Shapley Values

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Shapley Values

Coalitional game[1]

- Suppose there is a game played by a team (or "coalition") of players.
- A **coalition game** is
  - a set N consisting of n "players" and
  - a function $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$, assigning a value to any subset of players.
- Think of N as a team. Maybe they're trying to solve a puzzle together...
  - Says how well a subset of the team would have done, cooperating on the puzzle.
- Suppose the whole team plays and gets value $v(N)$.
- How should that value be allocated to the individuals on the team?
- Is there a fair way to do it that reflects the contributions of each individual?

[1]Based on Shapley value article in Wikipedia [Wik20] and [MP08].

- Where we're headed is that we're going to apply this approach of "value allocation" to "coalitions" of feature "working together" to produce the final output.
- Of course, it's not really clear what it means to use a subset of features with a specific prediction function $f(x)$.
- Various approaches to this will give us different interpretations.

Solutions to coalition games

- Let $\mathcal{G}(N)$ denote the set of all coalition games on set N.
  - i.e. a game for every possible $v : 2^N \to \mathbb{R}$.
- A **solution** to the allocation problem on the set $\mathcal{G}(N)$ is a map $\Phi : \mathcal{G}(N) \to \mathbb{R}^n$
  - gives the allocation to each of n players for any game $v \in \mathcal{G}(N)$.
- Next we'll give a particular solution, the Shapley value solution.
- Then we'll give various properties that seem desirable for a solution.
- Finally, we'll state a theorem that says the Shapley value solution
  - is the unique solution satisfying these properties.

The Shapley value solution

- The **Shapley value solution** is $\Phi(v) = (\phi_i(v))_{i=1}^n$ where

$$\phi_i(v) = \sum_{S \subset (N - \{i\})} k_{|S|,n} \left(v(S \cup \{i\}) - v(S)\right),$$

  where $k_{s,n} = s!(n-s-1)!/n!$.
- In words, for any game $v \in \mathcal{G}(N)$, player i receives $\phi_i(v)$.
  - You can show that $\sum_{i=1}^n \phi_i(v) = v(N)$.
- Equivalently,

$$\phi_i(v) = \frac{1}{n!} \sum_R \left[v(P_i^R \cup \{i\}) - v\left(P_i^R\right)\right],$$

  - where sum ranges over all $n!$ permutations R of the players in N.
  - P_i^R is the set of players in N that precede i in order R.

- The second version can be explained by the "room parable" [MP08, p. 6]: Players enter a room one at a time to form the team of n players. Each player receives the marginal contribution of their presence (could be negative). If all orders of entering the room have the same probability, then $\phi_i(v)$ is the expected value of how much player i receives.
- Yet another way to write the Shapley value is as

$$\begin{aligned}
\phi_i(v) &= \frac{1}{n}\sum_{s=0}^{n-1} \sum_{S\subset(N-\{i\}) \text{ and } |S|=s} \binom{n-1}{s}^{-1} [v(S\cup\{i\}) - v(S)] \\
&= \frac{1}{n} \sum_{s\text{:size of coalition}} \sum_{\text{coalition excluding } i \text{ of size } s} \frac{\text{marginal contribution of } i \text{ to the coalition}}{\text{number of coalitions of size } s \text{ excluding } i}.
\end{aligned}$$

Efficiency and symmetry properties

- **Efficiency**: For any $v \in \mathcal{G}(N)$,
$$\sum_{i \in N} \phi_i(v) = v(N).$$
- **Symmetry**: For any $v \in \mathcal{G}(N)$, if players i and j are equivalent in the sense that
$$v(S \cup \{i\}) = v(S \cup \{j\})$$
for every subset S of players that excludes i and j, then
$$\phi_i(v) = \phi_j(v).$$
- Also called "equal treatment of equals".

Linearity property

- **Linearity**: For any $v, w \in \mathcal{G}(N)$, we have

$$\phi_i(v+w) = \phi_i(v) + \phi_i(w)$$

  for every player i in N. Also, for any $a \in \mathbb{R}$,

$$\phi_i(av) = a\phi_i(v)$$

  for every player i in N.
- $v + w$ is the game resulting from summing the outcomes of v and w for each coalition.
- av is the game resulting from scaling up outcomes of v by the factor a
- (These can be useful for prediction functions that are linear combinations of other functions, such as gradient boosted regression trees.)

