Real and Complex Analysis

Lecture Notes

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Contents

Preface

These lecture notes serve as an introduction to undergraduate Real and Complex Analysis. Mathematics undergraduates usually take this course in their first year, focusing on the Real Field, sequences, and series. Advanced Economics undergraduates would need the same topics as preparation for graduate studies. Mathematics undergraduates then go on to the Riemann Integral and other topics throughout their second year. One would usually need to take a course on Set Theory and Formal Logic before learning Analysis. The author recommends Book of Proof by Richard Hammack and For all x by PD Magnus, Cathal Woods, and J Robert Loftis for those self-studying these notes.

A Mathematics undergraduate usually encounters Real and Complex Analysis as one's first formal study of familiar tools like sequences, Calculus and complex numbers. Proof-based Mathematics becomes the norm, with decreasing focus on computational and algorithmic exercises so common in lower education. This transition helps a student become rigorous—no more clinging onto intuitive facts or formulas or algorithms, more focus on trial and error, solutions and insight than answers. Only in Mathematics can proving a problem to be unsolvable be cause for celebration—consistency, sense, and rigor matter more than world-changing insights. Physicists stick to the physical world's stability. Engineers find comfort in tangible sense. Economists prefer splitting hairs on special cases than finding general principles and theorems. Computer Scientists wish to endlessly verify whether heuristics and algorithms work to solve problems. Mathematicians, however, seek escape from reality by performing useless work with symbols—the highest form of creative and artistic expression known to man.

1.0 Set-Theoretic Preliminaries

1.1 Introduction to Sets

We call a collection S of elements a set. The expression $x \in S$ means that the object x is an element of S. Likewise, $x \notin S$ means that x is not an element of S. As a general rule, we denote sets by big letters and their elements by small letters.

We may take some elements of S as their own set T. We call the set T a subset of S, which we denote by $T \subset S$.

Definition 1.1.1. A subset T of S, denoted by $T \subseteq S$, satisfies:

$$\forall x \in T(x \in S).$$

We may also find that two sets may be subsets of each other. One may show that

$$A \subseteq B \wedge B \subseteq A \implies A = B.$$

A proper subset $T \subset S$ satisfies

$$\exists x \in S(x \notin T)$$

so that S contains elements that are not elements of T.

Our first two operations on sets A and B comprise unions and intersections.

Definition 1.1.2. A union of sets A or B satisfies

$$A \cup B = \{x : x \in A \vee x \in B\}$$

while an intersection of sets A and B satisfies

$$A \cap B = \{x : x \in A \wedge x \in B\}.$$

Two sets are disjoint if

$$A \cap B = \emptyset.$$

From these two operations and standard rules of inference, we have the following theorems.

Theorem 1.1.3. The formula

$$(A \cup B) \cap C = A \cap C \cup B \cap C$$

holds for all sets A, B, and C.

Proof. For all sets A, B, and C, and for all elements x, we have that $(x \in A \cup B) \wedge (x \in C) \iff (x \in A \vee x \in B) \wedge x \in C$. By Distributive Rule of Inference, $(x \in A \wedge x \in C) \vee (x \in B \wedge x \in C) \iff (x \in A \cap C) \vee (x \in B \cap C)$, giving us our result of $(A \cup B) \cap C = A \cap C \cup B \cap C$. □

Theorem 1.1.4. The formula

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

holds for all sets A, B, and C.

Proof. For all sets A, B, and C, and for all elements x, we have that $x \in A \vee (x \in A \cap B) \iff x \in A \vee (x \in A \wedge x \in B)$. By distributivity, $(x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \iff (x \in A \cup B) \wedge (x \in A \cup C)$, giving us our result of $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. □

Corollary 1.1.5. For all sets A,

$$(A \cup A = A) \wedge (A \cap A = A).$$

Proof. For all sets A and elements x, $x \in A \vee x \in A$. Also, $x \in A \wedge x \in A$. By idempotency, $x \in A$ such that both statements above hold. □

Our next set operation comprises differences.

Definition 1.1.6. For sets A and B, their difference satisfies

$$A - B = \{x : x \in A \wedge x \notin B\}.$$

We introduce a new set operation relying on the definition of subsets and differences.

Definition 1.1.7. Let $A \subseteq U$. The complement of A, denoted by A^C or $\bar{A}$ satisfies

$$A^C = U - A.$$

In some contexts, we call U the universal set, the set of discourse, or simply the universe.

Theorem 1.1.8. For $B \subseteq A$, the formula

$$(B^C)^C = A - (A - B) = B$$
$$(A - B) \cup B = A.$$

for all sets A and B.

Proof. For all sets A and B, and for all elements x, $B^C = A - B = \{x : x \in A \wedge x \notin B\}$, so that $(B^C)^C = (A - B)^C = \{x : x \in B \wedge x \notin (A - B)\} = B = A - B^C$. Also, $(A - B) = B^C = \{x : x \in A \wedge x \notin B\} \iff B^C \cup B = \{x : x \in B^C \vee x \in B\} = \{x : x \in (A - B) \vee x \in B\} = \{x : x \in A \wedge (x \notin B \vee x \in B)\} = B$. □

We note that the formula $(A \cup B) - B = A$ holds only for disjoint A and B. One can verify this fact as an exercise.

One may also find that elements of a set may themselves be sets. Given a set A, one may consider sets whose elements are subsets of A. In particular, one may obtain the set of all subsets of A—the power set of A.

Definition 1.1.9. The power set of A is the set $\mathcal{P}(A) = \{X : x \in X \implies x \in A\}$.

One may find it convenient to denote multiple sets with subscripts. For sets $A_i, i = 1, 2, \ldots$, we have analogous definitions for unions and intersections.

Definition 1.1.10. For sets $A_i, i = 1, 2, \ldots$, we define their union by

$$\bigcup_{i=1} A_i = x : \exists A_i, 1 \leq i (x \in A_i)$$

and their intersection by

$$\bigcap_{i=1} A_i = x : \forall A_i, 1 \leq i (x \in A_i)$$

Theorem 1.1.11. The following formula holds for all sets A_i:

$$(\bigcup_i A_i)^C = \bigcap (A_i)^C.$$

Proof. For $(\bigcup_i A_i)^C$, the universe U, and element x, we have that $x \in (\bigcup_i A_i)^C \iff x \in U \wedge x \notin A_1 \wedge x \notin A_2 \wedge \ldots$. By Addition, Commutative, and Associative Rules of Inference, we can add true statements $x \in U$ repeatedly until we have $(x \in U \wedge x \notin A_1) \wedge (x \in U \wedge x \notin A_2) \wedge \ldots \iff x \in A_1^C \wedge x \in A_2^C \wedge \ldots$ to find that this satisfies $x \in \bigcap_i A_i^C$. □

Our next concepts involve ordered pairs and Cartesian Products, central to discussion about real numbers.

