

Limits

Informal definition

f has limit L at a , if $f(x)$ approaches L when x approaches a

$$\lim_{x \rightarrow a} f(x) = L$$

Example:

$$\lim_{x \rightarrow 5} \frac{x^2 + x - 2}{x - 1} = 7$$

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x - 1} = \frac{(x-1)(x+2)}{x-1} = \lim_{x \rightarrow 1} (x+2) = 3$$

we can cancel out $x-1$ because by definition, $x \neq 1$, so $x-1 \neq 0$

Formal definition

$$\lim_{x \rightarrow x_0} f(x) = L$$

$$\iff \forall \varepsilon > 0 \exists \delta > 0 : 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$$

Left-Right limits

$$\left. \begin{aligned} \lim_{x \rightarrow a^-} f(x) &= \lim_{x \uparrow a} f(x) \\ \lim_{x \rightarrow a^+} f(x) &= \lim_{x \downarrow a} f(x) \end{aligned} \right\} \begin{array}{l} \text{If these do not match,} \\ \text{then } \lim_{x \rightarrow a} f(x) \text{ does not} \\ \text{exist} \end{array}$$

Theorems

$$\lim_{x \rightarrow a} f(x) = L, \quad \lim_{x \rightarrow a} g(x) = M$$

$$\cdot \lim_{x \rightarrow a} [f(x) \pm g(x)] = L \pm M \quad \cdot \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{L}, L \geq 0$$

$$\cdot \lim_{x \rightarrow a} f(x) \cdot g(x) = L \cdot M$$

$$\cdot \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$$

Interesting limits

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{x}-1}{x-1} &= \lim_{x \rightarrow 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)} \\ &= \lim_{x \rightarrow 1} \frac{x-1}{(x-1)(\sqrt{x}+1)} \\ &= \lim_{x \rightarrow 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2} \end{aligned}$$

$$\lim_{x \rightarrow 1} |x-1| = 0$$

$$\lim_{x \rightarrow 1} \frac{|x-1|}{x-1}$$

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{|x-1|}{x-1} &= \frac{x-1}{x-1} = 1 \\ \lim_{x \rightarrow 1^-} \frac{|x-1|}{x-1} &= \frac{-(x-1)}{x-1} = -1 \end{aligned} \quad \left. \vphantom{\lim_{x \rightarrow 1^+} \frac{|x-1|}{x-1}} \right\} \begin{array}{l} \text{limit} \\ \text{doesn't} \\ \text{exist} \end{array}$$

$$\lim_{x \rightarrow 0} x^2 \cos \frac{1}{x} = 0 \quad \leftarrow \text{Squeeze Theorem}$$

$$f(x) = -x^2 \quad g(x) = x^2 \cos\left(\frac{1}{x}\right)$$

$$h(x) = x^2 \quad f(x) \leq g(x) \leq h(x) \Rightarrow \lim_{x \rightarrow 0} g(x) = 0$$

Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ on (a, b) , $x_0 \in (a, b)$

$$\text{and } \lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} h(x) = L$$

$$\Rightarrow \lim_{x \rightarrow x_0} g(x) = L$$

Limits at ∞

$$\lim_{x \rightarrow \infty} f(x) = L$$

$$\iff \forall \varepsilon > 0 \exists M > 0 : x > M$$

$$\implies |f(x) - L| < \varepsilon \quad x \in \text{dom}$$

Examples

$$\lim_{x \rightarrow \pm \infty} \frac{x}{\sqrt{x^2 + 1}}$$

$$= \lim_{x \rightarrow \pm \infty} \frac{x}{\sqrt{x^2 \left(1 + \frac{1}{x^2}\right)}}$$

$$= \lim_{x \rightarrow \pm \infty} \frac{x^{+\infty}}{|x| \sqrt{1 + \frac{1}{x^2}}} \rightarrow 0$$

L

$\pm \infty$

Note:

$$|x| = \sqrt{x^2}$$

$$\sqrt{x^2(a+b)}$$

$$= \sqrt{x^2} \cdot \sqrt{a+b}$$

$$= |x| \sqrt{a+b}$$

Asymptotes

- (1) Vertical
- (2) Horizontal
- (3) Oblique

Rational functions

$$\frac{P(x)}{Q(x)} \rightarrow \text{deg. } m$$

$$Q(x) \rightarrow \text{deg. } n$$

if $m \leq n$, there always is a H.A. of $y = 0$

Analysing complex polynomials

$$f(x) = \frac{x^3 + 4x - 5}{x^2 - 3x + 1} \rightarrow x^2 - 3x + 1 \overline{) x^3 + 4x - 5} \begin{array}{r} x+3 \\ \underline{x^3 - 3x^2 + x} \\ 3x^2 + 3x - 5 \end{array}$$

