

BOOK EXERCISES

Chapter 1.1

(1) (a) p and $q \rightarrow$ The number of students is increasing and the student grant is decreasing

(b) p or q //

(2) $p =$ Anne is a mathematician

$q =$ Anne won the race

✓ (a) $p \wedge q$ ✓ (d) $p \wedge \neg q$

✓ (b) $\neg p$ ✓ (e) $\neg p \wedge \neg q \equiv \neg(p \vee q)$

✓ (c) $p \vee q$

(3) $p \rightarrow$ students found the material too difficult

$q \rightarrow$ students thought the presentation was good

\neg → students enjoyed the course

(a) $p \wedge q$ (d) $\neg \neg \wedge p$

(b) $\neg \wedge \neg q$ (e) $p \wedge (\neg q \wedge \neg \neg)$

(c) $(p \vee q) \wedge \neg(p \wedge q)$

(4) $(p \wedge q) \wedge \neg(p \vee q) \rightarrow$ b always false

(5) $(p \vee q) \vee \neg(p \wedge q) \rightarrow$ b always true

In this case, positioning of the brackets doesn't matter, and they could be omitted.

Chapter 1.2

(2) (a) $p \vee (\neg p)$

p	$\neg p$	$p \vee (\neg p)$
0	1	1
1	0	1

Tautology

(b) $p \wedge (\neg p)$

p	$\neg p$	$p \wedge (\neg p)$
0	1	0
1	0	0

Contradiction

Chapter 1.3

(1) too easy

(2) Show that $p \Rightarrow q \equiv (\neg p) \vee q$

p	q	$\neg p$	$p \Rightarrow q$	$(\neg p) \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1 \square

(4) Ann is older than Betty

unless Betty is younger than Claire

Ann	Betty	Claire
10	11	13
10	9	14
10	6	5
10	11	9

$p \rightarrow$ Ann is older than Betty

$q \rightarrow$ Betty is younger than Claire

$$\neg q \Rightarrow p \equiv q \vee p$$

p	q	$q \vee p$
0	1	1
1	1	1
0	0	0
0	0	0

(3) $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

$$\equiv (\neg p \vee q) \wedge (\neg q \vee p)$$

Chapter 1.4

(2) (a) All fish seem

$(\forall \text{ fish } x)(x \text{ can swim})$

(b) Some newspapers exaggerate the truth

$(\exists \text{ newspaper } x)(x \text{ exaggerates the truth})$

(c) No students in mathematics are unable to use the computer

$(\forall \text{ student in math } x)(x \text{ cannot use a computer})$

(d) No presidents are economical with the truth

$(\forall \text{ president } x)(x \text{ is economical with the truth})$

(3) (a) $(\forall x > 0)(\exists y > 0)(x^2 = y)$

For every $x > 0$ there exists a $y > 0$
such that $x^2 = y$

(b) $(\forall x)(\forall y)(\exists z)(x + z = y)$

For every x and y exists a z
such that $x + z = y$

(c) $\neg[(\exists x)(\forall y)(y > x)]$

$\forall x \neg \forall y (y > x)$

$\forall x \exists y \neg (y > x)$

$(\forall x)(\exists y)(y \leq x)$

For every x , exists a y such
that $y \leq x$

(4) (a) All good students study hard

\hookrightarrow Some good students do not study hard

(b) No males give birth to their young

\hookrightarrow Some males give birth to their young true

(c) All of Shakespeare's plays are comedies

↳ Some of Shakespeare's plays are not comedies

(d) $(\forall \text{ integer } x)(\exists \text{ integer } y)(x^2 = y)$

$\neg(\forall \text{ integer } x)(\exists \text{ integer } y)(x^2 = y)$

$\equiv (\exists \text{ integer } x)(\forall \text{ integer } y)(x^2 \neq y) \rightarrow \text{False}$

(e) $(\exists \text{ integer } x)(\forall \text{ integer } y)(x^2 = y)$

$(\forall \text{ integer } x)(\exists \text{ integer } y)(x^2 \neq y)$

↳ True: $y = x^2 + 1 \neq x^2 \quad \checkmark$

Chapter 1.5

1. Prove that $\sqrt{5}$ is not a rational number. ← Proof by Contradiction

• Suppose $\sqrt{5}$ is rational

$\Rightarrow \sqrt{5} = \frac{a}{b}$, where $a > 0$, $b \neq 0$, $b > 0$

$\Rightarrow 5 = \frac{a^2}{b^2}$ ← Assume common factors of a and b have been cancelled out

