Integral, Measure, and Derivative

Lecture Notes

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Preface

1.0 Riemann Integral and Step Function

1.1 The Riemann Integral

Definition 1.1.1. (Parallelipiped). An n-dimensional parallelipiped is a set B of points $\vec{x} = (x_1, x_2, \ldots, x_n)$ where $B = \{\vec{x} : a_1 \leq x_1 \leq b_1, \ldots, a_n \leq x_n \leq b_n\}, \forall i(a_i < b_i)$.

It is convenient to call parallelipipeds blocks. The size of a block B is the maximum $\max(b_i - a_i), i \in \mathbb{N}$. The volume of B is the product $s(B) = \prod_{i=1}^{n}(b_i - a_i)$.

Suppose we take a partition of subblocks $B_1, \ldots, B_p$ of B. Then the volume $s(B)$ is additive:

$$s(B) = s(B_1) + \ldots + s(B_p).$$

This property is important for the following definition.

Definition 1.1.2. (Riemann Sum). If we are given a fixed block $\mathbb{B}$ (a basic block) and a real, bounded function $f(x)$ defined on $\mathbb{B}$, we can set Π to be a partition on $\mathbb{B}$ into p subblocks B_k. For all B_k there exists constant $c_k \in B_k$. We can thus construct the Riemann sum:

$$R_\Pi = \sum_{k=1}^{p} f(c_k)s(B_k).$$

Definition 1.1.3. (Riemann Integral). Let $d(\Pi)$ denote the largest size $d(\Pi) = \max(s(B_k)), k = 1, \ldots, p$. Also let $\Pi_1, \ldots, \Pi_p$ be a sequence of partitions such that $d(\Pi_q) \to 0$. If the limit of corresponding Riemann sums R_{Π_q} exists as $q \to \infty$ for arbitrary sequence (Π_q) and points $c_k \in B_k$, then we call the limit the Riemann integral:

$$\sum_{\mathbb{B}} f(\vec{x})d\vec{x} = \lim_{d(\Pi)\to 0} R_\Pi.$$

1.2 Lower and Upper Integrals

Definition 1.2.1. (Darboux Sums). Let Π be a partition on $\mathbb{B}$, $B_1, \ldots, B_p$ (hereafter called a p-partition). Set $m_k = \inf_{\vec{x}\in B_k} f(\vec{x})$ and $M_k = \sup_{\vec{x}\in B_k} f(\vec{x}), k = 1, \ldots, p$. Then, we call

$$D_{\Pi^-} = \sum_{k=1}^{p} m_k s(B_k)$$

the Lower Darboux Sum, and

$$D_{\Pi^+} = \sum_{k=1}^{p} M_k s(B_k)$$

the Upper Darboux Sum.

Clearly, for any constant $c_k \in B_k$,

$$\inf_{\vec{x}\in\mathbb{B}} f(\vec{x})s(\mathbb{B}) \leq R_\Pi \leq D_{\Pi^+} \leq \sup(\vec{x})f(\vec{x})s(\mathbb{B}).$$

Suppose that $\inf_{\vec{x}\in\mathbb{B}} f(\vec{x})s(\mathbb{B}) > D_{\Pi^-}$. Then for at least one B_k, $\inf_{\vec{x}\in\mathbb{B}} f(\vec{x})s(\mathbb{B}) > m_k s(B_k)$, which contradicts the lefthand value being the infimum for all values of $\mathbb{B}$.

Given a partition Π_1 of $\mathbb{B}$, construct Π_2 by further dividing subblocks of Π_1—a refinement of Π_1. Thus, we can replace every term of form $m_k s(B_k)$ of D_{Π^-} by $\sum_j m_{k_j} s(B_{k_j})$, where $B_k = \bigcup_j B_{k_j}$ and $m_{k_j} = \inf_{\vec{x}\in B_{k_j}} f(\vec{x})$.