

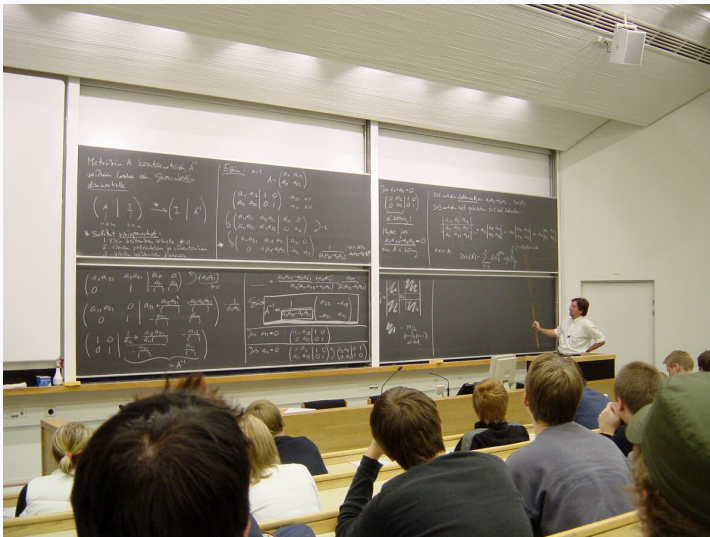
Calculus

Lecture 1: Functions and continuity

Otti D'Huys, Gijs Schoenmakers

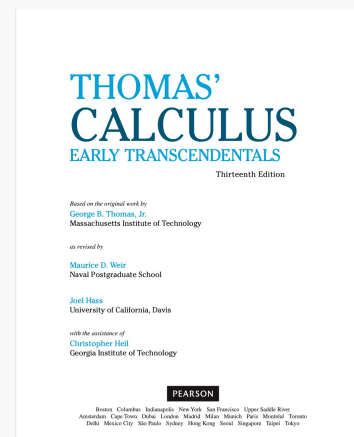
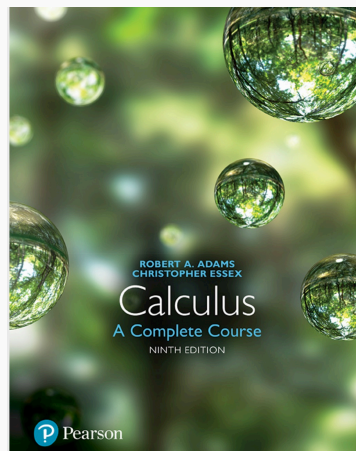
Calculus: Practicalities

- This class has 11 lectures, 8 tutorials, and a Q&A/revision lecture
 - In the tutorials you work on exercises and you have the opportunity to ask questions
- Lecturers: Otti D'Huys (also coordinator), Gijs Schoenmakers
- Teaching Assistants: Fivos Tzavellos, Ankie Fan, Ashkan Saber Karimi, Vitaly Tickovs, Spyridon Giagtzoglou, Riju Mukherjee, Zirui Wang, Jason Tsagkaris



Calculus: Practicalities

- Lecture materials (on Canvas):
 - (sometimes) a summary pdf
 - a scan of our handwritten preparation or lecture material
- Tutorial materials (on Canvas):
 - a list of tutorial exercises
 - pdf with their solutions
 - a checklist per course module, with all relevant concepts and an exhaustive list of useful exercises (no need to try them all...)
- Books: mainly Adams (9th ed) and Thomas (13th ed)



Calculus: Practicalities

- Evaluation = 100% final exam + 10% bonus quizzes

Exam:

- Closed book
- Formula sheet will be provided
- Calculators are not allowed

Quizzes

- 3 Canvas quizzes, worth 3.33 % each (bonus)
- Multiple choice and numerical questions
- The quiz remains open for a week, but once you start it, you have limited time to complete it.
- Every 2 weeks (more or less)
- How to get help?
 - **Anonymous** discussion boards on Canvas
 - tutorials!
 - No emails

