EXERCISES 4.3

Evaluate the limits in Exercises 1–32.

1. $\displaystyle\lim_{x\to 0}\frac{3x}{\tan 4x}$

2. $\displaystyle\lim_{x\to 2}\frac{\ln(2x-3)}{x^2-4}$

3. $\displaystyle\lim_{x\to 0}\frac{\sin ax}{\sin bx}$

4. $\displaystyle\lim_{x\to 0}\frac{1-\cos ax}{1-\cos bx}$

5. $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x}{\tan^{-1}x}$

6. $\displaystyle\lim_{x\to 1}\frac{x^{1/3}-1}{x^{2/3}-1}$

7. $\displaystyle\lim_{x\to 0}x\cot x$

8. $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{\ln(1+x^2)}$

9. $\displaystyle\lim_{t\to \pi}\frac{\sin^2 t}{t-\pi}$

10. $\displaystyle\lim_{x\to 0}\frac{10^x-e^x}{x}$

11. $\displaystyle\lim_{x\to \pi/2}\frac{\cos 3x}{\pi-2x}$

12. $\displaystyle\lim_{x\to 1}\frac{\ln(ex)-1}{\sin \pi x}$

13. $\displaystyle\lim_{x\to \infty}x\sin\frac{1}{x}$

14. $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x^3}$

15. $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}$

16. $\displaystyle\lim_{x\to 0}\frac{2-x^2-2\cos x}{x^4}$

17. $\displaystyle\lim_{x\to 0+}\frac{\sin^2 x}{\tan x-x}$

18. $\displaystyle\lim_{r\to \pi/2}\frac{\ln\sin r}{\cos r}$

19. $\displaystyle\lim_{t\to \pi/2}\frac{\sin t}{t}$

20. $\displaystyle\lim_{x\to 1-}\frac{\arccos x}{x-1}$

21. $\displaystyle\lim_{x\to \infty}x(2\tan^{-1}x-\pi)$

22. $\displaystyle\lim_{t\to(\pi/2)-}(\sec t-\tan t)$

23. $\displaystyle\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{te^{at}}\right)$

24. $\displaystyle\lim_{x\to 0+}x^{\sqrt{x}}$

25. $\displaystyle\lim_{x\to 0+}(\csc x)^{\sin^2 x}$

26. $\displaystyle\lim_{x\to 1+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$

27. $\displaystyle\lim_{t\to 0}\frac{3\sin t-\sin 3t}{3\tan t-\tan 3t}$

28. $\displaystyle\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$

29. $\displaystyle\lim_{t\to 0}(\cos 2t)^{1/t^2}$

30. $\displaystyle\lim_{x\to 0+}\frac{\csc x}{\ln x}$

31. $\displaystyle\lim_{x\to 1-}\frac{\ln\sin \pi x}{\csc \pi x}$

32. $\displaystyle\lim_{x\to 0}(1+\tan x)^{1/x}$

33. **(A Newton quotient for the second derivative)** Evaluate $\lim_{h\to 0}\dfrac{f(x+h)-2f(x)+f(x-h)}{h^2}$ if f is a twice differentiable function.

34. If f has a continuous third derivative, evaluate

$$\lim_{h\to 0}\frac{f(x+3h)-3f(x+h)+3f(x-h)-f(x-3h)}{h^3}.$$

35. **(Proof of the second l'Hôpital Rule)** Fill in the details of the following outline of a proof of the second l'Hôpital Rule (Theorem 4) for the case where a and L are both finite. Let $a<x<t<b$ and show that there exists c in (x,t) such that

$$\frac{f(x)-f(t)}{g(x)-g(t)}=\frac{f'(c)}{g'(c)}.$$

Now juggle the above equation algebraically into the form

$$\frac{f(x)}{g(x)}-L=\frac{f'(c)}{g'(c)}-L+\frac{1}{g(x)}\left(f(t)-g(t)\frac{f'(c)}{g'(c)}\right).$$

