Set theory: overview (so far)

- Concepts: set, elements of a set, cardinality
- Subsets: a set A is a subset of set B if every element of A is an element of B.
- Set operations: intersection, union, complement, difference
- Associative, distributive, de Morgan laws
- Proofs with sets

Today:

- Proofs with sets: another example
- Power sets
- Partitions
- Product sets

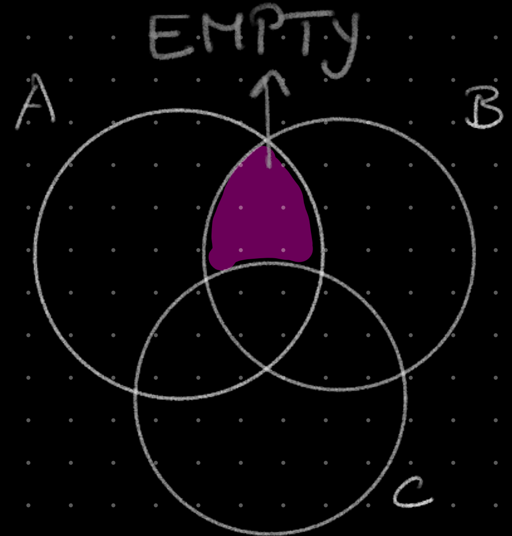
Book: Chapter 2, sections 2.4 and 2.6

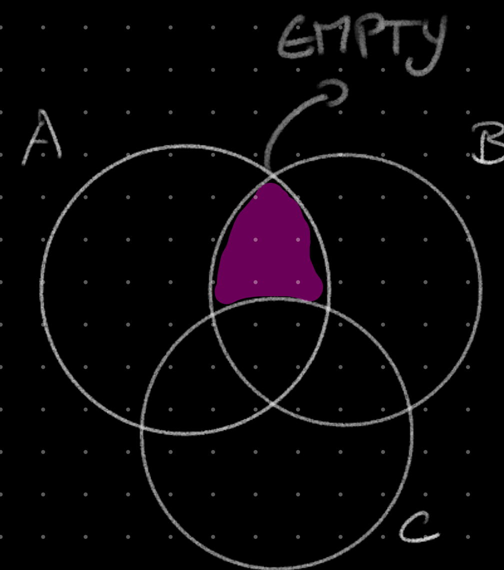
Proofs with sets: recap

- To prove that $A \subseteq B$, show that $\forall x \in A : x \in B$
- To prove that $A = B$, $A \subseteq B \land B \subseteq A$
- Statements can be proved by (a combination of)
  - using the definitions
  - using laws
  - converting to propositional logic.
- To get an intuition, draw a Venn diagram first.

For all sets A, B, C, $(B \cap (A^c \cup C)^c = \emptyset) \Leftrightarrow (A \subseteq B^c \cup C)$.

$B \cap (A \cap C^c) = \varphi$





↳ we try to prove that the statement is true (identical venn diagrams)

For all sets A, B, C, $(B \cap (A^c \cup C)^c = \emptyset) \Leftrightarrow (A \subseteq B^c \cup C)$.

$\Rightarrow B \cap A \cap C^c = \varphi \Rightarrow A \subseteq B^c \cup C$

Let $x \in A$

case 1: $x \in C$. Then $x \in C \vee x \notin B$

$\Rightarrow x \in B^c \cup C$

case 2: $x \notin C$

assume $x \in B$. Then $x \in A$, $x \in B \wedge x \notin C$

$x \in (B \cap A \cap C^c) \rightarrow$ empty!!

$\hookrightarrow$ CONTRADICTION

$\Rightarrow$ therefore $x \notin B \rightarrow x \in B^c \vee x \in C$

$\Rightarrow x \in B^c \cup C$

$\Leftarrow (A \subseteq B^c \cup C) \Rightarrow B \cap A \cap C^c = \emptyset$

Proof by contradiction: Assume $B \cap A \cap C^c \neq \emptyset$.

Then there is an $x \in B \cap A \cap C^c$

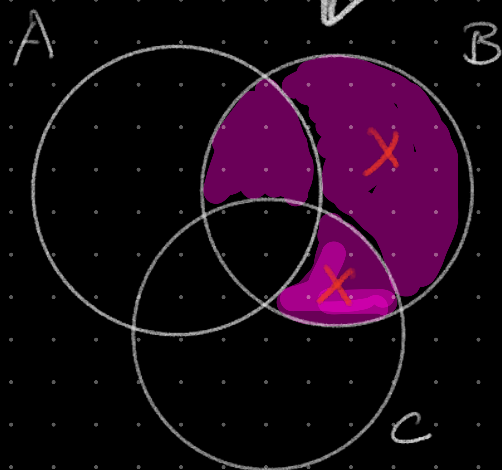
then $x \in B$, $x \in A$ and $x \notin C$

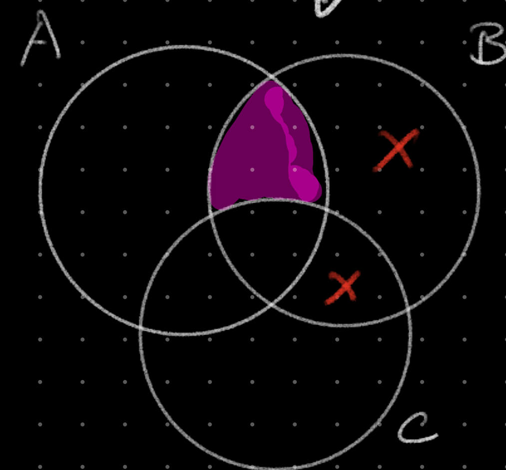
so $(x \in B^c \vee x \in C)$ and $(x \in B \wedge x \notin C)$

$\rightarrow$ contradiction! So, there is no $x \in B \cap A \cap C^c$

For all sets A, B, C, $(B \cap (A^c \cup C^c) = \emptyset) \Leftrightarrow (A \subseteq B^c \cup C)$.

empty parts are shaded.





