Lecture Notes on Social Choice Theory

Dave White

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1 Social Choice

1.1 Social preference

1.1.1 Introduction

Fundamental to the study of individual and social choice are the notions of **utility** and **preference**. Intuitively, the utility of a candidate for an individual or group is a measure of "how much" that individual or group "likes" the candidate in question. Preference is, in mathematical terms, a *(binary) relation* on the utilities of a set of candidates, i.e. a comparison between the utilities of two alternatives, the alternative with the higher utility being preferred (by definition) to the one with lower utility. Ascertaining the preferences of an individual is *relatively* simple: simply a matter of asking the individual to declare which of each pair of alternatives they prefer, and for this course we assume that this can be done and will not yield inconsistent results at the level of the individual. (In the "real world", complexities and paradoxes do arise even for individual preferences.) Doing this for groups is not nearly as straightforward, and finding methods for *aggregating* individual preferences to produce preferences that reflect the group's collective attitudes toward the given alternatives is the subject of the course at hand.

In general, finding precise numerical values for the utilities of various alternatives is, even in the case of a single individual, a tricky business, but a great deal of the theory can be developed without making utilities *explicit*: The preference relation between candidates' utilities—and thus between the candidates themselves—give us enough *implicit* knowledge of candidate utilities to work with. Down the line we will find that significant difficulties arise when trying to assure that collective preferences reflect individual ones in certain "reasonable" ways. Some ways of attacking these problems call for explicit utilities to be given and more complex comparisons to made among them. This more advanced level of the theory is beyond the scope of the current course.

1.1.2 Preference relations

For candidates x and y, the formal expression $x \sim y$ indicates that an individual or group considers the utilities of x and y to be equal, in which case we say that the individual or group are **indifferent** between x and y, or that x and y are indifferent. The expression $x \succeq y$ indicates that the they consider the utility of x to be *at least* as high as that of y. The **(weak) preference relation** "$\succeq$" is required to satisfy the following properties for any candidates x, y, z:

(i) Reflexivity: It is always the case that $x \succeq x$.

(ii) Antisymmetry: If $x \succeq y$ and $y \succeq x$, then $x \sim y$.

(iii) Transitivity: If $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

(iv) Connexity: At least one of $x \succeq y$ or $y \succeq x$ must hold.

Parenthetically, a relation satisfying conditions (i)–(iii) is called a *partial order*. When (iv) is required in addition, the result is called a *total order*. Whenever $x \succeq y$ and x and y are *not* indifferent, we may write $x \succ y$ and say that x is **(strictly) preferred to** y. This *strict preference relation* has the following properties:

(i′) Transitivity: If $x \succ y$ and $y \succ z$, then $x \succ z$.

(ii′) Trichotomy: Exactly one of $x \succ y$, $y \succ x$, and $x \sim y$ is true.

Notation:

- V denotes the (finite) set of voters, often called the **society of voters**.
- $N = \#V$ denotes the number of voters.
- We will identify voters by indexing them by the positive integers: v_1 denotes voter one, v_2 denotes voter two, and so on up to v_N. In general v_i denotes the i-th voter.

Definition 1. A **preference ballot** (or simply *ballot*) is an individual voter's rank-ordering of the candidates. A ballot is said to **cover** a candidate set C when all the candidates in C appear on the ballot. When the candidates of C *and only* those of C appear on the ballot, it is said to be **over** C. E.g. if $C = \{a, b, c, d\}$ then

$$b \succ a \succ d \succ c$$

is one possible preference ballot which ranks b in 1st place, a in 2nd place, d in 3rd place, and c in 4th place. Here the ballot is over C, and could be said to cover, E.g., the subset $\{b, c\}$. We will denote the ballot cast by the i-th voter by b_i.

In the general theory of individual/social choice, we typically allow individual voters to be indifferent between pairs of candidates. For the purposes of this course, however, it will be assumed that each voter applies a strict total order to the candidates; there are no ties on an individual ballot.

We will represent a preference ballot as an $n \times 1$ array, i.e. a single column with n rows/entries, with the voter's more preferred candidates above less preferred candidates. The preference ballot given above would be represented as

$$\begin{array}{c} b \\ a \\ d \\ c \end{array}$$

Definition 2. A **(preference) profile** is a *vector* $\mathbf{b} = (b_1, b_2, \ldots, b_N)$ of preference ballots, all over some common candidate set C. We say that a profile **covers**, resp. **is over**, a the candidate set C precisely when all its ballots cover, resp. are over, C.

We denote by $\mathfrak{U}$ the "universe" of candidates, i.e. a set containing everything that could conceivably be a member of some candidate set C (in the context of some social choice problem). With respect to $\mathfrak{U}$, the set of all possible ballots, i.e. ballots over any $C \subset \mathfrak{U}$, is denoted $\mathfrak{B}$. The full set of candidates C actually appearing in the profile are said to be **in contention**, while the candidates of $\mathfrak{U}$ which are not in C are said to be **out of contention**. When we speak of *removing candidates from contention*, we mean to consider the profile which results from deleting those candidates without any changes to the relative rankings of other candidates. The set of all possible profiles generated by N voters, with respect to some universe of candidates $\mathfrak{U}$ (which is usually implicit), is denoted $\mathfrak{B}^N$.

Over n candidates, we will represent a profile as an $n \times N$ array having the ballots as columns. E.g. suppose $N = 3$, $C = \{a, b, c\}$, and that both v_1 and v_2 rank the candidates $a \succ b \succ c$ while v_3 ranks them $b \succ c \succ a$. The associated array would be written

$$\begin{array}{ccc} a & a & b \\ b & b & c \\ c & c & a \end{array}$$

Such an array with all voters' ballots shown individually is called the **detailed preference schedule**.

The **reduced preference schedule** is the array of *distinct* preference ballots with the number of voters casting each ballot written above it. When used without qualification, *preference schedule* shall always refer to the reduced form. For the example above, the reduced preference schedule is

$$\begin{array}{cc} 2 & 1 \\ a & b \\ b & c \\ c & a \end{array}$$

Exercise 1. Determine the number of possible distinct preference ballots over a set of n candidates.

Exercise 2. Determine the number of possible distinct reduced preference schedules (up to rearranging columns) given a set of n candidates and N voters.

Definition 3. With respect to a universe of candidates $\mathfrak{U}$ and a society of N voters V, a **(social) welfare function** is a function $F : \mathfrak{B}^N \to \mathfrak{B}$ which maps a profile over C to a "ballot" also over C. This "ballot" represents the *social preferences* of the society of voters. The elements of the range of F, i.e. the "ballots" it produces from a given profile, are called **social rankings**.

The theory of social choice revolves around the creation of criteria that ought to be satisfied by a welfare function, based on what one might call "social values", "ethics", "fairness", etc., and the investigation into the implications of these criteria.

In many voting contexts, the entirety of the social ranking on candidates is not desired: all that matters is which candidate(s) *win* an election. To this end we consider a simplification of the welfare function which seeks only to identify which candidates come out "on top" when aggregating voter preferences.

1.1.3 Winner selection

Definition 4. A **winner selection method (WSM)** is a *deterministic* algorithm taking as its input a set of candidates C and a profile $\mathbf{b}$ covering C, and giving as its output a *nonempty* subset $W \subseteq C$, the members of which are the declared *winners* of the election. (By *deterministic* we mean that if the algorithm is run multiple times on the same input, it always ouputs the same result.) When we wish to distinguish that the winner set is determined from C, we write $W(C)$.

A WSM which always designates *exactly one* winner ($W = \{x\}$) is called a **single-winner selection method (SWSM)**.

