

## Exercises 2.10

$$(15) \int \cos(2x) dx = \frac{1}{2} \int 2\cos(2x) dx = \frac{1}{2} \sin(2x) + C$$
$$\frac{d}{dx} 2x = 2$$

$$(16) \int \sin\left(\frac{x}{2}\right) dx = 2 \int \frac{1}{2} \sin\left(\frac{x}{2}\right) dx = -2 \cos\frac{x}{2} + C$$
$$\frac{d}{dx} \frac{x}{2} = \frac{1}{2}$$

$$(17) \int \frac{1}{(1+x)^2} dx = \int (1+x)^{-2} dx = \frac{(1+x)^{-1}}{-1} = -\frac{1}{1+x} + C$$
$$\frac{d}{dx} (1+x) = 1$$

$$(18) \int \sqrt{2x+3} dx = \int (2x+3)^{1/2} dx = \frac{1}{2} \int 2(2x+3)^{1/2} dx$$
$$= \frac{1}{2} \left( \frac{(2x+3)^{3/2}}{3/2} \right) + C$$
$$= \frac{1}{2} \cdot \frac{2}{3} \sqrt{(2x+3)^3} + C$$
$$= \frac{1}{3} \sqrt{(2x+3)^3} + C$$

$$(20) \int \frac{2x}{\sqrt{x^2+1}} dx = \int 2x(x^2+1)^{-1/2} dx = 2\sqrt{x^2+1} + C$$

$$\frac{d}{dx}(x^2+1) = 2x \quad -\frac{1}{2} + 1 = \frac{1}{2}$$

$$(21) \int \sin x \cos x dx = \frac{1}{2} \int 2 \sin x \cos x dx = \frac{1}{2} \int \sin(2x) dx$$
$$\sin(2x) = 2 \sin x \cos x$$
$$= \frac{1}{4} \int 2 \sin(2x) dx$$
$$= -\frac{1}{4} \cos(2x) + C$$

## Exercises 5-6

$$(5) \int \frac{x}{(4x^2+1)^5} dx = \frac{1}{8} \int 8(4x^2+1)^{-5} dx = -\frac{1}{32} (4x^2+1)^{-4} + C$$

$$\frac{d}{dx}(4x^2) = 8x$$

$$(6) \int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx = 2 \int \frac{\sin(\sqrt{x})}{2\sqrt{x}} dx$$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} = -2\cos(\sqrt{x}) + C$$

$$(12) \int \frac{u+t}{t} dt = \int u du = \frac{1}{2}u^2 + C$$

$$u = ut \\ du = \frac{1}{t} dt \\ = \frac{1}{2}u^2 t + C$$

$$(39) \int_0^4 x^3(x^2+1)^{-1/2} dx = \frac{1}{2} \int_0^4 x^2(x^2+1)^{-1/2} 2x dx$$

$$u = x^2+1 \\ du = 2x dx$$

$$x^2 = u-1 \\ x^3 = x(x^2) =$$

$$\begin{array}{c|cc} x & u \\ 0 & 1 \\ \hline \pi & 1 \\ \end{array} \\ = \frac{1}{2} \left( u^{3/2} \cdot \frac{2}{3} - 2u^{1/2} \right) \Big|_1^{\pi}$$

$$(42) \int_{\frac{\pi}{4}}^{\pi} \sin^5 x dx$$

$$u = \cos x \\ du = -\sin x dx$$

$$= \int_{\pi/4}^{\pi} (1-\cos^2 x)^2 \sin x dx$$

$$= - \int_{\pi/4}^{\pi} (1-\cos^2 x)^2 \cdot (-\sin x) dx = - \int_{\pi/4}^{\pi} (1-u^2)^2 du$$

$$\begin{array}{c|cc} x & u \\ \hline \frac{\pi}{4} & 1 \\ \pi & \sqrt{2} \\ \end{array}$$

## Exercises 6.1

$$(1) \int x \cos x \, dx$$

$$\int \frac{d}{dx} [f(x)g(x)] \, dx = \int [f'(x)g(x) + g'(x)f(x)] \, dx$$

$$= \int f'(x)g(x) \, dx + \int g'(x)f(x) \, dx$$

↓

$$\int g'(x)f(x) \, dx = \int \frac{d}{dx} [f(x)g(x)] - \int f'(x)g(x) \, dx$$

$$\Rightarrow \int g'(x)f(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx$$

$$u = f(x) \quad v = g(x)$$

$$du = f'(x)dx \quad dv = g'(x)dx$$

$$\int u \, dv = uv - \int v \, du$$

$$\int x \cos x \, dx =$$

$$\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx$$

$$g(x) = \sin x \quad g'(x) = \cos x$$

$$f(x) = x \quad f'(x) = 1$$

$$= x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + C$$

$$(2) \int (x+3)e^{2x} dx$$

$$\begin{aligned} f(x) &= x+3 & f'(x) &= 1 \\ g(x) &= \frac{1}{2}e^{2x} & g'(x) &= e^{2x} \end{aligned}$$

