

Linear Algebra

Lecture Notes

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Contents

1	Vector Spaces	1
2	Determinants	2
2.1	Number Fields	2
2.2	Problems in the Theory of Linear Functions	4
2.3	Determinants of Order n	5
2.4	Properties of Determinants	7

1.0 Vector Spaces

2.0 Determinants

2.1 Number Fields

We suppose a set K of objects called *numbers*, subjected to four arithmetic operations which also give elements of K . These operations have four properties, called the *Field Axioms*.

Axiom 2.1. (Addition). To every $\alpha, \beta \in K$ corresponds a unique *sum* $\alpha + \beta \in K$, where

1. $\forall \alpha, \beta \in K, \alpha + \beta = \beta + \alpha$ (commutativity)
2. $\forall \alpha, \beta, \gamma \in K, (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ (associativity)
3. $\exists 0 \in K$ s.t. $0 + \alpha = \alpha, \forall \alpha \in K$ (identity element)
4. $\forall \alpha \in K \exists \gamma \in K$ s.t. $\alpha + \gamma = 0$ (negative element)

From Property 4, we can define the *difference* $\beta - \alpha$ as $\beta + \gamma$, with γ the solution to $\alpha + \gamma = 0$.

Axiom 2.2. (Multiplication). To every $\alpha, \beta \in K$ corresponds a unique *product* $\alpha\beta \in K$ (sometimes denoted $\alpha \cdot \beta$), where

1. $\forall \alpha, \beta \in K, \alpha\beta = \beta\alpha$ (commutativity)
2. $\forall \alpha, \beta, \gamma \in K, (\alpha\beta)\gamma = \alpha(\beta\gamma)$ (associativity)
3. $\exists (1 \neq 0) \in K$ s.t. $1 \cdot \alpha = \alpha, \forall \alpha \in K$ (identity element)
4. $\forall (\alpha \neq 0) \in K \exists \gamma \in K$ s.t. $\alpha\gamma = 1$ (reciprocal element)

From Property 4, we can define the *quotient* $\frac{\beta}{\alpha}$ as $\beta\gamma$, with γ the solution to $\alpha\gamma = 1$.

Axiom 2.3. (Distributive property).

1. $\forall \alpha, \beta, \gamma \in K, \alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

The set K with these operations defined on its elements is called a *field*.

Numbers $1, 1 + 1 = 2, 2 + 1 = 3$, etc are said to be in the set of *natural numbers* \mathbb{N} —none are equal to 0. The *integers* \mathbb{Z} in K are all naturals with 0 and their negative elements. The rationals \mathbb{Q} in K are the set of quotients $\frac{p}{q}, p, q \in \mathbb{Z}$ and $q \neq 0$.

The fields K and K' are *isomorphic* if one can set an injection between them such that one can associate to every sum (or product) of numbers in K another number that is the sum (or product) of corresponding numbers in K' .

We list the most commonly encountered fields:

- Rational numbers \mathbb{Q} : of quotients $\frac{p}{q}, (p, q \neq 0) \in \mathbb{Z}$.

The set of integers \mathbb{Z} is not a field since it fails to satisfy Multiplication Property 4. Every field K also has a *subfield*—subset satisfying field axioms—isomorphic to \mathbb{Q} .

- Real numbers \mathbb{R} : set of all points on the real line in geometry.

One may build an axiomatic treatment of \mathbb{R} by supplementing the field axioms with order axioms and the axiom of completeness.

- Complex Numbers \mathbb{C} : of form $a + ib, (a, b \in \mathbb{R})$, and i defined as $i^2 = -1$. The field \mathbb{C} has the defined operations:

– Addition: $(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$.

– Multiplication: $(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$.

Numbers of form $(a + i \cdot 0)$ reduce to operations on real numbers. Numbers of form $0 + ib$ are purely imaginary: $i^2 = i \cdot i = (0 + i1)(0 + i1) = -1$. The complex numbers also have subfields isomorphic to \mathbb{R} .

Remark. By the *Fundamental Theorem of Algebra*, any polynomial of form

$$z^n + a_1z^{n-1} + \dots + a_n = 0$$

has solutions in \mathbb{C} —not necessarily in \mathbb{R} . For example, $z^2 + 1 = 0$ has no real solution—the imaginary unit i solves it.

From now on, we will use K to denote any arbitrary field. Anything that holds for K also holds for \mathbb{R} and \mathbb{C} .

2.2 Problems in the Theory of Linear Functions

Definition 2.2.1. A system of equations takes the following form:

$$\begin{aligned}
 a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n &= b_1 \\
 a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n &= b_2 \\
 &\vdots \\
 a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n &= b_m
 \end{aligned} \tag{2.1}$$

The values x_i denote *unknowns*, or elements of K to be determined. Quantities a_i denote *coefficients* of system 2.1. Quantities b_i denote *constants* in the system. We assume both coefficients and constants to be known.

Definition 2.2.2. A solution to system 2.1 denotes the set $c_1, c_2, \dots, c_n \in K$ which, when substituted for the unknowns, turns the equations into identities.

