

# Feature Importance

David S. Rosenberg

NYU: CDS

November 24, 2021

# Contents

- 1 Introduction
- 2 Permutation feature importance
- 3 What can go wrong with permutation?
- 4 Partial dependence
- 5 Individual conditional expectations (ICE)

# Introduction

# Our approach to model interpretation

- “Model interpretation” is not a well-defined problem.
- Our goal will be to try to get some easy-to-understand explanation for how
  - our prediction function depends on each feature (or small sets of features).
- We’ll discuss:
  - partial dependence plots (PDP)
  - individual conditional expectation (ICE) plots
- And we’ll discuss caveats for the use of each.

## Feature importance

- Feature importance is closely related to model interpretation..
- Also not particularly well-defined.
- Many modeling packages give back “feature importance” scores for features.
- These are usually model specific, e.g.
  - Linear methods: absolute value of weights,  $p$ -values
  - Trees: Mean decrease in impurity
    - weighted combinations for tree ensembles
- We'll discuss methods that are model agnostic:
  - permutation feature importance
  - leave one covariate out (LOCO) importance
- As well as issues and limitations.

# Why bother with this stuff?

- Feature selection / domain understanding
- Sanity checking models
  - detecting leakage
  - detecting dependencies that won't generalize
- Fairness considerations
  - detecting dependencies that you **don't want**
- Explaining predictions

## Permutation feature importance



Machine Learning, 45, 5–32, 2001

© 2001 Kluwer Academic Publishers. Manufactured in The Netherlands.

## Random Forests

LEO BREIMAN

*Statistics Department, University of California, Berkeley, CA 94720*

- Leo Breiman introduced random forests in [Bre01] ([tech report](#) from Sept 1999).
- Still one of the most widely used machine learning methods.
- In a couple pages in Section 10, Breiman describes a procedure that is now referred to as **permutation feature importance**.

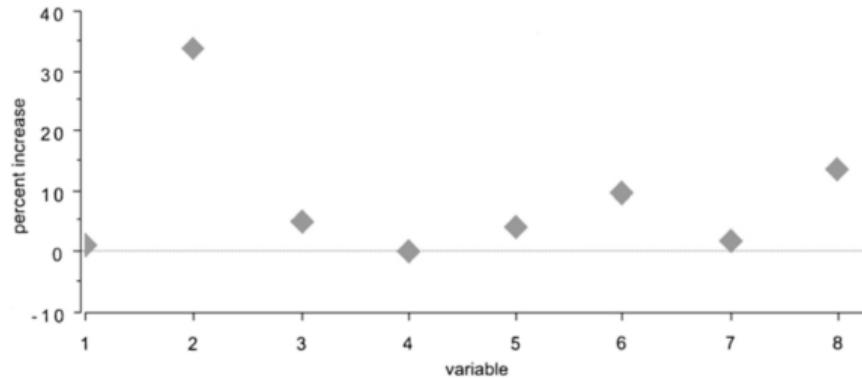
## Permutation feature importance

- [Bre01, Sec 10] describes method specifically for random forest.
  - Based on out-of-bag predictions. (A special attribute of RF models.)
- Here we'll present the obvious generalization.
- Given:
  - Prediction function  $f: \mathcal{X}_1 \times \dots \times \mathcal{X}_d \rightarrow \mathcal{A}$ . (Arbitrary type for each of  $d$  features.)
  - Data set  $\mathcal{D} = (x_1, \dots, x_d)$ , where  $x_i \in \mathcal{X}_i^n$  is the  $i$ th feature "column".
  - Performance metric:  $m: (f, \mathcal{D}) \rightarrow \mathbb{R}$
- Let  $\mathcal{D}_i$  be  $\mathcal{D}$  with  $x_i$  replaced by  $\text{Shuffle}(x_i)$ .
- **Permutation feature importance** of feature  $i$  is some comparison between  $m(f, \mathcal{D})$  and  $m(f, \mathcal{D}_i)$ 
  - e.g.  $m(f, \mathcal{D})/m(f, \mathcal{D}_i)$  or  $m(f, \mathcal{D}) - m(f, \mathcal{D}_i)$  or  $(m(f, \mathcal{D}_i) - m(f, \mathcal{D}))/m(f, \mathcal{D})$ .

# Discussion

- Shuffling feature  $i$ 
  - breaks any relationship between  $x_i$  and the response  $f(x)$
  - breaks any relationship between  $x_i$  and other features
    - may take us “off the data manifold” or “out of distribution” (more on this later)
- Some theory on what permutation feature importance corresponds to for additive models can be found in [GMSP15, Sec 2 and App A].
- Note that permutation feature importance requires a data set.
  - So we’re really evaluating  $f$  w.r.t. a particular covariate distribution.

## Breiman's example (I)



- Percent increase in misclassification rate from shuffling each variable.
- Variable 2 seems most important.
- Followed by 8 and then 6.

Figure 4 from [Bre01, p. 24].

## Breiman's example (II)

- Let's fit new models using different variable subsets.
- All variables: 23.1% error
- Only variable 2: 29.7% error
- Only variables 2 and 8: 29.4% error
- Conclusion: variable 8 doesn't add much information to what variable 2 provides
  - (at least that could be extracted by the random forest used for the model)
- Variables 2 and 6: 26.4% error

## Discussion

- So what does permutation importance tell us?
- First: it's a result about **the prediction function itself**
  - for a particular covariate distribution.
- We may get very different results for other prediction functions derived from
  - a different random training set, or
  - a different ML model or training algorithm.
- With a rather literal reading of permutation importance, it tells us
  - the effect of replacing a particular input value with a random draw from the marginal distribution of that variable.
- We can also take this as a proxy for the sensitivity to errors in that variable,
  - errors of a very particular distribution.

## Leave-one-covariate out (LOCO)

- Suppose we want to see if  $x_1$  is an important features.
- We do the following:
  - Remove  $x_1$  from the training set.
  - Retrain the model.
  - Compare performance of model with  $x_1$  and without  $x_1$ .
- This is a “leave-one-covariate-out” or LOCO method [LGR<sup>+</sup>18, Sec 6].
- Natural to use LOCO for iteratively removing features from model.
- What’s different compared to the feature importance methods described previously?

## Feature importance for what?

- We can think of the importance of a feature for
  - a particular prediction function  $\hat{f}(x)$
  - a particular learning method (e.g. linear models vs random forests)
  - the Bayes optimal prediction function
- Suppose  $x_1 = g(x_2)$  for some nonlinear function  $g$ .
- If  $\hat{f}(x)$  uses  $x_1$  but ignores  $x_2$ , then  $x_1$  is important for  $\hat{f}$  and  $x_2$  is not.
  - Permutation importance sampling will detect this.
- If our learning method cannot extract all the information needed from either  $x_1$  or  $x_2$ ,
  - LOCO would identify  $x_1$  and  $x_2$  as equally important.
- But if our learning method cannot extract the information needed from  $x_2$ ,
  - then LOCO would indicate that  $x_1$  is more important.
- The Bayes optimal prediction function will do at least as well with just  $x_2$  than just  $x_1$ .

