Resit January 2022 Discrete Mathematics DSAI

Question 1:

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(q \Rightarrow r) \wedge \neg r$	$((q \Rightarrow r) \wedge \neg r) \Rightarrow \neg p$	$(p \Rightarrow q) \Rightarrow (((q \Rightarrow r) \wedge \neg r) \Rightarrow \neg p)$
T	T	T	T	T	F	T	T
T	T	F	T	F	F	T	T
T	F	T	F	T	F	T	T
T	F	F	F	T	T	F	T
F	T	T	T	T	F	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	T	T	T

(Hence, the statement is a tautology.)

Question 2:

Base case: $\sum_{i=1}^{1} i^3 = 1^3 = 1 = 1/4 \cdot 1 \cdot 4 = 1/4 \cdot 1 \cdot 2^2 = 1/4 \cdot 1^2 \cdot (1+1)^2$ ✓

Inductive step: Let $n \in \mathbb{N}$.

Assume $\sum_{i=1}^{n} i^3 = 1/4 \cdot n^2 \cdot (n+1)^2$

Now, $\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3 = 1/4 \cdot n^2 \cdot (n+1)^2 + (n+1)^3$

$= (n+1)^2 \left(1/4 n^2 + (n+1)\right) = (n+1)^2 \cdot 1/4 \cdot (n^2+4n+4) = (n+1)^2 \cdot 1/4 \cdot (n+2)^2$

$= 1/4 (n+1)^2 \cdot ((n+1)+1)^2$ ✓ $\square$

Question 3:

ⓐ False, consider $A=\{1\}$, $B=\{2\}$ and $C=\{1,3\}$.
Then $A \not\subseteq B$ and $B \not\subseteq C$. However, $A \subseteq C$.

ⓑ False, consider the universe $U=\{1,2,3,4\}$, $A=\{1,2\}$, $B=\{1,2,3\}$ and $C=\{2\}$. Then, $A \subseteq B$. However, $A \cap (B \cap C)^c = A \cap (\{2\})^c = A \cap \{1,3,4\}$
$=\{1\} \neq \emptyset$.

Question 4:

* Reflexive:
Let $X \in P(\{a,b,c\})$, i.e., let $X \subseteq \{a,b,c\}$.
Since $X \setminus X = \emptyset$, we have $X \not R X$.
So, R is not reflexive.

* Symmetric:
Let $X \subseteq \{a,b,c\}$ and let $Y \subseteq \{a,b,c\}$.
Assume XRY.
So, $X \setminus Y \neq \emptyset$ and $Y \setminus X \neq \emptyset$.
(So, $Y \setminus X \neq \emptyset$ and $X \setminus Y \neq \emptyset$.)
Hence, also YRX.
So, R is symmetric.

* Transitive:
No, consider for example $X=\{a\}$, $Y=\{b\}$ and $Z=\{a,c\}$.
Then XRY, because $X \setminus Y = \{a\} \neq \emptyset$ and $Y \setminus X = \{b\} \neq \emptyset$.
Also YRZ, because $Y \setminus Z = \{b\} \neq \emptyset$ and $Z \setminus Y = \{a,c\} \neq \emptyset$.
However $X \not R Z$, since $X \setminus Z = \emptyset$.

* Anti-symmetry:
No, consider for example $X=\{a\}$ and $Y=\{b\}$.
Then XRY and YRX.
However $X \neq Y$.

Question 5:

(a) Rule of product: $2 \cdot 3 = 6$.

(b) There are 2^8 sequences of length 11 that start with 101.
→ $n=2$, $k=8$, repetition is allowed, order is important
Similarly: there are 2^8 sequences of length 11 that start with 010.
These two sets are disjoint, so rule of sum: $2^8 + 2^8 = 2 \cdot 2^8 = 2^9 = 512$.