Definition 1.1.12. An ordered pair (a, b) is defined by the set $\{\{a\}, \{a, b\}\}$ such that $(a, b) = (b, a) \iff a = b$.
The Cartesian Product of sets A and B is

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}.$$

Our definition has it so that $a = b \iff (a, b)$ contains one set, and $a \neq b \iff (a, b)$ contains two sets. We proceed with defining classes of sets with the same number of elements.

1.2 Functions

Definition 1.2.1. A rule of assignment is a subset r of the cartesian product $C \times D$, such that $c \in C$ appears as the first coordinate of at most one ordered pair belonging to r.

One consequence of the condition for $c \in C$ is that if $(c, d) \in r$ and $(c, d') \in r$ then $d = d'$.

One may also define a rule of assignment using its domain and image:

Definition 1.2.2. The domain of a rule of assignment $r \subseteq C \times D$ is the subset of C with all first coordinates of r

$$\text{dom}(r) = \{c : \exists d \in D \wedge (c, d) \in r\} \subseteq C$$

while the image of r is the subset of D with all second coordinates of r:

$$\text{img}(r) = \{d : \exists c \in C \wedge (c, d) \in r\} \subseteq D.$$

We can now discuss a special kind of rule of assignment.

Definition 1.2.3. A function f is a rule of assignment r, with a set B that includes $\text{img}(r)$.
The set $A = \text{dom}(r)$ is the domain of f, with $\text{img}(r) = \text{img}(A)$. The set B is called the codomain of f. Sometimes, we also call B the range.

We denote a function with domain A and codomain B by $f : A \to B$. One can visualize f as a geometric transformation from the points of A to those of B.

If $f : A \to B$ and $a \in A$, denote by $f(a) \in B$ the unique element that the rule f assigns to a. We call $f(a)$ the value of f at a. In terms of rules of assignment, $f(a) \in B$ is the unique element such that $(a, f(a)) \in r$.

Definition 1.2.4. Given functions $f : A \to B$ and $g : B \to C$, their composition $g \circ f$ is the function $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$.
Denote a composition by

$$\{(a, c) : \exists b \in B(f(a) = b \wedge g(b) = c)\}.$$

Physically, point a moves to point $f(a)$, then to point $g(f(a))$. If the composition $g \circ f(a)$ is defined, then the range of f equals the domain of g.

Example

The composite of functions $f : \mathbb{R} \to \mathbb{R}, f(x) = 3x^2 + 2$ and $g : \mathbb{R} \to \mathbb{R}, g(x) = 5x$ is

$$g \circ f : \mathbb{R} \to \mathbb{R} = g(f(x)) = 5(3x^2 + 2) = 15x^2 + 10.$$

Definition 1.2.5. We call a function $f : A \to B$ injective (one-to-one) if for all distinct elements $a \in A$, there exists a distinct element $b \in B$ such that each element a has a unique image:

$$f(a) = f(a') \implies a = a'$$

We call the function f surjective if for all $b \in B$, there exists $a \in A$ such that $b =$ img(a):

$$\forall b \in B \exists a \in A(b = f(a))$$

A function that is both injective and surjective is called bijective.

For a bijective function f, then the inverse function $f^{-1} : B \to A$ exists and is defined by $f^(b) = a$ such that $f(a) = b$. We now go to establish the cardinality of sets.

Definition 1.2.6. The cardinality of a set A, denoted by $|A|$, is the equivalence class of sets such that all sets are bijective to each other.

This definition has the effect that sets with the same number of elements all have the same cardinality.

Lemma 1.2.7. Let $f : A \to B$. If there exists functions $g : B \to A$ and $h : B \to A$ where for all $a \in A$ and $b \in B$, $g(f(a_i)) = a_i$ and $f(h(b_i)) = b_i$, then f is bijective and $g = h = f^{-1}$.

Proof. We first establish the existence of an inverse, then show that functions g and h satisfy its definition.
We note that for all $a_i \in A$ and $b_i \in B$, we have that $f(a_i) \in B$, $h(b_i) \in A$, and $g(b_i) \in A$. Since a_i is arbitary, we can set $a_0 = h(b_0)$ and take $g(f(a_0)) = g(f(h(b_0))) = g(b_0) = a_0$. We can then take $f(h(b_0)) = f(g(f(a_0))) = f(a_0) = b_0$. Note two results, $f(a_0) = b_0$ and $g(b_0) = a_0$ so that the inverse function exists, and f is bijective.
Lastly, set $f(a_0) = f(h(b_0)) = b_0$, which we can do since by hypothesis, $f(h(b_i)) = b_i$ for all b_i. Since f is bijective, we have by definition of injectivity that $h(b_0) = a_0$, and $g = h = f^{-1}$. □

Definition 1.2.8. Let $f : A \to B$. If $A_0 \subseteq A$, then $f(A_0) = \{b : \exists a \in A_0(b = f(a))\}$ is the set of all images of A_0 under the function f. Analogously, $f(A_0)$ is the image of A_0.
For $B_0 \subseteq B$, denote by $f^{-1}(B_0) = \{a : f(a) \in B_0\}$ the set of all elements of A whose images under f lie in B_0. We call $f^{-1}(B_0)$ the preimage of B_0 under f.

If no points $a \in A$ have images which lie in B_0, then $f^{-1}(B_0) = \emptyset$. For bijective f, then the preimage and image of B_0 are the same.

Exercises

2.0 Real Numbers

2.1 Real Number Axioms

Real Analysis studies concepts involving the real number system $\mathbb{R}$. Instead of real numbers themselves, we focus on their properties as undefined objects satisfying certain axioms.