Calculus: Course Contents

- Limits and continuity
- Differentiation
- Integration
- Sequences and series
- Basics of differential equations
- Basics of multivariate calculus

$\frac{\pi}{2}$, $y \geq 0$

p. 245 # 3, 5, 11, 15, 17, 19, 21, 23, 27, 31, 33, 35

(1) $D_x \ln \sqrt{x^4 + 4x + 1} = ?$ (Ans. $\frac{2x^3 + 2}{x^4 + 4x + 1}$)

(2) $D_x \ln |(x-2)^3(x-3)^5| = ?$ $\left(\frac{8x-19}{(x-2)(x-3)} \right)$

(3) $D_x \ln |x e^{-2x}| = ?$ $\left(\frac{1-2x}{x} \right)$

(4) $D_x \ln(\ln \sqrt{1+e^{2x}}) = ?$ $\left(\frac{2e^{2x}}{(1+e^{2x}) \ln(1+e^{2x})} \right)$

(5) $D_x (2x+1)^{\cos x} = ?$ $\left((2x+1)^{\cos x} \left[(-\sin x) \ln(2x+1) + \frac{\cos x}{2x+1} \cdot 2 \right] \right)$

(6) $D_x (|\ln x|^x) = ?$ $\left(|\ln x|^x \left[\ln |\ln x| + \frac{1}{\ln x} \right] \right)$

(7) If $x e^y = y e^x$, then $\frac{dy}{dx} = ?$
 $\left(\frac{y e^x - e^y}{x e^y - e^x} \right) :$

Why Calculus?

- Calculus was developed to describe motion (mainly by Isaac Newton and Gottfried Leibniz in the 17th century)
- It is a language to describe the world in a numerical way, in terms of functions and their rate of change.
 - Mathematical modelling, control theory, robotics,... all describe systems with differential equations.
 - Probability and statistics: a mathematical description of chance
 - Optimization: finding optimal (extreme) values
 - ...

Functions and continuity - Book chapters

Adams:

- P.1 Real numbers, intervals, absolute value
- P.2 Equation of a line
- P.3 Functions
- P.5. Combining functions
- P.6 Polynomials and Rational functions
- 1.4 Continuity


Real functions

- A function $f : D \rightarrow S$ on a set D into a set S is a rule that assigns a **unique** element $f(x) \in S$ to **each** element $x \in D$.

- Domain $D \rightarrow \mathbb{R}$, or $\mathbb{R} \setminus \{0\}$, $(1, 2)$

- Domain convention:

