BOOK EXERCISES

Chapter 1.1

(1) (a) p and q → The number of students is increasing and the student grant is decreasing

(b) p or q //

(2) p = Anne is a mathematician
q = Anne won the race

✓(a) $p \land q$ ✓(d) $p \land \neg q$

✓(b) $\neg p$ ✓(e) $\neg p \land \neg q \equiv \neg(p \lor q)$

✓(c) $p \lor q$

(3) p → students found the material too difficult

q → students thougth the presentation was good

z → students enjoyed the course

(a) $p \land q$

(b) $z \land \neg q$

(c) $(p \lor q) \land \neg(p \land q)$

(d) $\neg z \land p$

(e) $p \land (\neg q \land \neg z)$

In this case positioning of the brackets doesn't matter, and they could be omitted.

(4) $(p \land q) \land \neg(p \lor q)$ → b always false

(5) $(p \lor q) \lor \neg(p \land q)$ → b always true

Chapter 1.2

(2) (a) $p \lor (\neg p)$

p	¬p	p ∨ (¬p
0	1	1
1	0	1

Tautology

(b) $p \land (\neg p)$

p	¬p	p ∧ (¬p)
0	1	0
1	0	0

Contradiction

Chapter 1.3

(1) too easy

(2) Show that $p \Rightarrow q \equiv (\neg p) \vee q$

p	q	$\neg p$	$p \Rightarrow q$	$(\neg p) \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

$\square$

(4) Ann is older than Betty

unless Betty is younger than Claire

Ann	Betty	Claire
10	11	13
10	9	14
10	6	5
10	11	9

$p \rightarrow$ Ann is older than Betty

$q \rightarrow$ Betty is younger than Claire

$$\neg q \Rightarrow p \equiv q \vee p$$

p	q	$q \vee p$
0	1	1
1	1	1
0	0	0
0	0	0

(3) $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

$\equiv (\neg p \vee q) \wedge (\neg q \vee p)$

Chapter 1.4

(2) (a) All fish swim

$(\forall \text{ fish } x)(x \text{ can swim})$

(b) Some newspapers exaggerate the truth

$(\exists \text{ newspaper } x)(x \text{ exaggerates the truth})$

(c) No students in mathematics are unable to use the computer

$(\nexists \text{ student in math } x)(x \text{ cannot use a computer})$

(d) No presidents are economical with the truth

$(\nexists \text{ president } x)(x \text{ is economical with the truth})$

(3) (a) $(\forall x > 0)(\exists y > 0)(x^2 = y)$

For every $x > 0$ there exists a $y > 0$ such that $x^2 = y$

(b) $(\forall x)(\forall y)(\exists z)(x + z = y)$

For every x and y exists a z such that $x + z = y$

(c) $\neg[(\exists x)(\forall y)(y > x)]$

$\forall x \; \neg \forall y \; (y > x)$

$\forall x \; \exists y \; \neg(y > x)$

$(\forall x)(\exists y)(y \leq x)$

For every x, exists a y such that $y \leq x$

(4) (a) All good students study hard

↳ Some good students do not study hard

(b) No males give birth to their young

↳ Some males give birth to their young true

(c) All of Shakespear's plays are commedies

↳ Some of Shakespear's plays are not commedies

(d) $(\forall \text{ integer } x)(\exists \text{ integer } y)(x^2 = y)$

$\neg(\forall \text{ integer } x)(\exists \text{ integer } y)(x^2 = y)$

$\equiv (\exists \text{ integer } x)(\forall \text{ integer } y)(x^2 \neq y)$ → False

(e) $(\exists \text{ integer } x)(\forall \text{ integer } y)(x^2 = y)$

$(\forall \text{ integer } x)(\exists \text{ integer } y)(x^2 \neq y)$

↳ True ; $y = x^2 + 1 \neq x^2$ ✓

Chapter 1.5

1. Prove that $\sqrt{5}$ is not a rational number. ← Proof by Contradiction

- Suppose $\sqrt{5}$ is rational

$\Rightarrow \sqrt{5} = \frac{a}{b}$, where $a > 0$, $b \neq 0$, $b > 0$

$\Rightarrow 5 = \frac{a^2}{b^2}$ ← Assume common factors of a and b have been cancelled out