Axiom 2.1. (**Addition**). The sum $(x+y) \in \mathbb{R}$ of $x, y \in \mathbb{R}$ satisfies

1. $x+y=y+x$ (commutativity)
2. $\forall z \in \mathbb{R}[(x+y)+z=x+(y+z)]$ (associativity)
3. $\exists 0 \in \mathbb{R} \forall x \in \mathbb{R}(x+0=x)$ (zero element)
4. $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(x+y=0)$ (negative element)

Axiom 2.2. (**Multiplication**). The product $xy \in \mathbb{R}$ satisfies

1. $\forall x, y \in \mathbb{R}(xy=yx)$ (commutativity)
2. $\forall z \in \mathbb{R}[(xy)z=x(yz)]$ (associativity)
3. $\exists 1 \in \mathbb{R}, 1 \neq 0, \forall x \in \mathbb{R}(1x=x)$ (unitary element)
4. $\forall x \in \mathbb{R}, x \neq 0, \exists y \in \mathbb{R}(xy=1)$ (reciprocal element)
5. $x(y+z)=xy+xz$ (distrbutive property)

A set of objects x, y, z satisfying these axioms is called a field.

The real number system $\mathbb{R}$ is thus also called the real field. Other sets, such as the rationals $\mathbb{Q}$, are also fields.

Our next set of axioms concern order in sets.

Axiom 2.3. For all $x, y \in \mathbb{R}(x \geq y \vee x \leq y)$ with properties

1. $\forall x \in \mathbb{R}(x \leq x)$ (reflexivity)
2. $(x \leq y) \wedge (y \leq x) \implies x = y$ (antisymmetry)
3. $(x \leq y) \wedge (y \leq z) \implies x \leq z$ (transitivity)
4. $\forall z \in \mathbb{R}(x \leq y \implies x + z \leq y + z)$
5. $0 \leq x \wedge 0 \leq y \implies 0 \leq xy$

We call a set whose elements satisfy 1–3 a partially ordered set. The set $\mathbb{R}$ is a totally ordered set since it satisfies $x \geq y \vee x \leq y$ (connexity).

Before our last axiom, we need an important definition.

Definition 2.1.1. A set $E \subset \mathbb{R}$ is bounded above when

$$\exists z \in \mathbb{R} \forall x \in E(x \leq z).$$

We denote that is bounded above by $E \leq z$. We call the element z an upper bound of the set E.

We now posit that there exists a least element to the set of upper bounds.

Axiom 2.4. (Axiom of Completeness). For elements $z \in \mathbb{R}$, a set $E \leq z$ satisfies the axiom of completeness when

$$\forall E \subset \mathbb{R}, \exists z_0 \in \mathbb{R}(z_0 \leq z).$$

We call z_0 the supremum of E and denote it by $z_0 = \sup[E]$.

Exercises

1.

2.2 Consequences of Axioms

2.2.1 Addition

Theorem 2.2.1. The element $0 \in \mathbb{R}$ is unique.

Proof. Assume any two zero elements, 0_i and 0_j. By Addition Axioms 1 and 3:

$$0_i = 0_i + 0_j = 0_j + 0_i = 0_j.$$

□

Theorem 2.2.2. For all $x \in \mathbb{R}$, the negative element is unique.

Proof. Suppose $x \in \mathbb{R}$ has any two negative elements y_i and y_j. By Addition Axioms 1, 2, 3, and 4:

$$y_i = y_i + 0 = y_i + (x + y_j) = (y_i + x) + y_j = 0 + y_j = y_j.$$

We denote the unique negative element of $x \in \mathbb{R}$ by $-x \in \mathbb{R}$. We call the sum $x + (-y)$ the difference of x and y, and denote it by $x - y$. □

Theorem 2.2.3. The equation $a + x = b$ has a unique solution $x = b - a$.

Proof. Add the negative element $-a$ to both sides of the equality:

$$a + x + -a = x = b - a.$$

Verify whether this is the solution:

$$a + x = a + (b - a) = a + b - a = b + 0 = b.$$

□

2.2.2 Multiplication

Theorem 2.2.4. The set $\mathbb{R}$ contains a unique unit element 1.

Proof. Suppose two unit elements 1_i and 1_j. Then by Multiplication Axioms 1 and 3

$$1_i = 1_i 1_j = 1_j 1_i = 1_j.$$

□

Theorem 2.2.5. Every element $x \in \mathbb{R}$ has a unique reciprocal.

Proof. Suppose x has two reciprocals y_1 and y_2. Then

$$y_1 = y_1 1 = y_1(xy_2) = (y_1 x)y_2 = 1y_2 = y_2.$$

□

Definition 2.2.6. Numbers $1, 2 = 1+1, 3+2+1, \ldots$ are called the Natural Numbers, denoted $\mathbb{N}$. The set $\mathbb{N}$ may be defined as the smallest set containing 1 and $n+1$ whenever $n \in \mathbb{N}$.

One may need to show sometimes that a numerical set A—a set of all $n \in \mathbb{N}$ for which some property T holds—has all natural numbers. The method of mathematical induction verifies whether

$$(1 \in A) \wedge (n \in A \implies n+1 \in A)$$

The set of integers holds all $n \in \mathbb{N}$, reciprocals $-n$, and the number 0. We denote the set of integers by $\mathbb{Z}$. Mathematicians also differ in convention whether $0 \in \mathbb{N}$ or not. To remove ambiguity, one may use $\mathbb{Z}^+$ instead of our definition of $\mathbb{N}$. Additionally, suppose a number $m \in \mathbb{Z}$. We call the number $2m$ an even number, and $2m+1$ and odd number.

If we get all quotients of form $\frac{m}{n}, m, n \in \mathbb{Z}$ and $n \neq 0$, we get the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$. All other numbers are irrationals.

Theorem 2.2.7. The equation $ax = b, a \neq 0$ has the unique solution $\frac{b}{a} \in \mathbb{R}$.

Proof. Multiply both sides by $\frac{1}{a}$:

$$\frac{1}{a}ax = \frac{1}{a}b = 1x = x = \frac{b}{a}$$

Verify the solution:

$$ax = b = a(\frac{b}{a}) = a\frac{1}{a}b = 1b = b.$$

□

Define $x^n = \prod_n x, n \in \mathbb{N}$. We then have $x^n x^m = \prod_n x \prod_m x = x^{n+m}$. Additionally, $(x^n)^m = \prod_m(\prod_n x) = x^{mn}, m, n = 1, 2, \ldots$. We also define $x^0 = 1$, $x^{-m} = \frac{1}{x^m}, x \neq 0$.

