Lecture 8: Vector Spaces (part 2)
(book: 4.2, 4.3, 4.4, 4.5)

Today: basis, dimension, Nul, Col, Row.

A basis of a vector space V is a set $\{\underline{v}_1, \dots, \underline{v}_p\}$ of vectors in V such that:
- $*$ $\{\underline{v}_1, \dots, \underline{v}_p\}$ is lin. indep.
- $*$ it spans the whole vector space.

Example: A basis for $\mathbb{R}^3$, standard basis $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$.

But also $\left\{ \begin{bmatrix}3\\0\\-6\end{bmatrix}, \begin{bmatrix}-4\\1\\7\end{bmatrix}, \begin{bmatrix}-2\\1\\5\end{bmatrix} \right\}$. Why?

$$\begin{bmatrix}3 & -4 & -2\\0 & 1 & 1\\-6 & 7 & 5\end{bmatrix} \sim \dots \sim \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

- $*$ pivot in every column $\Rightarrow$ lin indep.
- $*$ pivot in every row $\Rightarrow$ spans $\mathbb{R}^3$.

Example: The standard basis for $\mathbb{P}_2$: $\{1, x, x^2\}$

But a basis is also: $\{1+x^2,\ x-3x^2,\ 1+x-3x^2\}$ (later).

Two views of a basis $\mathcal{B}$:
- $*$ it is maximal: adding any vector $\underline{v} \in V$ to $\mathcal{B}$, makes it lin dep.
- $*$ it is minimal: removing any vector $\underline{v}$ from $\mathcal{B}$, causes $\mathcal{B}$ to no longer span V.

dim(V): # vectors in a basis of V.

$\dim(\mathbb{R}^3) = 3$ $\qquad \dim(\mathbb{R}^n) = n$

$\dim(\mathbb{P}_n) = n+1$.

Let $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ be a basis for V.
For every $\underline{x} \in V$ we can find weights $\underbrace{c_1, \ldots, c_n}$ such that
$\underline{x} = c_1 \cdot \underline{b}_1 + \ldots + c_n \cdot \underline{b}_n$

coordinates of $\underline{x}$ wrt basis $\mathcal{B}$.

$$[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Note: this set of coordinates is unique.

Example: $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} \right\}$ basis for $\mathbb{R}^3$.

Consider $\underline{x} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$.

What are the coordinates of $\underline{x}$ wrt the basis $\mathcal{B}$?

$$\left[\begin{array}{ccc:c} 3 & -4 & -2 & 5 \\ 0 & 1 & 1 & 2 \\ -6 & 7 & 5 & 4 \end{array}\right] \sim \quad \ldots \quad \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

Matlab: rref

$$[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ 5 \\ -2 \end{bmatrix}$$

Example: $\mathcal{B} = \{1+x^2, x-3x^2, 1+x-3x^2\}$ is a basis for $\mathbb{P}_2$. (later).

Consider $p(x) = 6+3x-x^2$.
Find $[\underline{p}]_{\mathcal{B}}$.

$$6 + 3x - x^2 = c_1 \cdot (1+x^2) + c_2 \cdot (x-3x^2) + c_3 \cdot (1+x-3x^2).$$

SLE
$$\begin{cases} c_1 \phantom{{}+c_2} + c_3 = 6 \\ \phantom{c_1+{}} c_2 + c_3 = 3 \\ c_1 - 3c_2 - 3c_3 = -1 \end{cases} \qquad \left[\begin{array}{ccc:c} 1 & 0 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 1 & -3 & -3 & -1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

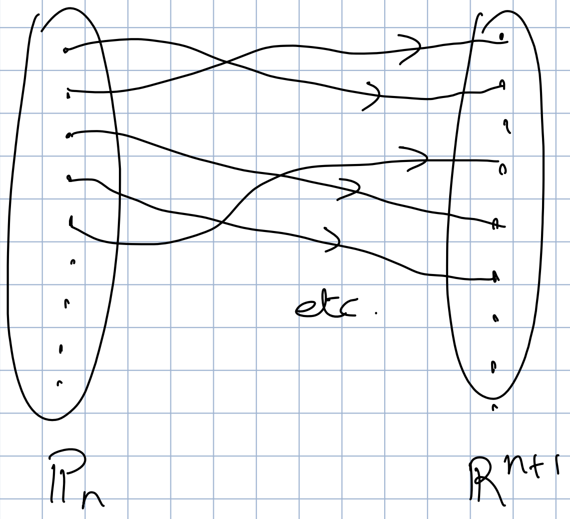
So, $[\underline{p}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ 5 \\ -2 \end{bmatrix}$.

Check: $p(x) = 8\,(1+x^2) + 5 \cdot (x-3x^2) + (-2) \cdot (1+x-3x^2)$. ✓

Theorem: Let $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ be a basis for V. Then, the linear transformation $T : V \longrightarrow \mathbb{R}^n$ defined by $T(\underline{x}) = [\underline{x}]_{\mathcal{B}}$ is a bijection.

