Section 3.1

A **relation(ship)** on a set A describes a connection between 2 (not necessarily different) elements of A. Notation: $x\mathbf{R}y$. Here x and y denote the elements out of the same set, whereas $\mathbf{R}$ denotes the relation.

Example 1 *Several relations:*

(i) $x\mathbf{R}_1y \Leftrightarrow y$ *is a multiple of* x *on* $\mathbb{N}$

(ii) $x\mathbf{R}_2y \Leftrightarrow x < y$ *on* $S = \{0, 1, 2, 3\}$

(iii) $x\mathbf{R}_3y \Leftrightarrow xy = y$ *on* $[0, 1]$

(iv) $x\mathbf{R}_4y \Leftrightarrow x - y$ *is even on* $\mathbb{N}$

(v) $x\mathbf{R}_5y \Leftrightarrow y > x - 1$ *on* $\mathbb{R}$

(vi) $(x, y)\mathbf{R}_6(a, b) \Leftrightarrow x^2 + y^2 \leq a^2 + b^2$ *on* $\mathbb{R}^2$ *(the set* $\mathbb{R}^2$ *is also called "the plane", because it can be identified with a plane, just like* $\mathbb{R}$ *can be identified with a line)*

(vii) $(x, y)\mathbf{R}_7(a, b) \Leftrightarrow x + y = a + b$ *on* $\mathbb{R}^2$

(viii) $X\mathbf{R}_8Y \Leftrightarrow X \subseteq Y$ *on* $\mathcal{P}(A)$ *for some nonempty set* A*. (In exercises about relation* $\mathbf{R}_8$ *you may assume that* $A = \{1, 2, 3\}$*, although that is not necessary.)*

Example 2 *The following collection of propositions is a relation on* $A = \{1, 2, 3\}$*:* $1\mathbf{R}1$*,* $1\mathbf{R}3$*,*$2\mathbf{R}2$*,*$3\mathbf{R}1$ *and* $3\mathbf{R}3$ *are true , whereas* $1\mathbf{R}2$*,*$2\mathbf{R}1$*,*$2\mathbf{R}3$ *and* $3\mathbf{R}2$ *are false.*

This allows us to write this relation by means of a table:

	1	*2*	*3*
1	*T*	*F*	*T*
2	*F*	*T*	*F*
3	*T*	*F*	*T*

Each row in the table corresponds to a possible value of $x \in A$ *in* $x\mathbf{R}y$ *and each column corresponds to a* y*.*

Apparently there doesn't have to be a specific formula for a relation. It is just a collection of propositions about tuples from A. So, mathematically speaking, we can describe a relation by giving the collection of propositions that are assigned the truth value 'TRUE'. In this case

$$\mathbf{R} = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\} \text{ on } \{1, 2, 3\}.$$

This example shows how we can mathematically define a relation:

Definition 1 *A relation on a set* A *is a collection of ordered pairs of elements of* A*. In other words: It is a subset of the product set* $A \times A$*.*

Exercise 1 *Let A be a set of size n. How many relations exist on A? (First solve: How many elements does the product set $A \times A$ have? Then: How many truth values for each element in $A \times A$?)*

Properties of relations:

A relation $\mathbf{R}$ is **symmetric** on S if the following proposition is true:

$$\forall x \in S\ \forall y \in S : x\mathbf{R}y \Rightarrow y\mathbf{R}x$$

Notice that x and y are not necessarily different.

In Example 1, only relations $\mathbf{R}_4$ and $\mathbf{R}_7$ are symmetric.

Example 3 *We prove that $\mathbf{R}_4$ is symmetric and that $\mathbf{R}_1$ is not symmetric.*

$\mathbf{R}_4$ *is symmetric.* ***Proof:*** *Let $x \in \mathbb{N}, y \in \mathbb{N}$ and suppose that $x\mathbf{R}_4y$ is true.*

$$\begin{aligned}
&\Rightarrow x - y \text{ is even}\\
&\Rightarrow x - y = 2k \text{ for some integer } k\\
&\Rightarrow y - x = 2 \cdot (-k) \text{ (and } -k \text{ is an integer)}\\
&\Rightarrow y - x \text{ is even}\\
&\Rightarrow y\mathbf{R}_4x \text{ is true,}
\end{aligned}$$

which completes the proof.

$\mathbf{R}_1$ *is not symmetric.* ***Proof:*** *(Counterexample) Take $x = 3$ and $y = 33$. Then y is a multiple of x, making $x\mathbf{R}_1y$ true, whereas x is not a multiple of y, so $y\mathbf{R}_1x$ is false.*

Exercise 2 *Find counterexamples to show that relations $\mathbf{R}_2$, $\mathbf{R}_3$, $\mathbf{R}_5$, $\mathbf{R}_6$ and $\mathbf{R}_8$ are not symmetric and prove that $\mathbf{R}_7$ is symmetric.*

A relation $\mathbf{R}$ is **transitive** on S if the following proposition is true:

$$\forall x \in S\ \forall y \in S\ \forall z \in S : (x\mathbf{R}y \wedge y\mathbf{R}z) \Rightarrow x\mathbf{R}z$$

Again notice that x, y and z are not necessarily different; they might all be the same element! (For instance if $|S| = 1$ this will always be the case.)