$\{x \in \mathbb{R} : f(x) \in \mathbb{R}\} \rightarrow$ if domain not mentioned

- Open interval: (a, b) $\{x \in \mathbb{R} : a < x < b\}$ 

- Closed interval: $[a, b]$ $\{x \in \mathbb{R} : a \leq x \leq b\}$ 

- Co-domain S : \mathbb{R}

- Range: $\{f(x) \mid x \in D\}$

Domain: check 3 things

* Domain $(\sqrt{x}) = [0, \infty)$

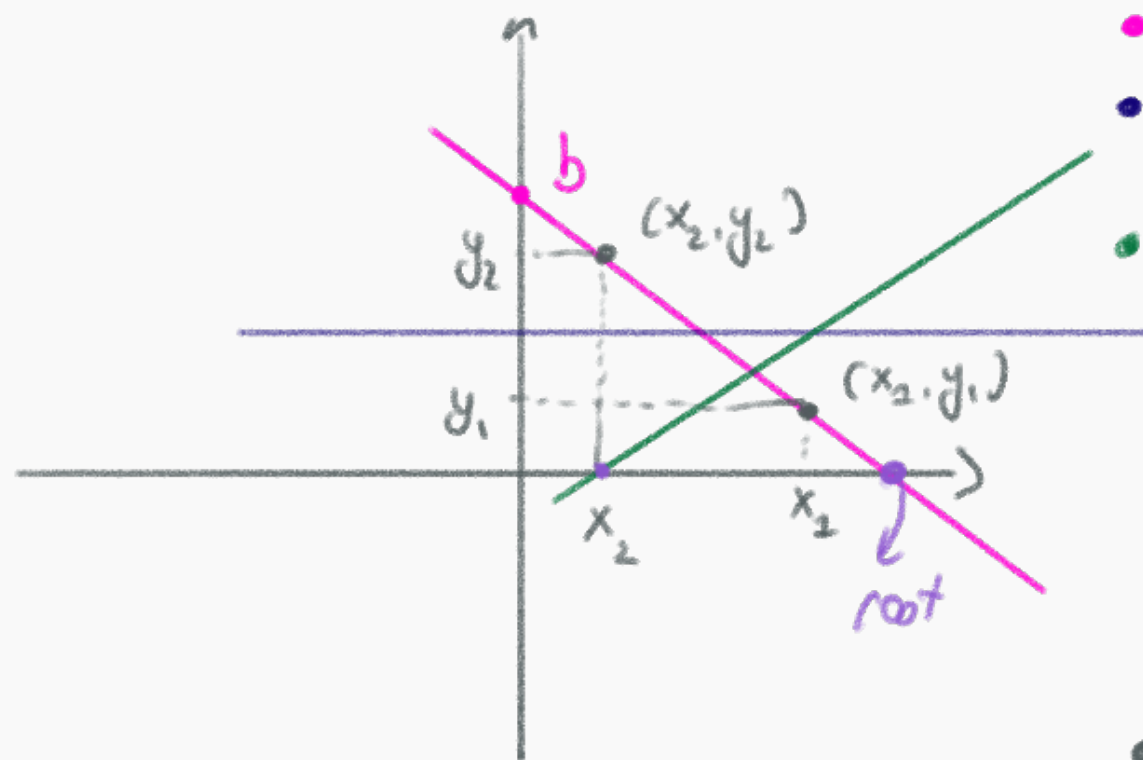
No roots of negative numbers

* Domain $(\frac{1}{x}) = \mathbb{R} \setminus \{0\}$
don't divide by 0

* Domain $(\ln(x)) = (0, \infty)$

Equation of a line

$$f(x) = \underset{\substack{\downarrow \\ \text{slope}}}{a}x + \underset{\substack{\rightarrow \\ \text{y-intercept}}}{b}$$



