

Exercises

1	2	3	4	5	6	7	8	9	10
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Surname, First name

Calculus Test

Example exam

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
7	7	7	7	7	7	7
8	8	8	8	8	8	8
9	9	9	9	9	9	9
0	0	0	0	0	0	0

a	<input checked="" type="radio"/>	c	d	e	f	→ b
a	b	<input checked="" type="radio"/>	d	e	f	→ c
<input checked="" type="radio"/>	b	c	<input checked="" type="radio"/>	e	f	→ a

Fill in your answer(s) to the multiple-choice questions as shown above (circles = one correct answer).

Program:

Course code: KEN/BCS 14460

Examiners: Otti D'Huys, Gijs Schoenmakers

Date/time:

Format: Written, closed book

Allowed aids: A formula sheet is attached to the exam.

Instructions to students:

- The exam consists of 9 questions on 12 pages.
- Fill in your name and student ID number on the cover page and tick the corresponding numerals of your student number in the table (top right cover page).
- Answer every question in the reserved space below the question. **Do not write outside the reserved space or on the back of pages, this will not be scanned and will NOT be graded!** As a last resort if you run out of space, use the extra answer space at the end of the exam.
- *In no circumstance write on or near the QR code at the bottom of the page!*
- Ensure that you properly motivate your answers.
- Only use black or dark blue pens, and write in a readable way. Do not use pencils.
- Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in the space reserved for the answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- You have to return all pages of the exam. You are not allowed to take any sheets, even blank, home.
- Good luck!

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Question 1

- 6p 1a For $f(x)$ and $g(x)$ differentiable functions, let $f(1) = 1$, $f(-2) = -2$, $f'(1) = -3$, $f'(-2) = 3$, $g(1) = -2$ and $g'(1) = 1$.

Find $\lim_{x \rightarrow 1} \frac{f(g(2-x)) - g(f(x^2))}{x-1}$

- (a) $-\infty$
 (b) -1
 (c) 0
 (d) 3
 (e) -6
 (f) $+\infty$
 (g) -9
 (h) The limit does not exist
 (i) None of the above

$$\lim_{x \rightarrow 1} \frac{f(g(2-x)) - g(f(x^2))}{x-1} =$$

$$\lim_{x \rightarrow 1} \frac{f'(g(1)) \cdot g'(1) \cdot (-1) - g'(f(1)) \cdot f'(1) \cdot 2}{1} =$$

$$3 \cdot 1 \cdot (-1) - 1 \cdot (-3) \cdot 2 = -3 - 6 = -9$$

- 6p 1b Calculate $\lim_{x \rightarrow 0^-} \frac{x}{\sqrt{x^2 - x}}$

- (a) 4
 (b) $-\frac{1}{2}$
 (c) $\frac{1}{2}$
 (d) -1
 (e) $-\infty$
 (f) 0
 (g) $+\infty$
 (h) 1
 (i) The limit does not exist
 (j) None of the above

$$\lim_{x \rightarrow 0^-} \frac{x}{\sqrt{x^2 - x}} = \lim_{x \rightarrow 0^-} \frac{x}{-x\sqrt{1 - \frac{1}{x}}} = 0$$

- 6p 1c Calculate $\int_0^1 x \ln(x) dx$

- (a) $-\frac{1}{2}$
 (b) This integral diverges to $-\infty$
 (c) 0
 (d) This integral diverges to $+\infty$
 (e) None of the above

$$\int_0^1 x \ln(x) dx = \lim_{a \rightarrow 0^+} \int_a^1 x \ln(x) dx$$

$$= \lim_{a \rightarrow 0^+} \left(\left[\frac{x^2}{2} \ln(x) \right]_a^1 - \int_a^1 \frac{x}{2} dx \right)$$

$$= \lim_{a \rightarrow 0^+} \left(\frac{a^2}{2} \ln(a) - \left[\frac{x^2}{4} \right]_a^1 \right)$$

$$= 0 - \frac{1}{4}$$

$\ln(x) = u$
 $du = \frac{1}{x} dx$
 $dv = x dx$
 $v = \frac{x^2}{2}$

$$\lim_{a \rightarrow 0^+} \frac{a^2}{2} \ln(a) = \lim_{a \rightarrow 0^+} \frac{1}{2} \cdot \frac{\ln(a)}{\frac{1}{a^2}}$$

$$\stackrel{H}{=} \lim_{a \rightarrow 0^+} \frac{1}{2} \cdot \frac{\frac{1}{a}}{-\frac{2}{a^3}} = 0$$

