Real and Complex Analysis

Lecture Notes

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Preface

These lecture notes serve as an introduction to undergraduate Real and Complex Analysis. Mathematics undergraduates usually take this course in their first year, focusing on the Real Field, sequences, and series. Advanced Economics undergraduates would need the same topics as preparation for graduate studies. Mathematics undergraduates then go on to the Riemann Integral and other topics throughout their second year. One would usually need to take a course on Set Theory and Formal Logic before learning Analysis. The author recommends Book of Proof by Richard Hammack and For all x by PD Magnus, Cathal Woods, and J Robert Loftis for those self-studying these notes.

A Mathematics undergraduate usually encounters Real and Complex Analysis as one's first formal study of familiar tools like sequences, Calculus and complex numbers. Proof-based Mathematics becomes the norm, with decreasing focus on computational and algorithmic exercises so common in lower education. This transition helps a student become rigorous—no more clinging onto intuitive facts or formulas or algorithms, more focus on trial and error, solutions and insight than answers. Only in Mathematics can proving a problem to be unsolvable be cause for celebration—consistency, sense, and rigor matter more than world-changing insights. Physicists stick to the physical world's stability. Engineers find comfort in tangible sense. Economists prefer splitting hairs on special cases than finding general principles and theorems. Computer Scientists wish to endlessly verify whether heuristics and algorithms work to solve problems. Mathematicians, however, seek escape from reality by performing useless work with symbols—the highest form of creative and artistic expression known to man.

1.0 Set-Theoretic Preliminaries

1.1 Introduction to Sets

We call a collection S of elements a set. The expression $x \in S$ means that the object x is an element of S. Likewise, $x \notin S$ means that x is not an element of S. As a general rule, we denote sets by big letters and their elements by small letters.

We may take some elements of S as their own set T. We call the set T a subset of S, which we denote by $T \subset S$.

Definition 1.1.1. A subset T of S, denoted by $T \subseteq S$, satisfies:

$$\forall x \in T (x \in S).$$

We may also find that two sets may be subsets of each other. One may show that

$$A \subseteq B \wedge B \subseteq A \implies A = B.$$

A proper subset $T \subset S$ satisfies

$$\exists x \in S (x \notin T)$$

so that S contains elements that are not elements of T.

Our first two operations on sets A and B comprise unions and intersections.

Definition 1.1.2. A union of sets A or B satisfies

$$A \cup B = \{x : x \in A \vee x \in B\}$$

while an intersection of sets A and B satisfies

$$A \cap B = \{x : x \in A \wedge x \in B\}.$$

Two sets are disjoint if

$$A \cap B = \emptyset.$$

From these two operations and standard rules of inference, we have the following theorems.

Theorem 1.1.3. The formula

$$(A \cup B) \cap C = A \cap C \cup B \cap C$$

holds for all sets A, B, and C.

Proof. For all sets A, B, and C, and for all elements x, we have that $(x \in A \cup B) \wedge (x \in C) \iff (x \in A \vee x \in B) \wedge x \in C$. By Distributive Rule of Inference, $(x \in A \wedge x \in C) \vee (x \in B \wedge x \in C) \iff (x \in A \cap C) \vee (x \in B \cap C)$, giving us our result of $(A \cup B) \cap C = A \cap C \cup B \cap C$. □

Theorem 1.1.4. The formula

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

holds for all sets A, B, and C.

Proof. For all sets A, B, and C, and for all elements x, we have that $x \in A \vee (x \in A \cap B) \iff x \in A \vee (x \in A \wedge x \in B)$. By distributivity, $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \iff (x \in A \cup B) \wedge (x \in A \cup C)$, giving us our result of $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. □

Corollary 1.1.5. For all sets A,

$$(A \cup A = A) \wedge (A \cap A = A).$$

Proof. For all sets A and elements x, $x \in A \vee x \in A$. Also, $x \in A \wedge x \in A$. By idempotency, $x \in A$ such that both statements above hold. □

Our next set operation comprises differences.

Definition 1.1.6. For sets A and B, their difference satisfies

$$A - B = \{x : x \in A \wedge x \notin B\}.$$

We introduce a new set operation relying on the definition of subsets and differences.

Definition 1.1.7. Let $A \subseteq U$. The complement of A, denoted by A^C or $\bar{A}$ satisfies

$$A^C = U - A.$$

In some contexts, we call U the universal set, the set of discourse, or simply the universe.

Theorem 1.1.8. For $B \subseteq A$, the formula

$$(B^C)^C = A - (A - B) = B$$
$$(A - B) \cup B = A.$$

for all sets A and B.

Proof. For all sets A and B, and for all elements x, $B^C = A - B = \{x : x \in A \wedge x \notin B\}$, so that $(B^C)^C = (A-B)^C = \{x : x \in B \wedge x \notin (A-B)\} = B = A - B^C$. Also, $(A-B) = B^C = \{x : x \in A \wedge x \notin B\} \iff B^C \cup B = \{x : x \in B^C \vee x \in B\} = \{x : x \in (A-B) \vee x \in B\} = \{x : x \in A \wedge (x \notin B \vee x \in B)\} = B$. □

We note that the formula $(A \cup B) - B = A$ holds only for disjoint A and B. One can verify this fact as an exercise.

One may also find that elements of a set may themselves be sets. Given a set A, one may consider sets whose elements are subsets of A. In particular, one may obtain the set of all subsets of A—the power set of A.

Definition 1.1.9. The power set of A is the set $\mathcal{P}(A) = \{X : x \in X \implies x \in A\}$.

One may find it convenient to denote multiple sets with subscripts. For sets $A_i, i = 1, 2, \ldots$, we have analogous definitions for unions and intersections.

Definition 1.1.10. For sets $A_i, i = 1, 2, \ldots$, we define their union by

$$\bigcup_{i=1} A_i = x : \exists A_i, 1 \leq i(x \in A_i)$$

and their intersection by

$$\bigcap_{i=1} A_i = x : \forall A_i, 1 \leq i(x \in A_i)$$

Theorem 1.1.11. The following formula holds for all sets A_i:

$$(\bigcup_i A_i)^C = \bigcap (A_i)^C.$$

Proof. For $(\bigcup_i A_i)^C$, the universe U, and element x, we have that $x \in (\bigcup_i A_i)^C \iff x \in U \wedge x \notin A_1 \wedge x \notin A_2 \wedge \ldots$. By Addition, Commutative, and Associative Rules of Inference, we can add true statements $x \in U$ repeatedly until we have $(x \in U \wedge x \notin A_1) \wedge (x \in U \wedge x \notin A_2) \wedge \ldots \iff x \in A_1^C \wedge x \in A_2^C \wedge \ldots$ to find that this satisfies $x \in \bigcap_i A_i^C$. □

Our next concepts involve ordered pairs and Cartesian Products, central to discussion about real numbers.

Definition 1.1.12. An ordered pair (a, b) is defined by the set $\{\{a\}, \{a, b\}\}$ such that $(a, b) = (b, a) \iff a = b$.
The Cartesian Product of sets A and B is

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}.$$

Our definition has it so that $a = b \iff (a, b)$ contains one set, and $a \neq b \iff (a, b)$ contains two sets. We proceed with defining classes of sets with the same number of elements.

1.2 Functions

Definition 1.2.1. A rule of assignment is a subset r of the cartesian product $C \times D$, such that $c \in C$ appears as the first coordinate of at most one ordered pair belonging to r.

One consequence of the condition for $c \in C$ is that if $(c, d) \in r$ and $(c, d') \in r$ then $d = d'$.

One may also define a rule of assignment using its domain and image:

Definition 1.2.2. The domain of a rule of assignment $r \subseteq C \times D$ is the subset of C with all first coordinates of r

$$\mathrm{dom}(r) = \{c : \exists d \in D \wedge (c, d) \in r\} \subseteq C$$

while the image of r is the subset of D with all second coordinates of r:

$$\mathrm{img}(r) = \{d : \exists c \in C \wedge (c, d) \in r\} \subseteq D.$$

We can now discuss a special kind of rule of assignment.

Definition 1.2.3. A function f is a rule of assignment r, with a set B that includes $\mathrm{img}(r)$.
The set $A = \mathrm{dom}(r)$ is the domain of f, with $\mathrm{img}(r) = \mathrm{img}(A)$. The set B is called the codomain of f. Sometimes, we also call B the range.

We denote a function with domain A and codomain B by $f : A \to B$. One can visualize f as a geometric transformation from the points of A to those of B.

If $f : A \to B$ and $a \in A$, denote by $f(a) \in B$ the unique element that the rule f assigns to a. We call $f(a)$ the value of f at a. In terms of rules of assignment, $f(a) \in B$ is the unique element such that $(a, f(a)) \in r$.

Definition 1.2.4. Given functions $f : A \to B$ and $g : B \to C$, their composition $g \circ f$ is the function $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$.
Denote a composition by

$$\{(a, c) : \exists b \in B(f(a) = b \wedge g(b) = c)\}.$$

Physically, point a moves to point $f(a)$, then to point $g(f(a))$. If the composition $g \circ f(a)$ is defined, then the range of f equals the domain of g.

Example

The composite of functions $f : \mathbb{R} \to \mathbb{R}, f(x) = 3x^2 + 2$ and $g : \mathbb{R} \to \mathbb{R}, g(x) = 5x$ is

$$g \circ f : \mathbb{R} \to \mathbb{R} = g(f(x)) = 5(3x^2 + 2) = 15x^2 + 10.$$

Definition 1.2.5. We call a function $f : A \to B$ injective (one-to-one) if for all distinct elements $a \in A$, there exists a distinct element $b \in B$ such that each element a has a unique image:

$$f(a) = f(a') \implies a = a'$$

We call the function f surjective if for all $b \in B$, there exists $a \in A$ such that $b = \text{img}(a)$:

$$\forall b \in B \exists a \in A(b = f(a))$$

A function that is both injective and surjective is called bijective.

For a bijective function f, then the inverse function $f^{-1} : B \to A$ exists and is defined by $f^(b) = a$ such that $f(a) = b$. We now go to establish the cardinality of sets.

Definition 1.2.6. The cardinality of a set A, denoted by $|A|$, is the equivalence class of sets such that all sets are bijective to each other.

This definition has the effect that sets with the same number of elements all have the same cardinality.

Proposition 1.2.7. Let $f : A \to B$. If there exists functions $g : B \to A$ and $h : B \to A$ where for all $a \in A$ and $b \in B$, $g(f(a_i)) = a_i$ and $f(h(b_i)) = b_i$, then f is bijective and $g = h = f^{-1}$.

Proof. We first establish the existence of an inverse, then show that functions g and h satisfy its definition.
We note that for all $a_i \in A$ and $b_i \in B$, we have that $f(a_i) \in B$, $h(b_i) \in A$, and $g(b_i) \in A$. Since a_i is arbitary, we can set $a_0 = h(b_0)$ and take $g(f(a_0)) = g(f(h(b_0))) = g(b_0) = a_0$. We can then take $f(h(b_0)) = f(g(f(a_0))) = f(a_0) = b_0$. Note two results, $f(a_0) = b_0$ and $g(b_0) = a_0$ so that the inverse function exists, and f is bijective.
Lastly, set $f(a_0) = f(h(b_0)) = b_0$, which we can do since by hypothesis, $f(h(b_i)) = b_i$ for all b_i. Since f is bijective, we have by definition of injectivity that $h(b_0) = a_0$, and $g = h = f^{-1}$. $\square$

Definition 1.2.8. Let $f : A \to B$. If $A_0 \subseteq A$, then $f(A_0) = \{b : \exists a \in A_0(b = f(a))\}$ is the set of all images of A_0 under the function f. Analogously, $f(A_0)$ is the image of A_0.
For $B_0 \subseteq B$, denote by $f^{-1}(B_0) = \{a : f(a) \in B_0\}$ the set of all elements of A whose images under f lie in B_0. We call $f^{-1}(B_0)$ the preimage of B_0 under f.

If no points $a \in A$ have images which lie in B_0, then $f^{-1}(B_0) = \emptyset$. For bijective f, then the preimage and image of B_0 are the same.

1.3 Relations

Definition 1.3.1. We call a subset C of the cartesian product $A \times A$ a relation on A.

For a relation C on A, we denote $\{x, y\} \in C$ by xCy.

A mapping r for a function $f : A \to A$ is also a subset of $A \times A$. The mapping r is special since one element of A appears as the first coordinate of r exactly once.

Example

Let P denote the set of all people in the world, and define $D \subseteq P \times P$ by $D = \{(x, y) : x \text{ descends from } y\}$ so that D is a relation on the set P.

Definition 1.3.2. An equivalence partition on a set A is a relation $\sim$ on A with the following properties:

1. Refelxivity: $\forall x \in A(x \sim x)$.
2. Symmetry: $x \sim y \implies y \sim x$.
3. Transitivity: $x \sim y \wedge y \sim z \implies x \sim z$.

Given an equivalence relation $\sim$ on a set A and an element $x \in A$, the subset $E \subseteq A$ is called an equivalence class, defined by $E = \{y : y \sim x\}$.

Lemma 1.3.3. (Equivalence Class Disjointedness Lemma). Two equivalence classes E_i and E_j are either disjoint or equal.

Proof. Suppose that $E_i \cap E_j \neq \emptyset$. Their intersection consists of elements $x_\sim \in E_i$ and $y_\sim \in E_j$ such that $x_\sim \sim y_\sim$. However, elements $x_i \in E_i$ and $y_j \in E_j$ have properties that $x_i \sim x_\sim$ and $y_j \sim y_\sim$, so that by transitivity $y_j \sim x_i$, causing both sets to be equal. Thus, if neither sets have the same elements, both become disjoint sets. □

Definition 1.3.4. (Partition of a Set). A partition of a set A is a collection of disjoint nonempty subsets of A such that $\bigcup_i A_i = A$.

Proposition 1.3.5. Given any partition $\mathcal{D}$ of A, there exists a unique equivalence relation derived from it.

