OPTIMIZATION
Course 2022-22. Fall Semester
Final Exam: January 9, 2023

Do the exercises in different sheets.

Exercici 1. Assume that the electricity supply is a duopoly market and the two companies act rationally (maximizers of the profit). In the suitable unities the demand function is given by $p(Q) = 100 - Q$ where p is the unitary price, and $Q := q_A + q_B$ where q_A and q_B are the unities produced of each company (A and B). Suppose that the cost functions are $C(q_A) = 2q_A$ and $C(q_B) = 3q_B$, respectively.

(a) Give the profit of the companies if the productions are $q_A = 10$ and $q_B = 20$.

(b) If company A knows *in advance* that company B is producing $q_B = 20$ unities which is the unities he will produce? Which is the profit of the two companies in this case?

(c) Find the Nash equilibrium i.e., the pair $(q_A^\star, q_B^\star)$ such that, if $\Pi_j(q_A, q_B)$ is the profit function of firm $j = A, B$, we have

$$\Pi_A(q_A^\star, q_B) \geq \Pi_A(q_A, q_B), \quad \forall q_A \geq 0, \quad \text{and}$$
$$\Pi_B(q_A, q_B^\star) \geq \Pi_B(q_A, q_B), \quad \forall q_B \geq 0.$$

In particular, no one of the companies wants to deviate unilaterally.

Exercici 2. Answer the following questions.

(a) Assume $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. Justify that if

$$\hat{p} := \min_{p \in \mathbb{R}^n} f(x) + (\nabla f(x))^T p \quad \text{s.t } \|p\| \leq r$$

then

$$\hat{p} = -\frac{r}{\|\nabla f(x)\|} \nabla f(x).$$

(b) Consider the function $f(x, y) = (x^2 + y^2 - 1) \exp(-x^2 - 2y^2)$. Set $x_0 = (1, 0)$. Prove that $p^T = (-1, 0)$ is a descent direction (of f at x_0). Find all minimizers of the problem

$$\min_{\alpha \in \mathbb{R}^+} f(x_0 + \alpha p^T).$$

Interpret the results.

Exercici 3. Explain the conjugate direction method. The equivalence with the solutions of linear systems, the meaning of conjugate systems and one main result of the theory.
Remark: A maximum of one side of a sheet.