$\Rightarrow 5b^2 = a^2 \Rightarrow a^2$ is divisible by 5

- We prove that a is also divisible by 5

↳ Suppose a is NOT divisible by 5, then

- $a = 5K + 1$
or $a = 5K + 2$
or $a = 5K + 3$
or $a = 5K + 4$

but then

$(5K+1)^2 \neq a$ because it is not divisible by 5
$(5K+2)^2 \neq a$ "
$(5K+3)^2 \neq a$ "
$(5K+4)^2 \neq a$ "

*

So a is divisible by 5 and $a = 5K$

- So $5b^2 = (5K)^2$

$\Rightarrow 5b^2 = 25K^2$

$\Rightarrow b^2 = 5K^2$ → So b^2 is divisible by 5 and b is also divisible by 5 *

$\Rightarrow a$ and b have a common factor of 5

contradiction □

(2) (a) If $n = p^2 + q^2$ (p and q are prime) then n is prime

$p = 2 \quad q = 2$

$n = (2)^2 + (2)^2 = 8 \longrightarrow$ n is NOT prime

(b) $a > b \implies a^2 > b^2$

- $a = 1, \; b = 1$
- $a^2 = 1, \; b^2 = 1$

then $a^2 \not> b^2$

(c) $x^4 = 1 \implies x = 1$

$x = -1 \implies x^4 = 1$

(3) x and y are odd int. $\implies$ $x + y$ is even

$x = 2k + 1$
$y = 2g + 1$

$$x + y = 2k + 1 + 2g + 1$$
$$= 2k + 2g + 2$$
$$= 2(k + g + 1)$$

$\implies$ $x + y$ is divisible by 2 $\square$

(4) x and y are odd $\implies$ xy is odd

$x = 2k + 1$

$y = 2g + 1$

$$xy = (2k+1)(2g+1)$$
$$= 4kg + 2k + 2g + 1$$
$$= 2(kg + k + g) + 1$$

$\implies$ xy is not diwsble by 2 $\square$

(5) $x^2 - 4 < 0 \implies -2 < x < 2$ $\longleftarrow$ Proof by contrapositive

$\neg(-2 < x < 2) \implies \neg(x^2 - 4 < 0)$

$x \leq -2 \vee x \geq 2 \Rightarrow x^2 - 4 \geq 0$

(i) $x \leq -2 \Rightarrow x^2 \geq 4$

$\rightarrow$ This is true for all x

(ii) $x \geq 2 \Rightarrow x^2 \geq 4$

$\rightarrow$ This is true for all x

(6) (a) If I study mathematics then I get a job

↳ If I don't get a job

$\Rightarrow$ I dont study mathematics

LECTURE NOTES EXERCISES

Chapter 1.1

(3) (c) $p \Longleftrightarrow \underbrace{(q \Rightarrow r)}_{B}$

p	q	r	$q \Rightarrow r$	$p \Longleftrightarrow B$
0	0	0	1	0
0	0	1	1	0
0	1	0	0	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
0	1	1	1	0
1	1	1	1	1

(d) $S = [((\neg(p \wedge \neg q) \wedge r) \Rightarrow p) \Leftrightarrow (q \wedge \neg r)]$

					A	B	C	D	
p	q	r	$\neg q$	$\neg r$	$(p \wedge \neg q)$	$(\neg A \wedge r)$	$B \Rightarrow p$	$q \wedge \neg r$	$C \Leftrightarrow D$
0	0	0	1	1	0	1	0	0	1
0	0	1	1	0	0	1	0	0	1
0	1	0	0	1	0	0	1	1	1
1	0	0	1	1	1	0	1	0	0
1	0	1	1	0	1	0	1	0	0
1	1	0	0	1	0	0	1	1	1
0	1	1	0	0	0	0	1	0	0
1	1	1	0	0	0	0	1	0	0

(6) Negate p. 2 & 4 of ex. 6

1 : $\exists x \in \mathbb{R} : x^2 \leq \frac{1}{\pi}$

2 : $\forall x \in \mathbb{R} : x^2 \leq \frac{1}{\pi}$

3 : $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z} : 3x + y \leq 4$

4 : $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z} : 3x + y \leq 4$

(2) $\exists x \in \mathbb{R} : x^2 > \frac{1}{\pi}$

(4) $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z}) : (3x + y \leq 4)$

$\neg(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z}) : (3x + y \leq 4)$

