Systems of linear equations

$$\begin{cases} 2x_1 + x_2 = 5 \\ 3x_1 - 2x_2 = 10 \end{cases} \longrightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}}_{\text{coef. matrix}} \quad \underbrace{\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 3 & -2 & 10 \end{array}\right]}_{\text{augmented matrix}}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 3 & -2 & 10 \end{array}\right] \xrightarrow{R_1 = 2R_1} \left[\begin{array}{cc|c} 2 & 2 & 10 \\ 3 & -2 & 10 \end{array}\right] \xrightarrow{R_1 = R_1 + R_2} \left[\begin{array}{cc|c} 5 & 0 & 20 \\ 3 & -2 & 10 \end{array}\right]$$

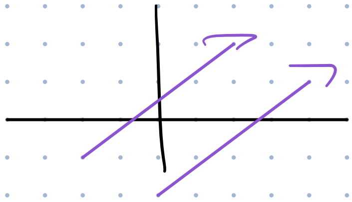
$$\xrightarrow{\frac{1}{5}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 3 & -2 & 10 \end{array}\right] \xrightarrow{R_2 = R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & -2 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 = -\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right] \qquad \begin{aligned} x_1 &= 4 \\ x_2 &= 1 \end{aligned}$$

Types of systems

Inconsistent

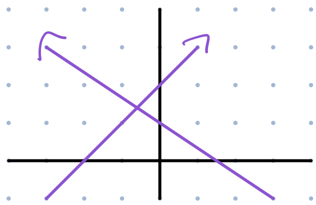
↳ NO SOLUTION

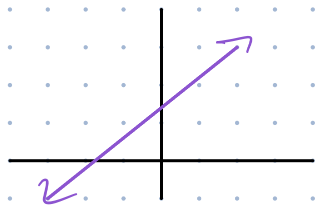


Consistent

↳ 1 SOLUTION

↳ INFINITE SOLUTIONS





Exercise

$$\left[\begin{array}{ccc|c} 1 & -2 & +1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & +5 & +9 & -9 \end{array}\right] \xrightarrow{R_3 = R_3 + 4R_1} \left[\begin{array}{ccc|c} 1 & -2 & +1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{R_2 = R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 28 & -12 \end{array}\right] \xrightarrow{R_1 = R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 11 & -2 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 28 & -12 \end{array}\right]$$

$$\xrightarrow{R_3 = \frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 11 & -2 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 7 & -3 \end{array}\right]$$

Existence & Uniqueness

- Consistent → solution exists
  - ↳ is it unique?

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array}\right] \longrightarrow \text{inconsistent system}$$

Row reduction algorithm

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array}\right] \xrightarrow[R_3 = R_3 - 3R_2]{R_2 = R_2 + 2R_1} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 8 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 11 \end{array}\right]$$

↑
inconsistent system

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

- Eigenvectors
- Orthogonality
- Bases
- Inverses
- Nul - Col
- Unit Vector
- Magnitude of matrix

$7 - 4(3) = 7 - 12 = -5$

$$1x_1 + 3x_2 + 0x_3 = -5$$
$$x_3 = 3$$

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

Computation of Ax

(1) $A = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(a) $A(\vec{u} + \vec{v}) = A \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \leftarrow x_1 \leftarrow x_2$

$= -1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} \quad 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \left[\begin{array}{cc|c} 2 & 1 & -1 \\ -4 & 2 & 1 \end{array}\right]$

$= \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

(b) $A\vec{u} + A\vec{v} = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} -12 \\ -4 \end{bmatrix} + \begin{bmatrix} 9 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

Homogeneous Systems

$Ax = 0$

$$\left[\begin{array}{ccc|c} 4 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 4 & 5 & 9 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 4 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 4 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 3/4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\searrow$ Consistent

$$\vec{x} = \begin{bmatrix} \frac{1}{4}x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1/4 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{cases} x_1 = \frac{1}{4}x_3 \\ x_2 = -2x_3 \end{cases} \quad x_3 \text{ is free}$$

Non-homogeneous systems