Proof. Set a relation xCy where for some $D_0 \in \mathcal{D}$, $x\ inD_0$ and $y \in D_0$.
Verify how the relation satisfies the equivalence partition properties and whether this equivalence relation is unique as an exercise. □

Definition 1.3.6. (Order Relation) An order relation on a set A is a relation $<$ on A with the following properties:

1. Comparability: $\forall x, y \in A(x < y \vee y > x)$.
2. Non-reflexivity: $\forall x \in A\ \not(x < x)$.
3. Transitivity: $x < y \wedge y < z \implies x < z$.

We sometimes call an order relation either a linear order or a simple order.Note that there exists a relation for $x, y \in A$ such that $x \sim y \vee x < y$, and is denoted by $x \leq y$. We use this relation in the following definition.

Definition 1.3.7. The set of all $x \in \mathbb{A}$ satisfying $a < x < b$ is called an open interval, with left-hand and right-hand endpoints a and b, respectively. If $a \leq x \leq b$, then we call the set a closed interval, with identical endpoints.

Definition 1.3.8. (Order Type). Suppose sets A and B with respective order relations $<_A$ and $<_B$. If there exists a bijection $f : A \to B$ such that $a_1 <_A a_2 \implies f(a_1) <_B f(a_2)$, then sets A and B have the same order type.

Definition 1.3.9. (Dictionary Order Relation). Suppose sets A and B with respective order relations $<_A$ and $<_B$. Define an order relation $<$ on $A \times B$ by $(a_1, b_1) < (a_2, b_2)$ if $a_1 <_A a_2$ or $a_1 = a_2 \wedge b_1 <_B b_2$. We call this ordering a dictionary order relation on $A \times B$.

Exercises

1. Finish the proof for Proposition 1.3.5, using Definition 1.3.2 and Lemma 1.3.3.

1.4 The Real Numbers

In this section, we construct the real numbers from elementary concepts known in Number Theory. We also prove that the real number operations follow from our construction. Instead of assuming their existence like most treatments, we explain the real numbers as a set satisfying certain properties related to more elementary constructions.

To add stuff definitions from Number Theory and theorems from Pugh's book eventually replacing section 1 of chapter 1.

We will discuss the consequences of operations defined on the real numbers in the next chapters. In the following sections, we will discuss certain types of sets including constructions made from real numbers in two or more dimensions. We will then construct the complex plane from two-dimensional real space by defining certain operations which complex numbers satisfy.

1.5 Finite and Infinite Sets

Definition 1.5.1. (Finite and Infinite Sets). If a bijection exists between a set A and a subset of the natural numbers $Z \subseteq \mathbb{N}$, then we call A countable. If $Z \neq \mathbb{N}$, then we call A finite. We call sets that are not finite as infinite. Sets with no bijection to the $\mathbb{N}$ are uncountable.

Theorem 1.5.2. The closed interval $[a, b]$ and open interval (a, b) are equivalent.

Proof. One may construct a bijection between these intervals. Suppose a sequence A of distinct points $x_1 = a, x_2 = b, x_3, x_4, \ldots \in A$. Clearly, points $x_3, x_4, \ldots$ and all points $y \notin A$ are elements of (a, b). We then make the rule $x_1 \to x_3, x_2 \to x_4, x_3 \to x_5, \ldots, x_n \to x_{n+2}, y \notin A \to y$ which establishes the existence of a bijection. $\square$

Theorem 1.5.3. Consider a set A with cardinality $|A| = n$. Then the power set of A has cardinality $|\mathcal{P}(A)| = 2^n$.

Proof. Set $|A| = n = 1$. Then we find two subsets: $\emptyset$ and A itself. Assume that this result holds for all n. Taking the case of $n + 1$, we can form two partitions, A_n with n elements and A_1 with 1 element. By induction step, $\mathcal{A}_\backslash$ has 2^n subsets. By base case, A_1 has $2 = 2^1$ subsets. Taking the union $A_n \cup A_1$, the number of subsets clearly becomes multiplied by 2, so that we have $2^n \dot{2}^1 = 2^{n+1}$. □

Lemma 1.5.4. Let $n \in \mathbb{N}$, A be a set, and $a_0 \in A$. Then there exists bijection $f : A \to \{1, \dots, n+1\}$ if and only if there exists a bijection $g : A - \{a_0\} \to \{1, \dots, n\}$.

Proof. Let there be a bijection $f : A \to \{1, \dots, n+1\}$. Both sets comprise an equivalence class of the same cardinality, such that $|A| = |\{1, \dots, n+1\}| = n + 1$. Clearly, the set $A - \{a_0\}$ has a cardinality of n, such that there exists an equivalence class between $A - \{a_0\}$ and $\{1, \dots, n\}$ with an existing bijection, which we denote by g. One can reverse these steps starting with a bijection $g : A - \{a_0\} \to \{1, \dots, n\}$ to prove the converse. □

Theorem 1.5.5. Suppose a set A, $n \in \mathbb{Z}$, $B \subset A$, and $B \neq \emptyset$. If there exists a bijection $f : A \to \{1, \dots, n\}$, then there exists no bijection $g : B \to \{1, \dots, n\}$, but there exists a bijection $h : B \to \{1, \dots, m\}$ where $m < n$.

Proof. Given that the bijection $f : A \to \{1, \dots, n\}$ exists, assume that the bijection $g : B \to \{1, \dots, n\}$ exists. Then the sets A, B, and $\{1, \dots, n\}$ form an equivalence class, contradicting our hypothesis that $B \subset A$. However, assume without loss of generality that $|B| = m$. By Lemma 1.5.4, we can take the difference of the last numerical $n - m$ elements in order from $\{1, \dots, n\}$ to construct a bijection $h : B \to \{1, \dots, m\}$. □

Corollary 1.5.6. For finite set A, there exists no bijection between A and any proper subset $B \subset A$.

Proof. Follows from Theorem 1.5.5. □

Corollary 1.5.7. The set of positive integers $\mathbb{Z}^+$ is not finite.

Proof. Prove this corollary as an exercise. □

Corollary 1.5.8. Every infinite subset B of a countable subset A is itself countable.

Proof. Prove this corollary as an exercise. □

Theorem 1.5.9. The union of countable sets is itself countable.

Proof. This theorem is so mind-numbingly obvious that we refuse to waste our time proving it. □

Theorem 1.5.10. The set of rational numbers is countable.

Proof. Take the union of sets $A_0 = \{n : n \in \mathbb{Z}\}, A_1 = \{\frac{n}{2} : n \in \mathbb{Z}\}, A_2 = \{\frac{n}{3} : n \in \mathbb{Z}\} \dots$ to obtain the set $\mathbb{Q}$. By Theorem 1.5.9, the set $\mathbb{Q}$ itself is countable. □

Exercises

1. Prove that $\mathbb{Z}^+$ is not finite.
2. Prove that every infinite subset B of a countable subset A is itself countable using Theorem 1.5.5.
3. Prove that the set of all polynomials $a_0 + a_1x + \ldots + a_nx^n$ is countable.
4. Given countable sets $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, prove that the set of all ordered pairs (a_k, b_n) is countable.
5. Prove Corollary 1.5.6

1.6 Mathematical Structures and Euclidean Space

A Mathematical Structure is a set with certain properties defined. Metric spaces, Chapter 5's subject, comprise a mathematical structure.

Definition 1.6.1. Two structures of the same kind are isomorphic if a bijection exists between them.

Every structure is isomorphic to itself through the identity mapping, such that all properties are satisfied by elements and subsets of the structure. Other nonidentical bijections also exist, called automorphisms.

Example

Suppose a linearly ordered set $E = \{x, y, \dots\}$ with the property that given any $x \neq y$, either $x < y$ or $x > y$, where this order relation is the defined property. Isomorphisms must preserve this ordering such that $x < y \implies f(x) < f(y)$.

Definition 1.6.2. For $n \in \mathbb{Z}$, the set $\mathbb{R}^n$ consists of all ordered n-tuples

$$\vec{x} = (x_1, x_2, \dots, x_n)$$

where $x_1, \dots, x_n \in \mathbb{R}$. We call the element $\vec{x}$ a vector with components $x_1, \dots, x_n \in \mathbb{R}$. Two vectors $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\vec{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ are equal if and only if $x_1 = y_1, \dots, x_n = y_n$. Lastly, we call the set $\mathbb{R}^n$ as Euclidean space, or n-dimensional real space.

We define addition on $\mathbb{R}^n$ by the formula

$$\vec{x} + \vec{y} = (x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

and multiplication with some $a \in \mathbb{R}$ by

$$a(x_1, \dots, x_n) = (ax_1, \dots, ax_n).$$

1.7 Complex Numbers

Apart from the defined operations, we can also define another multiplication operation on $\mathbb{R}^n$ with certain properties satisfied for $n = 2$. Let $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$. Write every vector $\vec{z} \in \mathbb{R}^2$ as $\vec{z} = (x, y) = x\vec{e}_1 + y\vec{e}_2$ where x and y are components of $\vec{z}$ with respect to the basis $\vec{e}_1, \vec{e}_2$.

Now, we set $\vec{e}_1^2 = \vec{e}_2$, $\vec{e}_2^2$, $\vec{e}_1\vec{e}_2 = \vec{e}_2\vec{e}_1$, and $\vec{e}_2\vec{e}_1 = \vec{e}_2$. This extends multiplication to all vectors $\vec{z} = x\vec{e}_1 + y\vec{e}_2$. Furthermore, if $\vec{w} = u\vec{e}_1 + v\vec{e}_2$, then we have that

$$\vec{z}\vec{w} = (x\vec{e}_1 + y\vec{e}_2)(u\vec{e}_1 + v\vec{e}_2) = xv\vec{e}_1^2 + xy\vec{e}_1\vec{e}_2 + yu\vec{e}_2\vec{e}_1 + yv\vec{e}_2^2 = (xu - yv)\vec{e}_1 + (xv + yu)\vec{e}_2.$$

We now define multiplication on $\mathbb{R}^2$ as

$$\vec{z}\vec{w} = (x, y)(u, v) = (xu - yv, xv + yu).$$

We note that the product $\vec{w}\vec{z}$ has the following properties:

1. $\vec{w}\vec{z} = (xu - yv, xv + yu) = (ux - vy, vx + uy) = \vec{z}\vec{w}$ (Commutativity).
2. $\vec{t} = (a, b) \implies (\vec{z}\vec{w})\,\vec{y} = (xua - yva - xvb - yub, xub - yvb + xva + yua) = (x(ua - vb) - y(va + ub), x(ub + va) + y(vb - yua)) = \vec{z}(\vec{w}\vec{t})$ (Associativity).
3. $\exists \vec{e} = (1, 0) \in \mathbb{R}^2 \forall \vec{x} = (a, b) \in \mathbb{R}^2 (\vec{e}\vec{x} = \vec{e}(a, b) = (a, b) = \vec{x})$ (Unit element).
4. $\forall (x, y) \in \mathbb{R}^2, (x, y) \neq (0, 0) \exists (u, v) \in \mathbb{R}^2 ((x, y)(u, v) = 1)$ (Reciprocal element).
5. $\vec{x} = (x_1, x_2), \vec{y} = (y_1, y_2), \vec{z} = (z_1, z_2) \implies \vec{z}(\vec{x} + \vec{y}) = \vec{x}\vec{z} + \vec{y}\vec{z}$.

From now, we refer to multiplying by a reciprocal elements as division.

Definition 1.7.1. The space $\mathbb{R}^2$ with the addition operation

$$(x, y) + (u, v) = (x + u, y + v)$$

and the multiplication operation

$$\vec{z}\vec{w} = (x, y)(u, v) = (xu - yv, xv + yu)$$

is called the complex plane, denoted by $\mathbb{C}$.

There exists an injection $f : \mathbb{R} \to \mathbb{C}$ where $f(x) = (x, 0)$. This injection preserves sums and products:

$$(x, 0) + (y, 0) = (x + y, 0)$$

$$(x, 0)(y, 0) = (xy, 0).$$

The injection thus embeds the real numbers $\mathbb{R}$ into $\mathbb{C}$. Furthermore, we can write the complex number $(x, 0)$ as x. Thus, we denote $\vec{e}_1 = (1, 0)$ by 1, and $\vec{e}_2 = (0, 1)$ by i, the imaginary unit. We also have from our definitions that $\vec{e}_2^2 = -\vec{e}_1 = i^2 = -1$. It follows that $i = +\sqrt{-1}$ and $-i = -\sqrt{-1}$. Both i and $-i$ are complex square roots of -1. Lastly, we write the multiplication rule in terms of i:

$$(x + iy)(u + iv) = (xu - yv) + i(xv + yu).$$

Definition 1.7.2. Suppose a complex number $z = x + iy$. We call x the real part of z, denoted by $\mathrm{Re}(z) = x$. We call y the imaginary part, denoted by $\mathrm{Im}(z) = y$.
For $\mathrm{Re}(z) = 0$ and $\mathrm{Im}(z) \neq 0$, we call z as purely imaginary. Meanwhile, the set of all z where $\mathrm{Im}(z) = 0$ as the real axis. A trivial bijection $f : \mathbb{R} \to \mathbb{C}$ exists such that $f(x) = (x, 0) = x$, letting us identify the real axis with the real numbers.

Definition 1.7.3. (Complex conjugates). We call complex numbers $x + iy$ and $x - iy$ as complex conjugates of each other. For one number z, we denote its conjugate by $\bar{z}$.

Theorem 1.7.4. The formula $z = \bar{z}$ holds if and only if $z \in \mathbb{R}$.

Proof. By definiiton of complex conjugates, we have that $z = x + iy$ and $\bar{z} = x - iy$ without loss of generality. Suppose that $z = \bar{z}$. Then, $x + iy = x - iy \iff x - x + iy = x - x - iy \iff iy = -iy$. However, we can divide both sides by the imaginary unit to obtain $y = -y$. This equation holds true only for $y = 0$, such that $z = \bar{z} = x \in \mathbb{R}$. □

Theorem 1.7.5. The formulas

1. $z_1 \bar{+} z_2 = \bar{z}_1 + \bar{z}_2$
2. $z_1 \bar{z}_2 = (\bar{z}_1)(\bar{z}_2)$

hold for all $z_1, z_2 \in \mathbb{C}$.

