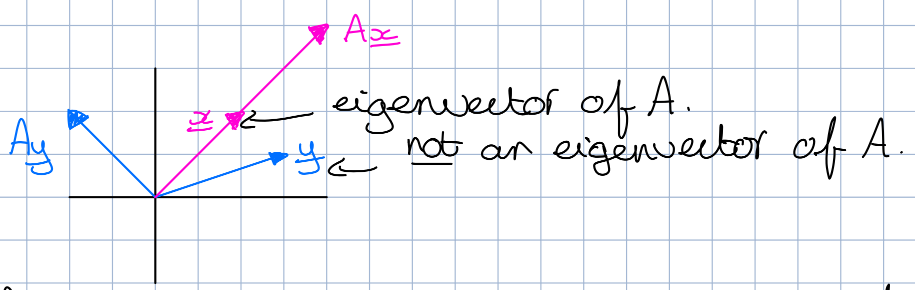
Lecture 9: Eigenvalues and Eigenvectors.
(book: 5.1, 5.2)

Previous episode: Vector Spaces.
Next episode: Diagonalization.

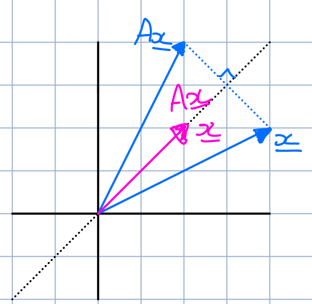
$n \times n$ matrix A $\qquad T: \underline{x} \mapsto A\underline{x}$



A non-zero $\underline{x} \in \mathbb{R}^n$ is an eigenvector of A when $A\underline{x} = \lambda \underline{x}$ for some scalar $\lambda \rightarrow$ eigenvalue.

in words: A produces a scalar multiple of $\underline{x}$ (the direction does not change).
(apart from a minus sign) !

Example: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\qquad A\underline{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$

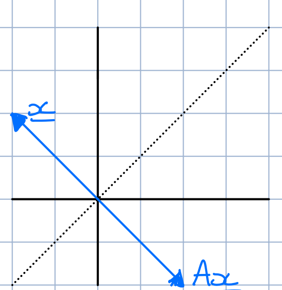


A doesn't change the direction of the vectors on the $x_2 = x_1$

$$A \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

So, $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is an eigenvector with eigenvalue 1.
$\lambda_1 = 1$.

$\hookrightarrow$ any vector of the form $\begin{bmatrix} t \\ t \end{bmatrix}$ with $t \neq 0$.



Each vector perpendicular to the line $x_2 = x_1$ is also an eigenvector.

$$A \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = -1 \cdot \begin{bmatrix} -3 \\ 3 \end{bmatrix}.$$

So, $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue -1.
$\lambda_2 = -1$.

$\hookrightarrow$ any vector of the form $\begin{bmatrix} -t \\ t \end{bmatrix}$ with $t \neq 0$.

Suppose $A\underline{x} = \underline{0}$ has a non-trivial solution.
$\Rightarrow \exists\, \underline{x} \neq \underline{0} : A\underline{x} = 0 \cdot \underline{x}$

So, each non-trivial solution is an eigenvector with eigenvalue 0.

Recall: A is invertible $\iff A\underline{x} = \underline{0}$ has only the trivial solution.
$\iff$ 0 is not an eigenvalue of A.

Example: Is $\underline{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$? No.

$A\underline{u} = \lambda \underline{u}$?

$$A\underline{u} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Is a scalar p an eigenvalue of A?

$A\underline{x} = p \cdot \underline{x}$ for some vector $\underline{x} \neq \underline{0}$?

$\iff A\underline{x} - p\underline{x} = \underline{0} \iff (A - pI)\underline{x} = \underline{0}$ has a nontrivial sol?

$\iff A - pI$ has a free variable?

Example: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Is 1 an eigenvalue?

$$A - 1 \cdot I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

x_2 is a free var
So, 1 is an eigenvalue.