It follows that

$$\left|\frac{f(x)}{g(x)}-L\right|$$
$$\le\left|\frac{f'(c)}{g'(c)}-L\right|+\frac{1}{|g(x)|}\left(|f(t)|+|g(t)|\left|\frac{f'(c)}{g'(c)}\right|\right).$$

Now show that the right side of the above inequality can be made as small as you wish (say, less than a positive number ϵ) by choosing first t and then x close enough to a. Remember, you are given that $\lim_{c\to a+}\left(f'(c)/g'(c)\right)=L$ and $\lim_{x\to a+}|g(x)|=\infty$.

EXERCISES 4.4

In Exercises 1–17, determine whether the given function has any local or absolute extreme values, and find those values if possible.

1. $f(x) = x + 2$ on $[-1, 1]$

2. $f(x) = x + 2$ on $(-\infty, 0]$

3. $f(x) = x + 2$ on $[-1, 1)$

4. $f(x) = x^2 - 1$

5. $f(x) = x^2 - 1$ on $[-2, 3]$

6. $f(x) = x^2 - 1$ on $(2, 3)$

7. $f(x) = x^3 + x - 4$ on $[a, b]$

8. $f(x) = x^3 + x - 4$ on (a, b)

9. $f(x) = x^5 + x^3 + 2x$ on $(a, b]$

10. $f(x) = \dfrac{1}{x-1}$

11. $f(x) = \dfrac{1}{x-1}$ on $(0, 1)$

12. $f(x) = \dfrac{1}{x-1}$ on $[2, 3]$

13. $f(x) = |x - 1|$ on $[-2, 2]$

14. $|x^2 - x - 2|$ on $[-3, 3]$

15. $f(x) = \dfrac{1}{x^2+1}$

16. $f(x) = (x + 2)^{2/3}$

17. $f(x) = (x - 2)^{1/3}$

In Exercises 18–40, locate and classify all local extreme values of the given function. Determine whether any of these extreme values are absolute. Sketch the graph of the function.

18. $f(x) = x^2 + 2x$

19. $f(x) = x^3 - 3x - 2$

20. $f(x) = (x^2 - 4)^2$

21. $f(x) = x^3(x - 1)^2$

22. $f(x) = x^2(x - 1)^2$

23. $f(x) = x(x^2 - 1)^2$

24. $f(x) = \dfrac{x}{x^2+1}$

25. $f(x) = \dfrac{x^2}{x^2+1}$

26. $f(x) = \dfrac{x}{\sqrt{x^4+1}}$

27. $f(x) = x\sqrt{2 - x^2}$

28. $f(x) = x + \sin x$

29. $f(x) = x - 2\sin x$

30. $f(x) = x - 2\tan^{-1} x$

31. $f(x) = 2x - \sin^{-1} x$

32. $f(x) = e^{-x^2/2}$

33. $f(x) = x\,2^{-x}$

34. $f(x) = x^2 e^{-x^2}$

35. $f(x) = \dfrac{\ln x}{x}$

36. $f(x) = |x + 1|$

37. $f(x) = |x^2 - 1|$

38. $f(x) = \sin |x|$

39. $f(x) = |\sin x|$

40. $f(x) = (x - 1)^{2/3} - (x + 1)^{2/3}$

In Exercises 41–46, determine whether the given function has absolute maximum or absolute minimum values. Justify your answers. Find the extreme values if you can.

41. $\dfrac{x}{\sqrt{x^2+1}}$

42. $\dfrac{x}{\sqrt{x^4+1}}$

43. $x\sqrt{4 - x^2}$

44. $\dfrac{x^2}{\sqrt{4-x^2}}$

45. $\dfrac{1}{x \sin x}$ on $(0, \pi)$

46. $\dfrac{\sin x}{x}$

47. If a function has an absolute maximum value, must it have any local maximum values? If a function has a local maximum value, must it have an absolute maximum value? Give reasons for your answers.

