular layers, which is more similar to the bcc reference lattice. The layers in CSD structure JEYSEL are nearly hexagonal and stacked vertically, rather than alternating. Therefore, it is associated with the simple hexagonal (sh) reference lattice. Thus the three structures in space group 62 are associated with different reference lattices.

3.5 Monoclinic Space Groups

Monoclinic structures are shown in Table ???. We adopt the unique b-axis orientation, cell choice one, origin choice two, and C centering for end centered lattices. All three lattice parameters and one lattice angle are able to relax so we indicate the closeness of the match for b/a , c/a , and β . The 22 structures fall into 16 distinct structural types [?]. In dividing into these types it is important to note that monoclinic space groups have multiple nonstandard settings. Also, during symmetry breaking from a high-symmetry parent phase to a monoclinic child phase, multiple domains may form creating unit cells that at first examination appear to be dissimilar. An example is CSD structure REKYUB and CSD structure BOGMEP. In a standard setting the important unit cell parameters are $(b/a, c/a, \beta) = (0.48, 1.00, 123)$ and $(0.28, 0.55, 145)$, respectively. The molecular positions also appear different. However, these are simply different domains of the same symmetry-breaking pathway with equivalent unit cell specifications. The phase transition is governed by the same IR, L_3^- , and order parameter direction, P7, using the notation of Stokes and Hatch [?]. Thus we group them in the same structural class.

Monoclinic structures have a greater diversity of reference lattices although those of 9 of the 16 structures are still canonical sphere packings. A good example of this hidden order in these low-symmetry groups is 15f,f,f,f which has 32 molecules per unit cell and four molecules per asymmetric unit, but still nearly packs in an fcc lattice.

Many of the structures belong to the same space group but have different spatial arrangements and so are grouped differently. This is particularly apparent for space groups 14 and 15, two of the most populous space groups in the CSD. A5', the distorted diamond lattice, fits two of the monoclinic structures, and simple hexagonal two more. One new packing, 70, is an orthorhombic distorted diamond lattice and has eight molecules at Wyckoff point a. The other new packing, 136, is a tetragonal structure with dimers of molecules at Wyckoff point f. Three additional dimer structures (TMSIAD, CAMPOV, MXSNOX) have dimers centered at high symmetry Wyckoff points without structural parameters.

3.6 Triclinic Space Groups

Triclinic structures are shown in Table ???. All lattice parameters are variable and are listed. The six structures fall in seven distinct structural types with four reference lattices [?]. There are two new reference lattices. A6 is a tetragonal distortion of fcc discussed in Sec. 3.3 and shown in Fig. ??. Ai is a rhombohedral distortion intermediate between sc and bcc. Its primitive cell a , b , and c parameters are all the same length and $\cos(\alpha) = \cos(\beta) = \cos(\gamma) = -1/4$. Two dimer structures in 136 and Ai also exist, the latter with dimer complexes at a high symmetry Wyckoff point so no structural parameters are listed in Table ?? for XAGXAE.

3.7 Reference Lattice Summary

The results of the reference lattice assignments of the experimental structures are summarized in Table ??. The Strukturbericht designation [?], Pearson symbol [?], and common name are shown when known for each reference lattice. The Strukturbericht designations beginning with an A indicate the reference lattices are also the crystal structures of various elements. The space group, occupied Wyckoff positions, and percentage of experimental structures that pertain to each lattice are also shown.

The fit of each experimental structure to its reference lattice is summarized in Fig. ??. In each box the experimental lattice vectors, angles, or Wyckoff positions are compared with their idealized values when inscribed in the corresponding reference lattice. These figures are a visual summary of Tables ??-??. If the fit were perfect all data points would lie directly on the line. The fit is quite good indicating that most low symmetry molecular crystals have molecules situated at or near lattice sites of high symmetry space groups. In the inorganic crystal structure database (ICSD), which is largely composed of atomic crystals, the most numerous structures are fcc, bcc, hcp, etc. [?] Originally this was considered to be distinct from the CSD which had primarily monoclinic and triclinic structures. Figure ?? challenges this distinction by showing molecular centers of mass, however, are nearly situated on lattice points of these archetypical high symmetry lattices so prevalent in the ICSD. The comparative number of monomer and dimer structures

is shown in Fig. ?? as is the breakdown of each into types of reference lattices. It is apparent that although there are 15 reference lattices, 76% of the structures are well represented by only 5 reference lattices.