Theorem 2.2.8. If $x \in \mathbb{R}$, then $0x = 0$.

Proof. Using the definition of x^2,

$$0x = (x - x)x = x^2 - x^2 = 0.$$

Alternatively, by Addition Axiom 3 and Multiplication Axiom 3

$$0x + 1x = (1 + 0)x = 1x = x \implies 0x = x - x = 0.$$

□

From the above theorem, we have the following useful corollary.

Corollary 2.2.9. If $xy = 0$ and $x \neq 0$, then

$$y = \frac{1}{x}xy = \frac{1}{x}(xy) = \frac{1}{x}0 = 0.$$

Thus if a product vanishes, then so does one of its factors.

Theorem 2.2.10. If $(u \neq 0) \wedge (v \neq 0)$, then $\frac{x}{u} + \frac{y}{v} = \frac{xv+yu}{vu}$.

Proof. Multiply the lefthand side by $\frac{vu}{vu} = 1$

$$\frac{vu}{vu}\left(\frac{x}{u} + \frac{y}{v}\right) = \frac{vu}{vu}\frac{x}{u} + \frac{vu}{vu}\frac{y}{v} = \frac{xv}{vu} + \frac{yu}{vu} = \frac{xv + yu}{vu}.$$

Alternatively, one may notice that

$$\frac{xv + yu}{vu} = \frac{1}{vu}(xv + yu) = \frac{xv}{vu} + \frac{yu}{vu} = \frac{x}{u} + \frac{y}{v}.$$

□

Theorem 2.2.11. If $x \in \mathbb{R}$, then $-x = (-1)x$.

Proof. By Multiplication Axiom 5,

$$0 = x - x = x(1-1) = x[1 + (-1)] = x + (-1)x.$$

□

Definition 2.2.12. The factorial of $n \in \mathbb{N}$, denoted by $n!$ is the term f_n of the sequence $f_0 = 1, f_n = nf_{n-1}, n \geq 1$.

Definition 2.2.13. For $n, k \in \mathbb{N}$ with $n \geq k$, the binomial coefficient $\binom{n}{k} \in \mathbb{N}$ is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Lemma 2.2.14. For natural numbers n and k with $n \geq k$,

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Proof. The proof is purely computational and is left as an exercise, if the reader pleases. □

Theorem 2.2.15. (Binomial Theorem). For nonzero $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof. For the base step, let $n = 1$. Then,

$$(a+b)^1 = a + b = \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k = \binom{1}{1} a^{1-1} b^1 + \binom{1}{0} a^{1-0} b^0 = a + b$$

For the induction step, assume that the theorem holds for n. For the case of $n+1$ we use the preceding lemma:

$$\begin{aligned}
(a+b)^{n+1} = (a+b)^n (a+b)^1 &= (a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \sum_{k=0}^{1} \\
&= a \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k + b \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1} + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=1}^{n} \binom{n}{k-1} a^{n+1-k} b^k + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} a^{n+1-k} b^k + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k + b^{n+1} \\
&= \binom{n+1}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k + \binom{n+1}{n+1} a^0 b^{k+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k
\end{aligned}$$

□

2.2.3 Consequences of Order Axioms

Lemma 2.2.16. If $x \leq y \wedge y \leq z \wedge x = z$, then $x = y = z$.

Proof.

$$y \leq z = x \implies y \leq x$$

since $x \leq y$, $x = y = z$. □

Lemma 2.2.17. If $x < y \wedge y \leq z$, then $x < z$. Similarly, if $x \leq y \wedge y < z$, then $x < z$.

Proof. If $y < z$, then by Order Axiom 3, $x < z$. If $y = z$, then by hypothesis $x < y = z \iff x < z$. One may show the second result through a similar way. □

Theorem 2.2.18. The following inequalities are equivalent:

- $x \leq y$
- $0 \leq y - x$
- $x - y \leq 0$

Proof. We take advantage of Order Axiom 4:

$$x \leq y \implies x - x \leq y - x \iff 0 \leq y - x$$
$$0 \leq y - x \implies 0 - y \leq y - x + (-y) \iff -y \leq -x$$
$$-y + x \leq -x + x \implies x - y \leq -x + x \implies x - y \leq 0.$$

□

Lemma 2.2.19. For all $z \in \mathbb{R}$, $x < y \implies x + z, y + z$

Proof.

$$x < y \implies x \leq y \implies x + z \leq y + z.$$

It must be that $x + z < y + z$ since $x + z = y + z \implies x = y$, a contradiction. □

Theorem 2.2.20. For all $i \in \mathbb{N}$, if $x_i \leq y_i$ then

$$\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} y_i.$$

The inequality becomes strict when $x_j < y_j$ for at least one pair x_j, y_j.

Proof. By Order Axiom 3:

$$\sum_{i=1}^{n} x_i \leq y_1 + \sum_{i=2}^{n} x_i \leq y_1 + y_2 + \sum_{i=3}^{n} x_i \leq \sum_{i=1}^{n} y_i.$$

Using Lemma 2.2.19,

$$x_j < y_j \implies \sum_{i=1}^{n} x_i < \sum_{i=1}^{n} y_i$$

for at least one pair x_j, y_j. □

One can note that $x_1 \leq 0, \dots, x_n \leq 0 \implies s = \sum_{i=1}^{n} x_i \leq 0$. If $x_j < 0$, then $s < 0$ for any j.

Theorem 2.2.21. The following inequalities are equivalent:

- $x < y$
- $0 < x - y$
- $x - y < 0$

Proof. One may apply Lemma 2.2.19 to Theorem 2.2.18 to obtain this theorem. □

Definition 2.2.22. A number $x \in \mathbb{R}$ is nonnegative if $x \geq 0$, positive if $x > 0$, nonpositive if $x \leq 0$, and negative if $x < 0$.
The number 0 is both nonnegative and nonpositive.

Definition 2.2.23. Suppose $x \leq y$ for $x, y \in \mathbb{R}$. We call x the minimum of numbers x and y. We denote this fact by $\min\{x, y\} = x$. Similarly, we call y the maximum, denoted by $\max\{x, y\} = y$.