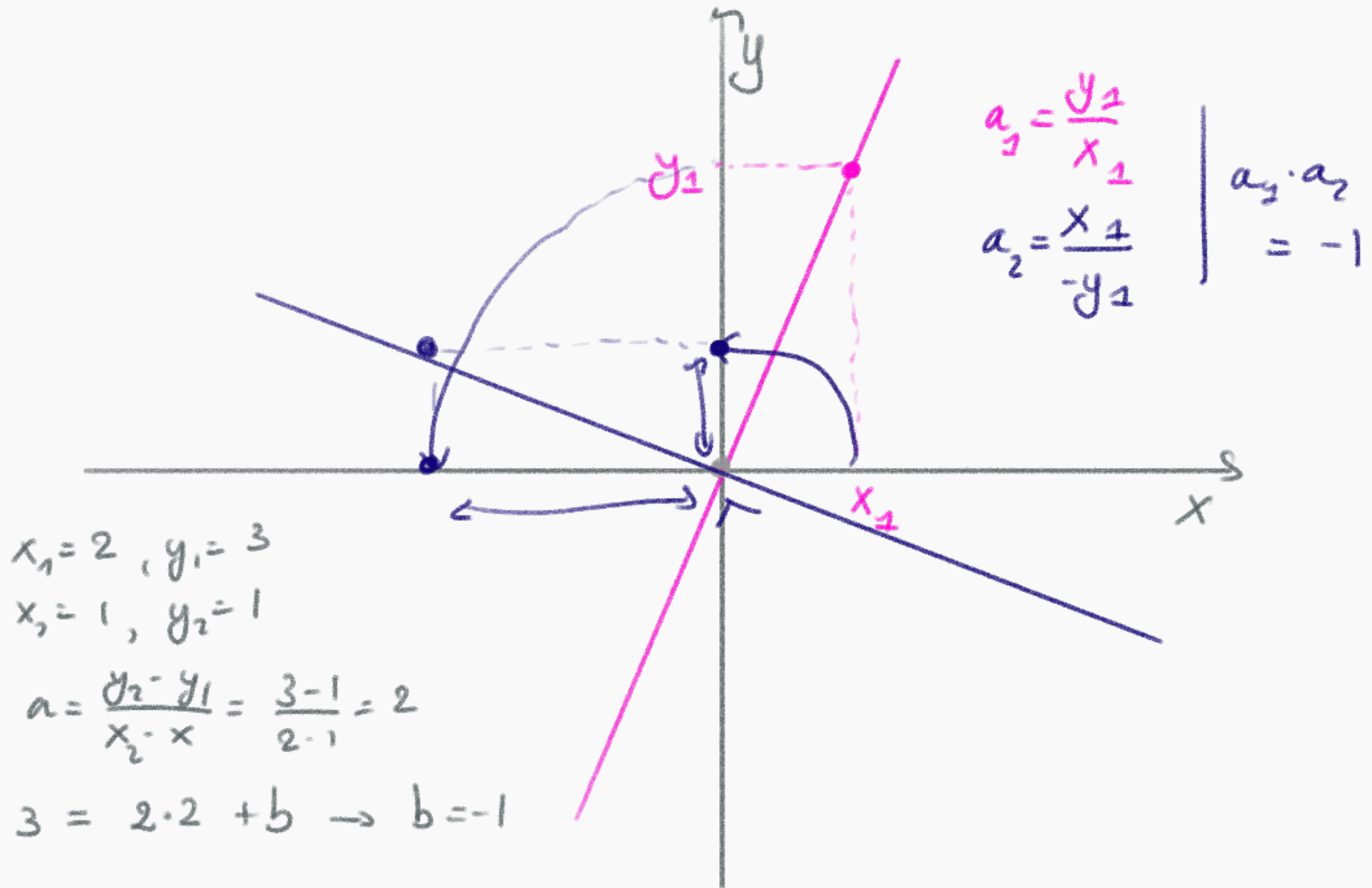
- negative slope $a < 0$
- $a = 0$ (zero slope constant function)
- $a > 0$

$$a = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad \rightarrow$$

$$-b = ax_1 - y_1$$

- lines with the same slope are parallel
- lines with $a_1 \cdot a_2 = -1$ are perpendicular.

Examples



Absolute value

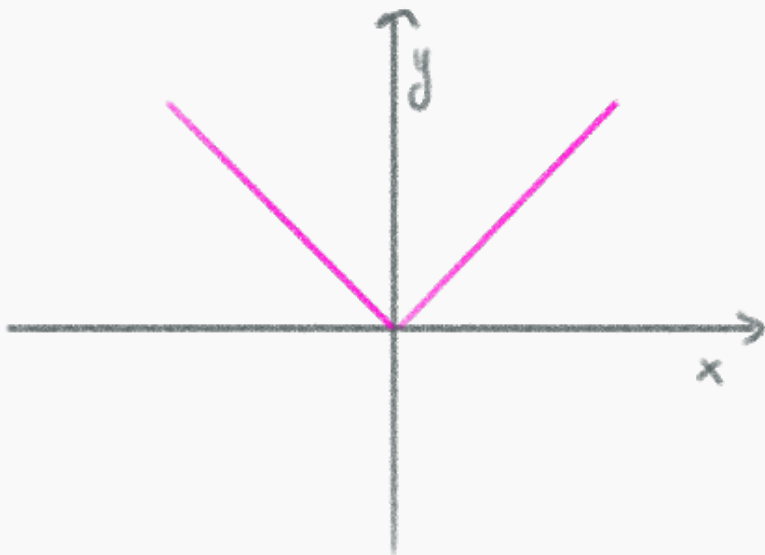
$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

