$\mathbb{MA}1506$ Cheat Sheet

Ordinary Differential Equations

Linear First-Order ODE

The standard form for linear first-order ODEs are:

$$\frac{dy}{dx} + p(x)y = r(x) \tag{1}$$

The integrating factor $\mu(t)$ is given by

$$\mu(t) = e^{\int p(x)dx} \tag{2}$$

Bernoulli Equations

Bernoulli equations have the standard form:

$$y' + p(x)y = q(x)y^n, n \in \mathbb{R} \tag{3}$$

When $n = 0, 1$, the equation is linear and we can solve it using the integrating factor. However, for other values of n, it is necessary to reduce the equation to linear form.
First, divide Equation 3 by y^n. We will use the substitution $v = y^{1-n}$, such that the derivative $\frac{dv}{dx} = (1-n)y^{-n}y'$. We then obtain the following:

$$\frac{1}{1-n}v' + p(x)v = q(x) \tag{4}$$

Second-Order ODE

The standard form for second-order ODEs is:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dy} + q(x)y = F(x) \tag{5}$$

If $F(x) \equiv 0$, the equation is *homogenous*. Otherwise, it is *nonhomogenous*.
A solution of a second-order ODE on some interval I is a function $y = h(x)$ with derivatives $y' = h'(x)$ and $y'' = h''(x)$ satisfying the ODE $\forall x$ in I.

Homogeous Second-Order ODEs

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dy} + q(x)y = 0 \tag{6}$$

Any linear combination of two solutions for Equation 6 on an open interval I is also a solution on I, i.e. sums and constant multiples of solutions are also themselves solutions.

Remark. The above is not true for nonhomogeous equations.

As it is a solution to $y' + ky = 0, k \in \mathbb{R}$, we thus find that $y = e^{\lambda x}$ is also a solution to Equation 6 if λ is a solution to Equation 7.

$$\lambda^2 + a\lambda + b = 0 \tag{7}$$

Case 1: two real solutions λ_1 and λ_2

In this case, the solutions to the ODE are $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ which are linearly independent solutions on any interval.
The corresponding general solution is thus given by

$$y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} \tag{8}$$

Case 2: one real solution λ

The corresponding general solution is given by

$$y = c_1e^{\lambda x} + c_2xe^{\lambda x} \tag{9}$$

Case 3: complex solutions $\lambda \pm \mu i$

The corresponding general solution is given by

$$y = c_1e^{\lambda x}\cos\mu x + c_2e^{\lambda x}\sin\mu x \tag{10}$$

Nonhomogenous Second-Order ODEs

Theorem. The general solution of the nonhomogenous differential equation $p(x)y'' + q(x)y' + r(x)y = G(x)$ can be written as:

$$y(x) = y_p(x) + y_c(x)$$

where $y_p(x)$ is a particular solution of Equation $p(x)y'' + q(x)y' + r(x)y = G(x)$ and $y_c(x)$ is the general solution of the *complementary* equation $p(x)y'' + b' + r(x)y = 0$.

Method of Undetermined Coefficients

$$y'' + p(x)y' + q(x)y = r(x)$$

Guess Table

$r(x)$	Guess
$k \in \mathbb{R}$	A
$5x + 7$	$Ax + B$
$3x^2 - 2$	$Ax^2 + Bx + C$
$\sin 4x$	$A\cos 4x + B\sin 4x$
$\cos 4x$	$A\cos 4x + B\sin 4x$
e^{5x}	Ae^{5x}
$(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
$5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x + (Ex^2+Fx+G)\sin 4x$
$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x + (Cx+E)e^{3x}\sin 4x$
$(5x+7)+\sin 4x$	$(Ax+B) + (C\cos 4x + D\sin 4x)$

Variation of Parameters

$$y'' + p(x)y' + q(x)y = r(x)$$

As above, the theorem for the general solution holds and we find the complementary solution $y_c(x) = c_1y_1(x) + c_2y_2(x)$.
Now let us define a pair of functions $u(x)$ and $v(x)$ such that

