Section 2.1

A **set** is a collection of specified objects. These objects are called the **elements** of the set.

Example 1 *$S = \{a, z, 33, \clubsuit\}$ is a set consisting of 4 elements (a, z, 33 and $\clubsuit$), placed between **braces** $\{\ \}$. The order of the elements does not matter, so*

$$\{a, z, 33, \clubsuit\} = \{33, \clubsuit, z, a\}$$

Some definitions: A is a **subset** of B, notation: $A \subseteq B$, if every element of A is also in B, or formally:

$$x \in A \quad \Rightarrow \quad x \in B$$

Notice that every set A is a subset of itself: $A \subseteq A$. A is a **proper subset** of B, notation: $A \subset B$, if $A \subseteq B$ and if there is an element in B that is not in A. The latter means that B is not a subset of A. This is denoted as follows: $B \nsubseteq A$

Sets A and B are equal if they are subsets of each other

$$A = B \quad \Leftrightarrow \quad A \subseteq B \ \wedge \ B \subseteq A.$$

Subsets of $\mathbb{R}$:

An **interval** is a set of real numbers with the property that any number that lies between two numbers in the set, is also in the set. For example, $\{x \in \mathbb{R} : -2 \leq x \leq 1\}$ is an interval. It includes -2 and 1 and also all numbers in between. An interval is called **closed** if it includes the *endpoints*. The above interval is closed: The endpoints (-2 and 1) are both in the set. A closed interval can be described by rectangular brackets [ ] wherein the endpoints are written, the smallest one first, separated by a comma. The above interval can be written as $[-2, 1]$. An interval is called **open** if it does not include any of the endpoints. An open interval is most often denoted by round brackets ( ). Also triangular brackets $\langle\ \rangle$ or rectangular brackets] [are sometimes used, but we will only use round brackets. An example: $(-2, 1) = \{x \in \mathbb{R} : -2 < x < 1\}$. If in an interval one endpoint is included and the other one is not, we call the interval **half-open**. Notation ( ] or [ ).

It is possible to describe a set in many different ways. The sets A and B below are described in several ways:

$$\begin{aligned} A &= \{n : n \text{ is a natural number between 3 and 18 inclusive}\} \\ &= \{n \in \mathbb{N} : 3 \leq n \leq 18\} \\ &= \{3, 4, 5, \ldots, 18\} \end{aligned}$$

or

$$\begin{aligned} B &= \left\{x : x \text{ is a real number such that } x^2 + x - 2 \leq 0\right\} \\ &= \left\{x \in \mathbb{R} : x^2 + x - 2 \leq 0\right\} \\ &= \{x \in \mathbb{R} : -2 \leq x \leq 1\} \\ &= [-2, 1] \text{ (the closed interval from -2 to 1)} \end{aligned}$$

Elements of sets may be sets themselves as the following example shows:

Example 2

$$A = \{1, 2, \{1,2\}, \{\{1,2\}, 1, 2\}\}.$$

The elements of A are $1,\ 2$, the set $\{1,2\}$ and the set $\{\{1,2\}, 1, 2\}$, which itself also contains a set. So:

$$\begin{aligned} 1 &\in A \\ 2 &\in A \\ \{1,2\} &\in A \\ \{\{1,2\}, 1, 2\} &\in A \end{aligned}$$

Some of the subsets of A are $\{1\},\ \{1,2\},\ \{\{1,2\}\}$ and $\{\{1,2\}, \{\{1,2\}, 1, 2\}\}$, where the last one consists of 2 elements, so

$$\begin{aligned} \{1\} &\subseteq A \\ \{1,2\} &\subseteq A \\ \{\{1,2\}\} &\subseteq A \\ \{\{1,2\}, \{\{1,2\}, 1, 2\}\} &\subseteq A \end{aligned}$$

Notice that $\{1,2\}$ is both an element and a subset of A!