One can use induction to define $\min\{x_1, \dots, x_n\}$ and $\max\{x_1, \dots, x_n\}$:

$$\max\{x_1, \dots, x_n\} = \max\{\max\{x_1, \dots, x_{n-1}\}, x_n\}$$

Definition 2.2.24. The number $|x| = \max\{x, -x\}$ is called the absolute value, or modulus, of x.
We have that $x > 0 \implies |x| = x$ and $x < 0 \implies |x| = -x$. Also,

$$\forall x > 0 (|x| > 0 \wedge |x| = |-x|).$$

Theorem 2.2.25. If $a > 0$, then the following are equivalent:

- $|x| \leq a$
- $x \leq a$
- $-x \leq a$

Proof. By definition of absolute value,

$$|x| \leq a \iff \max\{x, -x\} \leq a.$$

It follows that if $0 < x$, then $x \leq a$. Otherwise, if $x < 0$, then $-x \leq a$. Either way, both $x \leq a$ and $-x \leq a$. $\square$

By Theorem 2.2.18, $-x \leq a \iff -a \leq x$, so that $-a \leq x \leq a$.

Theorem 2.2.26. (Triangle inequality). For all $x, y \in \mathbb{R}$, we have that $|x + y| \leq |x| + |y|$

Proof. Let $x, y > 0$ or $x, y < 0$. Then, $|x + y| = \max\{(x + y), -(x + y)\}$.
If $x \geq 0$ and $y < 0$, then

$$x + y \leq x \leq x + |y| = |x| + |y|$$
$$-x - y \leq -y \leq -y + |x| = |x| + |y|$$

so that $|x + y| = \max\{(x + y), -(x + y)\} \leq |x| + |y|$. $\square$

One may use induction to show that

$$\left| \sum_{i=1}^{n} x_i \right| \leq \sum_{i=1}^{n} |x_i|.$$

Lemma 2.2.27. If $x > 0$ and $y > 0$, then $xy > 0$.

Proof. This lemma follows from Axiom 5 and Corollary 2.2.9:

$$xy = 0 \land x \neq 0 \implies y = 1y = \frac{1}{x}xy = \frac{1}{x}(xy) = \frac{1}{x}0 = 0.$$

$\square$

Theorem 2.2.28. If $x \leq y$ and $0 < z$, then $xz \leq yz$

Proof. By Axiom 4,

$$yz - xz = (y - x)z \geq 0.$$

$\square$

One may note that by Corollary 2.2.9, $x < z \land z > 0 \implies xz < yz$. Additionally,

$0 < x < 1 \implies x^2 < x$ and $1 < x \implies x < x^2$. One can also show that

$$0 < x \leq y \wedge 0 < z \leq u \implies xz \leq yz \leq yu.$$

We note that one may multiply inequalities under these conditions. In general, if $0 < x < y$, then for all $n \in \mathbb{N}$, we have that $x^n < y^n$.

Theorem 2.2.29. If $x \leq 0$ and $0 \leq y$, then $xy \leq 0$. Similarly, if $x \leq 0$ and $y \leq 0$, then $0 \leq xy$.

Proof. Since $x \leq 0$, we have that $0 \leq -x$. It follows from Axiom 5 and Theorem 2.2.11 that $0 \leq -xy \iff xy \leq 0$.
Similarly, $x \leq 0$ and $y \leq 0$ implies that $0 \leq -x$ and $0 \leq -y$, so that $0 \leq (-x)(-y) = -1(x) - 1(y) = -1(-1)xy = 1xy = xy$. $\square$

In particular, one may note that $x^2 = xx > 0 \forall x \neq 0$. One consequence of this fact is that $1 = 1(1) > 0$. Also, by Lemma 2.2.19, $2 = 1 + 1 > 0$, $3 = 2 + 1 > 2$, etc.

Theorem 2.2.30. For all $x, y \in \mathbb{R}$, we have that $|xy| = |x||y|$.

Proof. By definition of absolute value, $|xy| = \max\{xy, -xy\}$. Let $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$. Then, $\max\{xy, -xy\} = |x||y|$ by Theorem 2.2.29. Suppose without loss of generality that $x > 0$ and $y < 0$. Then, $xy < 0$. However, we now have that $\max\{xy, -xy\} = (-x)y = |x||y|$. If $x = 0$ or $y = 0$, then the absolute value is 0 by Corollary 2.2.9. $\square$

Theorem 2.2.31. If $x > 0$, then $\frac{1}{x} > 0$. Moreover, if $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

Proof. The quotient $\frac{1}{x}$ cannot be 0 since it is undefined for that value. Let $\frac{1}{x} < 0$. Then we would have $1 < 0$, which is impossible for $1 = 1^2 > 0$. Moreover, one can multiply the inequality $0 < x < y$ by $\frac{1}{xy}$ to get $0 < x\frac{1}{xy} < y\frac{1}{xy} \iff 0 < \frac{1}{y} < \frac{1}{x}$. $\square$

Theorem 2.2.32. Let $r \in \mathbb{R}$ and $r > 0$. For all r, if $z \geq 0$ and $z < r$, then $z = 0$.

Proof. If $z > 0$, then then $z > z \geq 0$, which is impossible by conexity. $\square$

2.2.4 Axiom of Completeness

Definition 2.2.33. A set $E \subset \mathbb{R}$ is bounded below if $\exists z \in \mathbb{R} \forall x \in E (z \leq x)$. We denote that E is bounded below by $z \leq E$.
All numbers z that satisfy this property are called lower bounds of E.

Proposition 2.2.34. If a set E is bounded above, then the set $-E = \{-x : x \in E\}$ is bounded below.

Proof. By definition of upper bounds, there exists $z \in \mathbb{R}$ such that for all $x \in E$, $x \leq z$. Taking their negative elements, one can find that for all $-x \in -E$, $-z \leq -x$. The number $-z \in \mathbb{R}$ easily satisfies the definition of a lower bound. □

One may also show that if E is bounded below by z, then $-E$ is bounded above by $-z$ in a similar way.

Theorem 2.2.35. Every set $E \subset \mathbb{R}$ bounded below has a greatest lower bound $-\sup(-E)$.

