Reminder: $f: S \to S$ function is also a relation

$S = \{s_1, s_2, s_3, s_4\}$

ex:

$f(s_1) = s_1 \iff f_{s_1 s_1}$

$f(s_2) = s_3 \iff f_{s_2 s_3}$

$f(s_3) = s_4$

$f(s_4) = s_1$

$f = \{(s_1, s_1), (s_2, s_3), (s_3, s_4), (s_4, s_1)\}$

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$R_a = \{(s_1, s_1), (s_2, s_1)\}$

$R_{?(p \vee q)} = \{(s_1, s_1), (s_3, s_3)\}$

$R_{?\neg(p \vee q)} = \{(s_2, s_2)\}$

$R_{?\neg(p \vee q);\, a;\, ?(p \vee q)} = R_{?\neg(p \vee q)} ; R_a ; R_{?(p \vee q)}$

$= \{(s_2, s_1)\}$

Prove $[\alpha^*]\varphi \rightarrow [\alpha][\alpha^*]\varphi$

| | | |
|---|---|---|
| 1. | $[\alpha^*]\varphi$ | Ass. |
| 2 | $[\alpha^*]\varphi \rightarrow (\varphi \wedge [\alpha][\alpha^n]\varphi)$ | Mix Axiom |
| 3. | $\varphi \wedge [\alpha][\alpha^n]\varphi$ | $E_\rightarrow(1,2)$ |
| 4 | $[\alpha][\alpha^*]\varphi$ | $E_\wedge(3)$ |
| 5. | $[\alpha^*]\varphi \rightarrow [\alpha][\alpha^*]\varphi$ | |

$$([\alpha](\varphi\to\psi)\wedge[\alpha]\varphi\wedge[\beta]\varphi)\to([\alpha\cup\beta]\varphi\wedge[\alpha]\psi)$$

| | | |
|---|---|---|
| 1. | $[\alpha](\varphi\to\psi)\wedge[\alpha]\varphi\wedge[\beta]\varphi$ | |
| 2. | $([\alpha]\varphi\wedge[\beta]\varphi)\to[\alpha\cup\beta]\varphi$ | Choice axiom |
| 3. | $[\alpha]\varphi\wedge[\beta]\varphi$ | $E_\wedge(1)$ |
| 4 | $[\alpha\cup\beta]\varphi$ | $E_\to(2,3)$ |
| 5 | $[\alpha](\varphi\to\psi)$ | $E_\wedge(1)$ |
| 6 | $[\alpha](\varphi\to\psi)\to([\alpha]\varphi\to[\alpha]\psi)$ | principle 2 |
| 7 | $[\alpha]\varphi\to[\alpha]\psi$ | $E_\to(5,6)$ |
| 8 | $[\alpha]\varphi$ | $E_\wedge(3)$ |
| 9 | $[\alpha]\psi$ | $E_\to(7,8)$ |
| 10 | $[\alpha\cup\beta]\varphi\wedge[\alpha]\psi$ | $I_\wedge(9,4)$ |
| 11 | Statement (too lazy to copy) | $I_\to(1,10)$ |