Question 2

- 16p 2 Find all the asymptotes of the function $f(x) = \frac{1}{\sqrt{x^2+3x}+x}$. For the horizontal and oblique asymptotes (if any), specify whether they are one-sided or two-sided. For the vertical asymptotes (if any), specify how the function approaches the asymptotes (i.e. whether the function goes to $+\infty$ or $-\infty$ on the left or right side of the asymptotes.)

Horizontal asymptotes

$$1) \lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{x^2+3x}+x} = 0$$

\rightarrow one-sided horizontal asymptote $y = 0$

$$\begin{aligned} 2) \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{1}{\sqrt{x^2+3x}+x} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+3x}-x}{(\sqrt{x^2+3x}+x)(\sqrt{x^2+3x}-x)} \\ &= \lim_{x \rightarrow -\infty} \frac{-x\sqrt{1+\frac{3}{x}}-x}{3x} = \lim_{x \rightarrow -\infty} \frac{-\sqrt{1+\frac{3}{x}}-1}{3} = -\frac{2}{3} \end{aligned}$$

\rightarrow one-sided horizontal asymptote $y = -\frac{2}{3}$

(so there are no oblique asymptotes)

Vertical asymptotes

$$\rightarrow \text{pdes? } \sqrt{x^2+3x}+x=0 \Leftrightarrow -x=\sqrt{x^2+3x}$$

$$\Rightarrow x^2 = x^2 + 3x \Rightarrow x = 0$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{x^2+3x}+x} = +\infty$$

$\lim_{x \rightarrow 0^-} f(x)$ does not exist, since $f(x)$ is not defined for

$$x^2+3x < 0 \Leftrightarrow -3 < x < 0$$

\rightarrow vertical asymptote $x = 0$

Question 3

- 10p 3 Does the function $f(x) = x \ln(x)$ have a global (absolute) minimum and maximum on $(0, \infty)$?
- If so, for which value(s) of x ?
 - If not, motivate why you can exclude the existence of a global extremum.

• Domain $(0, \infty)$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{\frac{1}{x}} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$$

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} x \ln(x) = +\infty$$

• Critical points? $f'(x) = \ln(x) + 1$

$$\Rightarrow f'(x) < 0 \text{ for } \ln(x) < -1 \Leftrightarrow 0 < x < e^{-1} \\ (f(x) \text{ decreases})$$

$$f'(x) > 0 \text{ for } \ln(x) > -1 \Leftrightarrow x > e^{-1} \\ (f'(x) \text{ increases})$$

$\rightarrow f(x)$ has a ^{local} minimum at $x = e^{-1}$, $f(e^{-1}) = -e^{-1}$

\hookrightarrow this is a global minimum, since $f(e^{-1}) < 0$
 $f(e^{-1}) < +\infty$

$\rightarrow f(x)$ does not have a global maximum, since

$$\lim_{x \rightarrow \infty} f(x) = +\infty$$

Question 4

10p 4 Evaluate the following integral: $\int_0^{\frac{\pi}{2}} (\sin(x))^3 dx$

$$\int_0^{\frac{\pi}{2}} \sin^3(x) = \int_0^{\frac{\pi}{2}} \sin^2(x) (\sin(x) dx) = - \int_1^0 (1-u^2) du = \int_0^1 (1-u^2) du$$

$$u = \cos(x) \quad u(0) = 1$$

$$du = -\sin(x) dx \quad u\left(\frac{\pi}{2}\right) = 0$$

$$= \left[u - \frac{u^3}{3} \right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

Question 5

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{e^{n+n}}{e^{2n}-n^2}$

$$\frac{e^{n+n}}{e^{2n}-n^2} = \frac{e^{n+n}}{(e^n-n)(e^n+n)} = \frac{1}{e^n-n}$$

1.5p 5a The series $\sum_{n=1}^{\infty} a_n$ diverges. We can conclude this by comparing to the series $\sum_{n=1}^{\infty} b_n$ with $b_n = \frac{1}{n}$ and $|a_n| \leq b_n$ for all $n \in \mathbb{N}$

