

Lecture 10 : Partial Derivatives

(Adams 12.1-12.5)

- * functions of multiple variables
- * limits and continuity - brief overview
- * partial derivatives
 - ↳ definition
 - ↳ normal line and tangent plane
 - ↳ higher order derivatives
 - ↳ chain rule

1) Functions of multiple variables

$$f(x_1, x_2, \dots, x_n)$$

Domain : subset of \mathbb{R}^n

- Domain convention : unless specified, the domain are all real values where
 - ↳ the denominator (if any) is nonzero
 - ↳ everything in a n th root is positive (in general $x^{\frac{1}{b}}$, b even)
 - ↳ the argument of a logarithm \ln is positive (and nonzero)
- representation :
 - 3D-plots for $f(x,y)$
 $z = f(x,y)$ is a surface
 - level (contour) plots (e.g. height lines)
↳ the levels represent curves $f(x,y) = c$
 - density (colour) plots
(see mathematical / slides)
- Continuity : $f(x,y)$ is continuous at (a,b) if
 - every neighbourhood of (a,b) contains points in the domain (no isolated point)
 - $\forall \epsilon > 0 \exists \delta > 0 : \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - f(a,b)| < \epsilon$
we use the Euclidean distance between points in \mathbb{R}^2
 - ↳ very similar definition as for single-variable functions : as (x,y) approaches (a,b) , $f(x,y)$ approaches $f(a,b)$
 - ↳ all "typical" functions (\sin , \cos , e^x , \ln), fractions, polynomials, Γ) are continuous on their domain
 - ↳ a discontinuity is like a stair/cliff in a height profile.

$p(x,y)$ (pressure) \rightarrow weather maps

Examples : $T(x,y)$ (temperature (location))

$h(x,y)$ (height maps)

• limits : $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$

• if every neighbourhood of (a,b) contains points in the domain of f
 \hookrightarrow we can approach (a,b)

• $\forall \epsilon > 0 \exists \delta > 0 : \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \epsilon$

• if f is defined at (a,b) , then $L = f(a,b)$

\hookrightarrow typically, we take limits towards the boundaries of the domain.

• in 1D, a limit does not exist when right and left limit differ.

eg. $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$, $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$, $\lim_{x \rightarrow 0} \frac{|x|}{x}$ DNE

$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$, $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$, $\lim_{x \rightarrow 0} \frac{1}{x}$ DNE

in 2D, a limit does not exist when it differs when approaching (a,b) in different directions

\hookrightarrow on a level plot, height lines near to intersect

\hookrightarrow very well visible in plots \rightarrow mathematical

(\hookrightarrow the calculations are optional for you)

• partial derivatives

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \rightarrow 0} \frac{f(x+h,y) - f(x,y)}{h} = f_1(x,y) = f_x(x,y) = D_x f(x,y) = D_1 f$$

\hookrightarrow the partial derivative with respect to x indicates how fast $f(x,y)$ changes, when varying x

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h \rightarrow 0} \frac{f(x,y+h) - f(x,y)}{h} = f_2(x,y) = f_y(x,y) = D_y f(x,y) = D_2 f$$

$$\left. \frac{\partial f}{\partial x} \right|_a^b = f_x(a,b) \quad (\text{notation in book})$$

Examples: $\frac{\partial}{\partial x}(xy) = y$, $\frac{\partial}{\partial y}(xy) = x$, $\frac{\partial}{\partial x}(x+y) = 1 = \frac{\partial}{\partial y}(x+y)$

$\frac{\partial}{\partial x}(x^2+y^2) = 2x$, $\frac{\partial}{\partial y}(x^2+y^2) = 2y$

• $\frac{\partial}{\partial x} \sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}}$ (Socratic)

• $\frac{\partial}{\partial x} \ln\left(\frac{x}{y}\right) = \frac{1}{x} \cdot \frac{1}{y} = \frac{1}{x}$, $\frac{\partial}{\partial y} \ln\left(\frac{x}{y}\right) = \frac{1}{x} \cdot \frac{-x}{y^2} = -\frac{1}{y}$

\hookrightarrow when you take a partial derivative with respect to x , you treat y as a constant (and vice versa)

- tangent plane and normal line
(2D)

↳ 1D: derivative = slope of tangent line to a curve

2D: partial derivatives = slopes of tangent plane to a surface.

for $z_0 = f(x_0, y_0)$, the tangent plane is given by

$$z = z_0 + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

↳ the normal line is the line perpendicular to the tangent plane

$$z - z_0 = - \frac{x - x_0}{\frac{\partial f}{\partial x}} = - \frac{y - y_0}{\frac{\partial f}{\partial y}} \quad (\text{note the similarity to 1D})$$

Example (Ex. 6)

for $f(x, y) = \sin(xy)$, $x_0 = \frac{\pi}{3}$, $y_0 = -1 \rightarrow z_0 = f(x_0, y_0) = \sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$