$$\int e^{2x}$$

$$= \frac{1}{2}e^{2x} + c$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

$$= \frac{1}{2}(x+3)e^{2x} - \int \frac{1}{2}e^{2x} dx$$

$$= \dots - \frac{1}{2} \int 2e^{2x} dx = \frac{1}{2}(x+3)e^{2x} - \frac{1}{4}e^{2x} + c$$

$$\frac{d}{dx} e^{2x} = 2e^{2x}$$

$$(5) \int x^3 \ln x dx$$

$$\begin{aligned} f(x) &= \ln x & f'(x) &= \frac{1}{x} \\ g(x) &= x^3 & g'(x) &= \frac{1}{4}x^4 \end{aligned}$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

$$= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 dx + c$$

$$(3) \int x^2 \cos(\pi x) dx$$

$$\begin{aligned} f(x) &= x^2 & f'(x) &= 2x \\ g'(x) &= \cos(\pi x) & g(x) &= \frac{1}{\pi}\sin(\pi x) = \frac{1}{\pi}\sin(\pi x) dx \end{aligned}$$

$$= \frac{x^2}{\pi} \sin(\pi x) - \int \frac{2x}{\pi} \sin(\pi x) dx \quad \frac{d}{dx} \frac{1}{\pi} \sin(\pi x)$$

$$= \frac{1}{\pi} \cos(\pi x)$$



$$\frac{d}{dx} -\frac{1}{\pi} \cos(\pi x) = -\frac{1}{\pi} (-\sin(\pi x)) \cdot \pi$$

$$\frac{2}{\pi} \int x \cdot \sin(\pi x) dx$$

$$f(x) = x$$

$$g'(x) = \sin(\pi x)$$

$$f'(x) = 1$$

$$g(x) = -\frac{1}{\pi} \cos(\pi x)$$

$$= \frac{2}{\pi} \left[ -\frac{x}{\pi} \cos(\pi x) - \int -\frac{1}{\pi} \cos(\pi x) dx \right]$$

$$= \frac{2}{\pi} \left[ \text{...} + \int \frac{1}{\pi} \cos(\pi x) dx \right]$$

$$= \frac{2}{\pi} \left[ \text{...} + \frac{1}{\pi} \sin(\pi x) \right]$$

$$= \frac{x^2}{\pi} \sin(\pi x) + \frac{2x}{\pi^2} \cos(\pi x) + \frac{2}{\pi^3} \sin(\pi x)$$

$\uparrow$

$$\frac{1}{\pi} \int \cos(\pi x) dx = \frac{1}{\pi} \sin(\pi x)$$

$$\frac{d}{dx} \sin(\pi x) = \pi \cos(\pi x)$$

$$(4) \int (x^2 - 2x) e^{kx} dx$$

$$f(x) = x^2 - 2x$$

$$g'(x) = e^{kx}$$

$$f'(x) = 2x - 2$$

$$g(x) = \frac{1}{k} e^{kx}$$

$$\int \frac{1}{k} e^{kx} dx$$

$$= \frac{1}{k^2} e^{kx}$$

$$= \frac{1}{k} (x^2 - 2x) e^{kx} - \frac{1}{k} \int (2x - 2) e^{kx} dx$$

$$= \text{(a)} - \frac{2}{k} \int x e^{kx} dx + \frac{2}{k} \int e^{kx} dx \quad \text{(b)}$$

$$(a) \quad f(x) = x$$

$$g'(x) = e^{kx}$$

$$f'(x) = 1$$

$$g(x) = \frac{1}{k} e^{kx}$$

$\nearrow$  (c)

$$= -\frac{2}{k} \int x e^{kx} dx = -\frac{2}{k} \left[ \frac{x}{k} e^{kx} - \int \frac{1}{k} e^{kx} dx \right]$$

$$= -\frac{2}{K} \left[ \frac{x}{K} e^{Kx} - \frac{1}{K^2} e^{Kx} \right] + C$$

$$(c) \frac{2}{K} \int e^{Kx} dx = \frac{2}{K} \left( \frac{1}{K} e^{Kx} \right) = \frac{2}{K^2} e^{Kx}$$

$$= (a) + (b) + (c)$$

$$= \frac{1}{K} (x^2 - 2x) e^{Kx} - \frac{2x}{K^2} e^{Kx} + \frac{2}{K^3} e^{Kx} + \frac{2}{K^2} e^{Kx}$$

$$= \frac{1}{K} (x^2 - 2x) e^{Kx} + \frac{2}{K^2} (1-x) e^{Kx} + \frac{2}{K^3} e^{Kx} \quad \checkmark$$

$$(6) \int x (\ln x)^3 dx \quad \frac{d}{dx} (\ln x)^3 = 3(\ln x)^2 \cdot \frac{1}{x}$$

$$\begin{aligned} f(x) &= (\ln^3 x) & f'(x) &= \frac{3 \ln^2 x}{x} \\ g'(x) &= x & g(x) &= \frac{x^2}{2} \\ &= \frac{x^2}{2} (\ln^3 x) - \frac{3}{2} \int x \ln^2 x dx & &= \frac{3}{2} \int x \ln^2 x \end{aligned}$$