Section 2.2

Several sets:

$\emptyset$	empty set
U	universal set (contains ”everything”). Could be $\mathbb{N}$, $\mathbb{R}$ etc.
$A^c = \{x \in U : x \notin A\}$	complement of A
$A \cup B = \{x \in U : (x \in A) \vee (x \in B)\}$	union of A and B
$A \cap B = \{x \in U : (x \in A) \wedge (x \in B)\}$	intersection of A and B
$A \setminus B = \{x \in U : (x \in A) \wedge (x \notin B)\}$	difference of A and B

Example 3 *We prove the results given in example 16 in the book. Notice first that* ***to prove that 2 sets are equal, we have to prove that each set is a subset of the other.***

I $\emptyset^c = U$. *To prove:* $\emptyset^c \subseteq U$ *and* $U \subseteq \emptyset^c$. *The former is trivial: As U contains everything, every set is a subset of U. The proof of the latter: Let $x \in U$. Then $x \notin \emptyset$. Hence $x \in \emptyset^c$.*

II $(A^c)^c = A$. *To prove (i)* $(A^c)^c \subseteq A$ *and (ii)* $A \subseteq (A^c)^c$. *(i): Let $x \in (A^c)^c$. Then $x \notin A^c$. But then $x \in A$. (ii): Let $x \in A$. Then $x \notin A^c$. But then $x \in (A^c)^c$.*

III $U^c = \emptyset$. *Proof:*

$$\begin{aligned} U^c &= (\emptyset^c)^c \ \textit{by I} \\ &= \emptyset \ \textit{by II} \end{aligned}$$

Exercise 1 *Prove the following proposition: If $A \subseteq B$, then $B^c \subseteq A^c$.*

Notice that

$$A \cup A^c = U$$
$$A \cap A^c = \emptyset$$

Example 22 in the book shows that $(A \subseteq B) \Rightarrow (A \cup B = B)$. It is left as an exercise to prove the following similar propositions:

Exercise 2 *Prove the following propositions:*

$$(A \subseteq B) \Rightarrow (A \cap B = A)$$
$$(A \subseteq B) \Rightarrow (A \setminus B = \emptyset)$$

Some theorems on intersections and unions (associative and distributive laws):

$$\begin{aligned}
\text{(i)}: &\ (A \cup B) \cup C = A \cup (B \cup C)\\
\text{(ii)}: &\ (A \cap B) \cap C = A \cap (B \cap C)\\
\text{(iii)}: &\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\\
\text{(iv)}: &\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)
\end{aligned}$$

We prove (i). The proofs of (ii) and (iii) can be found in the book; the proof of (iv) is left as an exercise.

Proof of (i): Recall that to prove that 2 sets are equal, we have to prove that they are subsets of each other. So we need to prove 2 things!

Proof of $(A \cup B) \cup C \subseteq A \cup (B \cup C)$:

$$\begin{aligned}
&x \in (A \cup B) \cup C\\
&\quad \Rightarrow (x \in A \cup B) \vee (x \in C)\\
&\quad \Rightarrow x \in A \vee x \in B \vee x \in C\\
&\quad \Rightarrow x \in A \vee x \in (B \cup C)\\
&\quad \Rightarrow x \in A \cup (B \cup C)
\end{aligned}$$

The proof of $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ is the same with the implication arrows in the opposite direction. You are allowed to just mention this instead of writing a whole new proof down, but do not forget it, since otherwise you only proved half of the proposition!

Exercise 3 *Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$*

As a result of these theorems, if in an exercise you encounter any of these sets, you are allowed to replace them by the other one, if you think that that facilitates the analysis (the theorems do not have to be proved again!).

Exercise 4 *Prove or disprove:* $A \subseteq B \Rightarrow A \cup (B \cap C) = B \cap (A \cup C)$

Exercise 5 *(exercise 2.2.2 in the book): Prove that in general* $A \cap (B \cup C) \neq (A \cap B) \cup C$.

Section 2.3

De Morgan's laws are the following set-theoretic theorems on complements of sets:

$$\begin{aligned} \text{(i)}: &\quad (A \cup B)^c = A^c \cap B^c \\ \text{(ii)}: &\quad (A \cap B)^c = A^c \cup B^c \end{aligned}$$

The proof of (i) can be found in the book; we will prove (ii):

$(A \cap B)^c \subseteq A^c \cup B^c$: Let $x \in (A \cap B)^c$. Then $x \notin (A \cap B)$. But then $(x \notin A) \vee (x \notin B)$, which means that $(x \in A^c) \vee (x \in B^c)$, from which we conclude that $x \in (A^c \cup B^c)$. The proof that $A^c \cup B^c \subseteq (A \cap B)^c$ is the same with the implication arrows in the opposite direction.

Again: You are allowed to apply De Morgan's laws directly (without proof) in any exercise.

Example 4 *Prove that* $(B \subseteq A) \Rightarrow (A^c \subseteq (A \cup B)^c)$.