# Feature importance for learning method vs prediction function

- Variable importance for the problem + learning method:
  - without  $x_1$ , we can't solve the problem with a particular learning method
  - LOCO methods are the most direct way to get to this.
- Variable importance for particular prediction function:
  - This particular prediction function needs  $x_1$  to get good performance.
  - This is what permutation importance is getting at most directly.
- In practice, LOCO is usually much more expensive than permutation importance
  - Requires retraining rather than just evaluating the prediction function.

## How to figure out which features to buy/generate?

- Suppose we're in a world where using more features has a cost
  - e.g. to buy, to compute, to keep in memory, to use in training
- This seems like a problem for LOCO.
- If LOCO is too expensive, can try permutation feature importance as an approximation
  - but this has issues, as discussed below...
- If features come in groups, leave out a group at a time.
  - we can do this with permutation importance as well.
  - This is called a **grouped variable importance measure** in [GMSP15, Sec 2]

What can go wrong with permutation?

## Please Stop Permuting Features An Explanation and Alternatives

Giles Hooker\*and Lucas Mentch<sup>†</sup>

- This paper [HM19] has a catchy title.
- It “advocates against permute-and-predict (PaP) methods for interpreting black box functions.”
- We need to dig in a bit to see when and why PaP methods may not be a good idea.

## Main argument in paper

- ➊ When features are not independent, PaP methods can end up evaluating  $\hat{f}$  on many out-of-distribution points.
- ➋ The PaP importance measures can be influenced significantly by what  $\hat{f}$  does out-of-distribution.
- ➌ For nonlinear  $\hat{f}$ , behavior can be quite different in-distribution and out-of-distribution.

For most ML applications, we don't care what  $\hat{f}$  does out-of-distribution. Yet our PaP importance measures can depend heavily on this.

To put it all together: When features are not independent, PaP importance measure can be influenced significantly by what  $\hat{f}$  does out of distribution. Robustness concerns aside, for most ML applications, we only care about what  $\hat{f}$  is like “in distribution”. For nonlinear  $\hat{f}$ , behavior can be quite different in-distribution and out-of-distribution.

## Possible mitigations

- If we can identify groups of dependent features, we can permute them together.
  - Reduces out-of-sample evaluation.
- There are also situations where considering  $\hat{f}(x)$  out of distribution may be reasonable:
  - Our model  $\hat{f}(x)$  is a **causal** model that we intend to use for interventions, which are almost definitionally “out-of-distribution.”
  - We’re interested in how robust  $\hat{f}(x)$  to errors in variables

# Partial dependence

# Friedman's gradient boosting paper (1999/2001)

*The Annals of Statistics*  
2001, Vol. 29, No. 5, 1189–1232

## 1999 REITZ LECTURE GREEDY FUNCTION APPROXIMATION: A GRADIENT BOOSTING MACHINE<sup>1</sup>

BY JEROME H. FRIEDMAN

*Stanford University*

- Jerry Friedman presented gradient boosting in 1999 (published 2001).
- The foundation of xgboost, lightGBM, catboost, and other SOTA tree-based methods.
- Gradient boosting is for learning a non-linear function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ .
- If  $d = 1$  or  $d = 2$ , we can interpret  $f$  by visualization.
- In [Fri01, Sec 8.2], Friedman presents **partial dependence plots**
  - as a way to use visualization to help interpret **high-dimensional** non-linear functions.

# The partial dependence function

- Suppose  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ .
- Let  $S \subset \{1, \dots, d\}$  index a target subset of features.
- Let  $C = \{1, \dots, d\} - S$  index the complement set of features.
- We'll write  $X_S \in \mathbb{R}^{|S|}$  for the subset of features indexed by  $S$ .
- Similarly for  $X_C \in \mathbb{R}^{|C|}$ .
- The **partial dependence function** of  $f$  on  $x_S$  is given by

$$f_S(x_S) = \mathbb{E}_{X_C} [f(x_S, X_C)] = \int f(x_S, x_C) dP(x_C).$$

- In words,  $f_S$  is the function  $f$  after integrating out over  $X_C$ 
  - with respect to the **marginal distribution** of  $X_C$ .
- Does partial dependence depend on the out-of-distribution behavior of  $f$ ?

- We can think of  $f_S(x_S)$  as an attempt to distill the original function  $f(x_S, x_C)$  into a function that only depends on  $x_S$ . In practice,  $S$  usually consists of a single variable at a time, occasionally two variables.
- Note that if  $X_S$  and  $X_C$  are dependent, then integrating out over the marginal distribution can lead to evaluating  $f$  out-of-distribution.

## Estimating the partial dependence function

- The obvious estimate of the partial dependence function is

$$\hat{f}_S(x_S) = \frac{1}{n} \sum_{i=1}^n f(x_S, x_{C_i}),$$

where  $(x_{C_1}, \dots, x_{C_n})$  are the  $n$  instantiations of  $x_C$  in a dataset  $\mathcal{D}$ .

- This can be used on any “black box” prediction function.
- When  $S$  consists of just one or two features, we can visualize  $\hat{f}_S(x_S)$ .
- Fix a set of points  $P$  of points in the domain of  $x_S$ 
  - e.g. the observed values of  $x_S$  themselves in dataset  $\mathcal{D}$ .
- Then plot  $\left\{ \left( x_S, \hat{f}_S(x_S) \right) : x_S \in P \right\}$ .

