Lecture 11 - Calculus

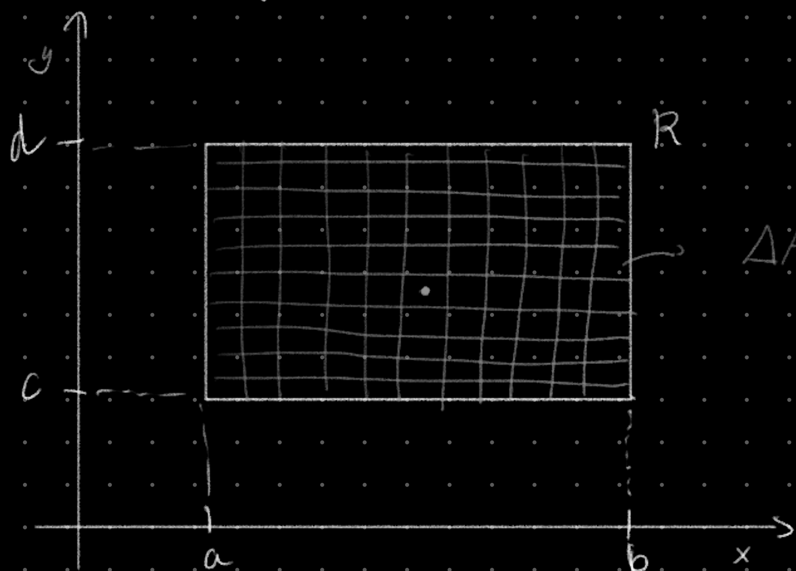
- **Limits and continuity**
- **Differentiation + applications**
- **Integration**
- **Sequences and series**
- **Differential equations**
- **Introduction to multivariate functions**
- **Double integrals: today** last lecture!

Thomas' Ch. 15.1-2 or Adams' Ch. 14.1-2

Double integrals

Let $f(x,y)$, continuous on a region R

R: $a \leq x \leq b$
$c \leq y \leq d$



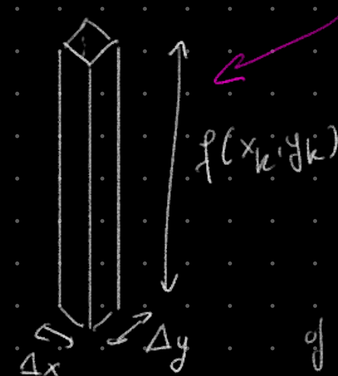
How to calculate $\iint_R f(x,y)\, dA$

↳ (signed) volume between R and the surface $f(x,y)$

ΔA_k partition of R into small rectangles ΔA_k, $\Delta A_k = \Delta x \Delta y$

→ Riemann sum

$$S_n = \sum_{k=1}^{n} \underbrace{\Delta A_k\, f(x_k, y_k)}$$



$f(x_k, y_k)\, \Delta x \Delta y$ is a volume element of the Riemann sum

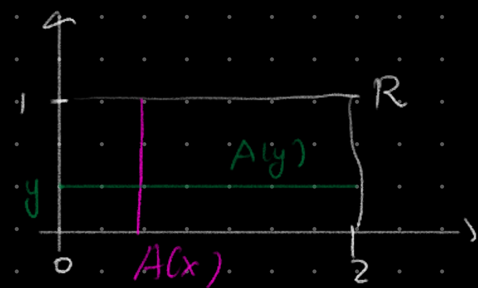
$f(x,y)$ is integrable if $\lim_{n \to \infty} S_n = I$

