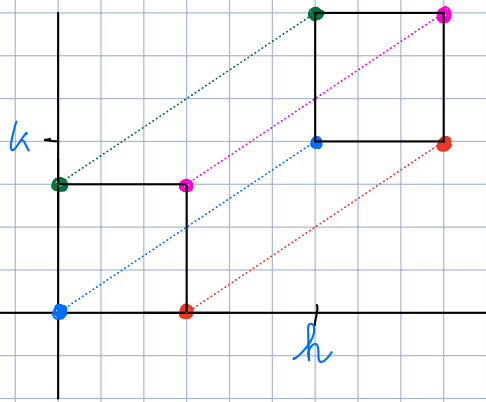


Lecture 5: Perspective projections + The Inverse of a Matrix (book: 2.2, 2.3, 2.7).

Previous lecture: Linear transformations, Matrix algebra

Applications to Computer Graphics.



This is a transformation, but not linear because $T(\underline{0}) \neq \underline{0}$.

We cannot find a 2×2 matrix A such that $A\underline{x} = T(\underline{x}) = \begin{bmatrix} x_1 + h \\ x_2 + k \end{bmatrix}$
 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \stackrel{\text{NOT possible}}{=} \begin{bmatrix} x_1 + h \\ x_2 + k \end{bmatrix}$

Homogeneous coordinates: work on a specific plane in \mathbb{R}^3 .
For example the plane with $x_3 = 1$.

(Note: a plane in \mathbb{R}^3 is not \mathbb{R}^2).

$$\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + h \\ x_2 + k \\ 1 \end{bmatrix}$$

LAB 2: Perspective projections

↳ project a 3-dimensional object on a 2-dimensional computer screen.

(not on the exam).

Recall the three row operations.

- * replacement
- * scaling
- * interchange.

$$A \sim A_2 \sim A_3 \sim A_4 \sim A_5.$$

Can we find an elementary matrix E such that $EA = A_5$.

Example: $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}_A \xrightarrow{\textcircled{1}} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}_{A_2} \xrightarrow{\textcircled{2}} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}_{A_3} \xrightarrow{\textcircled{3}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}_{A_4} \xrightarrow{\textcircled{4}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{A_5}$

- $\textcircled{1} R_1 \leftrightarrow R_2$
- $\textcircled{2} R_2: R_2 - 2 \cdot R_1$
- $\textcircled{3} R_2: R_2 * -1$
- $\textcircled{4} R_1: R_1 - 2 \cdot R_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\textcircled{1}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_1 \quad E_1 A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = A_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\textcircled{2}} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = E_2 \quad E_2 A_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = A_3$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\textcircled{3}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = E_3 \quad E_3 A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = A_4$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\textcircled{4}} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = E_4 \quad E_4 A_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A_5$$

$$\begin{aligned} \text{Hence, } A_5 &= E_4 A_4 = E_4 (E_3 A_3) = E_4 E_3 (E_2 A_2) = E_4 E_3 E_2 (E_1 A) \\ &= \underbrace{E_4 E_3 E_2 E_1}_E A = EA. \end{aligned}$$

$$\text{Check: } E = \dots = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \quad EA = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, E takes us to the reduced echelon form of A .

So, E is obtained by applying the same row operations on the identity matrix.

An $n \times n$ matrix A is invertible if there exists an $n \times n$ matrix C such that $CA = I_n$ and $AC = I_n$.

This C is the inverse of A .

The inverse of a matrix is unique.

Notation: A^{-1}

For a matrix A to be invertible:

* A must be square ($n \times n$)

* the RREF of A is the identity matrix: $A \sim \dots \sim I_n$ (Thm 7)
(A has a pivot position in every row/column; A has n pivot positions).

How to find the inverse of an invertible matrix A ?

We need to find E such that $EA = I_n$.

$$n[A; I_n] \sim E[A; I_n] = [EA; EI_n] = [I_n; \textcircled{E}] \text{ hey, there is } A^{-1} \text{ :D}$$

An algorithm for finding A^{-1} :

* Row reduce $[A; I_n]$

* If this leads to $[I_n; E]$, then A is invertible and $A^{-1} = E$.

* Otherwise, A is not invertible.

Example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ inverse of A ?

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2: R_2 - 2 \cdot R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad A \text{ is not invertible because we cannot reduce it to } I_2.$$

Example: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ inverse of A ?

$$\begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 1 & | & 0 & 1 \end{bmatrix} \xrightarrow{R_1: R_1 - 2R_2} \begin{bmatrix} 1 & 0 & | & 1 & -2 \\ 0 & 1 & | & 0 & 1 \end{bmatrix} \quad \text{So } A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

$$AA^{-1} = I_n \checkmark$$

Thm: The inverse of a matrix is unique.