$(\forall x \in \mathbb{Z})\neg(\forall y \in \mathbb{Z}) : (3x + y \leq 4)$

$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z}) : (3x + y > 4)$

(7) (1) $\exists x \in \mathbb{N} : x = 3$

$3 \in \mathbb{N} \Rightarrow \exists x \in \mathbb{N} : x = 3$

(2) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R}) : (x + y \geq 4)$

Let $x \in \mathbb{R}$

take $y = 4 - x$

What then happens to $x + y$?

then $x + y$

Show that this satisfies the inequality

$= x + (4 - x)$

$= 4 \quad (\geq 4)$ $\quad \square$

$\Rightarrow x + (4 - x) \geq 4$ is true $\quad x + y \geq 4$

$\Rightarrow \exists y \in \mathbb{R} : x + y \geq 4$

(3) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R}) : (y - x = -1)$

Let $x \in \mathbb{R}$

take $y = x-1$ $(\in \mathbb{R})$

then $y - x = ((x-1)-x) = -1$ $\square$

(4) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\exists z \in \mathbb{R})(x+y=2z)$

Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$

take $z = \dfrac{x+y}{2} \in \mathbb{R}$

then $x+y = 2\left(\dfrac{x+y}{2}\right) = 2z$ $\square$

(5) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(x < y \Rightarrow 2x < 2y)$

Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$

We assume that $x < y$

then $2x = x+x$ and $2y = y+y$

$x+x < x+y$ since $x<y$

so $x+x < y+y$

so $2x < 2y$ $\square$

(6) $(\forall x \in \mathbb{R}) : (x \geq 10 \Rightarrow x^5 \geq 3x^4 + 5x^2 + 2333)$

Let $x \in \mathbb{R}$

We assume that $x \geq 10$

then $x \cdot x \cdot x \cdot x \cdot x \geq 100000$ (since $x \geq 10$)

and $100000 \geq 3 \cdot (x \cdot x \cdot x \cdot x) + 5 \cdot (x \cdot x) + 2333$

$\geq 30000 + 500 + 2333$

≥ 32833

Let $x \geq 10$

then $x^5 = x \cdot x^4$

$\geq 10 \cdot x^4$ (since $x \geq 10$)

$= 3x^4 + 7x^4$

$\geq 3x^4 + 700x^2$ (since $x \geq 10$)

$= 3x^4 + 5x^2 + 695x^2$

$\geq 3x^4 + 5x^2 + 69500$ (since $x \geq 10$)

$> 3x^4 + 5x^2 + 2333 \quad \square$

(8) (1) $\forall n \in \mathbb{N} : n$ is odd $\Rightarrow$ n^2-1 is divisible by 8

Let $n \in \mathbb{R}$ be an odd number
then $n = 2k+1$ for some integer k
and
$$
\begin{aligned}
n^2-1 &= (n+1)(n-1) \\
&= (2k+1+1)(2k+1-1) \\
&= (2k+2)(2k) \\
&= 4(k+1)k
\end{aligned}
$$

- if k is even $\rightarrow$ $k = 2x$ for some integer x

  then $4(k+1)k = 4(2x)(2x+1)$
  $= 8x(2x+1)$

  So n^2-1 would be divisible by 8

- if $k+1$ is even $\rightarrow$ $k+1 = 2x$ for some integer x

  then $4(k+1)k = 4(2x)(2x-1)$
  $= 8x(2x-1)$

  So n^2-1 would be divisible by 8

So n^2-1 is divisible by 8

(2) $\forall n \in \mathbb{N} : (n$ odd and n is not divisible by 3 $)$

$\Rightarrow n^2-1$ is divisible by 6

Let $n \in \mathbb{N}$ be an odd number not divisible by 3

then
$$
\left.\begin{aligned}
n &= 2(3k+1)+1 \\
&= 6k+3
\end{aligned}\right\}
$$
this is not correct

so $n^2-1 = (6k+3)^2 - 1$

$$= 36k^2 + 36k + 9 - 1$$
$$= 36k^2 + 36k + 8$$
$$= 6(6k^2 + 6k) + 8$$

Let $n \in \mathbb{N}$ be an odd number

then $n = 2k + 1$ with k an integer

but then $n^2 - 1 = (n+1)(n-1)$
$$= 4k(k+1)$$

- $n^2 - 1$ to be divisible by 6 needs to be divisible by both 2 and 3

$4k$ is divisible by 3

(9) (4) $\forall x \in \mathbb{R}: x^2 - x - 6 \neq 0 \Rightarrow x \notin \{-2, 3\}$

