Solutions

Exercise 1: If $A \subseteq B$, then $B^c \subseteq A^c$.
Proof: Suppose that $A \subseteq B$. Then for all $x \in U$ we have:

$$(x \in A) \Rightarrow (x \in B)$$

The contrapositive of this proposition is:

$$(x \notin B) \Rightarrow (x \notin A)$$

Now let $y \in B^c$. Then $y \notin B$. But then by the above implication $y \notin A$. Hence $y \in A^c$, which completes the proof.

Exercise 2: Prove the following propositions:

$$\begin{aligned} \text{(i)}:&\ \ (A \subseteq B) \Rightarrow (A \cap B = A) \\ \text{(ii)}:&\ \ (A \subseteq B) \Rightarrow (A \setminus B = \emptyset) \end{aligned}$$

Proof of (i): Suppose that $A \subseteq B$. We have to prove 2 things:

1. $A \cap B \subseteq A$. This is trivial: If $x \in A \cap B$, then definitely $x \in A$.
2. $A \cap B \supseteq A$. Let $x \in A$. Since $A \subseteq B$ this means that also $x \in B$. But then $x \in A \cap B$.

Proof of (ii): Suppose that $A \subseteq B$. We have to prove 2 things:

1. $A \setminus B \supseteq \emptyset$. This is trivial, since the empty set is a subset of every set.
2. $A \setminus B \subseteq \emptyset$. Let $x \in A \setminus B$. Then $x \in A \ \wedge \ x \in B^c$. Since $A \subseteq B$ we have that $(x \in A) \Rightarrow (x \in B)$. So $x \in B \ \wedge \ x \in B^c$. But then $x \in (B \cap B^c) = \emptyset$.

Exercise 3: Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Proof:
"$\subseteq$":

$$\begin{aligned} & x \in A \cup (B \cap C) \\ & \quad \Rightarrow (x \in A) \vee (x \in B \cap C) \\ & \quad \Rightarrow (x \in A) \vee (x \in B \wedge x \in C) \\ & \quad \Rightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \\ & \quad \Rightarrow (x \in A \cup B) \wedge (x \in A \cup C) \\ & \quad \Rightarrow x \in (A \cup B) \cap (A \cup C) \end{aligned}$$

"$\supseteq$": The proof of "$\supseteq$" is the same as the proof of "$\subseteq$", but with the implication arrows in the opposite direction.

Exercise 4: Prove or disprove: $A \subseteq B \Rightarrow A \cup (B \cap C) = B \cap (A \cup C)$

Proof: We will prove the proposition. Notice first that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (by theorem 4 in section 2.2). Suppose that $A \subseteq B$. We have to prove 2 inclusions:

$$\begin{aligned} &" \subseteq " : \ (A \cup B) \cap (A \cup C) \subseteq B \cap (A \cup C) \text{ and} \\ &" \supseteq " : \ (A \cup B) \cap (A \cup C) \supseteq B \cap (A \cup C). \end{aligned}$$

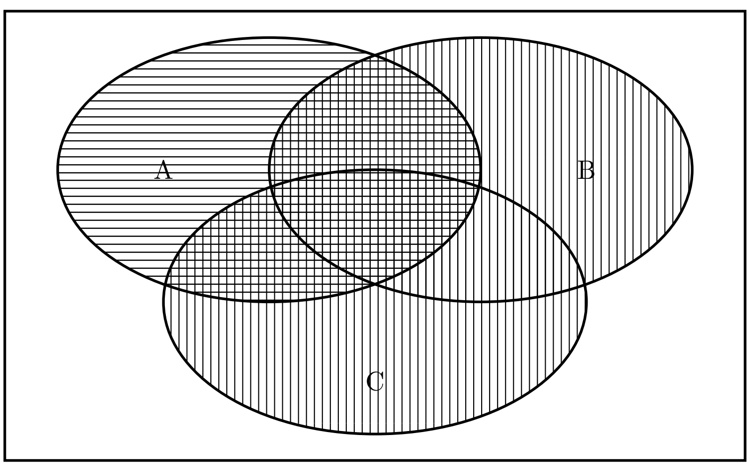
"$\subseteq$": Let $x \in (A \cup B) \cap (A \cup C)$ (by theorem 4 in section 2.2). Then

$$\begin{aligned} &x \in (A \cup B) \wedge x \in (A \cup C) \\ &\ \Rightarrow (x \in A \ \vee \ x \in B) \wedge (x \in A \ \vee \ x \in C) \\ &\ \Rightarrow x \in B \wedge (x \in A \ \vee \ x \in C) \text{ (since } A \subseteq B\text{: } x \in A \Rightarrow x \in B) \\ &\ \Rightarrow x \in B \cap (A \cup C) \end{aligned}$$

