Lecture 10 : Partial Derivatives

(Adams 12.1-12.5)

* functions of multiple variables
* limits and continuity - brief overview
* partial derivatives
  - ↳ definition
  - ↳ normal line and tangent plane
  - ↳ higher order derivatives
  - ↳ chain rule

1) Functions of multiple variables

Examples : $p(x,y)$ (pressure) → weather maps
$T(x,y)$ (temperature (location))
$h(x,y)$ (height maps)

$f(x_1, x_2, \dots, x_n)$

Domain : subset of $\mathbb{R}^n$

• Domain convention : unless specified, the domain are all real values where
↳ the denominator (if any) is nonzero
↳ everything in a squareroot is positive (in general $x^{\frac{a}{b}}$, b even)
↳ the argument of a logarithm / ln is positive (and nonzero)

• representation : • 3D-plots for $f(x,y)$
$z = f(x,y)$ is a surface
• level / contour plots (e.g. height lines)
↳ the levels represent curves $f(x,y) = c$
• density (colour) plots

(see mathematica / slides)

• Continuity : $f(x,y)$ is continuous at (a,b) if
• every neighbourhood of (a,b) contains points in the domain of f
(no isolated point)
• $\forall \epsilon > 0 \ \exists \delta > 0 : \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - f(a,b)| < \epsilon$

we use the Euclidean distance between points in $\mathbb{R}^2$

↳ very similar definition as for single-variable functions : as (x,y) approaches (a,b), $f(x,y)$ approaches $f(a,b)$

↳ all "typical" functions ($\sin()$, $\cos()$, $e^{()}$, $\ln()$, fractions, polynomials, $\sqrt{\ }$) are continuous on their domain

↳ a discontinuity is like a stair / cliff in a height profile.

- limits : $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$
  - if every neighbourhood of (a,b) contains points in the domain of f ↳ we can approach (a,b)
  - $\forall \epsilon > 0 \;\; \exists \delta > 0 \; : \; \sqrt{(x-a)^2+(y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \epsilon$
  - if f is defined at (a,b), then $L = f(a,b)$
    ↳ typically, we take limits towards the boundaries of the domain.

- in 1D, a limit does not exist when right and left limit differ.
  i.e. $\lim\limits_{x\to 0^+} \frac{|x|}{x} = 1$, $\lim\limits_{x\to 0^-} \frac{|x|}{x} = -1$, $\lim\limits_{x\to 0} \frac{|x|}{x}$ DNE
  $\lim\limits_{x\to 0^+} \frac{1}{x} = +\infty$, $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$, $\lim\limits_{x\to 0} \frac{1}{x}$ DNE

  in 2D, a limit does not exist when it differs when approaching (a,b) in different directions
  ↳ on a level plot, height lines seem to intersect
  ↳ very well visible in plots → mathematica
  (↳ the calculations are optional for you)

- partial derivatives

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = f_1(x,y) = f_x(x,y) = D_x f(x,y) = D_1 f$$

↳ the partial derivative with respect to x indicates how fast $f(x,y)$ changes, when varying x

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} = f_2(x,y) = f_y(x,y) = D_y f(x,y) = D_2 f$$

$$\left.\frac{\partial f}{\partial x}\right|_a^b = f_x(a,b) \quad \text{(notation in book)}$$

Examples: $\frac{\partial}{\partial x}(xy) = y$, $\frac{\partial}{\partial y}(xy) = x$, $\frac{\partial}{\partial x}(x+y) = 1 = \frac{\partial}{\partial y}(x+y)$

$\frac{\partial}{\partial x}(x^2+y^2) = 2x$, $\frac{\partial}{\partial y}(x^2+y^2) = 2y$

- $\frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}}$ (socrative)

- $\frac{\partial}{\partial x}\ln\left(\frac{x}{y}\right) = \frac{\cancel{y}}{x}\cdot\frac{1}{\cancel{y}} = \frac{1}{x}$, $\frac{\partial}{\partial y}\ln\left(\frac{x}{y}\right) = \frac{\cancel{y}}{\cancel{x}}\cdot\frac{-\cancel{x}}{y^{\cancel{2}}} = -\frac{1}{y}$

↳ when you take a partial derivative with respect to x, you treat y as a constant (and vice versa)

- tangent plane and normal line (2D)

↳ 1D : derivative = slope of tangent line to a curve
2D : partial derivatives = slopes of tangent plane to a surface.

for $z_0 = f(x_0, y_0)$, the tangent plane is given by

$$z = z_0 + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

↳ the normal line is the line perpendicular to the tangent plane

$$z - z_0 = -\frac{x-x_0}{\frac{\partial f}{\partial x}} = -\frac{y-y_0}{\frac{\partial f}{\partial y}}$$ (note the similarity to 1D)

