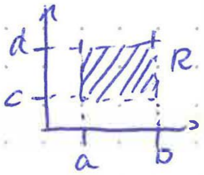
Calculus - lecture 11 Double / iterated integrals

\+ Thomas, 15.1-2
(or: Adams, 14.1-2)

I DOUBLE INTEGRALS

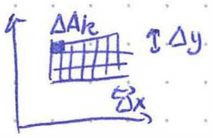
* for a function $f(x,y)$, continuous on a region R.

$R: \; a \leq x \leq b$
$\quad\; c \leq y \leq d$



How to calculate $\iint_R f(x,y)\,dA$

↳ we make a partition of R into small rectangles ΔA_k, $\Delta A = \Delta x \Delta y$



→ we can take a Riemann sum for points (x_k, y_k) in ΔA_k

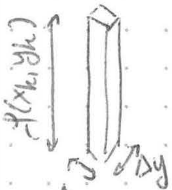
$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

* the norm $\|P\|$ of a partition is $\max\{\Delta x_k, \Delta y_k\}$
(maximal width / height)

↳ f is integrable over R if this limit exists and is finite

$$\cdot \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k = \iint_R f(x,y)\,dA$$

* an integral is the area under a curve (for $f(x) > 0$).
a double integral is the volume under a surface (for $f(x,y) > 0$)



$f(x_k, y_k)\Delta x \Delta y$ is a volume element of the Riemann sum,

* How to calculate double integrals?
as iterated integrals !!

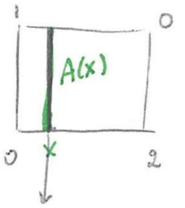
Example $f(x,y) = 4 - x - y$

$R: \; 0 \leq x \leq 2$
$\quad\;\; 0 \leq y \leq 1$

$$\iint_R f(x,y)\,dA = \overbrace{\int_0^2 \underbrace{\int_0^1 (4-x-y)\,dy}_{\text{inner integral} = A(x)} dx}^{\text{iterated integral}}$$

outer integral (number)

$$\iint_R f(x,y)\,dA = \int_0^2 \underbrace{\int_0^1 (4-x-y)\,dy}_{A(x)}\,dx$$



$$A(x) = \int_0^1 (4-x-y)\,dy = \left[4y - xy - \frac{y_2}{2}\right]_0^1 = 4 - x - \frac{1}{2} = \frac{7}{2} - x$$

↳ the outcome is a function of x

↳ x is treated as a constant in this integration

we first calculate a cross-section along y, for fixed x : A(x)

$$\iint_R f(x,y)\,dA = \int_0^2 \left(\int_0^1 (4-x-y)\,dy \right) dx = \int_0^2 \left(\frac{7}{2} - x\right) dx = \left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2 = 7 - 2 = 5$$

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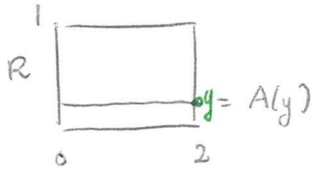
* we can also take a cross-section along x



$$\iint_R f(x,y)\,dA = \int_0^1 A(y)\,dy = \int_0^1 (6-2y)\,dy = \left[6y - y^2\right]_0^1 = 6 - 1 = 5$$

$$A(y) = \int_0^2 (4-x-y)\,dx = \left[4x - \frac{x^2}{2} - xy\right]_0^2 = (8 - 2 - 2y) = \boxed{6-2y}$$

↳ the outcome is a function of y

↳ the result of both calculations is - of course - the same

→ this is FUBINI's theorem (the order of integration does not matter)

* If f(x,y) is continuous on the rectangular region R : $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

* Example

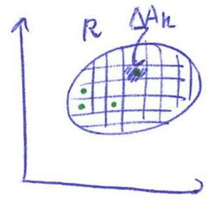
$$\iint_R (x+y)\,dA = \int_0^a \int_0^a (x+y)\,dx\,dy = \int_0^a \left[\frac{x^2}{2} + xy\right]_0^a dy = \int_0^a \left(\frac{a^2}{2} + ay\right) dy = \frac{a^2}{2}[y]_0^a + a\left[\frac{y^2}{2}\right]_0^a$$