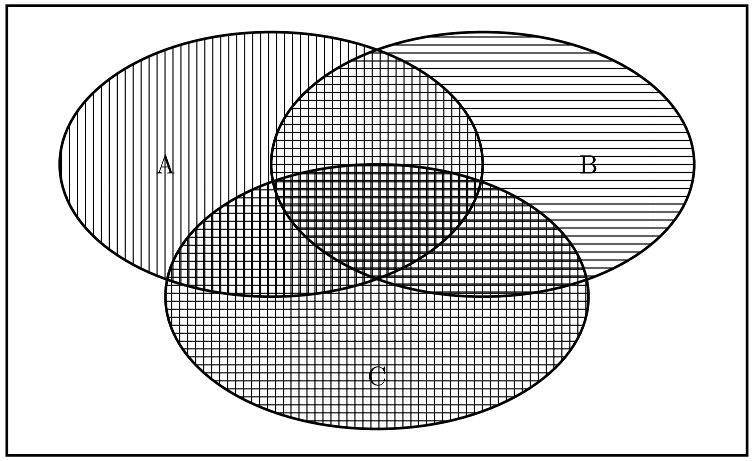
"$\supseteq$": Notice that $B \subseteq A \cup B$. So

$$\begin{aligned} &x \in B \cap (A \cup C) \\ &\ \Rightarrow x \in B \wedge (x \in A \ \vee \ x \in C) \\ &\ \Rightarrow x \in (A \cup B) \wedge (x \in A \ \vee \ x \in C) \text{ (since } B \subseteq A \cup B) \\ &\ \Rightarrow x \in (A \cup B) \cap (A \cup C) \end{aligned}$$

Exercise 5 *(exercise 2.2.2 in the book):* Prove that in general $A \cap (B \cup C) \neq (A \cap B) \cup C$. We need to find a counter example. We start by drawing the Venn-diagrams for both sets and observing where the difference is.



The above figure shows the Venn-diagram for the set $A \cap (B \cup C)$. It is the part in the drawing that is both horizontally and vertically striped. The part in the figure below that is both horizontally and vertically striped shows the set $(A \cap B) \cup C$:



Clearly the second set is bigger and any element that is in C but not in A will be in the second set and not in the first. So for a counterexample let for instance $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $C = \{3, 4, 5\}$. Then $(A \cap B) \cup C = \{2, 3, 4, 5\}$ and $A \cap (B \cup C) = \{2, 3\}$, making the sets unequal.

Exercise 6: Prove or disprove: $(A \cup B^c) \cap (A^c \cup B) = (A \cup B)^c \cup (A \cap B)$

Proof: We will prove the proposition. We have to prove 2 inclusions:

$$\text{"} \subseteq \text{"}: \ (A \cup B^c) \cap (A^c \cup B) \subseteq (A \cup B)^c \cup (A \cap B) \text{ and}$$
$$\text{"} \supseteq \text{"}: \ (A \cup B^c) \cap (A^c \cup B) \supseteq (A \cup B)^c \cup (A \cap B)$$

"$\subseteq$": Let $x \in (A \cup B^c) \cap (A^c \cup B)$. Then

$$\begin{aligned}
&x \in (A \cup B^c) \wedge x \in (A^c \cup B) \\
&\quad \Rightarrow ((x \in A) \vee (x \notin B)) \wedge ((x \notin A) \vee (x \in B)) \\
&\quad \Rightarrow ((x \in A) \wedge (x \notin A)) \vee ((x \in A) \wedge (x \in B)) \vee ((x \notin B) \wedge (x \notin A)) \vee ((x \notin B) \wedge (x \in B)) \\
&\quad \Rightarrow x \in (A \cap A^c) \vee x \in (A \cap B) \vee x \in (B^c \cap A^c) \vee x \in (B^c \cap B) \\
&\quad \Rightarrow x \in (A \cap B) \vee x \in (B^c \cap A^c) \\
&\quad \Rightarrow x \in (A \cap B) \vee x \in (A \cup B)^c \text{ (by De Morgan's law)} \\
&\quad \Rightarrow x \in (A \cup B)^c \cup (A \cap B)
\end{aligned}$$

"$\supseteq$": The proof of "$\supseteq$" is the same with the implication arrows in the opposite direction.