Example (Ex.6)

for $f(x,y) = \sin(xy)$, $x_0 = \frac{\pi}{3}$, $y_0 = -1$ → $z_0 = f(x_0,y_0) = \sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$

- the tangent plane is given by: $z + \frac{\sqrt{3}}{2} = -\frac{1}{2}(x - \frac{\pi}{3}) + \frac{\pi}{6}(y+1)$

$\frac{\partial f}{\partial x} = y\cos(xy)$, $\frac{\partial f}{\partial y} = x\cos(xy)$ OR $z = (\frac{\pi}{3} - \frac{\sqrt{3}}{2}) - \frac{x}{2} + \frac{\pi}{6}y$

↳ at $(\frac{\pi}{3},-1)$, $f_x(\frac{\pi}{3},-1) = -1 \cdot \cos(-\frac{\pi}{3}) = -\frac{1}{2}$

$f_y(\frac{\pi}{3},-1) = \frac{\pi}{3}\cos(-\frac{\pi}{3}) = \frac{\pi}{6}$

- the normal line is given by : $z + \frac{\sqrt{3}}{2} = -\frac{x - \frac{\pi}{3}}{\frac{1}{2}} = -\frac{y+1}{\frac{\pi}{6}}$

OR $z + \frac{\sqrt{3}}{2} = 2(\frac{\pi}{3} - x) = -\frac{6}{\pi}(y+1)$

or $z = (\frac{2\pi}{3} - \frac{\sqrt{3}}{2}) - 2x = -(\frac{6}{\pi} + \frac{\sqrt{3}}{2}) - \frac{6}{\pi}y$

→ note: if $\frac{\partial f}{\partial x} = 0$, the tangent plane is parallel to the x-axis
the normal line has a fixed x-component $x = x_0$

if $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, the tangent plane is horizontal $z = z_0$
the normal line is parallel to the z-axis , $x = x_0, y = y_0$

↳ example: normal / tangent at (0,0) for $f(x,y) = \frac{1}{1+x^2+y^2}$

- Higher order derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = f_{xx} = f_{11}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = f_{yx} = f_{21}$$

↳ if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous , $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Examples $\frac{\partial^2}{\partial x^2}(e^{xy}) = \frac{\partial}{\partial x}(y e^{xy}) = y^2 e^{xy}$ (socrative)

$$\frac{\partial^2}{\partial x \partial y}(e^{xy}) = \frac{\partial}{\partial x}(x e^{xy}) = e^{xy} + xy e^{xy}$$

$$\frac{\partial^2}{\partial x \partial y}(\sqrt{x^2+y^2}) = \frac{\partial}{\partial x} \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} = y \frac{\partial}{\partial x} \frac{1}{\sqrt{x^2+y^2}} = y\left(\frac{-1}{\cancel{2}} \frac{\cancel{2}x}{(x^2+y^2)^{3/2}}\right) = \frac{-xy}{(x^2+y^2)^{3/2}}$$

- chain rule

1D: $\frac{d}{dx}(f(g(x))) = f'(g(x))\, g'(x)$

↳ in 2D : for $z \doteq f(x,y)$, and x,y depend on t

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

(example $T(x,y)$ (temperature depends on position) $x(t), y(t)$, position depends on time)

→ if x and y depend on 2 variables $x(s,t)$, $y(s,t)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example : $z = \frac{1}{x^2+y^2}$, $x = r\cos(\varphi)$, $y = r\sin(\varphi)$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{-2x}{(x^2+y^2)^2}\cdot \cos\varphi + \frac{-2y}{(x^2+y^2)^2}\sin\varphi$$

$$= -2\frac{r\cos^2\varphi + r\sin^2\varphi}{(x^2+y^2)^2} = \frac{-2r}{r^4} = -\frac{2}{r^3}$$

↳ $z = \frac{1}{x^2+y^2} = \frac{1}{r^2} \to \frac{dz}{dr} = -\frac{2}{r^3}$

same result.

Example 2 : any function $f(x-vt)$ satisfies the PDE

$$\frac{\partial^2 f}{\partial x^2} = v^{-2}\frac{\partial^2 f}{\partial t^2}$$ (wave equation)

↳ $\frac{\partial f}{\partial x} = \frac{df}{d(x-vt)}\frac{d(x-vt)}{dx} = f'(x-vt)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} f'(x-vt) = \frac{df'}{d(x-vt)}\frac{\partial(x-vt)}{\partial x} = f''(x-vt)$$

↳ $\frac{\partial f}{\partial t} = \frac{df}{d(x-vt)}\frac{\partial(x-vt)}{\partial t} = -v\cdot f'(x-vt)$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(-v\cdot f'(x-vt)\right) = -v\cdot\frac{df'}{d(x-vt)}\frac{\partial(x-vt)}{\partial t} = v^2 f''(x-vt) = v^2\frac{\partial^2 f}{\partial x^2}$$