Proof: See the book (Theorem 9).

Hence, the vectors can be abstract objects, but using the coordinates we can translate them to real numbers.



bijection.

Example: Show that $\{1+x^2, x-3x^2, 1+x-3x^2\}$ forms a basis for $\mathbb{P}_2$.

The coordinate mapping of the standard basis $\{1, x, x^2\}$ produces the coordinate vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$.

Do those vectors form a basis in $\mathbb{R}^3$? Yes.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, $\{1+x^2, x-3x^2, 1+x-3x^2\}$ forms a basis for $\mathbb{P}_2$.

Three important subspaces associated with an $m \times n$ matrix A.

- $*$ $\text{Nul}(A)$
- $*$ $\text{Col}(A)$
- $*$ $\text{Row}(A)$

Recall from Monday: $\text{Nul}(A) = \{ \underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{0} \}$ is a subspace of $\mathbb{R}^n$.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 4 \end{bmatrix} \qquad \left[\begin{array}{ccc:c} 1 & -3 & 2 & 0 \\ 2 & -6 & 4 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc:c} \textcircled{1} & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underline{x} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \qquad \text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Note: $*$ # vectors in the spanning set = # free variables.

$*$ the vectors in the spanning set are always lin indep.

Basis for $\text{Nul}(A)$: $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. $\qquad \dim(\text{Nul}(A)) = 2$

So, in general:

$*$ $\dim(\text{Nul}(A))$ = # free vars.

$*$ the vectors from parametric vector form form a basis for $\text{Nul}(A)$.

$\text{Col}(A) = \text{Span}\{ \underline{a}_1, \dots, \underline{a}_n \}$ is a subspace of $\mathbb{R}^m$

How to find a basis for $\text{Col}\, A$?

Intermezzo.

The Spanning Set Theorem:

Let $\underline{v}_1, \dots, \underline{v}_p \in V$ and let $H = \text{Span}\{\underline{v}_1, \dots, \underline{v}_p\}$.

- If $\underline{v}_k$ is a lin comb. of the other vectors, then $\{\underline{v}_1, \dots, \underline{v}_p\} \setminus \{\underline{v}_k\}$ still spans H.
- If $H \neq \{\underline{0}\}$, some subset of $\{\underline{v}_1, \dots, \underline{v}_p\}$ spans H.

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* 1st strategy: keep deleting columns if they are dependent on others.

The columns that you end up with form a basis for $\text{Col}(A)$.

Easy when in reduced echelon form.

$$B = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\left.\begin{aligned} \underline{b}_2 &= 3 \cdot \underline{b}_1 \\ \underline{b}_4 &= -1 \cdot \underline{b}_1 - 2 \cdot \underline{b}_3 \end{aligned}\right\} \text{Remove } \underline{b}_2 \text{ and } \underline{b}_4.$$

So, $\underbrace{\{\underline{b}_1, \underline{b}_3\}}_{\hookrightarrow \text{ pivot columns.}}$ forms a basis and $\dim(\text{Col}(B)) = 2$. True in general.

Intermezzo.

Linear dependence relations between colums do not change under row operations.

If $A \sim B$, then $A\underline{x} = \underline{0}$ and $B\underline{x} = \underline{0}$ have the same set of solutions.

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 2 & 6 & -1 & 0 \\ 1 & 3 & 0 & -1 \end{bmatrix} \sim \dots \sim B$$

So, we also have $\underline{a}_2 = 3 \cdot \underline{a}_1$

$\underline{a}_4 = -1 \cdot \underline{a}_1 - 2 \cdot \underline{a}_3$

So, $\{\underline{a}_1, \underline{a}_3\}$ forms a basis for $\text{Col}(A)$. and $\dim(\text{Col}(A)) = 2$.

! be careful! of A! Not of REF!

* 2nd strategy: the pivot columns of A form a basis for $\text{Col}(A)$.

Hence, $\dim(\text{Col}(A)) = \#\text{pivot cols}$.

$\text{Row}(A) = \text{Span}\{\underline{r}_1, \dots, \underline{r}_m\}$ is a subspace of $\mathbb{R}^n$. $\qquad$ $\text{Row}(A) = \text{Col}(A^T)$.

Note : $\boxed{A \sim B \Rightarrow \text{Row}(A) = \text{Row}(B)}$ $\qquad$ (Warning $\text{Col}(A) \neq \text{Col}(B)$).

idea : $A \sim B$

$\Rightarrow$ rows of B are a lin comb. of the rows of A.

$\Rightarrow$ any lin comb of the rows of B are also a lin comb of the rows of A.

$\Rightarrow$ $\text{Row}(B) \leq \text{Row}(A)$.

$A \sim B \Rightarrow B \sim A \Rightarrow \dots \Rightarrow \text{Row}(A) \leq \text{Row}\, B$.

So, $\text{Row}(A) = \text{Row}(B)$.