Proof. Prove this theorem as an exercise. □

Exercises

1. Prove Theorem 1.7.5.
2. Prove that complex numbers $(1, 0) = 1$ and $(0, 1) = i$ satisfying the multiplication operation are unique (use the fact that $1 \in \mathbb{R}_2$ and the linear independence property from linear algebra).

2.0 Consequences of Real Number Properties

2.1 Real Number Axioms

Real Analysis studies concepts involving the real number system $\mathbb{R}$. Instead of real numbers themselves, we focus on their properties as undefined objects satisfying certain axioms.

Axiom 2.1. (Addition). The sum $(x + y) \in \mathbb{R}$ of $x, y \in \mathbb{R}$ satisfies

1. $x + y = y + x$ (commutativity)
2. $\forall z \in \mathbb{R}[(x + y) + z = x + (y + z)]$ (associativity)
3. $\exists 0 \in \mathbb{R} \forall x \in \mathbb{R}(x + 0 = x)$ (zero element)
4. $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(x + y = 0)$ (negative element)

Axiom 2.2. (Multiplication). The product $xy \in \mathbb{R}$ satisfies

1. $\forall x, y \in \mathbb{R}(xy = yx)$ (commutativity)
2. $\forall z \in \mathbb{R}[(xy)z = x(yz)]$ (associativity)
3. $\exists 1 \in \mathbb{R}, 1 \neq 0, \forall x \in \mathbb{R}(1x = x)$ (unitary element)
4. $\forall x \in \mathbb{R}, x \neq 0, \exists y \in \mathbb{R}(xy = 1)$ (reciprocal element)
5. $x(y + z) = xy + xz$ (distrbutive property)

A set of objects x, y, z satisfying these axioms is called a field.

The real number system $\mathbb{R}$ is thus also called the real field. Other sets, such as the rationals $\mathbb{Q}$, are also fields.

Our next set of axioms concern order in sets.

Axiom 2.3. For all $x, y \in \mathbb{R}(x \geq y \vee x \leq y)$ with properties

1. $\forall x \in \mathbb{R}(x \leq x)$ (reflexivity)
2. $(x \leq y) \wedge (y \leq x) \implies x = y$ (antisymmetry)
3. $(x \leq y) \wedge (y \leq z) \implies x \leq z$ (transitivity)
4. $\forall z \in \mathbb{R}(x \leq y \implies x + z \leq y + z)$
5. $0 \leq x \wedge 0 \leq y \implies 0 \leq xy$

We call a set whose elements satisfy 1–3 a partially ordered set. The set $\mathbb{R}$ is a totally ordered set since it satisfies $x \geq y \vee x \leq y$ (connexity).

Our last axiom requires a special definition.

Definition 2.1.1. (Upper Bounds). A set $E \subset \mathbb{R}$ is bounded above when

$$\exists z \in \mathbb{R} \forall x \in E(x \leq z).$$

We denote that E is bounded above by $E \leq z$. We call the element z an upper bound of the set E.

We now posit that there exists a least element to the set of upper bounds.

Axiom 2.4. (Axiom of Completeness). For elements $z \in \mathbb{R}$, a set $E \leq z$ satisfies the axiom of completeness when

$$\forall E \subset \mathbb{R}, \exists z_0 \in \mathbb{R}(z_0 \leq z).$$

We call z_0 the supremum of E and denote it by $z_0 = \sup[E]$.

2.2 Consequences of Axioms

2.2.1 Addition

Theorem 2.2.1. The element $0 \in \mathbb{R}$ is unique.

Proof. Assume any two zero elements, 0_i and 0_j. By Addition Axioms 1 and 3:

$$0_i = 0_i + 0_j = 0_j + 0_i = 0_j.$$

□

Theorem 2.2.2. For all $x \in \mathbb{R}$, the negative element is unique.

Proof. Suppose $x \in \mathbb{R}$ has any two negative elements y_i and y_j. By Addition Axioms 1, 2, 3, and 4:

$$y_i = y_i + 0 = y_i + (x + y_j) = (y_i + x) + y_j = 0 + y_j = y_j.$$

We denote the unique negative element of $x \in \mathbb{R}$ by $-x \in \mathbb{R}$. We call the sum $x + (-y)$ the difference of x and y, and denote it by $x - y$. □

Theorem 2.2.3. The equation $a + x = b$ has a unique solution $x = b - a$.

Proof. Add the negative element $-a$ to both sides of the equality:

$$a + x + -a = x = b - a.$$

Verify whether this is the solution:

$$a + x = a + (b - a) = a + b - a = b + 0 = b.$$

□

2.2.2 Multiplication

Theorem 2.2.4. The set $\mathbb{R}$ contains a unique unit element 1.

Proof. Suppose two unit elements 1_i and 1_j. Then by Multiplication Axioms 1 and 3

$$1_i = 1_i 1_j = 1_j 1_i = 1_j.$$

□

Theorem 2.2.5. Every element $x \in \mathbb{R}$ has a unique reciprocal.

Proof. Suppose x has two reciprocals y_1 and y_2. Then

$$y_1 = y_1 1 = y_1(xy_2) = (y_1 x)y_2 = 1y_2 = y_2.$$

□

Definition 2.2.6. Numbers $1, 2 = 1+1, 3+2+1, \ldots$ are called the Natural Numbers, denoted $\mathbb{N}$. The set $\mathbb{N}$ may be defined as the smallest set containing 1 and $n+1$ whenever $n \in \mathbb{N}$.

One may need to show sometimes that a numerical set A—a set of all $n \in \mathbb{N}$ for which some property T holds—has all natural numbers. The method of mathematical induction verifies whether

$$(1 \in A) \wedge (n \in A \implies n+1 \in A)$$

The set of integers holds all $n \in \mathbb{N}$, reciprocals $-n$, and the number 0. We denote the set of integers by $\mathbb{Z}$. Mathematicians also differ in convention whether $0 \in \mathbb{N}$ or not. To remove ambiguity, one may use $\mathbb{Z}^+$ instead of our definition of $\mathbb{N}$. Additionally, suppose a number $m \in \mathbb{Z}$. We call the number $2m$ an even number, and $2m+1$ and odd number.

If we get all quotients of form $\frac{m}{n}, m, n \in \mathbb{Z}$ and $n \neq 0$, we get the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$. All other numbers are irrationals.

Theorem 2.2.7. The equation $ax = b, a \neq 0$ has the unique solution $\frac{b}{a} \in \mathbb{R}$.

Proof. Multiply both sides by $\frac{1}{a}$:

$$\frac{1}{a}ax = \frac{1}{a}b = 1x = x = \frac{b}{a}$$

Verify the solution:

$$ax = b = a(\frac{b}{a}) = a\frac{1}{a}b = 1b = b.$$

□

Define $x^n = \prod_n x, n \in \mathbb{N}$. We then have $x^n x^m = \prod_n x \prod_m x = x^{n+m}$. Additionally, $(x^n)^m = \prod_m (\prod_n x) = x^{mn}, m, n = 1, 2, \ldots$. We also define $x^0 = 1$, $x^{-m} = \frac{1}{x^m}, x \neq 0$.

Theorem 2.2.8. If $x \in \mathbb{R}$, then $0x = 0$.

Proof. Using the definition of x^2,

$$0x = (x - x)x = x^2 - x^2 = 0.$$

Alternatively, by Addition Axiom 3 and Multiplication Axiom 3

$$0x + 1x = (1 + 0)x = 1x = x \implies 0x = x - x = 0.$$

□

From the above theorem, we have the following useful corollary.

Corollary 2.2.9. If $xy = 0$ and $x \neq 0$, then

$$y = \frac{1}{x}xy = \frac{1}{x}(xy) = \frac{1}{x}0 = 0.$$

Thus if a product vanishes, then so does one of its factors.

Theorem 2.2.10. If $(u \neq 0) \wedge (v \neq 0)$, then $\frac{x}{u} + \frac{y}{v} = \frac{xv+yu}{vu}$.

Proof. Multiply the lefthand side by $\frac{vu}{vu} = 1$

$$\frac{vu}{vu}\left(\frac{x}{u} + \frac{y}{v}\right) = \frac{vu}{vu}\frac{x}{u} + \frac{vu}{vu}\frac{y}{v} = \frac{xv}{vu} + \frac{yu}{vu} = \frac{xv+yu}{vu}.$$

Alternatively, one may notice that

$$\frac{xv+yu}{vu} = \frac{1}{vu}(xv+yu) = \frac{xv}{vu} + \frac{yu}{vu} = \frac{x}{u} + \frac{y}{v}.$$

□

Theorem 2.2.11. If $x \in \mathbb{R}$, then $-x = (-1)x$.

Proof. By Multiplication Axiom 5,

$$0 = x - x = x(1-1) = x[1 + (-1)] = x + (-1)x.$$

□

Definition 2.2.12. The factorial of $n \in \mathbb{N}$, denoted by $n!$ is the term f_n of the sequence $f_0 = 1, f_n = nf_{n-1}, n \geq 1$.

Definition 2.2.13. For $n, k \in \mathbb{N}$ with $n \geq k$, the binomial coefficient $\binom{n}{k} \in \mathbb{N}$ is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Lemma 2.2.14. For natural numbers n and k with $n \geq k$,

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Proof. The proof is purely computational and is left as an exercise, if the reader pleases. □

Theorem 2.2.15. (Binomial Theorem). For nonzero $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof. For the base step, let $n = 1$. Then,

$$(a+b)^1 = a + b = \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k = \binom{1}{1} a^{1-1} b^1 + \binom{1}{0} a^{1-0} b^0 = a + b$$

For the induction step, assume that the theorem holds for n. For the case of $n+1$ we use the preceding lemma:

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)^n (a+b)^1 = (a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \sum_{k=0}^{1} \\
&= a \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k + b \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1} + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=1}^{n} \binom{n}{k-1} a^{n+1-k} b^k + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} a^{n+1-k} b^k + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k + b^{n+1} \\
&= \binom{n+1}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k + \binom{n+1}{n+1} a^0 b^{k+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k
\end{aligned}$$

□

2.2.3 Consequences of Order Axioms

Lemma 2.2.16. If $x \leq y \wedge y \leq z \wedge x = z$, then $x = y = z$.

Proof.

$$y \leq z = x \implies y \leq x$$

since $x \leq y$, $x = y = z$. □

Lemma 2.2.17. If $x < y \wedge y \leq z$, then $x < z$. Similarly, if $x \leq y \wedge y < z$, then $x < z$.

Proof. If $y < z$, then by Order Axiom 3, $x < z$. If $y = z$, then by hypothesis $x < y = z \iff x < z$. One may show the second result through a similar way. □

Theorem 2.2.18. The following inequalities are equivalent:

- $x \leq y$
- $0 \leq y - x$
- $x - y \leq 0$

Proof. We take advantage of Order Axiom 4:

$$x \leq y \implies x - x \leq y - x \iff 0 \leq y - x$$
$$0 \leq y - x \implies 0 - y \leq y - x + (-y) \iff -y \leq -x$$
$$-y + x \leq -x + x \implies x - y \leq -x + x \implies x - y \leq 0.$$

□

Lemma 2.2.19. For all $z \in \mathbb{R}$, $x < y \implies x + z, y + z$

Proof.

$$x < y \implies x \leq y \implies x + z \leq y + z.$$

It must be that $x + z < y + z$ since $x + z = y + z \implies x = y$, a contradiction. □

Theorem 2.2.20. For all $i \in \mathbb{N}$, if $x_i \leq y_i$ then

$$\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} y_i.$$

The inequality becomes strict when $x_j < y_j$ for at least one pair x_j, y_j.

Proof. By Order Axiom 3:

$$\sum_{i=1}^{n} x_i \leq y_1 + \sum_{i=2}^{n} x_i \leq y_1 + y_2 + \sum_{i=3}^{n} x_i \leq \sum_{i=1}^{n} y_i.$$

Using Lemma 2.2.19,

$$x_j < y_j \implies \sum_{i=1}^{n} x_i < \sum_{i=1}^{n} y_i$$

for at least one pair x_j, y_j. □

One can note that $x_1 \leq 0, \ldots, x_n \leq 0 \implies s = \sum_{i=1}^{n} x_i \leq 0$. If $x_j < 0$, then $s < 0$ for any j.

Theorem 2.2.21. The following inequalities are equivalent:

- $x < y$
- $0 < x - y$
- $x - y < 0$

Proof. One may apply Lemma 2.2.19 to Theorem 2.2.18 to obtain this theorem. □

Definition 2.2.22. A number $x \in \mathbb{R}$ is nonnegative if $x \geq 0$, positive if $x > 0$, non-positive if $x \leq 0$, and negative if $x < 0$.
The number 0 is both nonnegative and nonpositive.

Definition 2.2.23. Suppose $x \leq y$ for $x, y \in \mathbb{R}$. We call x the minimum of numbers x and y. We denote this fact by $\min\{x, y\} = x$. Similarly, we call y the maximum, denoted by $\max\{x, y\} = y$.

One can use induction to define $\min\{x_1, \ldots, x_n\}$ and $\max\{x_1, \ldots, x_n\}$:

$$\max\{x_1, \ldots, x_n\} = \max\{\max\{x_1, \ldots, x_{n-1}\}, x_n\}$$

Definition 2.2.24. The number $|x| = \max\{x, -x\}$ is called the absolute value, or modulus, of x.
We have that $x > 0 \implies |x| = x$ and $x < 0 \implies |x| = -x$. Also,

$$\forall x > 0 (|x| > 0 \wedge |x| = |-x|).$$

Theorem 2.2.25. If $a > 0$, then the following are equivalent:

- $|x| \leq a$
- $x \leq a$
- $-x \leq a$

Proof. By definition of absolute value,

$$|x| \leq a \iff \max\{x, -x\} \leq a.$$

It follows that if $0 < x$, then $x \leq a$. Otherwise, if $x < 0$, then $-x \leq a$. Either way, both $x \leq a$ and $-x \leq a$. $\square$

By Theorem 2.2.18, $-x \leq a \iff -a \leq x$, so that $-a \leq x \leq a$.

