Calculus: lecture 4

- **L'Hôpital rules for calculating limits**
- **Extreme values**
  - Increasing and decreasing functions
  - Global and local extrema
  - Min-max theorem
- **Concavity and inflections**
- **Sketching functions**

Indeterminate forms

- $\lim_{x \to 0^+} x \ln(x)$ $[0 \cdot \infty]$
- $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x$ $[1^\infty]$
- $\lim_{x \to +\infty} \left(e^x - e^{(x^2)}\right)$ $[\infty - \infty]$
- $\lim_{x \to 64} \frac{\sqrt{x} - 8}{\sqrt[3]{x^2} - 16}$ $\left[\frac{0}{0}\right]$ l'Hôpital rules
- $\lim_{x \to +\infty} x^{\frac{1}{x}}$ $[\infty^0]$
- $\lim_{x \to \infty} \frac{\ln(x)}{x}$ $\left[\frac{\infty}{\infty}\right]$

1st l'Hôpital rule

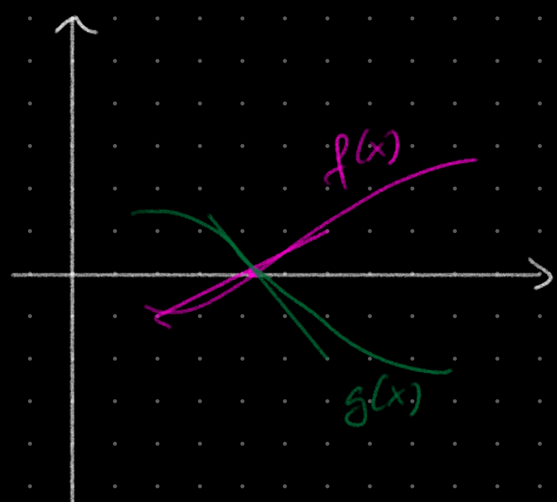
(can be $(-\infty, +\infty)$)

For f(x) and g(x) differentiable functions on (a,b), and g'(x)$\neq$0

IF
$$\begin{cases} \lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0 \text{ for some } x_0 \in [a,b] \\ \lim\limits_{x\to x_0} \dfrac{f'(x)}{g'(x)} = L \text{ (can be } \infty, \text{ but exists)} \end{cases} \underset{\text{then}}{\Longrightarrow} \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L$$

(Also true for left / right limits)

* this rule applies to indeterminate forms $\left[\frac{0}{0}\right]$



$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

$$\approx \frac{f(x)-f(x_0)}{x-x_0}$$

$$\Longrightarrow \frac{f(x)}{g(x)} \approx \frac{\overset{0}{\cancel{f(x_0)}} + \boxed{f'(x_0)}\cancel{(x-x_0)}}{\underset{0}{\cancel{g(x_0)}} + \boxed{g'(x_0)}\cancel{(x-x_0)}}$$

- $$\lim_{x \to 64} \frac{\sqrt{x} - 8}{\sqrt[3]{x^2} - 16} \overset{H}{=} \lim_{x \to 64} \frac{\frac{1}{2\sqrt{x}}}{\frac{2}{3} x^{-1/3}} = \lim_{x \to 64} \frac{3}{4} \cdot \frac{\sqrt[3]{x}}{\sqrt{x}} = \frac{3 \cdot \cancel{4}}{\cancel{4} \cdot 8}$$

- $$\lim_{x \to \infty} \frac{\ln(x)}{x} \overset{H}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

- $$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}}$$

- $$\lim_{x \to +\infty} x^{\frac{1}{x}} = \lim_{x \to +\infty} e^{\boxed{\ln(x) \cdot \frac{1}{x}} \to 0} = e^{0} = 1$$

$$\lim_{x \to +\infty} \ln(x) \cdot \frac{1}{x} = 0 \Rightarrow$$

2nd l'Hôpital rule

For f(x) and g(x) differentiable functions on (a,b), and $g'(x) \neq 0$

↳ can be $(-\infty, +\infty)$

IF $\begin{cases} \lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = \pm\infty \\ \lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)} = L \text{ (can be } \infty, \text{ but exists)} \end{cases}$ $\xRightarrow{\text{then}}$ $\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)} = L$

(Also true for $\lim\limits_{x\to b^-} \dfrac{f(x)}{g(x)}$)