What are the corresponding eigenvectors?

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenspace of λ is $\mathrm{Nul}\,(A - \lambda I) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$
$\hookrightarrow$ a vector space that consists of $\underline{0}$ and all eigenvectors corresponding to the eigenvalue λ.

How to find the eigenvalues?
λ is an eigenvalue $\iff (A - \lambda I)\underline{x} = \underline{0}$ has nontrivial sols.
$\iff A - \lambda I$ is not invertible.
$\iff \det(A - \lambda I) = 0$.

So, solve $\det(A-\lambda I)=0$ for λ.
$\Leftrightarrow$ polynomial of degree n. (characteristic equation) polynomial).

Example: Find the eigenvalues of $A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$.

$$A-\lambda I=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{bmatrix}$$

$$\det(A-\lambda I)=(2-\lambda)^2-1=\lambda^2-4\lambda+3.$$

$$\det(A-\lambda I)=0 \iff \lambda^2-4\lambda+3=0 \iff (\lambda-1)(\lambda-3)=0$$
$$\iff \lambda_1=1,\ \lambda_2=3.$$

And find the corresponding eigenvectors.

$$\lambda_1=1:\quad A-\lambda_1 I=\left[\begin{array}{cc|c}1 & 1 & 0\\ 1 & 1 & 0\end{array}\right]\sim\left[\begin{array}{cc|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right]\qquad \underline{x}=x_2\begin{bmatrix}-1\\ 1\end{bmatrix}\text{ with } x_2\neq 0.$$

$$\lambda_2=3:\quad A-\lambda_2 I=\left[\begin{array}{cc|c}-1 & 1 & 0\\ 1 & -1 & 0\end{array}\right]\sim\left[\begin{array}{cc|c}1 & -1 & 0\\ 0 & 0 & 0\end{array}\right]\qquad \underline{x}=x_2\begin{bmatrix}1\\ 1\end{bmatrix}\text{ with } x_2\neq 0.$$

Example: $A=\begin{bmatrix}5 & 5\\ 5 & 5\end{bmatrix}$

$$\underline{v}=\begin{bmatrix}1\\ 1\end{bmatrix}\qquad A\underline{v}=\begin{bmatrix}5 & 5\\ 5 & 5\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}10\\ 10\end{bmatrix}=10\cdot\begin{bmatrix}1\\ 1\end{bmatrix}\qquad \lambda_1=10.$$

$$\underline{u}=\begin{bmatrix}1\\ -1\end{bmatrix}\qquad A\underline{u}=\begin{bmatrix}5 & 5\\ 5 & 5\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}=0\cdot\begin{bmatrix}1\\ -1\end{bmatrix}\qquad \lambda_2=0.$$

Example: $A=\begin{bmatrix}1 & 2 & 5\\ 4 & 2 & 2\\ 6 & 1 & 1\end{bmatrix}$

$$\begin{bmatrix}1 & 2 & 5\\ 4 & 2 & 2\\ 6 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}=\begin{bmatrix}8\\ 8\\ 8\end{bmatrix}=8\cdot\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

Example: $A=\begin{bmatrix}5 & 1 & 0\\ 0 & -3 & 1\\ 0 & 0 & -3\end{bmatrix}$

$$A-\lambda I = \begin{bmatrix} 5-\lambda & 1 & 0 \\ 0 & -3-\lambda & 1 \\ 0 & 0 & -3-\lambda \end{bmatrix}$$

$$|A-\lambda I| = (5-\lambda)(-3-\lambda)(-3-\lambda) = 0$$

So, $\lambda_1 = 5$ and $\lambda_2 = -3$ (with multiplicity 2).

So, for triangular or diagonal matrices, the eigenvalues are the entries on the main diagonal!

Example $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\left(= \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \text{ with } \varphi = \pi/2 \right)$

Give me the eigenvalues.