$\Rightarrow 5b^2 = a^2 \Rightarrow a^2$ is divisible by 5

• We prove that a is also divisible by 5

↳ Suppose a is NOT divisible by 5, then

$$\bullet a = 5k + 1$$

$$\text{or } a = 5k + 2$$

$$\text{or } a = 5k + 3$$

$$\text{or } a = 5k + 4$$

but then

$$(5k+1)^2 \neq a \text{ because it is not divisible by 5}$$

$$(5k+2)^2 \neq a$$

$$(5k+3)^2 \neq a$$

$$(5k+4)^2 \neq a$$

so a is divisible by 5 and $a = 5k$

• So $5b^2 = (5k)^2 \Rightarrow$ so b^2 is divisible by 5 and b is also divisible by 5

$$\Rightarrow 5b^2 = 25k^2$$

$$\Rightarrow b^2 = 5k^2$$

$\Rightarrow a$ and b have a common factor of 5

contradiction □

(2) (a) If $n = p^2 + q^2$ (p and q are prime) then n is prime

$$p = 2 \quad q = 2$$

$$n = (2)^2 + (2)^2 = 8 \rightarrow n \text{ is NOT prime}$$

(b) $a > b \Rightarrow a^2 > b^2$

- $a = l, b = 1$

- $a^2 = l, b^2 = 1$

then $a^2 > b^2$

(c) $x^4 = 1 \Rightarrow x = 1$

$$x = -1 \Rightarrow x^4 = 1$$

(3) x and y are odd int. $\Rightarrow x+y$ is even

$$x = 2k+1$$

$$y = 2g+1$$

$$x+y = 2k+1 + 2g+1$$

$$= 2k+2g+2$$

$$= 2(k+g+1)$$

$\Rightarrow x+y$ is divisible by 2 \square

(4) x and y are odd $\Rightarrow xy$ is odd

$$x = 2k+1$$

$$y = 2g+1$$

$$xy = (2k+1)(2g+1)$$

$$= 4kg + 2k + 2g + 1$$

$$= 2(kg + k + g) + 1$$

$\Rightarrow xy$ is not divisible by 2 \square

(5) $x^2 - 4 < 0 \Rightarrow -2 < x < 2 \leftarrow \text{Proof by contrapositive}$

$$\neg(-2 < x < 2) \Rightarrow \neg(x^2 - 4 < 0)$$

$$x \leq -2 \vee x \geq 2 \Rightarrow x^2 - 4 \geq 0$$

$$(i) x \leq -2 \Rightarrow x^2 \geq 4$$

→ This is true for all x

$$(ii) x \geq 2 \Rightarrow x^2 \geq 4$$

→ This is true for all x

(6) (2) If I study mathematics then I get a job

↳ If I don't get a job

⇒ I don't study mathematics

LECTURE NOTES EXERCISES

Chapter 1.1

$$(3) (c) p \Leftrightarrow (q \Rightarrow r)$$

B

$$p \quad q \quad r \quad q \Rightarrow r \quad p \Leftrightarrow B$$

0	0	0	1	0
0	0	1	1	0
0	1	0	0	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
0	1	1	1	0
1	1	1	1	1

(d) $S = [((\neg(p \wedge \neg q) \wedge r) \Rightarrow p) \Leftrightarrow (q \wedge \neg r)]$

			A		B		C		D	
P	q	r	$\neg q$	$\neg r$	$(p \wedge \neg q)$	$\neg A \wedge r$	$B \Rightarrow p$	$q \wedge \neg r$	$C \Leftrightarrow D$	
0	0	0	1	1	0	1	0	0	1	
0	0	1	1	0	0	1	0	0	1	
0	1	0	0	1	0	0	1	1	0	
1	0	0	1	1	1	0	1	0	0	
1	0	1	1	0	1	0	1	0	0	
1	1	0	0	1	0	0	1	1	1	
0	1	1	0	0	0	0	1	0	0	
1	1	1	0	0	0	0	1	0	0	

(6) Negate p. 2 & 4 of ex. 6

1: $\exists x \in \mathbb{R} : x^2 \leq \frac{1}{\pi}$

2: $\forall x \in \mathbb{R} : x^2 \leq \frac{1}{\pi}$

3: $\forall x \in \mathbb{Z} \exists y \in \mathbb{Z} : 3x + y \leq 4$

4: $\exists x \in \mathbb{Z} \forall y \in \mathbb{Z} : 3x + y \leq 4$

(2) $\exists x \in \mathbb{R} : x^2 > \frac{1}{\pi}$

(4) $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z}) : (3x + y \leq 4)$

$\neg(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z}) : (3x + y \leq 4)$

$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z}) : (3x + y \leq 4)$

$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z}) : (3x + y > 4)$

(7) (1) $\exists x \in \mathbb{N} : x = 3$

$3 \in \mathbb{N} \Rightarrow \exists x \in \mathbb{N} : x = 3$

(2) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R}) : (x + y \geq 4)$

Let $x \in \mathbb{R}$

take $y = 4 - x$

then $x + y$

$= x + (4 - x)$

$= 4 \quad (\geq 4)$

What then happens to $x + y$?