Definition 5. Every WSM *induces* a welfare function F in the following manner:
Given a profile $\mathbf{b}$ over candidate set C we apply the WSM *recursively*:

1. Set $i = 1$ and $W_1 = W(C)$.
2. If $C \backslash W_i \neq \emptyset$, set $W_{i+1} = W(C \backslash W_i)$ and proceed to step 3; otherwise go to step 4.
3. Set $i = i + 1$ and go back to step 2.
4. Set $m = i$ and terminate.

Now define $F(\mathbf{b})$ by the following rule: If $x \in W_j$ and $y \in W_k$ where $1 \leq j < k \leq m$, then $x \succ y$, and $x \sim y$ if both $x, y \in W_\ell$ for $1 \leq \ell \leq m$. This process is called the **recursive ranking** of a given WSM.

Similarly, every welfare function F induces a WSM. Say $F(\mathbf{b})$ is defined by the ordering

$$a \sim b \sim c \succ d \succ e \sim f \succ \cdots .$$

Then the **induced WSM** selects $W = \{a, b, c\}$. In general the induced WSM selects as winners all those (necessarily pairwise indifferent) candidates appearing to the left of the first appearance of "$\succ$" when the ordering given by F is written as above (in descending order of preference left to right).

In general there will be many different welfare functions which induce the same WSM in the manner described.

Notation:

- When not specified otherwise, C is the set of candidates in contention, and $n = \#C$, the number of candidates.
- For each candidate x, the integer $1 \leq r_i(x) \leq n$, where $1 \leq i \leq N$, is the rank of x on the ballot b_i; E.g. $r_3(a) = 2$ when v_3 ranks candidate a in 2nd place on their ballot.

- For each candidate x, the integer $0 \leq R_k(x) \leq N$ is the number of ballots on which x is ranked in the k-th position, $1 \leq k \leq n$; i.e.

$$R_k(x) = \#\{i \mid r_i(x) = k\}\,.$$

- We use a subscript on the relation symbols to denote which ballot they apply to. Thus $x \succ_i y$ indicates that x is placed higher than y on b_i.

- We denote by $\Delta(x, y)$ the integer

$$\#\{i \mid x \succ_i y\} - \#\{i \mid x \prec_i y\}\,,$$

  i.e. the number of ballots on which x is ranked higher than y minus the number of ballots on which x is ranked lower than y. This number gives the (numerical) result of the *pairwise comparison* between x and y.

- If x wins or ties in the pairwise comparison with y (equivalently $\Delta(x, y) \geq 0$), then we write $x \trianglerighteq y$. If x wins ($\Delta(x, y) > 0$) we may write $x \triangleright y$, and if x ties with y ($\Delta(x, y) = 0$) we may write $x \triangleq y$.

Definition 6. A candidate x is said to have a **majority** (or to be a **majority candidate**) if $R_1(x) > N/2$.

Exercise 3. Show that if a majority candidate exists, it is unique.

Definition 7. The **(majoritarian) pairwise comparison** between any pair of candidates x and y is the determination of whether $x \triangleright y$, $y \triangleright x$, or $x \triangleq y$, i.e. the result of removing from contention all other candidates and then deciding between x and y by majority vote.

Definition 8. The **(majoritarian) pairwise-comparison (di)graph** is the directed graph (digraph) $G = (C, E)$ where $(x, y) \in E$ if and only if $x \triangleright y$.

Definition 9. A WSM, resp. welfare function, is said to satisfy the **principle of non-imposition**, also called **citizen sovereignty**, if every possible winner set, resp. social ranking, can be produced by some profile of preferences. (In this case the welfare function is said to be *surjective* or *onto*.)

Intuitively, this means that no restriction independent of the desires of the voters is imposed upon the election outcome.

Definition 10. A WSM is said to satisfy the **principle of one person, one vote (OPOV)** if the outcome depends *only* upon the *reduced* preference schedule.

Definition 11. Dictatorship of the k-th voter is the WSM whereby $W = \{x\}$ if and only if $r_k(x) = 1$.

In the context of welfare functions, the k-th voter is the dictator iff, for every possible profile $\mathbf{b}$, we have $F(\mathbf{b}) = b_k$, the ballot cast by the k-th voter. (In terminology introduced below, one says that v_k is *decisive* for every pair of candidates.)

Exercise 4. Convince yourself that dictatorship violates the principle of one person, one vote.

Definition 12. A candidate x is said to have a **plurality** if

$$R_1(x) = \max_{y \in C} R_1(y).$$

Exercise 5. Convince yourself that having a majority implies having a plurality, but not vice versa.

Definition 13. The **plurality method**: A candidate $x \in W$ iff x has a plurality.

This method is sometimes called **first past the post (FPTP)**.

Definition 14. The **plurality extension ranking** is the social ranking where the candidates are ranked in descending order of number of first-place votes.

Proposition 1. *The plurality extension ranking is* not *equivalent to the recursive ranking induced by the plurality method.*

Proof. Consider the following preference schedule.

3	2	1
a	b	c
c	c	a
b	a	b

The product of the extension ranking is $a \succ b \succ c$, while that of the recursive ranking is $a \succ c \succ b$. □

Note. Observe that the previous proposition demonstrates that WSM's can induce more than one social ranking, and these social rankings all necessary induce the given WSM. There is a "one-to-many" relationship between WSM's and welfare functions, resp.

Definition 15. The **runoff method**: Remove from contention all but the two candidates with the most first-place votes and then decide between these two by majority vote.

Definition 16. The **elimination (or instant runoff) method**: Remove from contention the candidate with the least first-place votes. Repeat this until some candidate wins by majority vote.

Note. The application of the runoff and elimination is ambiguous in the event of ties. There is no widespread agreement on how ties should be broken.

Exercise 6. Show that for the preference schedule

6	5	4	3
a	b	c	d
b	c	b	c
c	a	a	b
d	d	d	d

candidate a wins by plurality, b wins by runoff, and c wins by elimination.

Definition 17. Collectively the WSM's plurality, runoff, and elimination will be referred to as the **plurality variants**.

Definition 18. A WSM is called **majoritarian** if $W = \{x\}$ whenever x is a majority candidate.

Proposition 2. *The plurality variants are majoritarian.*

Proof. Exercise. $\square$

Definition 19. The **Borda count:** We assign to each candidate the *Borda score*

$$\mathcal{B}(x) = \sum_{k=1}^{n} R_k(x)(n-k+1) = \sum_{i=1}^{N} n - r_i(x) + 1.$$

(You should convince yourself that the two expressions given for the Borda score are indeed equivalent.) Then $x \in W$ if and only if

$$\mathcal{B}(x) = \max_{y \in C} \mathcal{B}(y)$$

Definition 20. The **Borda extension ranking** is the social ranking produced by ordering candidates by Borda score: the higher the Borda score, the higher the social preference, and equal Borda scores produce indifferences/ties.

Definition 21. The **Copeland/pairwise-comparison method**: For each $x \in C$ let the function h_x be defined on $C \backslash \{x\}$ by

$$h_x(y) = \begin{cases} 1 & \text{if } x \rhd y, \\ \frac{1}{2} & \text{if } x \triangleq y, \\ 0 & \text{if } x \lhd y. \end{cases}$$

We assign to each candidate x the *Copeland score*

$$\mathcal{C}(x) = \sum_{y \in C \backslash \{x\}} h_x(y).$$

Then $x \in W$ if and only if

$$\mathcal{C}(x) = \max_{y \in C} \mathcal{C}(y).$$

Note. An efficient way to compute the the Copeland scores is to enumerate and determine all of the *distinct* pairwise comparisons at the outset.

Exercise 7. Show that the number of pairwise comparisons is

$$\binom{n}{2} = \frac{n(n-2)}{2}.$$

Exercise 8. Show that $h_x(y) = \frac{1}{2} + \frac{1}{4}\left(|1 + \Delta(x,y)| - |1 - \Delta(x,y)|\right)$.

