Lecture 8: Sequences and series 2

- Recap: Sequences and Series
- Convergence tests for positive series
- Absolute and conditional convergence

Adams' Ch. 9.1-9.4, Thomas' Ch. 10.1

Sequences

A sequence $\{a_n\}$ is a list of numbers a_1, a_2, …, a_n, … in a given order

term   index

* a sequence can be seen as a function $f: \mathbb{N} \to \mathbb{R}$, $n \to a_n = f(n)$
* a sequence can converge to a limit L, $a_n \to L$ if the terms approach a constant value L

examples $\frac{1}{n} \to 0$ $n^{\frac{1}{n}} \to 1$

$\frac{n}{n+1} \to 1$ $\frac{1}{2^n} \to 0$

* otherwise, the sequence can diverge to $\pm\infty$, if the terms become arbitrarily large ($\sqrt{n} \to \infty$, $(1.5)^n \to \infty$)

* the sequence a_n diverges if $\lim_{n\to\infty} a_n$ DOES NOT EXIST

example: $(-1)^n$, $\cos(\pi n)$, $\sin(n)$

Infinite series

(Infinite) series = formal sum of infinitely many terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 \dots + a_n + \dots$$

summation index can change

+ a series can be seen as a sequence of partial sum $\{s_n\}$

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$s_n = \sum_{k=1}^{n} s_k$

the series converges to s if $s_n \to s$

$$\sum_{n=1}^{\infty} a_n = s$$

* SERIES ARE AN INDETERMINATE FORM (usually)

↳ we sum up infinitely many terms (that are infinitely small)

↳ usually, we cannot calculate the sum. We can only conclude whether they converge (the sum exists)

Important series

1. Geometric series

$$a_n = a r^{n-1}$$

$$r = \frac{a_{n+1}}{a_n}$$

↳ constant ratio between terms

→ convergence if $r^n \to 0$, $-1 < r < 1$ or $a = 0$

$$\sum_{n=1}^{\infty} a_n = \frac{a}{1-r}$$

→ divergence to $+\infty$ if $r \geq 1$, $a > 0$

$-\infty$ $r \geq 1$, $a < 0$

→ divergence if $a \neq 0$, $r < -1$

2. p-series

$$a_n = \frac{1}{n^p}$$

for $p = 1$, $a_n = \frac{1}{n}$ harmonic series

→ convergence for $p > 1$

divergence to $+\infty$ for $p \leq 1$

(shown by integral test)

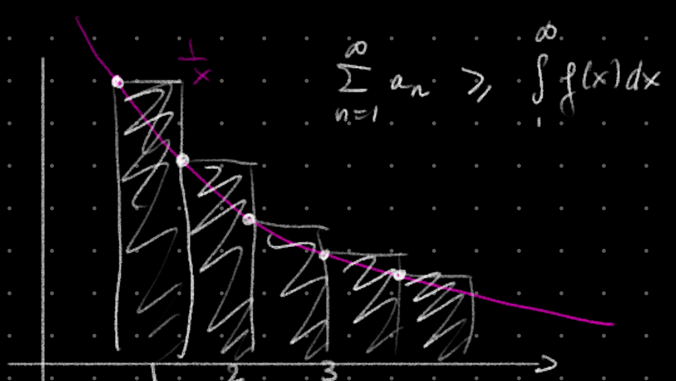
Integral test for positive series

* many convergence tests are for positive series only — adding up positive terms is conceptually easier than adding up positive and negative terms)

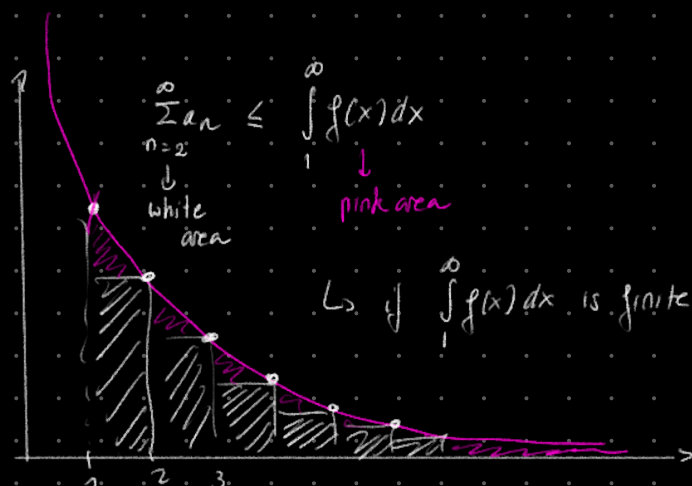
* if sequences compare to functions, series compare to improper integrals

↳ if $a_n = f(n)$, for f non-increasing on $[N, \infty)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ both converge or diverge to ∞

PROOF: the series $\sum a_n$ can be seen as both upper and lower Riemann sum.



if $\int_1^{\infty} f(x)\,dx = \infty$, then also $\sum_{n=1}^{\infty} a_n$



↳ if $\int_1^{\infty} f(x)\,dx$ is finite, so is $\sum_n^{\infty} a_n$

Comparison test (positive series)

for positive series $\sum a_n$ and $\sum b_n$

$$0 \leq a_n \leq b_n \cdot K \qquad (K > 0)$$

- if $\sum a_n$ diverges, then $\sum b_n$ diverges
- if $\sum b_n$ converges, then $\sum a_n$ converges

Examples: $\sum \frac{\ln(n)}{n}$, $\sum \frac{3n+1}{n^3+n}$

for $n \geq 3$, $\underbrace{\frac{\ln(n)}{n}}_{\text{diverges}} > \underbrace{\frac{1}{n}}_{\text{diverges}}$

$$\underbrace{\frac{3n+1}{n^3+n}}_{\text{converges}} \leq \frac{3n+1}{n^3} \leq \frac{3n+n}{n^3} = \underbrace{\frac{4}{n^2}}_{\text{converges}}$$

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$$

* we compare to $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ $\rightarrow$ this series DIVERGES, it is a p-series with

$p = \frac{1}{2} \rightarrow p < 1$

* $\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$ $\forall n \geq 2$, so, since the smaller series $\sum \frac{1}{\sqrt{n}}$ diverges, the larger series $\sum \frac{1}{\sqrt{n}-1}$ also diverges.