Exercise 7: Prove or disprove: $(B \cap C^c) \cup (A \setminus B) = (A \cup B) \cap C^c$

Proof: We will disprove the proposition by means of a counterexample. Take $A = \{1, 2, 4, 5\}$, $B = \{2, 3, 5, 6\}$ and $C = \{4, 5, 6, 7\}$ (and $U = \{1, 2, 3, 4, 5, 6, 7\}$). Then $C^c = \{1, 2, 3\}$, so $B \cap C^c = \{2, 3\}$. Furthermore $A \setminus B = \{1, 4\}$, so $(B \cap C^c) \cup (A \setminus B) = \{1, 2, 3, 4\}$. However, $A \cup B = \{1, 2, 3, 4, 5, 6\}$, so $(A \cup B) \cap C^c = \{1, 2, 3\}$, making the sets unequal.

We will now prove that $(A \cup B) \cap C^c \subseteq (B \cap C^c) \cup (A \setminus B)$ (this proof is of a slightly more than average degree of difficulty). Let $x \in (A \cup B) \cap C^c$. Then $x \in (A \cup B) \wedge x \in C^c$. This means that $(x \in A \ \vee \ x \in B) \wedge x \in C^c$. But then $(x \in A \ \wedge \ x \in C^c) \vee (x \in B \ \wedge \ x \in C^c)$, so $x \in A \cap C^c \ \vee \ x \in B \cap C^c$. If $x \in B \cap C^c$, then clearly $x \in (B \cap C^c) \cup (A \setminus B)$, so it remains to prove that this is also the case if $x \in A \cap C^c$. We now split the analysis into 2 parts, depending on whether x is also in B or not. (i) Suppose $x \in (A \cap C^c) \cap B$. Then $x \in A \ \wedge \ x \in C^c \ \wedge x \in B$. But then $x \in (B \cap C^c)$ and hence $x \in (B \cap C^c) \cup (A \setminus B)$. (ii) Suppose that $x \in (A \cap C^c) \cap B^c$. Then $x \in A \ \wedge \ x \in C^c \ \wedge x \in B^c$. But then $x \in A \setminus B$ and hence, again $x \in (B \cap C^c) \cup (A \setminus B)$, which completes the proof.

Exercise 8: The power set of A has 16 elements. I'm sure you can find them all yourself!

Exercise 9: Prove or disprove the following propositions:

1. $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ TRUE

   Proof:

   "$\subseteq$": Let $X \in \mathcal{P}(A \cap B)$. Then $X \subseteq (A \cap B)$. But then $(X \subseteq A) \wedge (X \subseteq B)$ (verify that this is true by making a Venn Diagram), which means that $X \in \mathcal{P}(A) \wedge X \in \mathcal{P}(B)$. Hence $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

   "$\supseteq$": The proof of "$\supseteq$" is the same with the implications in the opposite direction.

2. $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ FALSE

   Proof: Counterexample: Take $A = \{1\}$, $B = \{2\}$. Then $\{1, 2\} \in \mathcal{P}(A \cup B)$, but $\{1, 2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

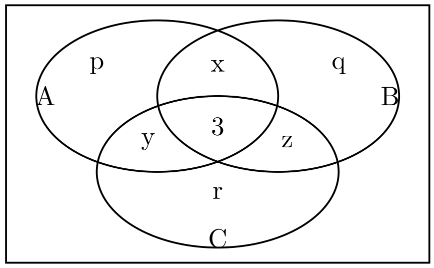
   We will prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

$$
\begin{aligned}
&X \in \mathcal{P}(A) \cup \mathcal{P}(B) \\
&\quad \Rightarrow (X \in \mathcal{P}(A)) \vee (X \in \mathcal{P}(B)) \\
&\quad \Rightarrow (X \subseteq A) \vee (X \subseteq B) \\
&\quad \Rightarrow X \subseteq A \cup B \text{ (this implication does not hold in the opposite direction!)} \\
&\quad \Rightarrow X \in \mathcal{P}(A \cup B)
\end{aligned}
$$

Exercise 10: If A has m elements then, in order to create a multiset B of size n that contains only elements from A, we essentially make n selections, where each time we select an element from A. So our group size is m and we make n selections, where (i) repetition is allowed (we can draw the same element from S multiple times) and (ii) order doesn't matter (the multiset $\{1, 1, 2\}$ is the same as the multiset $\{1, 2, 1\}$). So the total number of multisets of size n out of a set of m elements is $\binom{m-1+n}{n}$.

Exercise 11: Solve exercises 2.5.2, 2.5.3 and 2.5.4.