Calculation of a double integral

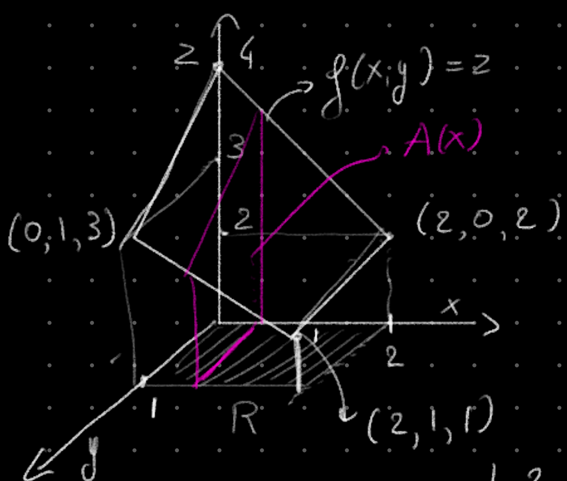
Example: $f(x,y) = 4 - x - y$

$R: \; 0 \le x \le 2$

$ 0 \le y \le 1$



$$\iint_R f(x,y)\,dA = \int_0^2 \underbrace{\int_0^1 (4-x-y)\,dy}_{A(x)}\,dx$$



$$A(x) = \int_0^1 (4-x-y)\,dy = 4 - x - \frac{1}{2} = \frac{7}{2} - x$$

$$\int_0^2 A(x)\,dx = \int_0^2 \left(\frac{7}{2} - x\right) dx = 7 - 2 = 5$$

$$\iint_R f(x,y)\,dA = \int_0^1 \underbrace{\int_0^2 (4-x-y)\,dx}_{A(y)}\,dy$$

$$A(y) = \int_0^2 (4-x-y)\,dx = \left[4x - \frac{x^2}{2} - xy\right]_0^2 = 8-2-2y = 6-2y$$

$$\int_0^1 A(y)\,dy = \int_0^1 (6-2y)\,dy = \left[6y - y^2\right]_0^1 = 6-1 = 5$$

Example : $\int_{-1}^{1}\int_0^1 (x+y)\,dy\,dx$

inner integral : $\int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}$

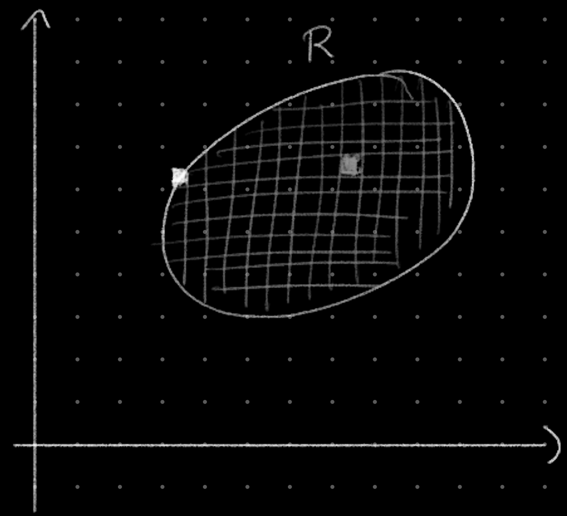
outer integral : $\int_{-1}^{1} (x+\frac{1}{2})\,dx = \left[\frac{x^2}{2} + \frac{x}{2}\right]_{-1}^{1} = \left(\frac{1}{2}+\frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{2}\right) = 1$

Fubini's theorem

If $f(x,y)$ is continuous on the rectangular area R: $a \le x \le b$, $c \le y \le d$,

then $\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$

Double integrals over general regions

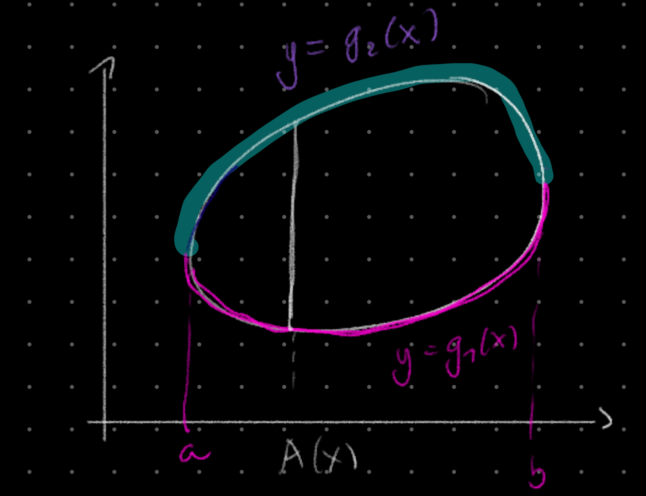


$\iint_R f(x,y)\,dA$ is the volume between $f(x,y)$ and R

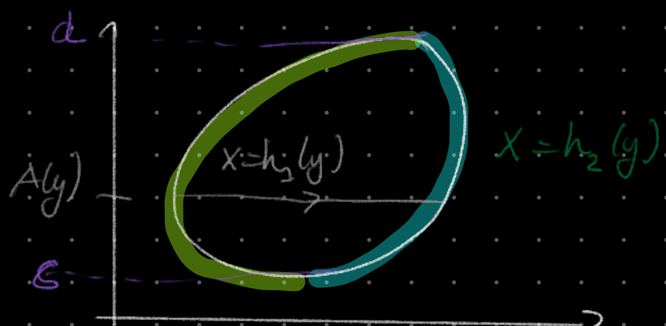
The double integral is the limit of the Riemann sums

