Solutions - Practice Exam Questions - Tutorial 4

1. (a)

   Anti-symmetric? No, e.g. $x = 0$ and $y = 1$. We get that both $2(0) - 2(1) \leq 3$ and $2(1) - 2(0) \leq 3$.

   (b)

   It is anti-symmetric.
   Assume $x \geqslant |y|$ and $y \geqslant |x|$.

   $$\left.\begin{array}{l} x \geqslant |y| \implies x \geq 0 \\ y \geqslant |x| \implies y \geqslant 0 \end{array}\right\} \implies \begin{array}{c} x \geqslant y \\ \wedge \\ y \geqslant x \end{array} \implies x = y. \checkmark$$

   (because $|x| = x$ and $|y| = y$)

   (c)

   Anti-symmetric

   Yes! Let x and y be arbitrary integers such that xRy and yRz. Need to show $x = y$.

   Now, $xRy \implies x \geqslant y$ (see transitivity proof)

   and $yRx \implies y \geqslant x$,

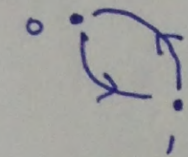
   So it follows that $xRy \wedge yRx \implies x = y$! ✓

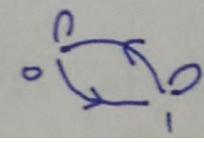
   So it is anti-symmetric.

2.

	A-S
a	×
b	✓
c	✓
d	✓

3.

if R is not anti-symmetric, this must happen: 

Now, if R is transitive too, then we need loops on both the 0 and 1, i.e. 

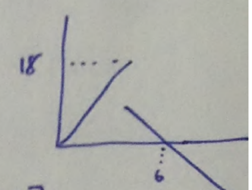
So R is reflexive. ✓

4.

Not injective, eg. $f(1) = 3 = f(9)$ (and $1 \neq 9$).

Not surjective eg. nothing maps to 20.

(So clearly not bijective).



Regarding the range, observe that the "$3x$" bit generates $y \in [0, 18]$.

The "$12-x$" bit can generate (amongst others) all numbers $y < 0$.

So the range is all real numbers $y \leq 18$.

5.

Recall: $A = \{x \in \mathbb{R} : x \neq 1\}$.

Let's prove injective.

Let x, y be arbitrary elements of A such that $f(x) = f(y)$. (We want to show $x = y$). ← the domain

So,
$$f(x) = f(y)$$
$$\Rightarrow \frac{x+1}{x-1} = \frac{y+1}{y-1} \qquad \left(\text{note that } \begin{matrix} x-1 \neq 0 \\ y-1 \neq 0 \end{matrix}\right)$$
$$\Rightarrow (x+1)(y-1) = (y+1)(x-1)$$
$$\Rightarrow xy - x + y - 1 = yx - y + x - 1$$
$$\Rightarrow 2y = 2x$$
$$\Rightarrow y = x \quad \square$$

Let's prove surjective.

Let y be an arbitrary element of the co-domain A.

We want ~~[illegible]~~ $x \in A$ such that,
$$f(x) = y$$

So we want x such that
$$\frac{x+1}{x-1} = y$$

Solve for x in terms of y
$$\frac{x+1}{x-1} = y$$
$$\Rightarrow x + 1 = y(x-1)$$
$$\Rightarrow 1 + y = yx - x$$
$$\Rightarrow 1 + y = x(y-1)$$
$$\Rightarrow x = \frac{1+y}{y-1}$$

We need to check that ↗ is in the domain A.

- Firstly, is it well-defined? $y \neq 1$ so there is no division by zero ✓
- We need to check that $\frac{1+y}{y-1} \neq 1$. Suppose for the sake of contradiction $\frac{1+y}{y-1} = 1$

$$\Rightarrow 1 + y = y - 1$$
$$\Rightarrow 1 = -1 \quad \text{FALSE}.$$

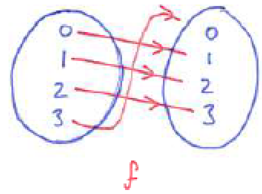
- So $\frac{1+y}{y-1}$ is indeed in A!
- So surjective ✓

So also bijective (because both injective & surjective).

6.

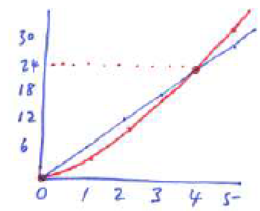
Q1

(a)



i.e. $f(0)=1$
$f(1)=2$
$f(2)=3$
$f(3)=0$

(b) Let's sketch this to get a feeling for what is going on.



