Solutions - Practice Exam Questions - Tutorial 2

1. (a)

Check base case: $n=1$

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} \stackrel{?}{=} \frac{1}{1+1}$$

Yes, $\frac{1}{1(1+1)} = \frac{1}{2}$ ✓

Inductive step. Assume the claim holds for n. Show it holds for $n+1$.

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+1+1)} \quad \text{(by algebra)}$$

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \quad \text{(by induction)}$$

$$= \frac{1}{(n+1)}\left[n + \frac{1}{n+2}\right]$$

$$= \frac{1}{(n+1)}\left[\frac{n(n+2)+1}{n+2}\right]$$

$$= \frac{1}{(n+1)}\left[\frac{n^2+2n+1}{n+2}\right]$$

$$= \frac{1}{(n+1)}\left[\frac{(n+1)^2}{n+2}\right]$$

$$= \frac{n+1}{n+2} \quad ✓ \quad \square.$$

(b)

check base case $n=1$:

$$7^1 - 4^1 = 3 \text{ which is trivially divisible by 3.}$$

inductive step.

assume it holds for n, prove $n+1$.

$$\begin{aligned} 7^{n+1} - 4^{n+1} &= 7 \cdot 7^n - 4 \cdot 4^n \\ &= 7 \cdot 7^n - 7 \cdot 4^n + 3 \cdot 4^n \\ &= \underbrace{7\Big[\underbrace{7^n - 4^n}_{\substack{\text{divisible by 3} \\ \text{by induction.}}}\Big] + \underbrace{3 \cdot 4^n}_{\text{divisible by 3.}}}_{\text{So divisible by 3.}} \end{aligned}$$

$\square$

2. (a)

$P(n)$ is the claim " $\sum_{i=1}^{n} i \times i! = (n+1)! - 1$ ".

Prove the base case,

ie. $P(1)$. Is it true that $\sum_{i=1}^{1} i \times i! \stackrel{?}{=} (1+1)! - 1$

$$1 \times 1! \stackrel{?}{=} 2! - 1$$

$$1 \stackrel{?}{=} 1$$ yes! Base case holds.

Now, let n be an arbitrary natural number ≥ 1.

We <u>assume</u> $P(n)$ holds ie. $\sum_{i=1}^{n} i \times i! = (n+1)! - 1$

and our <u>goal</u> is to prove $P(n+1)$, ie. that $\sum_{i=1}^{n+1} i \times i! = (\underbrace{(n+1)+1}_{n+2})! - 1$.

Now, note that $\sum_{i=1}^{n+1} i \times i! = \sum_{i=1}^{n} i \times i! + (n+1)(n+1)!$ (by algebra)

$$= (n+1)! - 1 + (n+1)(n+1)! \quad \text{(by induction)}$$

$$= (n+1)![1 + (n+1)] - 1$$

$$= (n+1)![n+2] - 1$$

$$= (n+2)! - 1. \quad \square$$

(b)

Check the base case, $n=1$.

Is $2^{3(1)} - 3^{(1)}$ divisible by 5?

Yes, because $2^3 - 3 = 8 - 3 = 5$ ✓

~~Now, assume the claim~~

Now, let n be an arbitrary natural ≥ 1. Assume the claim holds for n. Our goal is to show that it holds for $n+1$.

Now, consider $2^{3(n+1)} - 3^{(n+1)}$. [We need to prove this is divisible by 5].

By algebra, this is equal to $2^{3n+3} - 3^{n+1}$

$$= 8 \cdot 2^{3n} - 3 \cdot 3^n$$

$$= 8 \cdot 2^{3n} - 8 \cdot 3^n + 5 \cdot 3^n$$

$$= \underbrace{8[2^{3n} - 3^n]}_{1} + \underbrace{5 \cdot 3^n}_{2}.$$

Note that 1 is divisible by 5, by induction.

Also, 2 is divisible by 5, by algebra.

So $8[2^{3n} - 3^n] + 5 \cdot 3^n$ is divisible by 5 — So we are done! $\square$

3.

Basecase $(n=1)$

$$1(1+2) \stackrel{?}{=} \frac{1(1+1)(2(1)+7)}{6}$$

$$3 \stackrel{?}{=} \frac{1 \times 2 \times 9}{6}$$

$$3 \stackrel{?}{=} 3 \checkmark$$ holds.

Inductive step.