↳ distance from 0

→ domain \mathbb{R}

$$|a \cdot b| = |a| |b|$$

$$|a+b| \leq |a| + |b| \quad (\text{triangle inequality})$$



Polynomials

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

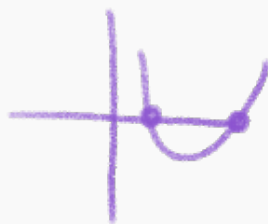
$P(x)$ $\hookrightarrow a_n \neq 0$

- Degree of a polynomial: n (highest degree)
- Root of a polynomial: $P(r) = 0$ $P(x) = (x-r)Q(x)$ \hookrightarrow degree $n-1$
- Number of (complex) roots of a polynomial: $\rightarrow n$ (degree)
 \hookrightarrow a root with multiplicity m appears m times

$n=1$ linear function

$n=0$ constant function

$n=2$ $a_2 x^2 + a_1 x + a_0 \rightarrow$ parabola



$$x^3 - 2x^2 + 2x - 1$$

$$= (x-1)Q(x)$$

$$P(x) = x^2 - 3x + 2 \quad r_1 = +2 \quad r_2 = +1$$

$$P(r_1) = P(+2) = 0$$

$$\rightarrow P(x) = (x-r_1)(x-r_2) = \underline{(x-2)(x-1)}$$

Rational functions

$$f(x) = \frac{P(x)}{Q(x)}$$

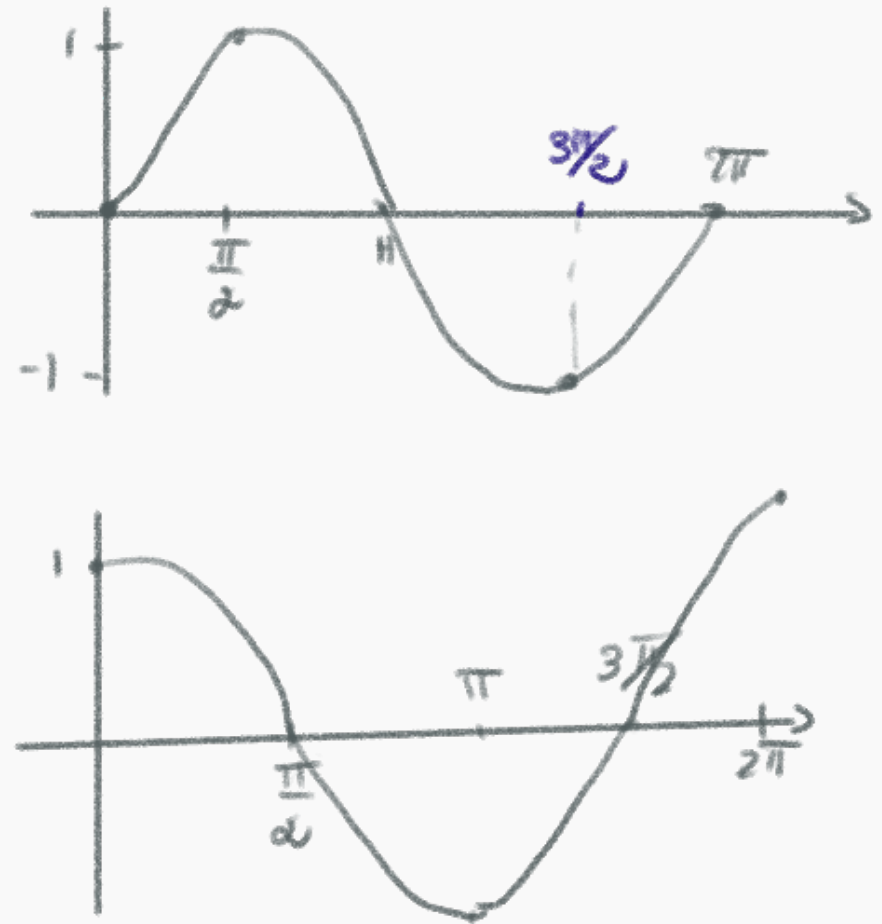
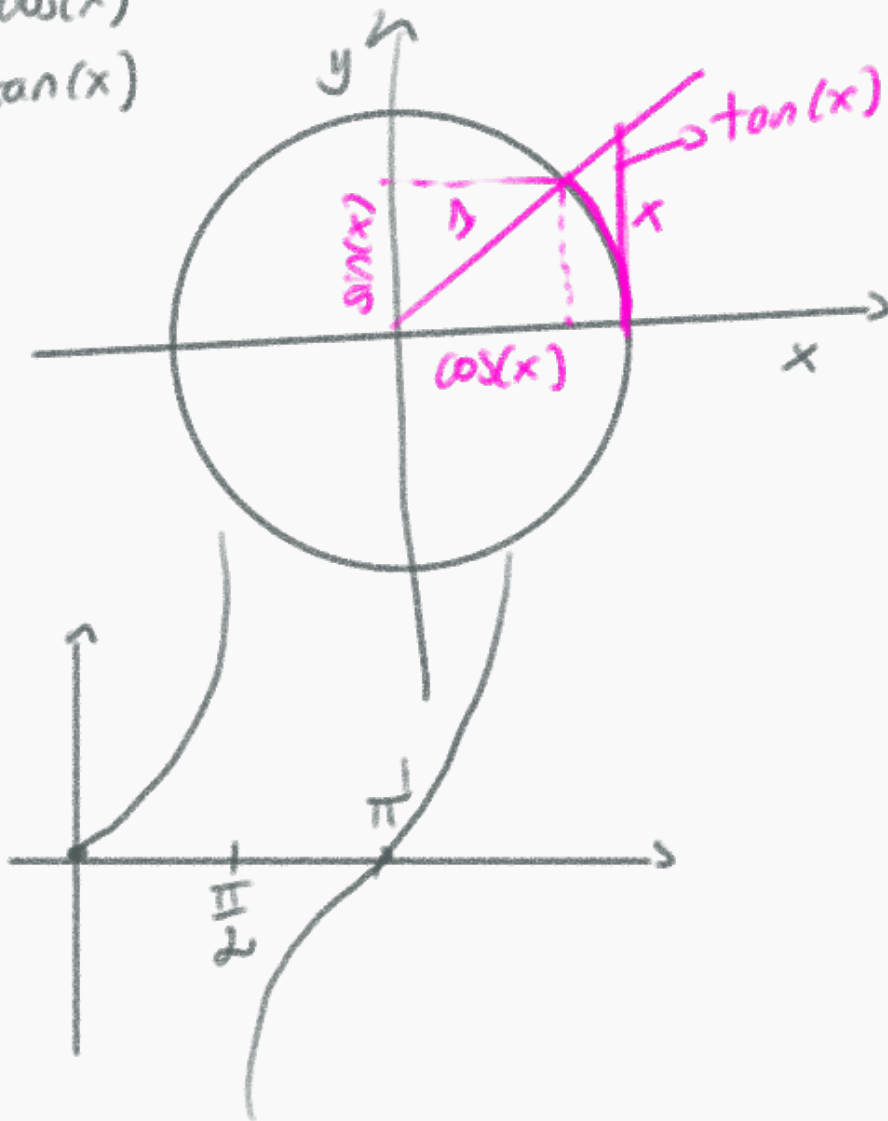
$P(x)$, $Q(x)$ polynomials

domain = $\mathbb{R} \setminus \{ \text{roots of } Q(x) \}$
↳ poles

Trigonometric functions

$\sin(x)$
 $\cos(x)$
 $\tan(x)$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$