The dimer structures, reference lattices, and dimer complex point group symmetries are listed in Table ?. Such structures can usually be identified by examining nearest neighbor distances. The presence of three or fewer nearest neighbors is evidence of dimerization. Many of the structures form D_{3d} or nearly D_{3d} dimers as illustrated in Fig. ?, the one exception being CSD structure LUFYEQ [?] whose molecules form C_{2v} complexes.

With the three techniques we have introduced here, (1) dimer screening by nearest neighbor inspection, (2) reference lattice assignment by large tolerance molecular center symmetry determination, (3) reference lattice recognition through template matching, one may determine translational packings or reference lattices for all structures in the CSD. While the second step has generally been done by hand, the first and third step have been automated in Python and MATHEMATICA scripts and are available upon request.

4 Discussion and Conclusion

We have shown that all of the single-component tetrahedral molecules in the CSD can be related to a reference lattice, or a high-symmetry starting point for GPD calculations. These reference lattices are generally high symmetry sphere packings or distortions thereof. These results also show the centers of mass of molecular crystals are not structurally distinct from atomic crystals, which also pack in high symmetry sphere packings such as fcc, bcc, hcp, sc, and so on [?]. From our experiences with tetrahedral molecules we hypothesize that the prevalence of low symmetry space groups for molecular crystals in the CSD [?] is due to the orientational ordering of molecules whose centers of mass are close to the lattice points of a higher symmetry space group. This reflects the close packing tendency of molecules [?].

We have shown many structures in our data set can be related to dimers situated on a high symmetry lattice. Understanding the prevalence of dimer structures in the CSD is important when designing configurational

search algorithms for crystal structure prediction. Also, since most crystal structures in our data set belong to only a few translational packings, or reference lattices, fewer GPD diagrams are necessary to describe crystallographic structural space for a given point group of molecules.

In conclusion, a molecular crystal global phase diagram (GPD) can summarize the experimental data using a modest number of reference lattices and IP parameters. In previous work [?, ?], we somewhat arbitrarily chose a single reference lattice (fcc) and truncated the intermolecular potential with three terms to illustrate the method. Here, we have used an experimental data set of crystals of tetrahedral molecules to determine reference lattices and an IP truncation sufficient to produce the observed phases. The data set is diverse enough to test the GPD’s ability to classify a wide range of space groups using a common intermolecular potential. Just as the van Konynenburg global phase diagram classification based on the simple van der Waals equation of state is nonetheless widely used to classify the phase behavior of real binary mixtures, molecular crystal global phase diagrams may be useful in elucidating phase behavior of a variety of real substances and, in turn, used to develop novel intermolecular potentials and materials.

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A Cartesian Rotations

The cartesian rotation matrix,

$$\mathbf{R}(a, \boldsymbol{\omega}) = a \begin{pmatrix} \cos(\alpha) \cos(\beta) \cos(\gamma) - \sin(\alpha) \sin(\gamma) & \cos(\beta) \cos(\gamma) \sin(\alpha) + \cos(\alpha) \sin(\gamma) & -\cos(\gamma) \sin(\beta), \\ -\cos(\gamma) \sin(\alpha) - \cos(\alpha) \cos(\beta) \sin(\gamma) & \cos(\alpha) \cos(\gamma) - \cos(\beta) \sin(\alpha) \sin(\gamma) & \sin(\beta) \sin(\gamma), \\ \cos(\alpha) \sin(\beta) & \sin(\alpha) \sin(\beta) & \cos(\beta) \end{pmatrix} \quad (2)$$

is produced by converting spherical coordinate Wigner functions to cartesian coordinates.