$\underbrace{\phantom{000}}_{(2)} \quad \underbrace{\phantom{000}}_{(1)}$

$$(2) \int x^m (\ln^2 x) dx$$

$$\begin{aligned} f(x) &= (\ln^2 x) & f'(x) &= \frac{2 \ln x}{x} \\ g'(x) &= x^m & g(x) &= \frac{x^{m+1}}{m+1} \\ &= \frac{x^{m+1}}{m+1} (\ln^2 x) - \int x^m \ln x dx \end{aligned}$$

$\underbrace{\phantom{000}}_{(2)} \quad \underbrace{\phantom{000}}_{(1)}$

$$(6) \int x^m (\ln x) dx =$$

$$\begin{aligned} f(x) &= (\ln x) & f'(x) &= \frac{1}{x} \\ g'(x) &= x^m & g(x) &= \frac{x^{m+1}}{m+1} \\ &= \frac{x^{m+1}}{m+1} \ln x - \int \frac{x^m}{m+1} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{x^2}{2} (\ln x - \frac{x^2}{4}) + c \\
&= \frac{x^2}{2} (\ln^3 x - \frac{3}{2} \left[ \frac{x^2}{2} \ln^2 x - \frac{2}{3} \left( \frac{x^2}{2} \ln x - \frac{x^3}{4} \right) \right]) + c \\
&= \frac{x^2}{2} (\ln^3 x - \frac{3}{2} \left[ \frac{x^2}{2} \ln^2 x - \frac{3x^2}{2} \ln x + \frac{2x^3}{4} \right]) + c \\
&= \frac{x^2}{2} (\ln^3 x - \frac{3x^2}{4} \ln^2 x + \frac{3x^3}{4} \ln x - \frac{3x^3}{8}) + c \\
&= \frac{x^2}{2} \left( (\ln^3 x - \frac{3}{2} \cancel{\ln^2 x} + \frac{3}{2} \cancel{\ln x} - \frac{3}{8} \cancel{x^3}) + c \right) \\
&= \frac{x^2}{2} \left( (\ln^3 x - \frac{3}{2} \cancel{x} \ln^2 x + \frac{2}{3} \cancel{x} \ln x - \frac{2}{9} \cancel{x}) + c \right) \\
&= \frac{x^2}{2} \left( (\ln^3 x - \frac{3}{2} \ln^2 x + \frac{3}{2} \ln x - \frac{3}{4}) + c \right)
\end{aligned}$$

$$(10) \quad \int x^5 e^{-x^2} dx = \int x^4 \cdot x e^{-x^2}$$

$$\begin{aligned}
f(x) &= x^4 & f'(x) &= 4x^3 \\
g(x) &= x e^{-x^2} & g'(x) &= -\frac{1}{2} e^{-x^2}
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{2} \int -2x e^{-x^2} dx &= -\frac{1}{2} e^{-x^2} \\
&= -\frac{1}{2} x^4 e^{-x^2} - \int 4x^3 \cdot -\frac{1}{2} e^{-x^2} dx
\end{aligned}$$

$$(12) = -\frac{1}{2} x^4 e^{-x^2} + 2 \int x^3 e^{-x^2} dx$$

$$\begin{aligned}
f(x) &= x^2 & f'(x) &= 2x \\
g(x) &= x e^{-x^2} & g'(x) &= -\frac{1}{2} e^{-x^2}
\end{aligned}$$

$$\int x^2 \cdot x e^{-x^2} dx = -\frac{x^2}{2} e^{-x^2} - \int -\frac{2}{2} x e^{-x^2} dx$$

$$(b) = -\frac{x^2}{2} e^{-x^2} + \int x e^{-x^2} dx$$

$$(c) -\frac{1}{2} \int -2x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} + C$$

$$\frac{d}{dx} \left( -\frac{1}{2} e^{-x^2} \right) = -\frac{1}{2} (-2x) e^{-x^2} \\ = x e^{-x^2}$$

$$= -\frac{1}{2} x^4 e^{-x^2} + 2 \left( -\frac{x^2}{2} e^{-x^2} - \frac{1}{2} e^{-x^2} \right) + C$$

$$= -\frac{1}{2} x^4 e^{-x^2} - x^2 e^{-x^2} - e^{-x^2} + C$$

$$= \left( -\frac{1}{2} x^4 - x^2 - 1 \right) e^{-x^2} + C \quad \checkmark$$

$$(13) \int e^{2x} \sin(3x) dx$$

$$f(x) = e^{2x} \quad f'(x) = 2e^{2x}$$

$$g'(x) = \sin(3x) \quad g(x) = \frac{1}{3} \cos(3x)$$

$$\int \sin(3x) dx = \frac{1}{3} \int 3 \sin(3x) dx = -\frac{1}{3} \cos(3x) + C$$

$$= -\frac{1}{3} e^{2x} \cos(3x) + \int 2e^{2x} \frac{1}{3} \cos(3x) dx$$

$$= -\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{3} \int e^{2x} \cos(3x) dx$$

