

Natural Deduction

Conditional Proof

$A \rightarrow C$

① Assume A

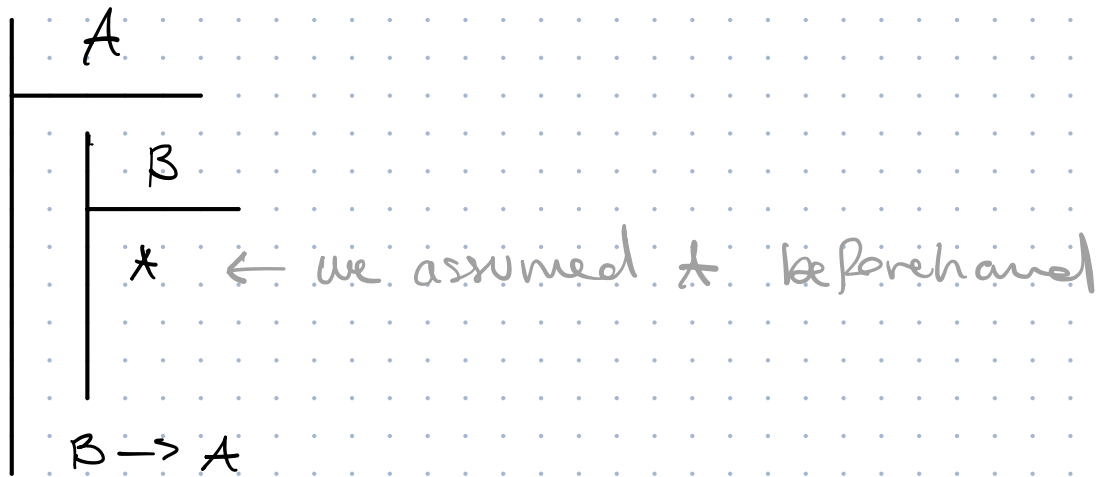
② Reasoning

③ Infer C

④ Conclude $A \rightarrow C \rightarrow$ cancel assumption

Examples

Prove $\vdash A \rightarrow (B \rightarrow A)$

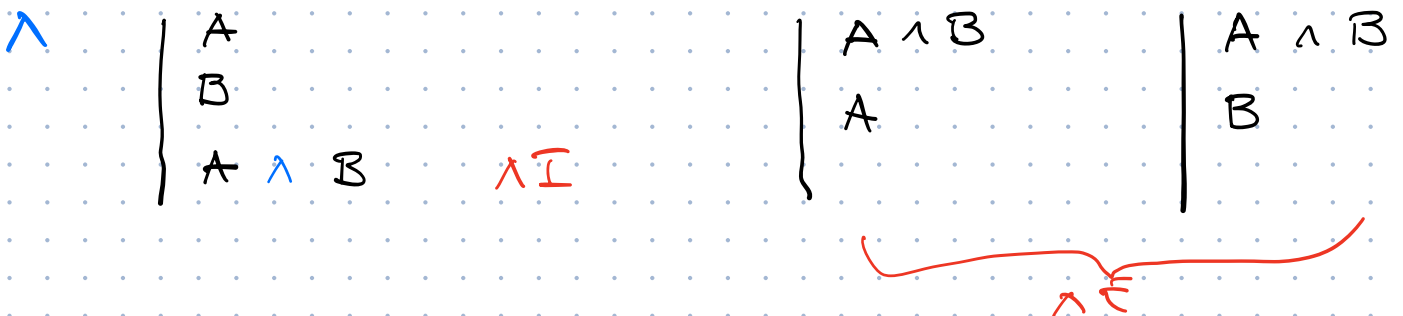


$A \rightarrow (B \rightarrow A)$

Rules

① Introduction rule \rightarrow introduce connectives into a proof

② Elimination rule \rightarrow take connective out



$$\checkmark \quad \left| \begin{array}{c} A \\ A \vee B \end{array} \right| \quad \left| \begin{array}{c} A \\ B \vee A \end{array} \right|$$

