Linear Algebra

Darren Wee
`darren.wee@u.nus.edu`

April 4, 2016

Contents

1 Matrices

1.1 Introduction

A system of linear algebraic equations in two variables may look like

$$2x + 7y = 3 \tag{1}$$
$$4x + 8y = 11 \tag{2}$$

This can be expressed as

$$\begin{bmatrix} 2 & 7 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix} \tag{3}$$

Each element in a matrix can be described using coordinates, e.g. (i, j).

A vector is a special case of a matrix where it has only one row or one column.

A matrix with only one row or one column is a scalar.

1.2 Matrix Operations

1.2.1 Addition and Subtraction

For two matrices X and Y both with dimensions $m \times n$, the operation $X - Y$ or $X + Y$ implies element-by-element subtraction/addition respectively.

1.2.2 Scalar Multiplication

Scalar multiplication is also possible; for a scalar $k \in \mathbb{R}$,

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ak & bk \\ ck & dk \end{bmatrix} \tag{4}$$

1.2.3 Matrix Transposition

For each element X at (i, j), swap the positions of X with its transposed element Y at (j, i).

The transposition of matrix A is denoted with A^T.

$$\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \tag{5}$$

It is trivial to see that the diagonal of the matrix is unaffected as $(i, j) = (j, i)$ along the diagonal since $i = j$.

In the case of vector transposition, a row vector is transposed into a column vector. The converse is also true.

Theorem 1. *A matrix A is said to be* symmetrical *if*

$$A^T = A \tag{6}$$

Theorem 2. *A matrix A can be said to be* anti-symmetric *if*

$$A^T = -A \tag{7}$$

1.2.4 Matrix Multiplication

For two matrices X and Y with dimensions $m \times n$ and $n \times t$, the matrix product XY results in a $m \times t$ matrix.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \tag{8}$$

Suppose the following:

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{9}$$

In general, the following is true for all matrices of any arbitrary size

$$c_{mn} = \sum_j a_{mj} b_{jn} \tag{10}$$

Note that $XY \neq YX$ usually, unless both matrices X and Y are filled with the same values.

In general, for matrices A and B of any time,

$$(AB)^T = B^T A^T \tag{11}$$

(don't forget to reverse the order of B and A)

1.2.5 Scalar and Vector Product in Matrices

Suppose you are given two column vectors $\vec{u}$ and $\vec{v}$ in R^3.

The dot product of $\vec{u}$ and $\vec{v}$ can be described as a matrix product,

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} \tag{12}$$

Similarly, the length of a vector $\vec{u}$ can be given by

$$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{\vec{u}^T \vec{u}} \tag{13}$$

Any vector in $\mathbf{R}^3$ can be used to define an anti-symmetrical 3×3 matrix.

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ defines } \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \tag{14}$$

1.2.6 Matrix Calculus

Calculus can be done on matrices as well. It is simple element-by-element integration/differentiation. The chain rule may be necessary depending on the variables inside the matrix and the variable of integration/differentiation.

1.3 Identity Matrix

The identity matrix is denoted by I for any size $n \times n$.

In I, the diagonal elements are always 1 with all other elements 0.

For example, I of size 2×2 is given by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Theorem 3. *A matrix B is said to be* orthogonal *if*

$$B^T B = I \tag{15}$$

An example of an orthogonal matrix is $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

2 Linear Transformations

2.1 Introduction

A linear transformation is a rule that turns vectors into other vectors.

Formally,

Theorem 4. *If T is a linear transformation mapping $\mathbf{R}^n$ to $\mathbf{R}^m$, and $\vec{x}$ is an n-dimensional column vector, then*

$$T(\vec{x}) = \mathbf{A}\vec{x} \tag{16}$$

where $\mathbf{A}$ is known as the transformation matrix *of T.*

Example 1. Suppose T is a linear transformation that turns 3-dimensional vectors into 2-dimensional ones. The rules of T are as follows:

$$T(\hat{i}) = 2\hat{i} \tag{17}$$

$$T(\hat{j}) = \hat{i} + \hat{j} \tag{18}$$

$$T(\hat{k}) = \hat{i} - \hat{j} \tag{19}$$

The transformation matrix of T is thus given by $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

2.1.1 Basic Properties

Suppose a linear transformation T, a scalar $c \in \mathbb{R}$ and vectors $\vec{u}$, $\vec{v}$.