$$f(x) = e^{2x} \quad f'(x) = 2e^{2x}$$

$$g'(x) = \cos 3x \quad g(x) = \frac{1}{3} \sin(3x)$$

$$\int \cos(3x) dx = \frac{1}{3} \sin(3x) + C$$

$$= -\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{3} \left( \frac{1}{3} e^{2x} \sin(3x) - \frac{2}{3} \int e^{2x} \sin(3x) dx \right)$$

$$= -\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{9} e^{2x} \sin(3x) - \frac{4}{9} \int e^{2x} \sin(3x) dx$$

$$= \int e^{2x} \sin(3x) dx$$

$$\int e^{2x} \sin(3x) dx + \frac{4}{9} \int e^{2x} \sin(3x) dx = -\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{9} e^{2x} \sin(3x)$$

$$\int e^{2x} \sin(3x) dx = \frac{3}{13} \cdot \frac{1}{3} \left( -e^{2x} \cos(3x) + \frac{2}{3} e^{2x} \sin(3x) \right) + C$$

$$= \frac{3}{13} \left( -e^{2x} \cos(3x) + \frac{2}{3} e^{2x} \sin(3x) \right) + C$$

$$\frac{1}{13} e^{2x} (2 \sin(3x) - 3 \cos(3x))$$

## Exercises 6.2

$$(1) \quad \int \frac{2}{2x-3} dx = \int \frac{2}{2x-3} dx = \ln|2x-3| + C$$

$$\frac{d}{dx}(2x-3) = 2$$

$$(4) \quad \int \frac{x^2}{x-4} dx = \int x dx + \int 4 dx + \int \frac{16}{x-4} dx$$

$$\frac{x^2}{x-4} = x+4 + \frac{16}{x-4} = \frac{x^2}{2} + 4x + 16 \ln|x-4| + C$$

$$(11) \quad \int \frac{x-2}{x^2+x} dx = -2 \int \frac{1}{x} dx + 3 \int \frac{1}{x+1} dx =$$

$$\frac{x-2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = -\frac{2}{x} + \frac{3}{x+1}$$

$$x-2 = A(x+1) + Bx$$

$$x-2 = Ax + A + Bx$$

$$= x(A+B) + A$$

$$\begin{cases} A+B=1 \\ A=-2 \end{cases} \quad \begin{cases} B=1-A \\ A=-2 \end{cases}$$

$$= 3 \ln|x+1| - 2 \ln|x+1|$$

$$(13) \int \frac{1}{9x^2 - 6x + 1} dx = \int \frac{1}{(3x-1)^2} dx = \frac{1}{3} \int 3(3x-1)^2 dx$$

$$9x^2 - 6x + 1 = (9x-3)(x-2) \quad \frac{d}{dx} = 3$$

$$\Delta/4 = 9 - 9 = 0$$

$$(3x-1)^2 = -\frac{1}{3} \frac{1}{(3x-1)}$$

$$(7) \int \frac{1}{a^2 - x^2} dx = \int \frac{1}{(a-x)(a+x)} dx$$

$$\frac{1}{(a-x)(a+x)} = \frac{A}{(a-x)} + \frac{B}{(a+x)} = \frac{1}{2a(a-x)} + \frac{1}{2a(a+x)}$$

$$1 = A(a+x) + B(a-x)$$

$$1 = Aa + Ax + Ba - Bx$$

$$1 = Ax + Ba + Ax - Bx$$

$$\begin{cases} A-B=0 \\ Aa+Ba=1 \end{cases} \quad \begin{cases} A=B \\ 2Aa=1 \end{cases} \quad \begin{cases} A=B \\ A=\frac{1}{2a} \end{cases}$$

$$= \frac{1}{2a} \left[ \int \frac{1}{a-x} dx + \int \frac{1}{a+x} dx \right]$$

$$= \frac{1}{2a} \left[ -\ln|a-x| + \ln|a+x| \right] + C$$

$$= \frac{1}{2a} \left( \ln \left| \frac{a+x}{a-x} \right| \right) + C$$

$$(10) \int \frac{x}{3x^2 + 8x - 3} dx$$

$$\frac{d}{dx}(3x^2 + 8x - 3) = 6x + 8$$

$$3x^2 + 8x - 3 =$$

$$\Delta/4 = 16 - 9 = 5^2$$

$$x_{1,2} = \frac{-4 \pm 5}{3} < \frac{-3}{3}$$

$$3(x - \frac{1}{3})(x + 3) \\ (3x - 1)(x + 3)$$

$$\frac{x}{3x^2 + 8x - 3} = \frac{A}{3x-1} + \frac{B}{x+3} = \frac{1}{10(3x-1)} + \frac{3}{10(x+3)}$$

$$x = A(x+3) + B(3x-1)$$

$$x = Ax + 3A + 3Bx - B$$

$$x = Ax + 3Bx + 3A - B$$

$$\begin{cases} A + 3B = 1 \\ 3A - B = 0 \end{cases} \quad \begin{cases} A = 1 - 3B \\ 3(1 - 3B) - B = 0 \end{cases}$$

$$\begin{cases} A = 1 - 3B \\ 3 - 9B - B = 0 \end{cases} \quad \begin{cases} A = 1 - 3B \\ 3 - 10B = 0 \end{cases} \quad \frac{10 - 3}{10} = \frac{1}{10}$$