Exercise 2.5.2. Let A be the set of students taking analysis, B the set of students taking algebra and C the set of students taking statistics. Then $|A| = 20$, $|B| = 30$, $|C| = 30$, $|A \cap B| = 5$, $|B \cap C| = 10$, $|A \cap C| = 4$ and $|A \cap B \cap C| = 3$. This corresponds to the following Venn-diagram:



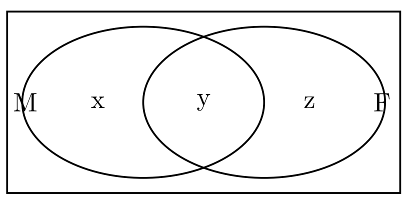
where $x + 3 = 5$, $y + 3 = 4$ and $z + 3 = 10$, so $x = 2$, $y = 1$ and $z = 7$. Furthermore $p + x + y + 3 = 20$, so $p = 14$; $x + q + 3 + z = 30$, so $q = 18$ and $y + 3 + z + r = 30$, so $r = 19$. To answer the questions: The number of students taking at least one course is

$$|A \cup B \cup C| = p + x + q + y + 3 + z + r = 64$$

and the number of students taking only analysis is

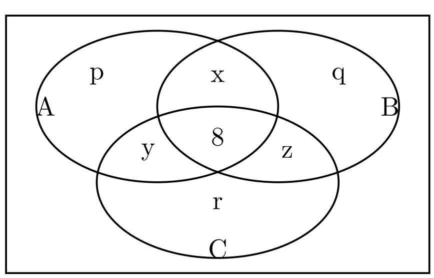
$$|A \cap B^c \cap C^c| = p = 14$$

Exercise 2.5.3. Let M be the set of students whose mother has been to university and F the set of students whose father has been to university. Then $|U| = 100$, $|M| = 20$, $|F| = 15$ and $|M^c \cap F^c| = 70$. This corresponds to the following Venn-diagram:



where $x + y = |M| = 20$, $y + z = |F| = 15$ and $x + y + z = |M \cup F| = 100 - 70 = 30$. This means that $x = 15$, $z = 10$ and $y = 5$. To answer the question: The number of students for whom both parents have been to university is $|M \cap F| = y = 5$.

Exercise 2.5.4. Let A be the set of people wearing glasses, B the set of people having false teeth and C the set of people voting Labour. Then $|U| = 500$, $|A| = 200$, $|B| = 50$, $|C| = 300$, $|A \cap B| = 40$, $|B \cap C| = 10$ and $|A \cap C| = 150$. This corresponds to the following Venn-diagram:



where $|A \cap B| = x+8 = 40$, so $x = 32$, $|B \cap C| = z+8 = 10$, so $z = 2$ and $|A \cap C| = y+8 = 150$, so $y = 142$. Furthermore $|A| = p + x + y + 8 = 200$, so $p = 118$, $|B| = x + q + 8 + z = 50$, so $q = 8$ and $|C| = 8 + y + z + r = 300$, so $r = 148$. Finally, since $|U| = 500$, we know that $|A^c \cap B^c \cap C^c| = 500 - p - q - r - x - y - z - 8 = 42$. To answer the questions: The number of people who voted Labour and did not have glasses or false teeth is $|A^c \cap B^c \cap C| = r = 148$ and the number of people who did not vote Labour and had false teeth and glasses is $|A \cap B \cap C^c| = x = 32$.

Exercise 12: An element from the product set $A \times B$ is constructed by taking and element from A and taking and element from B and then 'fusing' them into one element. Since A has $|A|$ elements and B has $|B|$ elements, the number of elements in $A \times B$ must be $|A| \cdot |B|$.

Exercise 13: Let $A = \{2, 3, 5, 7\}$, $B = \{1\}$ and $C = \{6, 8, 10\}$. Calculate $A \times B$, $B \times C$ and $B \times B \times C \times B \times B \times C$.

Solution:

$$\begin{aligned}
A \times B &= \{(2,1), (3,1), (5,1), (7,1)\} \\
B \times C &= \{(1,6), (1,8), (1,10)\} \\
B \times B \times C \times B \times B \times C &= \{(1,1,6,1,1,6), (1,1,6,1,1,8), (1,1,6,1,1,10), \\
&\quad (1,1,8,1,1,6), (1,1,8,1,1,8), (1,1,8,1,1,10), \\
&\quad (1,1,10,1,1,6), (1,1,10,1,1,8), (1,1,10,1,1,10)\}
\end{aligned}$$

Exercise 14: Prove or disprove the following proposition: $\mathbb{Q}$ (the set of rational numbers) is countable.

