Hand written ~~exercises~~ solutions to exercises in Chapter 2 of the book.

2.1.1

(a) $\{-1, \cancel{2}, 0, 1, 2, 3, 4\}$.

(b) {tuesday, thursday, friday, saturday}.

(c) $\{1, -1\}$

(d) $\{1, -1\}$.

2.1.2

(a) for example: $\{x \in \mathbb{Z} : x \geq 0 \text{ and } x \text{ is even}\}$

(b) for example: $\mathbb{Z}$

(c) for example: $\{x \in \mathbb{Z} : 0 \leq x \leq 3\}$.

(d) ~~for example:~~ $\cancel{\{3(x-1) : x \in \mathbb{N}\}}$

~~or~~ $\cancel{\{y \in \mathbb{Z} : y = 3x \text{ for some } x\}}$

for example:

$\{3(x-1) : x \in \mathbb{N}\}$

or $\{y \in \mathbb{Z} : y \geq 0 \text{ and there exists } x \in \mathbb{N} \text{ such that } y = 3(x-1)\}$. etc-etc.

2.1.3

(a) equal.

(b) not equal :- only solution to $x^2 + 2x + 1 = 0$ is $x = -1$.

(c) not equal :- prime numbers less than 8 are 2, 3, 5, 7.

2.1.4

(a) $x^2 + 4x - 12 = 0$.

so $(x+6)(x-2) = 0$.

So $x = -6$ or $x = 2$. So set is simply $\{2\}$.

(b) There is no even prime number greater than 3 so simply: $\emptyset$

2.1.5

(a) true. (b) true. (c) false, (d) true, (e) false, (f) true (g) false.

2.1.6

(a) True

(b) false: $A = \{1\}$ and $B = \{1, 2\}$

(c) True.

(d) True.

(e) False. $A = 2$. $B = \{3\}$. $C = \{2, 3\}$.

(2)

7. Man that's a dumb-ass question. They all contain a?

2.1.8 (a) $|X| = 5$

(b) $x^2 = x \Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0$ or $x = 1$ so $|X| = 2$.

(c)

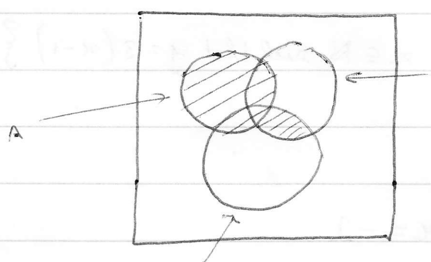
$x^2 \leq 25x \Rightarrow x^2 - 25x \leq 0$

$\Rightarrow x(x-25) \leq 0$

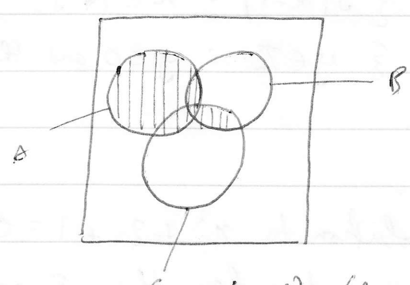
$\Rightarrow x \leq 25$ so $X = \{1, 2, \ldots 25\}$ so $|X| = 25$.

(d) we'll see later on that $|X| = |\mathbb{N}|$.

2.2.1

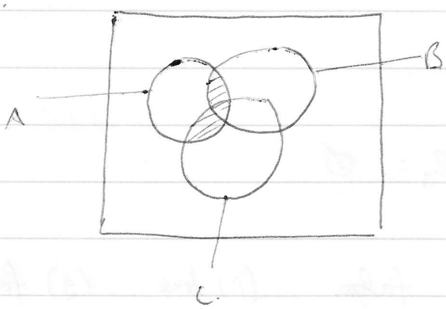


$A \cup (B \cap C)$

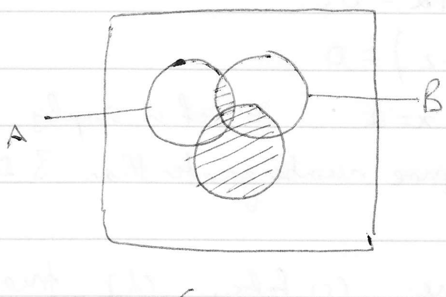


$(A \cup B) \cap (A \cup C)$.

2.2.2.



$A \cap (B \cup C)$



$(A \cap B) \cup C$.

Clearly if C contains something not in $A \cap B$, they are not the same.

