Practice Exam Questions Week 4, **Linear Algebra**

SOLUTIONS.

1. Consider the following matrix:

$$M = \begin{bmatrix} -2 & 6 & 0 & -2 & 2 \\ -1 & 3 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 \end{bmatrix}$$

Compute bases for the two subspaces Col(A) and Nul(A).

$$\begin{bmatrix} -2 & 6 & 0 & -2 & 2 \\ -1 & 3 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 \end{bmatrix} \overset{R_1 : R_1 * -1/2}{\sim} \begin{bmatrix} 1 & -3 & 0 & 1 & -1 \\ -1 & 3 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & -1 \end{bmatrix} \overset{R_2 : R_2 + R_1}{\sim}$$

$$\begin{bmatrix} 1 & -3 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 \end{bmatrix} \overset{R_2 <-> R_3}{\sim} \begin{bmatrix} 1 & -3 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_2 : R_2 * -1}{\sim}$$

$$\begin{bmatrix} 1 & -3 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_1 : R_1 + 3 * R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for Col(A) is $\left\{ \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ -1 \end{bmatrix} \right\}$.

$$\begin{cases} x_1 = -x_4 - 2x_5 \\ x_2 = -x_5 \\ x_3, x_4, x_5 \text{ are free} \end{cases}$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_4 - 2x_5 \\ -x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, a basis for Nul(A) is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2. True or false? If the given statement is true, briefly explain why. If it is false, give a counterexample.

a. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

False. $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$. The vectors in $\mathbb{R}^3$ all have three entries, whereas the vectors in $\mathbb{R}^2$ have only two entries.

b. A vector is an arrow in three-dimensional space.

False. An arrow in three-dimensional space is an example of a vector, but not every vector is an arrow in three-dimensional space.

c. A subset H of a vector space V is a subspace of V if the zero vector is in H.

False. H also needs to be closed under addition and scaling. (properties b. and c. on page 211).

d. A subspace is also a vector space.

True. Note that properties ⓐ, ⓑ and ⓒ on page 211 are Axioms 1, 4 and 6 on page 208. Axioms 2, 3 and 7-10 are automatically true for the subspace because they apply to all elements of the vector space, and thus also to all elements of the subspace. Axiom 5 is also true for the subspace, because if $\underline{u}$ is in the subspace, then $(-1)\underline{u}$ is also in the subspace by property ⓒ. And thus, by (3) on page 209, we know that $-\underline{u} = (-1)\underline{u}$. Hence, $-\underline{u}$ is in the subspace.

e. A vector space is also a subspace.

True. Every vector space is a subspace of itself.

f. The null space of an $m \times n$ matrix is in $\mathbb{R}^m$.

False. It is a subspace of $\mathbb{R}^n$.

g. The column space of an $m \times n$ matrix is in $\mathbb{R}^m$.

True, because the column space of a matrix is the set of all linear combinations of the columns of this matrix. Since the matrix has size $m \times n$, the columns of the matrix are in $\mathbb{R}^m$. And thus also every linear combination of the columns is in $\mathbb{R}^m$.

3 Determine all values of $p \in \mathbb{R}$ for which the null space of the following matrix has dimension 1.

$$G = \begin{bmatrix} 2 & -5 & 8 \\ -2 & -7 & p \\ 4 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -5 & 8 \\ -2 & -7 & p \\ 4 & 2 & 7 \end{bmatrix} \underset{R_3 : R_3 - 2R_1}{\overset{R_2 : R_2 + R_1}{\sim}} \begin{bmatrix} 2 & -5 & 8 \\ 0 & -12 & p+8 \\ 0 & 12 & -9 \end{bmatrix} \overset{R_3 : R_3 + R_2}{\sim} \begin{bmatrix} 2 & -5 & 8 \\ 0 & -12 & p+8 \\ 0 & 0 & p-1 \end{bmatrix}$$

The null space has dimension 1 if there is exactly one free variable. So, if one pivot is missing.
So, if $p-1=0 \Rightarrow p=1$.

4 True or false? If the given statement is true, briefly explain why. If it is false, give a counterexample.

a. If a 3×3 matrix A has rank 3, then its rows form a basis for $\mathbb{R}^3$.

True. If a 3×3 matrix has rank 3, then A has 3 pivots. Hence, the rows are three linearly independent vectors in $\mathbb{R}^3$, and they span all $\mathbb{R}^3$. As a consequence, they form a basis for $\mathbb{R}^3$.

b. If the first column of a 7×5 matrix B provides a basis for the column space, then the null space of B has dimension 4.

True. The first column provides a basis for the column space, so $\dim \mathrm{Col}(B) = 1$.
And, since B is a 7×5 matrix, we know
$\dim \mathrm{Col}(B) + \dim \mathrm{Nul}(B) = 5$.
Hence, $\dim \mathrm{Nul}(B) = 5 - 1 = 4$.

c. The number of variables in the equation $A\mathbf{x} = \mathbf{0}$ equals the dimensions of $\mathrm{Nul}(A)$.

False. The number of free variables equals the dimensions of $\mathrm{Nul}(A)$.

d. If $\dim V = n$ and if S spans V, then S is a basis of V.

False. The set S must also have n elements (see The Basis Theorem).