Exercise 9. Find an expression for $\sum_{x \in C} \mathcal{C}(x)$ that depends only on n.

Definition 22. A candidate x is called **Condorcet** if $x \rhd y$ for all $y \in C, y \neq x$.

Exercise 10. Prove that there can never be more than one Condorcet candidate and that it is possible for their to be no Condorcet candidate.

Exercise 11. Prove that any majority candidate is a Condorcet candidate, but not vice versa.

Definition 23. A WSM is said to be **Condorcet** if $W = \{x\}$ whenever x is a Condorcet candidate.

Proposition 3. *If a WSM is Condorcet, then it is majoritarian. However, the plurality variants are all majoritarian but not Condorcet.*

Proof. The first statement follows immediately from the fact that every majority candidate is Condorcet. (Convince yourself of this.)

Now for the following preference schedule, a is the Condorcet candidate but loses by the plurality, runoff, and elimination methods.

5	6	7
a	b	c
b	a	a
c	c	b

□

Proposition 4. *The Copeland/pairwise-comparison method is Condorcet (hence majoritarian).*

Proof. Suppose x is a Condorcet candidate. It is clear from the definition that the Copeland score of x is $N - 1$, one point for each other candidate, all of which x beats in pairwise comparison. For any candidate y, with $y \neq x$, again y loses to x in pairwise comparison, and hence has a *maximum* Copeland score of $N - 2$. Therefore x must be the sole candidate having the maximum Copeland score and is the sole winner. □

Proposition 5. *Borda count is non-majoritarian (hence non-Condorcet).*

Proof. For the following preference schedule, a is the majority candidate but b wins the Borda count, beating a 12:11.

3	2
a	b
b	c
c	a

□

The following theorem due to the Marquis de Condorcet (1743–94), the individual for whom Condorcet methods are named, was first published in 1785 in his work *Essay on the Application of Analysis to the Probability of Majority Decisions.* It represents one of the first attempts at justifying the principle of democracy in a mathematical fashion. Do you find it to be applicable to real human society? Do you think its conclusions really lend support to democratic ideals? What implications do you believe it may have for the manner in which a democratic society should conduct itself?

. *Condorcet's jury theorem Suppose a group of $N \geq 1$ voters, where N is odd, renders a dichotomous decision by majority vote. Suppose that one of the two options is "better" or "correct" and the other "worse" or "incorrect", and that any voter in the overall population has an independent probability p, called the voter's* judgmental competence *of voting for the correct option. Then*

(i) if $p > \frac{1}{2}$ then the probability of the group choosing the correct option increases monotonically as N is increased;

(ii) if $p < \frac{1}{2}$ then the probability of the group choosing the correct option monotonically decreases as N is increased.

(iii) if $p = \frac{1}{2}$ then the probability of the group choosing the correct option is $\frac{1}{2}$ for all N.

This theorem can be strengthened to require only that the mean $\bar{p}$ of voter judgmental competence be respectively greater than, less than, or equal to $\frac{1}{2}$.

C.f. Bernard Grofman; Guillermo Owen; Scott L. Feld (1983). "Thirteen theorems in search of the truth" (PDF). Theory & Decision. 15 (3): 261–78. doi:10.1007/BF00125672.

Definition 24. A nonempty subset $U \subseteq V$ of voters is called **decisive** for a pair x, y of candidates iff $x \succ y$ in the social ranking whenever $x \succ_i y$ for each i such that $v_i \in U$.

The following notion is named for the Italian polymath Vilfredo Pareto (1848–1923). He is perhaps most famous for what is sometimes called the *Pareto principle*, or the *80/20 rule*, versions of which find application across a broad range of fields, although he did not himself develop the general principle. *Very roughly speaking*, it states that 80% of the "capacity for some effect" is concentrated in 20% of the "potential causes". Pareto described a version of this principle in the context of income distribution, believing he had found that, as a general rule, 80% of income was distributed to the top 20% of the earning population. (Here, perhaps, income is the capacity for spending, and a human wealth-holder is of course the cause of any spending.) This principle could be called "the Fibonacci sequence of economics", for just as the Fibonacci sequence arises in an astounding number of natural and mathematical phenomena, the Pareto principle characterizes quite a number of phenomena in economics (and other more or less related fields).

Pareto has been controversial for seeming to propose that (economic) domination by an elite caste over the masses is inevitable in any society, with any attempt to produce greater equity merely producing inefficiency and corruption. The perception of this stance has lead some, though by no means all, to consider him a kind of intellectual apologist for fascism. Pareto only barely lived to see the rise of Mussolini and was never an explicit theorist of fascism, although he did comment—not altogether unfavorably—on it. Any conclusions on this controversy certainly must be based in a careful reading of his actual work and in a study of his historical context.

Definition 25. A WSM is **Pareto-efficient**, or **-optimal**, or is said to satisfy the **unanimity criterion**, if $y \notin W$ whenever there exists a candidate x such that $x \succ_i y$ for all $1 \leq i \leq N$, in which case $\Delta(x, y) = N$ and x is said to be **preferred unanimously** to y.

A welfare function F is said to be **Pareto-efficient**, or to satisfy the **unanimity criterion**, if $x \succ y$ in $F(\mathbf{b})$ whenever x is preferred unanimously to y, i.e. $x \succ_i y$ for all $1 \leq i \leq N$. This is to say that the set V of all voters is *decisive* for every pair of candidates x, y.

A WSM is Pareto-efficient iff it is induced by some Pareto-efficient welfare function.

Pareto-efficiency can be considered quite a weak requirement. As is shown below, when *independence of irrelevant alternatives* and non-imposition are assumed, the requirement of *monotonicity* will imply Pareto-efficiency, but *not* vice versa. Thus monotonicity is a stronger requirement; yet, as you may consider for yourself upon reading the definition below, monotonicity is generally considered a very reasonable requirement. Pareto-efficiency then could be said to be among the "bare minimum" criteria that an even halfway decent WSM or welfare function ought to satisfy.

Proposition 6. *Borda count is Pareto-efficient.*

Proof. We use the formula for the Borda score of a candidate z given by

$$\mathcal{B}(z) = \sum_{i=1}^{N} n - r_i(z) + 1.$$

Suppose y is preferred unanimously to x. Then $r_i(y) < r_i(x)$ for all $1 \le i \le N$, implying

$$n - r_i(y) + 1 > n - r_i(x) + 1, \quad \text{for all } 1 \le i \le N.$$

It follows immediately that $\mathcal{B}(y) > \mathcal{B}(x)$, and therefore that $\mathcal{B}(x)$ cannot be maximal. Hence, by the rule of the Borda count, $x \notin W$. ∎

Proposition 7. *The Copeland/pairwise-comparison method is Pareto-efficient.*

Proof. Suppose candidate x is preferred unanimously to candidate y. Take any third *distinct* candidate z. For each i there are three (mutually exclusive) possible cases:

1. $x \succ_i y \succ_i z$;
2. $x \succ_i z \succ_i y$;
3. $z \succ_i x \succ_i y$.