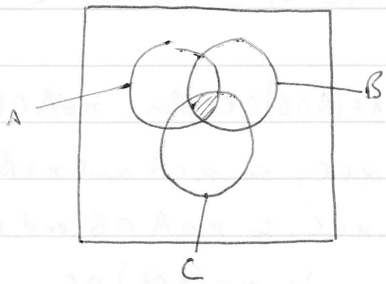
eg. $A = \{1, 2\}$ $B = \{2, 3\}$

$C = \{2, 3, 4\}$

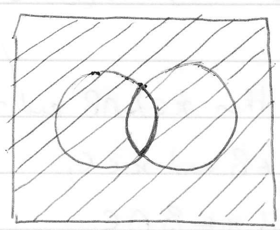
$A \cap (B \cup C) = \{2\}$. and $(A \cap B) \cup C = \{2, 3, 4\}$

2.2

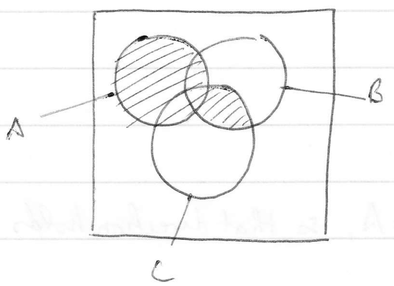
3 (a)



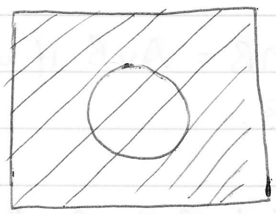
(c)



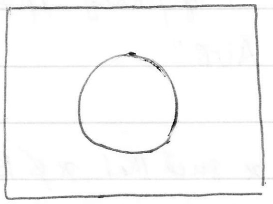
(b)



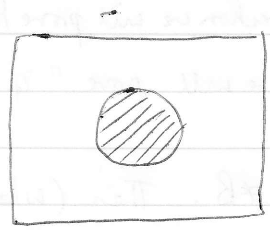
(d)



(e)



(f)



2.2.4 (a) $A^c = \{\ldots -6, -5, -4, -3, -2, -1, 4, 5, 6, 7, 8 \ldots\}$.

(b) $A^c = \{11, 12, 13, \ldots\}$.

(c). $A_1 = \{\cancel{1}, 2, 3, 4, \cancel{5}\}$. $A_2 = \{\cancel{5}, 6, 7, 8, 9, \cancel{10}\}$

$A_1 \cup A_2 = \{\cancel{1}, 2, 3, 4, \cancel{5}, 6, 7, 8, 9, \cancel{10}\}$.

$(A_1 \cup A_2)^c = \{\ldots, -4, -3, -2, -1, 0, 1, 5, 10, 11, 12, 13, 14 \ldots\}$.

2.2.6 (a) Prove that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (1)

and then $\supseteq$. (2)

First prove (1).

Let $x \in A \cup (B \cap C)$. If $x \in A$ then $x \in (A \cup B)$ and $x \in (A \cup C)$,

so $x \in (A \cup B) \cap (A \cup C)$.

~~if x~~ if $x \notin A$ then $x \in B \cap C$, so in that case $x \in B$ and $x \in C$.

But then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. ✓

First prove (2).

Let $x \in (A \cup B) \cap (A \cup C)$. So $x \in (A \cup B)$ and $x \in (A \cup C)$.

If $x \in A$ then $x \in A \cup (B \cap C)$ and we are done.

if $x \notin A$, then $x \in B$ and $x \in C$. So, $x \in B \cap C$ and thus $x \in A \cup (B \cap C)$.

(4)

(b) prove $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ and then $A \cap (B \cap C) \subseteq (A \cap B) \cap C$.

(1)

if $x \in (A \cap B) \cap C$ then $x \in A \cap B$ and $x \in C$,

so $x \in A$ and $x \in B$ and $x \in C$,

so $x \in A$ and $x \in B \cap C$,

so $x \in A \cap (B \cap C)$.

(2)

if $x \in A \cap (B \cap C)$, then $x \in A$ and $x \in B \cap C$.

So, $x \in A$ and $x \in B$ and $x \in C$.

So, $x \in A \cap B$ and $x \in C$.

So, $x \in (A \cap B) \cap C$.

2.2.7 Claim:- $A \cap B = A \cup B$ if and only if $A = B$.

$\Leftarrow$ Clearly, if $A = B$ then $A \cap B = A$ and $A \cup B = A$, so that direction holds.

$\Rightarrow$ to prove this direction we will prove the contrapositive. So instead of proving "if $A \cap B = A \cup B$ then $A = B$" we will prove "if $A \neq B$ then $A \cap B \neq A \cup B$".