$$\begin{cases} A = 1 - 3B \\ 3 - 10B = 0 \end{cases} \quad \begin{cases} \text{---} \\ 3 - 10B = 0 \end{cases}$$

$$\begin{cases} \text{---} \\ B = \frac{3}{10} \end{cases} \quad \begin{cases} A = 1 - \frac{3}{10} = \frac{1}{10} \\ B = \frac{3}{10} \end{cases}$$

$$\begin{aligned} &= \frac{1}{10} \left[ \frac{1}{3} \int \frac{3}{3x-1} dx + 3 \int \frac{1}{x+3} dx \right] \\ &= \frac{1}{10} \left[ \frac{1}{3} \ln |3x-1| + 3 \ln |x+3| \right] \end{aligned}$$

## Exercises 6.5

$$(1) \int_2^\infty \frac{1}{(x-1)^3} dx : \lim_{t \rightarrow +\infty} \int_2^t \frac{1}{(x-1)^3} dx$$

$$\int \frac{1}{(x-1)^3} dx = \int (x-1)^{-3} dx = -\frac{1}{2} (x-1)^{-2}$$

$$\lim_{t \rightarrow +\infty} \left[ -\frac{1}{2} (x-1)^{-2} \right]_2^t$$

$$-\frac{1}{2} \lim_{t \rightarrow +\infty} \left[ (t-1)^{-2} - (2-1)^{-2} \right]$$

$$= -\frac{1}{2} \lim_{t \rightarrow +\infty} \left[ \frac{1}{(t-1)^2} - 1 \right]$$

$$= -\frac{1}{2} (-1) = \frac{1}{2}$$

$$(10) \int_0^\infty x e^{-x} dx$$

$$= \lim_{t \rightarrow \infty} \int_0^t x e^{-x} dx$$

$$= \lim_{t \rightarrow \infty} \left[ -e^{-x}(x+1) \right]_0^t$$

$$= -\lim_{t \rightarrow \infty} \left[ e^{-t}(t+1) - e^0(1) \right]$$

$$= -\lim_{t \rightarrow \infty} \left[ e^{-t}(t+1) - 1 \right]$$

$$= 1$$

$$\int x e^{-x} dx =$$

$$f(x) = x \quad f'(x) = 1$$

$$g(x) = e^{-x} \quad g'(x) = -e^{-x}$$

$$\int -e^{-x} dx = -e^{-x}$$

$$-x e^{-x} - \int -e^{-x}$$

$$= -x e^{-x} + e^{-x}$$

$$= e^{-x}(1-x)$$

$$= -e^{-x}(x+1)$$

$$(15) \int_0^{\pi/2} \tan x dx$$

$$= -\lim_{t \rightarrow \pi/2} \left[ \ln |\cos t| - \ln |\cos 0| \right]$$

$$= -\lim_{t \rightarrow \pi/2} \left[ \ln |\cos t| + 1 \right]$$

$$= +\infty \quad (\text{diverges})$$

$$\int \tan x dx = \int \frac{\sin x}{\cos x}$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$= -\int \frac{-\sin x}{\cos x}$$

$$= -\ln |\cos x| + C$$

$$(9) \int_{-\infty}^{\infty} \frac{x}{1+x^2} dx \quad \text{split integral!} \quad \ln 1 = 0$$

$$= \lim_{t \rightarrow \infty} \int_{-t}^t \frac{x}{1+x^2} dx$$

$$= \frac{1}{2} \lim_{t \rightarrow \infty} \left[ \ln |x^2+1| \right]_{-t}^t$$

$$= \frac{1}{2} \lim_{t \rightarrow \infty} \left[ \ln |t^2+1| - \ln |(-t)^2+1| \right]_{-\infty}^{\infty}$$

$$= \frac{1}{2} \lim_{t \rightarrow \infty} \left[ \ln \left| \frac{t^2+1}{t^2+1} \right| \right]$$

$$= \frac{1}{2} \lim_{t \rightarrow \infty} \left[ \ln(1) \right] = \frac{e}{2}$$

$$\frac{1}{2} \int \frac{2x}{x^2+1} dx = \frac{1}{2} \ln |x^2+1|$$

$$= \int_{-\infty}^0 \frac{x}{1+x^2} dx + \int_0^\infty \frac{x}{1+x^2} dx$$

# Sequences and Series

## Notes

- Increasing / Dec.:  $a_{n+1} \geq a_n$   
if  $a_n = f(n)$ , show  $f$  is increasing  
on  $[1, \infty) \rightarrow f'(x) \geq 0$
- Convergence :  $\lim_{n \rightarrow \infty} a_n = L$       Bounded  
+ Non-Monotone
- If  $\{a_n\}$  converges, it is bounded      = DIVERGENT
- If  $\{a_n\}$  is bounded above/below and is increa./decre.  
then it is convergent