Let α be the number of ballots satisfying case 1, β the number satisfying case 2, and γ the number satisfying case 3. Then by definition

$$\Delta(x, z) = (\alpha + \beta) - \gamma \qquad \text{and} \qquad \Delta(y, z) = \alpha - (\beta + \gamma).$$

We find that $\Delta(x, z) - \Delta(y, z) = 2\beta \ge 0$; hence $\Delta(x, z) \ge \Delta(y, z)$. It follows immediately that $h_x(z) \ge h_y(z)$ for the arbitrary third candidate z. We can express the Copeland scores of x and y as

$$\mathcal{C}(x) = h_x(y) + \sum_{z \in C \setminus \{x,y\}} h_x(z) = 1 + \sum_{z \in C \setminus \{x,y\}} h_x(z)$$

and

$$\mathcal{C}(y) = h_y(x) + \sum_{z \in C \setminus \{x,y\}} h_y(z) = \sum_{z \in C \setminus \{x,y\}} h_y(z).$$

By the argument above,

$$\sum_{z \in C \setminus \{x,y\}} h_x(z) \ge \sum_{z \in C \setminus \{x,y\}} h_y(z),$$

which implies that

$$\mathcal{C}(x) = 1 + \sum_{z \in C \setminus \{x,y\}} h_x(z) > \sum_{z \in C \setminus \{x,y\}} h_y(z) = \mathcal{C}(y).$$

The Copeland score of y is clearly not maximal, since $\mathcal{C}(x) > \mathcal{C}(y)$, and therefore by the Copeland method $y \notin W$. $\square$

Definition 26. The **sequential comparison method/tournament**: Fix any ordering $x_1, x_2, \ldots, x_n$ of the candidates. For $2 \leq i \leq n$ let $w_1 = x_1$ and

$$w_i = \begin{cases} x_i & \text{if } x_i \rhd w_{i-1}, \\ w_{i-1} & \text{if } x_i \unlhd w_{i-1}. \end{cases}$$

Then $W = \{w_n\}$.

Definition 27. A WSM/welfare function is said to be **independent of candidate names** if its output does not depend on how the candidates are labeled or ordered.

Note. The notion above is closely related to the non-imposition criterion.

Exercise 12. Show that the sequential comparison tournament is *not* independent of candidate names.

Proposition 8. *The sequential comparison tournament is Condorcet but not Pareto-efficient.*

Proof. A Condorcet candidate wins every pairwise comparison, and thus the sequential comparison tournament.

For the following preference schedule, a is preferred unanimously to b, but b wins the sequential comparison tournament when the candidates are ordered a, c, d, b.

1	1	1
a	c	d
b	a	c
d	b	a
c	d	b

$\square$

1.1.4 Spoiler effects

Definition 28. A WSM is said to be **independent of irrelevant alternatives** if it is always possible to determine whether a candidate $x \in W$ based solely on the pairwise comparisons between x and each other candidate.

A welfare function is said to be **independent of irrelevant alternatives** if the following holds:

Let $\mathbf{b}_1$ be a profile over candidate set C_1 and $\mathbf{b}_2$ a profile over candidate set C_2, where $C_1 \cup C_2 \subset \mathfrak{U}$, and let $C \subset C_1 \cap C_2$. Suppose there is a one-to-one correspondence between the ballots of $\mathbf{b}_1$ and those of $\mathbf{b}_2$ such that, whenever $x, y \in C$, $x \succ y$ on a ballot of $\mathbf{b}_1$ iff $x \succ y$ on the corresponding ballot of $\mathbf{b}_2$. Then $x \succ y$ on $F(\mathbf{b}_1)$ iff $x \succ y$ on $F(\mathbf{b}_2)$.

A WSM is IIA if the recursive social ranking it induces is an IIA welfare function.

When a WSM is not IIA, there will be scenarios where placing some candidate x in or out of contention has an undesirable impact on the set of winners, leading us to call x a *spoiler*.

Definition 29. The candidate $x \in C$ is called a **spoiler** if *none* of the following hold:

(i) $W(C) = \{x\}$.

(ii) $W(C \backslash \{x\}) = W(C) \backslash \{x\}$.

(iii) $W(C \backslash \{x\}) = W(C)$.

Definition 30. A WSM is said to be **spoiler-free** if it does not allow spoilers.

Being spoiler-free (IIA) turns out to be a very strong requirement, stronger than what a great many "reasonable" WSM's can meet. This *incompatibility* between being spoiler-free and other, arguably more critical requirements is given concisely and rigorously by *Arrow's impossibility theorem* (see below). Hence we look for weaker requirements that avoid the most egregious instances of spoilers, viz. spoilers which are *dominated* by every other candidate.

Definition 31. Given two candidate subsets $A, B \subset C$, $A \neq \emptyset$, the subset A is said to **dominate** subset B, denoted $A \rhd B$, iff for every $a \in A$ and $b \in B$ we have $a \rhd b$.

A nonempty set $D \subset C$ is called **dominating** iff $D \rhd C \backslash D$.

Definition 32. Given a set of candidates C and a preference schedule on at least the candidates of C, there is a *unique minimal* dominating set $S(C)$, called the **Smith set**. When there is no ambiguity regarding the set of candidates, we write simply S in place of $S(C)$.

If $x \in S$ then x is called a **Smith candidate**.

Exercise 13. Prove that if x is a Condorcet candidate, then $S = \{x\}$.

Lemma 1. *If D_1 and D_2 are dominating sets, then $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$. I.e. dominating sets are **nested**.*

Proof. Suppose $x \in D_1 \backslash D_2$. Then $x \rhd y$ for any $y \notin D_1$, implying $y \notin D_2$, for otherwise D_2 could not be a dominating set as assumed. Hence $D_2 \subseteq D_1$. Swap indices for the other inclusion. ☐

Definition 33. We denote by D_x the (unique) minimal dominating set containing the candidate x. A dominating set D is called **primitive** if $D = D_x$ for some $x \in C$.

Proposition 9. *Every dominating set is primitive.*

Proof. Let $D = \{x_1, x_2, \ldots, x_m\}$ be a dominating set. By the nesting of dominated sets, we may assume that the elements of D were indexed in such a way that

$$D_{x_1} \subseteq D_{x_2} \subseteq \cdots D_{x_m} \subseteq D.$$

But clearly then $D = D_{x_m}$. ☐

Lemma 2. *If $x \in S$ then $S = D_x$*

Proof. Let $S = \{x_1, x_2, \ldots, x_m\}$. For each $1 \le i \le m$, the minimality of the Smith set and the fact that dominating sets are nested imply $S \subset D_{x_i}$, while the minimality of the primitive dominating set and nesting imply $D_{x_i} \subset S$. Therefore $S = D_{x_i}$. □

Proposition 10. *If x is designated winner by the sequential comparison method, then $x \in S$.*

Proof. In each pairwise comparison of the sequential tournament, if either of the two candidates involved is a Smith candidate, then the winner must be a Smith candidate. Since the Smith set is nonempty by definition, it follows that the ultimate winner must be a Smith candidate. □

Lemma 3. *For any $x \in C\backslash S$, $S(C\backslash \{x\}) = S(C)$; i.e. the removal of non-Smith candidates does not alter the Smith set.*

Proof. Let $S = S(C)$ and $S' = S(C\backslash \{x\})$. Clearly S remains dominating on $C\backslash \{x\}$, so since dominating sets are nested, $S' \subset S$. But then it is also clearly the case that S' is dominating on C, implying $S \subset S'$. Therefore $S = S'$. □

Definition 34. The **Smith method**: Set $W = S$.

Proposition 11. *The Smith method is Condorcet and does not permit losing spoilers.*

Proof. If there is a Condorcet candidate $x \in C$, then $S = \{x\}$, and thus for the Smith method $W(C) = S = \{x\}$.

Now take some losing $y \in C$; i.e. $y \notin W = S$. Then since removing a non-Smith candidate does not alter the Smith set (lemma above),

$$W(C\backslash \{y\}) = S(C\backslash \{y\}) = S(C) = W(C) = W(C)\backslash \{y\},$$

so that y is cannot be a spoiler. □

Proposition 12. *The Smith method is not Pareto-efficient.*

Proof. Consider the following preference schedule.

1	1	1
a	c	b
b	a	d
d	b	c
c	d	a

When the candidates are ordered a, b, c, d, the candidate d wins the sequential comparison tournament, hence by a previous proposition is a Smith candidate. However, b is unanimously preferred to d. □

Definition 35. A WSM is said to satisfy the **Smith criterion**, or to be **Smith-fair**, if it guarantees that all winners are Smith candidates.