Proof: Let A be an $n \times n$ matrix.

Suppose B and C are both an inverse of A and $B \neq C$.

$$B = BI_n = B(AC) = (BA)C = I_n C = C \quad \square$$

Thm: Let A be an $n \times n$ invertible matrix. Then, for each $\underline{b} \in \mathbb{R}^n$, the equation $A\underline{x} = \underline{b}$ has the unique solution $\underline{x} = A^{-1}\underline{b}$.

Proof:

* existence: A is invertible $\Rightarrow A$ has a pivot in every row $\Rightarrow A\underline{x} = \underline{b}$ is consistent for every $\underline{b} \in \mathbb{R}^n$.

* uniqueness: A is invertible $\Rightarrow A$ has a pivot in every col. \Rightarrow no free variables \Rightarrow solution must be unique.

* Show that $\underline{x} = A^{-1}\underline{b}$ is the unique solution:

$$A\underline{x} = \underline{b} \\ A^{-1}(A\underline{x}) = A^{-1}\underline{b}$$

$$A\underline{x} = A(A^{-1}\underline{b}) = (AA^{-1})\underline{b} = I_n \underline{b} = \underline{b} \checkmark$$

$$\begin{aligned}(A^{-1}A)\underline{x} &= A^{-1}\underline{b} \\ I_n \underline{x} &= A^{-1}\underline{b} \\ \underline{x} &= A^{-1}\underline{b} \quad \checkmark\end{aligned}$$

□

Thm: Let A be an $n \times n$ matrix.
If $CA = I_n$, then also $AC = I_n$.

Proof: Assume $CA = I_n$.

Claim/Lemma: For each $\underline{b} \in \mathbb{R}^n$, $A\underline{x} = \underline{b}$ has a solution.

Proof: Suppose $\exists \underline{b} \in \mathbb{R}^n$ st $A\underline{x} = \underline{b}$ has no solution.
Then, A does not have a pivot in every row.
So, A also does not have a pivot in every column.
So, there is a nontrivial solution $\underline{y}: A\underline{y} = \underline{0}$ with $\underline{y} \neq \underline{0}$.

Then, $\underline{y} = I_n \underline{y} = (CA)\underline{y} = C(A\underline{y}) = C\underline{0} = \underline{0}$ \downarrow
So, $A\underline{x} = \underline{b}$ has a solution for every $\underline{b} \in \mathbb{R}^n$.

Let $\underline{b} \in \mathbb{R}^n$. $A\underline{x} = \underline{b}$ has a sol.

$$\Rightarrow C(A\underline{x}) = C\underline{b}$$

$$\Rightarrow (CA)\underline{x} = C\underline{b}$$

$$\Rightarrow I_n \underline{x} = C\underline{b}$$

$$\Rightarrow \underline{x} = C\underline{b}$$

So, this is the sol of $A\underline{x} = \underline{b}$.

Hence, $\underline{b} = A\underline{x} = A(C\underline{b}) = (AC)\underline{b}$ So, we need $AC = I_n$ □

Properties of invertible matrices:

$$* (A^{-1})^{-1} = A$$

* If A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$ (verify yourself)

$$* (A^T)^{-1} = (A^{-1})^T$$

Summary (so far):

Let A be an $m \times n$ matrix with columns $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$.

$$m \geq n: \begin{bmatrix} A \end{bmatrix}$$

The following statements are equivalent:

- ① A has a pivot in every column
- ② A has n pivot positions
- ③ There are no free variables
- ④ $A\underline{x} = \underline{0}$ has only the trivial sol.
- ⑤ $\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\}$ is linearly indep.
- ⑥ $T: \underline{x} \mapsto A\underline{x}$ is one-to-one/injective

$$m \leq n: \begin{bmatrix} & A & \end{bmatrix}$$

The following statements are equivalent:

- Ⓐ A has a pivot in every row.
- Ⓑ A has m pivot positions.
- Ⓒ The echelon form of A does not contain a row of all zeros.
- Ⓓ $A\underline{x} = \underline{b}$ is consistent for every \underline{b} in \mathbb{R}^m .
- Ⓔ $\text{Span}\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\} = \mathbb{R}^m$.
- Ⓕ $T: \underline{x} \mapsto A\underline{x}$ is onto/surjective.

The Invertible Matrix Theorem:

If A is square ($n=m$), then statements ② and ⑥ are equivalent. Hence, the following statements are equivalent for square matrices

* ① - ⑥, Ⓐ - Ⓕ

* A is invertible

* There is a matrix C such that $CA = I_n$ and $AC = I_n$

* A is row equivalent to I_n .

* A^T is invertible.

* $\det A \neq 0$