# PDPs for 1-dimensional dependencies

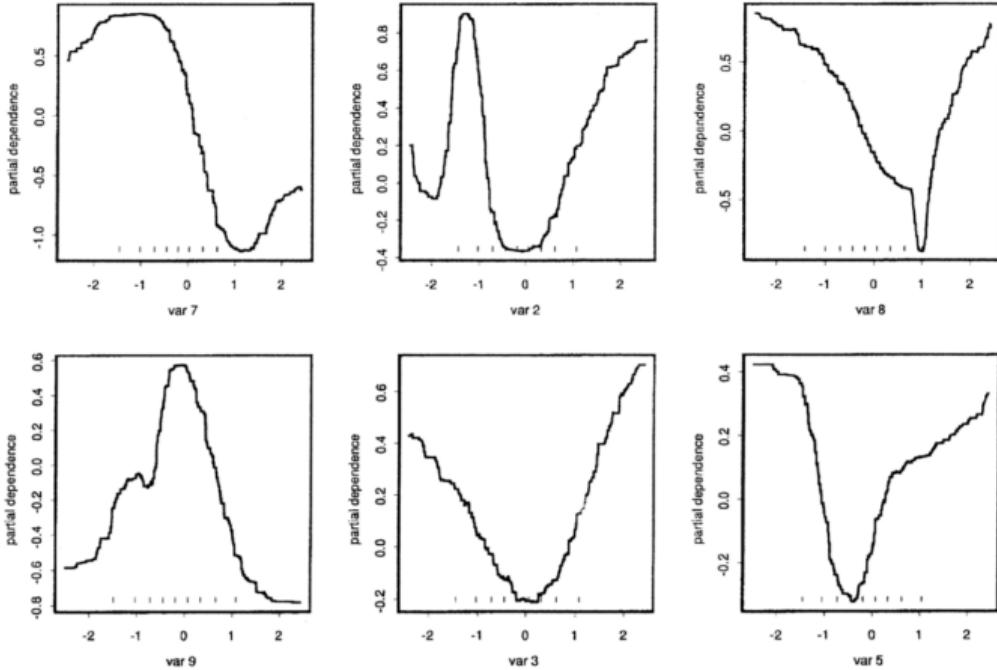


Figure 8 from [Fri01, p. 1223].

# PDPs for 2-dimensional dependencies

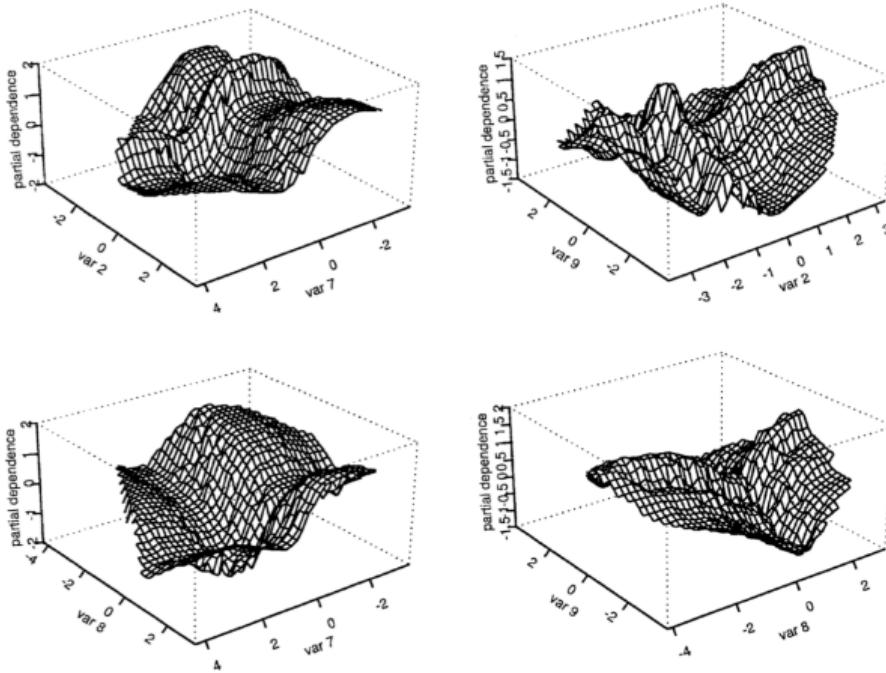


Figure 9 from [Fri01, p. 1224].

## Suppose $f$ is additive

- Suppose the function  $f$  decomposes additively as

$$f(x_S, x_C) = h_S(x_S) + h_C(x_C),$$

for some arbitrary  $h_S$  and  $h_C$ .

- Then the partial dependence function is

$$\begin{aligned} f_S(x_S) &= \mathbb{E}_{X_C} [h_S(x_S) + h_C(X_C)] \\ &= h_S(x_S) + \mathbb{E}[h_C(X_C)] \end{aligned}$$

- Thus  $f_S(x_S) - h_S(x_S) = \mathbb{E}[h_C(X_C)]$ .
- Since  $\mathbb{E}[h_C(X_C)]$  is a constant independent of  $x_S$ ,
  - the partial dependence function recovers  $h_S(x_S)$  up to an additive constant.

## Additive $f$ and GAMs

- Suppose the function  $f$  decomposes additively as

$$\begin{aligned} f(x_S, x_C) &= h_S(x_S) + h_C(x_C) \\ &= f_S(x_S) + h_C(x_C) + \text{constant} \end{aligned}$$

- Then the partial dependence  $f_S(x_S)$  tells us
  - how  $f(x_S, x_C)$  changes as a function of  $x_S$  with  $x_C$  fixed.
- So partial dependence plots are most useful when
  - $f$  decomposes additively on small subsets of features.
- Note that generalized additive models (GAMs) are exactly of this form [HT86]...

## Generalized additive models

- A generalized additive model is a model that looks like

$$f(x) = f_1(x_1) + f_2(x_2) + \cdots + f_d(x_d)$$

OR:  $f(x) = f_1(x_1) + f_{2,3}(x_2, x_3) + f_4(x_4) + \cdots + f_d(x_d)$

MORE GENERALLY:  $f(x) = \sum_{i=1}^G f_i(x_{S_i}),$

where each  $x_{S_i}$  is a subset of features – usually just one feature, sometimes two features.

- The  $f_i$ 's are estimated from the data.
- GAMs are often considered “white box” ML models, because they are directly interpretable.
  - Basically by looking at the PDPs that **exactly** represent the GAM.

# Interpretability, PDPs, and GAMs

- Using PDPs to explain a model fits into a broader approach to “interpretability”.
- Broader approach:
  - ① Find a black box model  $f(x)$  that has the performance you like.
  - ② Approximate  $f(x)$  with  $\hat{f}(x)$  using some “interpretable” ML model.
  - ③ Claim that  $f(x)$  is basically  $\hat{f}(x)$ , and so we now can interpret  $f(x)$ .
- The most common “white box” models used for this are
  - linear models (interpret by looking at coefficients)
  - shallow regression trees (interpret tree directly)
  - GAMs (interpret via partial dependence plots)

## Not the partial dependence function

- The **partial dependence function** of  $f$  on  $x_S$  is given by

$$f_S(x_S) = \mathbb{E}_{X_C} [f(x_S, X_C)] = \int f(x_S, x_C) dP(x_C).$$

- Note that this is **different from** a conditional variant:

$$f_S(x_S) = \mathbb{E}[f(X_S, X_C) | X_S = x_S] = \int f(x_S, x_C) dP(x_C | x_S).$$

- When  $f$  decomposes additively, the partial dependence function will recover the additive decomposition,
  - while the conditional form will not.
- When might we prefer one or the other?

