Solutions - Practice Exam Questions - Tutorial 1

1.

			A		B	answer:
p	q	r	(not p) ->r	(not q) <-> r	p or ((not q) <-> r)	A and B
T	T	T	T	F	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	F	F	F
F	T	F	F	T	T	F
F	F	T	T	T	T	T
F	F	F	F	F	F	F

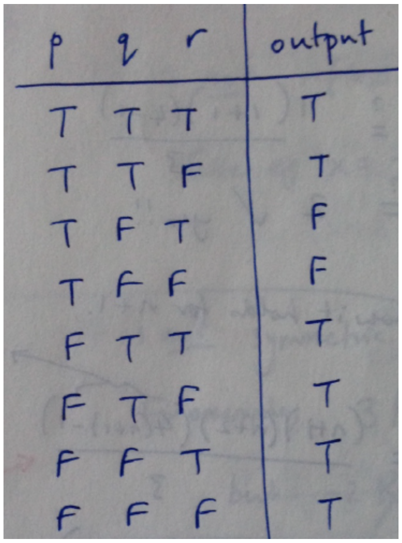
2.

			A	B	C	D	
p	q	r	r OR (not q)	q <=> p	A=>B	r OR p	**C AND D**
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	T	F	F	T	F
T	F	F	T	F	F	T	F
F	T	T	T	F	F	T	F
F	T	F	F	F	T	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	F

3.

			"x"	"y"	
p	q	r	$(q \wedge \neg p) \vee r$	$r \Rightarrow \neg q$	$x \Leftrightarrow y$
T	T	T	F	F	T
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	T	F
F	F	F	T	T	T

4.



p	q	r	output
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T

5. (a) **True.** The claim "all the numbers in the table are even" is obviously false, as is the claim "all the numbers in the table are odd", and false $\Leftrightarrow$ false (i.e. if and only if) evaluates to TRUE.

   (b) **True.** I simply go through all the possibilities, showing in each case that it holds: 11 is prime, 2 is even and prime, 10 is even, 7 is prime, 12 is even.

   (c) **False.** There are only 3 colours available (red, blue, green) and there is at least one red prime (5), at least one blue prime (2), and at least one green prime (3). So whatever colour you pick, it cannot be the case that no prime has that colour.

   (d) **False.** The second column is a counter-example. It contains three numbers but only two colours.

   (e) **False.** Take $x = 5$ and $y = 3$. Clearly here is no blue number z such that $5 < z < 3$.

   (f) **True.** We just have to find one single x and one single y with this property. One possiblity is: $x = 2$ and $y = 7$, both of which are blue.

   (g) **True.** I will simply list the possibilities (i.e. consider all possible even numbers and for each one give a prime number that is in the same row).

   - $8 \to 11$
   - $2 \to 2$ - note that x is allowed to be equal to y !
   - $4 \to 5$
   - $10 \to 5$
   - $6 \to 13$
   - $12 \to 7$
   - $14 \to 7$

   (h) **True.** No column has ≥ 4 numbers, so the "if" is always false, so the "if...then" is automatically true (for every column d).

(i) **True.** I'll just list all the possibilities. (Yes, I am that lazy. Of course, more elegant answers are also possible). Each line below consists of a value of x and then y that make the "if" true, followed by a blue number z such that $x < z < y$.

- 1, 3, 2
- 1, 8, 2
- 1, 9, 2
- 1, 13, 2
- 4, 8, 7
- 4, 9, 7
- 4, 13, 7
- 5, 8, 7
- 5, 9, 7
- 5, 13, 7
- 6, 8, 7
- 6, 9, 7
- 6, 13, 7

(j) **False.** Whatever x you try, taking $y = 10$ will break it, because 10 is in the rightmost column (i.e. 10 cannot be strictly to the left of x, whatever x you try). Note that here I am basically proving the negation: "for every odd x, there exists an even number y, such that y is in the same column as x or in a column to the right of the column x is in". To prove this: let x be an arbitrary odd number. Take $y = 10$. Clearly y is in the same column as x, or a column to the right of x, because 10 is in the rightmost column.

6. (a) This is **true**. Let x be an arbitrary natural number. Suppose x can be written as $3k$ for some natural number k (i.e. x is divisible by 3). In this case we take $y = 1$. This is sufficient because $(3k + 1)/3 = k + 1/3$ which is fractional and thus not a natural number. Another possibility is that x can be written as $3k + 1$ for some integer $k \geq 0$. In this case, again taking $y = 1$ gives $(3k + 2)/3 = k + 2/3$. Again, this is fractional, so not a natural number. The only other possibility is that x can be written as $3k + 2$ for some integer $k \geq 0$. In this case we take $y = 2$. This gives $(3k + 4)/3 = k + (4/3)$. Again, this is fractional, so not natural. $\square$

(b) This is **true**. Take $x = 1$. Now, let y be an arbitrary integer. $4y$ will always be even, so $3 + 4y$ will always be odd. 20 is an even number, so $3 + 4y$ cannot possibly be equal to 20. $\square$