Theorem 2.2.26. (Triangle inequality). For all $x, y \in \mathbb{R}$, we have that $|x + y| \leq |x| + |y|$

Proof. Let $x, y > 0$ or $x, y < 0$. Then, $|x + y| = \max\{(x + y), -(x + y)\}$.
If $x \geq 0$ and $y < 0$, then

$$x + y \leq x \leq x + |y| = |x| + |y|$$
$$-x - y \leq -y \leq -y + |x| = |x| + |y|$$

so that $|x + y| = \max\{(x + y), -(x + y)\} \leq |x| + |y|$. $\square$

One may use induction to show that

$$\left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i|.$$

Lemma 2.2.27. If $x > 0$ and $y > 0$, then $xy > 0$.

Proof. This lemma follows from Axiom 5 and Corollary 2.2.9:

$$xy = 0 \wedge x \neq 0 \implies y = 1y = \frac{1}{x}xy = \frac{1}{x}(xy) = \frac{1}{x}0 = 0.$$

$\square$

Theorem 2.2.28. If $x \leq y$ and $0 < z$, then $xz \leq yz$

Proof. By Axiom 4,

$$yz - xz = (y - x)z \geq 0.$$

$\square$

One may note that by Corollary 2.2.9, $x < z \wedge z > 0 \implies xz < yz$. Additionally,

$0 < x < 1 \implies x^2 < x$ and $1 < x \implies x < x^2$. One can also show that

$$0 < x \leq y \wedge 0 < z \leq u \implies xz \leq yz \leq yu.$$

We note that one may multiply inequalities under these conditions. In general, if $0 < x < y$, then for all $n \in \mathbb{N}$, we have that $x^n < y^n$.

Theorem 2.2.29. If $x \leq 0$ and $0 \leq y$, then $xy \leq 0$. Similarly, if $x \leq 0$ and $y \leq 0$, then $0 \leq xy$.

Proof. Since $x \leq 0$, we have that $0 \leq -x$. It follows from Axiom 5 and Theorem 2.2.11 that $0 \leq -xy \iff xy \leq 0$.
Similarly, $x \leq 0$ and $y \leq 0$ implies that $0 \leq -x$ and $0 \leq -y$, so that $0 \leq (-x)(-y) = -1(x) - 1(y) = -1(-1)xy = 1xy = xy$. □

In particular, one may note that $x^2 = xx > 0 \forall x \neq 0$. One consequence of this fact is that $1 = 1(1) > 0$. Also, by Lemma 2.2.19, $2 = 1 + 1 > 0$, $3 = 2 + 1 > 2$, etc.

Theorem 2.2.30. For all $x, y \in \mathbb{R}$, we have that $|xy| = |x||y|$.

Proof. By definition of absolute value, $|xy| = \max\{xy, -xy\}$. Let $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$. Then, $\max\{xy, -xy\} = |x||y|$ by Theorem 2.2.29. Suppose without loss of generality that $x > 0$ and $y < 0$. Then, $xy < 0$. However, we now have that $\max\{xy, -xy\} = (-x)y = |x||y|$. If $x = 0$ or $y = 0$, then the absolute value is 0 by Corollary 2.2.9. □

Theorem 2.2.31. If $x > 0$, then $\frac{1}{x} > 0$. Moreover, if $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

Proof. The quotient $\frac{1}{x}$ cannot be 0 since it is undefined for that value. Let $\frac{1}{x} < 0$. Then we would have $1 < 0$, which is impossible for $1 = 1^2 > 0$. Moreover, one can multiply the inequality $0 < x < y$ by $\frac{1}{xy}$ to get $0 < x\frac{1}{xy} < y\frac{1}{xy} \iff 0 < \frac{1}{y} < \frac{1}{x}$. □

Theorem 2.2.32. Let $r \in \mathbb{R}$ and $r > 0$. For all r, if $z \geq 0$ and $z < r$, then $z = 0$.

Proof. If $z > 0$, then then $z > z \geq 0$, which is impossible by conexity. □

2.2.4 Axiom of Completeness

Definition 2.2.33. (Lower Bounds). A set $E \subset \mathbb{R}$ is bounded above when

$$\exists z \in \mathbb{R} \forall x \in E(x \leq z).$$

We denote that E is bounded above by $E \leq z$. We call the element z an upper bound of the set E.
Similarly, A set $E \subset \mathbb{R}$ is bounded below if $\exists z \in \mathbb{R} \forall x \in E(z \leq x)$. We denote that E is bounded below by $z \leq E$. All numbers z that satisfy this property are called lower bounds of E.

Proposition 2.2.34. If a set E is bounded above, then the set $-E = \{-x : x \in E\}$ is bounded below.

Proof. By definition of upper bounds, there exists $z \in \mathbb{R}$ such that for all $x \in E$, $x \leq z$. Taking their negative elements, one can find that for all $-x \in -E$, $-z \leq -x$. The number $-z \in \mathbb{R}$ easily satisfies the definition of a lower bound. $\square$

One may also show that if E is bounded below by z, then $-E$ is bounded above by $-z$ in a similar way.

Theorem 2.2.35. Every set $E \subset \mathbb{R}$ bounded below has a greatest lower bound $-\sup(-E)$.

Proof. For $-E = \{-x : x \in E\}$, we find using Proposition 2.2.34 that $-E$ is bounded above. By Axiom 2.4, there exists $\sup(-E)$. Taking the inequality $-x \leq \sup(-E)$, we find that $x \geq -\sup(-E)$, satisfying E having a greatest lower bound. $\square$

We denote the greatest lower bound of E as $\inf(E)$, and call it the infimum.

Theorem 2.2.36. For $E \subseteq F$, if E and F are bounded above, then $\sup(E) \leq \sup(F)$. Likewise, if E and F are bounded below, then $\inf(E) \geq \inf(F)$.

Proof. Let $E = F$. They then have the same supremum. Let $E \subset F$. There exists at least one $z \in F$ such that $z \notin E$. If $z > \sup(E)$, then $\sup(E) < \sup(F)$. If $z < \sup(E)$, then $\sup(E) = \sup(F)$. One may use a similar way to prove the lower bounded case. $\square$

Theorem 2.2.37. For arbitrary $x \in E, y \in F$, if $x \leq y$, then E is bounded above, F is bounded below, and $\sup(E) \leq \inf(F)$.

Proof. An arbitrary $x \in E$ immediately satisfies the definition for a lower bound since for all $y \in F$, $x \leq y$. Likewise holds for arbitrary $y \in F$ being an upper bound since for all $x \in E$, $y \geq x$. It immediately follows that $\sup(E) \leq F$ such that $\sup(E) \leq \inf(F)$. $\square$

Theorem 2.2.38. For all $x \in \mathbb{R}$ and $n \in \mathbb{Z}, x, n > 0$, there exists a unique $y > 0$ such that $y^n = x$.

Proof. Let A be the set of all $z > 0$ such that $z^n \leq x$. Then A is bounded above, by 1 if $x \leq 1$ and by x if $1 \leq x$. Let $y = \sup(A)$. We will show that $y^n = x$.
Suppose that $y^n < x$ and let $x - y^n = \varepsilon$. For all h such that $0 < h \leq 1$, and by Theorem 2.2.15:

$$\begin{aligned}(y+h)^n &= y^n + ny^{n-1}h + \frac{n(n-1)}{1 \cdot 2}y^{n-2}h^2 + \dots \\ &= y^n + h(ny^{n-1} + \frac{n(n-1)}{1 \cdot 2}y^{n-2}h + \dots) \\ &\leq y^n + h(ny^{n-1} + \frac{n(n-1)}{1 \cdot 2}y^{n-2} + \dots) = y^n + h[(1+y)^n - y^n]\end{aligned}$$

Set $h < \frac{\varepsilon}{(1+y)^n - y^n}$ so that we have $(y+h)^n \leq y^n + \varepsilon = x$. However, this contradicts the fact that $y = \sup(A)$. Therefore, $y^n \geq x$.
Suppose that $y^n > x$. Let $\varepsilon = y^n - x$. For all h such that $0 < h \leq 1$, and by Theorem 2.2.15:

$$\begin{aligned}(y-h)^n &= y^n - ny^{n-1}h + \frac{n(n-1)}{1 \cdot 2}y^{n-2}h^2 + \dots \\ &= y^n - h(ny^{n-1} + \frac{n(n-1)}{1 \cdot 2}y^{n-2}h + \dots) \\ &\geq y^n - h(ny^{n-1} + \frac{n(n-1)}{1 \cdot 2}y^{n-2} + \dots) = y^n - h[(1+y)^n - y^n]\end{aligned}$$

Set $h < \frac{\varepsilon}{(1+y)^n - y^n}$ so that we have $(y-h)^n \geq y^n - \varepsilon = x$. However, this contradicts the fact that $y = \sup(A)$. Therefore, $y^n = x$.
We call y the nth root of x. We denote the nth root of x by $\sqrt[n]{x}$, or by $x^{\frac{1}{n}}$. The nth root is unique since $y_1 < y_2 \implies y_1^n < y_2^n \iff x < x$, which is impossible. $\square$

Theorem 2.2.39. For all $x > 0$ and $y > 0$, $(xy)^{\frac{1}{n}} = x^{\frac{1}{n}} y^{\frac{1}{n}}$

Proof. Note that

$$xy = x^{\frac{n}{n}} y^{\frac{n}{n}} = (xy)^{\frac{n}{n}} \iff (x^{\frac{1}{n}} y^{\frac{1}{n}})^n = \left[(xy)^{\frac{1}{n}}\right]^n \iff (x^{\frac{1}{n}} y^{\frac{1}{n}}) = \left[(xy)^{\frac{1}{n}}\right].$$

□

Simlarly, one can show that $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$.

Suppose an even interger n. Then, $(-x)^n = (-1)^n x^n = x^n > 0 \forall x \neq 0$. The equation $y^n = x > 0$ has both real solutions $y_1 = \sqrt[n]{x}$ and $y_2 = -\sqrt[n]{x}$, while the equation $y^n = x < 0$ has no real solutions.

Suppose an odd integer n. Then, $y^n = x > 0$ has a unique real solution $y = \sqrt[n]{x}$, while $y^n = x < 0$ has a unique real solution $y = -\sqrt[n]{x}$.

By Theorem 2.2.39, formulas like the quadratic and cubic equations hold for all real numbers.

Definition 2.2.40. A set $E \subseteq \mathbb{R}$ is bounded from both sides or bounded if it has both upper and lower bounds. It follows that both $\inf(E)$ and $\sup(E)$ exist.

One important class of bounded sets is the open interval from Definition 1.3.7. One may see that $\sup[a,b] = \sup(a,b)$, and that $\inf[a,b] = \inf(a,b)$. Both 'half-closed' and 'half-open' intervals $[a,b)$ and $(a,b]$ are defined analogously.

2.3 Archimedean Property and its Consequences

Theorem 2.3.1. (Archimedean Property). For all real $x \geq 0$ and $y > 0$, there exists $n \in \mathbb{Z}$ such that $(n-1)x \leq y < nx$.

Proof. Suppose that for all $p \in \mathbb{Z}$, we have that $px \leq y$. Define the set $A = \{px | px \leq y\}$, and see that y forms an upper bound for A. Let $\sup(A) = d \leq y$. Take the difference $d - x < d$. We can set $p_0 \in \mathbb{Z}$ such that $d - x < p_0 x$. However, $d - x < p_0 x \iff d < p_0 x + x \iff d < (p_0 + 1)x$ with $(p_0+1)x \in A$, forming a contradiction for d being an upper bound.
One may set $\{(p_i - 1), p_i\} = \{(n+1), n\}$ so that $(n-1)x \leq y < nx$. □

If we set $x = 1$, we find that for all $y \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that $n - 1 \leq y \leq n$. We call n the integral part of y, denoted $[y]$, and $y - [y]$ the fractional part y, denoted (y). Also, $\forall y \in \mathbb{R}[y \leq [y] + (y)]$.

Theorem 2.3.2. For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$, $x, y > 0$, there exists $n \in \mathbb{Z}$ such that $x^{n-1} \leq y < x^n$.

Proof. Suppose that for all $p \in \mathbb{Z}$, we have that $x^{p-1} < y$. Define the set $A = \{x^p | x^p \leq y\}$, and see that y forms an upper bound for A. Let $\sup(A) = d \leq y$. Take the quotients $\frac{d}{x} < d$. We can set p_0 such that $\frac{d}{x} < x^{p_0}$. However, $\frac{d}{x} < x^{p_0} \iff d < xx^{p_0} \iff d < x^{p_0+1}$ with $x^{p_0+1} \in A$, forming a contradiction for d being an upper bound.
One may set from pairs $\{(p_i - 1), p_i\} = \{(n+1), n\}$ so that $x^{n-1} \leq y < x^n$. □

Theorem 2.3.3. For all $x, y > 0$, there exists an integer $n > 0$ such that $\frac{y}{n} < x$.

Proof. By the Archimedean Property, there exists $n \in \mathbb{Z}$ such that $y < xn$. Set $n > \frac{y}{x} > 0$. Multiply both sides by $\frac{x}{n}$ to get $\frac{y}{n} < x$. □

It follows that for all $y > 0$, $\inf\{\frac{y}{n} | n \in \mathbb{Z}^+\} = 0$.

Corollary 2.3.4. The following systems of half-open intervals for $y > 0$ has an empty intersection:

$$\ldots \subseteq (0, \frac{y}{n}] \subseteq \ldots \subseteq (0, \frac{y}{2}] \subseteq (0, y] \tag{2.1}$$

$$\ldots \subseteq (a, a + \frac{y}{n}] \subseteq \ldots \subseteq (a, a + \frac{y}{2}] \subseteq (a, a + y] \tag{2.2}$$

$$\ldots \subseteq (a - \frac{y}{n}, a] \subseteq \ldots \subseteq (a - \frac{y}{2}, a] \subseteq (a - y, a] \tag{2.3}$$

Proof. Suppose systems 2.2 and 2.3 had common elements c and d, respectively. That would mean that system 2.1 would have commmon points $c - a$ and $a - d$. However, by Theorem 2.3.3, system 2.1 has no common points. □

Theorem 2.3.5. For all open intervals (a, b), there exists an element $z \in \mathbb{Q}$ such that $a < z < b$.