- What happens if  $f(x) = x_1$  but  $X_1$  and  $X_2$  are highly correlated? Suppose that we have access to the true  $f(x)$ , or that we have managed to fit it very well. Then

$$f_2(x_2) = \mathbb{E}_{X_1} [f(X_1, x_2)] = \mathbb{E}_{X_1} X_1,$$

which is a constant, showing no partial dependence on  $x_2$ . On the other hand, suppose we take the conditional form, and suppose that  $X_1$  and  $X_2$  are jointly distributed Gaussian r.v.'s with identical marginal distributions  $\mathcal{N}(0, 1)$  and correlation  $\rho$ . Then  $\mathbb{E}[X_1 | X_2] = \rho X_2$ . So

$$\begin{aligned}\mathbb{E}[f(X_1, x_2) | X_2 = x_2] &= \mathbb{E}[X_1 | X_2 = x_2] \\ &= \rho x_2.\end{aligned}$$

Now we see a linear dependence on  $x_2$ .

- The conditional form is basically just building another model. It's answering the question: what would we predict if we only had access to a particular variable?
- The partial dependence function is closer to answering the question: what does the residual dependence on a particular variable look like, after accounting for all the other variables? It's exactly answering this question for GAM models.

## Non-additive example

- Consider the following distribution

$$Y = 0.2X_1 - 5X_2 + 10X_2 \mathbb{1}[X_3 \geq 0] + \varepsilon,$$

where  $\varepsilon \sim \mathcal{N}(0, 1)$  and independent of  $X_1, X_2, X_3$ , which i.i.d.  $\text{Unif}(-1, 1)$ .

- A gradient boosting model is fit to an i.i.d. sample of size 1000 from this distribution.
- Note that  $f(x) := \mathbb{E}[Y | X = x]$  does not decompose additively for  $X_2$  or  $X_3$ .
- The partial dependence function for  $f$  on  $x_2$  is

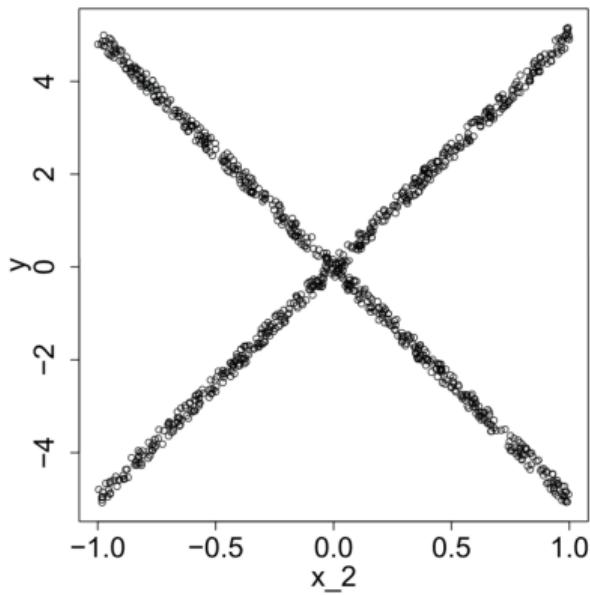
$$\begin{aligned} f_2(x_2) &= \mathbb{E}_{X_{\{1,3\}}} [f(x_2, X_1, X_3)] = \mathbb{E}[0.2X_1] - 5x_2 + 10x_2 \mathbb{P}(X_3 \geq 0) \\ &= 0 - 5x_2 + 10x_2 \left(\frac{1}{2}\right) = 0. \end{aligned}$$

- Yet  $f$  has a linear  $x_2$  term: either  $-5x_2$  or  $+5x_2$ , depending on the value of  $x_3$ .

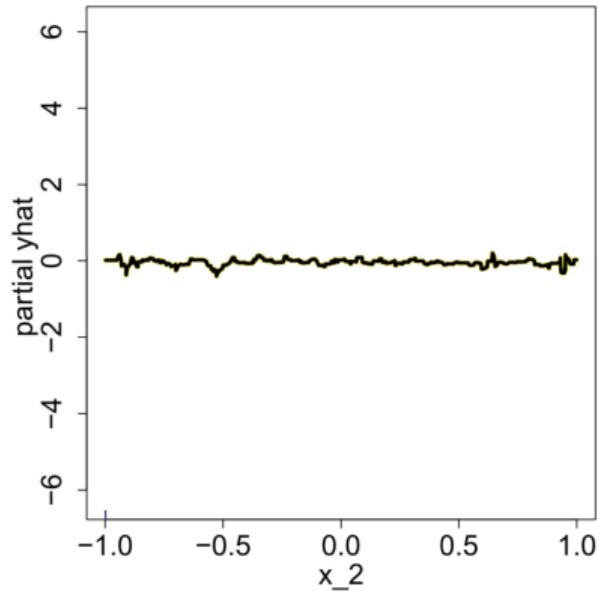
---

Example from [GKBP15, p. 47].

## Non-additive example



(a) Scatterplot of  $Y$  versus  $X_2$



(b) PDP

---

Figure 1 from [GKBP15, p. 47].

- On the left, we have a scatterplot of  $Y$  versus  $X_2$  for the sample of size 1000 from the data generating distribution. We can clearly see the two different linear relationships with  $x_2$ , which we know depends on the sign of  $x_3$ .
- On the right we see a plot of the estimated partial dependence on  $x_2$ , which is approximately flat. This agrees with our calculation on the previous slide.
- This is an example of how a partial dependence plot will fail to capture the relationship between a function value and a particular input when the function does not decompose additively with respect to the input.

## Individual conditional expectations (ICE)

# Peeking Inside the Black Box: Visualizing Statistical Learning With Plots of Individual Conditional Expectation

Alex GOLDSTEIN, Adam KAPELNER, Justin BLEICH, and Emil PITKIN

- Really expanded and improved on the idea of PDP.
- The rest of this module will be based primarily on this paper [[GKBP15](#)].

## Estimating the partial dependence function

- Recall our estimate for the partial dependence function:

$$\hat{f}_S(x_S) = \frac{1}{n} \sum_{i=1}^n f(x_S, x_{Ci}),$$

where  $(x_{C1}, \dots, x_{Cn})$  are the  $n$  instantiations of  $x_C$  in a dataset  $\mathcal{D}$ .

- Let's define the **individual conditional expectation** (ICE) for example  $i$ .
- Rather than integrating out over  $X_C$ , we condition on a specific value:

$$f_S^{(i)}(x_S) := \mathbb{E}[f(x_S, X_C) | X_C = x_{Ci}] = f(x_S, x_{Ci}).$$

- This tells us how  $f$  changes as we vary  $x_S$ , while keeping  $x_C = x_{Ci}$ .
- We can plot **all** the ICE functions  $f_S^{(1)}, \dots, f_S^{(n)}$  to get a better sense of the dependency.
- The partial dependence function estimate is just the average:  $\hat{f}_S(x_S) = \frac{1}{n} \sum_{i=1}^n f_S^{(i)}$ .

