Overview of the course

- **Continuity and limits**
- **Differentiation**
- **Integration**
- **Sequences and series**
- **Introduction to differential equations**
- **Introduction to multivariate calculus**

Today

- **Ordinary Differential Equations (ODE)**
- **Separable ODEs** (solution method)
- **Linear ODEs**
- **Parameter variation** (solution method)
- **Integral equations**

Adams' 2.10, 18.1, 18.2, 7.9

Differential equations

- What: An equation that involves derivatives of an unknown function $y(x)$
  - ↳ solution: explicit form $y(x)$
- ODE = Ordinary Differential equation
  - (↳ stochastic DE's, partial DEs, delay DEs also exist)
- Examples:

$$F = m a \longrightarrow m \frac{d^2}{dx^2} x = F(x,t) \longrightarrow \text{solution } x(t)$$

$$y' = 2y(3-y) \longrightarrow \text{solution } y(x)$$

$$y'' + 2y' + 3y = \cos(x)$$

- General form: $F(\underbrace{y^{(n)}}_{\text{order of ODE}}, y^{(n-1)}, \ldots, y', \underbrace{y}_{\text{dependent variable}}, \underbrace{x}_{\text{independent variable}}) = 0$

$\longrightarrow$ solution $y(x)$

Initial value problems (IVP)

- The solution $y(x)$ of a differential equation is not unique
  (c.f. integration constant)

- You need "initial conditions" $y(x_0) = y_0$ to find a unique solution $y(x)$
  ↳ you need as many initial conditions as the order o the ODE

Example:

$$\begin{cases} y' = \sin(x) & \longrightarrow \quad y(x) = -\cos(x) + C \\ y(0) = 1 & \longrightarrow \quad y(0) = -\cos(0) + C = 1 \implies C = 2 \end{cases}$$

$$y(x) = 2 - \cos(x)$$

$$\frac{d^2y}{dx^2} = 1 \longrightarrow \frac{dy}{dx} = x + C_1 \longrightarrow y(x) = \frac{1}{2}x^2 + C_1 x + C_2$$

↳ 2 initial values needed for a unique solution!

$$y'(0) = C_1, \quad y(0) = C_2$$

First order separable ODEs

A separable ODE has the form $\frac{dy}{dx} = f(x) \cdot g(y)$ (typically first order)

$\longrightarrow$ easily solved by $\int \frac{dy}{g(y)} = \int f(x)\,dx$

Example: $\frac{dy}{dx} = y \cdot x \Rightarrow \int \frac{dy}{y} = \int x \cdot dx \Rightarrow \ln|y| = \frac{1}{2}x^2 + C$

$$\Rightarrow y(x) = \pm e^{x^2/2} \cdot e^{C} = K \cdot e^{x^2/2} \quad (K \in \mathbb{R})$$

check: $y' = K \cdot \frac{\cancel{2}x}{\cancel{2}} e^{x^2/2} = x \cdot K e^{x^2/2} = x \cdot y$

Example 2: $y' = \frac{x}{y}$, $y(1) = 2$

$$\frac{dy}{dx} = \frac{x}{y} \Leftrightarrow \int y\,dy = \int x\,dx \Leftrightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$

$$y(x) = \pm\sqrt{x^2 + K} = +\sqrt{3 + x^2} \quad (K = 2C)$$

$$y(1) = \pm\sqrt{1 + K} = 2 \Rightarrow K = 3$$

Linear ODEs

- Linear: linear in $y(x)$ and all derivatives $y'(x), y''(x) \dots y^{(n)}(x)$

$$a_n(x) \cdot y^{(n)} + \dots + a_1(x) \cdot y' + a_0(x) \cdot y = f(x)$$

- Homogeneous: $f(x) = 0$

$\rightarrow y(x) = 0$ is a solution

Examples:

linear:

$$y'' - \sin(x)\, y' + 2x\, y = \cos(x)$$
$$y' = 0$$
$$y' = y\, x^2$$

nonlinear

$$y' = \sin(y)$$
$$y' = y^2 x$$
$$y'' = y \cdot y' + x$$

Linear homogeneous ODEs

Example: $\frac{d^2y}{dx^2} = -y(x)$

$y_1(x) = -\cos(x)$ $\qquad 2\sin(x)$

$y_2(x) = \sin(x)$

$-y_1 + y_2 = \sin(x) + \cos(x)$

$-y_1' + y_2' = +\cos(x) - \sin(x)$ $\quad \frac{d}{dx}$

$-y_1'' + y_2'' = -\sin(x) - \cos(x)$ $\quad \frac{d}{dx}$ $\quad = +y_1 - y_2$

* if $y_1(x)$ and $y_2(x)$ are solutions of a linear homogeneous ODE, then $a y_1(x) + b y_2(x)$ is a solution. $(a, b \in \mathbb{R})$

* number of independent solutions = order of ODE
* we can find all solutions by making linear combinations of independent solutions

Non-homogeneous linear ODEs

Form : $\frac{dy}{dx} + p(x)\, y = q(x)$

* assume a solution $y_H(x)$ of the homogeneous equation

$\longrightarrow \frac{dy_H}{dx} + p(x)\, y_H = 0$

↳ $C \cdot y_H$ is also a solution, since $\frac{d(C \cdot y_H)}{dx} + p(x)(C \cdot y_H) = 0$
$C \in \mathbb{R}$ *

* assume a solution $y_P(x)$ of the full ODE (particular solution)

$\longrightarrow \frac{dy_P}{dx} + p(x)\, y_P = q(x)$ **

(add up * and **) $\frac{d}{dx}(y_P + C \cdot y_H) + p(x)\,(y_P + C \cdot y_H) = q(x)$

$\Rightarrow y_P(x) + C y_H(x)$ is a solution of the full ODE !!

