Practice Exam Questions Week 2, **Linear Algebra**

SOLUTIONS.

1. Consider the following matrix A and vector $\mathbf{b}$:

$$A = \begin{bmatrix} -1 & 2 & 3 & -4 & 8 \\ 3 & -6 & -4 & 7 & -9 \\ 2 & -4 & 1 & 1 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 7 \\ -6 \\ 7 \end{bmatrix}$$

(a) Are the columns of A linearly independent?

No. A contains more columns than rows. Hence, A cannot have a pivot in every column. So, the columns of A are linearly dependent.

(b) Compute the solution set of the associated linear system of equations $A\mathbf{x} = \mathbf{b}$ and express it in parametric vector form.

$$\left[\begin{array}{ccccc|c} -1 & 2 & 3 & -4 & 8 & 7 \\ 3 & -6 & -4 & 7 & -9 & -6 \\ 2 & -4 & 1 & 1 & 5 & 7 \end{array}\right] \quad R_1 : R_1 * -1$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -2 & -3 & 4 & -8 & -7 \\ 3 & -6 & -4 & 7 & -9 & -6 \\ 2 & -4 & 1 & 1 & 5 & 7 \end{array}\right] \quad \begin{array}{l} R_2 : R_3 - 3*R_1 \\ R_3 : R_3 - 2*R_1 \end{array}$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -2 & -3 & 4 & -8 & -7 \\ 0 & 0 & 5 & -5 & 15 & 15 \\ 0 & 0 & 7 & -7 & 21 & 21 \end{array}\right] \quad R_2 : R_2 * 1/5$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -2 & -3 & 4 & -8 & -7 \\ 0 & 0 & 1 & -1 & 3 & 3 \\ 0 & 0 & 7 & -7 & 21 & 21 \end{array}\right] \quad R_3 : R_3 - 7R_2$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -2 & -3 & 4 & -8 & -7 \\ 0 & 0 & 1 & -1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 : R_1 + 3*R_2$$

$$\sim \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 2 + 2x_2 - x_4 - x_5 \\ x_2 \text{ is free} \\ x_3 = 3 + x_4 - 3x_5 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \end{cases}$$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 + 2x_2 - x_4 - x_5 \\ x_2 \\ 3 + x_4 - 3x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

2. True or false? If the given statement is true, briefly explain why. If it is false, give a counterexample.

(a) If the columns of an augmented matrix are linearly independent, then the associated linear system of equations is inconsistent. True.

Linearly independent columns means pivots in all columns. A pivot in the last column of an augmented matrix means to have an equality $0=\alpha$ with $\alpha \neq 0$, which is a contradiction. Hence, the system is inconsistent.

(b) Four different vectors in $\mathbb{R}^3$ always span $\mathbb{R}^3$.

False. Consider for example $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

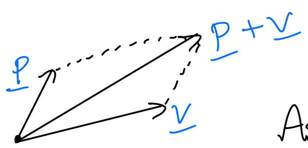
These are four different vectors in $\mathbb{R}^3$, but they all belong to the same line. Hence, the four vectors do not span $\mathbb{R}^3$.

Alternative answer: note that $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Hence, there is not a pivot position in every row and thus the vectors do not span $\mathbb{R}^3$.

(c) The effect of adding a vector $\mathbf{p}$ to a vector $\mathbf{v}$ is to move the vector $\mathbf{v}$ in a direction parallel to $\mathbf{p}$.

False.



As you can see, the direction of $\underline{v}+\underline{p}$ is not parallel to the direction of $\underline{p}$.

(d) If the augmented matrix of a linear system of equations has more rows than columns, then it cannot have infinitely many solutions.

False. Consider for example $[A \,\vdots\, \underline{b}] = \left[\begin{array}{cc:c} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

There is a free variable, so there are infinitely many solutions.

(e) If A and B are matrices for which the product AB and the sum $A+B$ are both well defined, then the product BA is also well defined.

$\begin{array}{cc} & n \\ m & \begin{bmatrix} \ A\ \ \end{bmatrix} \end{array} \quad \begin{array}{cc} & q \\ p & \begin{bmatrix} \ B\ \ \end{bmatrix} \end{array}$ True.

AB is well defined, so $n = p$.
Therefore,

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \;A\; \end{bmatrix} \end{array} \begin{array}{cc} & q \\ n & \begin{bmatrix} \;B\; \end{bmatrix} \end{array}$$

A+B is well defined, so $m = n$ and $n = q$.
Therefore,

$$\begin{array}{cc} & n \\ n & \begin{bmatrix} \;A\; \end{bmatrix} \end{array} \begin{array}{cc} & n \\ n & \begin{bmatrix} \;B\; \end{bmatrix} \end{array}$$

Hence, BA is well defined because

$$\underset{(n)}{\#\text{ columns of } B} = \underset{(n)}{\#\text{ rows of } A}.$$

(f) If all the rows of an augmented matrix have a pivot, then the associated linear system of equations is inconsistent.

False. Consider for example the reduced echelon form of an augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 3 & 6 \\ 0 & 0 & 1 & 4 & 7 \end{array}\right]$$

There is a pivot in every row. And the linear system is consistent (with infinitely many solutions).

(g) If S and T are 2×2 matrices such that $ST = 0$, then also $TS = 0$.

False. Consider for example $S = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Then, $ST = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but $TS = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$.

(h) If the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$ are such that $\{\mathbf{x}, \mathbf{y}\}$ is linearly independent, $\{\mathbf{x}, \mathbf{z}\}$ is linearly independent, and $\{\mathbf{y}, \mathbf{z}\}$ is linearly independent, then also $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent.

False. Consider for example $\underline{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\underline{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Then $\{\underline{x}, \underline{y}\}$ is linearly independent because neither of the vectors is a multiple of the other. Using the same argument, we also know that $\{\underline{x}, \underline{z}\}$ and $\{\underline{y}, \underline{z}\}$ are linearly independent. However, $\underline{z} = \underline{x} + \underline{y}$ and thus $\{\underline{x}, \underline{y}, \underline{z}\}$ is a linearly dependent set.