Note that there isn't really any expectation taking place in the ICE  $\mathbb{E}[f(x_S, X_C) | X_C = x_{Ci}]$ . This is more an instance of partial function application, where we define a new function by fixing a subset of the input variables to the original functions. We wrote it as  $\mathbb{E}[f(x_S, X_C) | X_C = x_{Ci}]$  to give at least some justification to the name "individual conditional expectation".

## ICE plots for our non-additive example

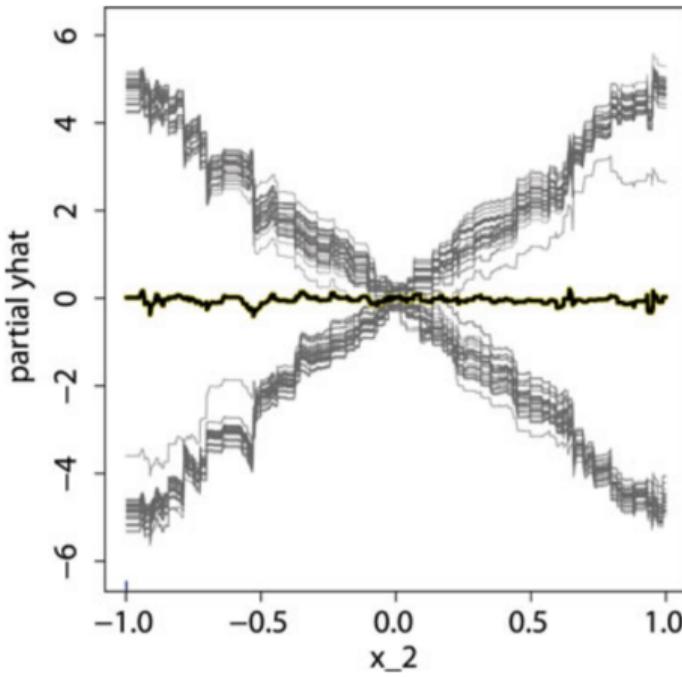
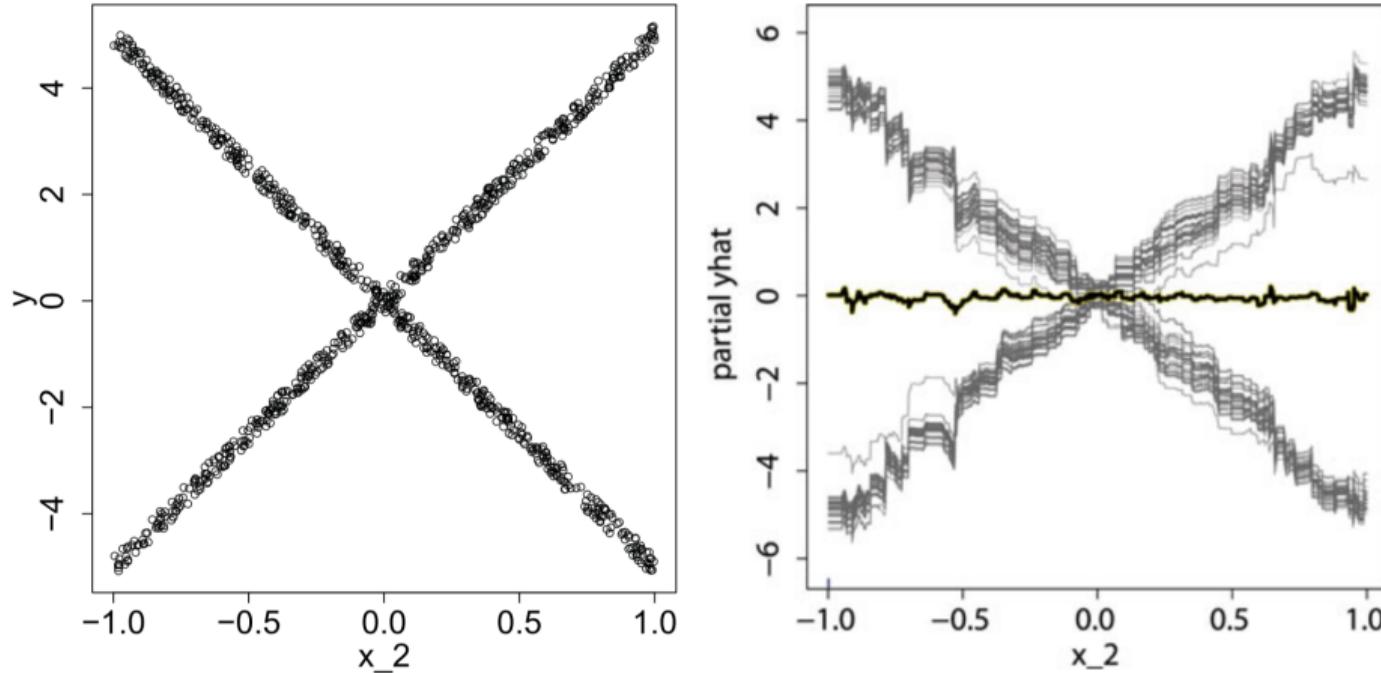


Figure 2 from [GKBP15, p. 48].

- Here we're taking  $f(x)$  to be a gradient boosting fit to a sample of size 1000. What's plotted are the ICE plots for  $x_2$  for each of the 1000 points:  $f_2^{(1)}(x_2), \dots, f_2^{(1000)}(x_2)$ . Let's review how  $f_2^{(i)}(x_2)$  is plotted. Suppose  $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$  is the input for example  $i$ .  $f_2^{(i)}(x_2)$  is the function that shows how  $f(x^{(i)}) = f(x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$  changes as we change  $x_2^{(i)}$  while leaving  $x_1^{(i)}$  and  $x_3^{(i)}$  fixed. That is,  $f_2^{(i)}(x_2) = f(x_1^{(i)}, x_2, x_3^{(i)})$ . So if  $x_3^{(i)} \geq 0$ , and if  $f(x) = \mathbb{E}[Y | X = x]$ , then  $f_2^{(i)}(x_2) = 0.2x_1^{(i)} + 5x_2$ , and if  $x_3^{(i)} < 0$ , then  $f_2^{(i)}(x_2) = 0.2x_1^{(i)} - 5x_2$ . Of course, in what's plotted, we're taking  $f(x)$  to be the gradient boosting fit to  $\mathbb{E}[Y | X = x]$ , so we just get an approximation.
- In the plots, note that some lines have a negative linear dependence on  $x_2$  and some have a positive linear dependence on  $x_2$ , depending on the value of  $x_3$ . Note also that there's some variation in the starting locations of the lines. For the idealized  $f(x) = \mathbb{E}[Y | X = x]$ , this variation in starting point will depend on the value of  $0.2x_1^{(i)}$ . Since  $X_1 \sim \text{Unif}(-1, 1)$ , the range of that variation should be about  $0.2 - (-0.2) = 0.4$ . The variation we see in the plot is at least twice as large as this. Presumably, this is coming from the fact that the gradient boosting fit to the data is not a perfect match for  $f(x)$ .
- The roughly horizontal line in the middle is the partial dependence plot, which is the pointwise average of the 1000 other functions plotted here.

