

Question 1:

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(q \Rightarrow r) \wedge \neg r$	$((q \Rightarrow r) \wedge \neg r) \Rightarrow \neg p$	$(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \wedge \neg r) \Rightarrow \neg p$
T	T	T	T	T	F	T	T
T	T	F	T	F	F	T	T
T	F	T	F	T	F	T	T
T	F	F	F	T	T	F	T
F	T	T	T	T	F	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

(Hence, the statement is a tautology.)

Question 2:

Base case: $\sum_{i=1}^1 i^3 = 1^3 = 1 = \frac{1}{4} \cdot 1 \cdot 4 = \frac{1}{4} \cdot 1 \cdot 2^2 = \frac{1}{4} \cdot 1^2 \cdot (1+1)^2$ ✓

Inductive step: Let $n \in \mathbb{N}$.

Assume $\sum_{i=1}^n i^3 = \frac{1}{4} \cdot n^2 \cdot (n+1)^2$

Now, $\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^n i^3 + (n+1)^3 = \frac{1}{4} \cdot n^2 \cdot (n+1)^2 + (n+1)^3$

$= (n+1)^2 \left(\frac{1}{4} n^2 + (n+1) \right) = (n+1)^2 \cdot \frac{1}{4} \cdot (n^2 + 4n + 4) = (n+1)^2 \cdot \frac{1}{4} \cdot (n+2)^2$

$= \frac{1}{4} (n+1)^2 \cdot ((n+1)+1)^2$ ✓

□

Question 3:

- Ⓐ False, consider $A = \{1\}$, $B = \{2\}$ and $C = \{1, 3\}$.
Then $A \not\subseteq B$ and $B \not\subseteq C$. However, $A \subseteq C$.
- Ⓑ False, consider the universe $U = \{1, 2, 3, 4\}$, $A = \{1, 2\}$, $B = \{1, 2, 3\}$ and $C = \{2\}$. Then, $A \subseteq B$. However, $A \cap (B \cap C)^c = A \cap (\{2\})^c = A \cap \{1, 3, 4\} = \{1\} \neq \emptyset$.

Question 4:

* Reflexive:

Let $X \in P(\{a, b, c\})$, i.e., let $X \subseteq \{a, b, c\}$.
Since $X \setminus X = \emptyset$, we have $X R X$.
So, R is not reflexive.

* Symmetric:

Let $X \subseteq \{a, b, c\}$ and let $Y \subseteq \{a, b, c\}$.
Assume $X R Y$.
So, $X \setminus Y \neq \emptyset$ and $Y \setminus X \neq \emptyset$.
(So, $Y \setminus X \neq \emptyset$ and $X \setminus Y \neq \emptyset$.)
Hence, also $Y R X$.
So, R is symmetric.

* Transitive:

No, consider for example $X = \{a\}$, $Y = \{b\}$ and $Z = \{a, c\}$.
Then $X R Y$, because $X \setminus Y = \{a\} \neq \emptyset$ and $Y \setminus X = \{b\} \neq \emptyset$.
Also $Y R Z$, because $Y \setminus Z = \{b\} \neq \emptyset$ and $Z \setminus Y = \{a, c\} \neq \emptyset$.
However $X \not R Z$, since $X \setminus Z = \emptyset$.

* Anti-symmetry:

No, consider for example $X = \{a\}$ and $Y = \{b\}$.
Then $X R Y$ and $Y R X$.
However $X \neq Y$.

Question 5:

a) Rule of product: $2 \cdot 3 = 6$.

b) There are 2^8 sequences of length 11 that start with 101.

→ $n=2$, $k=8$, repetition is allowed, order is important

Similarly: there are 2^8 sequences of length 11 that start with 010.

These two sets are disjoint, so rule of sum: $2^8 + 2^8 = 2 \cdot 2^8 = 2^9 = 512$.

c) There are $6!$ options for the first six books (the mathematics books).

→ $n=6$, $k=6$, repetition is not allowed, order is important.

Similarly: there are $6!$ options for the last six books (the economics books).

We can combine every order of the mathematics books with every order of the economics books, so rule of product: $6! \cdot 6!$

$$= 720 \cdot 720 = 518\,400.$$

Question 6:

a) True.

Let $x \in X$.

If $x = -2$, choose $y = 2 \in Y$

$x = -1$,

$y = 1 \in Y$

$x = 0$,

$y = 0 \in Y$

$x = 1$,

$y = -1 \in Y$

$x = 2$,

$y = -2 \in Y$.

Then, in any case we have $x+y=0$.

b) True.

Take $x=0 \in X$.

Let $y \in Y$. Then, $x+y=0+y=y$.

c) True.

Let $x \in \mathbb{R}$ and let $y \in \mathbb{R}$.

Take $z = \max \{ \lceil \sqrt{x^2 + y^2} \rceil, 5 \}$

Then, $z \in \mathbb{N}$ because z will be an integer that is ≥ 5 .

Note, that $z \geq \lceil \sqrt{x^2 + y^2} \rceil \geq \sqrt{x^2 + y^2}$ and thus $z^2 \geq (\sqrt{x^2 + y^2})^2 = x^2 + y^2$.

Moreover $z \geq 5$.

→ Hence, $(z^2 \geq x^2 + y^2) \wedge (z \geq 5)$.

Question 7:

- ① Injectivity: Let $x \in \mathbb{R} \setminus \{0\}$ and let $y \in \mathbb{R} \setminus \{0\}$.
Assume $f(x) = f(y)$.
So, $\frac{x+1}{x} = \frac{y+1}{y} \Rightarrow y(x+1) = x(y+1) \Rightarrow xy + y = xy + x \Rightarrow x = y \checkmark$

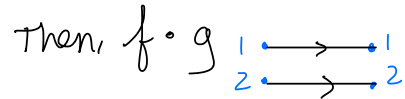
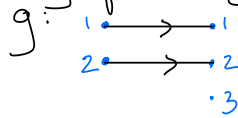
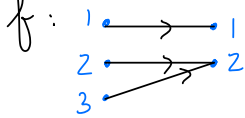
Surjectivity: Let $y \in \mathbb{R} \setminus \{1\}$.
Take $x = \frac{1}{y-1}$. Since $y \in \mathbb{R} \setminus \{1\}$, we know $x \in \mathbb{R}$.

We also need $x \neq 0$. So, we need $\frac{1}{y-1} \neq 0$. Is that true? Yes, because the numerator is always non-zero.

So, we have $x \in \mathbb{R} \setminus \{0\}$.

$$\begin{aligned} \text{Next, } f(x) &= f\left(\frac{1}{y-1}\right) = \frac{\frac{1}{y-1} + 1}{\frac{1}{y-1}} = \frac{\frac{1}{y-1} + \frac{y-1}{y-1}}{\frac{1}{y-1}} = \frac{\frac{1+y-1}{y-1}}{\frac{1}{y-1}} \\ &= \frac{(1+y-1)}{\frac{1}{y-1}} = \frac{y}{\frac{1}{y-1}} = \frac{y \cdot (y-1)}{1} = y \checkmark \end{aligned}$$

- ② Take $A = \{1, 2, 3\}$ and $B = \{1, 2\}$.
And consider the following f and g :



Then, f is not injective, since $f(2) = 2 = f(3)$.
But, g and $f \circ g$ are both injective.

Question 8:

- ① No, because the blocks are overlapping ($\{a, d, e\} \cap \{d, f\} \neq \emptyset$).
- ② $A \times B = \{1, 2\} \times \{2, 3\} = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$
 $A \times C = \{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$
 $(A \times B) \cap (A \times C) = \{(1, 3), (2, 3)\}$.