Example 4 *We prove the transitivity of relations $\mathbf{R}_2$, $\mathbf{R}_3$ and $\mathbf{R}_7$.*

$\mathbf{R}_2$ *is transitive.* ***Proof:*** *Let $x \in S$, $y \in S$, $z \in S$ and suppose that $x\mathbf{R}_2y$ and $y\mathbf{R}_2z$ are both true. Then $x < y$ and $y < z$. But then $x < z$, so $x\mathbf{R}_2z$ is true.*

$\mathbf{R}_3$ *is transitive.* ***Proof:*** *Let $x \in [0,1]$, $y \in [0,1]$, $z \in [0,1]$ and suppose that $x\mathbf{R}_3y$ and $y\mathbf{R}_3z$ are both true. Then $xy = y$ and $yz = z$. But then, since $xy = y$, we have: $(x = 1) \vee (y = 0)$ and since $yz = z$ we have: $(y = 1) \vee (z = 0)$. This implies that at least one of the equations*

$x = 1$ and $z = 0$ must be true, because clearly y can not be equal to 0 and 1 simultaneously. Formally: $(x = 1) \vee (z = 0)$. But then $xz = z$, so $x\mathbf{R}_3 z$ is true.

There is a much simpler proof: Let $x \in [0,1],\ y \in [0,1],\ z \in [0,1]$ and suppose that $x\mathbf{R}_3 y$ and $y\mathbf{R}_3 z$ are both true. Then $xy = y$ and $yz = z$. But then

$$\begin{aligned} xz &= xyz && \textit{since } yz = z \\ &= yz && \textit{since } xy = y \\ &= z && \textit{since } yz = z \end{aligned}$$

$\mathbf{R}_7$ *is transitive.* ***Proof:*** *Let $(x,y) \in \mathbb{R}^2,\ (a,b) \in \mathbb{R}^2,\ (c,d) \in \mathbb{R}^2$ and suppose that $(x,y)\,\mathbf{R}_7\,(a,b)$ and $(a,b)\,\mathbf{R}_7\,(c,d)$ are both true. Then $x+y = a+b$ and $a+b = c+d$. But then $x+y = c+d$, so $(x,y)\,\mathbf{R}_7\,(c,d)$ is true.*

Exercise 3 *For relations $\mathbf{R}_1$, $\mathbf{R}_4$, $\mathbf{R}_5$, $\mathbf{R}_6$ and $\mathbf{R}_8$ prove whether or not they are transitive.*

A relation $\mathbf{R}$ is **reflexive** on S if the following proposition is true:

$$\forall x \in S : x\mathbf{R}x \text{ is true}$$

Example 5 *We prove that in example 1 the relations $\mathbf{R}_1$ and $\mathbf{R}_4$ are reflexive and that $\mathbf{R}_3$ is not reflexive.*

$\mathbf{R}_1$ *is reflexive.* ***Proof:*** *Let $x \in \mathbb{N}$. Then $x = 1 \cdot x$, so x is a multiple of x and $x\mathbf{R}_1 x$ is true.*

$\mathbf{R}_3$ *is not reflexive.* ***Proof:*** *Take $x = \frac{1}{2} \in [0,1]$. Then $x \cdot x = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq \frac{1}{2} = x$, so $x\mathbf{R}_3 x$ is false.*

$\mathbf{R}_4$ *is reflexive.* ***Proof:*** *Let $x \in \mathbb{N}$. Then $x - x = 0 = 2 \cdot 0$, so $x - x$ is even and $x\mathbf{R}_4 x$ is true.*

Exercise 4 *For relations $\mathbf{R}_2$, $\mathbf{R}_5$, $\mathbf{R}_6$, $\mathbf{R}_7$ and $\mathbf{R}_8$ prove whether or not they are reflexive.*

Exercise 5 *For the relation in example 2, determine whether or not it is reflexive, symmetric and/or transitive.*

Exercise 6 *Below four relations on $\{1,2,3\}$ are given. However, some of the truth values are missing. For each relation find out what conditions on the truth values are required to make it reflexive, symmetric and/or transitive.*

$\mathbf{R}_1$	*1*	*2*	*3*
1	*F*	*T*	
2	*T*		*F*
3	*F*		*T*

$\mathbf{R}_2$	*1*	*2*	*3*
1	*T*	*T*	
2	*F*		*T*
3	*F*		*T*

$\mathbf{R}_3$	*1*	*2*	*3*
1	*T*	*F*	*F*
2		*T*	*T*
3	*F*		

$\mathbf{R}_4$	*1*	*2*	*3*
1	*F*	*F*	
2	*F*		*F*
3	*F*		*F*

Exercise 7 *Consider the following relation: $x\mathbf{R}y \Leftrightarrow x - 4y$ is divisible by 3 on $\mathbb{Z}$. Find out if $\mathbf{R}$ is reflexive, symmetric and/or transitive.*

Exercise 8 *How many of the relations in exercise 1 are reflexive? And symmetric? (Transitive is very difficult...)*

Section 3.2

A relation $\mathbf{R}$ that is symmetric, transitive and reflexive on S is called an **equivalence relation**. In Example 1 this is the case for relations $\mathbf{R}_4$ and $\mathbf{R}_7$.

For an equivalence relation $\mathbf{R}$ it is possible to make a partition $E_1, E_2, \ldots, E_m$ of the set S in the following way: A subset E_i of S consists of the following elements of S: If $a \in E_i$, then every $b \in S$ for which $a\mathbf{R}b$ is true are also E_i. Furthermore, every $b \in S$ for which $a\mathbf{R}b$ is false, is not in E_i; it must therefore be in another E_j. Each subset of the resulting partition of S is called an **equivalence class**. We will determine the equivalence classes for relations $\mathbf{R}_4$ and $\mathbf{R}_7$.

