The Laplace Transform

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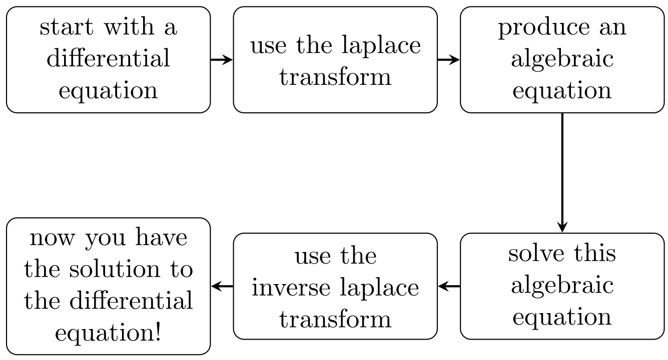
1 The Basic Idea



2 The Laplace Transform

2.1 Definition

Suppose $f(t)$ is a function of t. The Laplace transform of f is the following function of s where $s > 0$ is given by

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{1}$$

Theorem 1. *For any function h defined on $[0, \infty)$,*

$$\int_0^\infty h(t)\,dt = \lim_{w\to\infty} \int_0^w h(t)\,dt \tag{2}$$

The integral is said to converge *if this limit exists.*

Example 1. *Complete the Laplace transform of $f(t) = t$.*

Solution.

$$F(s) = \int_0^\infty te^{-st}\,dt \tag{3}$$

Here, we must integrate by parts[1]
Suppose we solve the *indefinite* integral first.

$$F(s) = \int te^{-st}\,dt \tag{4}$$
$$= t\Big(\frac{e^{-st}}{-s}\Big) - \int \frac{e^{-st}}{-s}\,dt \tag{5}$$
$$= t\Big(\frac{e^{-st}}{-s}\Big) - \frac{1}{s^2}e^{-st} + C \tag{6}$$

Now to solve for the *definite* integral, we take the limit

$$\lim_{w\to\infty}\Big[t\Big(\frac{e^{-st}}{-s}\Big) - \frac{1}{s^2}e^{-st}\Big]\Big|_{t=0}^{t=w} \tag{7}$$
$$= \lim_{w\to\infty}\Big[w\Big(\frac{e^{-sw}}{-s}\Big) - \frac{1}{s^2}(e^{-sw}) - \Big(0 - \frac{1}{s^2}\Big)\Big] \tag{8}$$
$$= \lim_{w\to\infty}\Big[-\frac{1}{s}(we^{-sw}) - \frac{1}{s^2}e^{-sw} + \frac{1}{s^2}\Big] \tag{9}$$

Applying L'Hopital's Rule, (10)

$$= \lim_{w\to\infty}\Big[\frac{1}{s^2}\Big] \tag{11}$$
$$= \frac{1}{s^2} \tag{12}$$

Therefore we see that $F(s) = \frac{1}{s^2}$.

2.2 Some Useful Identities

$$\mathcal{L}\left(e^{at}\right) = \frac{1}{s-a} \text{ for } s > a \tag{13}$$
$$\mathcal{L}\left(1\right) = \frac{1}{s} \text{ for } s > 0 \tag{14}$$

2.3 Basic Properties

A function $f(t)$ is of *exponential order* if there are constants C and a such that, for all $t > 0$,

$$|f(t)| \leq Ce^{at} \tag{15}$$

Remark. e^{t^2} is not of exponential order.

Theorem 2. *Suppose f is a piecewise continuous function defined on $[0,\infty)$ which is of exponential order. Then the Laplace transform $\mathcal{L}(f)(s)$ exists for large values of s. Specifically, if $|f(t)| \leq Ce^{at}$, then $\mathcal{L}(f)(s)$ exists for at least $s > a$.*

Theorem 3.

$$\mathcal{L}\left(af(t) + bg(t)\right) = a\,\mathcal{L}(f) + b\,\mathcal{L}(g) \tag{16}$$

The linearity property is also true for the inverse Laplace transform.

$$\mathcal{L}^{-1}(aF(t) + bG(t)) = a\,\mathcal{L}^{-1}(F) + b\,\mathcal{L}^{-1}(G) \tag{17}$$

[1] $\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$