Solutions

Exercise 1: We argue as follows:

1. If statement 2 is true then it is the first true statement and statement 1 must be false. However, if statement 2 is false then it is not the first false statement and statement 1 must be false. Either way, statement 1 must be false.
2. Since statement 1 is false, statement 9 and statement 10 must both be false.
3. If statement 6 were false, it would be the last true statement, which is obviously a contradiction; therefore statement 6 is true, which implies that at least one of statement 7 and statement 8 is true.
4. If statement 7 is false, then statement 8 is true, which implies that statement 5 is false, but also that statement 3 is false (if statement 3 is true, then there can not be 3 consecutive false statements, which contradicts statement 3 being true). But this means that there are no three consecutive true statements, contradicting the fact that statement 10 is false. So statement 7 is true.
5. Since statement 6 and statement 7 are true (implying that $N \geq 42$) and statement 1, statement 9 and statement 10 are false, we can conclude that statement 5 is false, since statement 5 true would imply that $N \leq 35$.
6. Also statement 8 is false, since if statement 8 would be true, then statement 7 says that N is divisible by 6, 7 and 8, making $N \geq 168$, contradicting that statement 8 is true.
7. From statement 8, statement 9 and statement 10 false we conclude that statement 3 is true and even better, from statement 10 false we conclude that also statement 2 and statement 4 are true.

Conclusion: Statements 2, 3, 4, 6 and 7 are true, and statements 1, 5, 8, 9 and 10 are false. Now to find N:

1. Since statement 4 is true, N must be divisible by $7 - 2 = 5$.
2. Since statement 7 is true, N must be divisible by 2, 3, 4, 6 and 7.
3. The smallest number satisfying properties 1 and 2 is 420.
4. It's easy to check that 420 has 22 divisors not including 1 and itself, and $2+3+4+6+7 = 22$, so indeed statement 9 is false.

So $N = 420$.

Exercise 2: Show that the proposition $\neg p \vee q$ is equivalent to the proposition $p \Rightarrow q$.

We do so by means of a truth table:

p	q	$\neg p$	$\neg p \vee q$	$p \Rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the truth values of both propositions are always the same, they are equivalent.

Exercise 3: Give the truth table for the propositions $p \Leftrightarrow (q \vee p)$, $p \Leftrightarrow (q \Rightarrow p)$ and $p \Leftrightarrow (q \Rightarrow r)$.

p	q	r	$q \vee p$	$p \Leftrightarrow (q \vee p)$	$q \Rightarrow p$	$p \Leftrightarrow (q \Rightarrow p)$	$q \Rightarrow r$	$p \Leftrightarrow (q \Rightarrow r)$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	F	F
T	F	T	T	T	T	T	T	T
T	F	F	T	T	T	T	T	T
F	T	T	T	F	F	T	T	F
F	T	F	T	F	F	T	F	T
F	F	T	F	T	T	F	T	F
F	F	F	F	T	T	F	T	F

Give the truth table for $S = [((\neg(p \wedge \neg q) \wedge r) \Rightarrow p) \Leftrightarrow (q \wedge \neg r)]$. For notational purposes we will call the proposition $((\neg(p \wedge \neg q) \wedge r) \Rightarrow p)$ t, whereas $(q \wedge \neg r)$ will be called u.

p	q	r	$p \wedge \neg q$	$\neg(p \wedge \neg q)$	$\neg(p \wedge \neg q) \wedge r$	t	u	$t \Leftrightarrow u$
T	T	T	F	T	T	T	F	F
T	T	F	F	T	F	T	T	T
T	F	T	T	F	F	T	F	F
T	F	F	T	F	F	T	F	F
F	T	T	F	T	T	F	F	T
F	T	F	F	T	F	T	T	T
F	F	T	F	T	T	F	F	T
F	F	F	F	T	F	T	F	F

Exercise 4: True or false: $1 + 1 = 3 \;\Rightarrow\; 2 + 2 = 5$

Solution: This is a conditional proposition that states that "$2 + 2 = 5$" must be true IF "$1 + 1 = 3$" is true. Since "$1 + 1 = 3$" is false, there is no condition on "$2 + 2 = 5$"; it does not matter if it is true or false. The proposition is therefore true.

Exercise 5: You have been selected to serve on jury for a criminal case. The attorney for the defence argues as follows: "If my client is guilty, then the knife was in the drawer. Either the knife was not in the drawer or Jason Pritchard saw the knife. If the knife was not there on 10 October, it follows that Jason Pritchard did not see the knife. Furthermore, if the knife was there on 10 October, then teh knife was in the drawer and also the hammer was in the barn. But we all know that the hammer was not in the barn. Therefore, ladies and gentlemen of the jury, my client is innocent."

Question: Is the attorney's argument sound? How should you vote?

Solution: The attorney is right; Jason Pritchard is innocent.

Exercise 6: Negate the following propositions:

- $\forall x \in \mathbb{R} : x^2 \leq \frac{1}{\pi}$. The negation is $\exists x \in \mathbb{R} : x^2 > \frac{1}{\pi}$.

- $\exists x \in \mathbb{Z}\ \forall y \in \mathbb{Z} : 3x + y \leq 4$. The negation is $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z} : 3x + y > 4$

Exercise 7: Prove the following propositions:

1. $\exists x \in \mathbb{N} : x = 3$.

   Proof: Take $x = 3\,(\in \mathbb{N})$. Then $x = 3$.

2. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} : x + y \geq 4$.

   Proof: Let $x \in \mathbb{R}$ and take $y = -x + 37$. Then

$$\begin{aligned} x + y &= x + (-x + 37) \\ &= 37 \\ &\geq 4. \end{aligned}$$

3. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} : y - x = -1$.

   Proof: Let $x \in \mathbb{R}$ and take $y = x - 1\,(\in \mathbb{R})$. Then $y - x = -1$.

4. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \exists z \in \mathbb{R} : x + y = 2z$

   Proof: Let $x \in \mathbb{R}$, let $y \in \mathbb{R}$ and take $z = \frac{x+y}{2}(\in \mathbb{R})$. Then

$$\begin{aligned} x + y &= 2 \cdot \tfrac{x+y}{2} \\ &= 2z. \end{aligned}$$

5. $\forall x \in \mathbb{N}\ \forall y \in \mathbb{N} : (x < y \Rightarrow 2x < 2y)$

   Proof: Let $x \in \mathbb{R}$, let $y \in \mathbb{R}$ and assume that $x < y$. Then

$$\begin{aligned} 2x &= x + x \\ &< x + y \ \ \text{(since } x < y\text{)} \\ &< y + y \ \ \text{(since } x < y\text{)} \\ &= 2y \end{aligned}$$

6. $\forall x \in \mathbb{R} : (x \geq 10 \Rightarrow x^5 \geq 3x^4 + 5x^2 + 2333)$

   Proof: Let $x \geq 10$. Then

$$\begin{aligned} x^5 &= x \cdot x^4 \\ &\geq 10x^4 \ \ \text{(since } x \geq 10\text{)} \\ &= 3x^4 + 7x^4 \\ &\geq 3x^4 + 700x^2 \ \ \text{(since } x \geq 10\text{)} \\ &= 3x^4 + 5x^2 + 695x^2 \\ &\geq 3x^4 + 5x^2 + 69500 \ \ \text{(since } x \geq 10\text{)} \\ &> 3x^4 + 5x^2 + 2333. \end{aligned}$$

Exercise 8:

1. $\forall n \in \mathbb{N} : n \text{ odd} \quad \Rightarrow \quad n^2 - 1$ is divisible by 8.

   Proof. Let n be an odd number. Then $n = 2k + 1$ for some integer k. But then

   $$\begin{aligned} n^2 - 1 = (n-1)(n+1) &= (2k+1-1)(2k+1+1) \\ &= 2k \cdot (2k+2) \\ &= 4k \cdot (k+1) \end{aligned}$$

   Now, either k or $k+1$ is even (and the other one is odd). This means that $k \cdot (k+1)$ is even and consequently that $n^2 - 1$ is divisible by 8. ∎

2. $\forall n \in \mathbb{N} : (n \text{ odd and } n \text{ is not divisible by } 3) \quad \Rightarrow \quad n^2 - 1$ is divisible by 6.

   Proof. Let n be an odd number that is not divisible by 3. Then $n = 2k + 1$ for some integer k (since n is odd). But then

   $$\begin{aligned} n^2 - 1 = (n-1)(n+1) &= (2k+1-1)(2k+1+1) \\ &= 2k \cdot (2k+2) \\ &= 4k \cdot (k+1) \end{aligned}$$

   For a number to be divisible by 6, it has to be divisible by 2 and by 3. Clearly $4k \cdot (k+1)$ is divisible by 2 (we actually already know that it is divisbile by 8 from the previous exercise). Remains to show that $k \cdot (k+1)$ is divisible by 3 (the factor 4 doesn't do anything with respect to divisibility by 3). Now we use the fact that n is not divisible by 3. Since $n = 2k + 1$ this means that $2k + 1$ is not divisible by 3. But then also $4k + 2$, which is $2 \cdot (2k + 1)$, is not divisible by 3. But then $4k + 2 - 3k = k + 2$ is not divisible by 3 either. But then either k or $k + 1$ must be divisible by 3, so $k \cdot (k + 1)$ is divisible by 3. Hence n is divisible by 6 (actually n is divisible by 24). ∎

3. $\forall n \in \mathbb{N} : n^5 - n$ is divisible by 30.

   Proof. For a number to be divisible by 30, it must be divisible by 2, 3 and 5. Let $n \in \mathbb{N}$. Then

$$\begin{aligned} n^5 - n = n(n^4 - 1) &= n(n^2 - 1)(n^2 + 1) \\ &= n(n - 1)(n + 1)(n^2 + 1) \end{aligned}$$

   Apparently $n^5 - n$ contains factors n, $n - 1$ and $n + 1$. One of these (consecutive) numbers must be divisible by 3 and at least one of them is divisible by 2. Remains to show that $n(n-1)(n+1)(n^2+1)$ is divisible by 5. If one of the factors n, $n-1$ and $n+1$ is divisible by 5, then we are done, so we assume that that is not the case (so none of the three numbers n, $n - 1$ and $n + 1$ are divisible by 5). Then either $n = 5k + 2$ or $n = 5k + 3$ for some integer k. We need to show that in either case $n^2 + 1$ is divisible by 5.

   Case 1: $n = 5k + 2$. Now $n^2 + 1 = (5k + 2)^2 + 1 = 25k^2 + 20k + 5 = 5(5k^2 + 4k + 1)$ is divisible by 5;

   Case 2: $n = 5k + 3$. Now $n^2 + 1 = (5k + 3)^2 + 1 = 25k^2 + 30k + 10 = 5(5k^2 + 6k + 2)$ is also divisible by 5.

   Conclusion: $n^5 - n$ is divisible by 2, 3 and 5, so it is divisible by 30. ■

4. $\forall n \in \mathbb{N} : n^7 - n$ is divisible by 42.

   Proof. For a number to be divisible by 42, it must be divisible by 2, 3 and 7. Let $n \in \mathbb{N}$. Then

$$\begin{aligned} n^7 - n = n(n^6 - 1) &= n(n^3 - 1)(n^3 + 1) \\ &= n(n - 1)(n + 1)(n^2 - n + 1)(n^2 + n + 1) \end{aligned}$$

   Apparently $n^7 - n$ contains factors n, $n - 1$ and $n + 1$. One of these (consecutive) numbers must be divisible by 3 and at least one of them is divisible by 2. Remains to show that $n(n - 1)(n + 1)(n^2 - n + 1)(n^2 + n + 1)$ is divisible by 7. If one of the factors n, $n - 1$ and $n + 1$ is divisible by 7, then we are done, so we assume that that is not the case (so none of the three numbers n, $n - 1$ and $n + 1$ are divisible by 7). Then we have that $n = 7k + 2$, $n = 7k + 3$, $n = 7k + 4$ or $n = 7k + 5$ for some integer k. We need to show that in either case one of the factors $(n^2 - n + 1)$ and $(n^2 + n + 1)$ is divisible by 7.