For relation $\mathbf{R}_4$: What elements of $\mathbb{N}$ are in the same equivalence class as 1? We will call this equivalence class E_1. We have:

$$\begin{aligned}
E_1 &= \{m \in \mathbb{N} : 1\mathbf{R}_4 m \text{ is true}\} \\
&= \{m \in \mathbb{N} : 1 - m \text{ is even}\} \\
&= \{m \in \mathbb{N} : 1 - m = 2 \cdot k \text{ for some integer } k\} \\
&= \{m \in \mathbb{N} : m = 2 \cdot (-k) + 1 \text{ for some integer } k\} \\
&= \{m \in \mathbb{N} : m = 2 \cdot l + 1 \text{ for some integer } l\} \\
&= \{m \in \mathbb{N} : m \text{ is odd}\}
\end{aligned}$$

so apparently the odd natural numbers form the equivalence class E_1. Now what elements of $\mathbb{N}$ are in the same equivalence class as 2? This equivalence class is called E_2. We have:

$$\begin{aligned}
E_2 &= \{m \in \mathbb{N} : 2\mathbf{R}_4 m \text{ is true}\} \\
&= \{m \in \mathbb{N} : 2 - m \text{ is even}\} \\
&= \{m \in \mathbb{N} : 2 - m = 2 \cdot k \text{ for some integer } k\} \\
&= \{m \in \mathbb{N} : m = 2 \cdot (-k) + 2 \text{ for some integer } k\} \\
&= \{m \in \mathbb{N} : m = 2 \cdot l \text{ for some integer } l\} \\
&= \{m \in \mathbb{N} : m \text{ is even}\}
\end{aligned}$$

so apparently the even natural numbers form the equivalence class E_2. Since $E_1 \cup E_2 = \mathbb{N}$, we have found all equivalence classes of $\mathbf{R}_4$.

For relation $\mathbf{R}_7$: What elements of $\mathbb{R}^2$ are in the same equivalence class as $(0, 0)$? We will call this equivalence class E_0 for reasons that become obvious soon. We have:

$$\begin{aligned}
E_0 &= \big\{(x, y) \in \mathbb{R}^2 : (0, 0)\, \mathbf{R}_7 (x, y) \text{ is true}\big\} \\
&= \big\{(x, y) \in \mathbb{R}^2 : 0 + 0 = x + y\big\} \\
&= \big\{(x, y) \in \mathbb{R}^2 : x + y = 0\big\}
\end{aligned}$$

so apparently all elements of $\mathbb{R}^2$ whose coordinates sum to 0 are in equivalence class E_0. Now what elements of $\mathbb{R}^2$ are in the same equivalence class as $(1, 0)$? This equivalence class is called

E_1. We have:

$$\begin{aligned} E_1 &= \big\{(x,y) \in \mathbb{R}^2 : (1,0)\,\mathbf{R}_7(x,y) \text{ is true}\big\} \\ &= \big\{(x,y) \in \mathbb{R}^2 : 1+0 = x+y\big\} \\ &= \big\{(x,y) \in \mathbb{R}^2 : x+y = 1\big\} \end{aligned}$$

so apparently all elements of $\mathbb{R}^2$ whose coordinates sum to 1 are in equivalence class E_1. This holds in general: For all $c \in \mathbb{R}$ let

$$E_c = \big\{(x,y) \in \mathbb{R}^2 : x+y = c\big\}$$

be the equivalence class of relation $\mathbf{R}_7$ consisting of all elements in $\mathbb{R}^2$ whose coordinates sum to c. Then every element of $\mathbb{R}^2$ is in exactly one equivalence class:

$$\begin{aligned} &E_p \cap E_q = \emptyset \text{ if } p \neq q \text{ and} \\ &\bigcup_{c\in\mathbb{R}} E_c = \mathbb{R}^2, \end{aligned}$$

making the sets E_c a partition of $\mathbb{R}^2$. Here $\bigcup\limits_{c\in\mathbb{R}} E_c$ is the union of all E_c's (notice that c can have any real value).

Exercise 9 *The relation in exercise 7 turned out to be reflexive, symmetric and transitive. Find its equivalence classes.*

Exercise 10 *Show that the following 2 relations are equivalence relations and determine their equivalence classes:*

(ix) $x\mathbf{R}_9y \Leftrightarrow x$ *has the same remainder on division by 5 as* y *on* $S = \{0,1,2,\ldots,20\}$*. Other formulations that mean the same are "*x *is congruent modulo 5 to* y*" and, mathematically, "*$x \equiv y \pmod 5$*" (also on* $\mathbb{N}$*) (example 2 in section 3.2)*

(x) $x\mathbf{R}_{10}y \Leftrightarrow x \cdot y$ *is a (perfect) square on* $S = \{1,2,\ldots,20\}$ *(also on* $\mathbb{N}$*) (exercise 3 in section 3.2)*

Section 3.3

A fourth property that a relation may have is antisymmetry: A relation $\mathbf{R}$ is **antisymmetric** on S if the following proposition is true:

$$\forall x \in S\ \forall y \in S : (x\mathbf{R}y \wedge y\mathbf{R}x) \Rightarrow x = y$$

This proposition is equivalent to the proposition (you can check this out by means of a truth table!):

$$\forall x \in S\ \forall y \in S : (x \neq y\ \wedge\ x\mathbf{R}y) \Rightarrow \neg(y\mathbf{R}x)$$

This second, alternative definition of antisymmetry is in some cases easier to apply when it comes to proving that $\mathbf{R}$ is antisymmetric. It is important to realize that a relation $\mathbf{R}$ being antisymmetric is not the same thing as $\mathbf{R}$ not being symmetric.

Example 6 *We prove that in example 1 relations* $\mathbf{R}_1$ *and* $\mathbf{R}_3$ *are antisymmetric and that* $\mathbf{R}_4$ *and* $\mathbf{R}_6$ *are not.*

$\mathbf{R}_1$ *is antisymmetric.* ***Proof:*** *Let* $x \in \mathbb{N}$, $y \in \mathbb{N}$ *and suppose that* $x \neq y$ *and that* $x\mathbf{R}y$ *is true. Then* $x \neq y$ *and* x *is a multiple of* y. *But then* $x \geq 2y$ *and hence* y *is not a multiple of* x.