## Scatterplot vs ICE plot



These two plots look quite similar in this case... will they always be similar?

Figures 1 and 2 from [GKBP15, p. 47-48].

- In the scatter plot, we're only ever plotting  $f(x) = f(x_1, x_2, x_3)$  for input points  $x^{(i)}$  that come from the data generating distribution.
- In the ICE plot on the right, we're plotting [our gradient boosting approximation to]  $f$  at points  $(x_1^{(i)}, x_2, x_3^{(i)})$ , which do not come from the data generating distribution.  $x_2$  is set to various values to create the plot. For this particular example, since  $X_1, X_2, X_3$  are i.i.d., as long as we're setting  $x_2$  to values from the marginal distribution of  $X_2$ , there is no difference. But in other data generating distributions, where  $X_1, X_2, X_3$  are not independent, this can lead to differences between the scatter plot and the ICE plot.
- For example, suppose  $X_2 = X_3$ . Then in the scatter plot, we'd only see the positive-sloped line of points for  $x_2 > 0$  and the negative slope for  $x_2 < 0$ . The dots would be in the form of a V. It's hard to predict what the gradient boosting fit would look like for this dataset. But if we plotted the partial dependence plot of the true regression function  $\mathbb{E}[Y | X = x]$ , it would again look like a full  $X$ .

## A real example

- Another example from [GKBP15, p. 48-50].
- Regression problem from Boston Housing Data (BHD).
- Trying to predict a census tract's median house price from features of the census tract.
- Fit with random forests.
- We want to examine the dependency of the fit  $\hat{f}$  on the feature "age".
  - "age" is the average age of homes in the census tract

# ICE Plot for RF on BHD

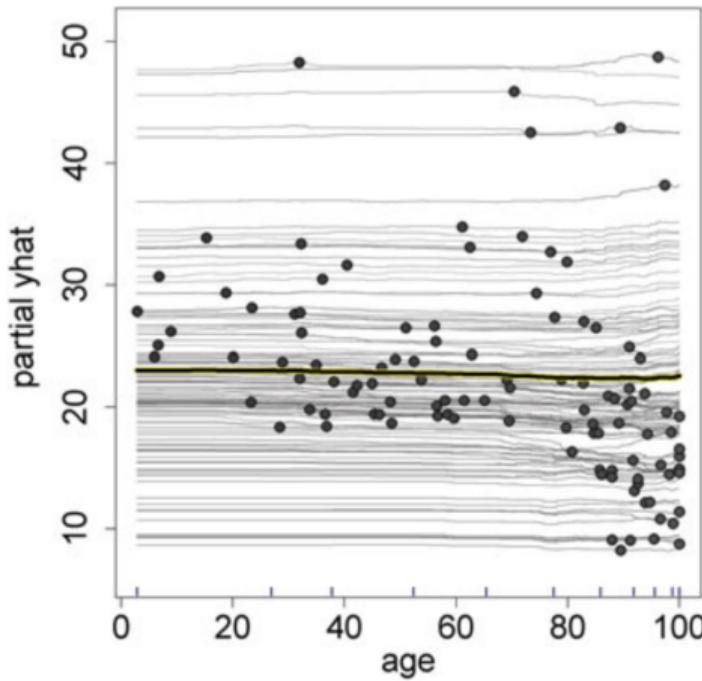


Figure 3 from [GKBP15, p. 49].

- The PDP is the highlighted line in the middle. It seems essentially flat.
- The black dot on each curve corresponds to the actual data point that the curve was built around.
- It doesn't seem like there's much of an effect of age on the response, even in the ICE plots... except, perhaps there's something going on in the high end of age? A lot of curves seem to have a slightly positive dependence on age starting around 85. But then other curves seem to have a slightly negative dependence on age, starting around 85. We'll revisit this.
- Also note the wide range of starting positions for these curves. This makes sense. There are many other factors that will have a large effect on the predicted price of the house besides age. These effects are visible in this range of starting values.

## The centered ICE plot [GKBP15, Sec 3.2]

- It can be easier to visually appreciate the relative shapes of the ICE plots
  - if we shift each plot so that they all intersect at a single point.
- They suggest having the curves intersect either at the minimum or maximum of the range of  $x_S$ .
- In doing this, we lose two things:
  - Seeing the range of  $y$  values.
  - How the eventual  $y$  value interacts with the shape of the ICE curve.
- So the original ICE plot can be used in conjunction with the centered ICE or “c-ICE” plot