Even and odd functions

- Even functions \rightarrow mirror around y-axis

$$f(x) = f(-x)$$

$$|x|, \cos(x), x^2$$

- Odd functions \rightarrow mirror around origin

$$f(x) = -f(-x)$$

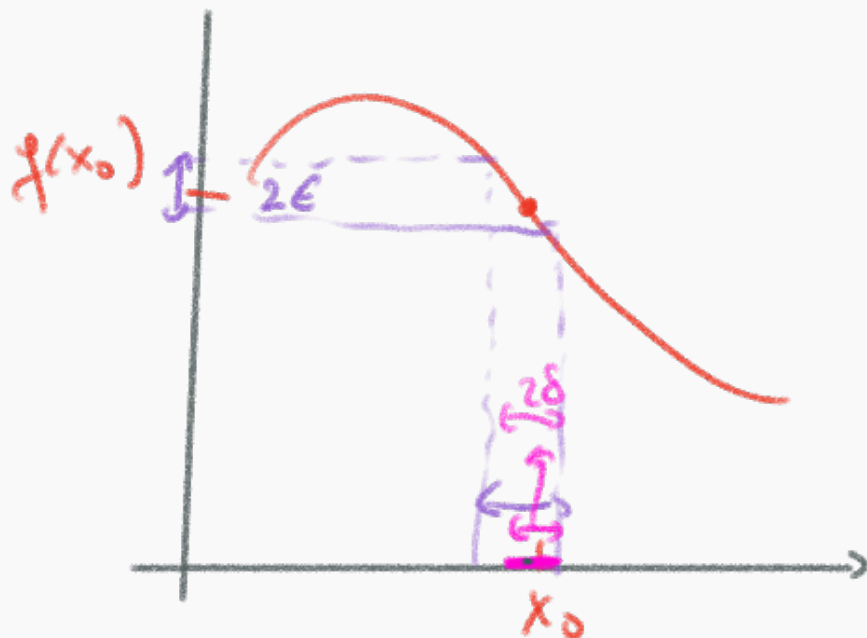
$$\sin(x), x^3, \sqrt[3]{x}$$

Continuity

A function $f(x)$ is **continuous** at an interior point x_0 of its domain if, for all points x in the domain,

$$\forall \epsilon > 0, \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

- (there are no jumps)
- if we x approaches x_0 , then $f(x)$ approaches $f(x_0)$