$\mathbf{R}_3$ *is antisymmetric.* ***Proof:*** *Let* $x \in [0,1]$, $y \in [0,1]$ *and suppose that* $x\mathbf{R}y$ *is true and that* $y\mathbf{R}x$ *is true. Then* $xy = y$ *and* $yx = x$. *But then* $x = xy = y$.

$\mathbf{R}_4$ *is not antisymmetric.* ***Proof:*** *(Counterexample): Take* $x = 1 \in \mathbb{N}$, $y = 3 \in \mathbb{N}$. *Then* $x - y$ *is even and* $y - x$ *is even, so* $x\mathbf{R}_4y$ *is true and* $y\mathbf{R}_4x$ *is true. However,* $x \neq y$.

$\mathbf{R}_6$ *is not antisymmetric.* ***Proof:*** *(Counterexample): Take* $(x,y) = (1,0) \in \mathbb{R}^2$, $(a,b) = (0,1) \in \mathbb{R}^2$. *Then* $x^2 + y^2 \leq a^2 + b^2$ *and* $a^2 + b^2 \leq x^2 + y^2$, *so* $(x,y)\mathbf{R}_6(a,b)$ *is true and* $(a,b)\mathbf{R}_6(x,y)$ *is true, but* $(x,y) \neq (a,b)$.

Exercise 11 *Let* $S = \{1,2,3\}$. *Find a relation on* S *that is both symmetric and antisymmetric.*

Exercise 12 *For relations* $\mathbf{R}_2$, $\mathbf{R}_5$, $\mathbf{R}_7$ *and* $\mathbf{R}_8$ *in example 1 as well as relations* $\mathbf{R}_9$ *and* $\mathbf{R}_{10}$ *in exercise 4, prove whether or not they are antisymmetric.*

Exercise 13 *For the relation in example 2, determine whether or not it is antisymmetric.*

Exercise 14 *For each relation in exercise 6 find out what conditions on the truth values are required to make it antisymmetric.*

Exercise 15 *Is the relation in exercise 7 antisymmetric?*

Exercise 16 *How many of the relations in exercise 1 are antisymmetric?*

Relations that are reflexive, antisymmetric and transitive are called **partial orders**, because in some way they order the elements in the set; the corresponding set is then called a **partially ordered set**. An example of such a relation is the $\leq$ -relation on e.g. $\mathbb{R}$. In example 1, relations $\mathbf{R}_1$ and $\mathbf{R}_8$ are partial orders.

Section 3.5

It is also possible to define a relation between two sets A and B. Similarly to definition 1, a relation between A and B is a subset of the product set $A \times B$. In the course we are only interested in relations $\mathbf{R}$ between A and B that satisfy the following property:

For every $a \in A$ there is *exactly* one $b \in B$ such that $a\mathbf{R}b$ is true.

A relation $\mathbf{R}$ that satisties this property, is called a **function**. We usually denote it by a small case letter, e.g. f. The set A is then called the **domain** of the function f and B is called the **codomain**. Notation:

$$f : A \to B$$

So, the function f assigns to every element x in its domain an element y in its codomain. We say that $f(x) = y$. Apparently, the codomain is the set of potential outcomes of f. The set of actual outcomes of f, which is a subset of the codomain, is called the **range** of f. A mathematical description of the range of a function f is $\{f(x) \in B : x \in A\}$.

Important: For every $x \in A$ its function value $f(x)$ **must** be in B; otherwise f is not a function! Furthermore, f can not assign more than one value out of the codomain to any x out of the domain.

Example 7 *Some examples of relations between two sets. Most of them are functions; some are not:*

1. $\mathbf{R_1} = \{(1,7),(3,3),(5,25)\}$ *on* $\{1,3,5\} \times \{3,7,25\}$ *is a function, say* f_1*, with domain* $\{1,3,5\}$ *and codomain* $\{3,7,25\}$*. We write* $f_1 : \{1,3,5\} \to \{3,7,25\}$*, where* $f_1(1) = 7$*,* $f_1(3) = 3$ *and* $f_1(5) = 25$*. Its range is* $\{3,7,25\}$*.*

2. $\mathbf{R_2} = \{(1,25),(3,3),(5,25)\}$ *on* $\{1,3,5\} \times \{3,25\}$ *is a function, say* f_2*, with domain* $\{1,3,5\}$ *and codomain* $\{3,25\}$*. We write* $f_2 : \{1,3,5\} \to \{3,25\}$*, where* $f_2(1) = 25$*,* $f_2(3) = 3$ *and* $f_2(5) = 25$*. Its range is* $\{3,25\}$*.*

3. $\mathbf{R_3} = \{(1,25),(3,3),(5,25)\}$ *on* $\{1,3,5\} \times \{3,7,25\}$ *is a function, say* f_3*, with domain* $\{1,3,5\}$ *and codomain* $\{3,7,25\}$*. We write* $f_3 : \{1,3,5\} \to \{3,7,25\}$*, where* $f_3(1) = 25$*,* $f_3(3) = 3$ *and* $f_3(5) = 25$*. Its range is* $\{3,25\}$*.*

4. $\mathbf{R_4} = \{(1,25),(3,3)\}$ *on* $\{1,3,5\} \times \{3,25\}$ *is not a function, because it assigns nothing to* 5*.*

5. $f_5 : \{1,2,3,4,5\} \to \mathbb{N}$*, where* $f_5(x) = 3x$ *is a function. It has domain* $\{1,2,3,4,5\}$*, codomain* $\mathbb{N}$ *and range* $\{3,6,9,12,15\}$*. In relation-form it looks like*

$$\mathbf{R_5} = \{(1,3),(2,6),(3,9),(4,12),(5,15)\} \text{ on } \{1,2,3,4,5\} \times \mathbb{N}.$$