Show that this satisfies the inequality



$\Rightarrow x + (4 - x) \geq 4$ is true $x + y \geq 4$

$\Rightarrow \exists y \in \mathbb{R} : x + y \geq 4$

(3) ($\forall x \in \mathbb{R}$) ($\exists y \in \mathbb{R}$): $y - x = -1$

Let $x \in \mathbb{R}$
take $y = x - 1 \in \mathbb{R}$

then $y - x = (x - 1) - x = -1 \quad \square$

(4) ($\forall x \in \mathbb{R}$) ($\forall y \in \mathbb{R}$) ($\exists z \in \mathbb{R}$) ($x + y = 2z$)

Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$

take $z = \frac{x+y}{2} \in \mathbb{R}$

then $x + y = 2 \left(\frac{x+y}{2} \right) = 2z \quad \square$

(5) ($\forall x \in \mathbb{R}$) ($\forall y \in \mathbb{R}$) ($x < y \Rightarrow 2x < 2y$)

Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$

We assume that $x < y$

then $2x = x + x$ and $2y = y + y$

$x + x < x + y$ since $x < y$

so $x + x < y + y$

so $2x < 2y \quad \square$

(6) ($\forall x \in \mathbb{R}$): $x \geq 10 \Rightarrow x^5 \geq 3x^4 + 5x^2 + 2333$)

Let $x \in \mathbb{R}$

We assume that $x \geq 10$

then $x \cdot x \cdot x \cdot x \cdot x \geq 100000$ (since $x \geq 10$)

$$\begin{aligned} \text{and } 100000 &\geq 3 \cdot (x \cdot x \cdot x \cdot x) + 5 \cdot (x \cdot x) + 2333 \\ &\geq 30000 + 500 + 2333 \\ &\geq 32833 \end{aligned}$$

Let $x \geq 10$

$$\text{then } x^5 = x \cdot x^4$$

$$\geq 10 \cdot x^4 \text{ (since } x \geq 10\text{)}$$

$$= 3x^4 + 7x^4$$

$$\geq 3x^4 + 400x^2 \text{ (since } x \geq 10\text{)}$$

$$= 3x^4 + 5x^2 + 695x^2$$

$$\geq 3x^4 + 5x^2 + 69500 \text{ (since } x \geq 10\text{)}$$

$$> 3x^4 + 5x^2 + 233 \quad \square$$

(8) (1) $\forall n \in \mathbb{N} : n \text{ is odd} \Rightarrow n^2 - 1 \text{ is divisible by 8}$

Let $n \in \mathbb{R}$ be an odd number

then $n = 2k + 1$ for some integer k

$$\text{and } n^2 - 1 = (n+1)(n-1)$$

$$= (2k + 1 + 1)(2k + 1 - 1)$$

$$= (2k + 2)(2k)$$

$$= 2(k+1)k$$

• if k is even $\rightarrow k = 2x$ for some integer x

$$\text{then } 4(k+1)k = 4(2x)(2x+1) \\ = 8x(2x+1)$$

So $n^2 - 1$ would be divisible by 8

• if $k+1$ is even $\rightarrow k+1 = 2x$ for some integer x

$$\text{then } 4(k+1)k = 4(2x)(2x-1) \\ = 8x(2x-1)$$

So $n^2 - 1$ would be divisible by 8

So $n^2 - 1$ is divisible by 8

(2) $\forall n \in \mathbb{N} : (n \text{ odd and } n \text{ is not divisible by 3})$

$\Rightarrow n^2 - 1$ is divisible by 6

Let $n \in \mathbb{N}$ be an odd number not divisible by 3

$$\text{then } n = 2(3k+1) + 1 \\ = 6k + 3$$

this is not correct

$$\text{so } n^2 - 1 = (6k+3)^2 - 1$$

$$\begin{aligned}
 &= 36K^2 + 36K + 9 - 8 \\
 &= 36K^2 + 36K + 1 \\
 &= 6(6K^2 + 6K) + 1
 \end{aligned}$$