Proof. For $-E = \{-x : x \in E\}$, we find using Proposition 2.2.34 that $-E$ is bounded above. By Axiom 2.4, there exists $\sup(-E)$. Taking the inequality $-x \leq \sup(-E)$, we find that $x \geq -\sup(-E)$, satisfying E having a greatest lower bound. □

We denote the greatest lower bound of E as $\inf(E)$, and call it the infimum.

Theorem 2.2.36. For $E \subseteq F$, if E and F are bounded above, then $\sup(E) \leq \sup(F)$. Likewise, if E and F are bounded below, then $\inf(E) \geq \inf(F)$.

Proof. Let $E = F$. They then have the same supremum. Let $E \subset F$. There exists at least one $z \in F$ such that $z \notin E$. If $z > \sup(E)$, then $\sup(E) < \sup(F)$. If $z < \sup(E)$, then $\sup(E) = \sup(F)$. One may use a similar way to prove the lower bounded case. □

Theorem 2.2.37. For arbitrary $x \in E, y \in F$, if $x \leq y$, then E is bounded above, F is bounded below, and $\sup(E) \leq \inf(F)$.

Proof. An arbitrary $x \in E$ immediately satisfies the definition for a lower bound since for all $y \in F$, $x \leq y$. Likewise holds for arbitrary $y \in F$ being an upper bound since for all $x \in E$, $y \geq x$. It immediately follows that $\sup(E) \leq F$ such that $\sup(E) \leq \inf(F)$. □

Theorem 2.2.38. For all $x \in \mathbb{R}$ and $n \in \mathbb{Z}, x, n > 0$, there exists a unique $y > 0$ such that $y^n = x$.

Proof. Let A be the set of all $z > 0$ such that $z^n \leq x$. Then A is bounded above, by 1 if $x \leq 1$ and by x if $1 \leq x$. Let $y = \sup(A)$. We will show that $y^n = x$.
Suppose that $y^n < x$ and let $x - y^n = \varepsilon$. For all h such that $0 < h \leq 1$, and by Theorem 2.2.15:

$$\begin{aligned}(y+h)^n &= y^n + ny^{n-1}h + \frac{n(n-1)}{1\cdot 2}y^{n-2}h^2 + \dots \\ &= y^n + h(ny^{n-1} + \frac{n(n-1)}{1\cdot 2}y^{n-2}h + \dots) \\ &\leq y^n + h(ny^{n-1} + \frac{n(n-1)}{1\cdot 2}y^{n-2} + \dots) = y^n + h[(1+y)^n - y^n]\end{aligned}$$

Set $h < \frac{\varepsilon}{(1+y)^n - y^n}$ so that we have $(y+h)^n \leq y^n + \varepsilon = x$. However, this contradicts the fact that $y = \sup(A)$. Therefore, $y^n \geq x$.
Suppose that $y^n > x$. Let $\varepsilon = y^n - x$. For all h such that $0 < h \leq 1$, and by Theorem 2.2.15:

$$\begin{aligned}(y-h)^n &= y^n - ny^{n-1}h + \frac{n(n-1)}{1\cdot 2}y^{n-2}h^2 + \dots \\ &= y^n - h(ny^{n-1} + \frac{n(n-1)}{1\cdot 2}y^{n-2}h + \dots) \\ &\geq y^n - h(ny^{n-1} + \frac{n(n-1)}{1\cdot 2}y^{n-2} + \dots) = y^n - h[(1+y)^n - y^n]\end{aligned}$$

Set $h < \frac{\varepsilon}{(1+y)^n - y^n}$ so that we have $(y-h)^n \geq y^n - \varepsilon = x$. However, this contradicts the fact that $y = \sup(A)$. Therefore, $y^n = x$.
We call y the nth root of x. We denote the nth root of x by $\sqrt[n]{x}$, or by $x^{\frac{1}{n}}$. The nth root is unique since $y_1 < y_2 \implies y_1^n < y_2^n \iff x < x$, which is impossible. □

Theorem 2.2.39. For all $x > 0$ and $y > 0$, $(xy)^{\frac{1}{n}} = x^{\frac{1}{n}}y^{\frac{1}{n}}$

Proof. Note that

$$xy = x^{\frac{n}{n}}y^{\frac{n}{n}} = (xy)^{\frac{n}{n}} \iff (x^{\frac{1}{n}}y^{\frac{1}{n}})^n = \left[(xy)^{\frac{1}{n}}\right]^n \iff (x^{\frac{1}{n}}y^{\frac{1}{n}}) = \left[(xy)^{\frac{1}{n}}\right].$$

□

Simlarly, one can show that $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$.

Suppose an even interger n. Then, $(-x)^n = (-1)^n x^n = x^n > 0 \forall x \neq 0$. The equation $y^n = x > 0$ has both real solutions $y_1 = \sqrt[n]{x}$ and $y_2 = -\sqrt[n]{x}$, while the equation $y^n = x < 0$ has no real solutions.

Suppose an odd integer n. Then, $y^n = x > 0$ has a unique real solution $y = \sqrt[n]{x}$, while

$y^n = x < 0$ has a unique real solution $y = -\sqrt[n]{x}$.

By Theorem 2.2.39, formulas like the quadratic and cubic equations hold for all real numbers.

Definition 2.2.40. A set $E \subseteq \mathbb{R}$ is bounded from both sides or bounded if it has both upper and lower bounds. It follows that both $\inf(E)$ and $\sup(E)$ exist.

One important class of bounded sets follows.

Definition 2.2.41. The set of all $x \in \mathbb{R}$ such that $a < x < b$ is called an open interval, with left-hand and right-hand endpoints a and b, respectively. If $a \leq x \leq b$, then we call the set a closed interval, with identical endpoints.

One may see that $\sup[a, b] = \sup(a, b)$, and that $\inf[a, b] = \inf(a, b)$. Both 'half-closed' and 'half-open' intervals $[a, b)$ and $(a, b]$ are defined analogously.

2.3 Archimedean Property and its Consequences

Theorem 2.3.1. (Archimedean Property). For all real $x \geq 0$ and $y > 0$, there exists $n \in \mathbb{Z}$ such that $(n-1)x \leq y < nx$.