6. $f_6 : \{1,2,3,4,5\} \to \mathbb{N}$*, where* $f_6(1) = 3$ *and* $f_6(x) = 3x$ *is a function. It has domain* $\{1,2,3,4,5\}$*, codomain* $\mathbb{N}$ *and range* $\{3,6,9,12,15\}$*. It does not matter that in the function description it says twice that* $f_6(1) = 3$*. In relation-form it looks like*

$$\mathbf{R_6} = \{(1,3),(2,6),(3,9),(4,12),(5,15)\} \text{ on } \{1,2,3,4,5\} \times \mathbb{N}.$$

7. $f_7 : \{0,1,2,3,4,5\} \to \mathbb{N}$*, where* $f_7(x) = 3x$ *is not a function. In fact it is not even a relation between* $\{0,1,2,3,4,5\}$ *and* $\mathbb{N}$*, because* $f_7(0) = 0 \notin \mathbb{N}$*.*

8. $f_8 : \{1,2,3,4,5\} \to \mathbb{N}$*, where* $f_8(x) = x^2$ *is a function. It has domain* $\{1,2,3,4,5\}$*, codomain* $\mathbb{N}$ *and range* $\{1,4,9,16,25\}$*. In relation-form it looks like*

$$\mathbf{R_8} = \{(1,1),(2,4),(3,9),(4,16),(5,25)\} \text{ on } \{1,2,3,4,5\} \times \mathbb{N}.$$

9. $f_9 : \mathbb{N} \to \mathbb{N}$*, where* $f_9(x) = (x-1)^2$ *is not a function. In fact it is not even a relation between* $\mathbb{N}$ *and* $\mathbb{N}$ *(or simply a relation on* $\mathbb{N}$*!), because* $f_9(1) = 0 \notin \mathbb{N}$*.*

10. $f_{10} : \mathbb{R} \to \mathbb{R}$, *where* $f_{10}(x) = (x-1)^2$ *is a function. It has domain* $\mathbb{R}$, *codomain* $\mathbb{R}$ *and range* $[0, \infty)$. *We can write it in relation-form as follows:* $x\mathbf{R}_{10}y \;\Leftrightarrow\; y = (x-1)^2$ *on* $\mathbb{R}$.

11. $f_{11} : \mathbb{N} \to \mathbb{N}$, *where* $f_{11}(1) = 3$ *and* $f_{11}(x) = x$ *is not a function, because it assigns 2 values to* $x = 1$. *It corresponds to the relation*

$$\mathbf{R}_{11} = \{(1,3),(1,1),(2,2),(3,3),(4,4),(5,5),\ldots\} \text{ between } \mathbb{N} \text{ and } \mathbb{N} \text{ (or on } \mathbb{N}).$$

12. $f_{12} : \mathbb{N} \to \mathbb{N}$, *where* $f(x) = \begin{cases} 3 & \text{if } x = 1 \\ x & \text{if } x \neq 1 \end{cases}$ *is a function. It has domain* $\mathbb{N}$, *codomain* $\mathbb{N}$ *and range* $\{2,3,4,5,\ldots\} = \mathbb{N}\backslash\{1\}$. *In relation-form it looks like:*

$$\mathbf{R}_{12} = \{(1,3),(2,2),(3,3),(4,4),(5,5),\ldots\} \text{ between } \mathbb{N} \text{ and } \mathbb{N} \text{ (or on } \mathbb{N}).$$

Of course the sets A and B do not necessarily consist of numbers. Below is an example of what in Game Theory is called the *value-function* or *characteristic function* of a 3-player *Cooperative Game*:

Example 8 $v : \mathcal{P}(\{1,2,3\}) \to \mathbb{R}$, *where*

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	1	2	3	4	5	6	10

In this example we have a set of players $\{1,2,3\}$ *and to each subset* S *of this set of players a value* $v(S)$ *is assigned. The function* v *has domain* $\mathcal{P}(\{1,2,3\})$, *codomain* $\mathbb{R}$ *and range* $\{0,1,2,3,4,5,6,10\}$.

Exercise 17 *What does the function in example 8 look like in relation-form?*

Exercise 18 *Let* A *and* B *be sets with* $|A| = m$ *and* $|B| = n$. *How many functions* $f : A \to B$ *exist?*

Section 3.6

We start this section by describing some basic mathematical tools that are useful to solve some of the exercises from this section as well as some later sections.

1. A simple observation: For all real numbers x and y and all natural numbers $n \geq 2$ we have:
$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^{n-1}) \tag{1}$$
You can see that this equation is true by expanding the product on the right hand side and then noticing that everything cancels out except the terms x^n and $-y^n$. Two specific cases: For $n = 2$ this gives the well-known equation
$$x^2 - y^2 = (x-y)(x+y);$$
for $n = 3$ we find
$$x^3 - y^3 = (x-y)(x^2 + xy + y^2).$$

It can also be used to find factors $(x-y)$ in expressions with roots. For example:

- $(\sqrt{x}-\sqrt{y}) = (x-y)\cdot\frac{1}{\sqrt{x}+\sqrt{y}}$, since $x-y = (\sqrt{x}-\sqrt{y})\cdot(\sqrt{x}+\sqrt{y})$;
- $(\sqrt[3]{x}-\sqrt[3]{y}) = (x-y)\cdot\frac{1}{\sqrt[3]{x^2}+\sqrt[3]{xy}+\sqrt[3]{y^2}}$, since $x-y = (\sqrt[3]{x}-\sqrt[3]{y})\cdot(\sqrt[3]{x^2}+\sqrt[3]{xy}+\sqrt[3]{y^2})$.

See example 10 *(iii), (iv^a) & (v^a)* for examples of how to use this in an exercise.