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x) \tag{11}$$

$$u'(x)y_1(x) + v'(x)y_2(x) = 0 \tag{12}$$

$$y_p'(x) = u(x)y_1'(x) + v(x)y_2'(x) \tag{13}$$

$$y_p''(x) = u'(x)y_1'(x) + u(x)y_1'(x) + v'(x)y_2'(x) + v(x)y_2''(x) \tag{14}$$

$$u'(x) = -\frac{y_2(x)r(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)} \tag{15}$$

$$v'(x) = \frac{y_1(x)r(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)} \tag{16}$$

$$\tag{17}$$

Thereafter, we can obtain $u(x)$ and $v(x)$ by integration. The constant of integration in $u(x)$ and $v(x)$ can be ignored.

Mathematical Modelling

Euler's Bending Equation

Suppose a cantilevered beam with a Young's modulus E, a distributed load across its length $w(x)$ and a deflection $v(x)$ as functions of horizontal position.

$$\frac{d^2}{dx^2}\left[EI_z\frac{d^2v}{dx^2}\right] = w(x) \tag{18}$$

Malthus Model of Population

For a population of initial size P_0 with size $P(t)$ at time t, and a population growth rate r = birth rate − death rate,

$$P(t) = P_0e^{rt} \tag{19}$$

$r < 0$	population collapse (more deaths than births per capita)
$r = 0$	remains stable (if and only if)
$r > 0$	population explosion (more births than deaths per capita)

Laplace Transform

Let f be a function defined for $t \geq 0$. The Laplace transform of f is the function $F(s)$, where

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st}f(t)\,dt \tag{20}$$

Theorem. For some $a, b \in \mathbb{R}$,

$$\mathcal{L}(af(t) + bg(t)) = a\,\mathcal{L}(f) + b\,\mathcal{L}(g)$$

This is also true for the inverse Laplace transform.

Remark. The Laplace transform is independent of whether the target function is continuous or not.

Theorem. Suppose the continuous function $f(t)$ has a well-defined Laplace transform on $[0,\infty)$ and $f'(t)$ is piecewise-continuous on $[0,\infty)$. Thus, $\mathcal{L}(f'(t))$ exists, and the following is true for $s > a$:

$$\mathcal{L}(f'(t)) = s\,\mathcal{L}(f) - f(0)$$

Some useful identities:

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2} \tag{21}$$

Trigonometric Identities

Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x}-1}{2e^x} = \frac{1-e^{-2x}}{2e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x}+1}{2e^x} = \frac{1+e^{-2x}}{2e^{-x}}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Pythagorean Identities

$$\sin^2 u + \cos^2 u = 1$$

$$1 + \tan^2 u = \sec^2 u$$

$$1 + \cot^2 u = \csc^2 u$$

Sum and Difference Formulae

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$
$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$
$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

Double Angle Formulae

$$\sin 2u = 2 \sin u \cos u$$
$$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$
$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

Half Angle Identities

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$
$$\cos^2 u = \frac{1 + \cos 2u}{2}$$
$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Sum → Product Identities

$$\sin u + \sin v = 2 \sin \frac{u+v}{2} \cos \frac{u-v}{2}$$
$$\sin u - \sin v = 2 \cos \frac{u+v}{2} \sin \frac{u-v}{2}$$
$$\cos u + \cos v = 2 \cos \frac{u+v}{2} \cos \frac{u-v}{2}$$
$$\cos u - \cos v = -2 \sin \frac{u+v}{2} \sin \frac{u-v}{2}$$

Product → Sum Identities

$$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$
$$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$$
$$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$
$$\cos u \sin v = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$$

Parity Identities

$$\sin(-u) = -\sin u$$
$$\cos(-u) = \cos u$$
$$\tan(-u) = -\tan u$$

$$\cot(-u) = -\cot u$$
$$\csc(-u) = -\csc u$$
$$\sec(-u) = \sec u$$

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