   Case 1: $n = 7k + 2$. Now $n^2 + n + 1 = (7k + 2)^2 + (7k + 2) + 1 = 49k^2 + 35k + 7 = 7(7k^2 + 5k + 1)$, which is divisible by 7;

   Case 2: $n = 7k + 3$. Now $n^2 - n + 1 = (7k + 3)^2 - (7k + 3) + 1 = 49k^2 + 35k + 7 = 7(7k^2 + 5k + 1)$, which is divisible by 7;

   Case 3: $n = 7k + 4$. Now $n^2 + n + 1 = (7k + 4)^2 + (7k + 4) + 1 = 49k^2 + 63k + 21 = 7(7k^2 + 9k + 3)$, which is divisible by 7;

   Case 4: $n = 7k + 5$. Now $n^2 - n + 1 = (7k + 5)^2 - (7k + 5) + 1 = 49k^2 + 63k + 21 = 7(7k^2 + 9k + 3)$, which is divisible by 7.

   Conclusion: We found that in all cases one of the factors $(n^2 - n + 1)$ and $(n^2 + n + 1)$ is divisible by 7. This means that for all n, $n^7 - n$ is divisible by 2, 3 and 7, so it is divisible by 42. ■

Exercise 9: Prove the following propositions using a contrapositive proof:

1 $\forall x \in \mathbb{R}: \ x^2 - x - 6 \neq 0 \Rightarrow x \notin \{-2, 3\}$

The contrapositive of this proposition is $\forall x \in \mathbb{R} : x \in \{-2, 3\} \Rightarrow x^2 - x - 6 = 0$.

Proof: Let $x \in \{-2, 3\}$. Then either $x^2 - x - 6 = (-2)^2 - (-2) - 6$ or $x^2 - x - 6 = 3^2 - 3 - 6$. In either case it is equal to 0.

2 For all nonnegative real numbers x and y we have: $3x + 4y < 12 \Rightarrow (x < 4) \wedge (y < 3)$

The contrapositive of this proposition is: $\forall x \geq 0 \ \forall y \geq 0 : (x \geq 4) \vee (y \geq 3) \Rightarrow 3x + 4y \geq 12$

Proof: Let $x \geq 0$ and $y \geq 0$ and assume that $(x \geq 4) \vee (y \geq 3)$. We distinguish between two cases: (i) If $x \geq 4$, then $3x + 4y \geq 3 \cdot 4 + 4 \cdot 0 = 12$. (ii) If $y \geq 3$, then we have $3x + 4y \geq 3 \cdot 0 + 4 \cdot 3 = 12$. In either case $3x + 4y \geq 12$.

3 $\forall x > 0 \ \forall y > 0 : x^2 + y^2 > 1 \Rightarrow x + y > 1$

The contrapositive of this proposition is: $\forall x > 0 \ \forall y > 0 : x + y \leq 1 \Rightarrow x^2 + y^2 \leq 1$

Proof: Let $x > 0$ and $y > 0$ and assume that $x + y \leq 1$. Then $x \leq 1$ and $y \leq 1$ and hence $x^2 = x \cdot x \leq 1 \cdot x = x$ and similarly $y^2 \leq y$. But then $x^2 + y^2 \leq x + y \leq 1$.

Exercise 10: Prove that $\forall a \in \mathbb{R} \ \forall b \in \mathbb{R} : [(\forall \varepsilon > 0 : b < a + \varepsilon) \Rightarrow b \leq a]$.
We use a contrapositive proof. The contrapositive of the above proposition is:

$$\forall a \in \mathbb{R} \ \forall b \in \mathbb{R} : [b > a \Rightarrow (\exists \varepsilon > 0 : b \geq a + \varepsilon)].$$

Proof: Let $a \in \mathbb{R}$ and let $b \in \mathbb{R}$ and assume that $b > a$. Then $b - a > 0$. Take $\varepsilon = \frac{1}{2}(b - a) > 0$. Then

$$\begin{aligned} a + \varepsilon &= a + \tfrac{1}{2}(b - a) \\ &= \tfrac{1}{2}b + \tfrac{1}{2}a \\ &\leq \tfrac{1}{2}b + \tfrac{1}{2}b \text{ (since } b > a) \\ &= b \end{aligned}$$

which completes the proof.

Exercise 11: Prove that for all real numbers x we have: $\left(x^2 + 5x - 6 = 0\right) \Leftrightarrow (x = -6 \vee x = 1)$.
We have to prove 2 implications here. We denote them by "$\Rightarrow$" and "$\Leftarrow$".
Proof: Let $x \in \mathbb{R}$.
"$\Rightarrow$" Assume that $x^2 + 5x - 6 = 0$. Since $x^2 + 5x - 6 = (x + 6)(x - 1)$ it follows that $(x + 6)(x - 1) = 0$. But then either $x + 6 = 0$ or $x - 1 = 0$, so $x = -6 \ \vee x = 1$.
"$\Leftarrow$" Assume that $x = -6 \ \vee x = 1$. Then either $x^2 + 5x - 6 = (-6)^2 + 5 \cdot (-6) - 6$ or $x^2 + 5x - 6 = 1^2 + 5 \cdot 1 - 6$. In either case $x^2 + 5x - 6 = 0$.

Exercise 12: $m \cdot n$ is odd $\Leftrightarrow$ m and n are both odd.

Proof: "⇒" Let m and n be odd numbers. Then there exist integers p and q such that $m = 2p + 1$ and $n = 2q + 1$. But then

$$\begin{aligned} m \cdot n &= (2p+1) \cdot (2q+1) \\ &= 4pq + 2p + 2q + 1 \\ &= 2(2pq + p + q) + 1, \end{aligned}$$

which is odd.

"⇐" This part of the proposition we prove contrapositively. In this case that means that we prove that if m and n are not both odd (so at least one of them is even), then $m \cdot n$ is even. So, let m and n be two integers that are not both odd. Assume without loss of generalization that m is even. Then $m = 2p$ for some integer p and $m \cdot n = 2pn$, which is even.