↳ in this way we can solve for all initial conditions,
(by adapting C) $\longrightarrow$ "general solution"

Example $y' + 2y = 3$

Homogeneous equation: $y' + 2y = 0$

$$\Rightarrow \frac{dy}{dx} = -2y \Leftrightarrow \int \frac{dy}{y} = \int -2\,dx \Rightarrow \ln|y| = -2x + C$$

$$\Rightarrow y_H(x) = K \cdot e^{-2x}$$

$y' + 2y = 3 \qquad y_p = \frac{3}{2}$

GENERAL SOLUTION $\quad y(x) = y_H(x) + y_p(x)$

$$= K \cdot e^{-2x} + \frac{3}{2}$$

Solving linear first order ODEs : parameter variation

$$y' + p(x)y = q(x)$$

1) Solve homogeneous equation $\frac{dy}{dx} + p(x)\,y(x) = 0$

* this is separable : $\int \frac{dy}{y} = -\int p(x)\,dx = -\mu(x) + C$

$$\Leftrightarrow \ln|y| = -\mu(x) + C$$

$$\Leftrightarrow y_H(x) = e^{-\mu(x)} \cdot e^{C} = k \cdot e^{-\mu(x)}$$

2) We make an assumption about the general solution :

→ we assume $y(x) = k(x)\,e^{-\mu(x)}$

↳ we change the constant in $y_H(x)$ into a function $k(x)$

→ we insert into the ODE

$$\left.\begin{aligned} y(x) &= k(x)\,e^{-\mu(x)} \\ y'(x) &= k'(x)\,e^{-\mu(x)} + k(x)\left(-\tfrac{d}{dx}\mu(x)\right)e^{-\mu(x)} \end{aligned}\right\} \text{product rule}$$

$$\rightarrow y'(x) = k'(x)\, e^{-\mu(x)} - p(x)\, k(x)\, e^{-\mu(x)} \quad (\text{def. } \mu(x))$$

insert in ODE:

$$y' + p(x)\, y = \underbrace{k'(x)\, e^{-\mu(x)} - \cancel{p(x)\, k(x)\, e^{-\mu(x)}}}_{y'} + \underbrace{\cancel{p(x)\, k(x)\, e^{-\mu(x)}}}_{p(x) \cdot y} = q(x)$$

$$\Rightarrow k'(x)\, e^{-\mu(x)} = q(x)$$

$$\Rightarrow k'(x) = q(x)\, e^{\mu(x)}$$

$$\Rightarrow k(x) = \int q(x)\, e^{\mu(x)}\, dx$$

$$\Rightarrow y(x) = k(x)\, e^{-\mu(x)} = e^{-\mu(x)} \int q(x)\, e^{\mu(x)}\, dx$$

→ this is the integrating factor formula ...

(do not learn by heart ... often leads to sign errors)

Example $\begin{cases} y' + \dfrac{y}{x} = 1 & (x > 0) \\ y(1) = 1 \end{cases}$

1) Homogeneous equation : $y' + \dfrac{y}{x} = 0 \iff \dfrac{dy}{dx} = -\dfrac{y}{x} \iff \int \dfrac{dy}{y} = -\int \dfrac{dx}{x}$

$\iff \ln|y| = -\ln|x| + c$

$\iff y_H(x) = e^{-\ln|x|} \cdot K = \dfrac{K}{e^{\ln(x)}} = \dfrac{K}{x}$

2) Parameter variation $y = \dfrac{K(x)}{x}$

$$\underbrace{y'(x) = \frac{K'(x)}{x} - \frac{K(x)}{x^2}}_{\text{product rule}} \underbrace{= 1 - \frac{y}{x}}_{\text{ODE}} = 1 - \frac{K(x)}{x^2}$$

$\Rightarrow \dfrac{K'(x)}{x} = 1 \Rightarrow K'(x) = x \Rightarrow K(x) = \dfrac{x^2}{2} + c$

$\Rightarrow y(x) = \dfrac{K(x)}{x} = \dfrac{1}{x}\left(\dfrac{x^2}{2} + c\right) = \dfrac{x}{2} + \dfrac{c}{x}$

3) Initial condition $y(1) = 1 = \dfrac{1}{2} + c \Rightarrow c = \dfrac{1}{2}$

$y(x) = \dfrac{x}{2} + \dfrac{1}{2x}$ end result.

Integral equations

$$y(x) = a + b\int_c^x F(y(t), t)\,dt$$

$$\begin{cases} y' = b\,F(y, x) & \text{ODE} \\ y(c) = a + \cancel{\int_c^c F(y, t)\,dt} = a \end{cases}$$

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Example: $y(x) = 3 + 2\int_1^x t\,y(t)\,dt$

$$\begin{cases} y' = 2xy \\ y(1) = 3 \end{cases} \longrightarrow \frac{dy}{dx} = 2xy \rightarrow \int \frac{dy}{y} = \int 2x\,dx$$

$$\ln|y| = x^2 + C$$

$$y(x) = e^{x^2+C} = K \cdot e^{x^2}$$

$$y(1) = K \cdot e = 3 \rightarrow K = \frac{3}{e}$$