Assume it holds for n. We need to prove it for $n+1$, ie. we need to prove,

$$\sum_{i=1}^{n+1} i(i+2) = \frac{(n+1)((n+1)+1)(2(n+1)+7)}{6}$$

$$= \frac{(n+1)(n+2)(2n+9)}{6}.$$

Now,

$$\sum_{i=1}^{n+1} i(i+2) = \sum_{i=1}^{n} i(i+1) + (n+1)(n+3) \quad \text{(by algebra)}$$

$$= \frac{n(n+1)(2n+7)}{6} + (n+1)(n+3) \quad \text{(by induction)}$$

$$= \frac{n(n+1)(2n+7) + 6(n+1)(n+3)}{6}$$

$$= \frac{(n+1)}{6}\left[n(2n+7) + 6(n+3)\right]$$

if we can show that $n(2n+7) + 6(n+3) = (n+2)(2n+9)$

then we are done.

$$\text{So,} \quad 2n^2 + 7n + 6n + 18 \stackrel{?}{=} 2n^2 + 9n + 4n + 18$$

$$2n^2 + 13n + 18 \stackrel{?}{=} 2n^2 + 13n + 18$$

yes!!!

$$\text{So} \quad \frac{(n+1)}{6}\left[n(2n+7) + 6(n+3)\right] = \frac{(n+1)(n+2)(2n+9)}{6}.$$

$\square$.

4.

Base case: $n = 1$.

Is $[2(1)-1][2(1)] \stackrel{?}{=} \dfrac{1(1+1)(4-1)}{3}$

$2 \stackrel{?}{=} 2$ ✓ yes!!

Let n be an arbitrary natural number. Assume it holds for n, and show it holds for $n+1$.

Our goal is to show

$$\sum_{i=1}^{n+1}(2i-1)(2i) = \frac{(n+1)(n+2)\big(\overbrace{4(n+1)-1}^{=4n+3}\big)}{3} \quad \leftarrow \text{goal RHS}$$

Now, $$\sum_{i=1}^{n+1}(2i-1)(2i) = \sum_{i=1}^{n}(2i-1)(2i) + (2n+1)(2n+2)$$

$$= \underbrace{\frac{n(n+1)(4n-1)}{3}}_{\text{by induction.}} + (2n+1)(2n+2)$$

$$= \frac{n(n+1)(4n-1)}{3} + \frac{3(2n+1)(2n+2)}{3}$$

$$= \frac{n(n+1)(4n-1)}{3} + \frac{6(2n+1)(n+1)}{3}$$

$$= \frac{(n+1)}{3}\Big[n(4n-1) + 6(2n+1)\Big]$$

$$= \frac{(n+1)}{3}\left[4n^2 - n + 12n + 6\right]$$

$$= \frac{(n+1)}{3}\left[4n^2 + 11n + 6\right]$$

$$= \frac{(n+1)}{3}(n+2)(4n+3) = \text{goal RHS} \checkmark$$

$\square$

5.

(a) FALSE. (note that $\{7\} \in A$, but $7 \notin A$)

(b) TRUE because $2 \in A$ and $4 \in A$.

(c) FALSE. (~~~~ $5 \notin A$ and $6 \notin A$)

(d) TRUE (see a).

(e) TRUE (empty set is a subset of everything)

(f) FALSE, because $\emptyset \notin A$.

(g) False, $|A| = 4$, because A contains four elements: $\{7\}$, 2, $\{4, \{5, 6\}\}$ and 4.

6.

(a) $A \setminus C = \{2,3\}$ so $(A \setminus C) \cup B = \{2,3,4,5,6,7\} = A \cup B$ ✓ TRUE.

(b) $C \setminus B = \emptyset$ so $(C \setminus B) \cup \{6,7\} = \{6,7\} \neq \emptyset$ FALSE.

(c) $B \cap A = \{4,5\} = C$ so TRUE

(d) $B \setminus C = \{6,7\}$, so $A \cap (B \setminus C) = \emptyset$. Now, $\emptyset \subseteq \emptyset$ always holds, so TRUE.

(e) $A \cap \{7,8\} = \emptyset$, but $\emptyset \notin \emptyset$, so FALSE.

(f) $\{2\} \notin A$ so FALSE.

7. *Proof.* This claim is *true*. Let's prove it! It's a $\Leftrightarrow$ (i.e. if and only if) proof so we need to prove that the implication holds in both directions. First we prove the $\Rightarrow$ direction i.e. $(B \cap (A^c \cup C)^c = \emptyset) \Rightarrow (A \subseteq B^c \cup C)$. To do this I will use the contrapositive i.e. I will prove $(A \nsubseteq B^c \cup C) \Rightarrow (B \cap (A^c \cup C)^c \neq \emptyset)$. Assume $(A \nsubseteq B^c \cup C)$. This means that there exists some element $x \in A$ such that $x \notin B^c \cup C$. So $x \notin B^c$ and $x \notin C$. So $x \in B$ and $x \notin C$. Now, observe that $x \notin A^c \cup C$ (because $x \in A$ and $x \notin C$). So $x \in (A^c \cup C)^c$. Combined with the fact that $x \in B$, we have that $x \in B \cap (A^c \cup C)^c$. So $(B \cap (A^c \cup C)^c \neq \emptyset)$.