Null player property

- A player i is **null** in v if $v(S \cup \{i\}) = v(S)$ for all coalitions $S \subset N$.
- If player i is null in a game v, then $\phi_i(v) = 0$.

- (In the context of machine learning, this is called the Dummy property. Presumably because if we replace players by features, a dummy feature has no information, and we want it to get importance 0?)

Shapley value theorem (Shapley, 1953)

Theorem

The Shapley value solution $\Phi(v) = (\phi_i(v))_{i=1}^{n}$ defined previously is the unique solution for $\mathcal{G}(N)$ that satisfies the

- *efficiency, symmetry, linearity, and null properties.*

- Proof: See references.

Example: Shapley values for constant game

- Suppose $v(S) \equiv c$ for any coalition $S \subset N$, except $v(\emptyset) = 0$.
- Then for any $i, j \in N$, $S \subset (N - \{i, j\})$, we have

$$v(S \cup \{i\}) = v(S \cup \{j\}) = c,$$

  which implies $\phi_1(v) = \cdots = \phi_n(v)$ by the symmetry property.
- By the efficiency property,

$$\sum_{i \in N} \phi_i(v) = v(N) = c.$$

- Therefore, $\phi_1(v) = \cdots = \phi_n(v) = c/n$.

Example: game plus a constant

- Suppose we have a game $v(S)$ on N
  - with Shapley values $\phi_1(v), \ldots, \phi_n(v)$.
- Suppose we shift the rewards, so $v'(S) := v(S) + c$.
- What are the Shapely values for $v'(S)$?
- Let $w(S) \equiv c$ for $S \subset N$, except $w(\emptyset) = 0$.
- Then $v'(S) = v(S) + w(S)$ and by linearity,

$$\phi_i(v') = \phi_i(v + w) = \phi_i(v) + \phi_i(w) = \phi_i(v) + \frac{c}{n}.$$

- So if we shift by a constant, the shift is divided equally among the players.

Shapley Values for Feature Importance

Shapley values for features

- Shapley values are about n-player games.
- In particular, they are about set functions on a set of n elements.
- How can we connect this to the feature importance in machine learning?
- Easy part: each "player" is a feature.
- Hard part: what's the set function?
- We have a prediction function,
  - but it doesn't naturally apply to subsets of features.
- What if we start earlier:
  - building a model with a subset of features

Attribute R^2 to features

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Analysis of regression in game theory approach

Stan Lipovetsky*,† and Michael Conklin

Custom Research Inc., 8401 Golden Valley Road, Minneapolis, MN 55427, U.S.A.

- An early application of Shapley values to machine learning [LC01].
- Applied Shapley values to allocate the R^2 performance measure to features
  - for linear regression, though we'll present the obvious generalization.
- Essentially the same approach was actually done much earlier,
  - without making the connection to Shapley values, e.g. [Kru87].

Attribute model performance to features

- Let $R(f)$ be some performance measure of a prediction function f.
- Let $\mathcal{A} : \mathcal{D} \mapsto f$ represent a model training algorithm that
  - takes a training dataset $\mathcal{D}$ and
  - produces a prediction function f.
- Let $\{1, \ldots, d\}$ index the features available for a problem.
- Let $\mathcal{D}_S$ denote the dataset with just the features indexed by $S \subset \{1, \ldots, d\}$.
- Define the set function $v(S) := R(\mathcal{A}(\mathcal{D}_S))$ and $v(\emptyset) = 0$.
- For any subset of features, $v(S)$ gives
  - the performance of the model trained on just that subset of features.

Lipovetsky and Conklin (2001)

- In [LC01],
  - performance measure was R^2
  - model class was linear models.
- They used only 7 features, and linear models train quickly,
  - so computation wasn't an issue.
- Generally speaking, need to train 2^d models.
- Not practical in most machine learning settings.

