Practice Exam Questions Week 7, **Linear Algebra**

SOLUTIONS

1. Let $V = \begin{bmatrix} 2 & -4 & 1 \\ -3 & -1 & 1 \\ 1 & 5 & 1 \end{bmatrix}$. Show that the columns of V are orthogonal to each other.

$$\begin{bmatrix} 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ -1 \\ 5 \end{bmatrix} = -8+3+5=0.$$

$$\begin{bmatrix} 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2-3+1=0$$

$$\begin{bmatrix} -4 & -1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -4-1+5=0$$

Hence, the columns of V are orthogonal to each other.

2. Consider the following matrix A and vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

a. Verify that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A.

$$A\underline{v}_1 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot \underline{v}_1$$

So, $\underline{v}_1$ is an eigenvector of A with associated eigenvalue $\lambda_1 = 1$.

$$A\underline{v}_2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = 4 \cdot \underline{v}_2$$

So, $\underline{v}_2$ is an eigenvector of A with associated eigenvalue $\lambda_2 = 4$.

b. Orthogonally diagonalize the matrix A.

$$A - \lambda_1 I = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, a basis for the eigenspace is $\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$.

$\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -1 \neq 0$, so these two vectors are not orthogonal.

The projection of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ onto $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is

$$\frac{\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}.$$

and the component of $\begin{bmatrix}1\\1\\0\end{bmatrix}$ orthogonal to $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ is $\begin{bmatrix}1\\1\\0\end{bmatrix}-\begin{bmatrix}1/2\\0\\-1/2\end{bmatrix}=\begin{bmatrix}1/2\\1\\1/2\end{bmatrix}$.

Hence, $\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}1/2\\1\\1/2\end{bmatrix}\right\}$ is an orthogonal set in the eigenspace for $\lambda=1$.

Since the eigenspace is two-dimensional, the orthogonal set $\left\{\underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{\underline{v}_1}, \underbrace{\begin{bmatrix}1/2\\1\\1/2\end{bmatrix}}_{\underline{v}_2}\right\}$ is an orthogonal basis for the eigenspace.

Likewise,

$$A-\lambda_2 I=\begin{bmatrix}-2&-1&1\\-1&-2&-1\\1&-1&-2\end{bmatrix}\sim\cdots\sim\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix} \quad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_3\\-x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

Hence, a basis for the eigenspace is $\left\{\underbrace{\begin{bmatrix}1\\-1\\1\end{bmatrix}}_{\underline{v}_3}\right\}$.

The vectors $\underline{v}_1, \underline{v}_2$ and $\underline{v}_3$ may be normalized to get the vectors

$$\underline{u}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}, \quad \underline{u}_2=\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix}, \quad \underline{u}_3=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}.$$

Let $P=\begin{bmatrix}-1/\sqrt{2}&1/\sqrt{6}&1/\sqrt{3}\\0&2/\sqrt{6}&-1/\sqrt{3}\\1/\sqrt{2}&1/\sqrt{6}&1/\sqrt{3}\end{bmatrix}$ and $D=\begin{bmatrix}1&0&0\\0&1&0\\0&0&4\end{bmatrix}$.

Then, P orthogonally diagonalizes A, and $A=PDP^{-1}$.

3. True or false? If the given statement is true, give a brief explanation. If it is false, give a counterexample.

a. If U and V are 3×3 orthogonal matrices, then their product $W=UV$ is also a 3×3 orthogonal matrix.

True.

U and V are orthogonal matrices, so $U^TU=I$ and $V^TV=I$

So, $W^TW=(UV)^T(UV)=(V^TU^T)(UV)=V^T\underbrace{U^TU}_{=I}V=\underbrace{V^TV}_{I}=I.$

So, W is also an orthogonal matrix.

b. If the columns of a 3×3 matrix Q are orthogonal to each other, then $Q^TQ=I$.

False. Consider for example $Q=\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}$. The columns of Q are orthogonal to each other, but $Q^TQ=\begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}\neq I$.

c. Every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.

False. Consider for example $\underline{x} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\underline{y} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

$\underline{x}$ and $\underline{y}$ are linearly independent, but $\underline{x} \cdot \underline{y} = 28 - 12 = 16 \neq 0$, so $\{\underline{x}, \underline{y}\}$ is not an orthogonal set.

d. If $A^T = A$ and if vectors $\mathbf{u}$ and $\mathbf{v}$ satisfy $A\mathbf{u} = 3\mathbf{u}$ and $A\mathbf{v} = 4\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.

True.

If $\underline{u}$ and $\underline{v}$ are both nonzero vectors, then $\underline{u}$ and $\underline{v}$ are two eigenvectors from different eigenspaces. Moreover, since A is symmetric, it follows that $\underline{u}$ and $\underline{v}$ are orthogonal (Theorem 1 in Section 7.1).

If one of the vectors (or both) is the zero vector, then automatically $\underline{u} \cdot \underline{v} = 0$.

e. There are symmetric matrices that are not orthogonally diagonalizable.

False, because an $n \times n$ matrix is orthogonally diagonalizable if and only if A is symmetric.