$\mathbb{R}^2 \to \mathbb{R}^2$.

$\mathbb{C}^2 \to \mathbb{C}^2$

$$|A-\lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2+1$$

$\lambda^2+1 = 0 \iff \lambda^2 = -1$

So, $\lambda_1 = i$ and $\lambda_2 = -i$

So, eigenvalues can also be complex numbers.

$\lambda_1 = i$: $A-\lambda_1 I = \begin{bmatrix} -i & -1 \\ -1 & -i \end{bmatrix} \sim \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$

$$\underline{x} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$\lambda_2 = -i$: ... $\underline{x} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

DIY: $A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$ Determine the eigenvalues and eigenvectors.

Properties:

* A is invertible $\iff$ 0 is not an eigenvalue of A.
* Exc 19, Ch 5.2: $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$.
* $\underbrace{\sum_{i=1}^{n} a_{ii}}_{\to\ \text{trace}(A)} = \lambda_1 + \lambda_2 + \dots + \lambda_n$.
* Thm 2: If $\underline{v}_1, \dots, \underline{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of a matrix A, then

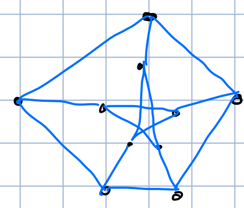
$\{\underline{v}_1, \dots, \underline{v}_r\}$ is linearly independent.

Applications to Graph Theory.

Given adjacency matrix A ($n \times n$, symmetric)

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$.

$\chi(G)$ chromatic number.



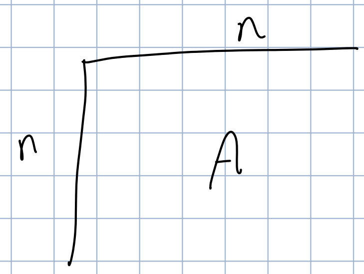
$\chi(G) = 3$.

Petersen graph.

* Hoffmann lower bound

$$\chi(G) \geq 1 + \frac{\lambda_1}{-\lambda_n}$$

* Wilf upper bound $\chi(G) \leq \lambda_1 + 1$.



More applications? Read the Google Page Rank algorithm paper.

Summary (so far):

Let A be an $m \times n$ matrix with columns $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$.

$m \geq n$: $\begin{bmatrix} A \end{bmatrix}$

The following statements are equivalent:

1. A has a pivot in every column
2. A has n pivot positions
3. There are no free variables
4. $A\underline{x} = \underline{0}$ has only the trivial sol.
5. $\{\underline{a}_1, \underline{a}_2, \ldots \underline{a}_n\}$ is linearly indep.
6. $T: \underline{x} \mapsto A\underline{x}$ is one-to-one/injective
7. $\text{Nul } A = \{\underline{0}\}$
8. $\dim \text{Nul } A = 0$
9. $\text{rank } A = n$

$m \leq n$: $\begin{bmatrix} A \end{bmatrix}$

The following statements are equivalent:

a. A has a pivot in every row.
b. A has m pivot positions.
c. The echelon form of A does not contain a row of all zeros.
d. $A\underline{x} = \underline{b}$ is consistent for every $\underline{b}$ in $\mathbb{R}^m$.
e. $\text{Span}\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\} = \mathbb{R}^m$.
f. $T: \underline{x} \mapsto A\underline{x}$ is onto/surjective.
g. $\text{Col } A = \mathbb{R}^m$
h. $\dim \text{Col } A = m$
i. $\text{rank } A = m$

If A is square $(n = m)$, then statements 2 and b are equivalent. Hence, the following statements are equivalent for square matrices.

* 1 – 9, a – i
* A is invertible
* There is a matrix C such that $CA = I_n$ <u>and</u> $AC = I_n$
* A is row equivalent to I_n.
* A^T is invertible.
* $\det A \neq 0$
* The columns of A form a basis for $\mathbb{R}^n$.
* 0 is not an eigenvalue of A.