So suppose $A \neq B$. Then (wlog) A contains an element x such that $x \notin B$. Clearly, $x \notin A \cap B$. But $x \in A \cup B$, so $A \cap B \neq A \cup B$. □

2.3.1 – 2.3.4: see the end of this document.

2.4.1

(a) $P(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$.

(b) $P(B) = \{\emptyset, \{\emptyset\}, \{1\}, \{2\}, \{\emptyset, 1\}, \{\emptyset, 2\}, \{1,2\}, \{\emptyset,1,2\}\}$.

(c) $P(C) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$

2.4.2 (a) $2^4 = 16$

(b) $2^6 = 64$

(c) $2^{10} = 1024$.

2.4.4 (a) False, eg. $A = \{1\}$, $P(A) = \{\emptyset, \{1\}\}$,

$$A \cup P(A) = \{1, \emptyset, \{1\}\} \neq P(A).$$

(b) False, same example, $A \cap P(A) = \emptyset$.

(c) $x \in P(A \cap B) \Rightarrow x \subseteq A \cap B.$

$\Rightarrow x \subseteq A \wedge x \subseteq B.$

$\Rightarrow x \in P(A) \wedge x \in P(B)$

$\cancel{\Rightarrow x \in P(A)\, A}$

$\Rightarrow x \in P(A) \cap P(B).$ this proves one direction!

$x \in P(A) \cap \cancel{x \in} P(B) \Rightarrow x \in P(A) \wedge x \in P(B)$

$\Rightarrow x \subseteq A \wedge x \subseteq B.$ because every element of x is in both A and B.

$\Rightarrow x \subseteq A \cap B$

$\Rightarrow x \in P(A \cap B).$ this proves the other direction, so the claim is TRUE.

(d) False. $A = \{1,2\}$, $B = \{3,4\}$.

$P(A)$ has 4 elements, $P(B)$ has 4 elements, so $P(A) \cup P(B)$ has at most 8 elements. But $A \cup B$ has 4 elements, so $P(A \cup B)$ has 16 elements, so

$P(A \cup B) \neq P(A) \cup P(B)$.

For example, $\{2, 3\} \in P(A \cup B)$ because $A \cup B = \{1,2,3,4\}$.

But $P(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

and $P(B) = \{\emptyset, \{3\}, \{4\}, \{3,4\}\}$

so $P(A) \cup P(B) = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3,4\}, \{3\}, \{4\}\}$

and $\{2,3\}$ is not in there.

2.4.5. $P(B) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

$P(P(B)) = \{ \emptyset, \{\emptyset, \{a\}\}, \{\emptyset, \{b\}\}, \{\emptyset, \{a,b\}\}$
$\{\{a\}, \{b\}\}, \{\{a\}, \{a,b\}\},$
$\{\{b\}, \{a,b\}\},$
$\{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a,b\}\},$
$\{\emptyset, \{a\}, \{b\}, \{a,b\}\},$
$\{\emptyset, \{a\}, \{b\}\},$
$\{\emptyset, \{a\}, \{a,b\}\},$
$\{\emptyset, \{b\}, \{a,b\}\},$
$\{\{a\}, \{b\}, \{a,b\}\} \}$

2.4.6 $|P_1| = 2^k$

$|P_2| = 2^{2^k}$ $|P_3| = 2^{2^{2^k}}$ etc.

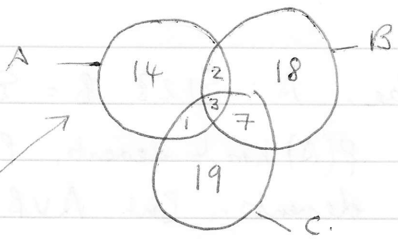
2.5.1. $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$
$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$
$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$
$- |A \cap B \cap C \cap D|$

2.5.2 A = Analysis $\quad |A| = 20$
B = Algebra $\quad |B| = 30$
C = statistics $\quad |C| = 30$

$|A \cap B| = 5 \qquad |A \cap B \cap C| = 3.$
$|B \cap C| = 10$
$|A \cap C| = 4.$

$|A \cup B \cup C| = 20 + 30 + 30 - 5 - 10 - 4 + 3$
$= 80 - 19 + 3$
$= 80 - 16 = 64.$

check :



So 64 take at least one course.