Proof. Suppose that for all $p \in \mathbb{Z}$, we have that $px \leq y$. Define the set $A = \{px | px \leq y\}$, and see that y forms an upper bound for A. Let $\sup(A) = d \leq y$. Take the difference $d - x < d$. We can set $p_0 \in \mathbb{Z}$ such that $d - x < p_0 x$. However, $d - x < p_0 x \iff d < p_0 x + x \iff d < (p_0 + 1)x$ with $(p_0 + 1)x \in A$, forming a contradiction for d being an upper bound.
One may set $\{(p_i - 1), p_i\} = \{(n+1), n\}$ so that $(n-1)x \leq y < nx$. $\square$

If we set $x = 1$, we find that for all $y \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that $n - 1 \leq y \leq n$. We call n the integral part of y, denoted $[y]$, and $y - [y]$ the fractional part y, denoted (y). Also, $\forall y \in \mathbb{R}[y \leq [y] + (y)]$.

Theorem 2.3.2. For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$, $x, y > 0$, there exists $n \in \mathbb{Z}$ such that $x^{n-1} \leq y < x^n$.

Proof. Suppose that for all $p \in \mathbb{Z}$, we have that $x^{p-1} < y$. Define the set $A = \{x^p | x^p \leq y\}$, and see that y forms an upper bound for A. Let $\sup(A) = d \leq y$. Take the quotients $\frac{d}{x} < d$. We can set p_0 such that $\frac{d}{x} < x^{p_0}$. However, $\frac{d}{x} < x^{p_0} \iff d < x x^{p_0} \iff d < x^{p_0+1}$ with $x^{p_0+1} \in A$, forming a contradiction for d being an upper bound.
One may set from pairs $\{(p_i - 1), p_i\} = \{(n+1), n\}$ so that $x^{n-1} \leq y < x^n$. □

Theorem 2.3.3. For all $x, y > 0$, there exists an integer $n > 0$ such that $\frac{y}{n} < x$.

Proof. By the Archimedean Property, there exists $n \in \mathbb{Z}$ such that $y < xn$. Set $n > \frac{y}{x} > 0$. Multiply both sides by $\frac{x}{n}$ to get $\frac{y}{n} < x$. □

It follows that for all $y > 0$, $\inf\{\frac{y}{n} | n \in \mathbb{Z}^+\} = 0$.

Corollary 2.3.4. The following systems of half-open intervals for $y > 0$ has an empty intersection:

$$\ldots \subseteq (0, \frac{y}{n}] \subseteq \ldots \subseteq (0, \frac{y}{2}] \subseteq (0, y] \tag{2.1}$$

$$\ldots \subseteq (a, a + \frac{y}{n}] \subseteq \ldots \subseteq (a, a + \frac{y}{2}] \subseteq (a, a + y] \tag{2.2}$$

$$\ldots \subseteq (a - \frac{y}{n}, a] \subseteq \ldots \subseteq (a - \frac{y}{2}, a] \subseteq (a - y, a] \tag{2.3}$$

Proof. Suppose systems 2.2 and 2.3 had common elements c and d, respectively. That would mean that system 2.1 would have commmon points $c - a$ and $a - d$. However, by Theorem 2.3.1, system 2.1 has no common points. □

Theorem 2.3.5. For all open intervals (a, b), there exists an element $z \in \mathbb{Q}$ such that $a < z < b$.

Proof. Take the difference $b - a = h$. By Theorem 2.3.1, there exists $n \in \mathbb{N}$ for $\frac{1}{h}$ such that $(n-1) \leq \frac{1}{h} < n \iff \frac{1}{n} < h$. By that same theorem, there exists $m \in \mathbb{N}$ for $\frac{1}{n}$ such that $\frac{m}{n} \leq a < \frac{m+1}{n}$. Add $\frac{1}{n}$ to both sides and subtract a to obtain $\frac{m}{n} + \frac{1}{n} - a \leq \frac{1}{n} + a - a \iff \frac{m+1}{n} - a \leq \frac{1}{n} < h = b - a \iff \frac{m+1}{n} < b$, showing that there exists $z = \frac{m+q}{n}$ so that $a < z < b$. □

One can show that infinitely many rational elements exist in (a, b) by applying the preceding theorem to the interval $(\frac{m+1}{n}, b)$, and so on.

Theorem 2.3.6. For all $c \in \mathbb{R}$, let N_c be the set of all $s \in \mathbb{Q}$ such that $s \leq c$, and P_c the set of all $r \in \mathbb{Q}$ such that $c \leq r$. Then, $\sup(N_c) = c = \inf(P_c)$.

Proof. The set N_c is bounded above by c, so that it has a supremum. Denote $\sup(N_c) = a$. By definition, $a \leq c$. Suppose that $a < c$. By Theorem 2.3.5, there exists a rational element p such that $p \in (a, c)$. Since $a = \sup(N_c)$, however, $p \leq a$, creating a contradiction. Therefore, $a = c = \sup(N_c)$. □

2.4 Nested Intervals Property

Definition 2.4.1. If a set Q of intervals on $\mathbb{R}$ has the property that given any two intervals $I, J \in Q$, either $I \subseteq J$ or $J \subseteq I$ holds, then we call Q a system of nested intervals.

By Corollary 2.3.4, a system of half-open intervals may well have an empty intersection. The same holds for nested open intervals. However, we now show that a system of closed intervals always has an intersection.

Theorem 2.4.2. **(Nested Intervals Property)**. For all systems of closed intervals $[a_i, b_i] \in Q$, there exists $c \in \mathbb{R}$ such that $c \in \bigcap_i [a_i, b_i]$. Specifically, their intersection comprises the interval $[\sup(A), \inf(B)]$ for sets $A = \{a_i : [a_i, b_i] \in Q\}$ and $B = \{b_i : [a_i, b_i] \in Q\}$.