Proof. Notice first that $(A \cup B)^c = A^c \cap B^c$ by De Morgan.
Let $B \subseteq A$. This means that $(x \in B) \Rightarrow (x \in A)$, or contrapositively:

$$(x \notin A) \Rightarrow (x \notin B). \tag{1}$$

We have to prove that as a logical consequence we have that $A^c \subseteq (A \cup B)^c$. So we suppose that $y \in A^c$. Then $y \notin A$. But then, by (1), $y \notin B$, which means that $y \in B^c$. So $y \in A^c \cap B^c$, which completes the proof. ■

Exercise 6 *Prove or disprove:* $(A \cup B^c) \cap (A^c \cup B) = (A \cup B)^c \cup (A \cap B)$

Exercise 7 *Prove or disprove:* $(B \cap C^c) \cup (A \setminus B) = (A \cup B) \cap C^c$

Section 2.4

A **power set** of a set S is the set consisting of ALL subsets of S and is denoted by $\mathcal{P}(S)$.

Example 5 *The power sets of* $\{\spadesuit, \diamondsuit\}$ *and* $\{1, 3, \{1, 3\}\}$*:*

$\mathcal{P}(\{\spadesuit, \diamondsuit\}) = \{\emptyset, \{\spadesuit\}, \{\diamondsuit\}, \{\spadesuit, \diamondsuit\}\}$

$\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Some more power sets:

$\mathcal{P}(\emptyset) = \{\emptyset\}$

$\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$

$\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

$\mathcal{P}(\{1, 3, \{1, 3\}\}) = \{\emptyset, \{1\}, \{3\}, \{\{1, 3\}\}, \{1, 3\}, \{1, \{1, 3\}\}, \{3, \{1, 3\}\}, \{1, 3, \{1, 3\}\}\}$

Notice that the empty set is always in $\mathcal{P}(S)$, as it is always a subset of S.

Exercise 8 *List* $\mathcal{P}(A)$ *for the set* A *in example 2.*

The **size** of a set S, notation: $|S|$, is the number of elements that is in the set. So $|\{\spadesuit, \diamondsuit\}| = 2$, $|\mathcal{P}(\{\spadesuit, \diamondsuit\})| = 4$, $|\{1, 3, \{1, 3\}\}| = 3$ and $|\mathcal{P}(\{1, 3, \{1, 3\}\})| = 8$.

Theorem 1 $|S| = n \Rightarrow |\mathcal{P}(S)| = 2^n$.

The argument of the proof is that if we consider an arbitrary subset T of S, then for every element in S we have that it is either in T or not in T. That gives us 2 possibilities per element of S and hence $2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ possibilities altogether. There is an alternative notation for the power set of a set S, which is related to theorem 1, namely 2^S.

Example 6 *(exercise 2.4.4 in the book): Are the following propositions true or false?*

1. $A \cup \mathcal{P}(A) = \mathcal{P}(A)$*: False. Counterexample:* $A = \{1\}$ *and notice that 1 is not an element of* $\mathcal{P}(A)$*.*
2. $A \cap \mathcal{P}(A) = A$*: False, same counterexample.*

Exercise 9 *Prove or disprove each of the other 2 propositions of exercise 2.4.4:*

1. $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$
2. $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$

Here we squeeze in some of the material from chapter 4 on combinatorics. Some notation:

- $n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$ (pronounce: n factorial)
- $\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$ is the number of combinations of k out of n. This number is positive for values of $k \in \{0, 1, 2, \ldots, n\}$.

Suppose that the following question is asked: In how many ways can we select k elements from a given set S of size n, (so $|S| = n$)? The answer to such a question depends on two things:

1. Whether we allow the same elements to be drawn more than once or not. Or: Is repetition allowed or not?
2. Whether the order in which we draw the elements matters or not.

This gives four different answers. In section 4.3 of the book we can find them, listed in the following table:

	order matters	order does not matter
repetition allowed	n^k	$\binom{n-1+k}{k}$
repetition not allowed	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Let's show again that theorem 1 is true. We consider the set S with $|S| = n$ and we want to know how many subsets it has. Let's create one: $A \subseteq S$. What are the possibilities? For each element $s \in S$ we know that it is either in A or it is not in A. So create A, we select from the set $\{\text{in } A, \text{not in } A\}$ and depending on the outcome we decide whether s is in A. We do this n times in total (namely for each $s \in S$). Now repetition is allowed (we may select 'in A' and 'not in A more than once) and order matters (since each selection concerns a different element of S). According to the table this can be done in 2^n ways, so there are 2^n subsets of S.

