Time Series and Discrete Dynamics

Lecture Notes

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May 26, 2021

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Preface

These lecture notes provide an exposition for analysis of Time Series and Discrete Dynamics on a formal level. Run of the mill forecasting and projection courses serve a general business audience for surface level analyses. Formal study of Time Series Analysis requires previous study in Probability Theory, Real Analysis, Mathematical Statistics. The author suggests reading Time Series Econometrics[1] by Justin Eloriaga as a quick guide before starting.

Time Series Analysis has many applications, from Economics to Finance, from Stochastic to Dynamic processes. Time series may be seen as random variables defined on integer values. They may serve as discrete approximations to stochastic processes, which take continuous time as their domains. One may also discretize differential equations as time series; here, they are known as difference equations.

These notes begin with exposition on difference equations and lag operators. The theory of ARIMA processes and derived concepts build upon these. There also exists exposition on estimation methods. These notes initially focus on linear models and stationary processes. Nonlinear models follow once we realize how prevalent nonstationary processes are.

[1]Eloriaga, J. (2020). Time series econometrics. Retrieved from: `https://justineloriaga.files.wordpress.com/2020/11/eloriaga_appliedtimeseries_lecturenotes.pdf`.

Real Analytic Preliminaries

Suppose two sets A and B. We may obtain a new set by defining the following operation.

Definition 0.1. An ordered pair (a, b) is defined by the set $\{\{a\}, \{a, b\}\}$ such that $(a, b) = (b, a) \iff a = b$.
The Cartesian Product of sets A and B is

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}.$$

Our definition has it so that $a = b \iff (a, b)$ contains one set, and $a \neq b \iff (a, b)$ contains two sets. We proceed with defining classes of sets with the same number of elements.

One may generalize the concept of Cartesian Products to multiple sets

$$A_1 \times A_2 \times \ldots \times A_n = (a_1, a_2, \ldots, a_n)$$

with $a_i \in A_i, i \in \mathbb{Z}_+$.

Using the concept of Cartesian products, we also derive definitions for mappings and functions.

Definition 0.2. A mapping is a subset r of the cartesian product $C \times D$, such that $c \in C$ appears as the first coordinate of at most one ordered pair belonging to r.

One consequence of the condition for $c \in C$ is that if $(c, d) \in r$ and $(c, d') \in r$ then $d = d'$.

One may also define a mapping using its domain and image:

Definition 0.3. The domain of a mapping $r \subseteq C \times D$ is the subset of C with all first coordinates of r

$$\text{dom}(r) = \{c : \exists d \in D \wedge (c, d) \in r\} \subseteq C$$

while the image of r is the subset of D with all second coordinates of r:

$$\text{img}(r) = \{d : \exists c \in C \wedge (c, d) \in r\} \subseteq D.$$

We can now discuss a special kind of mapping.

Definition 0.4. A function f is a mapping r, with a set B that includes $\mathrm{img}(r)$. The set $A = \mathrm{dom}(r)$ is the domain of f, with $\mathrm{img}(r) = \mathrm{img}(A)$. The set B is called the codomain of f. Sometimes, we also call B the range.

We denote a function with domain A and codomain B by $f : A \to B$. One can visualize f as a geometric transformation from the points of A to those of B.

If $f : A \to B$ and $a \in A$, denote by $f(a) \in B$ the unique element that the rule f assigns to a. We call $f(a)$ the value of f at a. In terms of rules of assignment, $f(a) \in B$ is the unique element such that $(a, f(a)) \in r$.

Definition 0.5. Given functions $f : A \to B$ and $g : B \to C$, their composition $g \circ f$ is the function $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$.
Denote a composition by

$$\{(a, c) : \exists b \in B(f(a) = b \wedge g(b) = c)\}.$$

Physically, point a moves to point $f(a)$, then to point $g(f(a))$. If the composition $g \circ f(a)$ is defined, then the range of f equals the domain of g.

Definition 0.6. We call a function $f : A \to B$ injective (one-to-one) if for all distinct elements $a \in A$, there exists a distinct element $b \in B$ such that each element a has a unique image:

$$f(a) = f(a') \implies a = a'$$

We call the function f surjective if for all $b \in B$, there exists $a \in A$ such that $b = \mathrm{img}(a)$:

$$\forall b \in B \exists a \in A(b = f(a))$$

A function that is both injective and surjective is called bijective.

We now define what a sequence is.

Definition 0.7. Let $S \subseteq \mathbb{Z}$ and $X \subseteq \mathbb{R}$. If there exists a bijective function $f : S \to X$, then we call $\mathrm{img}(f)$ a sequence.

In many practical applications, we let the set S represent time. If X is a random variable, we obtain a time series.

1.0 Difference Equations

1.1 First-Order Difference Equations

Suppose a random variable $\{y\}$ which takes values at date t. We denote each value of the sequence by y_t. We use dynamic equations to relate each value of $\{y\}$ at date t to another random variable w_t and to values of $\{y\}$ in previous dates.

$$y_t = \phi y_{t-1} + w_t \tag{1.1}$$

This equation is first-order since it relates y_t to only the immediately preceding date, or the first lag. We note that equation 1.1 expresses y_t as a linear function of y_{t-1} and w_t.

1.1.1 Solving by Recursive Substitution

Suppose that equation 1.1 governs the behavior of $\{y\}$ for all dates. If we know the value

Date	Equation
0	$y_0 = \phi y_{-1} + w_0$
1	$y_1 = \phi y_0 + w_1$
$\vdots$	$\vdots$
t	$y_t = \phi y_{t-1} + w_t$

of $\{y\}$ at date $t = -1$ and the value of $\{w\}$ for all dates $t = 0, 1, 2, \ldots$, then we can simulate a dynamic system to find all values of $\{y\}$. For example, if y_{-1} and w_0 are known, we can calculate y_0. Given y_0 and w_1, we can calculate y_1.

$$y_1 = \phi y_0 + w_1 = \phi(\phi y_{-1} + w_0) + w_1 = \phi^2 y_{-1} + \phi w_0 + w_1 \tag{1.2}$$

We call this procedure recursive substitution.

1.1.2 Dynamic Multipliers

Equation 1.2 expresses y_t as a linear function of y_{-1} and historical values of $\{w\}$. From this equation, we find that calculating the effects of w_0 on y_t becomes easier.

If w_0 changes with y_{t-1}, but $w_1, w_2, \ldots, w_t$ are unaffected, then

$$\frac{\partial y_t}{\partial w_0} = \phi^t \tag{1.3}$$

We arrive at the same calculation if our dynamic simulation began with date t, taking y_{t-1} as given. We find y_{t+j} to be a function of y_{t-1} and $w_t, w_{t+1}, \ldots, w_{t+j}$:

$$y_{t+j} = \phi^{j+1} y_{t-1} + \sum_{n=0}^{j} \phi^{j-n} w_{t+n}. \tag{1.4}$$

We find a similar effect of changes in w_t on y_{t+j}:

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi^j \tag{1.5}$$

so that dynamic multipliers in the form of 1.5 depend only on j, or how many dates separate a disturbance input w_t from observed output values y_{t+j}. Note that dynamic multipliers **do not** depend on the date t.

Different values of ϕ in equation 1.1 can produce different dynamic responses of $\{y\}$ to $\{w\}$. For $0 < \phi < 1$, the dynamic multiplier 1.5 geometrically vanishes. For $-1 < \phi < 0$, 1.5 alternates in sign for different dates—increases in w_t cause y_t to increase, y_{t+1} to decrease, and so on. The absolute value of 1.5 once again vanishes geometrically.