Exercise 13: Prove that there is no biggest number smaller than 1.

Proof: We prove this proposition by contradiction. Suppose that there is a biggest number smaller than 1. Let's call this number x. Now consider the number $y = \frac{x+1}{2}$. What can we tell about y? Firstly, as $x < 1$ we have that $y > \frac{x+x}{2} = \frac{2x}{2} = x$ and secondly we have that $y < \frac{1+1}{2} = 1$, so $x < y < 1$, which contradicts the assumption that x was the biggest number smaller than 1. Hence our assumption was wrong and there exists no biggest number smaller than 1.

Exercise 14: *(exercises 1.5.2 (a) and (b)):* Disprove the following propositions by means of a counterexample:

- For all prime numbers p and q we have: If $n = p^2 + q^2$, then n is prime.

  Proof: (Counterexample) Take $p = 3$, $q = 5$. Then p and q are primes, but $n = 3^2 + 5^2 = 34$ is no prime. (Obviously there are many more counterexamples for this exercise.)

- $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R} : \quad a > b \ \Leftrightarrow a^2 > b^2.$

  Proof: (Counterexample) Take $a = 0$ and $b = -1$. Then $a > b$, but $a^2 = 0 \not> 1 = b^2$.

Exercise 15: *(several exercises from section 1.5):* Prove or disprove:

1. $\sqrt{5}$ is a rational number. FALSE

   Proof: We disprove this proposition by means of contradiction. Suppose that $\sqrt{5}$ is a rational number. Then $\sqrt{5} = \frac{m}{n}$ for 2 natural numbers m and n and we take m and n in such a way that they do not contain any common factors. Squaring this equation yields $5 = \frac{m^2}{n^2}$, or $m^2 = 5n^2$. So m^2 is divisible by 5. But as m^2 is the square of a natural number, if we write m^2 as a product of prime factors, each prime will be in there an even number of times. But that means that the factor 5 must be in there 2, 4, 6, 8 or some other positive even number of times. But then m is also divisible by 5, say $m = 5p$ for some natural number p. Then we have $(5p)^2 = 5n^2$, or $25p^2 = 5n^2$, or $n^2 = 5p^2$. But this means that n^2 is divisible by 5 and hence, by the same argumentation as above, that n is divisible by 5. So apparently m and n are both divisible by 5, but that contradicts the fact that m and n have no common factors. We arrived at a logical contradiction without drawing any incorrect conclusions, so apparently out assumption was false and $\sqrt{5}$ is not a rational number.

2. $\forall x \in \mathbb{R} : x^4 = 1 \Rightarrow x = 1$ FALSE

   Proof: (Counterexample) Take $x = -1$. Then $x^4 = (-1)^4 = 1$, but $x \neq 1$.

3. $\forall x \in \mathbb{Z}\ \forall y \in \mathbb{Z} : x + y$ is even $\Leftrightarrow$ x and y are both even or x and y are both odd TRUE

   Proof: Let x and y be odd numbers. Then $x = 2k + 1$ for some integer k and $y = 2l + 1$ for some integer l. But then $x + y = (2k + 1) + (2l + 1) = 2(k + l + 1)$ is an even number since $k + l + 1$ is an integer.

4. $\forall x \in \mathbb{R} : x^2 - 4 < 0 \Rightarrow -2 < x < 2$ TRUE

   Proof: Let x be a real number such that $x^2 - 4 < 0$. Since $x^2 - 4 = (x - 2)(x + 2)$ it follows that $(x-2)(x+2) < 0$. But a product of 2 numbers is only negative if one of the factors is negative and the other is positive, so we need that either $x - 2 > 0 \wedge x + 2 < 0$ or $x - 2 < 0 \wedge x + 2 > 0$. The former is obviously impossible, whereas the latter implies that $-2 < x < 2$.

   Alternative proof: (Contrapositive) Let x be a real number such that $x \geq 2$ or $x \leq -2$. Then $x^2 \geq 4$ and hence $x^2 - 4 \geq 0$.

5. $\forall x \in \mathbb{N}\ \forall y \in \mathbb{N}\ \exists z \in \mathbb{N} : x + y = 2z$ FALSE

   Proof: (Counterexample) Take $x = 37,\ y = 38\ (\in \mathbb{N})$. Then $\frac{x+y}{2} = 37\frac{1}{2} \neq z\ \ \forall z \in \mathbb{N}$.

6. $\forall y \in \mathbb{R}\ \exists x \in \mathbb{R} : x^2 = y$ FALSE

   Proof: (Counterexample) Take $y = -1$. Then for all $x \in \mathbb{R}$ we have $x^2 \geq 0 \neq y$.

7. $\forall x \in \mathbb{N}\ \forall y \in \mathbb{N} :\ (x < y\ \Rightarrow\ x^2 < y^2)$ TRUE

   Proof: Let $x, y \in \mathbb{N}$ and assume that Let $x < y$. Then

$$\begin{aligned} x^2 &= x \cdot x \\ &< x \cdot y \\ &< y \cdot y \\ &= y^2. \end{aligned}$$

8. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R} :\ (x < y\ \Rightarrow\ x^2 < y^2)$ FALSE

   Proof: (Counterexample) Take $x = -1\ (\in \mathbb{R})$ and $y = 0\ (\in \mathbb{R})$. Then $x < y$, but $x^2 \not< y^2$.

9. $\forall x \in \mathbb{N}\ \forall y \in \mathbb{N}\ \forall z \in \mathbb{N}$: $((x < y) \wedge (y < z)\ \Rightarrow\ x \cdot y < y \cdot z)$ TRUE

   Proof: Let $x, y, z \in \mathbb{N}$ and assume that $(x < y)\ \wedge\ (y < z)$. Then

$$\begin{aligned} x \cdot y &< y \cdot y \ \ \text{(since } x < y\text{)} \\ &< y \cdot z \ \ \text{(since } y < z\text{)}. \end{aligned}$$

10. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ \forall z \in \mathbb{R}$: $((x < y) \wedge (y < z)\ \Rightarrow\ xy < yz)$ FALSE

    Proof: (Counterexample) Take $x = -3,\ y = -2,\ z = -1\ \ (\in \mathbb{R})$. Then $(x < y)\ \wedge\ (y < z)$, but $xy \not< yz$.