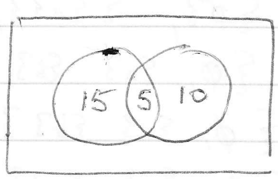
just taking analysis $= 14.$

$14 + 18 + 19 + 1 + 2 + 3 + 7 = 64.$

2.5.3. A = students whose mothers have been to university, $|A| = 20.$
B = students whose fathers have been to university, $|B| = 15.$

70 students had neither parent at university, so $|A \cup B| = 30.$

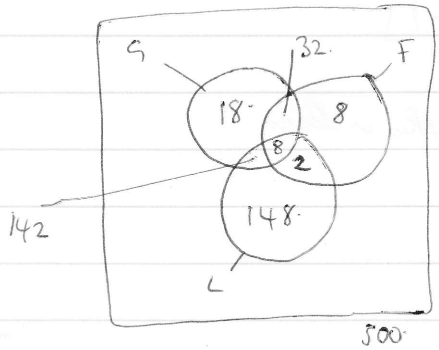
$|A \cup B| = |A| + |B| - |A \cap B|.$
$30 = 20 + 15 - |A \cap B|$
$30 = 35 - |A \cap B|$ so $|A \cap B| = 5$ so: 5 students had both parents at university ie.



total 100.

2.5.4.

G = people that where glasses, $|G| = 200$.
F = false teeth, $|F| = 50$
L = labour $|L| = 300$. $|G \cap L| = 150$.
$|F \cap L| = 10$ $|G \cap F| = 40$.
$|F \cap L \cap G| = 8$.



148 people voted labour who did not have glasses or false teeth

$= \cancel{|L| - |G| - |F| +}$

$= |L| - |G \cap L| - |F \cap L| + |G \cap F \cap L|$

$= 300 - 150 - 10 + 8$

$= 148$

did not vote labour who had false teeth and glasses = 32.

2.5.5 70 students in total.

15 joined 3 societies, so they paid $15 \times 5 \times 3 = 75 \times 3 = 210 + 15$
$= 225$ pounds.

34 joined at least 2 societies.

So $34 - 15$ joined exactly two societies $= 19$.

So 19 students paid $(2 \times 5) = 10$ pounds, 190 pounds.

So 415 was spent by the "2" and "3" society people.
rest of the 550 was spent by "1" society people.

So $550 - 415 = 135$ So $\frac{135}{5} = 5\overline{)135}$ 027 $\underline{27}$ students

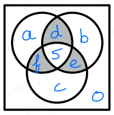
joined exactly 1 society.

So $70 - (15 + 19 + 27)$

$= 70 - (34 + 27)$

$= 70 - 61$

$= 9$ students didn't join any society

2.5.6. 

$b+c+e=10$
$a+c+f=15$
$a+b+d=20$ +
$2a+2b+2c+d+e+f=45$

$2a+2b+2c+d+e+f=45$
$a+b+c+d+e+f=40-5=35$ −
$a+b+c=10.$
$\Rightarrow d+e+f=35-10=25$
So, 25 students answered exactly two question.

8

2.6.1 M = main dishes
S = side dishes
F = {salt, no salt}.
All possible meal combinations are $M \times S \times F$.

2.6.2 $\{(a,l),(a,m),(a,n),(b,l),(b,m),(b,n)\}$

2.6.3. $(X=Y) \vee (X=\emptyset) \vee (Y=\emptyset)$ – we proved this in class.

2.6.4 There is 1 partition: $\{\{1\}\}$

2.6.5 There are 2 partitions. $\{\{1\},\{2\}\}$,
$\{\{1,2\}\}$

2.6.6 There are ~~3~~ 5 partitions: $\{\{1\},\{2\},\{3\}\}$,
$\{\{1,2\},\{3\}\}$,
$\{\{1,3\},\{2\}\}$,
$\{\{2,3\},\{1\}\}$.
$\{\{1,2,3\}\}$

2.7.1 If the first set can be written as a list $x_1, x_2, x_3, x_4 \ldots$ and the second as a list $y_1, y_2, y_3, y_4 \ldots$ then we can merge them into one list as follows:-

x_1
y_1
x_2
y_2
x_3
y_3
x_4
y_4
$\vdots$

(If you like:- map first set A to the even numbers, second set B to the odd numbers)

2.7.2 (a) I think this is true. ~~so~~ If you have two sets A and B then simply write A out as a list and delete those elements not in B. The elements left form a new list

eg. a_1, ~~a_2~~, a_3, ~~a_4~~, ~~a_5~~, a_6 $\longrightarrow$ a_1, a_3, a_6, $\vdots$

(b) False. We proved in class that the set of all subsets of $\mathbb{N}$ is uncountable, using a diagonalization argument (and encoding the subsets as binary strings).

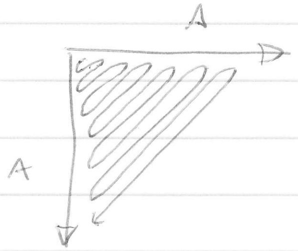
(c) True.