48. If the function f has an absolute maximum value and $g(x) = |f(x)|$, must g have an absolute maximum value? Justify your answer.

49. **(A function with no max or min at an endpoint)** Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$ but that it has neither a local maximum nor a local minimum value at the endpoint $x = 0$.

EXERCISES 4.9

In Exercises 1–10, find the linearization of the given function about the given point.

1. x^2 about $x = 3$

2. x^{-3} about $x = 2$

3. $\sqrt{4 - x}$ about $x = 0$

4. $\sqrt{3 + x^2}$ about $x = 1$

5. $1/(1 + x)^2$ about $x = 2$

6. $1/\sqrt{x}$ about $x = 4$

7. $\sin x$ about $x = \pi$

8. $\cos(2x)$ about $x = \pi/3$

9. $\sin^2 x$ about $x = \pi/6$

10. $\tan x$ about $x = \pi/4$

11. By approximately how much does the area of a square increase if its side length increases from 10 cm to 10.4 cm?

EXERCISES 4.6

1. Figure 4.43 shows the graphs of a function f, its two derivatives f' and f'', and another function g. Which graph corresponds to each function?

2. List, for each function graphed in Figure 4.43, such information that you can determine (approximately) by inspecting the graph (e.g., symmetry, asymptotes, intercepts, intervals of increase and decrease, critical and singular points, local maxima and minima, intervals of constant concavity, inflection points).

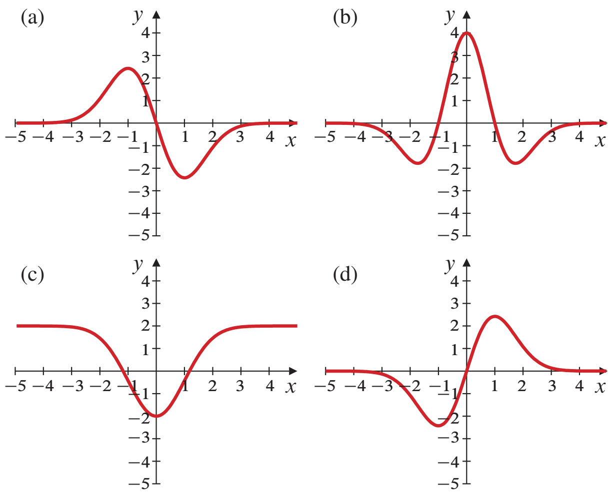

Figure 4.43

3. Figure 4.44 shows the graphs of four functions:

$$f(x) = \frac{x}{1-x^2}, \qquad g(x) = \frac{x^3}{1-x^4},$$

$$h(x) = \frac{x^3 - x}{\sqrt{x^6+1}}, \qquad k(x) = \frac{x^3}{\sqrt{|x^4-1|}}.$$

Which graph corresponds to each function?

4. Repeat Exercise 2 for the graphs in Figure 4.44.

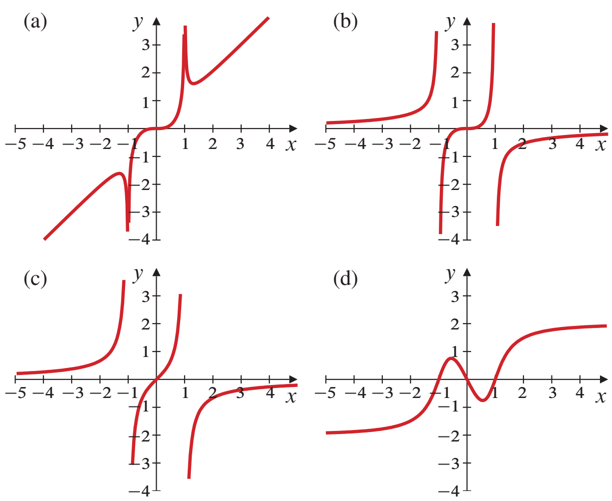

Figure 4.44

In Exercises 5–6, sketch the graph of a function that has the given properties. Identify any critical points, singular points, local maxima and minima, and inflection points. Assume that f is continuous and its derivatives exist everywhere unless the contrary is implied or explicitly stated.