Monte Carlo approach

- The Shapley values in our scenario are

$$\phi_i(v) = \frac{1}{d!}\sum_R \left[v(P_i^R \cup \{i\}) - v\left(P_i^R\right)\right],$$

  - where sum ranges over all $n!$ permutations R of the players in N.
  - P_i^R is the set of players in N that precede i in order R.
- We can approximate this by averaging a random sample of M permutations.
- This still requires training Md models, which may not be practical for large d.
- This whole approach is only realistic when d is small and training and evaluation are fast.

Connection to LOCO

- This approach is most related to LOCO from an earlier module.
- We're not saying anything about a particular prediction function.
- We're saying something about the importance of each feature
  - in a particular dataset,
  - for a particular model training procedure
- LOCO was about the effect of removing each single feature.
- Here we have something that seems a bit deeper,
  - though hard to say exactly what machine learning question it's answering.

Shapley values for prediction functions

Interpreting a prediction function

- Suppose we want to use Shapley values
  - to interpret a particular prediction function $f(x)$.
- It's not obvious what it means to evaluate f using a subset of features.
- This is not a standard operation in machine learning.
- Let's write x_S for the features corresponding to $S \subset \{1, \ldots, d\}$.
- Let's write x_C for the features corresponding to the complement $\{1, \ldots, d\} - S$.
- So if $f(x) = f(x_S, x_C)$, we need a definition for $f_S(x_S)$.

Two approaches to defining $f_S(x_S)$

- Two approaches, as described by [CJLL20, JMB19].
- **Conditional expectation** (or "**observational conditional expectation**")

$$f_S(x_S) := \mathbb{E}\left[f(x_S, X_C) \mid X_S = x_S\right].$$

- **Marginal expectation** (or "**interventional conditional expectation**")

$$\begin{aligned} f_S(x_S) &:= \mathbb{E}[f(x_S, X_C)] \\ &= \mathbb{E}\left[f(x_S, X_C) \mid do(X_S = x_S)\right], \end{aligned}$$

  where the do-operator is beyond our scope, but see [JMB19].
- Conditional expectation keeps our evaluations $f(x_S, x_C)$ on the data manifold.
- Marginal expectation will potentially evaluate $f(x_S, x_C)$ off the data manifold,
  - when we have dependencies between x_S and x_C.

(Note: we discussed exactly these definitions of $f_X(x_S)$ in the feature importance module.)

Independent features

- Note that when X_S and X_C are independent, we have

$$\mathbb{E}\left[f(x_S, X_C) \mid X_S = x_S\right] = \mathbb{E}\left[f(x_S, X_C)\right].$$

- So the conditional expectation and marginal expectation methods are equivalent.
- In the literature, some works (e.g. [LL17, AJL21])
  - see conditional expectation as the preferred approach,
  - but use marginal expectation as an approximation (e.g. KernSHAP in [LL17]).
- Others argue that marginal expectations is what we wanted in the first place.
  - Since it gives a more interventinal interpretation.

Estimating $f_S(x_S)$

- We generally don't know the joint distribution of X,
  - so we can't exactly compute the expectations needed for $f_S(x_S)$ (either version).
- For the marginal version, we can use the same estimate as for partial dependency:

$$\hat{f}_S(x_S) = \frac{1}{n}\sum_{i=1}^{n} f(x_S, x_{Ci}),$$

  where $(x_{C1}, \ldots, x_{Cn})$ are the n instantiations of x_C in a dataset $\mathcal{D}$.
- For consistency, we'll also define $\hat{f}_\emptyset = \frac{1}{n}\sum_{i=1}^{n} f(x_i)$.
- For conditional expectation, this estimation is much more challenging.
  - In general, seems to require training 2^d regression models.
  - (TreeSHAP uses a special property of trees to kinda approximate this.)