Exercise 10 *A* ***multiset*** *is a modification of the concept of a set that, unlike a set, allows for multiple instances of its elements. For example, given a set $A = \{a, b\}$ we can create several multisets using only the elements from A, like $\emptyset$, $\{a, b\}$, $\{a, a, b\}$ and $\{b, b, b, b\}$. Notice that the elements of A don't have to be present in the multiset and that the multiset $\{a, b\}$ and $\{a, a, b\}$ are different as they contain a different number of instances of the element a. Also, $\{a, a, b\}$ and $\{a, b, a\}$ are the same multiset, as like in a normal set, the order of the elements does not matter. The size (or cardinality) of a multiset, equals the total number of instances in the multiset. So the cardinality of $\{a, a, b\}$ is 3 and the cardinality of $\{b, b, b, b\}$ is 4.*
Let $B = \{1, 2, \ldots, m\}$ (or any set consisting of m elements). How many multisets with cardinality n can we make that contain only elements of B?

Section 2.5

The principle of **inclusion-exclusion** gives the following equation between the sizes of the sets A, B, $A \cap B$ and $A \cup B$:

$$|\, A \cup B \,| = |\, A \,| + |\, B \,| - |\, A \cap B \,|$$

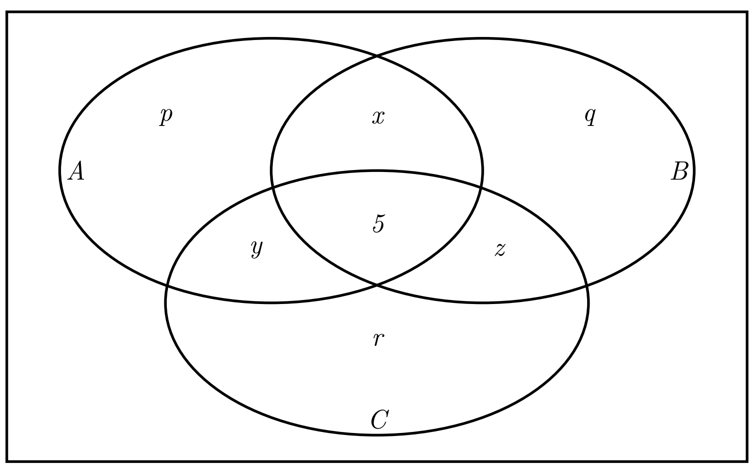
The intuition is simple. If, instead of counting the number of elements in $A \cup B$, you count the elements in A and the elements in B, then you counted the elements twice that were both in A and in B. So by subtracting $|\, A \cap B \,|$ you obtain the correct number. It may be useful to draw a Venn-Diagram to check this result. A similar argument can be applied to find that for 3 sets A, B and C we have:

$$|\, A \cup B \cup C \,| = |\, A \,| + |\, B \,| + |\, C \,| - |\, A \cap B \,| - |\, A \cap C \,| - |\, B \cap C \,| + |\, A \cap B \cap C \,|$$

To convince yourself draw a Venn-Diagram and check that each area of the set $A \cup B \cup C$ in the diagram has been counted exactly once!

For more than 3 sets the inclusion-exclusion rule says that you have to add all the numbers of the elements in the intersections of an odd number of sets, whereas you have to subtract all the numbers of elements in the intersection of an even number of sets.

Example 7 *(exercise 2.5.6 in the book): A good idea is to draw a Venn-Diagram. Let* A*,* B *and* C *be the sets of students who were able to do exercises 1, 2 and 3 respectively. Then* $|U| = 40$*,* $|A \cup B \cup C| = 40$*,* $|A^c| = 10$*,* $|B^c| = 15$*,* $|C^c| = 20$ *and* $|A \cap B \cap C| = 5$*. In the diagram this looks as follows:*



and notice that we are interested in $x + y + z$*. Avoiding the principle of inclusion-exclusion we can, from figure 3 and the data combined, conclude that*

$$\begin{cases} p + q + r + x + y + z = 35 \\ q + r + z = 10 \\ p + r + y = 15 \\ p + q + x = 20 \end{cases}$$