$6x$ in blue
$(x+1)^2-1$ in ~~black~~ red.

They crossover at $x=4$,
so it looks like $c=4$.

That is: $f(x) = \begin{cases} 6x & \text{if } 0 \le x \le 4 \quad \text{SD1} \\ (x+1)^2-1 & \text{if } x > 4 \quad \text{SD2} \end{cases}$

Let's prove this!

Let's prove <u>injective</u>.

Assume $f(x)=f(y)$. [We need to prove that $x=y$].

Observe that if x is from SD1, $0 \le f(x) \le 24$ and if y is from SD2, $f(x) > 24$. These regions are disjoint, so under the assumption that $f(x)=f(y)$, it must hold that both x and y are from SD1, or both from SD2.

both from SD1
$f(x)=f(y)$
$\Rightarrow 6x = 6y$
$\Rightarrow x=y$ ✓

both from SD2
$f(x)=f(y)$
$\Rightarrow (x+1)^2-1 = (y+1)^2-1$
$\Rightarrow (x+1)^2 = (y+1)^2$
$\Rightarrow x+1 = y+1$ (because $x+1$ and $y+1$ are both >0, which is a consequence of the fact that they are both in SD2).
$\Rightarrow x=y$ ✓

So it <u>is</u> injective.

Let's prove surjectivity.

For $0 \le y \le 24$ take $x = \frac{y}{6}$. (Why? $y = 6x \Rightarrow x = \frac{y}{6}$)

For $y > 24$ take $x = \sqrt{y+1} - 1$ (Why? $(x+1)^2 - 1 = y \Rightarrow x = \sqrt{y+1} - 1$)

take the positive root.

Need to check this really works. (Yes, you do need this step).

- For $0 \le y \le 24$, $0 \le \frac{y}{6} \le 4$ so $f\left(\frac{y}{6}\right) = 6\left(\frac{y}{6}\right) = y$ ✓

- for $y > 24$, $\sqrt{y+1} - 1 > \sqrt{25} - 1 = 4$.

$$\text{So } f\left(\sqrt{y+1} - 1\right) = \left[\sqrt{y+1} - 1 + 1\right]^2 - 1 = \left[\sqrt{y+1}\right]^2 - 1 = y + 1 - 1 = y \checkmark$$

So it is surjective.

So it is injective & surjective, and thus bijective (and thus invertible).

Inverse: $f^{-1}(x) = \begin{cases} \frac{x}{6} & \text{if } 0 \le x \le 24 \\ \sqrt{x+1} - 1 & \text{if } x > 24 \end{cases}$

make sure you get these right!

7.

x	1	2	3	4	5	6	7	8	9	10	11	12	13	...	24	25	26
$f(x)$	1	4	9	16	25	36	49	64	81	100	23	25	27		49	51	53
$g(x)$	3	~~5~~6	11	18	27	38	51	192	243	300	25	27	29		51	153	154
				1					2				3			4	

So the function behaves like this:

$$(g \circ f)(x) = \begin{cases} x^2+2 & 1 \le x \le 7. \quad (1) \\ 3x^2 & 8 \le x \le 10 \quad (2) \\ 2x+3 & 11 \le x \le 24. \quad (3) \\ 6x+3 & x > 24. \quad (4) \end{cases}$$

8.

(b) $f(3) = 3^2 - 10 = 9 - 10 = -1.$

So $g(f(3)) = 4(-1) + 7 = \underline{3}$

$f(4) = 4^2 - 10 = 16 - 10 = 6$

So $g(f(4)) = 5 \times 6^3 =$

$= \underline{1080}.$

(a) $\underline{c = 33}.$ [Not needed in answer, but why?
It's the solution to $6^2 = \sqrt{6+3} + c$

$\Rightarrow 36 = \sqrt{9} + c$

$\Rightarrow 36 = 3 + c$

$\Rightarrow c = 33.$]

$f:$

x^2 if $0 \le x \le 6$ "SD1"

$+\sqrt{x+3} + 33$ if $x > 6$ "SD2"

I need to prove injectivity & surjectivity.

(i) Injectivity.

Note that for $0 \le x \le 6$, $0 \le f(x) \le 36$,

but for $x > 6$, $f(x) > \sqrt{6+3} + 33$
$= 36.$

In other words, the two subdomains have disjoint output spaces.