2. **Completing the Square**: If $y = f(x)$ is a quadratic function, i.e. $y = ax^2+bx+c$ with $a \neq 0$, then we can manipulate the expression as follows: First we write

$$\begin{aligned}\tfrac{y}{a} &= x^2 + \tfrac{b}{a}x + \tfrac{c}{a}\\ &= x^2 + \tfrac{b}{a}x + \tfrac{b^2}{4a^2} - \tfrac{b^2}{4a^2} + \tfrac{c}{a}\\ &= (x+\tfrac{b}{2a})^2 - \tfrac{b^2}{4a^2} + \tfrac{c}{a},\end{aligned}$$

or $y = a(x+\frac{b}{2a})^2 - \frac{b^2}{4a} + c$. Notice that there is no longer a term of the form bx in the resulting expression and instead of x^2 we now have $(x+\frac{b}{2a})^2$. We have essentially 'completed' the square of $x+\frac{b}{2a}$, explaining the name of the technique.

Two examples: 1. $x^2-2x+5 = (x-1)^2+4$ (here $a=1$, $b=-2$ and $c=5$).
2. If $y = 3x^2+2x-6$, then $\frac{y}{3} = x^2+\frac{2}{3}x-2 = (x+\frac{1}{3})^2 - 2\frac{1}{9}$, so $y = 3\cdot(x+\frac{1}{3})^2 - 6\frac{1}{3}$.

The Completing-the-Square-technique will be used to write the equation $y = ax^2+bx+c$ in the form $x = g(y)$, which is often needed to prove that a function is surjective. It is done as follows: From $\frac{y}{a} = (x+\frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a}$ we deduce that $(x+\frac{b}{2a})^2 = \frac{y}{a} + \frac{b^2}{4a^2} - \frac{c}{a}$, which can be solved for x.

3. **Absolute value**: Let $x \in \mathbb{R}$. The absolute value of x, notation $|x|$, is defined as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0\\ -x & \text{if } x < 0\end{cases}$$

The absolute value essentially calculates the distance between the expression between the $|\ |$ signs and 0. Since it is a distance, it is of course always non-negative.

Two examples: 1. $|x+3|$: $x+3 \geq 0$ if $x \geq -3$, so $|x+3| = x+3$ if $x \geq -3$ and $|x+3| = -(x+3) = -x-3$ if $x < 3$.
2. $|x^2+7x-18|$: Notice first that $x^2+7x-18 = (x+9)(x-2)$. So, $x^2x+7x-18 \geq 0$ if $x \geq 2 \quad\vee\quad x \leq -9$. Consequently, $|x^2+7x-18| = x^2+7x-18$ if $x \geq 2 \quad\vee\quad x \leq -9$ and $|x^2+7x-18| = -(x^2+7x-18) = -x^2-7x+18$ if $-9 < x < 2$.

It is useful to notice that $|x| = \sqrt{x^2}$ (and be aware that the square root symbol represents the unique *positive* square root).

See example 12 *(iii)* for an example of how to use the completing the square technique and the absolute value in an exercise.

Exercise 19

a Write the following sets as intervals:

i $\{x \in \mathbb{R} \,:\, |x| < 3\}$
ii $\{x \in \mathbb{R} \,:\, |x - 2| \leq 2\}$
iii $\{x \in \mathbb{R} \,:\, |x + 4| < 3\}$
iv $\{x \in \mathbb{R} \,:\, |x + 7| \leq 9\}$

b Write the open interval $(6, 11)$ *and the closed interval* $[-2, 5]$ *by means of an absolute value.*

Two properties of functions:

A function f is called **injective** (or **one-one**) if different values of x always yield different values of $f(x)$, or, more formally:

$$\forall x \in A\ \forall y \in A : f(x) = f(y) \Rightarrow x = y$$

Notice that above proposition is the contrapositive of the more intuitive proposition

$$\forall x \in A\ \forall y \in A : x \neq y \Rightarrow f(x) \neq f(y)$$

This means that the two propositions are equivalent and both may be used to prove that a function is injective. The first definition is often the easiest to use, especially if the domain of a function is of infinite size.

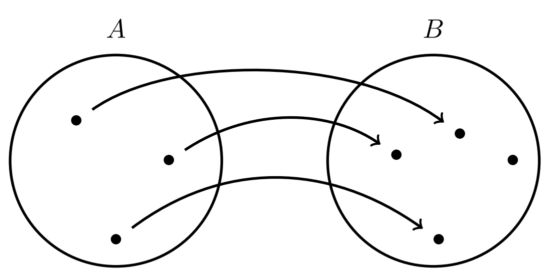
Example 9 *We will prove that in example 7 functions* f_1 *is injective and that* f_2 *is not.*

f_1 *is injective.* ***Proof:*** *Let* $x, y \in \{1, 3, 5\}$ *and assume that* $x \neq y$*. Then* $f(x) \neq f(y)$*, so* f_1 *is injective.*

f_2 *is not injective.* ***Proof:*** *(Counterexample) Take* $x = 1$ *and* $y = 5$*. Then* $f(x) = f(y)$*, but* $x \neq y$*, so* f_2 *is not injective.*

Exercise 20 *For the functions* f_3*,* f_5*,* f_6*,* f_8*,* f_{10} *and* f_{12} *in Example 7, determine whether or not they are injective.*

For a function $f : A \to B$ to be injective it is necessary that $|A| \leq |B|$, since otherwise we could not possibly assign a different element out of B to every element out of A. The following picture shows this:



Injective function

Example 10 *Some examples:*

(i) $f : \{1, 3, 5, 7, \ldots\} \to \{0, 1, 2, 3, 4, \ldots\}$, *where* $f(x) = \frac{1}{2}x - \frac{1}{2}$ *is injective.*