Proof: We prove that $\mathbb{Q}$ is countable. Notice that $\mathbb{Q} = \left\{\frac{m}{n} : m \text{ and } n \text{ are integers with } n \neq 0\right\}$. We show that $\mathbb{Q}^+$ is countable. $\mathbb{Q}$ is then countable as well (just put each negative number $-\frac{m}{n}$ in the list right after the positive number $\frac{m}{n}$). We do this by ordering the elements in

$$\mathbb{Q}^+ = \left\{\frac{m}{n} : m \text{ and } n \text{ are nonnegative integers with } n \neq 0\right\}$$

in such a way that $m + n$ increases all the time, where we erase elements that were already listed earlier (e.g. $\frac{6}{2} = \frac{3}{1}$ so $\frac{6}{2}$ will not be put in the list). The set $\mathbb{Q}^+$ then looks as follows:

$$\mathbb{Q}^+ = \left\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \ldots\right\}$$

and in this list for each element of $\mathbb{Q}^+$ we can find its exact location, making $\mathbb{Q}^+$ a countable set.

Exercise 15: Prove that the Cantor Set is uncountable.

Proof: This is a tricky exercise. The proof relies on the fact that we can write any real number not only in a decimal or binary notation, but in any numerical system. In a decimal representation of a number we use *decimals*, meaning that each digit is one of the numbers in the set $\{0, 1, ..., 9\}$; in a binary representation use *bits*, so each number is in $\{0, 1\}$. For our proof we make use of a different number system, namely the ternary number system, where any number is expressed using only the digits 0, 1 and 2. Now notice that any number in $(\frac{1}{3}, \frac{2}{3})$ in the ternery notation has the property that the first digit is a 1 (e.g. $\frac{1}{2} = 0.111111111$), whereas any number in $C_1 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3})$ can be written without a 1 as a first digit. (Notice that $\frac{1}{3} = 0.022222\ldots$). Now similarly the part that we delete in step 2 to obtain the set C_2 consists exactly of those numbers for which in ternary notation the second digit must be 1. This process continues to infinity, so the Cantor Set would eventually be the set of all numbers in $[0,1]$ for which there is a ternary representation that does not include any 1's. Now for every number in the Cantor Set let's replace every 2 by a 1 and consider the resulting number to be a number in binary notation. What is the set that we end up with? Looking closely we find ALL binary numbers in the set $[0,1]$. But that means that there is a one-to-one correspondence between the numbers in the Cantor Set and the numbers in $[0,1]$. But then the 2 sets must be of the same size and hence the Cantor Set is uncountable.

Exercise 16:

1. Let $x, y \in A_1 = [0,1]$ and let $z = \lambda \cdot x + (1-\lambda) \cdot y$ for some $\lambda \in [0,1]$. Then $z \geq \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0$ (since $x \geq 0$ and $y \geq 0$) and $z \leq \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1$ (since $x \leq 1$ and $y \leq 1$). So $z \in A_1$, which means that A_1 is convex.

2. Let $\mathbf{x} = (x_1, x_2) \in A_2 = [0,1] \times [3,4]$, $\mathbf{y} = (y_1, y_2) \in A_2$ and let $\mathbf{z} = \lambda \cdot \mathbf{x} + (1-\lambda) \cdot \mathbf{y}$ for some $\lambda \in [0,1]$. Then $z_1 = \lambda \cdot x_1 + (1-\lambda) \cdot y_1 \geq \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0$ (since $x_1 \geq 0$ and $y_1 \geq 0$) and $z_1 \leq \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1$ (since $x_1 \leq 1$ and $y_1 \leq 1$). Also, $z_2 = \lambda \cdot x_2 + (1-\lambda) \cdot y_2 \geq \lambda \cdot 3 + (1-\lambda) \cdot 3 = 3$ (since $x_2 \geq 3$ and $y_2 \geq 3$) and $z_2 \leq \lambda \cdot 4 + (1-\lambda) \cdot 4 = 4$ (since $x_2 \leq 4$ and $y_2 \leq 4$). But then $\mathbf{z} \in A_2$, so A_2 is convex.

3. Take $\mathbf{x} = (1,0)$, $\mathbf{y} = (0,1)$ and $\mathbf{z} = \frac{1}{2} \cdot \mathbf{x} + \frac{1}{2} \cdot \mathbf{y} = (\frac{1}{2}, \frac{1}{2})$. Then $x_1^2 + x_2^2 = 1^2 + 0^2 = 1$ and $y_1^2 + y_2^2 = 1$, so $\mathbf{x} \in A_3$ and $\mathbf{y} \in A_3$. However $z_1^2 + z_2^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2} \neq 1$, so $\mathbf{z} \notin A_3$. Hence A_3 is not convex.