Proposition 13. *The plurality method is* not *Smith-fair.*

Proof. Consider the following preference schedule.

2	1	1	1	1
a	b	c	d	e
b	e	e	c	d
c	d	b	b	c
d	c	d	e	b
e	a	a	a	a

Here a wins by plurality, but $a \lhd b, c, d, e$ (all 2:4), and therefore a cannot be a Smith candidate. Note that in this example $b \triangleq c$, $c \triangleq d$, and $d \triangleq e$, so that there is no Condorcet candidate. □

Proposition 14. *If a WSM is Smith-fair, then it is Condorcet, but not conversely.*

Proof. Suppose a WSM is Smith-fair. Whenever there is a Condorcet candidate, that candidate is the sole member of the Smith set, hence the winner. Therefore the WSM is Condorcet.

Now consider the WSM according to which any Condorcet candidate is declared the sole winner while, in the absence of a Condorcet candidate, the plurality method is used. This is Condorcet-fair by construction but not Smith-fair by proof of the previous proposition. □

Proposition 15. *The sequential comparison tournament is Smith-fair.*

Proof. This just restates a previous proposition, viz. that a Smith candidate always wins in the sequential comparison tournament, in light of the definition of Smith-fairness. □

Definition 36. A WSM is said to be **a priori Smith-fair** if it is always the case that $W(C \backslash E) = W(C)$ where $E \subset C \backslash S$ is any set of non-Smith candidates. This condition is also called **independence of Smith-dominated alternatives (ISDA)** or **weak independence of irrelevant alternatives (WIIA)**.

For every WSM there is a corresponding a priori Smith-fair variant implemented by applying the method to $C \backslash S$, i.e. *first* removing all non-Smith candidates from contention and *then* running the method.

If the winner selection is Smith-fair, but not a priori, then it is said to be **a posteriori Smith-fair**.

Definition 37. The **a posteriori Smith-fair Borda count** is the WSM whereby $x \in W(C)$ iff $x \in S$ and

$$\mathcal{B}(x) = \max_{y \in S} \mathcal{B}(y).$$

Note that here Borda scores are computed with all candidates, Smith or not, in contention.

Proposition 16. *A posteriori Smith-fair Borda count is* not *a priori Smith-fair.*

Proof. Given the preference schedule

2	1	1
b	a	a
a	b	c
c	c	b

the Smith set $S = \{a, b\}$, and a posteriori Smith-fair Borda count elects $W = \{a\}$. However, if the non-Smith candidate c were first removed from contention, then it elects $W = \{a, b\}$. □

Proposition 17. *The Copeland method is a priori Smith-fair.*

Proof. Exercise. □

The following definition makes precise the desired notion of a "particularly egregious" spoiler. Mind that this merely *a* mathematically precise description of what one *might* mean by a spoiler effect too bad to be permitted. Whether the requirement that a WSM be free from the kind of spoilers defined below is sufficiently strong, or perhaps still too strong, is a valid consideration, but not a *strictly mathematical* one.

Definition 38. A candidate $x \in C\backslash(W(C) \cup S)$ is called a **weak spoiler** if $W(C\backslash\{x\}) \neq W(C)$.

Definition 39. A WSM is said to be the **weak-spoiler-free** if it renders weak spoilers impossible.

Proposition 18. *A WSM is weak-spoiler-free if and only if it is a priori Smith-fair.*

Proof. This is more or less immediate from the definition of a prior Smith-fairness and is left as an exercise. □

1.1.5 Montonicity

Definition 40. Given a profile $\mathbf{b}$, suppose a new profile $\mathbf{b}'$ is produced from the given one by raising the ranking of a candidate x on some of the ballots without altering the *relative* rankings of any pair of candidates not including x. I.e.

1. The number $r_i(x)$ in $\mathbf{b}'$ is less than or equal to the number $r_i(x)$ in $\mathbf{b}$.
2. For two candidates y, z distinct from x, if $y \succ_i z$ in $\mathbf{b}$, then $y \succ_i z$ in $\mathbf{b}'$.

A WSM is **monotonic** iff $x \in W$ with respect to $\mathbf{b}'$ whenever $x \in W$ with respect to $\mathbf{b}$.

A welfare function is **monotonic** iff, for any candidate w, $x \succ w$ in $F(\mathbf{b}')$ whenever $x \succ w$ in $F(\mathbf{b})$.

A WSM is monotonic iff the recursive social ranking it induces is monotonic.

Lemma 4. *Take $x \in S$. Let S' be the Smith set after x is swapped on a ballot with the candidate y immediately above. One of the following holds:*

(i) $S' = S$;

(ii) $S' \subset S$, *but* $x \in S'$ *while* $y \in S\backslash S'$.

Proof. The only *possible* change in pairwise comparison outcomes induced by the given ballot alteration is that $y \trianglerighteq x$ becomes $x \trianglerighteq y$. If $x \in S$ and $y \trianglerighteq x$, then $y \in S$ as well, and so the aforementioned change can have only either of the two following outcomes: If y is dominated by every other Smith candidate, then y alone drops from the Smith set. Otherwise the Smith set must remain unchanged. □

Corollary 1. *The Smith method is monotonic.*

Proposition 19. *The a priori Smith-fair plurality method is* not *monotonic.*

Proposition 20. *The a priori Smith-fair runoff and elimination methods is* not *monotonic.*

Proposition 21. *The a priori Smith-fair Borda count is* not *monotonic.*

Proposition 22. *The Copeland method is monotonic.*

Proof. Exercise. □

Proposition 23. *Taken together, non-imposition, monotonicity, and IIA imply Pareto-efficiency (for both WSM's and welfare functions).*

Proof. We give a proof of the result for welfare functions, from which result for WSM's follows.

Suppose that a candidate x is preferred unanimously to candidate y, but that nevertheless $y \succeq x$ in the social ranking. By non-imposition there must be some way to alter the ballots to produce a profile $\mathbf{b}_x$ which yields $x \succ y$ in the social ranking, and by IIA this must involve switching the preference relation between x and y on some ballots. We produce $\mathbf{b}_x$ in the following manner: First move x and y to the bottom of every ballot, maintaining the relation $x \succ y$ on each one so that we continue to have $y \succeq y$ in the social ranking by IIA. Now swap the ballot positions of x and y on all those ballots which correspond to those of $\mathbf{b}_x$ on which $y \succ x$. By monotonicity this cannot hurt y relative to x, and so it is still the case that $y \succeq x$ in the social ranking. Now to complete the production of $\mathbf{b}_x$, clearly all that remains is to raise the positions of x and y appropriately, maintaining the current relative ranking of x versus y on each ballot. By assumption this should produce a profile giving the social ranking $x \succ y$, but by IIA this should not alter the preexisting social ranking $y \succeq x$, a contradiction. Therefore the hypothesized violation of Pareto-efficiency is not possible. □

We have seen that the Copeland/pairwise-comparison method satisfies almost all of the desirable qualities described so far. (It is not completely spoiler-free.) But the Copeland method frequently produces ties. Now we will give a WSM, the *Schulze (beatpath) method*, which produces ties less frequently and satisfies all of the criteria established above—Pareto-efficiency; monotonicity; non-imposition; the one person, one vote principle; independence of candidate names; non-dictatorship; etc.—except for IIA, but which satisfies the weakened ISDA form, i.e. being weak-spoiler-free, which at the very least implies being a priori Smith-fair, hence Smith-fair, hence Condorcet, hence majoritarian.

Parenthetically, the Schulze method is used by the development communities of numerous Linux distributions, including Debian and Gentoo, to reach collective decisions.