$R: 0 \le x, y \le a$

$$= \frac{a^3}{2} + \frac{a^3}{2} = a^3$$

II DOUBLE INTEGRALS OVER GENERAL REGIONS

double integrals can be defined on more general regions than rectangles



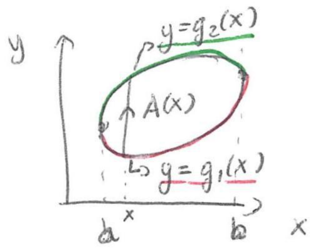
The double integral is the limit of the Riemann sums

$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_k f(x_k, y_k)\,\Delta A_k = \lim_{n\to\infty} \sum f(x_k, y_k)\,\Delta A_k$$

* formally, we only take into account rectangles ΔA_k that are FULLY inside R : as $\|P\| \to 0$, sufficiently regular areas R are filled

* for $f(x,y) > 0$, continuous on R, the integral $\iint_R f(x,y)\,dA$ is the VOLUME between R and the surface f(x,y)

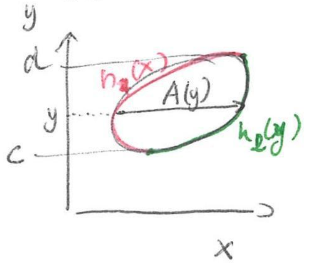
* to calculate this volume, we can parametrise the region R



then $\iint_R f(x,y)\,dA = \int_a^b A(x)\,dx$

with the cross-section $A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$

OR



then $\iint_R f(x,y)\,dA = \int_c^d A(y)\,dy$

with the cross-section $A(y) = \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx$

→ both results are the same ⇒ stronger form of Fubini's theorem.

* If $f(x,y)$ is continuous on a region R

→ if R is defined as $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with $g_1(x), g_2(x)$ continuous on $[a,b]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

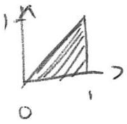
→ if R is defined as $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with $h_1(y), h_2(y)$ continuous on $[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Example

$f(x,y) = 3-x-y$

R :



1) R: $x: 0 \to 1$
$y: 0 \to x$

$$\iint_R f(x,y)\,dA = \int_0^1 \int_0^x (3-x-y)\,dy\,dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^1 \left(3x - x^2 - \frac{x^2}{2}\right)dx = \frac{3}{2}[x^2]_0^1 - \frac{1}{2}[x^3]_0^1 = 1$$

2) R: $y: 0 \to 1$
$x: y \to 1$

$$\iint_R f(x,y)\,dA = \int_0^1 \int_y^1 (3-x-y)\,dx\,dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_y^1 dx$$

$$= \int_0^1 \left(\left(3 - \frac{1}{2} - y\right) - \left(3y - \frac{y^2}{2} - y^2\right)\right)dy = \frac{5}{2} - 2[y^2]_0^1 + \frac{1}{2}[y^3]_0^1$$

$$= 1$$

* Properties of double integrals

For $f(x,y)$, $g(x,y)$ continuous on R (bounded)

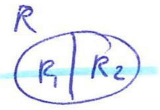
* $\iint_R c\,f(x,y)\,dA = c\iint_R f(x,y)\,dA$ for $c\in\mathbb{R}$

* $\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$

* if $f(x,y) \geq 0$ on R, then $\iint_R f(x,y)\,dA \geq 0$

if $f(x,y) \geq g(x,y)$ on R, then $\iint_R f(x,y)\,dA \geq \iint g(x,y)\,dA$

* if R is partitioned in R_1, R_2



$\rightarrow \iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

Fror * Triangle inequality $\left|\iint_R f(x,y)\,dA\right| \leq \iint_R |f(x,y)|\,dA$

* integrals correspond to volumes / negative integrals to volumes below the xy-plane

Examples

- $\int_0^1\int_0^x \frac{\sin(x)}{x}\,dy\,dx$
- $\int_0^1\int_{y^2}^{y} dx\,dy$
- $\int_0^2\int_{-x}^{x} xy^2\,dy\,dx$