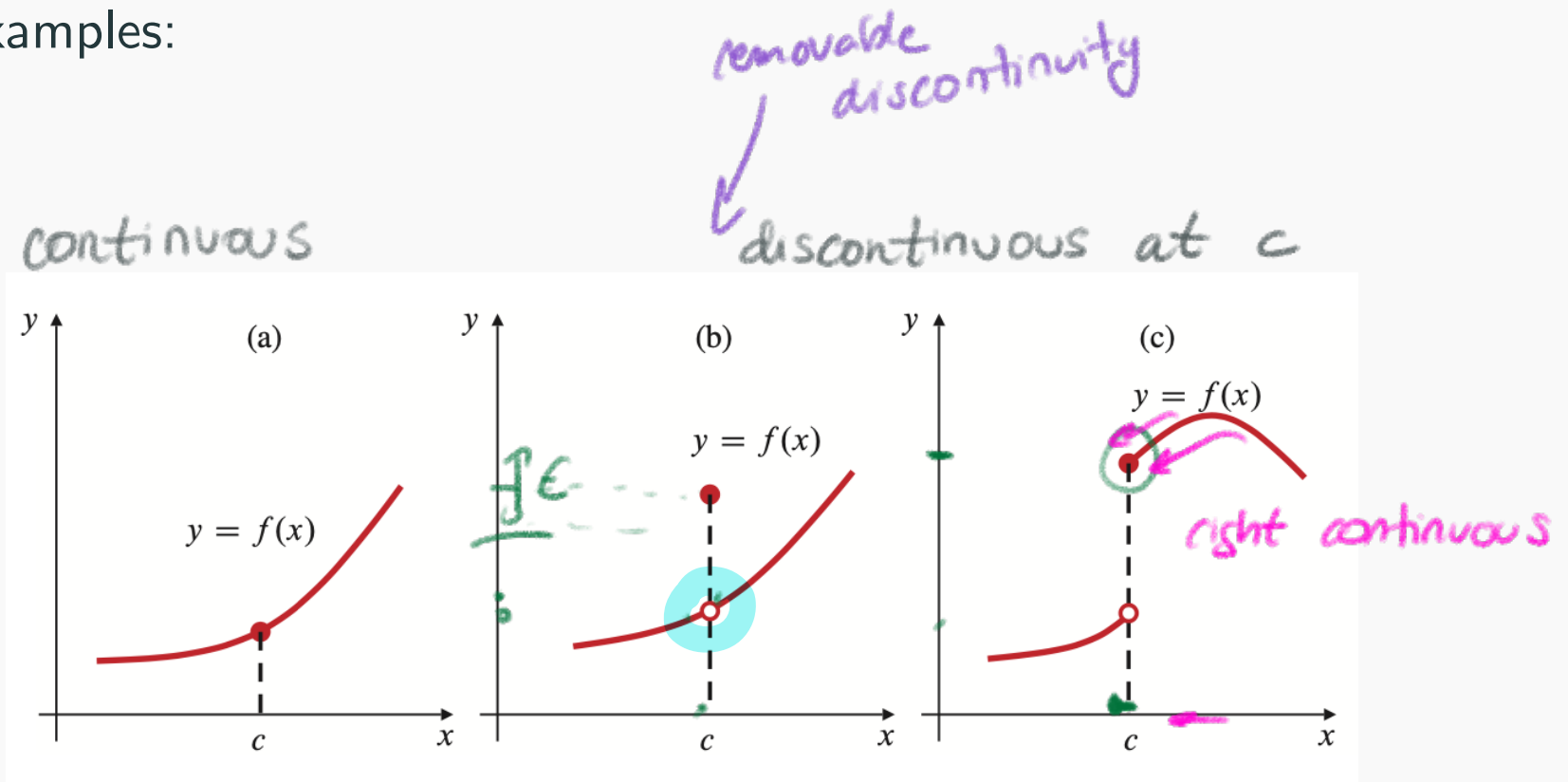


- Every "typical" function is continuous
- Every sum / difference / product / composition of continuous functions is continuous.

Continuity

- A function $f(x)$ is **discontinuous** at c if *there is a "jump" at $x=c$ (and $f(c)$ is defined)*

- Examples:



Note: (in this course) f can only be discontinuous at c if c is in the domain of f , i.e. if $f(c)$ exists (in this course). So $f(x) = \frac{1}{x}$ is not discontinuous at $x = 0$, it is undefined.

Left and right continuity

A function $f(x)$ is

- right continuous at c if $f(x)$ approaches $f(x_0)$ if x approaches x_0 from the right
- left continuous at c if $f(x)$ approaches $f(x_0)$ if x approaches x_0 from the left
- continuous at c if it is both right and left continuous at c .

• right continuous

$$\forall \epsilon > 0 \exists \delta > 0 : 0 < x - x_0 < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

• left continuous

$$\forall \epsilon > 0 \exists \delta > 0 : 0 < x_0 - x < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

Continuous functions on an interval

A function $f(x)$ is

- continuous on an interval $[a, b]$ if
 - it is continuous at all interior points
 - left continuous at b
 - right continuous at a
- piecewise continuous on $[a, b]$ if there are a finite number of discontinuities on $[a, b]$