(c) This is **false**. To see this, observe that $(\exists x \in \mathbb{N})(\exists y \in \mathbb{R})(2x - y \leq 10)$ is true: if we take $x = 1$ and $y = 0$, we get $2(1) - 0 \leq 10$ which is true. Hence, the original claim is false. $\square$

An alternative proof, which logically boils down to the same thing, is as follows. Observe that $\neg((\exists x \in \mathbb{N})(\exists y \in \mathbb{R})(2x - y \leq 10))$ is logically equivalent to $(\forall x \in \mathbb{N})(\forall y \in \mathbb{R})(2x - y > 10)$. This statement is false, because taking $x = 1, y = 0$ gives a counter-example.

(d) This is **true**. Take $x = 1$. Now, let y an arbitrary integer. Suppose $y \geq -32$. In this case simply take $z = 1$. This is sufficient, because then $1 + y + 1 \geq -32 + 2 \geq -32$. The other possibility is that $y < -32$. In this case, take $z = -y$. This will also work, because $x + y + z$ is then equal to $1 + y + (-y) = 1$ and this is ≥ -32. $\square$

7. (a) This is **true**. *Proof.* We select $x = 10$. Now, let y be an arbitrary integer y. Note that, if $y \neq 0$, then the right hand side of the logical-OR is true, so we are done. Otherwise, $y = 0$, and the left hand side of the logical-OR is true, because $10 = 0 + 10$ does indeed hold. Either way, the logical-OR is true. $\square$

   (b) This is **true**. *Proof.* Let x be an arbitrary integer. Now, we need to find a natural number y that satisfies the inequality. Here is one way of choosing y. Consider the expression $\frac{1}{2}x^2 - 20$. If this is < 1, we take $y = 1$, and we are done. If $\frac{1}{2}x^2 - 20$ is greater than or equal to 1, we take $y = \lceil \frac{1}{2}x^2 - 20 \rceil$ i.e. we round $\frac{1}{2}x^2 - 20$ upwards to the nearest integer that is greater than or equal to it; due to rounding upwards, this will be 1 or higher, and thus natural. Either way, the inequality holds. $\square$

   Comment. Writing "Let x be an arbitrary integer. We take $y = \frac{1}{2}x^2 - 20$" is definitely wrong! If you do that, you might end up with a negative and/or fractional value for y, neither of which can be a natural number!

   (c) This is *false*. **Proof.** One way of showing this is to prove that

$$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x^2 - x + y \text{ is odd})$$

   is true. (It will then follow by negation that the original claim is false). Let x be an arbitrary integer. We need to find some integer y (conditional on x) such that $x^2 - x + y$ is odd. Well, how about this. If $x^2 - x$ is even, choose $y = 1$. If $x^2 - x$ is odd, choose $y = 0$. Done! $\square$

   Comment 1. Some of you might have noticed that when x is an integer, $x^2 - x$ is actually *always* even, so in the above proof taking $y = 1$ irrespective of what x is, would also work. But in that case you should also argue briefly why $x^2 - x$ is always even, for integer x. Would such a proof be "better"? I think that's the wrong way of looking at it. Both are mathematically correct. The main difference is that in the first proof I made my life a bit easier by making use of the freedom that I am allowed to choose a different y, conditional on x.

   Comment 2. Another possible proof strategy would be to absorb the negation ($\neg$) into the original statement, to obtain a new (equivalent) statement...

$$(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})(x^2 - x + y \text{ is even})$$

   ...and then to directly prove that this new, equivalent statement is false. However, if I try to prove that this statement is false, I basically end up saying things like "This can't be true, because whatever integer x you choose you can always find some integer y that breaks the body of the claim...". Such a proof would in principle be fine, but there is a slight risk here that you degenerate into hand-waving, so be careful. In fact, my very first proof is a formalization of exactly this proof strategy, which has the advantage that it helps you to avoid degenerating into hand-waving.

   (d) This is *true*. **Proof.** Let x be an arbitrary integer. We take $y = x - 1$. If $x < 1$, take $z = 1$. Otherwise (i.e. $x \geq 1$), take $z = (x + 1)$. $\square$

8. (a)

This is <u>false</u>. I show this, by proving that the negation is <u>true</u>.

Negation is: $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{N})\left(x \not> 0 \ \vee \ x+y^2 \text{ is } \underline{\text{not}} \text{ even}\right)$

which is the same as, $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{N})\left(x \leq 0 \ \vee \ x+y^2 \text{ is odd}\right)$.

Let x be an arbitrary element of $\mathbb{Z}$.

if $x \leq 0$, choose $y = 1$. Clearly $x \leq 0$, so we are good.

if $x > 0$
(ie. $x \geq 1$), we distinguish two cases:

– x is odd? pick $y=2$:
we have odd $+ \ 2^2$
$=$ odd $+$ even
$=$ odd ✓

– x is even? pick $y = 1$:
we have even $+ \ 1^2$
$=$ even $+$ odd
$=$ odd ✓

So the negation is indeed true! □

(b)

This is <u>true</u>. [An important observation here is that if $z \in \mathbb{N}$, then $z - 10 \geqslant -9$ (because $z \geqslant 1$).] *

Let x be an arbitrary integer.