If B is RREF, then the nonzero rows of B are lin indep and thus form a basis for both $\text{Row}(B)$ and $\text{Row}(A)$.

$$B = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, $\dim(\text{Row}(A)) = \#$ pivot rows.

$$\boxed{\text{Rank}(A)} = \boxed{\dim(\text{Col}(A))} = \#\text{pivot cols} = \#\text{pivots} = \#\text{ pivot rows} = \boxed{\dim(\text{Row}(A))}$$

$\hookrightarrow = n - \#$ nonpivot cols.

$= n - \#$ free variables.

$\boxed{= n - \dim(\text{Nul}(A))}$

Recall a linear transformation $T : \underline{x} \longrightarrow A\underline{x}$ $\qquad$ $A : m \times n$

domain : $\mathbb{R}^n$

codomain : $\mathbb{R}^m$.

$*$ Range $= \{T(\underline{x}) : \underline{x} \in \mathbb{R}^n\} = \{A\underline{x} : \underline{x} \in \mathbb{R}^n\} = \text{Col}(A)$

So, T is surjective $\Leftrightarrow$ range = codomain $\Leftrightarrow$ $\text{Col}(A) = \mathbb{R}^m$

$\Leftrightarrow \dim(\text{Col}(A)) = m \Leftrightarrow \text{Rank}(A) = m$

$*$ Kernel $= \{\underline{x} \in \mathbb{R}^n : T(\underline{x}) = \underline{0}\} = \{\underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{0}\} = \text{Nul}(A)$.

So T is injective $\Leftrightarrow$ $T(\underline{x}) = \underline{0}$ has only the trivial sol.

$\Leftrightarrow$ Kernel $= \{\underline{0}\} \Leftrightarrow \text{Nul}(A) = \{\underline{0}\} \Leftrightarrow \dim(\text{Nul}(A)) = 0$.

$\Leftrightarrow \text{Rank}(A) = n$

Summary (so far):

Let A be an $m \times n$ matrix with columns $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$.

$m \geq n$: $\begin{bmatrix} A \\ \\ \\ \end{bmatrix}$

The following statements are equivalent:

① A has a pivot in every column

② A has n pivot positions

③ There are no free variables

④ $A\underline{x} = \underline{0}$ has only the trivial sol.

⑤ $\{\underline{a}_1, \underline{a}_2, \dots \underline{a}_n\}$ is linearly indep.

⑥ $T: \underline{x} \mapsto A\underline{x}$ is one-to-one/injective

⑦ $\text{Nul } A = \{\underline{0}\}$

⑧ $\dim \text{Nul } A = 0$

⑨ $\text{rank } A = n$

$m \leq n$: $\begin{bmatrix} & A & \end{bmatrix}$

The following statements are equivalent:

ⓐ A has a pivot in every row.

ⓑ A has m pivot positions.

ⓒ The echelon form of A does not contain a row of all zeros.

ⓓ $A\underline{x} = \underline{b}$ is consistent for every $\underline{b}$ in $\mathbb{R}^m$.

ⓔ $\text{Span}\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\} = \mathbb{R}^m$.

ⓕ $T: \underline{x} \mapsto A\underline{x}$ is onto/surjective.

ⓖ $\text{Col } A = \mathbb{R}^m$

ⓗ $\dim \text{Col } A = m$

ⓘ $\text{rank } A = m$

If A is square ($n = m$), then statements ② and ⓑ are equivalent.
Hence, the following statements are equivalent for square matrices.

- ① - ⑨, ⓐ - ⓘ
- A is invertible
- There is a matrix C such that $CA = I_n$ <u>and</u> $AC = I_n$
- A is row equivalent to I_n.
- A^T is invertible.
- $\det A \neq 0$
- The columns of A form a basis for $\mathbb{R}^n$.

A: m×n matrix.

	definition	subspace of	equal to $\mathbb{R}^n$ / $\mathbb{R}^m$ / $\mathbb{R}^n$ if	equal to $\{\underline{0}\}$ if
Nul A	$\{\underline{x} : A\underline{x} = \underline{0}\}$	$\mathbb{R}^n$	A is the zero matrix	A has a pivot in every column
Col A	Span$\{\underline{a}_1, \dots, \underline{a}_n\}$	$\mathbb{R}^m$	A has a pivot in every row	A is the zero matrix
Row A	Span$\{\underline{r}_1, \dots, \underline{r}_m\}$	$\mathbb{R}^n$	A^T has a pivot in every row	A is the zero matrix

	dimension
Nul A	# free var's in the equation $A\underline{x}=\underline{0}$ = # nonpivot columns in A = n - rank A
Col A	# pivot columns in A. = # pivot columns in A = rank A
Row A	# non-zero rows in the echelon form of A = # pivot columns in A = rank A

	finding a basis
Nul A	Find the general sol. of $A\underline{x}=\underline{0}$. Write the solution in parametric vector form where the weights are the free var's. The corresponding vectors form a basis for Nul A.
Col A	The pivot columns of A (so, of A itself, and thus not the pivot columns of a reduced form of A)
Row A	The nonzero rows of an echelon form of A.