11. $\exists x \in \mathbb{N}\, \forall y \in \mathbb{N} : |x| \le \frac{1}{2}|y|$ FALSE

    Proof: (Counterexample) Let $x \in \mathbb{N}$ and take $y = 1$. Then $|x| \ge 1 > \frac{1}{2} = \frac{1}{2}|y|$.

12. $\exists x \in \mathbb{R}\, \forall y \in \mathbb{R} : |x| \le \frac{1}{2}|y|$ TRUE

    Proof: Take $x = 0$. Then $\forall y \in \mathbb{R} : \frac{1}{2}|y| \ge \frac{1}{2} \cdot 0 = 0 = |x|$.

13. $\forall x \in \mathbb{N}\, \forall y \in \mathbb{N} : (x^2 + y^2 = 1 \quad\Rightarrow\quad x + y = 1)$ TRUE

    Proof: Let $x, y \in \mathbb{N}$. Then $x \ge 1$ and $y \ge 1$, so $x^2 \ge 1$ and $y^2 \ge 1$ and hence $x^2 + y^2 \ge 2$, which means that the proposition on the left hand side of the implication arrow is always false. Therefore the implication is true.

14. $\forall x \in \mathbb{R}\, \forall y \in \mathbb{R} : (x^2 + y^2 = 1 \quad\Rightarrow\quad x + y = 1)$ FALSE

    Proof: (Counterexample) Take $x = \frac{4}{5},\ y = \frac{3}{5}$. Then $x^2 + y^2 = 1$, but $x + y = \frac{7}{5} \ne 1$.

15. $\forall x \in \mathbb{N}\, \forall y \in \mathbb{N} : (x^2 - y^2 = 1 \quad\Rightarrow\quad x = 2)$ TRUE

    Proof: Let $x, y \in \mathbb{N}$. Suppose without loss of generality (i.e. this does not affect the logical arguments in the proof) that $x \ge y$. Then $x^2 - y^2 = (x-y)(x+y)$, which equals 0 if $x = y$ and otherwise $(x-y)(x+y) \ge x + y \ge 2$. Hence $x^2 - y^2 \ne 1$ and the implication is true.

16. $\forall x \in \mathbb{R}\, \forall y \in \mathbb{R} : (x^2 - y^2 = 1 \quad\Rightarrow\quad x = 2)$ FALSE

    Proof: (Counterexample) Take $x = 1,\ y = 0$. Then $x^2 - y^2 = 1$, but $x \ne 2$.

17. $\exists x \in \mathbb{R}\, \forall y \in \mathbb{R} : (x^2 - y^2 = 1 \quad\Rightarrow\quad x = 2)$ TRUE

    Proof: Take $x = 0$. Then $\forall y \in \mathbb{R}$ we have that $x^2 - y^2 \le 0$, so $x^2 - y^2 \ne 1$. Consequently the proposition is true.

18. $\forall x \in \mathbb{N}\, \exists y \in \mathbb{N}\, \exists z \in \mathbb{N} :\ x^2 + y^2 = z^2$ FALSE

    Proof: (Counterexample) Take $x = 1$. Then $\forall y, z \in \mathbb{N}$ the equation can be rewritten as $z^2 - y^2 = 1$, which, by the same arguments used in exercise 15.15, is always false.

19. $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, \exists z \in \mathbb{R} :\ x^2 + y^2 = z^2$ TRUE

    Proof: Let $x \in \mathbb{R}$. Take $y = 0$ and $z = x$. Then $x^2 + y^2 = x^2 = z^2$.

Exercise 16: Show that $4^n - 1$ is divisible by 3 for all $n \in \mathbb{N}$.

Proof: By induction where $P(n)$ is the proposition: $4^n - 1$ is divisible by 3.

Step 0: $4^1 - 1 = 3$, which is divisible by 3.

Step 1: Suppose $P(n)$ is true. Then $4^n - 1$ is divisible by n. But then

$$\begin{aligned} 4^{n+1} - 1 &= 4 \cdot 4^n - 1 \\ &= 4 \cdot (4^n - 1) + 3 \end{aligned}$$

By the induction hypothesis $4 \cdot (4^n - 1)$ is divisible by 3 and clearly 3 is divisible by 3, so $4 \cdot (4^n - 1) + 3$ is divisible by 3. Hence $P(n+1)$ is true, which completes the proof.
Alternatively we could have written:

$$\begin{aligned} 4^{n+1} - 1 &= 4 \cdot 4^n - 1 \\ &= 3 \cdot 4^n + 4^n - 1 \end{aligned}$$

where clearly $3 \cdot 4^n$ is divisible by 3 and by the induction hypothesis $4^n - 1$ is divisible by 3. The rest of the proof then remains unchanged.

Exercise 17:

1. $\forall n \geq 4 : 2^n \geq n^2$. (Notice that the proposition $P(n+1)$ in this case states that $2^{n+1} \geq (n+1)^2$, where $(n+1)^2 = n^2 + 2n + 1$).

   Proof:

   Basic step: The proposition $P(4)$ states that $2^4 \geq 4^2$, which is true.

   Inductive step: Let $n \geq 4$ and suppose that $P(n)$ is true (so $2^n \geq n^2$). Then

   $$\begin{aligned} 2^{n+1} &= 2 \cdot 2^n \\ &\geq 2 \cdot n^2 \text{ (by the induction hypothesis)} \\ &= n^2 + n^2 \\ &= n^2 + n \cdot n \\ &\geq n^2 + 4n \text{ (since } n \geq 4) \\ &= n^2 + 2n + 2n \\ &\geq n^2 + 2n + 2 \cdot 4 \text{ (since } n \geq 4) \\ &> n^2 + 2n + 1 \\ &= (n+1)^2 \end{aligned}$$

2. $\forall n \in \mathbb{N} : 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{1}{2}n(n+1)\right)^2$. (Notice that the proposition $P(n+1)$ in this case states that $1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1)^3 = \left(\frac{1}{2}(n+1)(n+2)\right)^2$).