Limit comparison test (positive series)

for $\{a_n\}$, $\{b_n\}$ positive sequences

- if $\lim_{n \to \infty} \frac{a_n}{b_n} = L$ (exists, can be ∞)
  - if $0 < L < \infty$, $\sum a_n$ and $\sum b_n$ both converge or both diverge
  - if $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ also converges
  - if $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

- if $\lim_{n \to \infty} \frac{a_n}{b_n} = L$, $0 < L < \infty$, it means that a_n behaves like a multiple of b_n

  ↳ they do the same

- if $L = 0$, $\sum a_n$ is a lot smaller than $\sum b_n$. If the larger series converges, so does the smaller series.

- if $L = \infty$, $\sum a_n$ is a lot larger than $\sum b_n$. If the smaller series diverges, so does the larger one.

Example: $\sum_{n=5}^{\infty} \frac{n+5}{n^3-2n+3}$

compare with $\sum \frac{1}{n^2}$ (converging p-series with $p=2$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+5}{n^3-2n+3} \cdot n^2 = \lim_{n\to\infty} \frac{n^3+5n^2}{n^3-2n+3} = 1$$

↳ both series behave the same : they both converge.

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$$

compare with $\sum \frac{1}{n^{3/2}}$ (converging p-series with $p=\frac{3}{2}$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(n)}{n^2} \cdot n^{3/2} = \lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}}$$

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = 0$$

since $\sum b_n$ converges, $\sum a_n$ also converges

note: the integral test is also possible here.

Ratio test (positive series)

for a positive sequence $\{a_n\}$,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho \quad (\text{can be } \infty)$$

if $\rho < 1 \Rightarrow \sum a_n$ converges

if $\rho > 1 \Rightarrow \sum a_n$ diverges (and $a_n \to \infty$)

if $\rho = 1$ NO CONCLUSION !!

Example $\sum \frac{3^n}{n!}$ $\quad a_n = \frac{3^n}{n!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{\cancel{n}+1}}{(n+1)\cancel{!}} \cdot \frac{\cancel{n!}}{3^{\cancel{n}}} = \lim_{n\to\infty} \frac{3}{n+1} = 0$$

$\Rightarrow$ CONVERGENCE

$$\sum \frac{n^n}{n!}$$

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n \to \infty} \frac{1}{\cancel{n+1}} \cdot \left(\frac{n+1}{n}\right)^n \cdot \cancel{(n+1)}$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

DIVERGENCE!

Absolute and conditional convergence

The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum |a_n|$ is convergent

↳ absolute convergence implies convergence

↳ a series that is convergent, but not absolutely convergent, is called CONDITIONALLY convergent

* $\sum \frac{(-1)^n}{n^2}$ is absolutely convergent, since $\sum |a_n| = \sum \frac{1}{n^2}$ converges
* $\sum \frac{(-1)^n}{n}$ is NOT absolutely convergent, since $\sum |a_n| = \sum \frac{1}{n}$ diverges

Alternating series test

If 1) $\{a_n\}$ is an alternating sequence

2) $|a_{n+1}| < |a_n|$ for $n \geq N$ (decreasing in absolute value)

3) $a_n \to 0$

Then $\sum a_n$ converges

Note: first test for absolute convergence with another test. If $\sum a_n$ is not absolutely convergent, $\{a_n\}$ is alternating, use the alternating series test for to check for conditional convergence

$\sum \frac{(-1)^n}{n}$ is conditionally convergent.

Example: for what values of $x \in \mathbb{R}$ does $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n \cdot 2^n}$ converge absolutely, converge conditionally, or diverge?

1) $\sum_{n=1}^{\infty} \frac{(x-5)^n}{2^n \cdot n} = \sum_{n=1}^{\infty} \left(\frac{x-5}{2}\right)^n \frac{1}{n}$

2) Apply ratio test to $\sum \left|\frac{x-5}{2}\right|^n \frac{1}{n}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x-5}{2}\right|^{n+1} \cdot \left|\frac{x-5}{2}\right|^{-n} \cdot \frac{n}{n+1} = \left|\frac{x-5}{2}\right|$$

$\Rightarrow$ absolute convergence for $\left|\frac{x-5}{2}\right| < 1 \iff 3 < x < 7$

$\Rightarrow$ divergence (also for the sequence) for $\left|\frac{x-5}{2}\right| > 1$

$\longrightarrow$ if $x > 7$, divergence to ∞

$x < 3$, divergence (alternating sequence)

- if $x = 7$, $\sum \frac{(x-5)^n}{2^n \cdot n} = \sum \frac{1}{n} \to$ divergence to ∞
- if $x = 3$, $\sum \frac{(x-5)^n}{2^n n} = \sum \frac{(-1)^n}{n} \longrightarrow$ conditional convergence

↳ alternating
↳ $a_n \to 0$
↳ $|a_{n+1}| < |a_n|$

| alternating series test implies convergence.