* this rule applies to indeterminate forms $\left[\frac{\infty}{\infty}\right]$

Increasing and decreasing functions

- increasing function: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

  e^x, $\ln(x)$, $\sqrt{x}$, $ax+b$ $(a>0)$

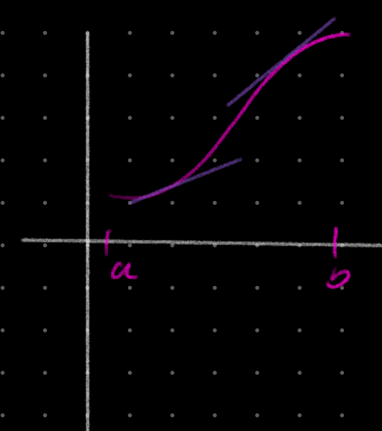
- decreasing function: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

  x^2, on $[-\infty, 0]$, $\frac{1}{x}$, on $(0, +\infty)$, ...

- non-increasing: $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ (decreasing or flat)

- For a differentiable function f on (a,b)

  if $f'(x) > 0$ on (a,b), then $f(x)$ is increasing



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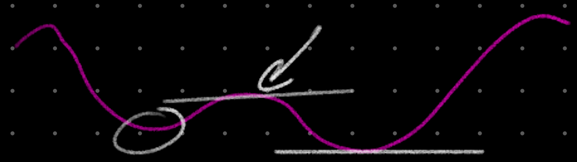
Extreme values

- Absolute minimum/maximum: $f(x)$ has a maximum at x_0 if for all points in the domain, $f(x) \leq f(x_0)$

- Min-max theorem: a continuous function on a closed, bounded domain reaches an absolute minimum and maximum.

  $\to [a,b]$

- Local minimum/maximum



f has a local maximum/minimum at x_0 if there is an open interval (x_0-h, x_0+h) on which $f(x_0)$ is an extremum.

↳ if f is differentiable at a local extremum x_0, then $f'(x_0) = 0$

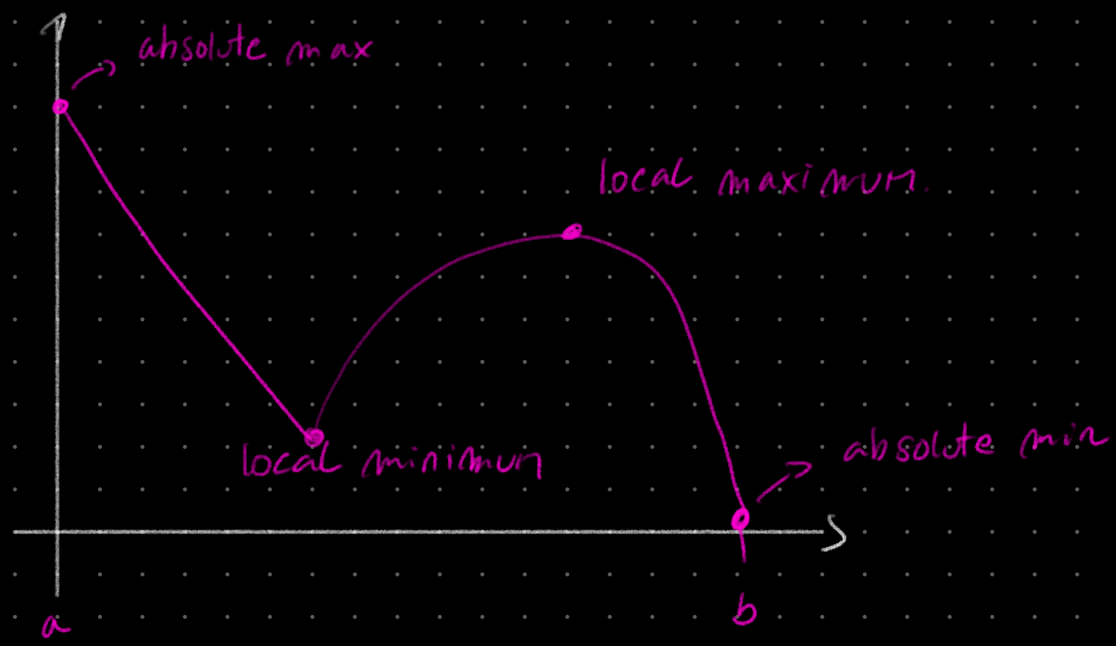
$$f'(x_0) = \lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h}$$