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k$$

Calculating integrals over general regions



$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Fubini's theorem (stronger form)

If $f(x,y)$ is continuous on a region R

$\longrightarrow$ if R is defined as $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$

with $g_1(x)$ and $g_2(x)$ continuous on $[a, b]$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

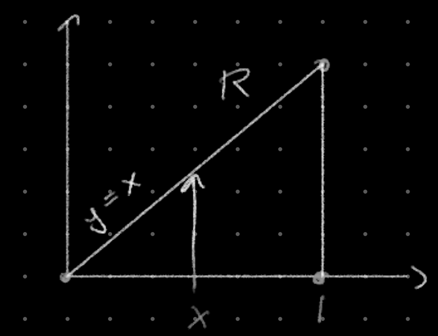
$\longrightarrow$ if R is defined as $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$

with $h_1(y)$ and $h_2(y)$ continuous on $[c, d]$, then

$$\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Example

$f(x,y) = 3-x-y \qquad R = \text{triangle with vertices } (0,0), (1,0), (1,1)$



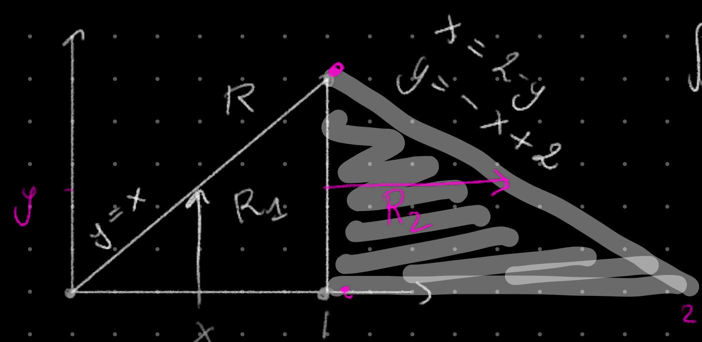
$x : 0 \to 1$

$y : 0 \to x$

$$\iint_R f(x,y)\,dA = \int_0^1 \underbrace{\int_0^x (3-x-y)\,dy}_{A(x)}\,dx$$

$$A(x) = \int_0^x (3-x-y)\,dy = \left[3y - xy - \frac{y^2}{2}\right]_0^x = 3x - x^2 - \frac{x^2}{2} = 3x - \frac{3x^2}{2}$$

$$\int_0^1 A(x)\,dx = \int_0^1 \left(3x - \frac{3x^2}{2}\right)dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_0^1 = \frac{3}{2} - \frac{1}{2} = 1$$



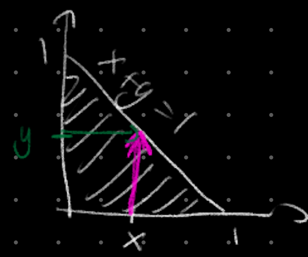
$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

R_2: $y : 0 \to 1$

$x : 1 \to 2-y$

$$\iint_R (x+y)\, dA$$

R:



$x : 0 \longrightarrow 1$
$y : 0 \longrightarrow 1-x$

OR

$y : 0 \longrightarrow 1$
$x : 0 \longrightarrow 1-y$

$$\int_0^1 \int_0^{1-x} (x+y)\, dy\, dx$$

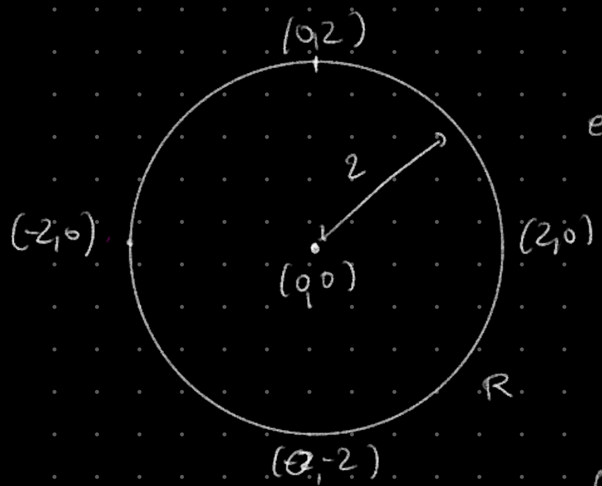
$$A(x) = \int_0^{1-x} (x+y)\, dy = \left[xy + \frac{y^2}{2} \right]_0^{1-x} = x(1-x) + \frac{(1-x)^2}{2} = \frac{1}{2} - \frac{x^2}{2}$$

$$= \cancel{x} - x^2 + \frac{1}{2} - \cancel{x} + \frac{x^2}{2}$$

$$\int_0^1 A(x)\, dx = \int_0^1 \left(\frac{1}{2} - \frac{x^2}{2} \right) dx = \frac{1}{2} - \left[\frac{x^3}{6} \right]_0^1 = \frac{1}{3}$$

An example with a more complex region R

(the example in class was too complicated)



equation of circle : $x^2+y^2=2^2 \Rightarrow y=\pm\sqrt{4-x^2}$

R : x : $-2 \to 2$
y : $-\sqrt{4-x^2} \to +\sqrt{4-x^2}$

$$f(x,y) = x+y^2$$

$$\iint_R f(x,y)\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x+2y^2)\,dy\,dx$$

$$A(x) = \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x+2y^2)\,dy = \left[xy+\frac{2}{3}y^3\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} = 2x\sqrt{4-x^2} + \frac{4}{3}\sqrt{4-x^2}^{\,3}$$

$$\int_{-2}^{2} A(x)\,dx = \int_{-2}^{2} 2x\sqrt{4-x^2}\,dx + \frac{4}{3}\int_{-2}^{2}\sqrt{4-x^2}^{\,3}\,dx = \frac{8}{3}\int_{-\pi/2}^{\pi/2}\sqrt{4-\sin^2(u)}^{\,3}\cos(u)\,du$$

0

odd function, symmetric domain.

$x = 2\sin(u)$
$dx = 2\cos(u)\,du$

$x = \pm 2 \to u = \pm\frac{\pi}{2}$

$$= \frac{16}{3} \int_{-\frac{\pi}{2}}^{\pi/2} \cos^4(u)\, du = \frac{4}{3} \int_{-\frac{\pi}{2}}^{\pi/2} (1 + \cos(2u))^2 du = \frac{4}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + 2\cos(2u) + \left(\frac{1}{2} + \frac{\cos(4u)}{2}\right)\right) du$$

$$\cos^4(u) =$$

$$\cos(2u) = 2\cos^2 u - 1 \rightarrow \cos^2 u = \frac{1}{2}(1 + \cos(2u))$$

$$\cos^2(2u) = \frac{1}{2}(1 + \cos(4u))$$

$$= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du + \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2u)\, du + \frac{2}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(4u)\, du = 2\pi + \frac{4}{3}\left[\sin(2u)\right]_{-\frac{\pi}{2}}^{\pi/2} + \frac{1}{6}\left[\sin(4u)\right]_{-\frac{\pi}{2}}^{\pi/2}$$

$$= \boxed{2\pi}$$

THIS INTEGRAL IS A DEMONSTRATION THAT DOUBLE INTEGRALS CAN BECOME VERY DIFFICULT QUICKLY

→ such an integral is way too long and complex for the exam!!

Properties of double integrals

For $f(x,y)$, $g(x,y)$ continuous on a bounded region R

* $\iint_R c \cdot f(x,y)\,dA = c \cdot \iint_R f(x,y)\,dA$ for $c \in \mathbb{R}$

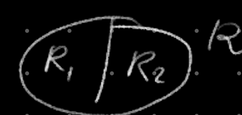
* $\iint_R (f(x,y) \pm g(x,y))\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$

* if $f(x,y) \geq 0$ on R, then $\iint_R f(x,y)\,dA \geq 0$

  ↳ choose parametrization correctly!

* if $f(x,y) \geq g(x,y)$ on R, then $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$

* if R is partitioned into R_1 and R_2



  then $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

Examples