Contrapositive:

$\forall x \in \mathbb{R}: x \in \{-2, 3\} \Longrightarrow x^2 - x - 6 = 0$

Let $x = -2$

then $(-2)^2 - (-2) - 6 =$
$$= 4 + 2 - 6 = 0$$

Let $x = 3$

then $(+3)^2 - (3) - 6 =$
$$= 9 - 3 - 6 = 0$$

In either case, it is equal to 0 $\square$

(2) $\forall x, y > 0 : 3x + 4y < 12 \Rightarrow (x < 4) \wedge (y < 3)$

$\neg[(x<4) \wedge (y<3)] \equiv \neg(x<4) \vee \neg(y<3)$

$\equiv x \geq 4 \vee y \geq 3$

Contrapositive:

$\forall x, y > 0 : (x \geq 4) \vee (y \geq 3) \Rightarrow 3x + 4y \geq 12$

Let $x, y \in \mathbb{R}$ and $x, y > 0$

We assume $x \geq 4$ or $y \geq 3$

(i) $x \geq 4$ then $3x \geq 12$ (since $x \geq 4$)

(ii) $x \geq 3$ then $4y \geq 12$ (since $x \geq 3$)

therefore, if either (i) or (ii) is true,
we get that $3x + 4y \geq 12$ $\square$

(3) $\forall x > 0 \ \forall y > 0 : x^2 + y^2 > 1 \Rightarrow x + y > 1$

Contrapositive:

$\forall x > 0 \ \forall y > 0 : x + y \leq 1 \Rightarrow x^2 + y^2 \leq 1$

Let $x > 0$ and $y > 0$

We assume that $x + y \leq 1$

then $x \leq 1$ and $y \leq 1$

hence $x^2 = x \cdot x \leq 1 \cdot x = x$ since $x \leq 1$

and similarly $y^2 = y \cdot y \leq 1 \cdot y = y$ since $y \leq 1$

so $x^2 + y^2 \leq x + y \leq 1$ $\square$

(10) $a, b \in \mathbb{R} : (\forall \varepsilon > 0 : b < a + \varepsilon) \Rightarrow b \leq a$

Contrapositive *

$\forall a \in \mathbb{R} \ \forall b \in \mathbb{R} : b > a \Rightarrow \exists \varepsilon > 0 : b \geq a + \varepsilon$

Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and assume $b > a$

then $b - a > 0$

take $\varepsilon = b - a \longrightarrow$ We just need a specific positive value for ε

* We need to prove that there exists a positive value of ε such that $b > a + \varepsilon$

Consider $b - (\varepsilon + a)$

$$= b - a - \varepsilon$$

$$= b - a - b + a = 0$$

therefore, $b \geq a + \varepsilon$, where $\varepsilon = b - a$

This proves that if $b > a$, there exists a positive ε ($\varepsilon = b - a$) such that $b \geq a + \varepsilon$

$$\left\{\begin{aligned} a + \varepsilon &= a + b - a, \text{ with } \varepsilon = b - a \\ &= b \\ \text{therefore } & b \geq a + \varepsilon \end{aligned}\right\}$$

Biconditional proofs

(ii) $\forall x \in \mathbb{R} : (x^2 + 5x - 6 = 0) \Longleftrightarrow (x = -6 \vee x = 1)$

(1) $x^2 + 5x - 6 = 0 \Rightarrow x = -6 \vee x = 1$

We assume that $x^2 + 5x - 6 = 0$

$$x^2 + 5x - 6 = (x - 1)(x + 6)$$

(2) $x = -6 \vee x = 1 \Rightarrow x^2 + 5x - 6 = 0$