Not all systems of the above form have solutions. For example, take the system

$$\begin{aligned}
 2x_1 + 3x_2 &= 5 \\
 2x_1 + 3x_2 &= 6
 \end{aligned}$$

No solution exists at all.

Definition 2.2.3. Systems of form 2.1 with solutions are called *compatible*. Otherwise, they are *incompatible*.

Compatible systems may have one or more solutions. We can distinguish multiple solutions with superscripts: $c_1^{(1)}, c_2^{(1)}, \dots, c_n^{(1)}, c_1^{(2)}, c_2^{(2)}, \dots, c_n^{(2)}$.

If multiple solutions have it so that $c_i^{(1)} \neq c_i^{(2)}$ for any $i = 1, 2, \dots, n$, then we call them *distinct*.

For example, the system

$$\begin{aligned}
 2x_1 + 3x_2 &= 0 \\
 4x_1 + 6x_2 &= 0
 \end{aligned}$$

has distinct solutions $c_1^{(1)} = c_2^{(1)} = 0$ and $c_1^{(2)} = 3, c_2^{(2)} = -2$.

Definition 2.2.4. We call systems of form 2.1 with a unique solution *determinate*. Systems with multiple solutions are *indeterminate*.

We have the following basic problems in studying a system of linear equations:

1. Ascertain whether the system is compatible or incompatible
2. If compatible, whether it is determinate
3. If determinate, find the unique solution
4. If indeterminate, describe the set of solutions

The basic tool of doing such is the theory of determinants.

2.3 Determinants of Order n

We shall primarily work with square matrices.

Definition 2.3.1. A square matrix is an array of n^2 numbers $(a_{ij}), i, j = 1, 2, \dots, n$, and $a_{ij} \in K, \forall i, j$.

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix} \quad (2.2)$$

We call its number of rows and columns the matrix *order*. The numbers a_{ij} are its *elements*. The first index i denotes row, the second index j column. Elements $a_{ij}, i = j$ form a matrix's *principal diagonal*.

Consider any product of n elements appearing in different rows and columns of matrix 2.2:

$$a_{\alpha_1,1} a_{\alpha_2,2}, \dots, a_{\alpha_n,n} \quad (2.3)$$

We can always choose the element in the first column of matrix 2.2 for our first factor a_{α_1} . Here we denote by α_1 the element's row. We get by hypothesis that factors $a_{\alpha_i,i}$ appear in different rows of matrix 2.2. The row indices α_i then represent some permutation of $1, 2, \dots, n$.

Definition 2.3.2. We mean by an *inversion* of the sequence (α_i) an arrangement of two indices such that larger indices come before smaller ones.

We count the total number of inversions by $N(\alpha_1, \alpha_2, \dots, \alpha_n)$.

Take the sequence $(2, 1, 4, 3)$. Two inversions exist: $(2, 1)$, and $(4, 3)$. We then have $N(2, 1, 4, 3) = 2$.

Consider also the sequence $(4, 3, 1, 2)$. We count five inversions: $(4, 3)$, $(4, 1)$, $(4, 2)$, $(3, 1)$, and $(3, 2)$, so that $N(4, 3, 1, 2) = 5$.

2 Determinants

We go back now to the product of n elements 2.3. We multiply the product by $(-1)^{N(\alpha_1, \dots, \alpha_n)}$, so that even inversions put a plus sign and odd inversions put a minus.

The total number of products from elements of order n matrices is equal to the number of permutations for numbers $1, 2, \dots, n$, which is $n!$. We now arrive to the definition of a determinant:

Definition 2.3.3. The *determinant* D of a matrix with form 2.2 is the algebraic sum of $n!$ products of form 2.3. Each product is multiplied by a rule-determined sign:

$$D = \sum (-1)^{N(\alpha_1, \dots, \alpha_n)} a_{\alpha_1, 1} a_{\alpha_2, 2} \dots a_{\alpha_n, n}$$

For order 2 determinants,

$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$$

Order 3 determinants,

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1}a_{2,2}a_{3,3} - a_{2,1}a_{1,2}a_{3,3} + a_{2,1}a_{3,2}a_{1,3} \\ - a_{3,1}a_{2,2}a_{1,3} + a_{3,1}a_{1,2}a_{2,3} - a_{1,1}a_{3,2}a_{2,3}$$

Determinants become useful in solving systems like 2.2

$$a_{1,1}x_1 + a_{1,2}x_2 = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 = b_2$$

by obtaining the formulas

$$x_1 = \frac{b_1a_{2,2} - b_2a_{1,2}}{a_{1,1}a_{2,2} - a_{2,1}a_{1,2}}, x_2 = \frac{b_2a_{1,1} - b_1a_{2,1}}{a_{1,1}a_{2,2} - a_{2,1}a_{1,2}}$$

Assuming a nonzero denominator, we can represent the numerators and denominator as

$$b_1a_{2,2} - b_2a_{1,2} = \begin{vmatrix} b_1 & a_{1,2} \\ b_2 & a_{2,2} \end{vmatrix}, a_{2,1}b_2 - a_{2,1}b_1 = \begin{vmatrix} a_{1,1} & b_1 \\ a_{2,1} & b_2 \end{vmatrix} \\ a_{1,1}a_{2,2} - a_{2,1}a_{1,2} = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}$$

Similar formulas hold for solutions of systems with arbitrary number of unknowns.