$$f(x) = x+3 + \frac{12-8}{x^2-3x+1}$$

Completing the square

$$\sqrt{x^2 + 2x} = \sqrt{x^2 + 2x + 1 - 1}$$

$$= \sqrt{(x+1)^2 - 1}$$

$$= \sqrt{(x+1)^2 \left(1 - \frac{1}{(x+1)^2}\right)} = |x+1| \sqrt{1 - \frac{1}{(x+1)^2}}$$

$$3x^2 + 3x - 5$$

$$3x^2 - 9x + 3$$

$$12x - 8$$

Limits using the definition

Prove that $\lim_{x \rightarrow 3} (x^2 + x - 6) = 6$

Proof:

• Let $\varepsilon > 0$

• take $\delta = \min(1,$

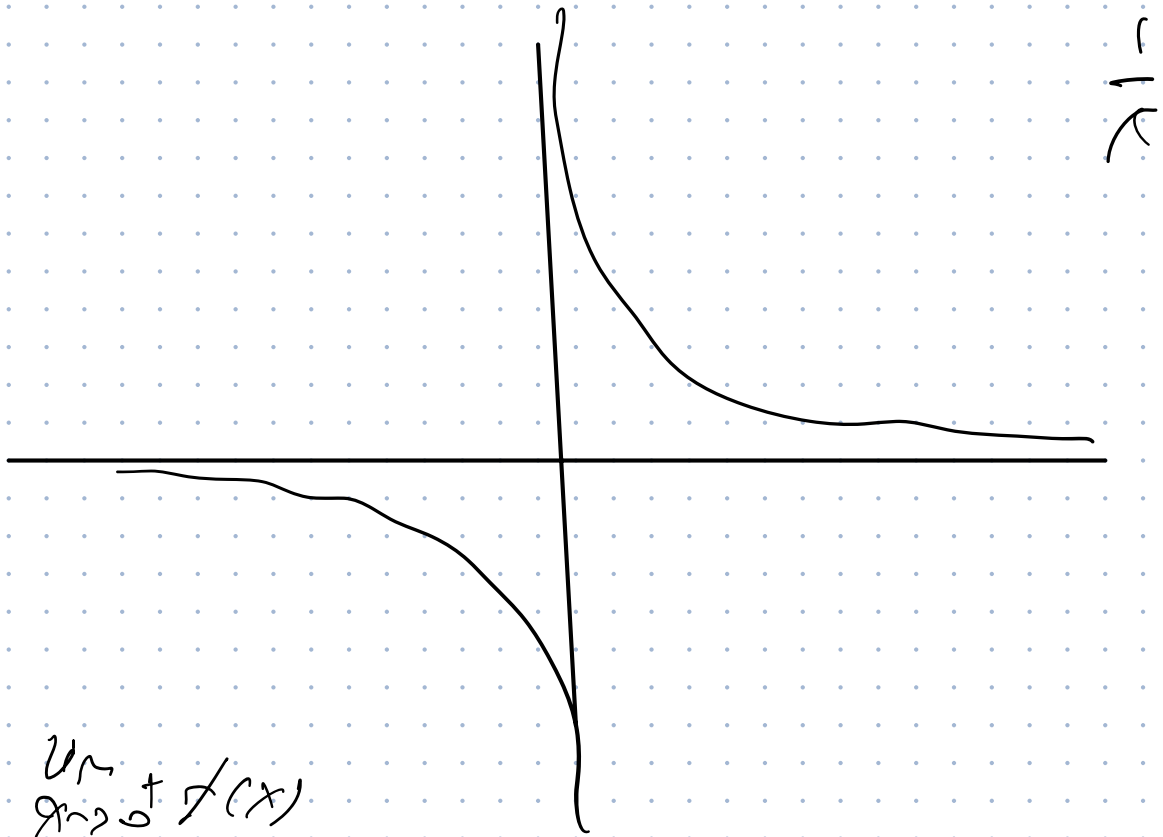
• Assume that $0 < |x - 3| < \delta$

↗ distance between x and 3 is less than δ

• then $|x^2 + x - 6 - 6| = |x^2 + x - 12|$

$$= |(x-3)(x+4)| = |x-3||x+4|$$

$$< \delta |x+4| < 8\delta \leq 6 \frac{\varepsilon}{\delta} = \varepsilon \quad \square$$