4. Let $\mathbf{x}, \mathbf{y} \in A_4$ (so $x_1^2 + x_2^2 \leq 1$ and $y_1^2 + y_2^2 \leq 1$) and let $\mathbf{z} = \lambda \mathbf{x} + (1-\lambda)\mathbf{y}$ for some $\lambda \in [0,1]$. Then

$$\begin{aligned}
z_1^2 + z_2^2 &= (\lambda x_1 + (1-\lambda) y_1)^2 + (\lambda x_2 + (1-\lambda) y_2)^2 \\
&= \lambda^2 (x_1^2 + x_2^2) + (1-\lambda)^2 (y_1^2 + y_2^2) + 2\lambda(1-\lambda)(x_1 y_1 + x_2 y_2)) \\
&= (\lambda^2 + \lambda - \lambda)(x_1^2 + x_2^2) + ((1-\lambda)^2 + (1-\lambda) - (1-\lambda))(y_1^2 + y_2^2) \\
&\quad + 2\lambda(1-\lambda)(x_1 y_1 + x_2 y_2)) \\
&= \lambda(x_1^2 + x_2^2) + (1-\lambda)(y_1^2 + y_2^2) - \lambda(1-\lambda)(x_1^2 + x_2^2 + y_1^2 + y_2^2) \\
&\quad + 2\lambda(1-\lambda)(x_1 y_1 + x_2 y_2) \\
&= \lambda(x_1^2 + x_2^2) + (1-\lambda)(y_1^2 + y_2^2) - \lambda(1-\lambda)(x_1^2 + y_1^2 - 2x_1 y_1 + x_2^2 + y_2^2 - 2x_2 y_2) \\
&= \lambda(x_1^2 + x_2^2) + (1-\lambda)(y_1^2 + y_2^2) - \lambda(1-\lambda)((x_1 - y_1)^2 + (x_2 - y_2)^2) \\
&\leq \lambda(x_1^2 + x_2^2) + (1-\lambda)(y_1^2 + y_2^2) \\
&\leq \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1,
\end{aligned}$$

so $\mathbf{z} \in A_4$ and A_4 is convex!

5. Let $\mathbf{x} = (x_1, x_2) \in A_5$, $\mathbf{y} = (y_1, y_2) \in A_5$ and let $\mathbf{z} = \lambda \cdot \mathbf{x} + (1-\lambda) \cdot \mathbf{y}$ for some $\lambda \in [0, 1]$. Then $x_1 + 2x_2 \leq 4$ and $y_1 + 2y_2 \leq 4$ (and $z_1 = \lambda \cdot x_1 + (1-\lambda) \cdot y_1$ and $z_2 = \lambda \cdot x_2 + (1-\lambda) \cdot y_2$). So $z_1 + 2z_2 = \lambda(x_1 + 2x_2) + (1-\lambda)(y_1 + 2y_2) \leq \lambda \cdot 4 + (1-\lambda) \cdot 4 = 4$. This means that $\mathbf{z} \in A_5$ and hence A_5 is convex.

6. Let $\mathbf{x} = (x_1, x_2) \in A_6$, $\mathbf{y} = (y_1, y_2) \in A_6$ and let $\mathbf{z} = \lambda \cdot \mathbf{x} + (1-\lambda) \cdot \mathbf{y}$ for some $\lambda \in [0, 1]$. Then $x_1 \geq 0$, $y_1 \geq 0$, $x_2 \geq 0$, $y_2 \geq 0$, $x_1 + 2x_2 \leq 4$ and $y_1 + 2y_2 \leq 4$ (and $z_1 = \lambda \cdot x_1 + (1-\lambda) \cdot y_1$ and $z_2 = \lambda \cdot x_2 + (1-\lambda) \cdot y_2$). So $z_1 \geq \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0$, $z_2 \geq \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0$ and $z_1 + 2z_2 = \lambda(x_1 + 2x_2) + (1-\lambda)(y_1 + 2y_2) \leq \lambda \cdot 4 + (1-\lambda) \cdot 4 = 4$. That means that $\mathbf{z} \in A_6$ and hence A_6 is convex.

7. Take $\mathbf{x} = (0, 0)$, $\mathbf{y} = (1, 1)$ and $\mathbf{z} = \frac{1}{2} \cdot \mathbf{x} + \frac{1}{2} \cdot \mathbf{y} = (\frac{1}{2}, \frac{1}{2})$. Then $x_2 = x_1^2$ and $y_2 = y_1^2$, so $\mathbf{x} \in A_7$ and $\mathbf{y} \in A_7$. However $z_2 \neq z_1^2$, so $\mathbf{z} \notin A_7$. Hence A_7 is not convex.