Definition 41. Schulze (beatpath) method: Given a preference schedule on candidates C, let $G = (C, E, \Delta)$ be the pairwise-comparison digraph equipped with the weight function $\Delta : E \to \mathbb{N}$, where $\Delta(x, y)$ is as previously defined. We say that x has a *beatpath* to y if there exists a *directed* path P, considered as a sequence of distinct edges, from x to y in G. The *strength* of the beatpath P is the number

$$\sigma(P) = \min_{(u,v)\in P} \Delta(u, v),$$

i.e. the minimum of edge weights along the path.

If there is a beatpath P from x to y such that $\sigma(P) > \sigma(P')$ for any beatpath P' from y to x, then x is said to have an *unmatched* beatpath to y, and generally that there is an unmatched beatpath *against* y, and we write $x \to y$.

Now $x \in W$ iff there is no unmatched beatpath from y to x for each other candidate y.

Proposition 24. *The unmatched beatpath relation is* antisymmetric*; i.e. if $x \to y$ and $y \to x$, then $x = y$.*

Proof. Exercise. □

Proposition 25. *The unmatched beatpath relation is* transitive*; i.e. if $x \to y$ and $y \to z$, then $x \to z$.*

Proof. This is proven by a technique of *composing* beatpaths. □

Proposition 26. *There always exists a candidate against whom there is no unmatched beatpath.*

Proof. Suppose that there is an unmatched beatpath against every candidate. Then there is an ordering of the candidates such that

$$x_1 \to x_2 \to x_3 \to \cdots .$$

If a candidate x appears twice in this chain, then for some other candidate $y \neq x$, it must hold both that $x \to y$ and that $y \to x$, by transitivity. But this contradicts that the relation is antisymmetric. Hence the chain the must be infinite. This is not possible since there are assumed to be finite candidates. Therefore there must be some candidate against whom there is no unmatched beatpath. □

Proposition 27. *The Schulze method is a priori Smith-fair.*

Proof. Exercise. □

Proposition 28. *The Schulze method is Pareto-efficient.*

Proof. Exercise. □

Proposition 29. *The Schulze method is monotonic.*

Proof. Exercise. □

Proposition 30. *A priori Smith-fair Borda count is monotonic if there are fewer than five candidates in the Smith set.*

Definition 42. A winner selection method is said to be **strongly monotonic** if it is monotonic and satisfies the IIA criterion.

Proposition 31. *Strong monotonicity is equivalent to the following property: Take a fixed set of candidates C and a fixed number of voters N. Let $W \subset C$ be the set of winners resulting from preference ballots b_i, and let $W' \subset C$ be the winners resulting from preference ballots b_i'. Let $>_i$ denote the preference relation on ballots b_i and $>_i'$ that on ballots b_i'. If $x \in W$ and $x >_i' y$ whenever $x >_i y$, then $x \in W'$.*

1.1.6 Strategy

Definition 43. A voter is said to have engaged in **strategic voting** if they cast a *dishonest* ballot, i.e. a ballot which does not reflect their true preferences, with the effect that the set of winners changes in a manner which reflects their true preferences.

Definition 44. A winner selection method is said to be **strategy-proof** if it renders strategic voting impossible.

. *Gibbard-Satterthwaite theorem When $n > 2$, the only SWSM which is weakly Pareto-efficient and strategy-proof is dictatorship.*

Proof. One proof is simply a modification of the proof of Arrow's theorem, and this appears alongside the latter in the text linked below the statement of Arrow's theorem below. □

Theorem 1. *When $N = n > 2$, any strategy-proof SWSM is non-majoritarian.*

1.1.7 Arrow's theorem

. *Arrow's impossibility theorem Suppose $n \geq 3$ and $N \geq 2$. Then there is no WSM or welfare function which simultaneously satisfies all of the following criteria:*

(i) ***Universality****—It is defined for all possible preference profiles.*

(ii) ***Non-Dictatorship****—No individual voter is decisive for all pairs of candidates.*

(iii) ***Independence of irrelevant alternatives (IIA)***

(iv) ***Pareto-efficiency****—The set of all voters is decisive for all pairs of candidates.*

Proof. Cf. Reny, Philip J. *Arrow's theorem and the Gibbard-Satterthwaite theorem: a unified approach* (2000). Economics Letters 70 (2001) 99–105. □

Definition 45. A winner selection method is said to be **weakly Pareto-efficient** if $W = \{x\}$ whenever $R_1(x) = N$.

Exercise 14. Show that Pareto-efficiency implies weak Parento-efficiency.

In light of the weakened form of the Pareto-efficiency requirement defined above, we have the following strengthened version of Arrow's theorem specifically for SWSM's.

. *Muller-Satterthwaite theorem When $n > 2$, the only SWSM which is weakly Pareto-efficient, monotonic, and IIA is dictatorship.*

1.1.8 Weighted voting

In certain contexts it is deemed desirable for some "voting entity" in an election to be given more votes than another. A real-world example is the U.S. Electoral College: Each state sends a number of electors to the College roughly proportional to its population, and since the electors from an individual state in expectation and practice—although not in theory—must all vote the same way (with the exception of Nebraska and Maine, which allow vote splitting), one can consider the state's whole delegation as a homogeneous voting entity with a certain number of votes at its disposal. Such a system is referred to as **weighted voting**.

Definition 46. A **weighted voting system** is characterized by the following:

- A finite set V of voting entities, which we will call **players**, $p_1, p_2, \ldots, p_N$.
- A **weight function** $w : V \to \mathbb{R}$, where for brevity we write $w_i = w(p_i)$, $1 \leq i \leq N$. Often, and in this course, always, the range of w will be restricted to the positive integers.
- A **quota**, the number of votes required for a particular decision to be carried, which will be denoted q. Clearly the quota must be no greater than the total number of votes available, i.e.

  $$q \leq \sum_{i=1}^{N} w_i.$$

  It is most common for the quota to be required to greater than half the number of available. (It is considered to be defined this way in WebAssign.) Symbolically:

  $$q > \frac{1}{2} \sum_{i=1}^{N} w_i.$$

This characterization of a weighted voting system can be denoted concisely by

$$[q : w_1, w_2, \ldots, w_N],$$

where the players, hence weights, are indexed so that $w_1 \geq w_2 \geq \cdots \geq w_N$.

Definition 47. A **dictator** in a weighted voting system is a player p such that $w(p) \geq q$. Whenever p votes for some decision, that decision must be carried.

Definition 48. A *nonempty* subset of players all of whom vote the same way in a particular election are said to be in a **coalition**.

Proposition 32. *The number of possible coalitions over N players is $2^N - 1$.*

Proof. Exercise. $\square$

Definition 49. A **coalition** Q is called **winning** (or **decisive**) iff

$$\sum_{p \in Q} w(p) \geq q.$$

A coalition which is not winning is called, predictably, **losing**.

Definition 50. Given a *winning* coalition Q, a player $p \in Q$ is called **critical** for Q iff $Q \setminus \{p\}$ is *losing*.

Definition 51. A player p is called a **dummy** iff p is *never* critical for any winning coalition. In this case p can never affect the outcome of a vote.

Definition 52. A player p which is not a dictator is said to have **veto power** iff p is critical for *every* winning coalition.

Exercise 15. Show that a player p is either a dictator or has veto power iff

$$\sum_{x \in V \setminus \{p\}} w(x) < q.$$

Definition 53. The **Banzhaf power index** of a player p, which we will denote $\beta(p)$, is a measure of the probability that p determines the outcome of a vote, computed in the following manner:

1. Enumerate all of the possible *winning* coalitions.
2. For each player x, let M_x be the number of winning coalitions Q for which x is *critical*.
3. Let $M = \sum_{x \in V} M_x$.

