Introduction to Modern Finance

Lecture Notes

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1.0 Binomial Pricing with No Arbitrage

1.1 One-Period Model

Suppose an equity with price $S_0 > 0$ at time 0. Our goal this section is to determine how much of our initial wealth $X_0 > 0$ to invest in the stock and money markets. For time 1, let $S_1(H) > 0$ be the equity's price with probabiity p, and $S_1(T) > 0$ be the price with probability $q = 1 - p$. Suppose two values $u = \frac{S_1(H)}{S_0}$ and $d = \frac{S_1(T)}{S_0}$, called the up and down factors, respectively. These values show how worth our investment in stock was. They also depend on the outcome of a coin toss, not necessarily fair. Now let $r > -1$ be our interest rate. We have our first axiom from these values.

Axiom 1.1. (No Arbitrage). For all d, u, and r, we have that

$$0 < d < 1 + r < q.$$

Contrary assertions $d \geq 1 + r$ and $1 + r \geq q$ lead to risk-free gain, called arbitrage. The first assertion lets us borrow money to invest in stocks. The second lets us short-sell stocks, or borrowing stocks to sell before a price fall and repurchase, to invest gains in the money market. No arbitrage models focus on accounting for risk in decision making. Any arbitrage in a real-world market would be quickly spotted, taken advantage of, and removed.

One type of option easy to model is the European call option, where at time 1, a holder has a right (but not obligation) to buy a stock for a strike price K. We thus have our second axiom.

Axiom 1.2. (Strike Pricing). For all $S_1(T)$, $S_1(H)$, and K, we have that

$$S_1(T) < K < S_1(H).$$

This second axiom holds that a tail on a coin toss yields a worthless option not to be exercised, and a head yields a profit of $S_1(H) - K$. A coin toss on heads thus gives an option worth $\max(0, S_1 - K)$. Our problem now is to valuate the option price at time 1 before deciding at time 0.

At time 0, we have wealth X_0 and we wish to buy Δ_0 shares of stock. In time 1, we thus have wealth

$$X_1 = \Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_1) = (1 + r)X_0 + \Delta_0(S_1 - (1 + r)S_0). \quad (1.1)$$

Note that our wealth in time 1 depends on the coin toss. Let our total payoffs conditional on the result be $V_1(H) = X_1(H)$ and $V_1(T) = X_1(T)$. We now note that values $V_1(H)$ and $V_1(T)$ are known in time 0 as derivatives of possible asset price movements. We replicate these values by alotting Δ_0, such that we price the derivative relative to other financial instruments.

We have two equations representing each coin toss result.

$$\begin{aligned} V_1(H) &= \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_1(H)) = (1+r)X_0 + \Delta_0(S_1(H) - (1+r)S_0) \\ V_1(T) &= \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_1(T)) = (1+r)X_0 + \Delta_0(S_1(T) - (1+r)S_0) \end{aligned}$$

Dividing by $(1+r)$ and factoring out Δ_0, we obtain

$$X_0 + \Delta_0 \left(\frac{1}{1+r_0} S_1(H) - S_0 \right) = \frac{1}{1+r} V_1(H) \tag{1.2}$$

$$X_0 + \Delta_0 \left(\frac{1}{1+r_0} S_1(T) - S_0 \right) = \frac{1}{1+r} V_1(T). \tag{1.3}$$

We now obtain the desired value Δ_0 by subtracting equation 1.3 from 1.2: We now obtain the desired value Δ_0 by subtracting equation 1.3 from 1.2:

$$\begin{aligned} \frac{V_1(H) - V_1(T)}{1+r} &= X_0 + \Delta_0 \left(\frac{1}{1+r} S_1(H) - S_0 \right) - X_0 - \Delta_0 \left(\frac{1}{1+r} S_1(T) - S_0 \right) \\ \iff \Delta_0 &= \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} \end{aligned} \tag{1.4}$$

giving us the delta hedging formula.

Our next task is to determine what value to assign our derivative at time 1. Equations 1.2 and 1.3 represent different outcomes such that we can multiply the first by p and the second by q then add them to obtain an expected value. Substituting the delta-hedging formula Δ_0, we obtain

$$X_0 + \Delta_0 \left(\frac{1}{1+r} [pS_1(H) + qS_1(T)] - S_0 \right) = \frac{1}{1+r} (pV_1(H) + qV_1(T)).$$

Let p satisfy

$$S_0 = \frac{1}{1+r} (pS_1(H) + qS_1(T)). \tag{1.5}$$

Then, we obtain

$$X_0 = \frac{1}{1+r} (pV_1(H) + qV_1(T)). \tag{1.6}$$

Note that we can solve for p using equation 1.5. We take

$$S_0 = \frac{1}{1+r} (pS_1(H) + qS_1(T)) = \frac{S_0}{1+r} [(u-d)p + d]$$

letting us obtain

$$\begin{aligned} p &= \frac{1+r-d}{u-d} \\ q &= \frac{u-1-r}{u-d}. \end{aligned} \tag{1.7}$$

We thus price the derivative security which pays V_0 at

$$V_0 = \frac{1}{1+r}(pV_1(H) + qV_1(T)) = \frac{1}{1+r}\mathbb{E}(V_1). \tag{1.8}$$

At no arbitrage, this formula represents the price of hedging the short position of derivative security.

Remark. We note that we call values p and q risk-neutral probabilities. They represent no real-world values, called physical probabilties. Instead, risk-neutral probabilities implicitly assume that investors and speculators require no compensation or give no extra payment for risk. Thus, the model assumes that stocks and money market grow at the same rate.