Proof. Let A be the set of all left endpoints $A = \{a_i : [a_i, b_i] \in Q\}$, and B the set of all right endpoints $B = \{b_i : [a_i, b_i] \in Q\}$. Since for any two intervals $[a_i, b_i]$ and $[a_j, b_j]$, $i \leq j$, we have that $[a_j, b_j] \subseteq [a_i, b_i]$, it follows that $a_i \leq a_j$ and $b_j \leq b_i$. One may also note that all b_i serve as upper bounds to A, and all a_i serve as lower bounds to B. By Completeness, $\sup(A)$ and $\inf(B)$ both exist. Since for all $[a_i, b_i] \in Q$ we have $a_i \leq \sup(A) \leq \inf(B) \leq b_i$, it follows that $[\sup(A), \inf(B)] \subseteq \bigcap_i [a_i, b_i]$. Moreover, suppose an x where $x \notin [\sup(A), \inf(B)]$ such that $x < \sup(A)$. Then, we can always find an a_i where $x < a < \sup(A)$. A similar result holds for some $x > \inf(B)$, so that $\bigcap_i [a_i, b_i] = [\sup(A), \inf(B)]$. The case where $\sup(A) = \inf(B)$ gives the value for c. □

Theorem 2.4.3. For all systems of closed intervals $[a_i, b_i] \in Q$ there exists a single point c that serves as their only intersection if and only if for all $\varepsilon > 0$ there exists an interval $[a, b] \in Q$ such that $b - a < \varepsilon$.

Proof. From Theorem 2.4.2, the intersection of nested closed intervals consists of a single point if and only if $\sup(A) = \inf(B)$. For all $\varepsilon > 0$, there exists $[a_1, b_1]$ and $[a_2, b_2]$ with $[a_2, b_2] \subseteq [a_1, b_1]$ such that

$$a_1 > \sup(A) - \frac{\varepsilon}{2} \wedge b_2 < \inf(B) + \frac{\varepsilon}{2}.$$

Set $c = \sup(A) = \inf(B)$. Then,

$$c - \frac{\varepsilon}{2} < a_1 \leq a_2 \iff b_2 - \frac{\varepsilon}{2} < c < a_1 + \frac{\varepsilon}{2} \leq a_2 + \frac{\varepsilon}{2} \iff b_2 - a_2 < c + \frac{\varepsilon}{2} < a_1 + \varepsilon \leq \varepsilon$$

giving us $b_2 - a_2 < \varepsilon$ as the required interval.
For the converse, let there be an interval $[a_\varepsilon, b_\varepsilon]$ that depends on some $\varepsilon > 0$ such that $b_\varepsilon - a_\varepsilon < \varepsilon$. Taking them as part of their respective endpoint sets, we have that $a_\varepsilon \leq \sup(A)$ and $b_\varepsilon \geq \inf(B)$. Since $b_\varepsilon - a_\varepsilon \geq \sup(A) - \inf(B)$, succesively taking smaller values of ε would cause the righthand side to reduce to 0, such that $\sup(A) = \inf(B)$. □

2.5 The Extended Real Number System

Definition 2.5.1. The extended real number system $\bar{\mathbb{R}}$ consists of the real number line $\mathbb{R}$ and two infinity elements $-\infty$ and $+\infty$. The usual order relations get extended by the following rules:

1. $\forall x \in \mathbb{R}(-\infty < x)$.
2. $\forall x \in \mathbb{R}(x < \infty)$.
3. $-\infty < \infty$.

Ordinary order axioms continue to hold in $\bar{\mathbb{R}}$. Elements $x \in \mathbb{R}$ are called finite, contrasting with infinity elements $-\infty$ and ∞.

For nonempty $E \subseteq \mathbb{R}$, we define $\sup(E) = \infty$ if $\infty \in E$ or there exists no x such that $E \leq x$. Likewise, $\inf(E) \neq -\infty$ if $-\infty \in E$ or there exists no x such that $E \geq x$.

For two points $a, b \in \bar{\mathbb{R}}, a < b$, then the set $[a, b] = \{x \in \bar{\mathbb{R}} : a \leq x \leq b\}$ is a closed interval with endpoints a and b. Meanwhile, the set $(a, b) = \{x \in \bar{\mathbb{R}} : a < x < b\}$ is called an open interval with the same endpoints.

We can also generalize the Nested Interval Property to $\bar{\mathbb{R}}$. For a system of closed intervals $Q = \{[a_i, b_i] : a, b \in \bar{\mathbb{R}} \wedge [a_j, b_j] \subseteq [a_i, b_i]\}$. Then, there exists $c \in \bar{\mathbb{R}}$ such that for all $[a_i, b_i]$, $c \in [a_i, b_i]$. This intersection corresponds to the interval $\sup(A), \inf(B)$ where $A = \{a_i : [a_i, b_i \in Q]\}$ and $B = \{b_i : [a_i, b_i]\}$, so that $\sup(A) \leq \inf(B)$.

Exercises

3.0 Metric Spaces

3.1 Equivalence and Cardinality

Definition 3.1.1. We call a set as countable if a bijection exists between it and a subset of $\mathbb{N}$.

Theorem 3.1.2. The closed interval $[a, b]$ and open interval (a, b) are equivalent.

Proof. One may construct a bijection between these intervals. Suppose a sequence A of distinct points $x_1 = a, x_2 = b, x_3, x_4, \ldots \in A$. Clearly, points $x_3, x_4, \ldots$ and all points $y \notin A$ are elements of (a, b). We then make the rule $x_1 \to x_3, x_2 \to x_4, x_3 \to x_5, \ldots, x_n \to x_{n+2}, y \notin A \to y$ which establishes the existence of a bijection. □

Theorem 3.1.3. Consider a set A with cardinality $|A| = n$. Then the power set of A has cardinality $|\mathcal{P}(A)| = 2^n$.

Proof. Set $|A| = n = 1$. Then we find two subsets: $\emptyset$ and A itself. Assume that this result holds for all n. Taking the case of $n + 1$, we can form two partitions, A_n with n elements and A_1 with 1 element. By induction step, $\mathcal{A}_\backslash$ has 2^n subsets. By base case, A_1 has $2 = 2^1$ subsets. Taking the union $A_n \cup A_1$, the number of subsets clearly becomes multiplied by 2, so that we have $2^n \dot{2}^1 = 2^{n+1}$. □

3.2 Mathematical Structures

A Mathematical Structure is a set with certain properties defined. Metric spaces, this chapter's topic, comprise a mathematical structure, as we will see later.

Definition 3.2.1. Two structures of the same kind are isomorphic if a bijection exists between them.

Every structure is isomorphic to itself through the identity mapping, such that all properties are satisfied by elements and subsets of the structure. Other nonidentical bijections also exist, called automorphisms.

Example

Suppose a linearly ordered set $E = \{x, y, \ldots\}$ with the property that given any $x \neq y$, either $x < y$ or $x > y$, where

3.3 Open Sets

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