   Proof:

   Basic step: $P(1)$ states that $1^3 = (\frac{1}{2} \cdot 1 \cdot 2)^2$, which is true.

   Inductive step: Let $n \geq 1$ and suppose that $P(n)$ is true (so $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{1}{2}n(n+1)\right)^2$). Then

   $$\begin{aligned} 1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1)^3 &= \left(\tfrac{1}{2}n(n+1)\right)^2 + (n+1)^3 \text{ (ind. hyp.)} \\ &= \tfrac{1}{4}n^2 \cdot (n+1)^2 + (n+1) \cdot (n+1)^2 \\ &= (n+1)^2 \left(\tfrac{1}{4}n^2 + (n+1)\right) \\ &= \tfrac{1}{4}(n+1)^2(n^2 + 4n + 4) \\ &= \tfrac{1}{4}(n+1)^2(n+2)^2 \\ &= \left(\tfrac{1}{2}(n+1)(n+2)\right)^2 \end{aligned}$$

3. $\forall n \in \mathbb{N} : 6^n - 5n + 4$ is divisible by 5. (Notice that the proposition $P(n+1)$ in this case states that $6^{n+1} - 5(n+1) + 4$ is divisible by 5).

   Proof:

   Basic step: $P(1)$ states that $6^1 - 5 \cdot 1 + 4$ is divisible by 5, which is true.

   Inductive step: Let $n \geq 1$ and suppose that $P(n)$ is true (so $6^n - 5n + 4$ is divisible by 5). Then

$$\begin{aligned} 6^{n+1} - 5(n+1) + 4 &= 6 \cdot 6^n - 5(n+1) + 4 \\ &= (5+1) \cdot 6^n - 5(n+1) + 4 \\ &= 5 \cdot 6^n - 5 + 6^n - 5n + 4. \end{aligned}$$

   Now $6^n - 5n + 4$ is divisible by 5 by the induction hypothesis and $5 \cdot 6^n - 5$ is also divisible by 5, so $6^{n+1} - 5(n+1) + 4$ is divisible by 5 and $P(n+1)$ is true.

4. $\forall n \in \mathbb{N} : 1 + a + a^2 + a^3 + \ldots + a^n = \frac{a^{n+1}-1}{a-1}$. (Notice that the proposition $P(n+1)$ in this case states that $1 + a + a^2 + a^3 + \ldots + a^n + a^{n+1} = \frac{a^{n+2}-1}{a-1}$).

   Proof:

   Basic step: $\frac{a^2-1}{a-1} = \frac{(a-1)\cdot(a+1)}{a-1} = 1 + a$, so the proposition $P(1)$, which states that $1 + a = \frac{a^2-1}{a-1}$, is true.

   Inductive step: Let $n \geq 1$ and suppose that $P(n)$ is true (so $1 + a + a^2 + a^3 + \ldots + a^n = \frac{a^{n+1}-1}{a-1}$). Then

$$\begin{aligned} 1 + a + a^2 + a^3 + \ldots + a^n + a^{n+1} &= \tfrac{a^{n+1}-1}{a-1} + a^{n+1} \\ &= \tfrac{a^{n+1}-1}{a-1} + a^{n+1} \cdot \tfrac{a-1}{a-1} \\ &= \tfrac{a^{n+1}-1+a^{n+2}-a^{n+1}}{a-1} \\ &= \tfrac{a^{n+2}-1}{a-1}, \end{aligned}$$

   so $P(n+1)$ is true.

5. $\forall n \in \mathbb{N} : (1+2+3+\ldots+n)^2 = 1^3+2^3+3^3+\ldots+n^3$. (Notice that the proposition $P(n+1)$ in this case states that $(1+2+3+\ldots+n+(n+1))^2 = 1^3+2^3+3^3+\ldots+n^3+(n+1)^3$.

   Proof:

   Basic step: The proposition $P(1)$ states that $1^2 = 1^3$, which is true.

   Inductive step: Let $n \geq 1$ and suppose that $P(n)$ is true (so $(1+2+3+\ldots+n)^2 = 1^3 + 2^3 + 3^3 + \ldots + n^3$). Then

$$\begin{aligned} &(1+2+3+\ldots+n+(n+1))^2 \\ &= (1+2+3+\ldots+n)^2 + 2\cdot(1+2+3+\ldots+n)\cdot(n+1) + (n+1)^2 \\ &= 1^3+2^3+3^3+\ldots+n^3 + 2\cdot(1+2+3+\ldots+n)\cdot(n+1) + (n+1)^2 \\ &= 1^3+2^3+3^3+\ldots+n^3 + n(n+1)\cdot(n+1) + (n+1)^2 \text{ (by example 13)} \\ &= 1^3+2^3+3^3+\ldots+n^3 + (n+1)^3 \end{aligned}$$

so $P(n+1)$ is true.

Exercise 18:

1. $\forall n \in \mathbb{N} : 7^n - 1$ is divisible by 6

   Proof:

   Basic Step: $P(1)$ states that $7^1 - 1$ is divisible by 6, which is obviously true.

   Inductive Step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true (so $7^n - 1$ is divisible by 6). Then

$$\begin{aligned} 7^{n+1} - 1 &= 7 \cdot 7^n - 1 \\ &= 6 \cdot 7^n + 1 \cdot 7^n - 1. \end{aligned}$$

   Now, $6 \cdot 7^n$ is obviously divisible by 6 and $7^n - 1$ is divisible by 6 by the induction hypothesis. Hence $P(n+1)$ is true.

2. $\forall n \in \mathbb{N} : 7^n + 2n - 1$ is divisible by 4

   Proof:

   Basic Step: $P(1)$ states that $7^1 + 2 \cdot 1 - 1$ is divisible by 4, which is obviously true.