So in the injection proof we can focus on $f(x) = f(y)$ when x and y are both from the same subdomain.

Let x and y be arbitrary numbers $0 \le x, y \le 6$ such that $f(x) = f(y)$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = y$ (because both x, y assumed to be ≥ 0) ✓

Let x and y be arbitrary numbers such that $x, y > 6$ and $f(x) = f(y)$

$\Rightarrow +\sqrt{x+3} + 33 = +\sqrt{y+3} + 33$

$\Rightarrow +\sqrt{x+3} = +\sqrt{y+3}$

$\Rightarrow x = y$ ✓

So injective!!

Surjective?

- Let y be an arbitrary element of the co-domain $\mathbb{R}^+$, i.e. $y \geq 0$.

– If $y > 36$, take $x = (y-33)^2 - 3$.

Check: for such y, $y - 33 > 3$,

so $(y-33)^2 > 9$,

so $9 - 3 > 6$,

so $x > 6$,

$$\text{So } f\left((y-33)^2-3\right) = +\sqrt{(y-33)^2 - 3 + 3} + 33$$

$$= +\sqrt{(y-33)^2} + 33$$

$$= y - 33 + 33$$

$$= y \checkmark$$

– if $0 \leq y \leq 36$,
take $x = +\sqrt{y}$.

Check: for such y, $+\sqrt{y}$ is in the interval $[0, 6]$.

$$\text{So } f(+\sqrt{y}) = \left[+\sqrt{y}\right]^2 = y. \checkmark$$

So surjective!

$$\text{Inverse: } f^{-1}(y) = \begin{cases} (y-33)^2 - 3 & \text{if } y > 36 \\ +\sqrt{y} & \text{if } 0 \leq y \leq 36. \end{cases}$$

9.

I "guess" $c=6$. I will verify this by showing that for this choice the function is invertible.

Injective: Assume $f(x)=f(y)$ and prove $x=y$.

Note that for $0 \le x < 6$, $0 \le f(x) \le \frac{1}{2}(6)^2 = 18$.

for $x \geq 6$, $f(x) = 2x + c = 2x + 6 \geq 18$.

ie. outputs of the 2 subdomains are disjoint

So, if $f(x)=f(y)$, then x and y are both $0 \le x, y < 6$ or are both $\geqslant 6$.

case $0 \le x, y < 6$:

$$f(x) = f(y) \Rightarrow \tfrac{1}{2}x^2 = \tfrac{1}{2}y^2 \Rightarrow x^2 = y^2 \Rightarrow x = y \quad (\text{because } x, y \geqslant 0)$$ ✓

case $x, y \geqslant 6$:

$$f(x) = f(y) \Rightarrow 2x + 6 = 2y + 6 \Rightarrow x = y$$ ✓

So it <u>is</u> injective.

Surjective: for $0 \le y < 18$, take $x = +\sqrt{2y}$.

for $y \geqslant 18$, take $x = \frac{y-6}{2}$.

Observe that in this area, $0 \le 2y < 36$

$$\Rightarrow 0 \le \sqrt{2y} < 6$$

So $f(\sqrt{2y}) = \frac{1}{2}\left[\sqrt{2y}\right]^2 = \frac{1}{2} \times 2y = y$ ✓

Observe that here, $\frac{y-6}{2} \geqslant 6$, So $f\left(\frac{y-6}{2}\right) = 2\left[\frac{y-6}{2}\right] + 6 = y$ ✓ So it <u>is</u> surjective.

Inverse:

$$f^{-1}(x) = \begin{cases} +\sqrt{2x} & 0 \le x < 18 \\ \frac{x-6}{2} & x \geqslant 18. \end{cases}$$

answer!

(b) $g(0) = 4$
$g(1) = 3$
$g(2) = 2$
$g(3) = 1$
$g(4) = 0$.

So take
$$\boxed{\begin{aligned} f(0) &= 0 \\ f(1) &= 2 \\ f(2) &= 4 \\ f(3) &= 1 \end{aligned}}$$

(Check: $(g \circ f)(0) = g(f(0)) = g(0) = 4$ ✓
$(g \circ f)(1) = g(f(1)) = g(2) = 2$ ✓
$(g \circ f)(2) = g(f(2)) = g(4) = 0$ ✓
$(g \circ f)(3) = g(f(3)) = g(1) = 3$ ✓)