The Banzhaf power index of p is the ratio

$$\beta(p) = M_p / M.$$

When the players are indexed $V = \{p_1, p_2, \ldots, p_N\}$, we may use the following alternate notation for the quantities above: for $1 \leq i \leq N$,

- $\beta_i = \beta(p_i)$
- $M_{p_i} = M_i$

By the second of these we can rewrite the definition of M as $M = \sum_{i=1}^{N} M_i$.

Exercise 16. Prove that, for a set of players $V = \{p_1, \ldots, p_N\}$, the Banzhaf power index satisfies

$$\sum_{i=1}^{N} \beta_i = 1.$$

Exercise 17. Find $\beta(p)$ when p is a dummy.

Find $\beta(p)$ when p is a dictator.

Definition 54. A **voting sequence** is some order in which the players could cast their votes one player at a time. Given a set of players $V = \{p_1, \ldots, p_N\}$ we can denote a voting sequence either as an ordered N-tuple of players. For example if $V = \{p_1, p_2, p_3\}$, one particular voting sequence can be denoted (p_2, p_1, p_3). When we wish denote an *arbitrary* sequence, this would result in subscripts on subscripts, as in $(p_{i_1}, p_{i_2}, p_{i_3})$. For the sake of readability, we may denote the latter simply by the sequence of subscripts, as in (i_1, i_2, i_3), where it is to be understood that this is a sequence of integers which are the indices of players.

Exercise 18. Given N players the number of voting sequences is $N! = N(N-1)(N-2)\cdots(2)(1)$.

Definition 55. Given a voting sequence $(i_1, i_2, \ldots, i_N)$, the **pivotal** player is the player with index i_k satisfying the following: k is the *minimum* integer such that $(i_1, i_2, \ldots, i_k)$ represents a winning coalition of players.

Definition 56. Another measure of the voting power of a player is the **Shapley-Shubik power index** of a player p, which we will denote $\varsigma(p)$, and which is calculated as follows:

1. Enumerate all of the $N!$ possible **voting sequences**.
2. For each voting sequence, identify the **pivotal player**.
3. Let L_p be the number of sequences in which player p is pivotal.

The Shapley-Shubik power index of p is the ratio

$$\varsigma(p) = L_p/N!.$$

As in the case of the Banzhaf index, for player p_i we can alternatively denote L_{p_i} by L_i and the its Shapley-Shubik power index by ς_i.

1.2 Division

We now move to a topic of social choice which represents a bit of a departure from the winner selection and rank-ordering we have considered up till now. Here we consider the problem of how to divide limited resources among individuals. The resources are characterized broadly by two dichotomies: **divisible vs. indivisible** and **homogeneous vs. heterogeneous**.

A collection of indivisible resources is one composed of items which cannot individually be cut up and distributed in pieces, such as books, furniture, and individual stocks in a company. Divisible resources by contrast are those that (at least for the purposes of the theory) can be divided as finely as one pleases; E.g. money (when fractions of a cent are permitted), land territory, and pie.

Resources are said to be heterogeneous when some pieces or items are qualitatively different from one another, and hence may be valued differently by an individual. Homogeneous resources are just the opposite: the whole mass or all the items in the collection are indistinct, and therefore no one would value one item differently than another, nor one piece differently from a piece of the same size.

Just as in the case of determining social preferences over some set of candidates, the theory of fair division relies on the notion of **utility**. Given a portion of a resource, its utility to a specific individual is a measure of how valuable the individual considers it to be.

One of the more significant difficulties in using utilities to solve problems of social choice is that of comparing the utilities of different individuals. Suppose for instance that two individuals wish to split a pie. If individual 1 enjoys pie twice as much individual 2, is slicing the pie right down the middle really the best solution? Assuming for the moment that this division can be done with indisputable precision, certainly both individuals would feel they received half of the value available in the pie, which has the ring of fairness about it. Yet if we consider their valuations of pie to be directly comparable as we did at the outset, then we conclude that what individual 1 receives is of twice the value of what individual 2 gets. Ostensibly individual 1 is getting twice the enjoyment of individual 2, which individual 2 might have cause to envy. An alternative division would be to give individual 1 a third of the pie and individual 2 two thirds. This time each should get the same utility from their share, but in this case individual 1 has cause to feel envious. Why should individual 1 be penalized for liking pie?

One straightforward way to give such a measure is by pricing: asking how much money the individual would pay for the given portion; and this method has usefulness in many applications. It does however have flaws, among which is the following: Not everyone places the same value on money itself. For example, a billionaire ostensibly values \$100 significantly less than does someone getting by on minimum wage, and so we would probably not assert that two such individuals place the same value on some object just because both of these individuals would be willing to pay \$100 for it. This illustrates that money is itself a resource having utility which varies in its amount and among individuals.

1.2.1 Divisible resources

Our metaphor for the division of a divisible resource will be cutting up a cake. It is assumed that the entirety of the cake is of equal value to each of the parties, and we represent that value numerically by 1. A **cake-cutting** has the following components:

- The whole cake C.
- The parties $a_1, \dots, a_N$.
- For each party a_i, a **utility function** $u_i : 2^C \to [0,1]$. Alternatively for parties $a, b, c, \dots$, their respective utility functions are $u_a, u_b, u_c, \dots$. Given a portion $P \subseteq C$ of the cake, $u_i(P)$ is the fraction of the cake which a_i perceives P to be, in terms of *value*. We assume that utility here is **additive**: given two **disjoint** portions $P, Q \subset C$, meaning $P \cap Q = \emptyset$ (no overlap), $u(P \cup Q) = u(P) + u(Q)$ for any utility function u. Remember that by assumption $u_i(C) = 1$, and clearly $u_i(\emptyset) = 0$ for all $1 \leq i \leq N$.
- A **partition** $\{P_i\}_{i=1}^N$ of the cake, i.e. a collection of disjoint portions $P_i \subseteq C$ the union of which is C. In general P_i is the portion going to party a_i. If the parties are given rather as $a, b, c, \dots$, we give their respective portions as $P_a, P_b, P_c, \dots$.

In the proceeding 2^C denotes the **power set** of C, which is the set of all possible subsets (portions) of C. The **interval** $[0,1]$ is the set of all real numbers between 0 and 1, inclusive.

Definition 57. A **piecewise homogeneous cake** is one which is composed of n **components** $C_1, C_2, \dots, C_n$ each of which is considered to be completely uniform in character by all parties. For each party the value of a piece of cake taken from one component C_k depends *only* on the size of the piece and how that party values C_k as a whole.

The following are some possibilities for what it might mean for a cake-cutting to be "fair":

Definition 58. A cake-cutting is called **proportional** iff $u_i(P_i) \geq 1/N$ for all $1 \leq i \leq N$. That is, every party receives an amount cake which *they perceive* to be at least $\frac{1}{N}$-th of the total value in the cake.

Definition 59. A cake-cutting is called **envy-free** iff $u_i(P_i) \geq u_i(P_j)$ for all $1 \leq i, j \leq N$. That is, for all i the i-th party considers its portion of cake to be of no lesser value than that of the portion given to any other party.

Definition 60. Let a cake-cutting $\{P_i\}_{i=1}^N$ be given, and suppose there is an alternate cutting $\{Q_i\}_{i=1}^N$ such that $u_i(Q_i) \geq u_i(P_i)$ for *all* $1 \leq i \leq N$ *and* $u_k(Q_k) > u_k(P_k)$ for *some* $1 \leq k \leq N$. Then the alternate cake-cutting is called an **objective improvement** over the initially given one.

A cake-cutting is called **Pareto-efficient** iff it admits *no* objective improvement.

Definition 61. A cake-cutting $\{P_i\}_{i=1}^N$ is called **equitable** iff $u_i(P_i) = u_j(P_j)$ for all $1 \leq i, j \leq N$.