Shapley values for prediction function

- Suppose we have an estimate $\hat{f}_S(x_S)$ for each $S \subset \{1, \ldots, d\}$.
- Then we can define the set function for our "game" on $\{1, \ldots, d\}$ as

$$
\begin{aligned}
v(S) &:= \hat{f}_S(x_S) \\
v(\emptyset) &:= 0.
\end{aligned}
$$

- Frequently it's defined as

$$
\begin{aligned}
v(S) &:= \hat{f}_S(x_S) - \hat{f}_\emptyset \\
v(\emptyset) &:= 0.
\end{aligned}
$$

- That way, Shapley values indicate how each feature
  - pulls the prediction away from the mean / "no information" prediction.

Linear case[2]

- Consider a linear regression model $f(x) = \beta_0 + \sum_{j=1}^{M} \beta_j x_j$.
- Consider the Shapley values based on marginal expectation,
  - or equivalently, based on conditional expectation with independent features.
- Let β_S be the vector of coefficients corresponding to X_S, and similarly for β_C.

$$
\begin{aligned}
f_S(x_S) &= \mathbb{E}\left[f(x_S, X_C)\right] \\
&= \mathbb{E}\left[\beta_0 + \beta_S^T x_S + \beta_C^T X_C\right] \\
&= \beta_0 + \beta_S^T x_S + \beta_C^T \mathbb{E}\left[X_C\right]
\end{aligned}
$$

- Let's define the set function as

$$
\begin{aligned}
v(S) &= f_S(x_S) - \mathbb{E}\left[f(X)\right] \\
v(\emptyset) &= 0
\end{aligned}
$$

[2] Based on [AJL21, App B.1].

Shapley value linear case

- Let $C_i = C - \{i\}$. Then we have

$$
\begin{aligned}
v(S+\{i\}) - v(S) &= \beta_0 + \beta_S^T x_S + \beta_i x_i + \beta_{C_i}^T \mathbb{E}[X_{C_i}] - \mathbb{E}[f(X)] \\
&\quad - \left(\beta_0 + \beta_S^T x_S + \beta_i \mathbb{E}[X_i] + \beta_{C_i}^T \mathbb{E}[X_{C_i}] - \mathbb{E}[f(X)]\right) \\
&= \beta_i (x_i - \mathbb{E}[X_i]).
\end{aligned}
$$

- Note this is independent of S.
- Let's verify that this makes sense when $S = \emptyset$.
- Let C consist of all features except i:

$$
\begin{aligned}
v(\{i\}) - v(\emptyset) &= \beta_0 + \beta_i x_i + \beta_C^T \mathbb{E}[X_C] - \mathbb{E}[f(X)] \\
&= \beta_0 + \beta_i x_i + \beta_C^T \mathbb{E}[X_C] - \mathbb{E}\left[\beta_0 + \beta_i X_i + \beta_C^T X_C\right] \\
&= \beta_i (x_i - \mathbb{E}[X_i]).
\end{aligned}
$$

- Note that if we had defined $v(S) = f_S(x_S)$, without subtracting off $\mathbb{E}[f(X)]$, then

$$v(S + \{i\}) - v(S) = \beta_i (x_i - \mathbb{E}[X_i])$$

  would only hold for $S \neq \emptyset$. Which would have complicated the subsequent calculations.

- Once we know the Shapley values for $v(S) = f_S(x_S) - \mathbb{E}[f(X)]$As we know, all the Shapley values would just increase by β_0/M if we used $v(S) = f_S(x_S)$.

Shapley value linear case

- So the Shapley value is

$$\begin{aligned}\phi_i(v) &= \sum_{S \subset (N - \{i\})} k_{|S|,n} \left(v \left(S \cup \{i\} \right) - v(S) \right) \\ &= \beta_i \left(x_i - \mathbb{E}\left[X_i \right] \right) \sum_{S \subset (N - \{i\})} k_{|S|,n} \\ &= \beta_i \left(x_i - \mathbb{E}\left[X_i \right] \right),\end{aligned}$$

where the last step we leave as an easy exercise.

References

Resources

- The most common citation for the proof of the Shapley value theorem is Shapley's paper [Sha53]. These slides provide a proof of the Shapley value theorem, and I think the first few sections of [MP08] are easier to read than Shapley's paper.
- The result of SHAP See [AJL21, App B.1]

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