• the tangent plane is given by: $z + \frac{\sqrt{3}}{2} = -\frac{1}{2}(x - \frac{\pi}{3}) + \frac{\pi}{6}(y + 1)$

$$\frac{\partial f}{\partial x} = y \cos(xy), \quad \frac{\partial f}{\partial y} = x \cos(xy) \quad \text{OR} \quad z = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) - \frac{x}{2} + \frac{\pi}{6}y$$

↳ at $(\frac{\pi}{3}, -1)$, $\frac{\partial f}{\partial x}(\frac{\pi}{3}, -1) = -1 \cdot \cos(-\frac{\pi}{3}) = -\frac{1}{2}$

$$\frac{\partial f}{\partial y}(\frac{\pi}{3}, -1) = \frac{\pi}{3} \cos(-\frac{\pi}{3}) = \frac{\pi}{6}$$

• the normal line is given by: $z + \frac{\sqrt{3}}{2} = - \frac{x - \frac{\pi}{3}}{\frac{1}{2}} = - \frac{y + 1}{\frac{\pi}{6}}$

OR $z + \frac{\sqrt{3}}{2} = 2(\frac{\pi}{3} - x) = -\frac{6}{\pi}(y + 1)$

or $z = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) - 2x = -\left(\frac{6}{\pi} + \frac{\sqrt{3}}{2}\right) - \frac{6}{\pi}y$

→ note: if $\frac{\partial f}{\partial x} = 0$, the tangent plane is parallel to the x-axis
the normal line has a fixed x-component $x = x_0$

if $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, the tangent plane is horizontal $z = z_0$
the normal line is parallel to the z-axis, $x = x_0, y = y_0$

↳ example: normal/tangent at $(0,0)$ for $f(x, y) = \frac{1}{1+x^2+y^2}$

- Higher order derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \frac{\partial f}{\partial x} = f_{xx} = f_{11}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} = f_{yx} = f_{21}$$

↳ if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Examples $\frac{\partial^2}{\partial x^2} (e^{xy}) = \frac{\partial}{\partial x} (y e^{xy}) = y^2 e^{xy}$ (Socratic)

$$\frac{\partial^2}{\partial x \partial y} (e^{xy}) = \frac{\partial}{\partial x} (x e^{xy}) = e^{xy} + xy e^{xy}$$

$$\frac{\partial^2}{\partial x \partial y} (\sqrt{x^2 + y^2}) = \frac{\partial}{\partial x} \frac{y}{\sqrt{x^2 + y^2}} = y \frac{\partial}{\partial x} \frac{1}{\sqrt{x^2 + y^2}} = y \left(-\frac{1}{2} \frac{2x}{(x^2 + y^2)^{3/2}} \right) = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

• chain rule

1D: $\frac{d}{dx} (f(g(x))) = f'(g(x)) g'(x)$

↳ in 2D: for $z = f(x, y)$, and x, y depend on t

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

(example $T(x, y)$ (temperature depends on position)
 $x(t), y(t)$, position depends on time)

→ if x and y depend on 2 variables $x(s, t), y(s, t)$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

Example: $z = \frac{1}{x^2 + y^2}$, $x = r \cos(\varphi)$, $y = r \sin(\varphi)$

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = \frac{-2x}{(x^2 + y^2)^2} \cos \varphi + \frac{-2y}{(x^2 + y^2)^2} \sin \varphi \\ &= -2 \frac{r \cos^2 \varphi + r \sin^2 \varphi}{(x^2 + y^2)^2} = -\frac{2r}{r^4} = -\frac{2}{r^3} \end{aligned}$$

(same result.)

↳ $z = \frac{1}{x^2 + y^2} = \frac{1}{r^2} \rightarrow \frac{dz}{dr} = -\frac{2}{r^3}$

Example 2: any function $f(x-vt)$ satisfies the PDE

$$\frac{\partial^2 f}{\partial x^2} = v^2 \frac{\partial^2 f}{\partial t^2} \quad (\text{wave equation})$$

↳ $\frac{\partial f}{\partial x} = \frac{df}{d(x-vt)} \frac{d(x-vt)}{dx} = f'(x-vt)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} (f'(x-vt)) = \frac{df'}{d(x-vt)} \frac{\partial(x-vt)}{\partial x} = f''(x-vt)$$

↳ $\frac{\partial f}{\partial t} = \frac{df}{d(x-vt)} \frac{\partial(x-vt)}{\partial t} = -v \cdot f'(x-vt)$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t} (-v \cdot f'(x-vt)) = -v \cdot \frac{df'}{d(x-vt)} \frac{\partial(x-vt)}{\partial t} = v^2 f''(x-vt) = v^2 \frac{\partial^2 f}{\partial x^2}$$