We will now discuss **methods** of cutting a cake. Each of the qualifiers defined above is applied to a method when it always produces a cake-cutting to which that qualifier can be applied. To begin with we will assume that the cutting can be done in a *geometrically* *objective* manner. That is to say, there is never any disagreement among the parties as to what fraction of the *physical substance* of the cake a portion/component constitutes, though we allow that they may perceive different values in a given portion.

Proposition 33. *When $N = 2$ any proportional method is envy-free.*

Proof. Let party a_i receive portion P_i for $i = 1, 2$. By proportionality we have

$$u_i(P_i) \geq \frac{1}{2} \qquad \text{for } i = 1, 2.$$

By additivity these imply

$$u_i(P_{2-i+1}) < \frac{1}{2} \qquad \text{for } i = 1, 2.$$

Envy-freeness follows immediately. □

Definition 62. Cutter-chooser method for $N = 2$ parties: One party is arbitrarily chosen as the cutter, a, and the other as chooser, b. The cutter divides the cake into two portions P_1 and P_2 (indexed arbitrarily) such that

$$u_a(P_1) = u_a(P_2) = \frac{1}{2}.$$

Then b chooses which of P_1 and P_2 to take: b chooses P_1 iff

$$u_b(P_1) \geq u_b(P_2).$$

Party a takes whichever of the portions was not chosen by b.

Proposition 34. *The cutter-chooser method for $N = 2$ is proportional and therefore envy-free, but is neither Pareto-efficient nor equitable.*

Definition 63. Proportional division method: Given a piecewise homogeneous cake, each of the N parties receives the fraction $1/N$ of each homogeneous component.

Proposition 35. *The proportional division method is envy-free and equitable, but not Pareto-efficient.*

Proposition 36. *Any Pareto-efficient, equitable cutting of a piecewise homogeneous cake is proportional.*

Proof. Suppose the cutting is not proportional but is equitable, so that $u_i(P_i) = s < 1/N$ for all i. Then a proportional division is an objective improvement over this cutting, and therefore the original cutting cannot be Pareto-efficient. □

Lone-divider method for $N = 3$ parties: One of the three parties is arbitrarily designated the divider a and the other two as choosers b and c. The divider cuts the cake into three portions P_1, P_2, P_3 such that

$$u_a(P_i) = \frac{1}{3} \qquad \text{for } i = 1, 2, 3.$$

Each chooser then designates which of the portions they consider to be worth at least one third the value of the cake. Note that each will necessarily designate at least one portion. There are two possible cases:

Case 1: Between the two of them, b and c designate at least two portions. We can assume by relabeling if necessary that b designated P_2 and c designated P_3. Then a receives P_1, b receives P_2, and c receives P_3.

Case 2: Between the two of them, b and c designate exactly one of the pieces, which again we may assume is P_3. We let a take P_1 and then have b and c apply the two-party cutter-chooser method to $P_2 \cup P_3$.

Proposition 37. *The lone-divider method for $N = 3$ is proportional, but is neither envy-free nor Pareto-efficient nor equitable.*

Lone-chooser method for $N = 3$ parties: Arbitrarily, one party is designated the first cutter a, a second the second cutter b, and the third the chooser c. The first cutter divides the cake into two portions P_1 and P_2 such that $u_a(P_1) = u_a(P_2) = \frac{1}{2}$. We may assume by relabeling if necessary that $u_b(P_2) \geq \frac{1}{2}$. Then a cuts P_1 into three subportions P_1^1, P_1^2, P_1^3, and b cuts P_2 into three subportions P_2^1, P_2^2, P_2^3, such that

$$u_a(P_1^1) = u_a(P_1^2) = u_a(P_1^3) \qquad \text{and} \qquad u_b(P_2^1) = u_a(P_2^2) = u_a(P_2^3).$$

Again we may assume that

$$u_c(P_1^3) = \max_{1 \leq j \leq 3} \left\{ u_c(P_1^j) \right\} \qquad \text{and} \qquad u_c(P_2^3) = \max_{1 \leq j \leq 3} \left\{ u_c(P_2^j) \right\}.$$

The final cake-cutting is given by

$$P_a = P_1^1 \cup P_1^2, \qquad P_b = P_2^1 \cup P_2^2, \qquad P_c = P_1^3 \cup P_2^3.$$

Proposition 38. *The lone-chooser method for $N = 3$ is proportional, but is neither envy-free nor Pareto-optimal nor equitable.*

Both of the methods above for dividing a cake among 3 parties suffer from the inadequacy of satisfying just one of our proposed "fairness criteria", viz. proportionality. Below we give a method which is both proportional and envy-free.

Selfridge-Conway method: Given three parties we arbitrarily designate one the divider a, another the trimmer b, and the last the chooser c. First a cuts the cake into three portions P_1, P_2, P_3 such that $u_a(P_i) = \frac{1}{3}$ for $i = 1, 2, 3$. We assume that these are indexed so that $u_b(P_1) \geq u_b(P_2) \geq u_b(P_3)$. Now b trims P_1, i.e. cuts a subportion T off of P_1, leaving the subportion $P_1' = P_1 \backslash T$ such that $u_b(P_1') = u_b(P_2)$. The chooser c then selects one of the portions P_1', P_2, P_3. If c selects P_1', then b selects P_2, and if c selects either of the untrimmed portions, then b selects the trimmed portion P_1'.

Let x be the party which selects P_1' and y the party which chose either P_2 or P_3. Party y cuts the trimmings T into three portions T_1, T_2, T_3 such that $u_y(T_1) = u_y(T_2) = u_y(T_3)$, and where we assume that these are indexed so that $u_x(T_1) \geq u_x(T_2), u_x(T_3)$ and $u_a(T_2) \geq u_a(T_3)$. Then x takes T_1, a takes T_2, and y takes T_3.

We now give a method of dividing a piecewise homogeneous cake between $N = 2$ parties which is Pareto-efficient and equitable, hence also proportional.

Adjusted winner method: Consider a cake C with homogeneous components $C_1, \dots, C_n$, which is to be divided between parties a and b. We adopt the notation $\alpha_i = u_a(C_i)$ and $\beta_i = u_b(C_i)$ for $1 \le i \le n$. The ratio α_i/β_i is called the *a-to-b valuation ratio for the i-th component.* By reindexing the components as necessary, we can assume that the ratios are in (weakly) decreasing order:

$$\frac{\alpha_1}{\beta_1} \ge \frac{\alpha_2}{\beta_2} \ge \cdots \ge \frac{\alpha_n}{\beta_n}.$$

We can divide the cake in the following manner, called a **threshold division**: Choose $r \in \mathbb{R}$, called the **threshold ratio**. If $\alpha_i/\beta_i > r$ then C_i goes to a, and if $\alpha_i/\beta_i < r$ then C_i goes to b. If $\alpha_i/\beta_i = r$ then C_i may be given to a or b, or may be split arbitrarily between them. We make the following claim:

Proposition 39. *The threshold divisions are* precisely *the Pareto-optimal cake-cuttings between $N = 2$ parties.*

We now seek to find the threshold r such that the corresponding threshold division is equitable. This is accomplished by trial-and-error along with some basic algebra. First we try values of r which are strictly between two a-to-b valuation ratios. Either we will find a threshold producing an equitable division, or we will find some $1 \le k \le n$ and thresholds $r_1 > \alpha_k/\beta_k > r_2$ such that r_1 gives b a higher value fraction than a and r_2 gives a a higher value fraction than b. In this latter case we set $r = \alpha_k/\beta_k$ and next determine how to divide component C_k. Let p be the fraction of C_k going to a (so that necessarily $1 - p$ is the fraction going to b). Then we simply solve the equation

$$\alpha_1 + \alpha_2 + \cdots \alpha_{k-1} + p\alpha_k = (1-p)\beta_k + \beta_{k+1} + \beta_{k+2} + \cdots + \beta_n.$$