$\vee I$

$$\left| \begin{array}{c} A \vee B \\ \hline A \\ \hline C \end{array} \right| \quad \left| \begin{array}{c} B \\ \hline C \end{array} \right| \quad \left. \begin{array}{c} A \\ B \end{array} \right\} \text{Assume } A \text{ or } B$$

$\vee E$

Conditional Proof

$$\rightarrow \quad \left| \begin{array}{c} \hline A \\ \hline B \end{array} \right|$$

$A \rightarrow B \quad \rightarrow I$

Modus Ponens

$$\left| \begin{array}{c} A \rightarrow B \\ A \\ B \end{array} \right|$$

$\rightarrow E$

$$\leftrightarrow \quad \left| \begin{array}{c} A \rightarrow B \\ B \rightarrow A \\ A \leftrightarrow B \end{array} \right|$$

$\leftrightarrow I$

$$\left| \begin{array}{c} A \leftrightarrow B \\ (A \rightarrow B) \wedge (B \rightarrow A) \end{array} \right|$$

$\leftrightarrow E$

$\perp \rightarrow$ Arbitrary contradiction (False)

Reductio ad Absurdum

$$\neg \quad \left| \begin{array}{c} \hline A \\ \hline \perp \end{array} \right|$$

$\neg A \quad \neg A$

Indirect Proof

$$\perp \quad \left| \begin{array}{c} \hline \neg A \\ \hline \perp \end{array} \right|$$

A

$$\left| \begin{array}{c} \neg A \\ A \\ \perp \end{array} \right|$$

$\neg E (II)$

$$\left| \begin{array}{c} \perp \\ A \end{array} \right|$$

can be anything
 $\perp E (X)$

Examples

$$(1) \quad p \rightarrow (q \rightarrow r) \vdash q \rightarrow (p \rightarrow r)$$

$$\begin{array}{|l}
 \hline
 p \rightarrow (q \rightarrow r) \\
 \hline
 \begin{array}{|l}
 \hline
 q \\
 \hline
 \begin{array}{|l}
 \hline
 p \\
 \hline
 q \rightarrow r \\
 r \\
 \hline
 p \rightarrow r \\
 \hline
 \end{array} \\
 \hline
 \end{array} \\
 \hline
 \end{array}$$

assume q (purple arrow from q to the outer scope)
 assume p to get $p \rightarrow r$ (purple arrow from p to the inner scope)
 prove this using the premise (grey arrow from $p \rightarrow r$ to the text)

$\rightarrow E$
 $\rightarrow E$
 $\rightarrow I$
 $\rightarrow I$

$$\begin{array}{|l}
 \hline
 A \rightarrow B \\
 \hline
 A \\
 B \\
 \hline
 \end{array}$$

$q \rightarrow (p \rightarrow r)$

$$(2) \quad p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$$

$$\begin{array}{|l}
 \hline
 p \rightarrow (q \rightarrow r) \\
 \hline
 \begin{array}{|l}
 \hline
 p \wedge q \\
 \hline
 p \\
 q \\
 \hline
 q \rightarrow r \\
 \hline
 r \\
 \hline
 \end{array} \\
 \hline
 \end{array}$$

Assume this one (grey arrow from $p \rightarrow (q \rightarrow r)$ to the text)

$\wedge E$
 $\wedge E$
 $\rightarrow E$ (Modus Ponens)
 $\rightarrow E$ (Modus Ponens)
 $\rightarrow I$ Conditional Proof

$(p \wedge q) \rightarrow r$

Strategies

(1) Look at the target

$$A \vdash B \rightarrow C$$

(1) A as premise

(2) Assume B

(3) Use premises (1) + derived to infer C

(4) Infer $B \rightarrow C$

More Examples

(1) $p \rightarrow (q \rightarrow r) \vdash (p \rightarrow q) \rightarrow (p \rightarrow r)$

0	$p \rightarrow (q \rightarrow r)$	
1	$p \rightarrow q$	
2	p	
3	$q \rightarrow r$	MP (2, 0)
4	q	MP (2, 1)
5	r	MP (4, 3)
6	$p \rightarrow r$	$\rightarrow I(2, 5)$
7	$(p \rightarrow q) \rightarrow (p \rightarrow r)$	$\rightarrow I(1, 6)$