Proof: *Let* $x, y \in \{1, 3, 5, 7, \ldots\}$. *Then*

$$\begin{aligned} f(x) = f(y) \quad &\Rightarrow \quad \tfrac{1}{2}x - \tfrac{1}{2} = \tfrac{1}{2}y - \tfrac{1}{2} \\ &\Rightarrow \quad \tfrac{1}{2}x = \tfrac{1}{2}y \\ &\Rightarrow \quad x = y, \end{aligned}$$

so f *is injective.*

(ii) $f : \{2, 4, 6, 8, \ldots\} \to \{\ldots, -4, -3, -2, -1\}$, *where* $f(x) = -\frac{1}{2}x$ *is injective.*

Proof: *Let* $x, y \in \{2, 4, 6, 8, \ldots\}$. *Then*

$$\begin{aligned} f(x) = f(y) \quad &\Rightarrow \quad -\tfrac{1}{2}x = -\tfrac{1}{2}y \\ &\Rightarrow \quad x = y, \end{aligned}$$

so f *is injective.*

(iii) $f : [1, \infty) \to [-1, \infty)$, *where* $f(x) = x^2 - 2x$ *is injective.*

Proof: *Let* $x \in [1, \infty)$, $y \in [1, \infty)$. *Then*

$$\begin{aligned} f(x) = f(y) \quad &\Rightarrow \quad x^2 - 2x = y^2 - 2y \\ &\Rightarrow \quad x^2 - 2x - y^2 + 2y = 0 \\ &\Rightarrow \quad x^2 - y^2 - 2x + 2y = 0 \\ &\Rightarrow \quad (x - y)(x + y) - 2(x - y) = 0 \\ &\Rightarrow \quad (x - y)\,(x + y - 2) = 0 \\ &\Rightarrow \quad x - y = 0, \end{aligned}$$

since $x \geq 1$ *and* $y \geq 1$. *Hence* f *is injective.*

(iv^a) $f : [1, 2] \to \mathbb{R}$, *where* $f(x) = x^3 - x + 1$ *is injective.*

Proof: *Let* $x, y \in [1, 2]$. *Then*

$$\begin{aligned} f(x) = f(y) \quad &\Rightarrow \quad x^3 - x + 1 = y^3 - y + 1 \\ &\Rightarrow \quad x^3 - y^3 - x + y = 0 \\ &\Rightarrow \quad (x - y)(x^2 + xy + y^2) - x + y = 0 \qquad \textit{(by equation (1))} \\ &\Rightarrow \quad (x - y)(x^2 + xy + y^2 - 1) = 0 \end{aligned}$$

which means that $x - y = 0 \lor (x^2 + xy + y^2 - 1) = 0$. *Now, because* $x, y \in [1, 2]$, *we have* $x \geq 1, y \geq 1$. *Consequently,* $x^2 + xy + y^2 - 1 \geq 1^2 + 1 \cdot 1 + 1^2 - 1 = 2$, *so it is never equal to 0. Hence* $x - y = 0$, *or* $x = y$.

(iv^b) $f : [0, 1] \to \mathbb{R}$, *where* $f(x) = x^3 - x + 1$ *is not injective.*

((Draft: Let $x, y \in [0, 1]$. *Doing the same calculation as in (iv^a), again gives* $x - y = 0 \lor (x^2 + xy + y^2 - 1) = 0$. *In this case, however, it is possible to find values of* x *and* y *such that* $(x^2 + xy + y^2 - 1) = 0$, *for instance* $x = 0$ *and* $y = 1$.*))*

Proof: *(Counterexample) Take* $x = 0$ *and* $y = 1$. *Then* $f(x) = f(y)$ *(both are equal to 1), but* $x \neq y$.

(v^a) *$f : [1,2] \to \mathbb{R}$, where $f(x) = \sqrt[3]{x} - 2x$ is injective.*

Proof: *Let $x, y \in [1,2]$. Then*

$$\begin{array}{rcll} f(x) = f(y) & \Rightarrow & \sqrt[3]{x} - 2x = \sqrt[3]{y} - 2y & \\ & \Rightarrow & \sqrt[3]{x} - \sqrt[3]{y} - 2x + 2y = 0 & \\ & \Rightarrow & (x-y)\cdot \frac{1}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}} - 2x + 2y = 0 & \text{(by equation (1))} \\ & \Rightarrow & (x-y)\cdot (\frac{1}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}} - 2) = 0 & \end{array}$$

which means that $x - y = 0 \vee \frac{1}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}} - 2 = 0$. Now, because $x, y \in [1,2]$, we have $x \geq 1, y \geq 1$. Consequently, $\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2} \geq 3$, so $\frac{1}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}} - 2 \leq \frac{1}{3} - 2 = -1\frac{2}{3}$, so it is never equal to 0. Hence $x - y = 0$, or $x = y$.

(v^b) *$f : [0,1] \to \mathbb{R}$, where $f(x) = \sqrt[3]{x} - 2x$ is not injective.*

Proof: *We are going to find two numbers in $[0,1]$ that have the same function values. Notice first that $f(0) = 0$. But $\sqrt[3]{x} - 2x = x \cdot (\frac{1}{\sqrt[3]{x^2}} - 2)$, which is equal to 0 if and only if $x = 0 \vee \frac{1}{\sqrt[3]{x^2}} - 2 (= x^{-2/3} - 2) = 0$. The solution of the second equation is $x = \frac{1}{4}\sqrt{2}$ which is in $[0,1]$. So our proof by counterexample goes as follows:*
Let $x = 0 \in [0,1]$ and $y = \frac{1}{4}\sqrt{2} \in [0,1]$. Then $f(x) = f(y) = 0$, but $x \neq y$.