(a) True ☒ False

you can only prove divergence if $a_n \geq k \cdot b_n$ and $\sum b_n$ diverges

1.5p 5b The series $\sum_{n=1}^{\infty} a_n$ converges. We can conclude this by comparing to the series $\sum_{n=1}^{\infty} b_n$ with $b_n = e^{-n}$ and $a_n \leq b_n$ for all $n \in \mathbb{N}$

(a) True ☒ False

$$a_n > b_n$$

1.5p 5c The series $\sum_{n=1}^{\infty} a_n$ converges. We can conclude this because $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = e^{-1}$

☒ True (b) False

1.5p 5d The series $\sum_{n=1}^{\infty} a_n$ converges. We can conclude this because $\lim_{n \rightarrow \infty} a_n = 0$

(a) True ☒ False

you cannot conclude anything about $\sum a_n$ if $a_n \rightarrow 0$

Question 6

- 10p 6 Determine which values of $x \in \mathbb{R} \setminus \{0\}$, the given series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} n \left(\frac{2}{x} \right)^n$$

1) Check for absolute convergence, consider $\sum n \left| \frac{2}{x} \right|^n$

2) Ratio test

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \left| \frac{2}{x} \right|^{n+1-n} = \lim_{n \rightarrow \infty} \frac{n+1}{n} \left| \frac{2}{x} \right| = \left| \frac{2}{x} \right|$$

\Rightarrow if $\left| \frac{2}{x} \right| < 1$, the series is absolutely convergent

$$\Leftrightarrow x < -2 \quad \text{or} \quad x > 2$$

\Rightarrow if $\left| \frac{2}{x} \right| > 1$, the series (and sequence) diverge.

$$\Leftrightarrow -2 < x < 2$$

\Rightarrow if $\left| \frac{2}{x} \right| = 1$, no conclusion

\rightarrow we treat $x=2$ and $x=-2$ separately

3) $x=2 \rightarrow \sum n$ diverges

$x=-2 \rightarrow \sum n(-1)^n$ diverges

4) Summary $\sum n \left(\frac{2}{x} \right)^n$ is absolutely convergent for

$$x < -2 \quad \text{and} \quad x > 2$$

$\sum n \left(\frac{2}{x} \right)^n$ diverges for $-2 \leq x \leq 2$

Question 7

10p 7 Find the solution $y = y(x)$ to the given initial value problem.

$$\begin{cases} y' + y = x \\ y(0) = 3 \end{cases}$$

This is a linear, first order, non homogeneous ODE.

1) Homogeneous equation.

$$y' + y = 0 \Rightarrow y' = -y \Rightarrow \int \frac{dy}{y} = -\int dx \Rightarrow \ln|y| = -x + C \\ \Rightarrow y_H(x) = h \cdot e^{-x}$$

2) Parameter variation

$$\text{assume } y(x) = h(x) \cdot e^{-x}$$

$$y' + y = h'(x)e^{-x} - \cancel{h(x)e^{-x}} + \cancel{h(x)e^{-x}} = x$$

$$\Rightarrow h'(x) = xe^x$$

$$\Rightarrow h(x) = \int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

$$\begin{matrix} u = x & du = dx \\ dv = e^x dx & v = e^x \end{matrix}$$

$$\Rightarrow y(x) = h(x)e^{-x} = (xe^x - e^x + C) \cdot e^{-x} = x - 1 + C \cdot e^{-x}$$

$$y(0) = -1 + C = 3 \Rightarrow C = 4$$

$$\boxed{y(x) = x - 1 + 4 \cdot e^{-x}}$$

$$y' + y = x \quad y' = 1 - 4e^{-x} \quad y' + y = (x - 1 + 4e^{-x}) + (1 - 4e^{-x}) = x$$

Question 8

10p **8** For the function $f(x, y) = \sin(2x - y)$, give the equation of the tangent plane at $(0, 0, f(0, 0))$

Tangent plane equation.