# The c-ICE plot for RF on BHD

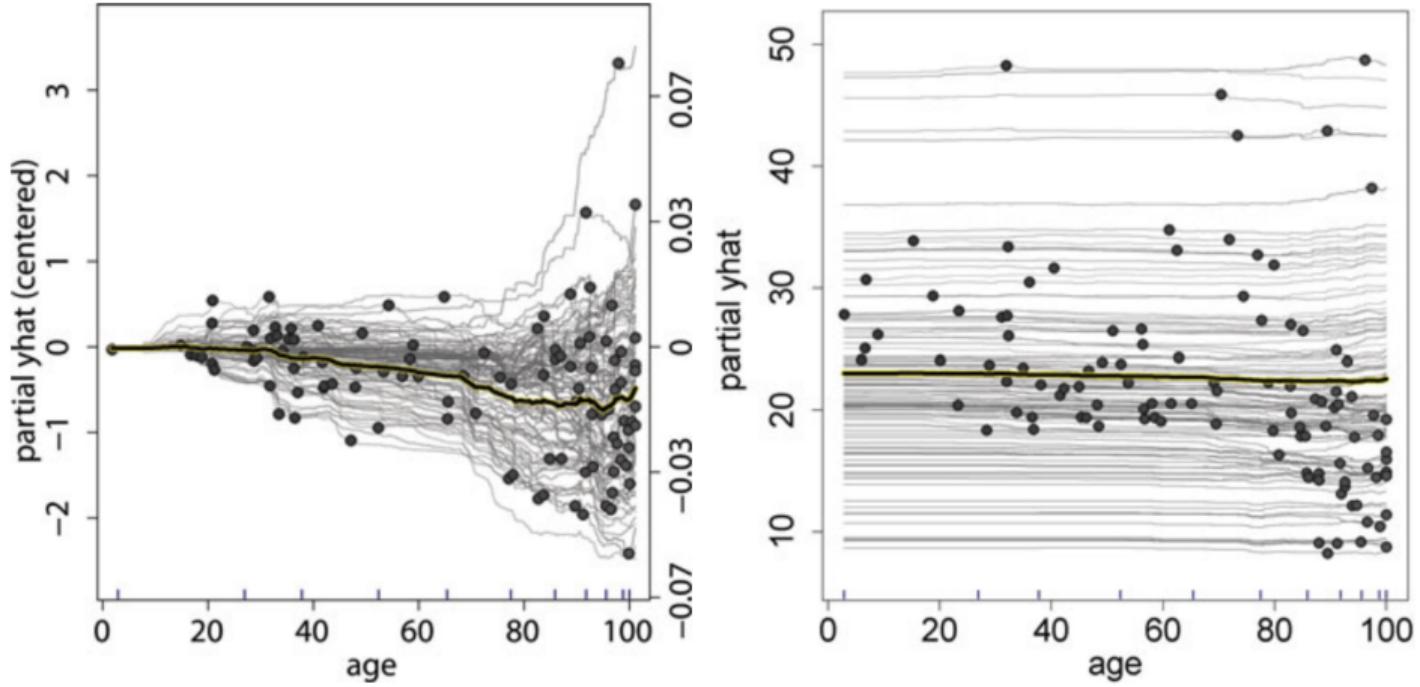


Figure 4 from [GKBP15, p. 50].

- With the “centering” at the left end of age, the heterogeneity of the age effect becomes quite apparent.
- On the other hand, looking at the scale of the y-axis on the left, compared to the scale of the y-axis in the original ICE plot, you can see that the effect of age isn’t that large, after accounting for everything else. But 1) there is an effect, and 2) the effect is completely different depending on the settings of other variables.
- The scale on the right side of the c-ICE plot is described as displaying “changes in  $\hat{f}$  over the baseline as a fraction of  $y$ ’s observed range.” Here’s how they get it: find the difference between the maximum and minimum values of  $\hat{f}$  over all points evaluated. Then the right scale is just the left scale divided by that difference.
- How could one figure out what determines the relation between drivers the

## Color / shade to visualize a second feature

- “rm” denotes the mean number of rooms per house in a census tract.
  - Color lines red if  $rm > \text{median}(rm)$ , blue otherwise.

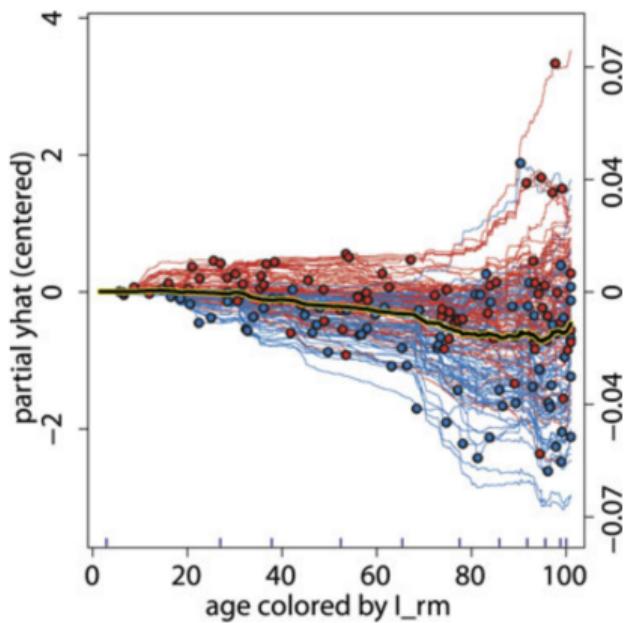


Figure 6 from [GKBP15, p. 52].

- Visually, we seem to have good support for  $\mathbb{1}[rm > \text{median}(rm)]$  being correlated with the type of partial dependence we have between the response and age.
- Could we discover this correlation automatically?
- If we can make a simple rule labeling curves by the pattern of partial dependence we see, perhaps we can try to predict that relationship based on simple, interpretable models, such as decision stumps or sparse linear models.

## ICE plots and additivity

- Suppose the function  $f$  decomposes additively as

$$f(x_S, x_C) = h_S(x_S) + h_C(x_C),$$

for some arbitrary  $h_S$  and  $h_C$ .

- The ICE plot corresponding to the  $i$ th point  $X_C^{(i)}$  is

$$\left\{ (x_S, h_S(x_S) + h_C(X_C^{(i)})) : x_S \in \mathcal{X}_S \right\}.$$

- So all the ICE plots are shifted versions of  $h_S(x_S)$ .
- We can check for additive decomposition by “parallel” ICE curves.

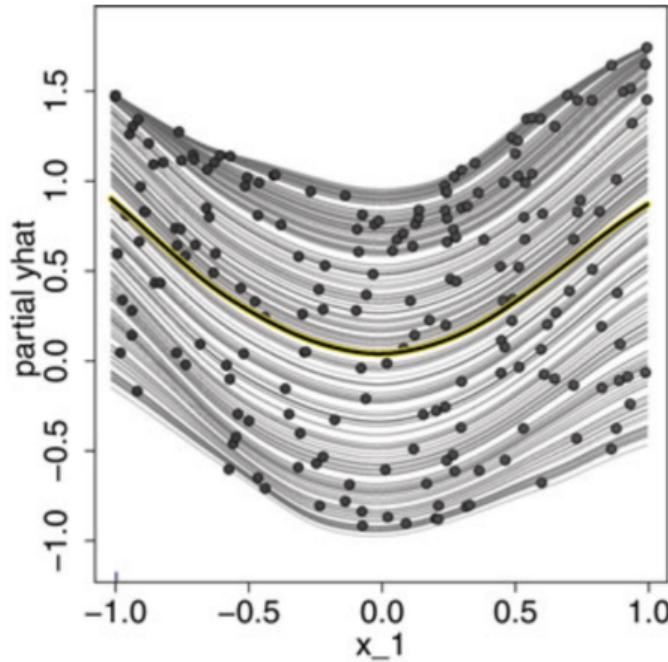
## Checking for additivity

- Consider the following data-generating model:

$$Y = X_1^2 + X_2 + \varepsilon, \quad X_1, X_2 \sim \text{Unif}(-1, 1), \quad \varepsilon \sim \mathcal{N}(0, 1)$$

- Suppose we get a sample  $(X^{(1)}, Y^{(1)}), \dots, (X^{(n)}, Y^{(n)})$  i.i.d. from this distribution.
- Using the sample, we want to figure out whether  $\mathbb{E}[Y | X = x]$  decomposes additively,
  - or whether there's an interaction.
- We fit a model to these points.
- We can check for additivity with the ICE plots...
- Obviously, we can't conclude anything about  $\mathbb{E}[Y | X = x]$ .
- But it suggests how we can restrict whatever model class we're using.