Then, the following equations are true:

$$T(c\vec{u}) = cT(\vec{u}) \tag{20}$$

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \tag{21}$$

A linear transformation is thus some mapping that preserves the operations of addition and scalar multiplication.

Example 2. Suppose we know the following for some linear transformation T:

$$T(\hat{i}) = \hat{i} + \frac{1}{4}\hat{j} \tag{22}$$

$$T(\hat{j}) = \frac{1}{4}\hat{i} + \hat{j} \tag{23}$$

We can calculate the linear transformation of some other vector easily.

$$T(2\hat{i} + 3\hat{j}) = 2T(\hat{i}) + 3T(\hat{j}) \tag{24}$$

$$= 2(\hat{i} + \frac{1}{4}\hat{j}) + 3(\frac{1}{4}\hat{i} + \hat{j}) \tag{25}$$

$$= \frac{11}{4}\hat{i} + \frac{7}{2}\hat{j} \tag{26}$$

This works because we know what the rule/mapping T does to the basis of the vector space (in this case the basis is $\hat{i}$ and $\hat{j}$ since our vector is in $\mathbf{R}^2$).

2.2 Identity Linear Transformation

Theorem 5. *The linear identity transform I is the rule*

$$I\vec{u} = \vec{u} \tag{27}$$

for all vectors $\vec{u}$.

2.3 Applications of Simple Linear Transformations

2.3.1 Shear Forces

Suppose you have a block of rubber and shear it. There is no change in volume but the top of the block shifts a distance $\tan\theta$ in the direction of the shear relative to the base.

The shearing transformation S can be thought of as

$$S(\hat{i}) = \hat{i} \tag{28}$$

$$S(\hat{j}) = \hat{i}\tan\theta + \hat{j} \tag{29}$$

$$S = \begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix} \tag{30}$$

Shear does not add up linearly.

Example 3. Suppose you take a piece of rubber and shear it parallel to the x-axis by θ degrees, and then shear it again by ϕ degrees.

$$\begin{bmatrix} 1 & \tan\phi \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \tan\phi \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan\phi + \tan\theta \\ 0 & 1 \end{bmatrix} \tag{31}$$

The shear angles θ and ϕ do not add up linearly.

2.3.2 Plane Rotations

Suppose you want to rotate an entire plane through an angle θ counter-clockwise.

The rule for plane rotation R can be defined as

$$R(\hat{i}) = \hat{i}\cos\theta + \hat{j}\sin\theta \tag{32}$$

$$R(\hat{j}) = -\hat{i}\sin\theta + \hat{j}\cos\theta \tag{33}$$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{34}$$

2.4 Composite Transformations

The order of priority of composite transformations is left to right.

It is necessary to ensure that the transformations on the right are able to take in the output of whatever transformation went before it as input.

Example 4. Suppose the linear transformation A eats vectors in $\mathbf{R}^3$ and spits out a vector in $\mathbf{R}^2$, while another linear transformation B does the opposite[1]. The composite transform AB would thus not make sense, but BA would be okay.

---

[1] eats vectors in $\mathbf{R}^2$ and spits out a vector in $\mathbf{R}^3$

Theorem 6. *Suppose a_{ij} is the matrix of a linear transformation A relative to $\hat{i}$, $\hat{j}$ and $\hat{k}$. Suppose b_{ij} is the matrix of a linear transformation B relative to $\hat{i}$, $\hat{j}$ and $\hat{k}$. And suppose that the composite transformation AB is sensible.*

Then the transformation matrix AB relative to $\hat{i}$, $\hat{j}$ (and $\hat{k}$) is simply the matrix product of a_{ij} and b_{ij}.

2.5 Determinants

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{35}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ h & l & m \end{vmatrix} \tag{36}$$