   Inductive Step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true (so $7^n + 2n - 1$ is divisible by 4). Then

$$\begin{aligned} 7^{n+1} + 2(n+1) - 1 &= 7 \cdot 7^n + 2n + 2 - 1 \\ &= 8 \cdot 7^n - 7^n + 2n + 1 \\ &= 8 \cdot 7^n - 7^n - 2n + 4n + 1 \\ &= 8 \cdot 7^n + 4n - 1 \cdot (7^n + 2n - 1). \end{aligned}$$

   Now, $8 \cdot 7^n + 4n$ is obviously divisible by 4 and $-1 \cdot (7^n + 2n - 1)$ is divisible by 4 by the induction hypothesis. Hence $P(n+1)$ is true.

3. $\forall n \in \mathbb{N} : 6^{2n-1} + 5^{2n-1}$ is divisible by 11

   Proof:

   Basic Step: $P(1)$ states that $6^1 + 5^1$ is divisible by 11, which is obviously true.

   Inductive Step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true (so $6^{2n-1} + 5^{2n-1}$ is divisible by 11). Then

$$\begin{aligned} 6^{2(n+1)-1} + 5^{2(n+1)-1} &= 6^{2n+1} + 5^{2n+1} \\ &= 6^2 \cdot 6^{2n-1} + 5^2 \cdot 5^{2n-1} \\ &= 36 \cdot 6^{2n-1} + 25 \cdot 5^{2n-1} \\ &= 11 \cdot 6^{2n-1} + 25 \cdot (6^{2n-1} + 5^{2n-1}). \end{aligned}$$

   Now, $11 \cdot 6^{2n-1}$ is obviously divisble by 11 and $25 \cdot (6^{2n-1} + 5^{2n-1})$ is divisible by 11 by the induction hypothesis. Hence $P(n+1)$ is true.

4. $\forall n \in \mathbb{N} : 4^{2n-1} + 3^{n+1}$ is divisible by 13

   Proof:

   Basic Step: $P(1)$ states that $4^1 + 3^2$ is divisible by 13, which is obviously true.

   Inductive Step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true (so $4^{2n-1} + 3^{n+1}$ is divisible by 13). Then

$$\begin{aligned} 4^{2(n+1)-1} + 3^{(n+1)+1} &= 4^{2n+1} + 3^{n+2} \\ &= 16 \cdot 4^{2n-1} + 3 \cdot 3^{n+1} \\ &= 13 \cdot 4^{2n-1} + 3 \cdot (4^{2n-1} + 3^{n+1}). \end{aligned}$$

   Now, $13 \cdot 4^{2n-1}$ is obviously divisible by 13 and $3 \cdot (4^{2n-1} + 3^{n+1})$ is divisible by 13 by the induction hypothesis. Hence $P(n+1)$ is true.

5. $\forall n \in \mathbb{N} : 7^n + 3^n - 2$ is divisible by 8

   Proof:

   Basic Step: $P(1)$ states that $7^1 + 3^1 - 2$ is divisible by 8, which is obviously true.

   Inductive Step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true (so $7^n + 3^n - 2$ is divisible by 8). Then

$$\begin{aligned} 7^{n+1} + 3^{n+1} - 2 &= 7 \cdot 7^n + 3 \cdot 3^n - 2 \\ &= 4 \cdot 7^n + 3 \cdot 7^n + 3 \cdot 3^n - 2 \\ &= 3 \cdot (7^n + 3^n - 2) + 4 + 4 \cdot 7^n \\ &= 3 \cdot (7^n + 3^n - 2) + 4 \cdot (1 + 7^n) \end{aligned}$$

   Now, 7^n is odd (it it a product of odd numbers), so $1+7^n$ is even. Consequently, $4 \cdot (1+7^n)$ is 4 times an even number, so it is divisible by 8. Furthermore, $3 \cdot (7^n + 3^n - 2)$ is divisible by 8 by the induction hypothesis. We conclude that $P(n+1)$ is true.

6. $\forall n \in \mathbb{N} : 1 + 3 + 5 + \ldots + (2n - 1) = n^2$

   Proof:

   Basic Step: $P(1)$ states that $2 \cdot 1 - 1 = 1^2$, which is obviously true.

   Inductive Step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true (so $1+3+5+\ldots+(2n-1) = n^2$). Then

$$\begin{aligned} 1 + 3 + 5 + \ldots + 2n - 1 + 2(n+1) - 1 &= n^2 + 2(n+1) - 1 && \text{(ind. hyp.)} \\ &= n^2 + 2n + 1 \\ &= (n+1)^2, \end{aligned}$$

   so $P(n+1)$ is true.

7. $\forall n \in \mathbb{N} : 5 + 8 + 11 + \ldots + (3n + 2) = \frac{1}{2}n(3n + 7)$

   Proof:

   Basic Step: $P(1)$ states that $3 \cdot 1 + 2 = \frac{1}{2} \cdot 1 \cdot 10$, which is obviously true.

   Inductive Step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true (so $5+8+11+\ldots+(3n+2) = \frac{1}{2}n(3n+7)$). Then

$$\begin{aligned} 5 + 8 + \ldots + (3n + 2) + 3(n + 1) + 2 &= \tfrac{1}{2}n(3n + 7) + 3(n + 1) + 2 && \text{(ind. hyp.)} \\ &= \tfrac{1}{2}(3n^2 + 7n + 6n + 6 + 4) \\ &= \tfrac{1}{2}(3n^2 + 13n + 10) \\ &= \tfrac{1}{2}(n + 1)(3n + 10) \\ &= \tfrac{1}{2}(n + 1)(3(n + 1) + 7), \end{aligned}$$

   so $P(n+1)$ is true.

8. $\forall n \in \mathbb{N} : 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \ldots + n \cdot (n+2) = \frac{1}{6}n(n+1)(2n+7)$

   Proof:

   Basic Step: $P(1)$ states that $1 \cdot 3 = \frac{1}{6} \cdot 1 \cdot 2 \cdot 9$, which is obviously true.