## Exercises 9.1

(a) bounded

(b) positive / negative

(c) increasing / dec / alternating

(d) convergent / divergent / divergent to  $\pm\infty$

every converging sequence is bounded

every bounded monotonic sequence converges

$$(1) \left\{ \frac{2n^2}{n^2+1} \right\} \quad \begin{matrix} \leftarrow \text{Num degree grow faster than} \\ \uparrow \text{denominator} \end{matrix}$$

same degree  $\rightarrow$  converges

$$\begin{aligned} (c) \frac{2n^2}{n^2+1} &\leq \frac{2(n+1)^2}{(n+1)^2+1} \\ &\equiv \leq \frac{2(n^2+2n+1)}{(n^2+2n+1)+1} \end{aligned} \quad \begin{aligned} " &\leq \frac{2n^2+4n+2}{n^2+2n+2} \\ 2n^2(n^2+2n+2) &\leq (n^2+1)(2n^2+4n+2) \\ 2n^4+4n^3+2n^2 &\leq 2n^4+4n^3+2n^2 \\ &+ 2n^2+4n+2 \end{aligned}$$

$$0 \leq 4n+2$$

$$-\frac{1}{2} \leq n \quad \checkmark$$

increasing

$$(d) \lim_{x \rightarrow +\infty} \frac{2n^2}{n^2} = 2 \rightarrow \text{convergent}$$

(e) Bounded

$$(f) \forall n \in \mathbb{N}, \frac{2n^2}{n^2+1} \geq 0, \text{ positive}$$

$$(a) \left\{ \sin \frac{1}{n} \right\}$$

$$(d) \lim_{x \rightarrow \infty} \sin \frac{1}{x} = 0 \rightarrow \text{convergent}$$

(c) bounded

(b) positive

$$(c) \sin \left( \frac{1}{n} \right) \leq \sin \left( \frac{1}{n+1} \right)$$

$$f(x) = \sin \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2} \sin \frac{1}{x}$$

$$-\frac{1}{x^2} \sin \frac{1}{x} \geq 0$$

$$-\frac{1}{x^2} \geq 0 \quad \exists x \in \mathbb{R}$$

$$\sin \frac{1}{x} \geq 0$$

$$\sin t \geq 0$$

$$0 < t \leq \pi$$

$$0 < \frac{1}{x} \leq \pi \quad \forall x \in \mathbb{N}$$

Decreasing

$$(8) \left\{ \frac{(-1)^n n}{e^n} \right\}$$

$$(d) \lim_{x \rightarrow \infty} \frac{(-1)^n n}{e^n} = \lim_{x \rightarrow \infty} \left( \frac{1}{e^{-n}} \cdot n \right) = 0$$

$$\frac{(-1)^n}{e^{-n}} \cdot n = \left( -\frac{1}{e} \right)^n n = e^{-n} n$$

converges to 0

linearly converging  
gives us a function

(a) Bounded

(c) Alternating

(11)  $\{ \text{functions} \left( \frac{n\pi}{2} \right) \}$

(1)  $\lim_{x \rightarrow \infty} n \cos\left(\frac{n\pi}{2}\right)^{\alpha} \rightarrow$  limit doesn't exist

↳ divergent sequence

(19)  $a_n = \frac{e^n - e^{-n}}{e^n + e^{-n}} = \frac{e^n(1 - e^{-2n})}{e^n(1 + e^{-2n})} = \frac{1 - e^{-2n}}{1 + e^{-2n}}$

$$\lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow \infty} \frac{e^x(1 - e^{-2x})}{e^x(1 + e^{-2x})} = 1$$

(23)  $a_n = \sqrt{n+1} - \sqrt{n}$

$$\lim_{x \rightarrow \infty} (\sqrt{n+1} - \sqrt{n})$$

$$= \lim_{x \rightarrow \infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{x \rightarrow \infty} \frac{n+1 - n}{\sqrt{n+1} + \sqrt{n}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

## Exercises 3-2

### Theory

$$\sum_{n=0}^{\infty} ar^{n-1} = a + ar + \dots = \frac{a_{n+1}}{r}$$

$$r = \frac{a_{n+1}}{a_n} = \frac{ar^n}{ar^{n-1}}$$

$$S_n = a + ar + ar^2 + \dots = \frac{a(1 - r^n)}{1 - r}$$

$$\sum_{n=1}^{\infty} ar^{n-1}$$

$\begin{cases} a = 0 \rightarrow \text{converges to } 0 \\ |r| < 1 \rightarrow \text{converges to } \frac{a}{1-r} \\ r \geq 1 \wedge a > 0 \rightarrow \text{diverges to } \infty \\ r \geq 1 \wedge a < 0 \rightarrow \text{diverges to } -\infty \\ r \leq -1 \wedge a \neq 0 \rightarrow \text{diverges} \end{cases}$

$$(1) \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

$$= \frac{1}{3} \left( 1 + \frac{1}{3} + \left( \frac{1}{3} \right)^2 + \dots \right)$$

$$r = \frac{1}{3} \rightarrow \text{converges to } \frac{a}{1-r}$$

$$(3) \sum_{n=5}^{\infty} \frac{1}{(2+\pi)^{2n}} = \frac{1}{(2+\pi)^{10}} + \frac{1}{(2+\pi)^{12}} + \frac{1}{(2+\pi)^{14}} \dots$$