## ICE plot check for additivity



- These curves look fairly parallel – i.e. fairly close to additive.
- Not sure why they don't do a c-ICE plot so it would be more clear.

Figure 7 from [GKBP15, p. 53].

- If you find that your fit is essentially additive, what can you do?
- You could use a model that's explicitly additive, such as a GAM.
- You could build an additive model "by hand", by fitting a model on one set of features, and then [in the case of regression] fitting the residual using the other set of features. Something similar can be done for generalized additive models as well.

## Extrapolation experiment

- Consider the following model (from [GKBP15, Sec 4.3]):

$$\begin{aligned} Y &= 10X_1^2 + \mathbb{1}[X_2 \geq 0] + \varepsilon \\ \varepsilon &\sim \mathcal{N}(0, 0.1^2) \\ (X_1, X_2) &\sim \text{Unif}(S), \end{aligned}$$

where  $S = (-1, 1) \times (-1, 1) - (0, 1) \times (0, 1)$ .

- That is,  $S \subset \mathbb{R}^2$  is a square with the upper right quadrant removed.
- So  $\mathbb{P}(X_1 > 0, X_2 > 0) = 0$ .
- For  $f(x) = \mathbb{E}[Y | X = x]$ , we'd expect the ICE plots for  $x_1$  to be
  - two parabolas:  $10x_1^2$  and  $10x_1^2 + 1$ .
- We draw 1000 observations and fit a random forest model  $\hat{f}(x)$ .
- Let's look at the ICE plots for  $\hat{f}$  w.r.t.  $x_1$ ...

## All observations

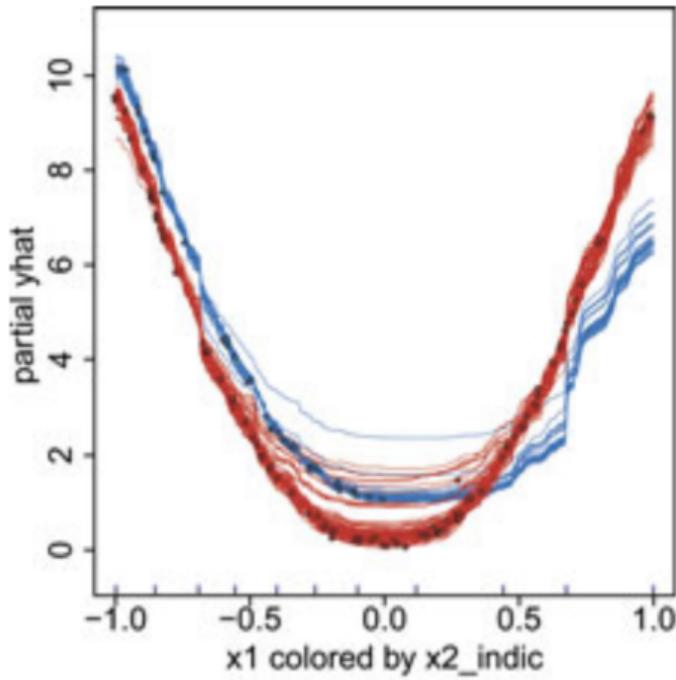


Figure 9(a) from [GKBP15, p. 55].

- Here the ICE plots are colored according to whether or not  $x_2 < 0$ .
- For  $x_1 < 0$ , we see two sets of parabolas, one set basically one unit higher than the other.
- For  $x_1 > 0$ , the curves actually cross – completely different from what we'd expect for the ICE plot of  $f(x) = \mathbb{E}[Y | X = x]$ .
- If you look closely, you can see that there are no black dots on the blue curves to the right of  $x_1 = 0$ . This is because  $\mathbb{P}(X_1 > 0, X_2 > 0) = 0$ . So the blue curves are based on  $\hat{f}$  in a region of pure extrapolation – that is, there are no training examples in that region.
- What's the takeaway? If we look at the plots naively, one may be tempted to conclude that the function  $\mathbb{E}[Y | X = x]$  has some interaction between  $X_1$  and  $X_2$ , since the ICE curves are not parallel. However, this is the wrong conclusion: all we can really conclude is that the fit  $\hat{f}$  has an interaction. By noting that  $\hat{f}$  is extrapolating in the region that's suggesting an interaction, we realize that we don't have much evidence for an interaction in  $\mathbb{E}[Y | X = x]$ . Indeed, we know that there are interactions between  $x_1$  and  $x_2$  in  $\mathbb{E}[Y | X = x]$ .

- Some recent work on interpreting GAMs (and by extension, partial dependency plots) in the multiclass setting can be found in [ZTK<sup>+</sup>18].
- The ESL book has a note on partial dependence plots [HTF09, Sec 10.13.2].
- [Limitations of Interpretable Machine Learning Methods](#) has some interesting and relevant chapters.
- Of course, all the papers referenced throughout this module.

## References I

- [Bre01] Leo Breiman, *Random forests*, Mach. Learn. **45** (2001), no. 1, 5–32.
- [Fri01] Jerome H. Friedman, *Greedy function approximation: A gradient boosting machine.*, Ann. Statist. **29** (2001), no. 5, 1189–1232.
- [GKBP15] Alex Goldstein, Adam Kapelner, Justin Bleich, and Emil Pitkin, *Peeking inside the black box: Visualizing statistical learning with plots of individual conditional expectation*, Journal of Computational and Graphical Statistics **24** (2015), no. 1, 44–65.
- [GMSP15] Baptiste Gregorutti, Bertrand Michel, and Philippe Saint-Pierre, *Grouped variable importance with random forests and application to multiple functional data analysis*, Computational Statistics & Data Analysis **90** (2015), no. nil, 15–35.
- [HM19] Giles Hooker and Lucas Mentch, *Please stop permuting features: an explanation and alternatives*, CoRR (2019).

## References II

- [HT86] Trevor Hastie and Robert Tibshirani, *Generalized Additive Models*, Statistical Science 1 (1986), no. 3, 297 – 310.
- [HTF09] Trevor Hastie, Robert Tibshirani, and Jerome H. Friedman, *The elements of statistical learning: Data mining, inference, and prediction, 2nd edition*, Springer Series in Statistics, Springer, 2009.
- [LGR<sup>+</sup>18] Jing Lei, Max G'Sell, Alessandro Rinaldo, Ryan J. Tibshirani, and Larry Wasserman, *Distribution-free predictive inference for regression*, Journal of the American Statistical Association 113 (2018), no. 523, 1094–1111.
- [ZTK<sup>+</sup>18] Xuezhou Zhang, Sarah Tan, Paul Koch, Yin Lou, Urszula Chajewska, and Rich Caruana, *Axiomatic interpretability for multiclass additive models*, CoRR (2018).