Summing the last 3 equation yields

$$\begin{aligned} x+y+z &= 45 - 2\cdot(p+q+r) \\ &= 45 - 2\cdot(35 - x - y - z) \text{ (from the first equation)} \\ &\Rightarrow -1\cdot(x+y+z) = 45 - 2\cdot 35 = -25 \\ &\Rightarrow x+y+z = 25 \end{aligned}$$

Notice that from the data it is not possible to calculate x, y and z separately! It is possible however, to obtain the solution a bit quicker, making use of the principle of inclusion-exclusion: Notice that $|A| = 30$, $|B| = 25$ and $|C| = 20$. We obtain

$$\begin{aligned} 40 &= 30 + 25 + 20 - |A\cap B| - |A\cap C| - |B\cap C| + 5, \\ &\Rightarrow |A\cap B| + |A\cap C| + |B\cap C| = 40 \\ &\Rightarrow x+y+z = 40 - 3\cdot 5 = 25 \end{aligned}$$

Exercise 11 *Make exercises 2, 3 and 4 of section 2.5.*

Section 2.6

For any 2 sets A and B the **product set** $A\times B$ is defined as follows:

$$A\times B = \{(a,b) : a\in A,\, b\in B\}$$

The pair (a,b), being an element of the set $A\times B$, is called a **tuple**.

Example 8 *Let $A=\{1,3\}$, $B=\{1,2,6\}$, $C=\{4,5\}$ and $D=\{3\}$. Then*

$$\begin{aligned} A\times A &= \{(1,1),(1,3),(3,1),(3,3)\} \\ B\times C &= \{(1,4),(1,5),(2,4),(2,5),(6,4),(6,5)\} \\ D\times B &= \{(3,1),(3,2),(3,6)\} \\ D\times A\times C &= \{(3,1,4),(3,1,5),(3,3,4),(3,3,5)\} \end{aligned}$$

The product set $A\times A$ is also denoted A^2 (pronounce A-square). The most common product set is $\mathbb{R}^2$, the set of all tuples of real numbers.

Example 9 *Let $A=\{\spadesuit,\heartsuit,\diamondsuit,\clubsuit\}$ and let $B=\{2,3,4,\ldots,9,10,J,Q,K,A\}$. Then $A\times B$ is a deck of cards!!*

The number of elements in the product set $A\times B$ is equal to the product of the numbers of elements in A and B, or

$$|A\times B| = |A|\cdot|B|$$

Exercise 12 *Argue why the above equation is correct. You may assume that the sets are finite (see section 2.7 for infinite sets).*

Exercise 13 *Let* $A = \{2, 3, 5, 7\}$*,* $B = \{1\}$ *and* $C = \{6, 8, 10\}$*. Calculate* $A \times B$*,* $B \times C$ *and* $B \times B \times C \times B \times B \times C$*.*

A **partition** of a set A is a collection of nonempty subsets A_i, $i = 1, 2, \ldots, m$, such that

$$\begin{aligned} (1) &: \ A_i \cap A_j = \emptyset \text{ for all } i, j \in \{1, 2, \ldots, m\}, i \neq j \\ (2) &: \ A_1 \cup A_2 \cup \ldots \cup A_m = A \end{aligned}$$

so all subsets are **disjoint** (the formal term to denote that the intersections are empty, even though none of the sets A_i is empty) and the union of all the subsets is the entire set A. In example 8 the sets B, C and D form a partition of $\{1, 2, 3, 4, 5, 6\}$.

Product sets and partitions will be used in chapter 3 with respect to relations.

Section 2.7

Two sets are of the same size if their elements can be put in a one-to-one correspondence. For finite sets this clearly means that both sets are of size N (some natural number). For infinite sets we say that a set is **countable** if it is of the same size as $\mathbb{N}$ (the set of natural numbers). This means that we should be able to put the elements of the set in a list such that we can give the exact position of each element in our list. Every infinite set that is not of the same size as $\mathbb{N}$ (which means that it is bigger than $\mathbb{N}$) is called **uncountable**.