(c) There are $6!$ options for the first six books (the mathematics books).
→ $n=6$, $k=6$, repetition is not allowed, order is important.
Similarly: there are $6!$ options for the last six books (the economics books).
We can combine every order of the mathematics books with every order of the economics book, so rule of product: $6! \cdot 6!$
$= 720 \cdot 720 = 518\,400$.

Question 6:

(a) True.
Let $x \in X$.
If $x = -2$, choose $y = 2 \in Y$
$x = -1$, $y = 1 \in Y$
$x = 0$, $y = 0 \in Y$
$x = 1$, $y = -1 \in Y$
$x = 2$, $y = -2 \in Y$.
Then, in any case we have $x + y = 0$.

(b) True.
Take $x = 0 \in X$.
Let $y \in Y$. Then, $x + y = 0 + y = y$.

(c) True.
Let $x \in \mathbb{R}$ and let $y \in \mathbb{R}$.

Take $z = \max\left\{\left\lceil\sqrt{x^2+y^2}\right\rceil, 5\right\}$
Then, $z \in \mathbb{N}$ because z will be an integer that is ≥ 5.

$\left\{\begin{array}{l}\text{Note, that } z \geq \left\lceil\sqrt{x^2+y^2}\right\rceil \geq \sqrt{x^2+y^2} \text{ and thus } z^2 \geq \left(\sqrt{x^2+y^2}\right)^2 = x^2+y^2. \\ \text{Moreover } z \geq 5.\end{array}\right.$
→ Hence, $(z^2 \geq x^2+y^2) \wedge (z \geq 5)$.

Question 7:

ⓐ Injectivity: Let $x \in \mathbb{R}\setminus\{0\}$ and let $y \in \mathbb{R}\setminus\{0\}$
Assume $f(x) = f(y)$.
So, $\frac{x+1}{x} = \frac{y+1}{y} \Rightarrow y(x+1) = x(y+1) \Rightarrow \cancel{xy} + y = \cancel{xy} + x \Rightarrow x = y$ ✓

Surjectivity: Let $y \in \mathbb{R}\setminus\{1\}$
Take $x = \frac{1}{y-1}$. Since $y \in \mathbb{R}\setminus\{1\}$, we know $x \in \mathbb{R}$.
We also need $x \neq 0$. So, we need $\frac{1}{y-1} \neq 0$. Is that true? Yes, because the numerator is always non-zero.
So, we have $x \in \mathbb{R}\setminus\{0\}$.
Next, $f(x) = f\left(\frac{1}{y-1}\right) = \dfrac{\frac{1}{y-1} + 1}{\frac{1}{y-1}} = \dfrac{\frac{1}{y-1} + \frac{y-1}{y-1}}{\frac{1}{y-1}} = \dfrac{\frac{1+y-1}{y-1}}{\frac{1}{y-1}}$

$= \frac{(1+y-1)}{\cancel{y-1}} \cdot \frac{\cancel{y-1}}{1} = \frac{1+y-1}{1} = 1+y-1 = y$ ✓

ⓑ Take $A = \{1,2,3\}$ and $B = \{1,2\}$.
And consider the following f and g:

$f: \begin{array}{l} 1 \to 1 \\ 2 \to 2 \\ 3 \to 2 \end{array}$ $\quad g: \begin{array}{l} 1 \to 1 \\ 2 \to 2 \\ \quad\ \ 3 \end{array}$ $\quad$ Then, $f \circ g$ $\begin{array}{l} 1 \to 1 \\ 2 \to 2 \end{array}$

Then, f is not injective, since $f(2) = 2 = f(3)$.
But, g and $f \circ g$ are both injective.

Question 8:

ⓐ No, because the blocks are overlapping ($\{a,d,e\} \cap \{d,f\} \neq \emptyset$).

ⓑ $A \times B = \{1,2\} \times \{2,3\} = \{(1,2),(1,3),(2,2),(2,3)\}$

$A \times C = \{1,2\} \times \{3,4\} = \{(1,3),(1,4),(2,3),(2,4)\}$

$(A \times B) \cap (A \times C) = \{(1,3),(2,3)\}$.