$$= \frac{1}{(2+\pi)^{10}} \left[ 1 + \frac{1}{(2+\pi)^2} + \left( \frac{1}{(2+\pi)^2} \right)^2 \right]$$

$$\left. \begin{array}{l} a = (2+\pi)^{10} \quad 0 < a < 1 \\ r = \frac{1}{(2+\pi)^2} \quad 0 < r < 1 \end{array} \right\} |r| < 1 \text{ converges to } \frac{a}{1-r}$$

$$\left( \frac{1}{(2+\pi)^{10}} \right) / \left( 1 - \frac{1}{(2+\pi)^2} \right) = \frac{1}{(2+\pi)^{10}} \left/ \frac{(2+\pi)^2 - 1}{(2+\pi)^2} \right.$$

$$= \frac{1}{(2+\pi)^{10}} \cdot \frac{(2+\pi)^2}{4 + 4\pi + \pi^2 - 1}$$

$$= \frac{1}{(2+\pi)^8 (4 + 4\pi + \pi^2 - 1)} \quad \checkmark$$

$$(7) \sum_{n=0}^{\infty} \frac{2^{n+3}}{e^{n+3}}$$

$$= \frac{2^3}{e^{-3}} + \frac{2^4}{e^{-2}} + \frac{2^5}{e^{-1}} + \frac{2^6}{e^0} + \frac{2^7}{e^1} \dots$$

$$= \frac{2^3}{e^{-3}} \left( 1 + \frac{2}{e} + \frac{2^2}{e^2} \dots \right)$$

$$r = \frac{2}{e} \quad a = \frac{2^3}{e^{-3}} = \frac{8}{\frac{1}{e^3}} \quad r < 1, \quad a > 1$$

$$\text{converges to } \frac{a}{1-r} = \frac{8e^3}{1 - \frac{2}{e}}$$

$$(15) \sum_{n=1}^{\infty} \frac{1}{2n-1} = \frac{1}{2-1} + \frac{1}{4-1} + \frac{1}{6-1}$$
$$= \frac{1}{1} + \frac{1}{3} + \frac{1}{5}$$

$\geq 0$

Converges to 0

• • •

# Double Integrals

$$(1) \int_0^1 dx \int_0^x (xy + y^2) dy$$

$$= \int_0^1 dx \cdot \left[ \frac{xy^2}{2} + \frac{y^3}{3} \right]_0^x$$

$$= \int_0^1 dx \left[ \left( \frac{x^3}{2} + \frac{x^3}{3} \right) - (0) \right]$$

$$= \int_0^1 \frac{5}{6} x^3 dx$$

$$= \frac{5}{6} \int_0^1 x^3 dx = \frac{5}{6} \left[ \frac{x^4}{4} \right]_0^1$$

$$\frac{5x^3 + 2x^3}{6} = \frac{5x^3}{6}$$

$$= \frac{5}{6} \left( \frac{1}{4} \right) = \frac{5}{24}$$

$$(2) \int_0^1 \int_0^y (xy + y^2) dx dy$$

$$= \int_0^1 \left[ \frac{yx^2}{2} + y^2 x \right]_0^y dy$$

$$= \int_0^1 \left( \frac{y^3}{2} + y^3 \right) dy$$

$$= \frac{3}{2} \int_0^1 y^3 dy$$

$$= \frac{3}{2} \left( \frac{y^4}{4} \right)_0^1$$

$$= \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8}$$

$$\frac{y^3 + 2y^3}{2} = \frac{3}{2} y^3$$

$$(3) \int_0^{\pi} \int_{-x}^x \cos y dy dx$$

$$= \int_0^{\pi} \left[ \sin y \right]_{-x}^x dx$$

$$= \int_0^{\pi} [\sin x - \sin(-x)] dx$$

$$= \int_0^{\pi} [2 \sin(x)] dx$$

$$= 2 \int_0^{\pi} \sin x dx$$

$$\begin{aligned} & \frac{d}{dx} - \cos(-x) \\ &= + \sin(-x) \end{aligned}$$

$$= 2 \left[ -\cos x \right]_0^\pi$$

$$= 2 \left[ -\cos \pi + \cos 0 \right]$$

$$= 2(-1 + 1) = 0$$

## Fubini's Theorems

(1)  $f(x, y)$  continuous on rectangular area  $R$   $a \leq x \leq b$   $c \leq y \leq d$

$$\Rightarrow \iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx$$

$$= \int_a^b \int_c^d f(x, y) dx dy$$

(2)  $f(x, y)$  continuous on region  $R$

$\rightarrow R$  is  $a \leq x \leq b$ ,  $g_1(x) \leq y \leq g_2(x)$

$$\Rightarrow \iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

$\rightarrow R$  is  $a \leq y \leq b$ ,  $h_1(y) \leq x \leq h_2(y)$

$$\Rightarrow \iint_R f(x, y) dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

