$$G = \begin{pmatrix} g_{11} & & \\ g_{21} & g_{22} & \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \qquad A = \begin{pmatrix} g_{11}^2 & & \\ g_{21}g_{11} & g_{21}^2 + g_{22}^2 & \\ g_{31}g_{11} & g_{31}g_{21} + g_{32}g_{22} & g_{31}^2 + g_{32}^2 + g_{33}^2 \end{pmatrix}$$

Master in Foundations of Data Science — 2022-2023

NUMERICAL LINEAR ALGEBRA

Final exam, 16 January 2023 from 15:00h till 18:00h

$g_{11} = 1$
$\cancel{g_{21} = -2}$
$g_{31} = 0$
$g_{22} = (3-4)^{1/2}$

Exercises should be delivered in separated pages. All answers should be suitably justified.

1. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & -2 \end{bmatrix}.$$

(1) Compute the PLU factorization of A produced by the GEPP algorithm.
(2) Compute the $LDL^\top$ factorization of A.
(3) Explain how you would solve the equation $Ax = b$ for the vector $b = (1, 0, 1)$ using each of these factorizations, and compute the solution vector x using one of them.

2. A Schur factorization of the matrix in Exercise 1 is $A = UTU^\top$ with

$$U = \begin{bmatrix} 0.46 & -0.87 & 0.20 \\ -0.85 & -0.37 & 0.38 \\ 0.25 & 0.34 & 0.90 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 4.68 & 0 & 0 \\ 0 & 0.15 & 0 \\ 0 & 0 & -2.83 \end{bmatrix}$$

up to two decimal digits.

(1) Compute the eigenvalues and eigenvectors of A from this factorization.
(2) Suppose that you apply the power method to the matrix A with initial vector $x_0 = (1, 0, 0)$. Which would be the eigenpair approached by this iterative algorithm? Determine its rate of convergence.
(3) Since A is a symmetric matrix, would it be a good idea to apply Cholesky's algorithm to solve the linear equation $Ax = b$?

3. Let A be the matrix in Exercise 1.

(1) Discuss differents algorithms to compute the QR factorization of A. Taking into account the structure of A, which of these algorithms would be your choice?
(2) Briefly detail the QR iteration applied to any square matrix, and prove that all the matrices produced by the QR iteration applied to A are Hessenberg.
Hint: Show first that the orthogonal matrix Q is Hessenberg.

4. Let A be again the matrix in Exercise 1.

(1) Compute a singular value decomposition (SVD) of A using the data obtained in Exercise 2(1).
(2) Using this SVD, compute the condition numbers of A with respect to the 2-norm and the Frobenius norm.
(3) Briefly explain the role of the condition number of A with respect to an arbitrary vector norm $\|\cdot\|$ when solving the linear equation $Ax = b$.

$-2x_1 + 3x_2 - 2x_3 = 0$

$2x_1 = 3x_2 - 2x_3$

$-2x_2 - 2x_3 = 1$

$\begin{pmatrix} -1 & 0 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{matrix} -1 & -2 \\ -2 & -2 \end{matrix}$

$\cancel{+2x}$

$2x_2 = -2x_3 - 1$