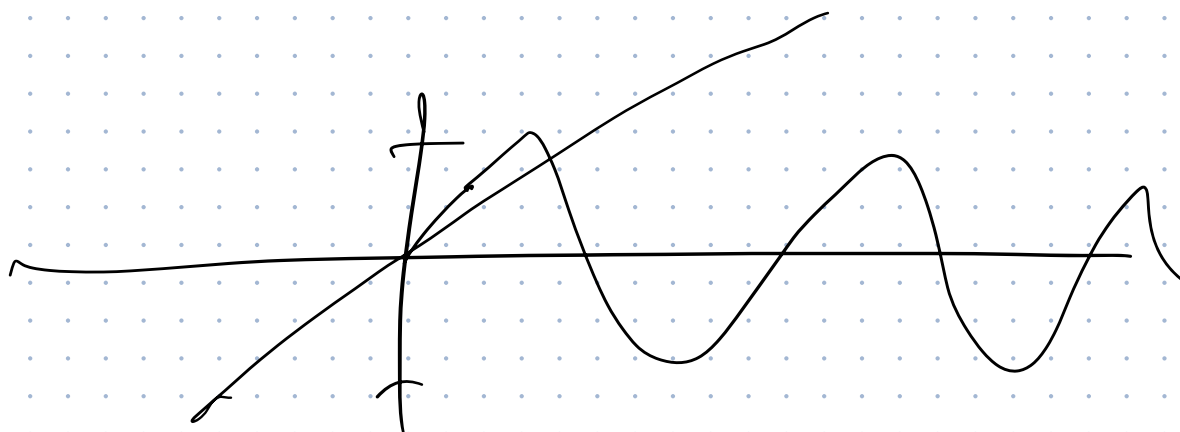
$$(1) \lim_{x \rightarrow x_0} f(x) = L \quad \leftarrow$$

$$(2) \lim_{x \rightarrow x_0} f(x) = \infty$$

$$(3) \lim_{x \rightarrow \infty} f(x) = L \quad \leftarrow$$

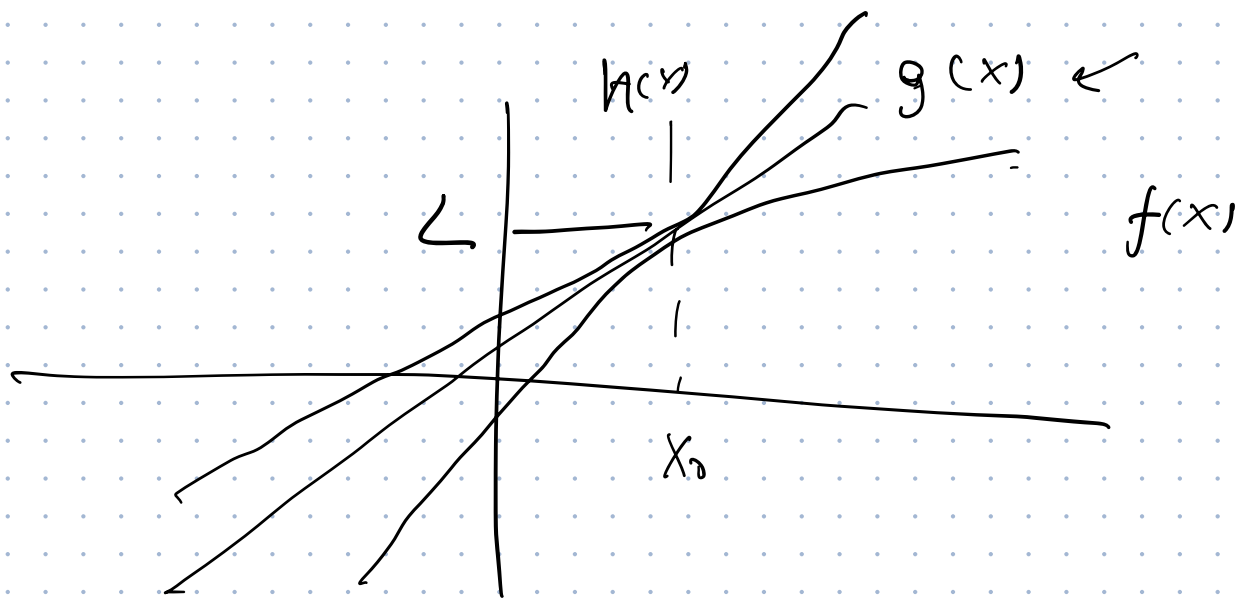
$$(4) \lim_{x \rightarrow \infty} f(x) = \infty$$

$$\lim_{x \rightarrow x_0} \sqrt{x^2 + 2} = \lim_{x \rightarrow x_0} \underset{\uparrow}{|x|} \sqrt{2}$$



$$\lim_{x \rightarrow \infty} x^2 \sin x$$

$$-x^2 \leq x^2 \sin x \leq x^2$$



$$f(x) \geq g(x) \geq h(x)$$

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} h(x) = L$$

$$\lim_{x \rightarrow x_0} g(x) = L$$