Equivalence

$A \dashv\vdash B$ (Proof both ways)

Examples

$$(A \rightarrow B) \vdash \neg(A \wedge \neg B)$$

$$0 \quad (A \rightarrow B)$$

$$1 \quad A \wedge \neg B$$

$$2 \quad A$$

$$3 \quad B$$

$$\neg B$$

$$\perp$$

$$\neg(A \wedge \neg B)$$

MP (0,2)

(1)

$\downarrow I$

$$\neg(A \wedge \neg B)$$

$$A$$

$$\neg B$$

$$A \wedge \neg B$$

$$\perp$$

$$B$$

$$A \rightarrow B$$

$$(2) \quad \neg A \wedge \neg B \vdash \neg(A \vee B)$$

$$0 \quad \neg A \wedge \neg B$$

$$1 \quad A \vee B$$

$$2 \quad A$$

$$3 \quad \neg A$$

$$4 \quad \perp$$

$$5 \quad B$$

$$6 \quad \neg B$$

$$7 \quad \perp$$

$$8 \quad \neg(A \vee B)$$

0

~~RAX~~

$$\neg(A \vee B)$$

$$A$$

$$A \vee B$$

$$\perp$$

$$\neg A$$

$$B$$

$$A \vee B$$

$$\perp$$

$$\neg B$$

$$\neg A \wedge \neg B$$

$\downarrow I$

Quantifier Rules

Universal Elimination

$\forall x Fx$	$\forall x A(\dots x \dots x \dots)$	
Fa	$A(\dots c \dots c \dots)$	$\forall E$

Existential Introduction

Fa	$A(\dots c \dots c \dots)$	
$\exists x Fx$	$\exists x A(\dots c \dots x \dots)$	x must not be in A not be

Universal Introduction

Fa	$\leftarrow a$ is a new name	
$Fa \vee Ga$		
$Fa \rightarrow (Fa \vee Ga)$		$A(\dots c \dots c \dots)$
$\forall x (Fx \rightarrow (Fx \vee Gx))$		$\forall x A(\dots x \dots x \dots)$

Existential Elimination

$\exists x A(\dots x \dots x \dots)$	
$A(\dots c \dots c \dots)$	
B	
B	

Examples

$$(1) \exists x Fx \rightarrow Ga \vdash \forall x (Fx \rightarrow Ga)$$

0	$\exists x Fx \rightarrow Ga$	
1	Fb	← assume F for arbitrary value
2	$\exists x Fx$	\exists intro
3	Ga	$\rightarrow E$ (Modus Ponens) (0, 2)
4	$Fb \rightarrow Ga$	$\rightarrow I$ (1, 3)
	$\forall x (Fx \rightarrow Ga)$	arbitrary instance of target $\forall I$

$$(2) \exists x (Fx \wedge Gx) \vdash \exists x Fx \wedge \exists x Gx$$

	$\exists x (Fx \wedge Gx)$	take arbitrary
	$Fa \wedge Ga$	
	Fa	$\wedge E$
	$\exists x Fx$	\exists Intro
	Ga	$\wedge E$
	$\exists x Gx$	$\exists I$
	$\exists x Fx \wedge \exists x Gx$	$\wedge I$
	$\exists x Fx \wedge \exists x Gx$	

More Examples

11) $q \rightarrow (p \rightarrow r) \quad \vdash (s \rightarrow p) \rightarrow (q \rightarrow (s \rightarrow r))$

0	$q \rightarrow (p \rightarrow r)$		
1	$s \rightarrow p$		
2	q		
3	s	3	s
4	r	4	$p \rightarrow r$
5	$s \rightarrow r$	5	p
6	$q \rightarrow (s \rightarrow r)$	6	r
7			
8			
9	$(s \rightarrow p) \rightarrow (q \rightarrow (s \rightarrow r))$		

MP (e, 2)

MP (1, 3)

MP (4, 5)

assume \vdash WS if, then get that