Exam Linear Algebra DSAI.

$\boxed{\begin{array}{l}\text{Marieke: } 1, 2, 4, 5 \\ \text{Stefan: } 3, 6.\end{array}}$

Question 1:

$$\det A = \begin{vmatrix} 2 & 1 & 1 \\ 0 & p & 3 \\ 5 & 3 & p \end{vmatrix} = 2 \cdot \begin{vmatrix} p & 3 \\ 3 & p \end{vmatrix} + 5 \begin{vmatrix} 1 & 1 \\ p & 3 \end{vmatrix} = 2(p^2-9)+5(3-p)$$

$$= 2p^2 - 18 + 15 - 5p = 2p^2 - 5p - 3 = 0.$$

A is not invertible $\iff \det A = 0 \iff p = \frac{5 \pm \sqrt{(-5)^2 - 4 \cdot 2 \cdot -3}}{4} \iff p = 3$ or $p = -\frac{1}{2}$.

Question 2:

False. Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. A is symmetric because $A = A^T$

However $\det(A) = 1 \cdot 1 - 1 \cdot 1 = 0$ and thus A is not invertible.

Question 3:

$$\begin{vmatrix} \sqrt{2} & \sqrt{3} \\ \sqrt{3} & \sqrt{2} \end{vmatrix} = \sqrt{2}\sqrt{2} - \sqrt{3}\sqrt{3} = 2 - 3 = -1 \neq 0.$$

Hence, the vectors $\underline{u}$ and $\underline{v}$ do form a basis for $\mathbb{R}^2$.

Question 4:

A is upper triangular, so $\lambda_1 = 1$ (with mult. 2) and $\lambda_2 = -1$

eigenspace of $\lambda_1 = 1$: $A - \lambda_1 I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

So, $\dim(\text{Nul}(A - \lambda_1 I)) = 1 < 2 =$ mult. of λ_1.

Hence, A is not diagonalizable.

Question 5:

Denote $\underline{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}$. Then, $\|\underline{v}\| = \sqrt{1^2 + 0^2 + (-2)^2 + 3^2} = \sqrt{14}$.

So, $\begin{bmatrix} 1/\sqrt{14} \\ 0 \\ -2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$ and $\begin{bmatrix} -1/\sqrt{14} \\ 0 \\ 2/\sqrt{14} \\ -3/\sqrt{14} \end{bmatrix}$.

Question 6:

(a) $A - 1I = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} \underset{R_3 : R_3 - \frac{1}{2}R_1}{\overset{R_2 : R_2 + \frac{1}{2}R_1}{\sim}} \begin{bmatrix} -2 & 1 & -1 \\ 0 & -1\frac{1}{2} & -1\frac{1}{2} \\ 0 & -1\frac{1}{2} & -1\frac{1}{2} \end{bmatrix} \underset{R_3 : R_3 - R_2}{\sim}$

$\begin{bmatrix} -2 & 1 & -1 \\ 0 & -1\frac{1}{2} & -1\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \underset{R_2 : R_2 * -2/3}{\overset{R_1 : R_1 * -1/2}{\sim}} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1 : R_1 + \frac{1}{2}R_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

So, basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

(b) $A - (-2)I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \underset{R_3 : R_3 + R_1}{\overset{R_2 : R_2 - R_1}{\sim}} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So, basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) $\underline{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\underline{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\underline{v}_2$ and $\underline{v}_3$ are not orthogonal.

Projection of $\underline{v}_3$ onto $\underline{v}_2$ is $\dfrac{\underline{v}_3 \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2} \underline{v}_2 = \dfrac{-1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$.

And the component of v_3 orthogonal to $\underline{v}_2$ is $\underline{v}_3 - \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$.

Normalizing these vectors results in $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$.

Normalizing $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ results in $\begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

Hence, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $P = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/ \end{bmatrix}$

Question 7:

(a) $2A + 3C$ cannot be performed because A is a 2x2 matrix and C is a 2x3 matrix.
So, the answer is $\boxed{C}$.

(b) $\left[\begin{array}{ccc|c} 2 & 2 & -4 & 3 \\ 1 & 3 & -2 & 4 \\ -4 & k & 8 & -6 \end{array}\right] \underset{R_3: R_3+2R_1}{\overset{R_2: R_2-\frac{1}{2}R_1}{\sim}} \left[\begin{array}{ccc|c} 2 & 2 & -4 & 3 \\ 0 & 2 & 0 & 2\frac{1}{2} \\ 0 & k+4 & 0 & 0 \end{array}\right]$

Hence, if $k=-4$, then x_3 is a free variable and thus the SLE has infinitely many solutions.
So, the answer is $\boxed{b}$.

(c) $\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$
$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ Hence, $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 5 \\ 0 \end{bmatrix}$

So, the answer is $\boxed{c}$

(d) $\begin{bmatrix} 1 \\ 2 \\ a \\ b \end{bmatrix} = c_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 1 \\ 0 \\ 5 \\ -1 \end{bmatrix}$.

From the second row it follows that $2 = c_1 \cdot 1 + c_2 \cdot 0 \Rightarrow c_1 = 2$.
From the first row it follows that $1 = c_1 \cdot 1 + c_2 \cdot 1 \Rightarrow 1 = 2 \cdot 1 + c_2 \cdot 1$
$\Rightarrow c_2 = 1-2 = -1$.
Hence, $a = c_1 \cdot 3 + c_2 \cdot 5 = 2 \cdot 3 + (-1) \cdot 5 = 6-5 = 1$.
and $b = c_1 \cdot 1 + c_2 \cdot (-1) = 2 \cdot 1 + (-1) \cdot (-1) = 2+1 = 3$.
So, the answer is $\boxed{d}$.

(e) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

Hence, the standard matrix is $\begin{bmatrix} -1 & 2 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} \underset{R_2: R_2+R_1}{\sim} \begin{bmatrix} -1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

There is a pivot in every column $\Rightarrow$ T is injective
There is not a pivot in every row $\Rightarrow$ T is not surjective.
So, the answer is $\boxed{b}$

(f) Note that $\mathrm{Nul}(\underline{u}^T) = \{\underline{x} \in \mathbb{R}^3 : \underline{u}^T \underline{x} = 0\} = \{\underline{x} \in \mathbb{R}^3 : \underline{x} \cdot \underline{u} = 0\}$

Since $\mathrm{Nul}(\underline{u}^T)$ is a subspace of $\mathbb{R}^3$, $\{\underline{x} \in \mathbb{R}^3 : \underline{x} \cdot \underline{u} = 0\}$ is also a subspace of $\mathbb{R}^3$.

Hence, the solution is $\boxed{b}$.

(g) If a polynomial has the property $p(0)=0$, then we know $a_0 = 0$.
Hence, H consists of all polynomials of the form $p(t) = a_1 t + a_2 t^2 + a_3 t^3$.
As a result, $\{t, t^2, t^3\}$ is a basis for H.
Hence, the solution is $\boxed{a}$.

(h) $\begin{bmatrix} 1 & 1 & -2 & -1 \\ -1 & 4 & -3 & 1 \\ 0 & 7 & 1 & -8 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So, $\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in Nul(A)$.

Hence, the answer is $\boxed{c}$.

(i) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \underset{R_3 : R_3 - R_1}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \underset{R_3 : R_3 - R_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Hence, $Col(A) = Span\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} = \left\{ c_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in Col(A)$? 1st row implies $c_1 = 1$
2nd row implies $c_2 = 2$
Then, the 3rd row becomes $c_1 + c_2 = 1 + 2 = 3$ ☺

So, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and thus $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in Col(A)$.

Hence, the answer is (a).