2.4 Properties of Determinants

Definition 2.4.1. The *transposition operation* $(a_{ij})^T = (a_{ji})$ obtained from a matrix of form 2.2 comes from interchanging rows and columns with identical indices. One may see that transposes have identical determinants.

Taking the transpose of a determinant is geometrically equivalent to a 180° rotation in space about the principal diagonal. All inversions remain, so term signs also remain.

Since this property establishes equivalence between rows and columns of a matrix, further theorems will be proved for columns only.

Proposition 2.4.2. (Antisymmetry). Interchanging two adjacent columns of a determinant changes its sign.

Proof. Interchanging two adjacent columns changes the number of determinant inversions by one, changing whether it is even or odd. The sign of the determinant then changes. \square

Remark. Interchanging nonadjacent columns with m columns in between causes $m + (m + 1)$ sign changes in the determinant. The resulting sign will always be opposite the original, since $m + (m + 1) = 2m + 1$, an odd number.

Corollary 2.4.3. A determinant with two identical columns vanishes.

Proof. A determinant with two identical columns has at least one pair of identical terms. Interchanging two identical columns doesn't change the determinant D since two identical terms exist. However, the determinant's sign must change with column interchanges. Since $D = -D$, it must be that $D = 0$. \square

Definition 2.4.4. A matrix column a_{ij} is a *linear combination* of two columns if for all elements of the column,

$$a_{ij} = \lambda b_i + \mu c_i, i = 1, 2, \dots, n$$

for fixed numbers λ and μ .

Theorem 2.4.5. (Linear Property of Determinants). If a matrix column a_{ij} is a linear combination $a_{ij} = \lambda b_i + \mu c_i, i = 1, 2, \dots, n$ of two columns, then the determinant D is a linear combination of two determinants:

$$D = \lambda D_1 + \mu D_2 \quad (2.4)$$

with D_1 and D_2 having the same columns as D except for the j -th; determinant D_1 has elements of b_i while D_2 has elements of c_i for the j -th column.

Proof. We can write each term of the determinant D as

$$\begin{aligned} a_{\alpha_1,1} a_{\alpha_2,1} \dots a_{\alpha_j,j} \dots a_{\alpha_n,n} &= a_{\alpha_1,1} a_{\alpha_2,1} \dots (\lambda b_{\alpha_j} + \mu c_{\alpha_j}) \dots a_{\alpha_n,n} \\ &= \lambda b_{\alpha_j} a_{\alpha_1,1} a_{\alpha_2,1} \dots a_{\alpha_n,n} + \mu c_{\alpha_j} a_{\alpha_1,1} a_{\alpha_2,1} \dots a_{\alpha_n,n} \end{aligned}$$

All the first terms of the determinant add to λD_1 , while the second terms add to μD_2 . □

We can write the determinant formula more conveniently. Let D be a fixed determinant, and $D_i(p_i)$ the determinant obtained by replacing the j -th column of D by $p_i, i = 1, \dots, n$. The determinant 2.4 takes the form

$$D_j(\lambda b_i + \mu c_i) = \lambda D_i(b_i) + \mu D_i(c_i).$$

One may easily extend the linear property of determinants to cases where every element of the j -th column is a linear combination of any other term

$$a_{ij} = \lambda b_i + \mu c_i + \dots + \tau f_i$$

so that we obtain the determinant

$$\begin{aligned} D_j a_{ij} &= D_j(\lambda b_i + \mu c_i + \dots + \tau f_i) \\ &= \lambda D_j(b_i) + \mu D_j(c_i) + \dots + \tau D_j(f_i). \end{aligned}$$

Corollary 2.4.6. Any common factor of a column of a determinant can be factored out of the determinant.

Proof. If $a_{i,j} = \lambda b_{i,j}$, then we have that $D_j(a_{i,j}) = D_j(\lambda b_{i,j}) = \lambda D_j(b_{i,j})$ □

Corollary 2.4.7. If a determinant column consists entirely of 0s, then the determinant vanishes.

Proof. We can factor the 0 out of the determinant column

$$D_j(0b_{i,j}) = 0D_j(b_{i,j}) = 0.$$

□

Theorem 2.4.8. A determinant value is not changed by adding elements of a column multiplied by an arbitrary number to elements of another column.

Proof. Suppose that we multiply the k th column by λ and add it to the j th column. By Linear Property of Determinants, we can write D_j as

$$D_j(\lambda a_{ik} + a_{ij}) = \lambda D_j(a_{ik}) + D_j(a_{ij})$$

The column j consists of elements a_{ik} and is identical to the k th column, causing the determinant to vanish. □