$f(x_0+h) - f(x_0) \mapsto$ negative if $f(x_0)$ is max

$h \to$ negative

$$f'_+(x_0) \leq 0 \qquad f'_-(x_0) \geq 0 \quad \Rightarrow \quad f'(x_0) = 0$$

Extreme values



we can find extreme values at

- end points
- critical points ($f' = 0$)
- singular points (f' DNE)

- f is continuous on (a, b) (can be $\mathbb{R}$)

if $\lim_{x \to a^+} f(x) = L$ if $\exists\, x \in (a,b)$ $f(x) > L$, $f(x) > M$

$\lim_{x \to b^-} f(x) = M$ then f has a maximum on (a,b)

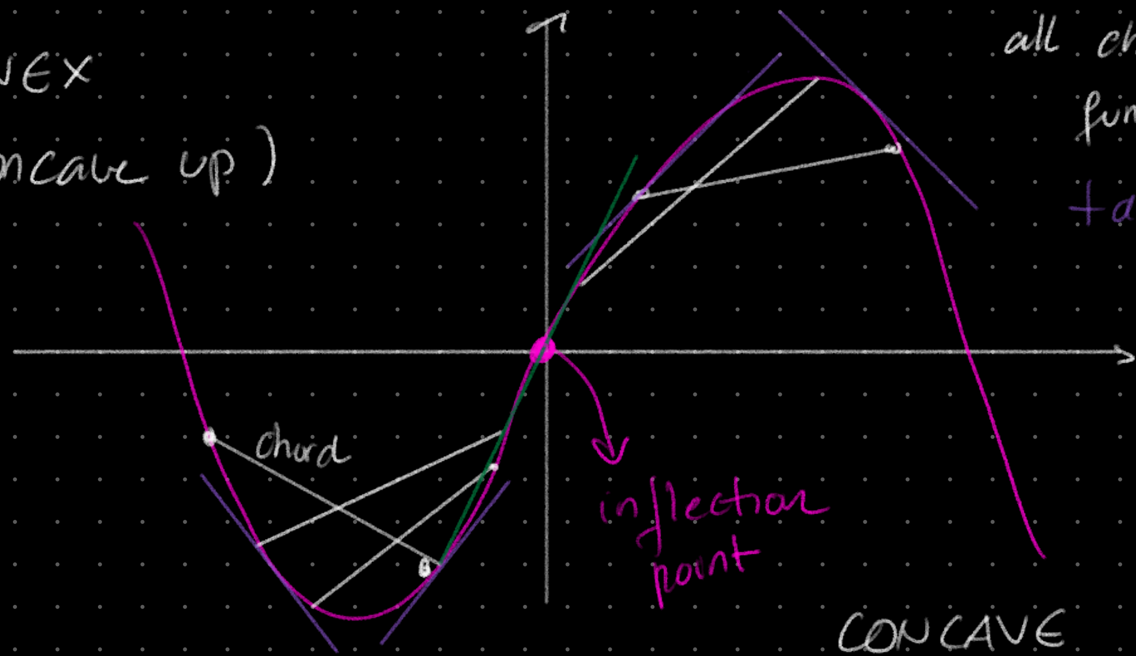
Concavity and inflections

CONVEX

(concave up)

all chords below function

tangent lines above function.



CONCAVE

(concave down)

all chords above function

tangent lines below function

$f''(x) < 0$

$f''(x) > 0$

inflection point: if $f''(x_0)$ exists $f''(x_0) = 0$

Sketching functions

- domain
- even / odd
- asymptotes
- $f'(x) \longrightarrow$ increasing / decreasing intervals

  $f''(x) \longrightarrow$ concave / convex

$$f(x) = \frac{1}{\sqrt{x^2-1}}$$

- domain? $x^2 - 1 > 0 \iff (x+1)(x-1) > 0$

  $\Rightarrow x > 1 \quad \vee \quad x < -1$

- $f(x) \stackrel{?}{=} f(-x) \rightarrow$ yes! the function is even.

