Exam 22 October 2021 Discrete Mathematics DSAI

Question 1:

p	q	r	$q \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$	$p \wedge q$	$(p \wedge q) \Rightarrow r$	$(\ldots) \Leftrightarrow (\ldots)$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

(Hence, the two statements are equivalent).

Question 2:

Base case: $\sum_{i=1}^{1} (-1)^i i^2 = (-1)^1 \cdot 1^2 = -1 = \frac{-1 \cdot 2}{2} = \frac{(-1)^1 \cdot 1 \cdot (1+1)}{2}$ ✓

Inductive step: Let $n \in \mathbb{N}$.

Assume $\sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$.

Now, $\sum_{i=1}^{n+1} (-1)^i i^2 = \sum_{i=1}^{n} (-1)^i i^2 + (-1)^{n+1}(n+1)^2 = \frac{(-1)^n n(n+1)}{2} + (-1)^{n+1}(n+1)^2$

$= \frac{(-1)^n n(n+1)}{2} + \frac{2 \cdot (-1)^n \cdot (-1) \cdot (n+1)(n+1)}{2} = \frac{(-1)^n (n+1)}{2} \left(n + 2 \cdot (-1) \cdot (n+1)\right)$

$= \frac{(-1)^n (n+1)}{2} (n - 2n - 2) = \frac{(-1)^n (n+1)}{2} (-n-2) = \frac{(-1)^n (n+1)(-1)(n+2)}{2}$

$= \frac{(-1)^{n+1} (n+1)(n+2)}{2} = \frac{(-1)^{n+1} (n+1)((n+1)+1)}{2}$ ✓ $\square$

Question 3:

ⓐ False, consider $A=\{1,2,3,4\}$, $B=\{2,4\}$ and $C=\{3,4\}$.
Then $B \cap C = \{4\} \subseteq \{1,2,3,4\} = A$
and $(A \setminus B) \cap (A \setminus C) = \{1,3\} \cap \{1,2\} = \{1\} \neq \emptyset$.

ⓑ True.

Proof:

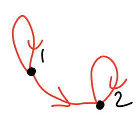
Let $(x,y) \in (A \times B) \cup (C \times D)$.
Then $(x,y) \in A \times B$ or $(x,y) \in C \times D$.
So, ($x \in A$ and $y \in B$) or ($x \in C$ and $y \in D$).
Distinguish between those two cases.

Case 1: $x \in A$ and $y \in B$.
Then, $x \in A \cup C$ and $y \in B \cup D$.
As a result, $(x,y) \in (A \cup C) \times (B \cup D)$.

Case 2: $x \in C$ and $y \in D$.
Then, also $x \in A \cup C$ and $y \in B \cup D$.
As a result, $(x,y) \in (A \cup C) \times (B \cup D)$. $\square$

Question 4:

ⓐ For example,



ⓑ R has four equivalence classes, namely:

* $\emptyset$ (the subsets with cardinality 0)
* $\{1\}, \{2\}, \{3\}$ (the subsets with cardinality 1)
* $\{1,2\}, \{1,3\}, \{2,3\}$ (the subsets with cardinality 2)
* $\{1,2,3\}$ (the subsets with cardinality 3)

Question 5:

ⓐ $n = 3$
$k = 6$
repetition is allowed
order is important

$\left.\right\}$ $n^k = 3^6 = \boxed{729}$.

ⓑ Since there are 6 elements in the domain and only 3 elements in the co-domain, it's not possible to have an invertible function.
So, $\boxed{0}$

ⓒ Since every grandchild gets at least 1 chocolate bar, we can already give every grandchild a chocolate bar.
As a result, there are only 10 chocolate bars left.
To any of these chocolate bars, we will assign a grandchild.

$n = 10$ (# grandchildren)
$k = 10$ (# chocolate bars)
repetition is allowed (a grandchild can get more than 1 chocolate bar)
order is not important (all chocolate bars are identical)

$\binom{(n-1)+k}{k} = \binom{19}{10} = \boxed{92\,378}$

Question 6:

ⓐ True.
Proof: Let $x \in \mathbb{R}$.
Distinguish between two cases: $x \neq 0$ and $x = 0$.
* Case 1: If $x \neq 0$, then take $y = -1/x$. (Note that y exists because $x \neq 0$. Also note that $y \in \mathbb{R}$ because $x \in \mathbb{R}$).
As a result, $x^2 y + 2x = x^2 \cdot \frac{-1}{x} + 2x = -x + 2x = x$ ✓

* Case 2: If $x = 0$, then take (for example) $y = 1$.
Hence, $x^2 y + 2x = 0^2 \cdot 1 + 2 \cdot 0 = 0 = x$. ✓ $\square$

ⓑ False, consider $x = 4$ and $y = 4$.
Then we need an $z \in \mathbb{N}$ such that $\underline{z^2 \geq 4^2 + 4^2 = 16 + 16 = 32}$
$z < 5$.
$\rightarrow z \geq \sqrt{32}$, so $z \geq 6$ (because $z \in \mathbb{N}$)
So, we need $z \geq 6$ and $z < 5$, which is impossible.

Question 7:

(a) Injective: Let $x\in\mathbb{R}\setminus\{3\}$, let $y\in\mathbb{R}\setminus\{3\}$ and assume $f(x)=f(y)$.

$\frac{x+2}{x-3}=\frac{y+2}{y-3} \Rightarrow (x+2)(y-3)=(y+2)(x-3)$

$\Rightarrow \cancel{xy}-3x+2y\cancel{-6}=\cancel{xy}-3y+2x\cancel{-6}$

$\Rightarrow 5y=5x$

$\Rightarrow x=y$

Surjective: Let $y\in\mathbb{R}\setminus\{1\}$.

Take $x=\frac{2+3y}{y-1}$. Since $y\in\mathbb{R}\setminus\{1\}$, we know $x\in\mathbb{R}$. We also need $x\neq 3$.

So we need $\frac{2+3y}{y-1}\neq 3$. Is that true? $2+3y\neq 3(y-1)$? $2+\cancel{3y}\neq\cancel{3y}-3$?

Yes, because $2\neq -3$.

So, we have $x\in\mathbb{R}\setminus\{3\}$.

Next, $f(x)=f\left(\frac{2+3y}{y-1}\right)=\dfrac{\frac{2+3y}{y-1}+2}{\frac{2+3y}{y-1}-3}=\dfrac{\frac{2+3y+2(y-1)}{y-1}}{\frac{2+3y-3(y-1)}{y-1}}=\frac{2+3y+2(y-1)}{2+3y-3(y-1)}$

$=\frac{\cancel{2}+3y+2y\cancel{-2}}{2+\cancel{3y}-\cancel{3y}+3}=\frac{5y}{5}=y$

(b) $f\circ g$ is well-defined, because range$(g)\subseteq$ co-domain$(g)=\mathbb{R}=$ domain(f)

$g\circ f$ is not well-defined, because (for example)

$f(0)=-1+0^2=-1+0=-1\notin\mathbb{N}$.

Question 8:

(a) $A\setminus B=\{1\}$

$B\cap C=\{3,4\}$

So, $(A\setminus B)\times(B\cap C)=\{(1,3),(1,4)\}$

And, thus $((A\setminus B)\times(B\cap C))\setminus\emptyset=\{(1,3),(1,4)\}$.

(b) $A\times A=\{(1,1)\}$

So, $P(A\times A)=\{\emptyset,\{(1,1)\}\}$