Proof. Take the difference $b - a = h$. By Theorem 2.3.1, there exists $n \in \mathbb{N}$ for $\frac{1}{h}$ such that $(n-1) \leq \frac{1}{h} < n \iff \frac{1}{n} < h$. By that same theorem, there exists $m \in \mathbb{N}$ for $\frac{1}{n}$ such that $\frac{m}{n} \leq a < \frac{m+1}{n}$. Add $\frac{1}{n}$ to both sides and subtract a to obtain $\frac{m}{n} + \frac{1}{n} - a \leq \frac{1}{n} + a - a \iff \frac{m+1}{n} - a \leq \frac{1}{n} < h = b - a \iff \frac{m+1}{n} < b$, showing that there exists $z = \frac{m+q}{n}$ so that $a < z < b$. □

One can show that infinitely many rational elements exist in (a, b) by applying the preceding theorem to the interval $(\frac{m+1}{n}, b)$, and so on.

Theorem 2.3.6. For all $c \in \mathbb{R}$, let N_c be the set of all $s \in \mathbb{Q}$ such that $s \leq c$, and P_c the set of all $r \in \mathbb{Q}$ such that $c \leq r$. Then, $\sup(N_c) = c = \inf(P_c)$.

Proof. The set N_c is bounded above by c, so that it has a supremum. Denote $\sup(N_c) = a$. By definition, $a \leq c$. Suppose that $a < c$. By Theorem 2.3.5, there exists a rational element p such that $p \in (a, c)$. Since $a = \sup(N_c)$, however, $p \leq a$, creating a contradiction. Therefore, $a = c = \sup(N_c)$. $\square$

Definition 2.3.7. If a set Q of intervals on $\mathbb{R}$ has the property that given any two intervals $I, J \in Q$, either $I \subseteq J$ or $J \subseteq I$ holds, then we call Q a system of nested intervals.

By Corollary 2.3.4, a system of half-open intervals may well have an empty intersection. The same holds for nested open intervals. However, we now show that a system of closed intervals always has an intersection.

Theorem 2.3.8. (Nested Intervals Property). For all systems of closed intervals $[a_i, b_i] \in Q$, there exists $c \in \mathbb{R}$ such that $c \in \bigcap_i [a_i, b_i]$. Specifically, their intersection comprises the interval $[\sup(A), \inf(B)]$ for sets $A = \{a_i : [a_i, b_i] \in Q\}$ and $B = \{b_i : [a_i, b_i] \in Q\}$.

Proof. Let A be the set of all left endpoints $A = \{a_i : [a_i, b_i] \in Q\}$, and B the set of all right endpoints $B = \{b_i : [a_i, b_i] \in Q\}$. Since for any two intervals $[a_i, b_i]$ and $[a_j, b_j]$, $i \leq j$, we have that $[a_j, b_j] \subseteq [a_i, b_i]$, it follows that $a_i \leq a_j$ and $b_j \leq b_i$. One may also note that all b_i serve as upper bounds to A, and all a_i serve as lower bounds to B. By Completeness, $\sup(A)$ and $\inf(B)$ both exist. Since for all $[a_i, b_i] \in Q$ we have $a_i \leq \sup(A) \leq \inf(B) \leq b_i$, it follows that $[\sup(A), \inf(B)] \subseteq \bigcap_i [a_i, b_i]$. Moreover, suppose an x where $x \notin [\sup(A), \inf(B)]$ such that $x < \sup(A)$. Then, we can always find an a_i where $x < a < \sup(A)$. A similar result holds for some $x > \inf(B)$, so that $\bigcap_i [a_i, b_i] = [\sup(A), \inf(B)]$. The case where $\sup(A) = \inf(B)$ gives the value for c. $\square$

Theorem 2.3.9. (Nested Intervals Property). For all systems of closed intervals $[a_i, b_i] \in Q$ there exists a single point c that serves as their only intersection if and only if for all $\varepsilon > 0$ there exists an interval $[a, b] \in Q$ such that $b - a < \varepsilon$.

Proof. From Theorem 2.3.8, the intersection of nested closed intervals consists of a single point if and only if $\sup(A) = \inf(B)$. For all $\varepsilon > 0$, there exists $[a_1, b_1]$ and $[a_2, b_2]$ with $[a_2, b_2] \subseteq [a_1, b_1]$ such that

$$a_1 > \sup(A) - \frac{\varepsilon}{2} \wedge b_2 < \inf(B) + \frac{\varepsilon}{2}.$$

Set $c = \sup(A) = \inf(B)$. Then,

$$c - \frac{\varepsilon}{2} < a_1 \leq a_2 \iff b_2 - \frac{\varepsilon}{2} < c < a_1 + \frac{\varepsilon}{2} \leq a_2 + \frac{\varepsilon}{2} \iff b_2 - a_2 < c + \frac{\varepsilon}{2} < a_1 + \varepsilon \leq \varepsilon$$

giving us $b_2 - a_2 < \varepsilon$ as the required interval.
For the converse, let there be an interval $[a_\varepsilon, b_\varepsilon]$ that depends on some $\varepsilon > 0$ such that $b_\varepsilon - a_\varepsilon < \varepsilon$. Taking them as part of their respective endpoint sets, we have that $a_\varepsilon \leq \sup(A)$ and $b_\varepsilon \geq \inf(B)$. Since $b_\varepsilon - a_\varepsilon \geq \sup(A) - \inf(B)$, succesively taking smaller values of ε would cause the righthand side to reduce to 0, such that $\sup(A) = \inf(B)$. $\square$

2.4 The Extended Real Number System

Definition 2.4.1. The extended real number system $\bar{\mathbb{R}}$ consists of the real number line $\mathbb{R}$ and two infinity elements $-\infty$ and $+\infty$. The usual order relations get extended by the following rules:

1. $\forall x \in \mathbb{R}(-\infty < x)$.
2. $\forall x \in \mathbb{R}(x < \infty)$.
3. $-\infty < \infty$.

Ordinary order axioms continue to hold in $\bar{\mathbb{R}}$. Elements $x \in \mathbb{R}$ are called finite, contrasting with infinity elements $-\infty$ and ∞.

For nonempty $E \subseteq \mathbb{R}$, we define $\sup(E) = \infty$ if $\infty \in E$ or there exists no x such that $E \leq x$. Likewise, $\inf(E) \neq -\infty$ if $-\infty \in E$ or there exists no x such that $E \geq x$.

For two points $a, b \in \bar{\mathbb{R}}, a < b$, then the set $[a, b] = \{x \in \bar{\mathbb{R}} : a \leq x \leq b\}$ is a closed interval with endpoints a and b. Meanwhile, the set $(a, b) = \{x \in \bar{\mathbb{R}} : a < x < b\}$ is called an open interval with the same endpoints.

We can also generalize the Nested Interval Property to $\bar{\mathbb{R}}$. For a system of closed intervals $Q = \{[a_i, b_i] : a, b \in \bar{\mathbb{R}} \wedge [a_j, b_j] \subseteq [a_i, b_i]\}$. Then, there exists $c \in \bar{\mathbb{R}}$ such that for all $[a_i, b_i]$, $c \in [a_i, b_i]$. This intersection corresponds to the interval $\sup(A), \inf(B)$ where $A = \{a_i : [a_i, b_i \in Q]\}$ and $B = \{b_i : [a_i, b_i]\}$, so that $\sup(A) \leq \inf(B)$.

Exercises

1. Prove that there exist bijections $f : \mathbb{R} \to (0, 1)$ and $g : \bar{\mathbb{R}} \to [0, 1]$, but not $\mathbb{R} \to [0, 1]$ and $\bar{\mathbb{R}} \to (0, 1)$.

3.0 Topology of Metric Spaces

3.1 Metric Spaces

Definition 3.1.1. (Metric Spaces). A metric space M is a set of points together with a unique metric $d : M \to \mathbb{R}$ satisfying the following axioms:

1. $\forall x, y \in M(d(x, y) \geq 0)$ where $d(x, y) = 0$ if and only if $x = y$ (Positive Definiteness).

2. $\forall x, y \in M(d(x, y) = d(y, x))$ (Symmetry).

3. $d(x, z) \leq d(x, y) + d(y, z)$ Triangle Inequality.

The triangle inequality generalizes the geometric fact that the length of one triangle side never exceeds the sum of the other sides' lengths.

Note that subsets of M with the same metric attached are considered metric subspaces of M. Lastly, one can form separate metric spaces from the same set E by specifying different metrics d_1 and d_2, such that we have metric spaces E_1 and E_2

We now prove two useful results concerning the triangle inequality.

Theorem 3.1.2. The inequality

$$d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n)$$

holds for arbitrary $x_1, \ldots, x_n \in M$.

Proof. We prove this theorem by successively applying the triangle inequality:

$$\begin{aligned} d(x_1, x_n) &\leq d(x_1, x_2) + d(x_2, x_n) \\ &\leq d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_n) \\ &\leq \ldots \leq d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n). \end{aligned}$$

□

Theorem 3.1.3. (Quadrilateral Inequality). For all points $x, y, z, v \in M$, the inequality

$$|d(x,y) - d(z,v)| \leq d(x,z) + d(y,v)$$

holds.

Proof. By triangle inequality, we have that $d(x,y) \leq d(x,z) + d(z,v) + d(v,y) \leq (z,y)$ and $d(v,z) \leq d(v,y) + d(y,x) + d(x,z)$. Take the differences:

$$d(x,y) - d(z,v) \leq d(x,z) + d(v,y)$$
$$d(v,z) - d(y,x) \leq d(v,y) + d(x,z).$$

Since $|a - b| = |b - a|$, and by symmetry and addition commutativity, we thus have

$$|d(x,y) - d(z,v)| \leq d(x,z) + d(y,v).$$

□

Note that setting $v = y$ gives $|d(x,y) - d(z,v)| \leq d(x,z)$, which in geometry means that the difference between to triangle side lengths may never exceed the length of the third side.

The triangle inequality and its consequences see much use not just in real analysis, but in numerous fields. One will experience its importance in the following sections.

Exercises

1. Prove that the absolute value function satisfies properties of a metric for points on the real line.

2. Demonstrate that the absolute value function satisfies theorems for metric.

3.

3.2 Open Sets

Definition 3.2.1. (Interior points and Open Sets). A point x_0 in a subset E of a metric space M is called an interior point of E when $d(x_0, x) < r \implies x \in E$, for a radius $r > 0$.
The set that includes x and x_0 satisfying the definition is called an open ball or neighborhood, denoted by $B(x_0)$, and with center x_0. A set where all elements are interior points is called an open set.

Note that this definition implies that any open set must contain an open ball. We now prove that all open balls themselves are open sets.

Theorem 3.2.2. Let M be a metric space. Then the open ball centered at $x_0 \in M$ such that $B(x_0) = \{x \in M : d(x_0, x) < r\}$ is an open set.

Proof. There exists some arbitrary $y \in B(x_0)$ such that $y \neq x_0$ and that satisfies the distance $d(x_0, y) = \theta < r$. For an open ball $B_1(y)$ with radius $r_1 < r - \theta$, we have for all $x_i \in B_1$ that $d(x_i, y) \leq d(x_i, x_0) + d(x_0, y) < r_1 + \theta < r$. Since the point y was arbitrary, all points in the open ball are thus interior points. □

Theorem 3.2.3. The union of any family and the intersection of a finite family of open sets are themselves open sets.

Proof. Suppose open sets $A_i, i \in \mathbb{N}$, which could be infinitely many. Open balls $B(x_i)$ exist for all $x_i \in A_i$, such that all points k_i satisfying $d(x_i, k_i) < r$, for each $r > 0$, are also elements of A_i. Taking the union of all sets A_i thus also includes all points k_i, such that all open balls $B(x_i)$ are contained in the union.
Now suppose n open sets A_i with a nonempty intersection. For points belonging to all sets A_i, an open ball of some radius must also be contained in all sets. For arbitrary common point $x_i \in \bigcap_{i=1}^n A_i$, the specific open ball is the one that satisfies $B(x_i) = \{x \in \bigcap_{i=1}^n A_i : d(x, x_i) < r_{\min}\}$. The radius $r_{\min}$ satisfies $r_{\min} = \min(r_1, r_2, \ldots, r_n)$, where r_i is the radius of $B_i(x_i)$ for each set A_i. □

Theorem 3.2.4. All open sets G on the real line are finite or countable unions of non-intersecting open intervals.

Proof. We first note that the open interval (a, b) is an open set (prove this as an exercise). As such, we need only to construct the specific intervals. Set $G = (a, b)$ without loss of generality, then take endpoints $a < a_1 < a_2 < \dots$ and $\dots < b_2 < b_1 < b$. We thus have component intervals $(a, a_1), (a_1, a_2), \dots, (b_3, b_2), (b_2, b_1), (b_1, b)$. One may repeat this step again with new endpoints $a < a_4 < a_1 < a_5 < a_2 < a_6 < \dots$ and $\dots < b_6 < b_2 < b_5 < b_1 < b_4 < b$, and again interatively and indefinitely. Taking the union of all the component intervals gives the set G. Since a bijection exists from the natural numbers to each interval as numbered, the sets are clearly countable. $\square$

Exercises

1. Provide a counterexample that shows how the intersection of infinitely many open sets may itself not be an open set (Hint: what happens when $d(x_0, x) = 0$?).

2. Prove that the open interval (a, b) is an open set in $\mathbb{R}$ (Hint: use the midpoint of the interval).

3. Is the closed interval $[a, b] \subset X$ where $[a, b] = X$ an open set in the context of X? (Hint: we assume that the real numbers don't exist for this item).

3.3 Sequences and Convergence

Definition 3.3.1. For a metric space M, we call an ordered list of points $p_1, p_2, \dots \in M$ a sequence $(p_n), n \in \mathbb{N}$:

$$f : \mathbb{N} \to M$$

where $f(n) = p_n$. Points can repeat in the sequence, and only some points of M need to appear in the list.