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

Here : $f(0, 0) = 0$

$$a = 0, b = 0$$

$$\frac{\partial f}{\partial x} = 2 \cos(2x - y) \rightarrow \frac{\partial f}{\partial x}(0, 0) = 2$$

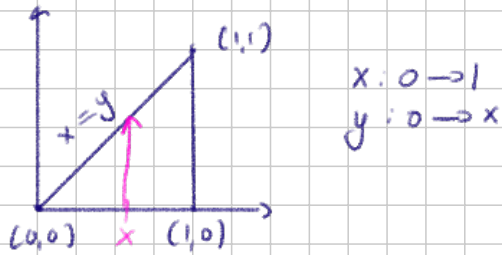
$$\frac{\partial f}{\partial y} = -\cos(2x - y) \rightarrow \frac{\partial f}{\partial y}(0, 0) = -1$$

$$\rightarrow z = 2x - y$$

Question 9

10p 9 Evaluate the double integral over a triangle T with vertices $(0,0)$, $(1,0)$, $(1,1)$:

$$\iint_T e^{2x+y} dA$$



$$\iint_T e^{2x+y} dA = \int_0^1 \int_0^x e^{2x+y} dy dx$$

$$A(x) = \int_0^x e^{2x+y} dy = e^{2x} \int_0^x e^y dy = e^{2x} [e^y]_0^x = e^{3x} - e^{2x}$$

$$\int_0^1 A(x) dx = \int_0^1 (e^{3x} - e^{2x}) dx = \frac{1}{3} [e^{3x}]_0^1 - \frac{1}{2} [e^{2x}]_0^1$$

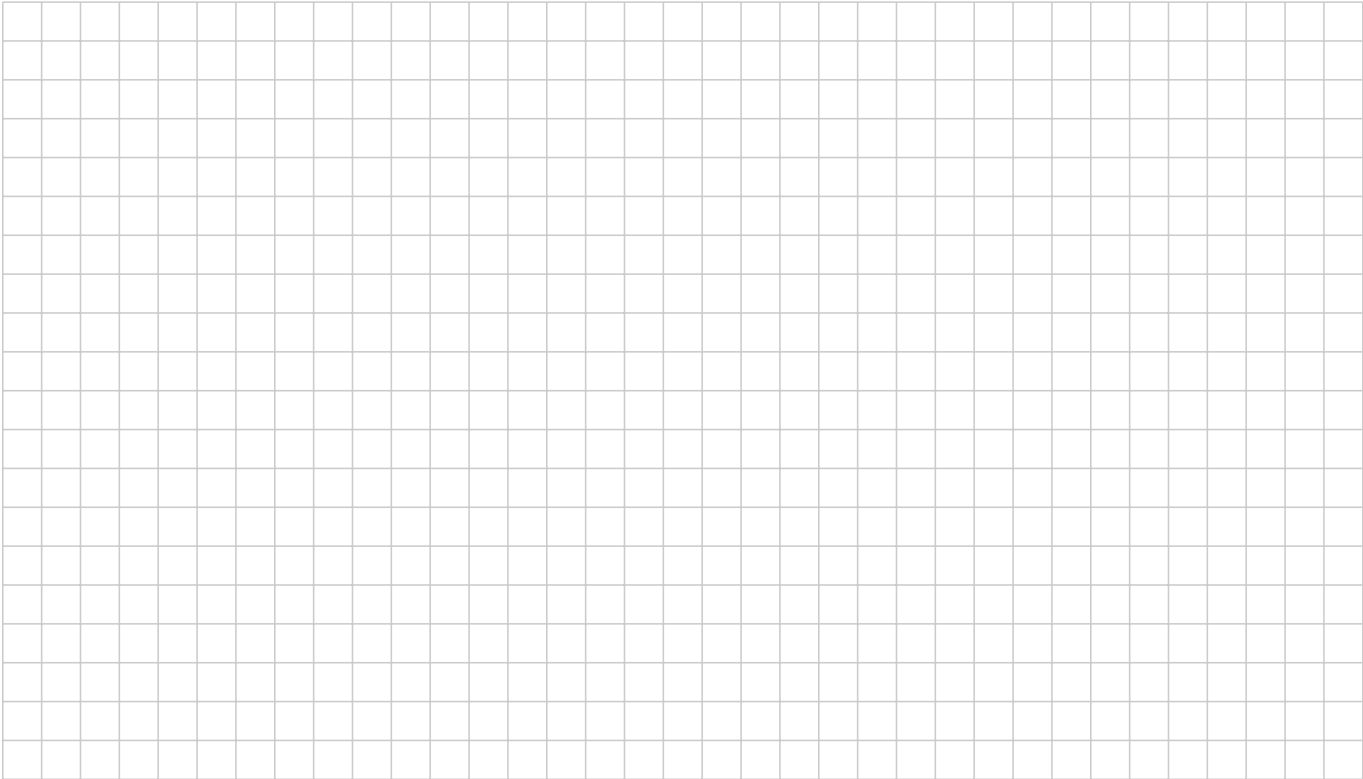
$$= \frac{1}{3} (e^3 - 1) - \frac{1}{2} (e^2 - 1) = \frac{e^3}{3} - \frac{e^2}{2} + \frac{1}{6}$$