Let $n \in \mathbb{N}$ be an odd number

then $n = 2k + 1$ with $k \in \mathbb{N}$ an integer

$$\begin{aligned}
 \text{but then } n^2 - 1 &= (n+1)(n-1) \\
 &= 4k(2k+1)
 \end{aligned}$$

• $n^2 - 1$ to be divisible by 6
needs to be divisible by both 2 and 3

4k is divisible by 3

•

$$(8) (\forall) \forall x \in \mathbb{R} : x^2 - x - 6 \neq 0 \Rightarrow x \notin \{-2, 3\}$$

Controposition:

$$\exists x \in \mathbb{R} : x \in \{-2, 3\} \Rightarrow x^2 - x - 6 = 0$$

Let $x = -2$

$$\begin{aligned}
 \text{then } (-2)^2 - (-2) - 6 &= \\
 &= 4 + 2 - 6 = 0
 \end{aligned}$$

Let $x = 3$

$$\begin{aligned}
 \text{then } (+3)^2 - (-3) - 6 &= \\
 &= 9 - 3 - 6 = 0
 \end{aligned}$$

In either case, it is equal to 0 \square

$$(2) \forall x, y > 0 : 3x + 4y < 12 \Rightarrow (x < 4) \wedge (y < 3)$$

$$\begin{aligned}\neg[(x < 4) \wedge (y < 3)] &\equiv \neg(x < 4) \vee \neg(y < 3) \\ &\equiv x \geq 4 \vee y \geq 3\end{aligned}$$

Contrapositive:

$$\forall x, y > 0 : (x \geq 4) \vee (y \geq 3) \Rightarrow 3x + 4y \geq 12$$

Let $x, y \in \mathbb{R}$ and $x, y > 0$

We assume $x \geq 4$ or $y \geq 3$

(i) $x \geq 4$ then $3x \geq 12$ (since $x \geq 4$)

(ii) $x \geq 3$ then $4y \geq 12$ (since $x \geq 3$)

Therefore, if either (i) or (ii) is true, we get that $3x + 4y \geq 12$ \square

$$(3) \forall x > 0 \forall y > 0 : x^2 + y^2 > 1 \Rightarrow x + y > 1$$

Contrapositive:

$$\forall x > 0 \forall y > 0 : x + y \leq 1 \Rightarrow x^2 + y^2 \leq 1$$

Let $x > 0$ and $y > 0$

We assume that $x + y \leq 1$

then $x \leq 1$ and $y \leq 1$

$$\text{hence } x^2 = x \cdot x \leq 1 \cdot x = x$$

since $x \leq 1$

since $y \leq 1$

$$\text{and similarly } y^2 = y \cdot y \leq 1 \cdot y = y$$

$$\text{so } x^2 + y^2 \leq x + y \leq 1 \quad \square$$

$$(10) a, b \in \mathbb{R} : (\forall \varepsilon > 0 : b < a + \varepsilon) \Rightarrow b \leq a$$

Contrapositive

*

$$\forall a \in \mathbb{R} \forall b \in \mathbb{R} : b > a \Rightarrow \exists \varepsilon > 0 : b \geq a + \varepsilon$$

Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and assume $b > a$

then $b - a > 0$

take $\varepsilon = b - a \rightarrow$ we just need a specific positive value for ε

* We need to prove that there exists a positive value of ε such that $b > a + \varepsilon$

Consider $b - (a + \varepsilon) = 0$

$$= b - a - \varepsilon$$

$$= b - a - b + a = 0$$

therefore, $b \geq a + \varepsilon$, where $\varepsilon = b - a$

This proves that if $b > a$, there exists a positive ε ($\varepsilon = b - a$) such that $b \geq a + \varepsilon$

$$\left. \begin{aligned} a + \varepsilon &= a + b - a, \text{ with } \varepsilon = b - a \\ &= b \end{aligned} \right\}$$

therefore $b \geq a + \varepsilon$

Biconditional proofs

(1) $\forall x \in \mathbb{R} : (x^2 + 5x - 6 = 0) \Leftrightarrow (x = -6 \vee x = 1)$

(1) $x^2 + 5x - 6 = 0 \Rightarrow x = -6 \vee x = 1$

We assume that $x^2 + 5x - 6 = 0$

$$x^2 + 5x - 6 = (x - 1)(x + 6)$$

(2) $x = -6 \vee x = 1 \Rightarrow x^2 + 5x - 6 = 0$