8. Take $\mathbf{x} = (1, 1)$, $\mathbf{y} = (\frac{1}{2}, 2)$ and $\mathbf{z} = \frac{1}{2} \cdot \mathbf{x} + \frac{1}{2} \cdot \mathbf{y} = (\frac{3}{4}, 1\frac{1}{2})$. Then $x_1 \cdot x_2 = 1$ and $y_1 \cdot y_2 = 1$, so $\mathbf{x} \in A_8$ and $\mathbf{y} \in A_8$. However $z_1 \cdot z_2 = \frac{3}{4} \cdot 1\frac{1}{2} = 1\frac{1}{8} > 1$, so $\mathbf{z} \notin A_8$. Hence A_8 is not convex.

9. Let $\mathbf{x}, \mathbf{y} \in A_9$ (so $x_1 \cdot x_2 \geq 1$ and $y_1 \cdot y_2 \geq 1$) and let $\mathbf{z} = \lambda \mathbf{x} + (1-\lambda)\mathbf{y}$ for some $\lambda \in [0, 1]$. We split the analysis in two cases. **Case 1:** $x_1 \geq y_1$ and $x_2 \geq y_2$ (or, similarly, $x_1 \leq y_1$ and $x_2 \leq y_2$). Then $z_1 \geq y_1$ and $z_2 \geq y_2$ so $z_1^2 + z_2^2 \geq y_1^2 + y_2^2 = 1$ and we are done. **Case 2:** $x_1 > y_1$ and $x_2 < y_2$ (or, similarly, $x_1 < y_1$ and $x_2 > y_2$). For both cases the following calculation holds:

$$
\begin{aligned}
z_1 \cdot z_2 &= (\lambda x_1 + (1-\lambda)y_1)(\lambda x_2 + (1-\lambda)y_2) \\
&= \lambda^2 x_1 x_2 + (1-\lambda)^2 y_1 y_2 + \lambda(1-\lambda)(x_1 y_2 + x_2 y_1) \\
&= (\lambda^2 + \lambda - \lambda)x_1 x_2 + ((1-\lambda)^2 + (1-\lambda) - (1-\lambda))y_1 y_2 + \lambda(1-\lambda)(x_1 y_2 + x_2 y_1) \\
&= \lambda x_1 x_2 + (1-\lambda)y_1 y_2 + \lambda(1-\lambda)(-x_1 x_2 - y_1 y_2 + x_1 y_2 + x_2 y_1) \\
&= \lambda x_1 x_2 + (1-\lambda)y_1 y_2 + \lambda(1-\lambda)(x_1 - y_1)(y_2 - x_2) \\
&\geq \lambda x_1 x_2 + (1-\lambda)y_1 y_2 \quad \text{since } x_1 > y_1 \text{ and } x_2 < y_2 \\
&\geq \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1,
\end{aligned}
$$

so $\mathbf{z} \in A_9$. We conclude that A_9 is convex.

10. Take $x = 1$, $y = 2$ and $z = \frac{1}{2}x + \frac{1}{2}y = 1\frac{1}{2}$. Then $x \in \mathbb{N}$ and $y \in \mathbb{N}$, but $z \notin \mathbb{N}$. This means that $\mathbb{N}$ is not convex.

11. Let $x, y \in \mathbb{R}$ and let $z = \lambda \cdot x + (1-\lambda) \cdot y$ for some $\lambda \in [0, 1]$. Then $z \in \mathbb{R}$, so $\mathbb{R}$ is convex.

12. Take $x = 0$, $y = 2$ and $z = \frac{1}{2}\sqrt{2} \cdot y + (1 - \frac{1}{2}\sqrt{2}) \cdot x = \sqrt{2}$. Then $x \in A_{12}$ and $y \in A_{12}$, but $z \notin A_{12}$ (since $\sqrt{2}$ is not rational). This means that A_{12} is not convex.

13. The empty set is convex, essentially due to the fact that it is not possible to choose an $x \in \emptyset$.

14. Let $x, y \in A_{14} = \{(e, \pi)\}$ and let $z = \lambda \cdot x + (1-\lambda) \cdot y$ for some $\lambda \in [0, 1]$. Then $x = (e, \pi)$ and $y = (e, \pi)$, so $z = \lambda \cdot (e, \pi) + (1-\lambda) \cdot (e, \pi) = (e, \pi) \in \{(e, \pi)\}$. So $\{(e, \pi)\}$ is convex.