10a

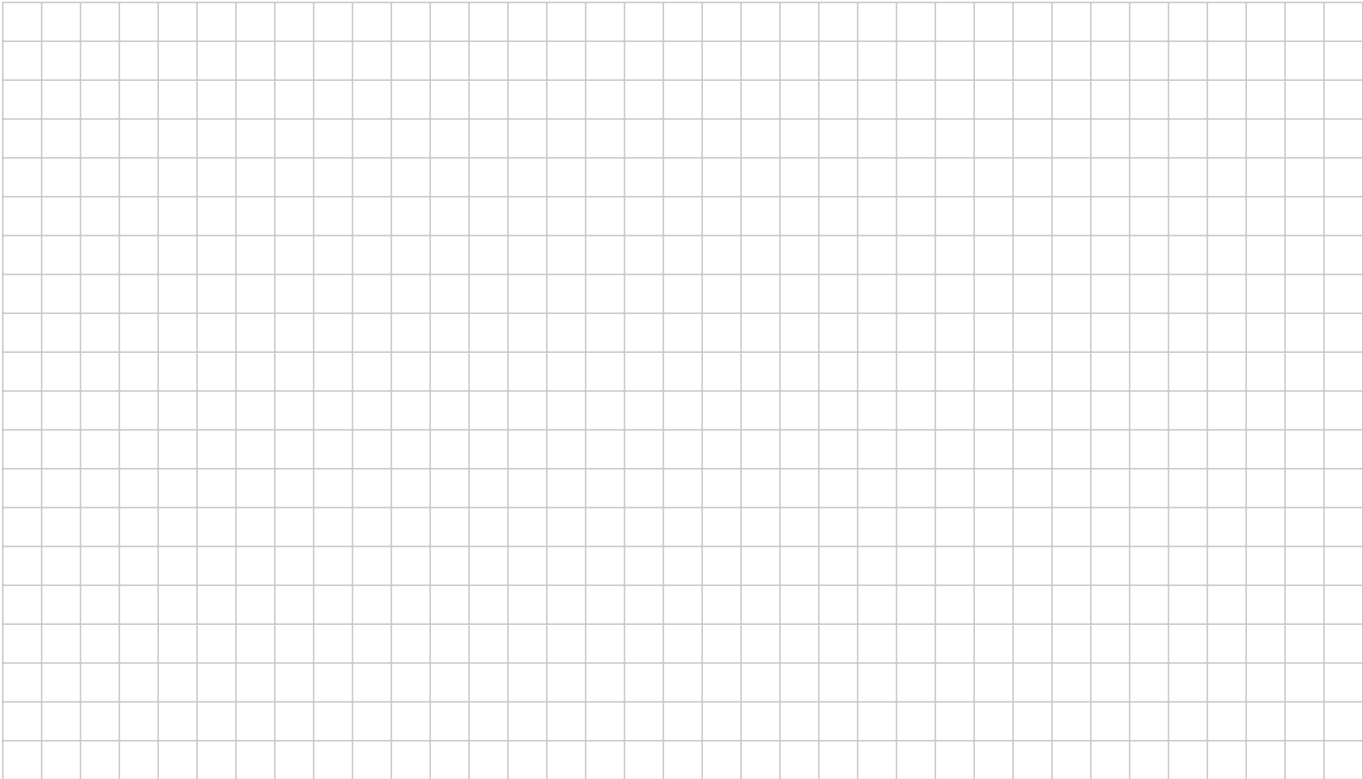
10b



10c



10d



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