Next we prove the $\Leftarrow$ direction i.e. $(A \subseteq B^c \cup C) \Rightarrow (B \cap (A^c \cup C)^c = \emptyset)$. Once again I will use the contrapositive: I will prove $(B \cap (A^c \cup C)^c \neq \emptyset) \Rightarrow (A \nsubseteq B^c \cup C)$. So, assume $(B \cap (A^c \cup C)^c \neq \emptyset)$. This means that there exists some element x such that $x \in B$ and $x \in (A^c \cup C)^c$. So $x \in B$ and $x \notin A^c \cup C$. Putting this all together, this means $x \in B, x \in A$ and $x \notin C$. Hence $x \in A$ and $x \notin B^c \cup C$. So $A \nsubseteq B^c \cup C$. $\square$

8.

True. Proof:

"$\Leftarrow$" Assume $A \cap C = \emptyset$ and $B^c \subseteq C^c$.
So, $A \cap C = \emptyset$ and $(C^c)^c \subseteq (B^c)^c$.
So, $A \cap C = \emptyset$ and $C \subseteq B$.

We need to prove $C \subseteq B \setminus A$.
Hence, let $x \in C$.
Since $A \cap C = \emptyset$, we know $x \notin A$.
Since $C \subseteq B$, we know $x \in B$.
So, $x \in B$ and $x \notin A$.
So, $x \in B \setminus A$. ✓

"$\Rightarrow$" Assume $C \subseteq B \setminus A$.
We need to prove two things: $\underbrace{A \cap C = \emptyset}_{1}$, and $\underbrace{B^c \subseteq C^c}_{2}$.

1 Suppose for the sake of contradiction that $A \cap C \neq \emptyset$.
Then, there exists an element x such that $x \in A$ and $x \in C$.
As $x \in C$, it follows from the assumption that $x \in B \setminus A$.
So, $x \in B$ and $x \notin A$.
So, $x \in A$ and $x \notin A$, which is a contradiction.
So, we have $A \cap C = \emptyset$. ✓

2 Proving $B^c \subseteq C^c$ is equivalent to proving $C \subseteq B$.
Hence, let $x \in C$.
Then, it follows from the assumption that $x \in B \setminus A$.
So, $x \in B$. ✓

$\square$

9.

Let us call $B \subseteq A^c \cup C$ "X"

and $((A \cap B) \setminus (A \cap C) = \emptyset)$ "Y".

We need to show $X \iff Y$ for all sets A, B, C.

Let A, B, C be arbitrary sets.

I will prove (1) $X \Rightarrow Y$ and (2) $Y \Rightarrow X$.

I will prove the contrapositive, ie. I will prove $\neg Y \Rightarrow \neg X$.

So assume $\neg Y$, ie. we assume

$$(A \cap B) \setminus (A \cap C) \neq \emptyset.$$

So there exists an element x such that

$x \in A \cap B$ and $x \notin A \cap C$.

So $x \in A$, $x \in B$, and from this we conclude that $x \notin C$ (because $x \notin A \cap C$).

So $x \notin A^c$, $x \in B$, $x \notin C$.

So $x \in B$, and $x \notin A^c \cup C$

So $B \nsubseteq A^c \cup C$

ie. $\neg X$ is true.

Again I prove the contrapositive.

I assume $\neg X$, ie. I assume $B \nsubseteq A^c \cup C$.

So there exists an element x such that

$x \in B$ but $x \notin A^c \cup C$,

So $x \in B$, $x \notin A^c$, $x \notin C$,

So $x \in B$, $x \in A$, $x \notin C$,

So $x \in A \cap B$

and $x \notin A \cap C$,

So $x \in (A \cap B) \setminus (A \cap C)$

So $(A \cap B) \setminus (A \cap C) \neq \emptyset$.

ie. $\neg Y$ is true.

done! $\square$

10.

This is **false**!

Counter-example:

$A = \{1\}$

$B = \{1\}$

$C = \emptyset$ (So $C^c = U$. Take any universe you like, it doesn't matter).

Observe: $1 \in A \cup (C^c \setminus B)$

because $1 \in A$.

<u>But</u> $1 \notin (A \cup C^c) \setminus B$,

because $1 \in B$.

So $A \cup (C^c \setminus B) \neq (A \cup C^c) \setminus B$ in this specific example.