- if $x \leq 10$, choose $y = 20$. Then, $x - y \leq 10 - 20 = -10$.

  Combined with (*), we see that $x - y < z - 10$, whatever natural z we pick.

- if $x > 10$, choose $y = x + 10$. Clearly, $y \in \mathbb{N}$.

  Now, $x - y = x - (x + 10) = -10$. Again, by (*), this is $< z - 10$, for every natural z. □

9. (a)

This is <u>false</u>. I prove this by proving that $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{N})(-y < x < y)$ is <u>true</u>.

Let x be an arbitrary integer.

<u>if $x \geqslant 0$,</u> take $y = x + 1$.
This is clearly natural, and $-y < x < y$ clearly holds because $-y$ is strictly negative and $x < x + 1$.

<u>if $x < 0$,</u> take $y = |x - 1|$.
We need $-y < x < y$ to hold. The $x < y$ bit holds because $x < 0$ and $y \geqslant 0$.
The $-y < x$ bit holds because
$$-|x - 1| = x - 1$$
and $x - 1 < x$ ✓. □

(b)

This is false.

I prove the negative is true.

The negative is, $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{N})(\exists z \in \mathbb{N})(x+y=z)$

$$\Updownarrow$$

$$x = z-y.$$

Let x be an arbitrary integer.

if $x=0$, take $y=z=1$. Clearly, $z-y=0$ ✓

if $x \geq 1$, take $\cancel{\left.\begin{array}{l} z = x+1 \\ y = x \end{array}\right\} \Rightarrow z-y}$

$$\left.\begin{array}{l} z = x+1 \\ y = 1 \end{array}\right\} \Rightarrow z-y = x \ ✓$$

if $x<0$, take $\left.\begin{array}{l} z = 1 \\ y = |x|+1 \end{array}\right\} \Rightarrow z-y = -|x| = x$ ✓ $\square$

10.

This statement is equivalent to $(\forall x \in \mathbb{R})\Big[\big(x \in \mathbb{N} \Rightarrow (2x \in \mathbb{N}) \wedge (3x \in \mathbb{N})\big)$
$\wedge$
$\big((2x \in \mathbb{N}) \wedge (3x \in \mathbb{N}) \Rightarrow x \in \mathbb{N}\big)\Big]$

I say this is <u>true</u>.

– one direction is easy (ie. "$\Rightarrow$").

Let x be an arbitrary real number.

Assume $x \in \mathbb{N}$. It is clear that $2x$ and $3x$ are both in $\mathbb{N}$ ✓ — I really think this is obvious

– the other direction is a bit more tricky. ("<=")

Let x be an arbitrary natural number.

Assume $2x \in \mathbb{N}$. So $x \in \{0.5, 1, 1.5, 2, 2.5, \ldots\}$.

Also, assume $3x \in \mathbb{N}$. So $x \in \{\frac{1}{3}, \frac{2}{3}, 1, 1\frac{1}{3}, 1\frac{2}{3}, 2, \ldots\}$.

Observe: these two sets <u>only</u> have the numbers $\{1, 2, \ldots\}$ in common.

So $x \in \mathbb{N}$ ✓

this is absolutely fine!

alternative proof – done by a student last year:

$3x - 2x$ is an integer.

Also, $3x - 2x \geq 1$ because $x \geq 1$.

So $3x - 2x$ is an integer that is ≥ 1 ie. a natural.

Now, $3x - 2x = x$, So x is natural! ($\ddot{o}$)

– No, I would never have thought of this either!

11.

This is <u>true</u>.

Let m and n be arbitrary integers.

We need to prove two things: (1) mn even $\Rightarrow$ m even $\vee$ n even

(2) m even $\vee$ n even $\Rightarrow$ mn even.

We prove (2) first. Suppose m is even - then we can write $m = 2k$ for some integer k. So $mn = 2kn$, which is even.

Suppose n is even: symmetrical argument to above.

Next we prove (1).

I use the contrapositive for this. That is, I aim to prove not(m even $\vee$ n even) $\Rightarrow$ not (mn even).

which is the same as m odd $\wedge$ n odd $\Rightarrow$ mn odd.

So assume that m and n are both odd. (If they are not, the IF becomes vacuously true).

So $m = 2k+1$ for some integer k.

and $n = 2\ell+1$ for some integer ℓ.

$$
\begin{aligned}
\text{So } mn &= (2k+1)(2\ell+1) = 4k\ell + 2k + 2\ell + 1 \\
&= 2[2k\ell + k + \ell] + 1 \\
&= \text{even} + \text{odd} \\
&= \text{odd} \quad ☺ \qquad \square.
\end{aligned}
$$