↳ since the Venn diagrams are different, we try to show that the statement is false.

Choose a counter-example with an element in the area that is different in both Venn-diagrams (x)

$A = \{1\}$, $B = \{2,3\}$, $C = \{3,4\}$ ($U = A \cup B \cup C$ here)

then $B^c = \{1,4\}$, $A^c = \{2,3,4\}$, $C^c = \{1,2\}$

so $A \subseteq B^c \cup C$, but $B \cap (A^c \cup C^c) = B \cap \{1,2,3,4\} = \{2,3\}$
$\neq \emptyset$

Power sets

The **power set** of a set A is the set of all subsets of A

DEF:

$$B \in \mathbb{P}(A) \iff B \subseteq A$$

$$A = \{1, 2, 3\}$$

$$\mathbb{P}(A) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{\}, \{1,2,3\}\}$$

$$|\mathbb{P}(A)| = 8$$

$$B = \{1, \emptyset\} \quad \overset{\mathbb{P}(B)}{=} \{\varphi, \{1, \emptyset\}, \{1\}, \{\varphi\}\} \qquad |\mathbb{P}(B)| = 4$$

$$|\mathbb{P}(A)| = 2^{|A|}$$

1	2	3	
IN	IN	IN	$\{1, 2, 3\}$
IN	IN	OUT	$\{1, 2\}$
IN	OUT	IN	$\{1, 3\}$

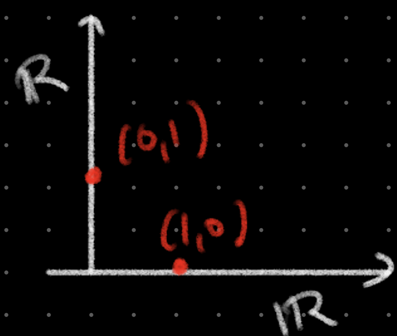
Product sets

For two sets A and B, the product set A×B is defined as

$\{(a,b)$ with $a \in A$ and $b \in B\}$

ordered paire!

ex. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$



$A = \{$steak, risotto, paella$\}$

$B = \{$tiramisu, dame blanche$\}$

$A \times B = \{$(paella, dame blanche), ( ) . . . $\}$

$|A \times B| = |A| \cdot |B|$

$A \times B \stackrel{?}{=} B \times A$ → when $A = B$

→ when $A = \emptyset$, $A \times B = B \times A = \emptyset$
$B = \emptyset$ $A \times B = B \times A = \emptyset$

$A = \{1, 2, 3\}$

$B = \{0, 1\}$

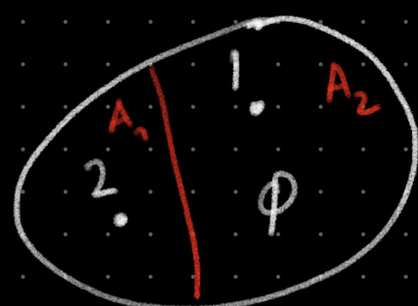
$A \times B = \{(1,0), (1,1), (2,0), (2,1), (3,0), (3,1)\}$

Set partitions

A partition of a set A is a set of subsets of A
$\{A_1, A_2, A_3, \ldots\}$ such that

1) $A_i \neq \varphi \quad \forall i$
2) $A_i \cap A_j = \varphi$ for $i \neq j$
3) $A_1 \cup A_2 \cup \ldots \cup A_n = A$

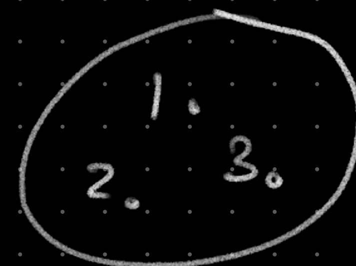
$A = \{1, 2, 3\}$

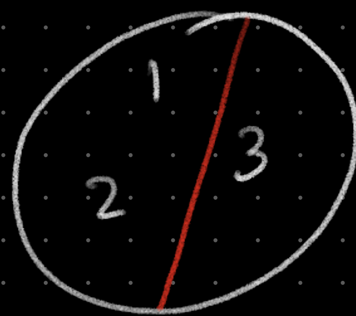


$\{\{2\}, \{1, \varphi\}\}$

$\{\{1\}, \{2, 3\}\}$

$\{\{1\}, \{2\}, \{3\}\}$





$\{\{1,2,3\}\}$

* NOT partitions. $\{\varphi, \{1\}, \{2,3\}\}$

$\{\{1,2\}, \{2,3\}\}$

$\{\{1\},\{2\}\}$

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

↳ not exam material!

$B_0 = 1$, $\{\}$

→ How many partitions? B_n (number of partitions of a set with n elements.)

$B_1 = 1$ $B_3 = 5$

$B_2 = 2$

Checklist: set theory

- Do you know how set membership works?
- Do you understand the meaning of the set operators (complement, intersection, union, difference)
- Do you know how to use Venn diagrams to develop an intuition
- Do you understand the concept and definition of subset?
- Do you know how to prove that two sets are equal?
- Do you understand how to use the associative, distributive and de Morgan laws? Can you prove them?
- Do you know how to use power sets? Can you formulate the power set of a (finite) set?
- Do you understand how set product works?
- Do you know what a partition is?