   Inductive Step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true (so $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \ldots + n \cdot (n+2) = \frac{1}{6}n(n+1)(2n+7)$). Then

$$\begin{aligned}
1 \cdot 3 + 2 \cdot 4 + \ldots + n \cdot (n+2) + (n+1)(n+3) &= \tfrac{1}{6}n(n+1)(2n+7) + (n+1)(n+3) \\
&\text{(ind. hyp.)} \\
&= \tfrac{1}{6}(n+1)\left(n(2n+7) + 6(n+3)\right) \\
&= \tfrac{1}{6}(n+1)(2n^2 + 13n + 18) \\
&= \tfrac{1}{6}(n+1)(n+2)(2n+9),
\end{aligned}$$

   so $P(n+1)$ is true.

9. $\forall n \in \mathbb{N} : 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \ldots + n \cdot (n+3) = \frac{1}{3}n(n+1)(n+5)$

   Proof:

   Basic Step: $P(1)$ states that $1 \cdot 4 = \frac{1}{3} \cdot 1 \cdot 2 \cdot 6$, which is obviously true.

   Inductive Step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true (so $1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \ldots + n \cdot (n+3) = \frac{1}{3}n(n+1)(n+5)$). Then

$$\begin{aligned}
1 \cdot 4 + 2 \cdot 5 + \ldots + n \cdot (n+3) + (n+1)(n+4) &= \tfrac{1}{3}n(n+1)(n+5) + (n+1)(n+4) \\
&\text{(ind. hyp.)} \\
&= \tfrac{1}{3}(n+1)(n(n+5) + 3(n+4)) \\
&= \tfrac{1}{3}(n+1)(n^2 + 8n + 12) \\
&= \tfrac{1}{3}(n+1)(n+2)(n+6),
\end{aligned}$$

   so $P(n+1)$ is true.

10. $\forall n \in \mathbb{N} : 2^2 + 4^2 + 6^2 + \ldots + (2n)^2 = \frac{2}{3}n(n+1)(2n+1)$

   Proof:

   Basic Step: $P(1)$ states that $2^2 = \frac{2}{3} \cdot 1 \cdot 2 \cdot 3$, which is obviously true.

   Inductive Step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true (so $2^2 + 4^2 + 6^2 + \ldots + (2n)^2 = \frac{2}{3}n(n+1)(2n+1)$). Then

$$\begin{aligned}
2^2 + 4^2 + 6^2 + \ldots + (2n)^2 + (2(n+1))^2 &= \tfrac{2}{3}n(n+1)(2n+1) + (2(n+1))^2 \\
&\text{(ind. hyp.)} \\
&= (n+1)(\tfrac{2}{3}n(2n+1) + 4(n+1)) \\
&= \tfrac{2}{3}(n+1)(n(2n+1) + 6(n+1)) \\
&= \tfrac{2}{3}(n+1)(2n^2 + 7n + 6) \\
&= \tfrac{2}{3}(n+1)(n+2)(2n+3),
\end{aligned}$$

   so $P(n+1)$ is true.

11. $\forall n \in \mathbb{N} : \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n\cdot(n+1)} = 1 - \frac{1}{n+1}$

    Proof:

    Basic Step: $P(1)$ states that $\frac{1}{1\cdot 2} = 1 - \frac{1}{2}$, which is obviously true.

    Inductive Step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true (so $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n\cdot(n+1)} = 1 - \frac{1}{n+1}$). Then

    $$\begin{aligned} \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n\cdot(n+1)} + \frac{1}{(n+1)(n+2)} &= 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} && \text{(ind. hyp.)} \\ &= 1 - \frac{n+2}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} \\ &= 1 - \frac{n+1}{(n+1)(n+2)} \\ &= 1 - \frac{1}{n+2}, \end{aligned}$$

    so $P(n+1)$ is true.

12. $\forall n \in \mathbb{N} : \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

    Proof:

    Basic Step: $P(1)$ states that $\frac{1}{2!} = 1 - \frac{1}{(1+1)!}$, which is obviously true.

    Inductive Step: Let $n \in \mathbb{N}$ and assume that $P(n)$ is true (so $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$). Then

    $$\begin{aligned} \frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!} &= 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} && \text{(ind. hyp.)} \\ &= 1 - \frac{n+2}{(n+2)!} + \frac{n+1}{(n+2)!} \\ &= 1 - \frac{1}{(n+2)!}, \end{aligned}$$

    so $P(n+1)$ is true.

Exercise 19: In a square all prime factors appear an even number of times.

Proof: Let s be a square. Then $s = n^2$ for some $n \in \mathbb{N}$. But by the fundamental theorem of arithmetic we uniquely have $n = p_1^{m_1} \cdot p_2^{m_2} \cdot p_3^{m_3} \cdot \ldots \cdot p_k^{m_k}$, where the p's are the prime factors of n. But then

$$\begin{aligned} s = n \cdot n &= (p_1^{m_1} \cdot p_2^{m_2} \cdot p_3^{m_3} \cdot \ldots \cdot p_k^{m_k}) \cdot (p_1^{m_1} \cdot p_2^{m_2} \cdot p_3^{m_3} \cdot \ldots \cdot p_k^{m_k}) \\ &= p_1^{m_1} \cdot p_1^{m_1} \cdot p_2^{m_2} \cdot p_2^{m_2} \cdot p_3^{m_3} \cdot p_3^{m_3} \cdot \ldots \cdot p_k^{m_k} \cdot p_k^{m_k} \\ &= p_1^{2m_1} \cdot p_2^{2m_2} \cdot p_3^{2m_3} \cdot \ldots \cdot p_k^{2m_k}, \end{aligned}$$

so indeed all prime factors appear an even number of times.

Exercise 20: p is the proposition: "$1 + 2 + 3 + \ldots + n + (n+1) = \frac{1}{2}(n+1)(n+2)$" and q is "$\frac{1}{2}n^2 + \frac{3}{2}n + 1 = \frac{1}{2}n^2 + \frac{3}{2}n + 1$".

Exercises 21 and **22** are not relevant for the course and the solutions are omitted.