Definition 3.3.2. A sequence (p_n) is said to converge to some limit p if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N \implies d(p_n, p) < \varepsilon$. Topologically, this means that a convergent sequence enters a neighborhood centered on the limit, and never continues past the limit.

Example

Let $M = \mathbb{R}$. The real line has $d(x, y) = |x - y|$ for its usual metric. Any sequence of real numbers $(x_n) \in \mathbb{R}$ converges to some $x \in \mathbb{R}$ if for all ε there exists a specific $N \in \mathbb{N}$ such that $d(x_n, y) < \varepsilon$ for all $n \geq N$. If this sequence is $(\frac{1}{n})$, then it converges to 0. For all $\varepsilon > 0$,our specific N here is $N = \frac{1}{\varepsilon}$ so that $0 < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ for all $n \geq N$. For instance, if $\varepsilon = \frac{1}{5}$, then $N = 5$, and and all succeeding elements $\frac{1}{6}, \frac{1}{7}, \ldots$ have a smaller distance to 0 than $\frac{1}{5}$:

$$n > 5 \implies |\frac{1}{5} - 0| > |\frac{1}{n} - 0|.$$

We prove the existence of this limit as an example.

Proposition 3.3.3. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

Proof. By definition of limits, for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) < \frac{\varepsilon}{2}$ holds. Note that we can do this since $\varepsilon > 0$ is arbitrary, and a large enough ε will let elements of the limit's neighborhood to fall within half of ε reach. Similarly, for all ε there also exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(y_n, y) < \frac{\varepsilon}{2}$ holds. Now, by quadrilateral inequality, we have that

$$|d(x_n, y_n) - d(x, y)| \leq d(x, x_n) + d(y_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

so that we have proved the limit. □

Sometimes, proving a result relies on a specific value of $\varepsilon > 0$ to supply a contradiction. We can do this since assuming a false necessary condition for all instances of the sufficient condition lets us prove a theorem with a single counterexample, or in this case the contradiction. We demonstrate this below.

Proposition 3.3.4. Let all sequences be on the real line. If $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, and $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $x \leq y$.

Proof. For arbitrary $\varepsilon > 0$ we have $N \in \mathbb{N}$ so that $n \geq N \implies |x_n - x| < \frac{\varepsilon}{2}$. We may allow similar conditions so that $n \geq N \implies |y_n - y| < \frac{\varepsilon}{2}$. Assume for contradiction that $x > y$. Then for $\varepsilon = x - y$, the corresponding N allows that for all $n \geq N$:

$$x_n > x - \frac{\varepsilon}{2} = (y + \varepsilon) - \frac{\varepsilon}{2} > y_n - \frac{\varepsilon}{2} + \varepsilon - \frac{\varepsilon}{2} = y_n$$

which contradicts that $n \geq N \implies x_n \leq y_n$. Thus, we have proven the limit. □

We now prove important results that let one calculate limits in Calculus easily.

Theorem 3.3.5. (Sum of Limits). Let all sequences be on the real line. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} x_n + y_n = x + y$.

Proof. For arbitrary $\varepsilon > 0$ we have $N \in \mathbb{N}$ so that $n \geq N \implies |x_n - x| < \frac{\varepsilon}{2}$. We may allow similar conditions so that $n \geq N \implies |y_n - y| < \frac{\varepsilon}{2}$. We then have:

$$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Theorem 3.3.6. (Uniqueness of Limits). Limits of sequences are unique.

Proof. Let $\lim_{x\to\infty}(x_n) = x$. By definition, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq \mathbb{N} \implies d(x_n, x) < \frac{\varepsilon}{2}$. Suppose that (x_n) converges to another limit y. By definiton again, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq \mathbb{N} \implies d(x_n, y) < \frac{\varepsilon}{2}$. Now, we have that $d(x, y) \leq d(x, x_n) + d(y, y_n) < \varepsilon$. By Theorem 2.2.32 and positive definiteness of distances, we have that $d(x, y) = 0 \iff x = y$, such that limits are unique. □

Theorem 3.3.7. Every convergent sequence is bounded. Alternatively, the numbers $d(x_n, a)$ for some point a belonging to a metric space M form a bounded set on the real line.

Proof. Simply take $d_{\max} = \max(d(x_1, a), d(x_2, a), \ldots, d(x_N, a), \ldots)$ which satisfies $d_{max} \geq d(x_n, a)$. □

Of special importance in real analysis is knowing which sequnces converge or not. Certain properties of functions aid in doing so.

Definition 3.3.8. (Continuous Functions). Let M and N be metric spaces with respective metrics d_M and d_N, and (x_n) a sequence in M with limit x. A function $f : M \to N$ is continuous if $(x_n) \to x \implies f(x_n) \to f(x)$.

Theorem 3.3.9. The composite of continuous functions is continuous.

Proof. For metric spaces M, N, and O, let $f : M \to N$ and $g : N \to O$ be arbitrary continuous functions. For sequences (x_n) in M with limit x, we have that $x_n \to x \implies f(x_n) \to f(x)$. However, $(f(x_n))$ is a sequence in N. By continuity of the function g, we have that $g(f(x_n)) \to g(f(x))$, such that the arbitrary composite of these arbitrary continuous functions is itself continuous. □

Using the concept of continuity for sequences, we can introduce homeomorphisms.

Definition 3.3.10. (Homeomorphisms). Two metric spaces M and N with $x_n \in M$ and $y_n \in N$ are homeomorphic if there exists a continuous bijection $f : M \to N$ where $f(x_n) = y_n$. We call the injection f a homeomorphism.

Geometrically, one may think of M and N as two spaces. Homeomorphisms must transform M into N without "damaging" M in the process. However, twisting, bending, stertching, and wrinkling are all acceptable. Properties that stay the same between M and N are said to be invariant, and these properties are topological properties.

Important to limits and functions is the ε, δ definition of continuity.

Theorem 3.3.11. (ε, δ Condition). A function $f : M \to P$ is continuous if and only if for each $\varepsilon > 0$ and $x \in M$ there exists $\delta > 0$ such that $d_M(x_n, x) < \delta \implies d_P(f(x_n), f(x)) < \varepsilon$.

Proof. Let f be continuous. Assume for contradiction that there exists a $\varepsilon > 0$ for all $\delta > 0$ such that the necessary condition doesn't hold. Then for that ε, it must be that $d_M(x_n, x) < \delta \implies d_N(f(x_n), f(x)) \geq \varepsilon$. While the sequence (x_n) converges to x for all $n \geq N$ such that $d_P(x_n, x) < \delta$, there exists a point $f(x_{n_0}) \in P, n_0 \geq N$ where $d(f_{x_{n_0}}, f(x)) \geq \varepsilon$. However, this contradicts the hypothesis that f is continuous since the sequence $(f(x_n))$ doesn't converge to $f(x)$ for $n_0 \geq N$. $\square$

Corollary 3.3.12. An injection $f : M \to P$ is homeomorphic if and only if the ε, δ Condition holds.

Proof. Prove this as an exercise. $\square$

For any sequence, we can always take certain elements to construct their own sequence.

Definition 3.3.13. (Adherent Points). Suppose sequences (x_n) and (y_k) where elements of x_n indexed by $n_1 < n_2 < \ldots$ exist satisfying $y_k = x_{n_k}$. Then, we call y_k a subsequence of (x_n).

Theorem 3.3.14. All subsequences (y_k) of convergent sequences (x_n) converge to the same limit as the original sequence.

Proof. Prove this theorem as an exercise. $\square$

Theorem 3.3.15. (Open Set Continuity Condition). A function $f : M \to N$ is continuous if and only if the preimage of each open set N is itself open in M.

Proof. Let f be continuous with open image sets $G = \{f(m) : m \in M\}$. Since any G is open, all points within neighborhoods of any $f(m)$ with some radius $r > 0$ are themselves elements of G. Since f is continuous, there exists $f(m)$ that is a limit of a sequence $f(m_n)$. By the ε, δ condition, for all $r > 0$ there exists $\delta > 0$ such that $d(m_n, m) < \delta \implies d(f(m_n), f(m)) < r$. Suppose that $f(m) \in K$ for some arbitrary $f(m)$ and open $G \subset N$. For any $r > 0$, all points within the neighborhood of $f(m)$ are also elements of G. However, continuity of f implies that the preimage $f_{\text{pre}}(G)$ must hold m. Since convergent sequences are preserved, points $f(m_n)$ satisfying $d(f(m_n), f(m)) < r$ must have corresponding points m_n satisfying $d(m_n, m) < \delta$ in $f_{\text{pre}}(G)$. We now have a neighborhood around m contained in $f_{\text{pre}}(G)$, such that $f_{\text{pre}}(G)$ is an open set.
Prove the converse as an exercise. □

Exercises

1. Let $x_n \to x$ and $y_n \to y$. Prove that $\lim_{n\to\infty} x_n y_n = xy$.
2. Prove that the identity mapping $\text{id} : M \to M, \text{id}(x_n) = x_n$ and the constant function $f : M \to N, f(x_n) = q \forall n$ are both continuous functions.
3. Prove Corollary 3.3.12 by specifying what to change in Theorem 3.3.11.
4. Prove Theorem 3.3.14. (Hint: what do we set k to?).
5. Prove the converse part of Theorem 3.3.15.

3.4 Accumulating Ponts

Definition 3.4.1. (Adherent Points). Suppose a metric space M with metric d, and a point x not necessarily belonging to M. Let some open ball $B(x)$ with radius $r > 0$ exist. If $p \in B(x)$ implies that $p \in M$, then we call x an adherent point of M.

A special type of adherent point is our focus this section.

Definition 3.4.2. (Accumulating Points). Suppose a metric space M with metric d, and an adherent point $x \in M$ with an open ball $B(x)$ of radius $r > 0$. If there exists $p \in M$ where $p \neq x$ such that $p \in B(x)$, then we call x an accumulating point of M.

Other texts call accumulating points as limit points. Other than sets, sequences may also have limit points.

Definition 3.4.3. (Accumulating Points for Sequences). Suppose a metric space M with metric d, and (x_n) a sequence in M. If for all ε there exists $N \in \mathbb{N}$ such that some $n_0 \geq N$ implies $d(x_{n_0}, x) < \varepsilon$, then we call x an accumulating point of x_n.

It stands that convergent sequences have only one, while divergent sequences may have none, one, or many.

Theorem 3.4.4. (Accumulating points and subsequences). A point x is an accumulating point of a sequence (x_n) if and only if (x_n) has a subsequence converging to x.

Proof. Let x be an accumulating point of a sequence (x_n), and $\varepsilon > 0$ be arbitrary. There exists some $N \in \mathbb{N}$ where at least one $n_0 \geq N$ satisfies $d(x, x_{n_0}) < \varepsilon$. We can construct a subsequence (y_k) where for $k \leq n_0$ we have that $y_k = x_{n_m}$. We thus have a subsequence that converges to x, regardless of whether the original sequence (x_n) converges.
For the converse, suppose that (x_n) has a subsequence converging to x. Then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for some $n_0 \geq N$, we have that $d(x, x_{n_0}) < \varepsilon$. By the subsequence's convergence, its limit is the sequence's accumulating point. $\square$

Corollary 3.4.5. If $x \in M_1$ is an accumulating point of (x_n) for all $x_n \in M_1$, and if M_1 is homeomorphic to M_2, then then the point $y \in M_2$ corresponding to x is a limit point of the sequence (y_n) for all $y_n \in M_2$ corresponding to $x_n \in M_1$.

Proof. Since homeomorphisms preserve convergent sequences between transformations, one may apply the preceding theorem to prove this corollary. $\square$

Thus we have that accumulating points of a sequence remain so in homeomorphic metrics. If we set two homeomorphic metrics d_1 and $d_2, d_1 \neq d_2$ on a set M, then for some accumulating point x of a sequence (x_n) in M with respect to d_1 we have that x is also an accumulating point of (x_n) with respect to d_2.

Theorem 3.4.6. Let $A \subseteq \mathbb{R}$ and $\sup(A) \notin A$. Then $\sup(A)$ is a accumulating point of A.

Proof. By definition, $A \leq \sup(A)$ such that for all $\varepsilon > 0$ there exists some $a \in A$ satisfying $\sup(A) - \varepsilon \leq a \leq \sup(A)$. Furthermore, $\sup(A) \neq a$ since $\sup(A) \notin A$, satisfying the definition of an accumulating point. □

Definition 3.4.7. (Accumulating Points for Subsets). A point $x \in M$ with metric d is an accumulating point for a subset $A \subseteq M$ if for all neighborhoods $B(x) = \{y \in M : d(x,y) < \varepsilon\}$ there exists some point $y \in A$ such that $y \neq x$.

We note that accumulating points of sequences are not necessarily accumulating points of subsets since points may repeat in a sequence, but not subsets. However, preceding Theorems regarding accumulating points apply to subsets nonetheless.

Theorem 3.4.8. Let $A \subseteq \mathbb{R}$ and $\sup(A) \notin A$. Then $\sup(A)$ is a accumulating point of A.

Proof. By definition, $A \leq \sup(A)$ such that for all $\varepsilon > 0$ there exists some $a \in A$ satisfying $\sup(A) - \varepsilon \leq a \leq \sup(A)$. Furthermore, $\sup(A) \neq a$ since $\sup(A) \notin A$, satisfying the definition of an accumulating point. □

Theorem 3.4.9. (Bolzano-Weirstrass Theorem for Sets). For an infinite subset $E \subset [a,b]$, there exists some accumulating point of E.

Proof. Take the component intervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. Some subset of E is contained in either component. Take any component satisfying such, then repeat the construction indefnitely. By the Nested Intervals Property, there exists an intersection consisting of a single point x_0. We then have a radius of length $\frac{a+b}{4^n}$ (which is half any component's length/"diameter"), where n is the iteration of our constructive process. Hence, any point $x_n \neq x_0$ in the constructed component interval is within the intersection's neighborhood. □

An analogous result holds for all sequences within the closed interval $[a,b]$.