In the formulas given above, p represents the ratio of money market gain over stock gain. This formula shows that there is no free lunch—higher stock gain comes at the expense of low probability. This value also reflects market derivative prices, which lets us replicate said derivative using a portfolio of stock and money market assets.

Hedging risk thus requires no real-world probability, and only the up-down factors u, d and interest rate r.

1.2 Multiperiod Binomial Model

For repeated coin-tosses, we have an analogous model. With intial stock price S_0, heads and tails outcomes $S_1(H) = uS_0, dS_1(T)$ for $t = 1$, we have the following values for $t = 2$:

$$S_2(HH) = uS_1(H) = u^2S_0, S_2(HT) = uS_1(T) = udS_0 \tag{1.9}$$

$$S_2(TH) = dS_1(H) = duS_0 S_2(TT) = dS_1(T) = d^2S_0. \tag{1.10}$$

Like the one-period case, we can analyze a European call option. This time, suppose a derivative security with strike price K which expires at $t = 2$:

$$V_2 = \max(0, S_2 - K).$$

Once again, the derivative security V_2 and stock price S_2 depend on two coin toss outcomes. We now determine the no-arbitrage price at $t = 0$.

For the option sold at $t = 0$ for V_0, we invest $V_0 - \Delta_0$ into the money market after buying Δ_0 stocks. At $t = 1$, we have a portfolio valued at

$$X_1 = \Delta_0 S_1 + (1+r)(V_0 - \Delta_0 S_0).$$

We can now readjust our hedge by holding Δ_1 shares, depending on the first coin toss. Our remaining wealth, $X_1 - \Delta_1 S_1$, goes to the money market. We thus have our wealth in the next period as

$$V_2 = \Delta_1 S_1 + (1+r)(X_1 - \Delta_1 S_1). \tag{1.11}$$

We have four possible outcomes depending on our coin toss outcomes, with four corresponding equations.

Since we tackled the case of $t = 1$ in the previous section, we deal only with calculations for $t = 2$. We solve a system of 6 equations with 6 unknowns, starting with the difference

$$\begin{aligned} V_2(TH) - V_2(TT) &= \Delta_1(T)S_2(TH) + (1+r)(X_1(T) - \Delta_1(T)S_1(T)) \\ &\quad - \Delta_1(T)S_2(TT) + (1+r)(X_1(T) - \Delta_1(T)S_1(T)) \end{aligned}$$

conditional on our first coin toss being tails. We thus have another delta hedging formula:

$$\Delta_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)}. \tag{1.12}$$

Given risk-neutral probabilities p and $q = 1 - p$, we substitute $\Delta_1(T)$ back to obtain

$$X_1(T) = \frac{1}{1+r}[pV_2(TH) + qV_2(TT)].$$

We then take the sum

$$pV_2(TH) + qV_2(TT) = p\Delta_1(T)S_2(TH) + p(1+r)X_1(T) - p(1+r). \tag{1.13}$$

One can derive the sum by noting that $pS_2(TH) + qS_2(TT) = (1+r)S_1(T)$, so that all terms involving $\Delta_1(T)$ cancel out.

We call our equation for $X_1(T)$ the option price at $t = 1$, for tails. Denote this price at $V_1(T)$ such that

$$V_1(T) = \frac{1}{1+r}[pV_2(TH) + qV_2(TT)].$$

We thus have another instance of risk-neutral pricing.

An analogous delta-hedging formula exists for $V_2(HH)$ and $V_2(HT)$:

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)}.$$

Our option price at $t = 1$ is $V_1(H) = X_1(H)$ for replicating the derivative security conditional on a heads outcome. Our risk neutral price now is

$$V_1(H) = \frac{1}{1+r}[pV_2(HH) + qV_2(HT)].$$

Similar to the one-period model, we can set $X_1(H) = V_1(H)$ and $X_1(T) = V_1(T)$ to derive solutions for X_1.

We thus have our first example of a stochastic process.

Definition 1.2.1. A stochastic process is a random variable indexed by time.

Values of a stochastic process are random since they depend on coin tosses. Our subscript is the number of coin tosses done. In our case, we have three: (Δ_0, Δ_1), (X_0, X_1, X_2), and (V_1, V_2, V_3). We also define the value of our portfolio recursively starting with X_0:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

which we call our wealth equation. In fact, we can obtain for all times a specific value just by specifying X_0 then setting our Δ_n. Our actual wealth is determined by coin tosses, which we can compute using the wealth equation.

Theorem 1.2.2. Set the derivative security price at time N to be a random variable

$$V_N(w_1 w_2 \dots w_N) = \frac{1}{1+r}(pV_{n+1}(w_1 w_2 \dots w_n H) + qV_{n+1}(w_1 w_2 \dots w_n T))$$

where $0 \leq n \leq N-1$. Additionally, set the delta-hedging formula

$$\Delta_n(w_1 \dots w_n) = \frac{V_{n+1}(w_1 \dots w_n H) - V_{n+1}(w_1 \dots w_n T)}{S_{n+1}(w_1 \dots H) - S_{n+1}(w_1 \dots T)}.$$

Then, we can set $X_0 = V_0$ to recursively obtain portfolio values

$$X_N(w_1 w_2 \dots w_N) = V_N(w_1 w_2 \dots w_n) \tag{1.14}$$

for all coin toss outcomes $w_1, w_2, \dots, w_N$.

Proof. For the base case, set $n = 0$. Our discussion above shows this to be true. For the induction step, let $X_n(w_1 w_2 \dots w_n) = V_n(w_1 w_2 \dots w_n)$ for all $t \leq n$. Given coin toss outcomes $w_1, w_2, \dots, w_n, w_{n+1}$, let us consider $w_{n+1} = H$. □