(d). Not sure what they mean by "one to one" here but we know that for a finite set A, $|P(A)| = 2^{|A|}$ so they do not have the same ~~set~~ number of elements and thus there cannot exist a bijection between them. We also saw that e.g. $|P(\mathbb{N})| > \mathbb{N}$.

(e). True. Each positive number $n \in \{0, 2, 4, 6\}$ is mapped to natural number $\frac{n}{2} + 1$.

2.7.3. I would argue like this. Let $A = \mathbb{N} \cup \{0\}$.

Consider an $A \times A$ table navigated with a zig-zag argument.



this constructs a list of all possible pairs of integer values $m, n \geq 0$. Now, for a given pair of values $m=a$, $n=b$, we actually have four cases:

$a \quad b$

$-a \quad b$

$a \quad -b$

$-a \quad -b$.

So for each element in $\triangledown$ we immediately list all 4 possibilities and then delete duplicates →

		$\mathbb{N}$
~~0 0~~	0 0	← 1
~~-0 0~~	~~-0 0~~	
~~0 -0~~	~~0 -0~~	
~~-0 -0~~	~~-0 -0~~	
~~1 0~~	1 0	← 2
~~-1 0~~	-1 0	← 3
~~0 -1~~	~~1 -0~~	
	-1 -0	← 4.
	⋮ etc.	

2.8. Fuck that!

Answers - Book - Chapter 2 - Exercise 2.3.

① Statement: $(A \cap B)^c = A^c \cup B^c$

Proof:

"$\subseteq$" Let $x \in (A \cap B)^c$.
Then, $x \notin (A \cap B)$.
So, $x \notin A$ or $x \notin B$.
So, $x \in A^c$ or $x \in B^c$
So, $x \in A^c \cup B^c$.

"$\supseteq$" Let $x \in A^c \cup B^c$.
Then, $x \in A^c$ or $x \in B^c$
So, $x \notin A$ or $x \notin B$.
So, $x \notin (A \cap B)$
So, $x \in (A \cap B)^c$ □

② Statement: $A \cup (B \cup C) = (A \cup B) \cup C \Rightarrow A \cap (B \cap C) = (A \cap B) \cap C$.

Proof:

Assume $A \cup (B \cup C) = (A \cup B) \cup C$.
Then, by replacing A with A^c, replacing B with B^c and replacing C with C^c, we get

$$A^c \cup (B^c \cup C^c) = (A^c \cup B^c) \cup C^c.$$

Hence, by taking the complements on both sides, we get

$$(A^c \cup (B^c \cup C^c))^c = ((A^c \cup B^c) \cup C^c)^c$$

Now, by using De Morgan's law, we get

$$A \cap (B^c \cup C^c)^c = (A^c \cup B^c)^c \cap C.$$

Again, by using De Morgan's law, ~~get~~ ~~wet~~ we get

$$A \cap (B \cap C) = (A \cap B) \cap C.$$ □

③ Statement: $A \cup (B \cap A^c)^c = A \cup B^c$.

Proof:

"$\subseteq$" Let $x \in A \cup (B \cap A^c)^c$
Then, $x \in A$ or $x \notin (B \cap A^c)$
So, $x \in A$ or $(x \notin B$ or $x \notin A^c)$
So, $x \in A$ or $(x \notin B$ or $x \in A)$
So, $x \in A$ or $x \notin B$
So, $x \in A$ or $x \in B^c$
So, $x \in A \cup B^c$

"$\supseteq$" Let $x \in A \cup B^c$
So, $x \in A$ or $x \in B^c$
So, $x \in A$ or $x \notin B$
So, $x \in A$ or $(x \notin B$ or $x \in A)$
So, $x \in A$ or $(x \notin B$ or $x \notin A^c)$
So, $x \in A$ or $x \notin (B \cap A^c)$
So, $x \in A \cup (B \cap A^c)^c$ □

(4) Statement: $((A \cap B)^c \cup B)^c = \emptyset$.

Proof: (by contradiction)

Suppose, for the sake of contradiction, that $((A \cap B)^c \cup B)^c \neq \emptyset$.

Then, $\exists x : x \in ((A \cap B)^c \cup B)^c$.

So, $x \notin ((A \cap B)^c \cup B)$

So, $x \notin (A \cap B)^c$ and $x \notin B$.

So, $x \in A \cap B$ and $x \notin B$.

So, $(x \in A$ and $x \in B)$ and $x \notin B$.

This is a contradiction.

Hence, our assumption $((A \cap B)^c \cup B)^c \neq \emptyset$ was incorrect.

Hence, $((A \cap B)^c \cup B)^c = \emptyset$ $\square$