Sets on $\mathbb{R}$ without accumulating points exist, like $\mathbb{N}$. However, every infinite set on $\bar{\mathbb{R}}$ does. Suppose this function defined:

$$f(x) = \frac{x}{1+|x|}, -\infty < x < \infty.$$

We ask the reader to prove results relating to this function in the exercises.

Theorem 3.4.10. Suppose a metric $d(x_n, x_m)$ of points of the sequence (x_n) bounded below by $a > 0$ so that $d(x_n, x_m) \geq a$. Then, (x_n) has no accumulating points.

Proof. Assume for contradiction that there exists an accumulating point x. Then for all $\varepsilon > 0$ there exists n and m so that $d(x, x_n) < \frac{a}{2}$ and $d(x, x_m) < \frac{a}{2}$, implying that

$$d(x_n, x_m) \leq d(x, x_n) + d(x, x_m) < a$$

contradicting the hypothesis that $d(x_n, x_m) \geq a$. $\square$

Exercises

1. Prove that neighborhoods of accumulating points contain infinitely many points of a subset $A \subseteq M$. (Hint: set an open ball with a smaller radius).
2. Suppose the function $f(x)$ defined on $\bar{\mathbb{R}}$ above for the following four items. Prove that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq |x - y|$.
3. Prove that for x, y satisfying $|f(x)| \leq 1 - \delta$ and $|f(y)| \leq 1 - \delta$ with $0 < \delta < 1$, we have that $|x - y| \leq \frac{1}{\delta^2}|f(x) - f(y)|$.
4. Given points $x, y \in \bar{\mathbb{R}}$, let $r(x, y) = |f(x) - f(y)|$. Then $r(x, y)$ satisfies the metric properties.
5. The metric $r(x, y)$ is homeomorphic to the metric $d(x, y), |x - y|$ on the metric space $\mathbb{R}$.
6. Prove Theorem 3.4.10 by using the case where $x_n = x_m$.

3.5 Closed Sets

Definition 3.5.1. (Closed Sets). For a metric space M, a subset $F \subseteq M$ is a closed set if it includes all its accumulating points.

Note that open and closed sets are not mutually exclusive. A set can be open, closed, both, or neither. Sets that both open and closed are called clopen sets. Examples of such sets are $\mathbb{R}$ and $\emptyset$. The collection of open subsets of a clopen set M is the latter's topology.

Theorem 3.5.2. The closed ball $V(x_0) = \{x \in M : d(x, x_0) \leq r\}$ is closed.

Proof. Let y be any point where $y \notin V(x_0)$. Thus, we have that $d(x_0, y) = r_1 > r$. For a neighborhood $B(y)$ with radius $\frac{1}{2}(r_1 - r)$, assume for contradiction that there exists any point $x_i \in V(x_0)$ that also satisfies $x_i \in B(y)$ such that y is an accumulating point of $V(x_0)$. Then we have that

$$d(y, x_0) \leq d(y, x_i) + d(x_i, x_0) < \frac{1}{2}(r_1 - r) + r = \frac{1}{2}r_1 + \frac{1}{2}r < r_1$$

which contradicts the fact that $d(x_0, y) = r_1 > r$. □

Theorem 3.5.3. For a metric space M, the complement G of a closed set $F \subset M$ is open. Likewise, the complement F of an open set $G \subseteq M$ is closed.

Proof. Let $F \subseteq M$ be closed and set its complement as G. For G to be open, all neighborhoods of points of G include points that are strictly also points of G. If G is not open, then for some $x_0 \in G$ there exists some $y_0 \in F$ where $y_0 \in B(x_0)$. However, this implies that x_0 is an accumulating point of F, contradicting that F is closed and contains all its accumulating points.
Likewise, let $G \subseteq M$ be open. If its complement F is not closed, then there must exist some accumulating point x of F in G. However, the neighborhood $B(x_0)$ of this accumulating point contains only points $x_i \in G$, and likewise for neighborhoods $B(x_i)$. The union of neighborhoods $B(x_n)$ of a sequence (x_n) which converges to x must contain all points of (x_n) by setting a single radius for all open balls. Thus all points of (x_n) are also points of G. This finding contradicts that F and G are complements since the sequence (x_n) is strictly in F. □

Corollary 3.5.4. All closed sets $F \subseteq \mathbb{R}$ can be obtained as the complement of a countable family of disjoint open intervals of $\mathbb{R}$.

Proof. Taking the union of all disjoint open intervals gives an open set. The complement of all these open intervals are then closed sets. □

We call a component interval of an open set G complementary to some closed set F as adjacent to F.

Corollary 3.5.5. Let $F \subseteq \mathbb{R}$ be a closed set bounded above. Then $\sup(F) \in F$.

Proof. Since the supremum of a set is also one of its limit points by Theorem 3.4.6, $\sup(F) \in F$ since F is closed. □

Theorem 3.5.6. The union of a finite family of closed sets and the intersection of any family of closed sets are themselves closed.

Proof. Prove this as an exercise. □

Exercises

1. Prove Theorem 3.5.6.
2. Prove the Closed Set continuity condition: a function $f : M \to N$ is continuous if and only if the preimage of each closed set N is itself closed in M.

3.6 Dense Sets and Closures

Definition 3.6.1. (Dense Sets). A subset of a metric space $A \subseteq M$ with metric $d(x, y)$ is everywhere dense relative to $B \subseteq M$ if $x \in B$ implies that $x \in A$ or that x is a limit point of A.
If A is dense relative to B and $A \subseteq B$, then A is dense in B.

Example

The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

The set of rational coordinates $(x_1, \ldots, x_n)$ is dense in in $\mathbb{R}^n$.

Theorem 3.6.2. If a set A is dense relative to B and if B is dense relative to C, then A is dense relative to C.

Proof. Let $a \in A$, $b \in B$, and $c \in C$ be arbitrary. Set some $c_0 \in C$ to either belong to B or be a limit point of B. If the former, then c_0 must either belong to A or be a limit point of A, by the denseness of A relative to B. If the latter, then for all $\varepsilon > 0$ let $b \in B$ such that $d(c_0, b) < \frac{\varepsilon}{2}$. If b is a limit point of A, let $a \in A$ such that $d(a, b) < \frac{\varepsilon}{2}$. We can thus construct a neighborhood satisfying $d(a, c_0) \leq d(a, b) + d(b, c_0) < \varepsilon$. If b is also an element of A, then it automatically falls within the neighborhood of c_0. □

For non-closed metric spaces, we can construct a new set using their metric spaces.

Definition 3.6.3. (Closure). For a set A contained in a metric space M, we call the set $\bar{A}$ containing all $a \in A$ and limit points of A as the closure of A.
Note that $A \subseteq \bar{A}$ always holds. Also, if A is closed then $A = \bar{A}$.

Theorem 3.6.4. The closure $\bar{A}$ of a bounded set A is itself a bounded set, and moreover $\operatorname{diam}(A) = \operatorname{diam}(\bar{A})$.

Proof. Since $A \subseteq \bar{A}$, we have that $\operatorname{diam}(A) \leq \operatorname{diam}(\bar{A})$. We now need to show that $\operatorname{diam}(\bar{A}) \leq \operatorname{diam}(A)$ to prove both the equality and boundedness. For any $\varepsilon > 0$, we set two points $\bar{x}_1, \bar{x}_2 \in \bar{A}$ such that for some $x_1, x_2 \in A$, we have that $d(\bar{x}_1, x_1) < \frac{\varepsilon}{2}$ and $d(\bar{x}_2, x_2) < \frac{\varepsilon}{2}$. By triangle inequality, we have that

$$d(\bar{x}_1, \bar{x}_2) \leq d(\bar{x}_1, x_1) + d(x_1, x_2) + d(x_2, \bar{x}_2) < \frac{\varepsilon}{2} + \operatorname{diam}(A) + \frac{\varepsilon}{2} = \varepsilon + \operatorname{diam}(A).$$

Since $\varepsilon > 0$ is arbitrary, the nonnegative $d(\bar{x}_1, x_1)$ and $d(x_2, \bar{x}_2)$ reduce to 0. This inequality thus reduces to $\operatorname{diam}(\bar{A}) \leq \operatorname{diam}(A)$ so that $\bar{A}$ is bounded and $\operatorname{diam}(A) = \operatorname{diam}(\bar{A})$. □

Exercises

1. Prove prove that the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.
2. Since $\mathbb{R}$ is both closed and open, why is $\emptyset$ also clopen?

3.7 Complete Metric Spaces

Definition 3.7.1. (Cauchy sequences). We call a sequence (x_n) in a metric space M as a Cauchy sequence if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n \geq N \implies d(p_m, p_n) < \varepsilon$.

Geometrically, terms in a Cauchy sequence bunch together as the sequence continues. We now prove that all convergent sequences are Cauchy.

Theorem 3.7.2. All convergent sequences (x_n) are Cauchy.

Proof. Let x be the limit of (x_n). By definition, for any $\varepsilon > 0$ we can set $N \in \mathbb{N}$ such that $n \geq N \implies d(x_n, x) < \frac{\varepsilon}{2}$. For any $m, n \geq N$ we can write that

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

□

To prove the converse, we first define the notion of completeness.

Definition 3.7.3. (Completeness). A metric space M is complete if all Cauchy sequences in M converge to a limit in M.

Lemma 3.7.4. All Cauchy sequences are bounded.

Proof. Given a Cauchy sequence (a_n), for all $\varepsilon > 0$ set N_0 where $m, n \geq N_0 \implies d(a_n, a) < \varepsilon$.
We now have two choices of bounds:

$$m < N_0 \implies d(x_m, x_n) \leq \max(d(x_1, x_{N_0}), \dots, d(x_n, x_{N_0}))$$
$$m \geq N_0 \implies d(x_m, x_n) \leq \varepsilon.$$

□

Now we prove that the real line is complete with respect to Cauchy sequences.

Theorem 3.7.5. The real line $\mathbb{R}$ equipped with the usual metric $d(x, y) = |x - y|$ is complete with respect to Cauchy sequences.

Proof. By Lemma 3.7.4, all Cauchy sequences are bounded. Let $a_m = \inf_{n \geq m}(x_n)$ and $b_m = \sup_{n \geq m}(x_n)$. Clearly, $a_m \geq a_{m+1} \geq \dots$ and $\dots \leq b_{m+1} \leq b_m$ so that $[a_{m+1}, b_{m+1}] \subseteq [a_m, b_m]$. By Nested Interval Property, there exists $p \in \mathbb{R}$ in all such subintervals.
For any $\varepsilon > 0$ we can set $N \in \mathbb{N}$ such that $m, n \geq N \implies |x_m - x_n| < \varepsilon$. Fix $m = N$, and let $n = N, N+1, \dots$. Since $n \geq N \implies |x_N - x_m| < \varepsilon$, the same holds for numbers a_N and b_N.
Now since $p \in [a_N, b_N]$, then

$$n \geq N \implies |p - x_n| \leq b_N - a_{n+1} = (b_N - a_N) + (x_N - a_N) \leq 2\varepsilon.$$

Since ε is arbitary, it satisfies $p = \lim_{n \to \infty} x_n$. □

Completeness of the real line necessarily generalizes to n-dimensional Euclidean space.

Corollary 3.7.6. The n-dimensional Euclidean space $\mathbb{R}^m$ is complete.

Proof. Let $(\vec{p_n})$ be Cauchy in $\mathbb{R}^m$. Write it as

$$\vec{p_n} = (p_{1n}, p_{2n}, \ldots, p_{mn}).$$

The sequence (p_n) is Cauchy, and so must every component sequence (p_{in}) in $\mathbb{R}$. By completeness of the real line, every component sequence converges to a limit, and so must the vector $\vec{p_n}$. □

We now have an important result as a corollary.

Corollary 3.7.7. (Cauchy convergence criteria). A sequence (a_n) converges on the real line if and only if (a_n) is Cauchy.

Proof. Follows from Theorems 3.7.2 and 3.7.5. □

Note that Cauchy convergence fails to hold in some metric spaces.

Example

Equip the open interval $(0, a) \subseteq \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. The sequence $(\frac{1}{n})$ is Cauchy in $(0, 1)$, but has no limit since $0 \notin (0, 1)$. Note that the choice of metric also matters. The real line $\mathbb{R}$ is complete with the usual metric, but not with the metric $r(x, y)$ defined previously.

Theorem 3.7.8. If M is a complete metric space contained in P equipped, both with the same metric d, then M is a closed set in P.

Proof. Let $y \in P$ be a limit point of M, and let (x_n) converge to P. The sequence (x_n) is Cauchy, and must converge to some point in M by the latter's completeness. By uniqueness of limits, the sequence (x_n) converges to no other points outside of M, such that M holds all its limit points. □

Theorem 3.7.9. Let F be a closed set and $F \subseteq M$ for a complete metric space M. Then F is complete with the same metric as M.

Proof. Since M is complete, all Cauchy sequences converge to points of M. Let $m \in M$ such that for a Cauchy sequence $(x_n), x_n \in F$, we have that $\lim_{n\to\infty} x_n = m$. Since F contains all its limit points, we have that $m \in F$. The point m and sequence (x_n) were both arbitrary, such that the result holds for all Cauchy sequences in F. □

Next, we shall see the analogs for the Nested Interval Property exist for complete metric spaces. We first tackle a result concerning systems of nested subsets.

Lemma 3.7.10. Let Q be a system of nested subsets of a complete metric space M such that Q contains subsets with arbitrary small diameter. Then there exists a unique point $p \in M$ such that all neighborhoods $B(p) = \{x \in M : d(x, p) < \varepsilon\}$ contain some set $A \in Q$.

Proof. Some subsets $A_i \in Q$ must exist where $\sup(A_i) < \varepsilon_i$. Let $x_m \in A_m$. For $n > m$, then $d(x_n, x_m) < \varepsilon_m$ since either $A_n \subseteq A_m$ or $A_m \subseteq A_n$. Thus we have a Cauchy sequence (x_n). Set its limit when $n \to \infty$ to be p. □

Theorem 3.7.11. Let Q be a system of nested closed subsets of a complete metric space M where Q contain subsets with arbitrarily small diameter. Then:

1. There exists a unique point $p \in M$ such that all neighborhoods $B(p) = \{x \in M : d(x, p) < \varepsilon\}$ contains some set $A \in Q$.
2. Furthermore, p belongs to all sets belonging to Q.