5. $f(0) = 1$, $f(\pm 1) = 0$, $f(2) = 1$, $\lim_{x\to\infty} f(x) = 2$, $\lim_{x\to-\infty} f(x) = -1$, $f'(x) > 0$ on $(-\infty, 0)$ and on $(1, \infty)$, $f'(x) < 0$ on $(0, 1)$, $f''(x) > 0$ on $(-\infty, 0)$ and on $(0, 2)$, and $f''(x) < 0$ on $(2, \infty)$.

6. $f(-1) = 0$, $f(0) = 2$, $f(1) = 1$, $f(2) = 0$, $f(3) = 1$, $\lim_{x\to\pm\infty}(f(x) + 1 - x) = 0$, $f'(x) > 0$ on $(-\infty, -1)$, $(-1, 0)$ and $(2, \infty)$, $f'(x) < 0$ on $(0, 2)$, $\lim_{x\to -1} f'(x) = \infty$, $f''(x) > 0$ on $(-\infty, -1)$ and on $(1, 3)$, and $f''(x) < 0$ on $(-1, 1)$ and on $(3, \infty)$.

In Exercises 7–39, sketch the graphs of the given functions, making use of any suitable information you can obtain from the function and its first and second derivatives.

7. $y = (x^2-1)^3$

8. $y = x(x^2-1)^2$

9. $y = \dfrac{2-x}{x}$

10. $y = \dfrac{x-1}{x+1}$

11. $y = \dfrac{x^3}{1+x}$

12. $y = \dfrac{1}{4+x^2}$

13. $y = \dfrac{1}{2-x^2}$

14. $y = \dfrac{x}{x^2-1}$

15. $y = \dfrac{x^2}{x^2-1}$

16. $y = \dfrac{x^3}{x^2-1}$

17. $y = \dfrac{x^3}{x^2+1}$

18. $y = \dfrac{x^2}{x^2+1}$

19. $y = \dfrac{x^2-4}{x+1}$

20. $y = \dfrac{x^2-2}{x^2-1}$

21. $y = \dfrac{x^3-4x}{x^2-1}$

22. $y = \dfrac{x^2-1}{x^2}$

23. $y = \dfrac{x^5}{(x^2-1)^2}$

24. $y = \dfrac{(2-x)^2}{x^3}$

25. $y = \dfrac{1}{x^3-4x}$

26. $y = \dfrac{x}{x^2+x-2}$

27. $y = \dfrac{x^3-3x^2+1}{x^3}$

28. $y = x + \sin x$

29. $y = x + 2\sin x$

30. $y = e^{-x^2}$

31. $y = xe^x$

32. $y = e^{-x}\sin x, \ (x \ge 0)$

33. $y = x^2 e^{-x^2}$

34. $y = x^2 e^x$

35. $y = \dfrac{\ln x}{x}, \ (x > 0)$

36. $y = \dfrac{\ln x}{x^2}, \ (x > 0)$

37. $y = \dfrac{1}{\sqrt{4-x^2}}$

38. $y = \dfrac{x}{\sqrt{x^2+1}}$

39. $y = (x^2-1)^{1/3}$

40. What is $\lim_{x\to 0+} x\ln x$? $\lim_{x\to 0} x\ln|x|$? If $f(x) = x\ln|x|$ for $x \ne 0$, is it possible to define $f(0)$ in such a way that f is continuous on the whole real line? Sketch the graph of f.

41. What straight line is an asymptote of the curve $y = \dfrac{\sin x}{1+x^2}$? At what points does the curve cross this asymptote?