Example 10 *Prove that* $\mathbb{Z}$ *(the set of integers) is countable.*

Proof: Write $\mathbb{Z}$ *as follows:* $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots\}$*. Then for every integer we can find its exact position in the list, thereby making it a countable set.*

Example 11 *Other examples of countable sets:*

1. The set of prime numbers: $\{2, 3, 5, 7, 11, 13, \ldots\}$
2. The set of squares of integers: $\{0, 1, 4, 9, 16, 25, \ldots\}$
3. The set of square roots of natural numbers: $\left\{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \ldots\right\}$
4. The product set $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N} = \{(m, n) : m \in \mathbb{N}, n \in \mathbb{N}\}$
   $= \{(1, 1), (2, 1), (1, 2), (3, 1), (2, 2), (1, 3), (4, 1), (3, 2), \ldots\}$

Exercise 14 *Prove or disprove the following proposition:* $\mathbb{Q}$ *(the set of rational numbers) is countable.*

Example 12 *Prove that $\mathbb{R}$ (the set of real numbers) is uncountable.*

Proof. We prove that a subset of $\mathbb{R}$, namely the closed interval $[0,1]$ is uncountable. Since $\mathbb{R}$ includes this set, it must be uncountable as well. The proof is by contradiction. Notice that each number in $[0,1]$ can be written as an infinite sequence of digits (integers from 0 to 9). For instance $\frac{1}{2}\sqrt{2} = 0.707106781\ldots$ Suppose that $[0,1]$ is countable. Then we can put the elements of $[0,1]$ into a list, such that for each element we can find its exact location in the list. Such a list would then look as follows: $\{a_1, a_2, a_3, a_4, \ldots\}$. Now we put the list in a table (here a_{ij} denotes the j-th digit of the number a_i (the i-th number in the list):

$$\begin{array}{ccccccccc}
a_1 & = & 0. & a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & \ldots \\
a_2 & = & 0. & a_{21} & a_{22} & a_{23} & \ldots & a_{3n} & \ldots \\
a_3 & = & 0. & a_{31} & a_{32} & a_{33} & \ldots & a_{2n} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots \\
a_n & = & 0. & a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots
\end{array}$$

Notice that by the assumption that $[0,1]$ is countable, ALL elements of $[0,1]$ are listed in the table above. Now we construct a number $b \in [0,1]$ (so $b = 0.\,b_1\; b_2\; b_3\; \ldots$) that is definitely not in the list. We do that as follows: The number b_n, which is the n^{th} term in the sequence, is constructed as follows:

$$b_n = \begin{cases} a_{nn} - 1 & \text{if } a_{nn} > 0 \\ 3 & \text{if } a_{nn} = 0 \end{cases}$$

Notice that b is indeed in $[0,1]$. The question now is: Is b in the list? The answer: NO, because of the following argument: b can not be equal to a_1, because they differ in their first digit ($b_1 \neq a_{11}$). Furthermore $b \neq a_2$, because they differ in their second digit: $b_2 \neq a_{22}$. But this argument holds in general: For any arbitrary element a_n in our list, we see that by construction the n^{th} digit of b is unequal to a_n, which means that b is unequal to all elements in the list $\{a_1, a_2, a_3, a_4, \ldots\}$. But that means that it must have been impossible to construct a list that contained all elements of the set $[0,1]$; apparently our assumption that we could make such a list was false. Hence $[0,1]$ is uncountable. ■

Notice that we could have taken a lot of different b's that are not in our list. For instance taking $b_n = 4$ or $b_n = 7$ instead of $b_n = 3$ would also suffice. Also we could have made sure that for all n the $(n+1)^{st}$ digit of b would differ from the $(n+1)^{st}$ digit of a_n (instead of making the n^{th} digit different) and there are a lot of other possibilities. This is the reason why uncountable sets are really a lot bigger than countable sets.

As we saw from the definition a real number can be represented by an infinite sequence of digits. It is not necessary that these digits are all in the set $\{0, 1, 2, \ldots, 9\}$. Binary numbers, for instance, are numbers that are represented by an infinite sequence of only 0's and 1's. Here each digit tells us that we add 0 or 1 times 2^n to our total. A few examples:

$$41 = 1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 101001 \text{ in binary notation}$$

$$\frac{4}{3} = 1 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 0 \cdot 2^{-3} + 1 \cdot 2^{-4} + \ldots = 1.010101\ldots \text{ in binary notation}$$

Clearly every real number can be expressed in a (not necessarily unique) way in binary notation. A number b in the set $[0,1]$ is then expressed as $0.b_1\, b_2\, b_3\, \ldots$. We use this information to prove the following theorem.