15. Take $x = e$, $y = \pi$ and $z = \frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}(e + \pi)$. Then $x \in A_{15}$ and $y \in A_{15}$, but $z \notin A_{15}$. This means that A_{15} is not convex.

16. Let $x, y \in A_{16} = [e, \pi]$ and let $z = \lambda \cdot x + (1 - \lambda) \cdot y$ for some $\lambda \in [0, 1]$. Then $z \geq e$ (since both $x \geq e$ and $y \geq e$), and $z \leq \pi$ (since both $x \leq \pi$ and $y \leq \pi$). So $z \in [e, \pi]$ and $[e, \pi]$ is convex.

17. Let $\mathbf{x} = (x_1, x_2) \in A_{17}$, $\mathbf{y} = (y_1, y_2) \in A_{17}$ and let $\mathbf{z} = \lambda \cdot \mathbf{x} + (1 - \lambda) \cdot \mathbf{y}$ for some $\lambda \in [0, 1]$. Then $|x_1 - x_2| \leq 1$ and $|y_1 - y_2| \leq 1$, so $|z_1 - z_2| = |\lambda x_1 + (1 - \lambda)y_1 - \lambda x_2 + (1 - \lambda)y_2| = |\lambda(x_1 - x_2) + (1 - \lambda)(y_1 - y_2)|$. The biggest possible value for $\lambda(x_1 - x_2) + (1 - \lambda)(y_1 - y_2)$ is $\lambda \cdot 1 + (1 - \lambda) \cdot 1 = 1$; the smallest possible value is $\lambda \cdot -1 + (1 - \lambda) \cdot -1 = -1$. This means that $|\lambda x_1 + (1 - \lambda)y_1 - \lambda x_2 + (1 - \lambda)y_2| \leq 1$, so A_{17} is convex.

18. Take $\mathbf{x} = (2, 0)$, $\mathbf{y} = (0, 2)$ and $\mathbf{z} = \frac{1}{2} \cdot \mathbf{x} + \frac{1}{2} \cdot \mathbf{y} = (1, 1)$. Then $|x_1 - x_2| = 2 \geq 1$ and $|y_1 - y_2| = 2 \geq 1$, so $\mathbf{x} \in A_{18}$ and $\mathbf{y} \in A_{18}$. However, $|z_1 - z_2| = 0 \ngeq 1$, so $\mathbf{z} \notin A_{18}$. Hence A_{18} is not convex.

19. Let $\mathbf{x} = (x_1, x_2) \in A_{19}$, $\mathbf{y} = (y_1, y_2) \in A_{19}$ and let $\mathbf{z} = \lambda \cdot \mathbf{x} + (1 - \lambda) \cdot \mathbf{y}$ for some $\lambda \in [0, 1]$. Then $|x_1| + |x_2| \leq 1$ and $|y_1| + |y_2| \leq 1$. Notice that for an absolute value of a number a we always have $a \leq |a|$ and also $-a \leq |a|$. We will use this in the remainder of the proof. We have:

$$\begin{aligned} |z_1| + |z_2| &= |\lambda x_1 + (1 - \lambda)y_1| + |\lambda x_2 + (1 - \lambda)y_2| \\ &= \pm(\lambda x_1 + (1 - \lambda)y_1) \pm (\lambda x_2 + (1 - \lambda)y_2), \end{aligned}$$

depending on if the expressions are positive or negative. In either case

$$\begin{aligned} |z_1| + |z_2| &= \lambda(\pm x_1 \pm x_2) + (1 - \lambda)(\pm y_1 + \pm y_2) \\ &\leq \lambda(|x_1| + |x_2|) + (1 - \lambda)(|y_1| + |y_2|) \\ &\leq \lambda \cdot 1 + (1 - \lambda) \cdot 1 = 1, \end{aligned}$$

so $\mathbf{z} \in A_{19}$ and hence A_{19} is convex.

20. Take $\mathbf{x} = (5, 5)$, $\mathbf{y} = (5, -5)$ and $\mathbf{z} = \frac{1}{2} \cdot \mathbf{x} + \frac{1}{2} \cdot \mathbf{y} = (5, 0)$. Then $|x_1| - |x_2| = |5| - |5| = 0 \leq 1$ and $|y_1| - |y_2| = |5| - |-5| = 0 \leq 1$, so $\mathbf{x} \in A_{20}$ and $\mathbf{y} \in A_{20}$. However, $|z_1| - |z_2| = |5| - |0| = 5 > 1$, so $\mathbf{z} \notin A_{20}$. Hence A_{20} is not convex.