- asymptotes

  $$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{\sqrt{x^2-1}} = 0$$ HA $y = 0$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{\sqrt{x^2-1}} = +\infty$$

$$\bullet\ f'(x) = \frac{-1}{x^2-1} \cdot \frac{1}{\cancel{2}\sqrt{x^2-1}} \cdot (\cancel{2}x) = \frac{-x}{(x^2-1)^{3/2}}$$

$$f(x) = \frac{1}{\sqrt{x^2-1}}$$

$\to f'(x) \overset{?}{=} 0$ no, 0 is not in the domain

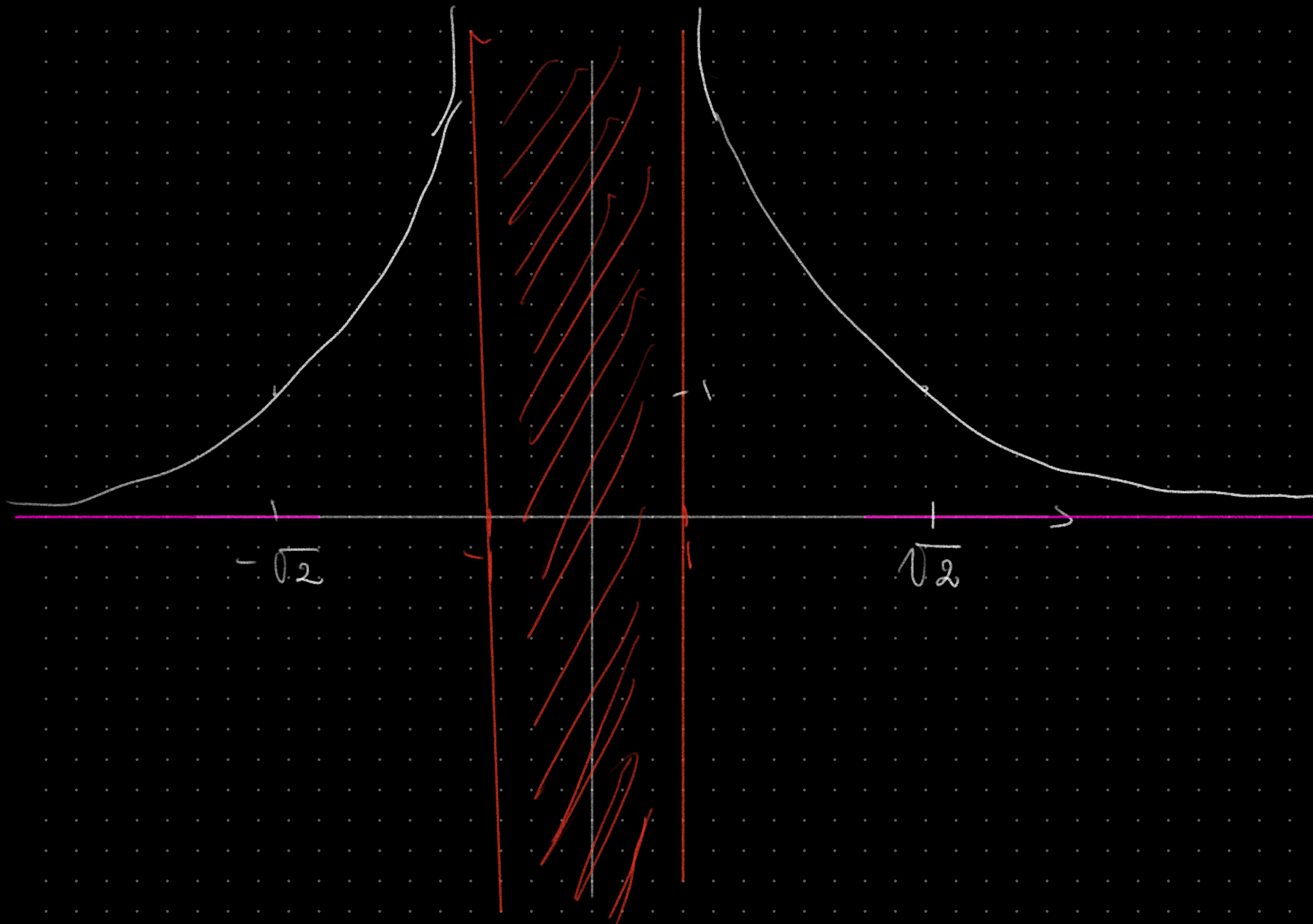
$f'(x) > 0$ for $x < 0$ increasing

$f'(x) < 0$ for $x > 0$ decreasing

$$f''(x) = \frac{(x^2-1)^{3/2}(-1) - (-x)\frac{3}{\cancel{2}}(x^2-1)^{1/2}\cdot(\cancel{2}x)}{(x^2-1)^3}$$

$$= \frac{-(x^2-1) + 3x^2}{(x^2-1)^{5/2}} = \frac{2x^2+1}{(x^2-1)^{5/2}} > 0$$

$\hookrightarrow$ always convex



$-\sqrt{2}$
-1
1
$\sqrt{2}$