Example 13 *$\mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$, is uncountable.*

Proof. The proof, again by contradiction, is very similar to the proof that $[0,1]$ is uncountable. We consider the following countable binary representation $x = (x_1, x_2, x_3, \ldots)$ of each element $X \in \mathcal{P}(\mathbb{N})$ (recall that this means that $X \subseteq \mathbb{N}$):

$$\text{for all } m \in \mathbb{N}: \begin{cases} x_m = 1 \text{ if } m \in X \\ x_m = 0 \text{ if } m \notin X \end{cases}$$

So X is represented by an infinite sequence of 0's and 1's. Now suppose that $\mathcal{P}(\mathbb{N})$ is countable. Then we can put the elements of $\mathcal{P}(\mathbb{N})$ into a list, such that for each element we can find its exact location in the list. Such a list would then look as follows: $\{a_1, a_2, a_3, a_4, \ldots\}$. Now we put the list in a table:

$$\begin{array}{ccccccccc}
a_1 & = & a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & \ldots \\
a_2 & = & a_{21} & a_{22} & a_{23} & \ldots & a_{3n} & \ldots \\
a_3 & = & a_{31} & a_{32} & a_{33} & \ldots & a_{2n} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ldots \\
a_n & = & a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots
\end{array}$$

Notice that by the assumption that $\mathcal{P}(\mathbb{N})$ is countable, ALL elements of $\mathcal{P}(\mathbb{N})$ are listed in the table above. Now we construct a binary sequence $b = b_1\ b_2\ b_3\ \ldots$ that is definitely not in the list. We do this as follows: The number b_n, which is the n^{th} term in the sequence b, is constructed as follows:

$$b_n = \begin{cases} 0 & \text{if } a_{nn} = 1 \\ 1 & \text{if } a_{nn} = 0 \end{cases}$$

Notice that b is indeed an infinite sequence of 0's and 1's and therefore that it represents a subset B of $\mathbb{N}$. As in the previous theorem, b is definitely not in the list, because its n^{th} term is unequal to the n^{th} term of the n^{th} item in the list, a_n. Hence $\mathcal{P}(\mathbb{N})$ is uncountable. ■

The same technique can obviously not be used to prove that $\mathbb{N}$ is uncountable (since it is not!). Therefore consider the following list where the numbers in $\mathbb{N}$ are written in a binary notation:

	2^0	2^1	2^2	2^3	....
1	1	0	0	0	....
2	0	1	0	0	....
3	1	1	0	0	....
4	0	0	1	0	....
5	1	0	1	0	....

Constructing the number b where for all numbers n we write

$$b_n = \begin{cases} 0 & \text{if the } n^{th} \text{ digit of } 2^n \text{ is } 1 \\ 1 & \text{if the } n^{th} \text{ digit of } 2^n \text{ is } 0 \end{cases}$$

leads to $b = 0011111111\ldots$, where the number of 1's in the list is infinite. This means that $b = \infty \notin \mathbb{N}$. Yes, there may be an infinite number of numbers in $\mathbb{N}$, but none of these numbers equals ∞, they are all finite! The problem that we encounter here is that no matter how we list the elements of $\mathbb{N}$, if we do it properly, i.e. in such a way that we can find the exact position of every number in the list, then the list you construct to obtain b always contains an infinite number of 1's.

The Cantor Set is named after Georg Cantor, the "inventor" of the mathematical branch called set theory (cf. page 28 of the book). The set can be constructed as follows: Take the closed interval $[0,1]$. We know that this is an uncountable set. Let's call this set C_0. Now we delete the open interval that is right in the middle: $\left(\frac{1}{3}, \frac{2}{3}\right)$. The remaining set is then $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, which we shall call C_1. Now in each of the intervals in C_1 we delete the middle third, leaving us with $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. We continue this procedure a countably infinite number of times, thus obtaining the set C_∞: the Cantor Set. Obviously this set is nonempty; it even is infinite, as can be concluded from the following argument: All endpoints of any closed interval in any C_k are in C_∞, since at every step only the middle part of each interval is deleted. However, it seems like the Cantor Set consists only of a collection of separate points; it can not include any intervals, can it? Therefore this set must definitely be countable. So now the shocking exercise:

Exercise 15 *Prove that the Cantor Set is uncountable!*

Extra Section: Convex Sets

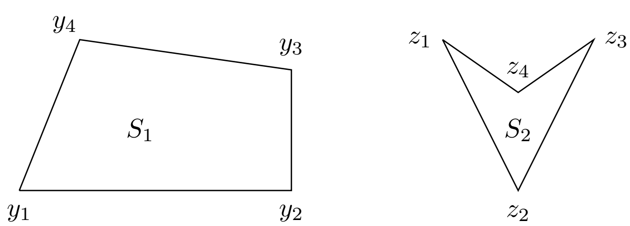
In this section we discuss a property that a set may have, called **convexity**, that is more related to topics within Calculus than Discrete Mathematics. It gives rise to some nice proofs though...