(5)  $\iint_R (x^2 + y^2) dA$   $R: 0 \leq x \leq a$   $0 \leq y \leq b$  } Rectangle

$$= \int_0^a \int_0^b (x^2 + y^2) dy dx$$

$$= \int_0^a \left[ x^2 y + \frac{y^3}{3} \right]_0^b dx =$$

$$= \int_0^a \left( b x^2 + \frac{b^3}{3} \right) dx$$

$$= \left[ b \frac{x^3}{3} + \frac{b^3}{3} x \right]_0^a$$

$$= \left( \frac{1}{3} a^3 b + \frac{1}{3} a b^3 \right)$$

$$(8) \iint_T (x - 3y) dA$$

$0 \leq x \leq a$

$0 \leq y \leq b - \frac{b}{a}x$

$T: (0,0), (a,0), (0,b)$

$y = mx + b$

$m = \frac{0-b}{a-0} = -\frac{b}{a}$

$y = -\frac{b}{a}x + b$

$y = b - \frac{b}{a}x$

$$\iint_T (x - 3y) dA = \int_0^a \int_0^{b - \frac{b}{a}x} (x - 3y) dy dx$$

$$(11) \iint_D \ln x dA$$

$D$  is bounded by  $2x + 2y = 5$  and  $xy = 1$

(11) Find intersections

$$\begin{cases} 2x + 2y = 5 \\ xy = 1 \end{cases}$$

$$\begin{cases} y = -x + \frac{5}{2} \\ y = \frac{1}{x} \end{cases}$$

$$-x + \frac{5}{2} = \frac{1}{x}$$

$$-\frac{2x+5}{2} = \frac{1}{x} \rightarrow -2x^2 - 5x - 2 = 0$$

$$2x^2 + 5x + 2 = 0$$

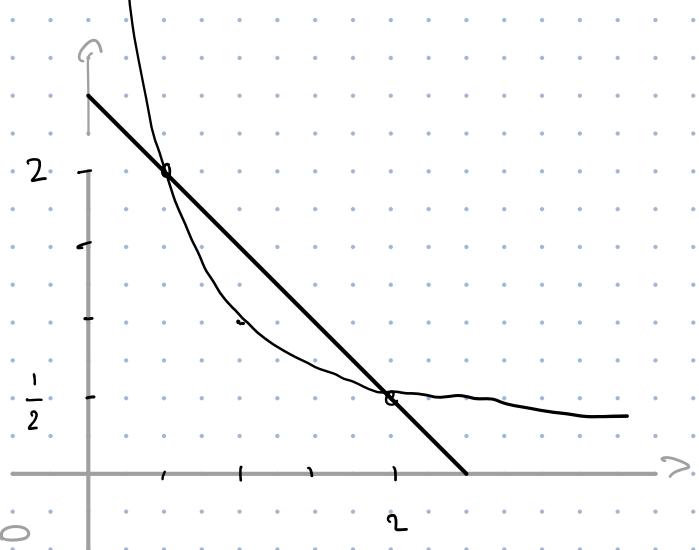
$$\Delta = 25 - 4(4) = 25 - 16 = 3^2$$

$$x_{1,2} = \frac{-5 \pm 3}{4} \quad \begin{matrix} 2 \\ \frac{1}{2} \end{matrix}$$

$$(2, \frac{1}{2}) \vee (\frac{1}{2}, 2)$$

$$\iint_D \ln x dA = \int_{1/2}^2 \int_{1/x}^{5/2-x} \ln x dy dx$$

$$= \int_{1/2}^2 [y \ln x]_{1/x}^{5/2-x} dx$$



$$\int \ln x = \int_1^x \ln x$$

$$f(x) = \ln x \quad f'(x) = \frac{1}{x}$$

$$g'(x) = 1 \quad g(x) = x$$

$$= x \ln x - \int_1^x dx$$

$$= x \ln x - x + c$$

$$= x(\ln x - 1)$$

$$= \int_{1/2}^2 \left[ \left( \frac{5}{2} - x \right) \ln x - \frac{1}{x} \ln x \right] dx$$

$$= \int_{1/2}^2 \frac{5}{2} \ln x dx - \int_{1/2}^2 x \ln x dx - \int_{1/2}^2 \frac{1}{x} \ln x dx$$

$$= \frac{5}{2} \left[ x(\ln x - 1) \right]_{1/2}^2 - \left[ \frac{x^2}{2} (\ln x - 1) \right]_{1/2}^2 - \left[ \frac{\ln^2 x}{2} \right]_{1/2}^2$$

$$\int x \ln x = \frac{x^2}{2} \ln x - \int \frac{x}{2} dx$$

$$f(x) = \ln x \quad f'(x) = \frac{1}{x} \quad \frac{x^2}{2} \cdot \frac{1}{x} = \frac{x}{2}$$

$$g'(x) = x \quad g(x) = \frac{x^2}{2}$$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} x^2$$

$$= \frac{x^2}{2} (\ln x - 1)$$

$$\int \frac{1}{x} \ln x dx$$

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

$$\int u du = \frac{u^2}{2} = \frac{\ln^2 x}{2} + c$$

$$= \frac{5}{2} \left[ (2 \ln 2 - 2) - \left( \frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} \right) \right] - \left[ (2 \ln 2 - 2) - \left( \frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} \right) \right]$$