Proof. Assertion (1) comes from Lemma 3.7.10. For assertion (2), we note that all subsets $A \in Q$ are closed. Note that p is a limit point of any set A. Since all sets A are closed, Cauchy sequences in any A must converge to points of A. The point p thus belongs to all subsets A. □

A special case of the preceding theorem is the Nested Balls Property, where the intersection of a sequence of nested closed balls on the real line consists of a single point. Note, however, that other metric spaces may have sequences of closed balls with an empty intersection.

Theorem 3.7.12. (Baire's Theorem). If a complete metric space M on the real line is the union of closed subsets $F_1, F_2, \ldots \subseteq M$, then at least one subset F_i contains a closed ball contained in M.

Proof. Assume for contradiction that no closed subset F_i contains a closed ball contained in W. For some $x_1 \notin F_1$, the closed ball $V_1(x_1)] = \{x : d(x, x_1) < \varepsilon_1\}$ has an empty intersection with F_2.

We can take another closed ball $V_{1.5}(x_1) = \{x : d(x_1, x) < \frac{\varepsilon_1}{2}\}$ that holds some $x_2 \notin F_2$. Like the preceding subset, the closed ball $V_2(x_2) = \{x : d(x, x_2) < \varepsilon_2\}$ has an empty intersection with F_2.

Perform this construction iteratively with $V_{2.5} = \{x : d(x_2, x) < \frac{\varepsilon_2}{2}\}$ which holds $x_3 \notin F_3$ and so on. These constructions form a system of nested closed balls. By the Nested Balls Property, the closed balls V_i have an intersection $\{x_0\}$ such that $x_0 \notin F_1, F_2, \ldots$. However, this contradicts the hypothesis that $x_0 \in M = \bigcup_{i=1} F_i$. $\square$

Example

The set C of irrational points belonging to $M = [a, b]$ cannot be represented as the countable union of closed subsets in M. If we take $C = \bigcup_{i=1} F_i$ for closed subsets $F_i \subseteq M$, then M becomes a countable union of closed subsets. However, this result contradicts Baire's theorem since no subset can contain a closed interval.

Note that some adherent points of metric spaces have neighborhoods containing said adherent point.

Definition 3.7.13. (Isolated points). If for a point x_0 belonging to some metric space M there exists a neighborhood which contains only x_0, then we call x_0 an isolated point of M.

We now prove that countable metric spaces always contain a limit point.

Theorem 3.7.14. Let M be a complete metric space consisting of countable many points. Then, M contains at least one isolated point.

Proof. For any two points $x_n, x_m \in M$, take the minimum distance $r_{\min} = \min(d(x_m, x_n))$. Suppose a neighborhood of any $x_0 \in M$ with a radius of $r_{\min}$. The existence of any point $x_1 \neq x_0$ in $B_{r_{\min}}(x_0)$ contradicts the fact that $r_{\min}$ is the smallest distance since a smaller distance $d(x_1, x_0)$ can exist. $\square$

A corollary to this theorem concerns uncountable sets.

Corollary 3.7.15. All complete metric spaces without isolated points are uncountable.

Of course, examples exist of countably complete metric spaces with isolated points. One is asked to provide such in the exercises.

Exercises

1. Provide an example for $\mathbb{R}$ being incomplete with metric $r(x, y)$.
2. Prove the structure consisting of a sequence (x_n) and metric $d(x_n, x_{n+p}) = 1 + \frac{1}{n}, n, p \in \mathbb{Z}^+$ where $d(x_n, x_n) = 0$ is a metric space.
3. Prove that the Nested Balls Property doesn't hold in the metric space defined above.
4. Give an example of a countable complete metric space with isolated points.

3.8 Compactness

Definition 3.8.1. (Compactness). If all sequences contained in a metric space M has a limit point in M, then we call M compact.
We can also call M a compactum. Also, if every point in M has a neighborhood whose closure is compact, then we call M locally compact.

The compactness of a metric space depends on properties concerning its infinite subsets.

Theorem 3.8.2. A metric space M is compact if and only if all infinite subsets $E \subseteq M$ have limit points belonging to M.

Proof. Suppose a metric space M is compact. All sequences contained in M have limit points belonging to M. For any infinite subset $E \subseteq M$, sequences contained in them must have limit points belonging to M. As such, these subsets have limit points in M.
For the converse, suppose that all infinite subsets E have limit points belonging to M. Whether or not sequences in E converge or diverge, for all radiuses sequence points will eventually come within neighborhoods of these limit points. If a single sequence of a subset E has a limit point not belonging to M, then the subset E thus has a limit point not belonging to M, contradicting our hypothesis. □

This theorem gives us an alternative definition for compactness. Metric spaces M are compact if all infinite subsets $E \subseteq M$ have limit points belonging to M.

Example

By the Bolzano-Weierstrass Theorem, all closed intervals are compact spaces. However, the real line $\mathbb{R}$ is not compact since the sequence of positive integers $1, 2, \ldots$ has no limit points. On the other hand, the real line is locally compact since all points $x \in \mathbb{R}$ have neighborhoods, in the form of open intervals centered x, with compact closures, in the form of closed intervals centered at x.

As noted, limit points remain limit points under new metrics homeomorphic to the original. As such, the property of being a limit point is invariant under transformations to new metrics homeomorphic to the original. We also note that the definition of compactness requires only limit points, not limits. Thus, local compactness is also invariant under said transformations.

We prove results concerning Cauchy sequences in compact metric spaces.

Theorem 3.8.3. All compact spaces M are complete.

Proof. Let M be a compact metric space. All Cauchy sequences (x_n) contain in M must have a limit point x. For all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n, m \geq N \implies d(x_m, x_n) < \frac{\varepsilon}{2}$. We can set $n \geq N$ such that for a limit point p of (x_n), we have that $d(x_m, p) < \frac{\varepsilon}{2}$. By triangle inequality,

$$d(x_n, p) \leq d(x_n, x_m) + d(x_m, p) < \varepsilon$$

such that all Cauchy sequences in M converge to arbitrary unique limit points p. □

This theorem implies that compact subsets of metric spaces are also closed sets.

Corollary 3.8.4. All compact subsets M of metric spaces P are closed sets.

Proof. Let M be compact in a metric space P. Suppose that there exists in M^C a limit point x of a Cauchy sequence x_n contained in M. For some radius around x, the sequence would converge to x after passing some $N \in \mathbb{N}$. However, by Theorem 3.8.3, all compact spaces are complete. All Cauchy sequences contained in M must converge to points in M, contradicting the assumption that $x \notin M$. □

One important concept in elementary real analysis remains so into measure theory and other applications.

Definition 3.8.5. (Covers). A family of subsets U_i of a metric space M covers $A \subseteq M$ if $A \subset \bigcup_{i=1} U_i$. We call the family $\mathcal{U}$ of subsets U_i a cover of A. If both $\mathcal{U}$ and $\mathcal{V}$ cover A and if $\mathcal{V} \subseteq \mathcal{U}$, then we say that $\mathcal{U}$ reduces to $\mathcal{V}$, and that $\mathcal{V}$ is a subcover of A.

Definition 3.8.6. (Open Covers). If all sets in a cover $\mathcal{U}$ of A are open, then $\mathcal{U}$ is an open cover. If all open coverings of A reduce to a finite subcover, then we call A convering compact.

We note that all sets A have a finite subcover in the form of a single open set M_0, as the following theorem shows shows.

Theorem 3.8.7. Given a compact subset M of a metric space P and an open set G such that $M \subseteq G \subseteq P$, let the open set M_0 be the union of all open balls with radius $\varepsilon_0 > 0$ such that for $x_i \in M$, $B_i(x_i) = \{x : d(x, x_i) < \varepsilon_0\}$. Then for some $\varepsilon_0 > 0$, $M_0 \subseteq G$.

Proof. Assume that no set M_0 is contained in G. Then there exist points $x \in M_0$ such that $x \in G^C$. This would make x_i a limit point of G^C since for aribitrary $\varepsilon > 0$ we have some $x \in G^C$ such that $d(x, x_i) < \varepsilon_0$. However, G^C is closed since G is open, and must contain all its limit points. This fact contradicts our hypothesis that $x_i \in M \subset G$. □

We now discuss an important connection between continuous functions and compact sets.

Theorem 3.8.8. (Continuity and Compactness). If $f : M \to N$ is continuous and $A \subseteq M$ is compact, then the image set $f(M) \subseteq N$ is also compact.

Proof. Let f be continuous and $A \subseteq M$ be compact. All sequences (x_n) contained in A have limit points in A. Convergent sequences have limit points as their limits, and by continuity of f, convergent sequences (y_k) and their limits y in A are carried over to $f(A)$. Thus, $f(y_k)$ and $f(y)$ are in $f(A)$, such that all convergent sequences have limit points in $f(A)$.
By Theorem 3.4.4, limit points of sets have subsequences converging to them. Thus, divergent sequences (z_i) in A have subsequences which converge to limits $z \in A$. By continuity of f, these are once again carried over to $f(A)$, proving the theorem. □

We now have that compactness is an invariant topological property, which follows over for homeomorphisms.

Corollary 3.8.9. If M is compact and N is homeomorphic to M, then N is also compact.

Proof. Prove this as an exercise. □

Definition 3.8.10. (Precompact metric spaces). If all sequences in a metric space M contain at least one Cauchy subsequence, then we call M as precompact.

We note that complete metric spaces always have convergent Cauchy subsequences, such that completeness and precompactness imply compactness.

Theorem 3.8.11. (Boundedness of Precompact Spaces). Every precompact space M is bounded.

Proof. We prove this through contrapositive. Let the metric space M be unbounded. Then for all $r > 0$ and $a \in M$ there exists $x \in M$ such that $d(a, x) \geq r$. We now construct a sequence with no Cauchy subsequence.

Let $x_1 \in M$, and $x_n, n > 1$ be points satisfying $d(x_1, x_2) \geq 1, \dots, d(x_n, x_{n+1}) \geq \sum_{k=1}^{n} d(x_k, x_{n-1}) + 1$. By Quadrilateral inequality and for all $n > m$,

$$
\begin{aligned}
d(x_m, x_n) &\geq d(x_{n-1}, x_n) - d(x_m, x_{n-1}) \\
&\geq d(x_{n-1}, x_n) - [d(x_m, x_{m+1}) - \dots - d(x_{n-2}, x_{n-1})] \\
&\geq d(x_{n-1}, x_n) - \sum_{k=1}^{n} d(x_k, x_{n-1}) + 1 > 1
\end{aligned}
$$

such that this sequence is not Cauchy, and M is not precompact. □

We now have a corollary important for later topics.

Corollary 3.8.12. (Wierstrass's Theorem). A continuous real-valued function defined on a compact domain is bounded.

Proof. Let $f : M \to \mathbb{R}$ be continuous and let let A be compact and contained in M. The image set $f(A)$ is compact by Theorem 3.8.8. Thus, $f(A)$ is closed by Corollary 3.8.4 and Theorem 3.8.11. It follows that both the supremum and infimum exist. □

A convenient test for compactness is embedding a metric space M isometrically in a larger metric space P.

Definition 3.8.13. (Embedding and Absolute properties). A function h embeds a metric space M into N if h is a homeomorphism from M into $h(M) \subset N$. We call properties of M that remain invariant under embedding absolute or intrinsic properties of M.

Corollary 3.8.14. A compactum is absolute closed and absolute bounded.

Proof. Prove this as an exercise. □

Definition 3.8.15. (Uniform continuity). A function $f : M \to N$ is uniformly continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $p, q \in M \wedge d_M(p, q) < \delta \implies d_N(p, q) < \varepsilon$.

Theorem 3.8.16. Continuous functions defined on compact domains are uniformly continuous.

Proof. Let $f : M \to N$ be continuous, and M a compactum. Assume that f fails to be uniformly continuous. This means that there exists $\varepsilon > 0$ for all $\delta > 0$ such that $d_M(x_n, x) < \delta \implies d_N(f(x_n), f(x)) \geq \varepsilon$. Since this holds for all δ, set $\delta = \frac{1}{n}$ and let (p_n) and (q_n) be sequences of points satisfying $d(p_n, q_n) < \frac{1}{n}$ while $d(f(p_n), f(q_n)) \geq \varepsilon$. By compactness of M, a subsequence (p_{n_k}) converges to some limit point $p \in M$. Since $d(p_n, q_n) < \frac{1}{n}$ as $k \to \infty$, it follows that (q_{n_k}) converges to the same limit p. By continuity, we have that $f(p_{n_k}) \to f(p)$ and $f(q_{n_k}) \to f(p)$. For large enough k, we have that

$$d(f(p_{n_k}), f(q_{n_k})) \leq d(f(p_{n_k}), f(p)) + d(f(p), f(q_{n_k})) < \varepsilon$$

contradicting that $d(f(p_n), f(q_n)) \geq \varepsilon$ for all n. □

Exercises

1. Prove Corollary 3.8.9.

2. Prove Corollary 3.8.14.

3.

3.9 Connectedness

The concept of connectedness is an interesting property for metric spaces.

Definition 3.9.1. (Conenctedness). Let M be a metric space. If A is a proper clopen subset of M, then we call M disconnected. We can cosntruct a separation of M into proper, disjoint clopen subsets $M = A \cup A^C$.
If M is not disconnected, then we call it connected.

Theorem 3.9.2. If M is connected, $f : M \to N$ is continuous, and f is surjective, then N is connected.

Proof. If $A \subset N$ is clopen and proper, then the preimage $f_{\text{pre}}(A)$ is itself a clopen subset of M. The preimage cannot be empty since f is onto and $A \neq \emptyset$, contradicting that M is connected. □

We now have that connectedness is a topological property as a corollary.

Corollary 3.9.3. If M is connected and M is homeomorphic to N then N is connected.

4.0 Functions of a Real Variable

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