In Euclidean space ($\mathbb{R}^n$, the product set $\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$), a convex set is a region such that, for every pair of points within the region, *every point on the straight line segment that joins the pair of points, is also within the region.* The mathematical formulation of that property is as follows: A set S is convex if

$$\forall x \in S \;\; \forall y \in S \;\; \forall \lambda \in [0,1] \,:\, \lambda x + (1-\lambda) y \;\in S$$

A convex set is, as long as it is bounded (i.e. not infinitely big), characterized by its extreme points; the points that are not in between two other points in the set.

Example 14 *The set S_1 is convex, S_2 is not:*



The extreme points of S_1 are y_1, y_2, y_3 and y_4; all other points in S_1 are in between these four points. In S_2 the point z_4 is not an extreme point.

Even if a convex set is bounded, it may still have an infinite number of extreme points. A disk-shaped set (with a circle as its perimeter), like $\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1^2 + x_2^2 \leq 1\}$, is a nice example of that: Every point on the circle is an extreme point.

Example 15 *For those of you who have followed the Game Theory course: It can be shown (and this isn't particularly hard) that the following sets are always convex:*

1. *The core of a cooperative game;*
2. *The set of optimal mixed actions for players 1 and 2 in a matrix game;*
3. *The set of feasible rewards and the set of equilibrium rewards in a repeated game.*

Example 16 *The set $S = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \geq 0,\, x_2 \geq \frac{1}{2}x_1,\, x_2 - x_1 \leq 1\}$ is convex.*

Proof. *Let $\mathbf{x} = (x_1, x_2) \in S$, $\mathbf{y} = (y_1, y_2) \in S$ and let $\mathbf{z} = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ for some $\lambda \in [0,1]$. (We need to show that $\mathbf{z} \in S$). We know all of the following:*

1. $z_1 = \lambda \cdot x_1 + (1-\lambda) \cdot y_1$ *and* $z_2 = \lambda \cdot x_2 + (1-\lambda) \cdot y_2$
2. $x_1 \geq 0$, $x_2 \geq \frac{1}{2}x_1$ *and* $x_2 - x_1 \leq 1$
3. $y_1 \geq 0$, $y_2 \geq \frac{1}{2}y_1$ *and* $y_2 - y_1 \leq 1$

But then we have that

(a) $z_1 = \lambda \cdot x_1 + (1-\lambda) \cdot y_1 \geq \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0$;

(b) $z_2 = \lambda \cdot x_2 + (1-\lambda) \cdot y_2 \geq \lambda \cdot \frac{1}{2}x_1 + (1-\lambda) \cdot \frac{1}{2}y_1 = \frac{1}{2}(\lambda \cdot x_1 + (1-\lambda) \cdot y_1) = \frac{1}{2}z_1$;

(c) $z_2 - z_1 = \lambda(x_2 - x_1) + (1-\lambda)(y_2 - y_1) \leq \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1$.

This means that $x \in A$ and hence A is convex. ∎

Exercise 16 *For each of the following sets prove or disprove that it is convex. The starred exercises are more complicated than the other ones.*

1. $A_1 = [0,1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$
2. $A_2 = [0,1] \times [3,4] = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 3 \leq x_2 \leq 4\}$
3. $A_3 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$
4. * $A_4 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$
5. $A_5 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1 + 2x_2 \leq 4\}$
6. $A_6 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 + 2x_2 \leq 4\}$
7. $A_7 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 = (x_1)^2\}$
8. $A_8 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 \cdot x_2 \leq 1\}$
9. * $A_9 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 \cdot x_2 \geq 1\}$
10. $A_{10} = \mathbb{N}$
11. $A_{11} = \mathbb{R}$
12. $A_{12} = \mathbb{Q} \cap [0,2]$
13. $A_{13} = \emptyset$
14. $A_{14} = \{(e, \pi)\}$
15. $A_{15} = \{e, \pi\}$
16. $A_{16} = [e, \pi]$
17. $A_{17} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_2| \leq 1\}$
18. $A_{18} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_2| \geq 1\}$
19. * $A_{19} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_1| + |x_2| \leq 1\}$
20. $A_{20} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_1| - |x_2| \leq 1\}$