(vi) *$f : \mathbb{N} \to \mathbb{Z}$, where $f(x) = \begin{cases} \frac{1}{2}x - \frac{1}{2} & \text{if } x \text{ is odd} \\ -\frac{1}{2}x & \text{if } x \text{ is even} \end{cases}$ is injective.*

Proof: *Notice first that this example is a combination of examples (i) and (ii). Let $x \in \mathbb{N}$, $y \in \mathbb{N}$. We split the analysis in 3 parts. First suppose that x and y are both odd. Then*

$$\begin{array}{rcl} f(x) = f(y) & \Rightarrow & \frac{1}{2}x - \frac{1}{2} = \frac{1}{2}y - \frac{1}{2} \\ & \Rightarrow & \frac{1}{2}x = \frac{1}{2}y \\ & \Rightarrow & x = y. \end{array}$$

Now suppose that x and y are both even. Then

$$\begin{array}{rcl} f(x) = f(y) & \Rightarrow & -\frac{1}{2}x = -\frac{1}{2}y \\ & \Rightarrow & x = y. \end{array}$$

Now suppose that x is odd and y is even. Then $x \neq y$, but also $f(x) = \frac{1}{2}x - \frac{1}{2} \geq 0$ and $f(y) = -\frac{1}{2}y < 0$, so $f(x) \neq f(y)$. Consequently the implication $(x \neq y) \Rightarrow (f(x) \neq f(y))$ is true. The same argument can be applied if x is even and y is odd. Hence f is injective.

(vii) *$f : \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\} \to \mathbb{N}$, where $f(x) = \frac{1}{x} - 1$ is injective.*

Proof: *Let $x, y \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$. Then*

$$\begin{array}{rcl} f(x) = f(y) & \Rightarrow & \frac{1}{x} - 1 = \frac{1}{y} - 1 \\ & \Rightarrow & \frac{1}{x} = \frac{1}{y} \\ & \Rightarrow & x = y, \end{array}$$

so f is injective.

(viii) $f : (0,1) \to [1,\infty)$, *where* $f(x) = \begin{cases} \frac{1}{x} - 1 & \text{if } x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\} \\ \frac{1}{x} & \text{if } x \notin \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\} \end{cases}$ *is injective.*

Proof: *Notice first that this example is an extension of example (vii). Let* $x \in (0,1)$, $y \in (0,1)$. *We split the analysis into 3 parts. First suppose that* $x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$, $y \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$. *Then*

$$\begin{aligned} f(x) = f(y) \quad &\Rightarrow \quad \frac{1}{x} - 1 = \frac{1}{y} - 1 \\ &\Rightarrow \quad \frac{1}{x} = \frac{1}{y} \\ &\Rightarrow \quad x = y. \end{aligned}$$

Now suppose that $x \notin \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$, $y \notin \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$. *Then*

$$\begin{aligned} f(x) = f(y) \quad &\Rightarrow \quad \frac{1}{x} = \frac{1}{y} \\ &\Rightarrow \quad x = y. \end{aligned}$$

Now suppose that $x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$, $y \notin \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$. *Then* $x \neq y$, *but also* $f(x) = \frac{1}{x} - 1 \in \mathbb{N}$, *and* $f(y) = \frac{1}{y} \notin \mathbb{N}$, *so* $f(x) \neq f(y)$. *Consequently the implication* $(x \neq y) \Rightarrow (f(x) \neq f(y))$ *is true. The same argument can be applied if* $x \notin \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$ *and* $y \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$. *Hence* f *is injective.*

Exercise 21 *For the following functions prove whether or not they are injective.*

a) $g_1 : (0,1] \to [1,\infty)$, *where* $g_1(x) = \frac{1}{x}$.

b) $g_2 : [2,4] \to [5,9]$, *where* $g_2(x) = 2x+1$.

c) $g_3 : [0,3] \to [0,7]$, *where* $g_3(x) = 2x+1$.

d) $g_4 : (0,1) \to (1,4)$, *where* $g_4(x) = \frac{8}{6x+2}$.

e) $g_5 : [5,7] \to [1,5]$, *where* $g_5(x) = \frac{x+1}{-2x+16}$.

f) $g_6 : [0,5] \to [-4,\infty)$, *where* $g_6(x) = x^2 + x - 4$.

g) $g_7 : [0,5] \to [-5,\infty)$, *where* $g_7(x) = x^2 - x - 4$.

h) $g_8 : [1,5] \to [1,3]$, *where* $g_8(x) = \sqrt{2x-1}$.

i) $g_9 : [-3,1) \to (0,1]$, *where* $g_9(x) = \frac{1}{x^2+1}$.

j) $g_{10} : (0,2] \to [\frac{1}{5},1)$, *where* $g_{10}(x) = \frac{1}{x^2+1}$.

k) $g_{11} : [0,5] \to \mathbb{R}$, *where* $g_{11}(x) = x^4 - 3x^3 + 1$.

l) $g_{12} : [3,4] \to \mathbb{R}$, *where* $g_{12}(x) = x^4 - 3x^3 + 1$.

m) $g_{13} : [0,3] \to \mathbb{R}$, *where* $g_{13}(x) = \sqrt[3]{4x} - x$.

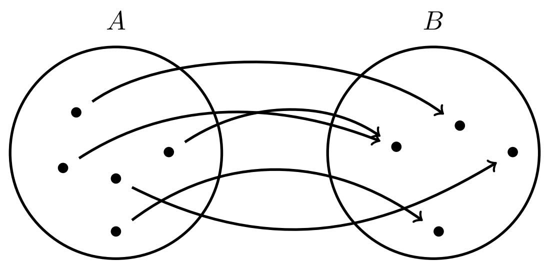
n) $g_{14} : [1,2] \to \mathbb{R}$, *where* $g_{14}(x) = \sqrt[3]{4x} - x$.

A function $f : A \to B$ is called **surjective** (or **onto**) if the range is equal to the codomain or, more formally:

$$\forall y \in B \; \exists x \in A : f(x) = y$$

None of the functions in the examples of section 3.5 is surjective, as we can easily find natural numbers that can not be obtained as outcomes of the mentioned functions. The functions in examples (i) and (ii) of this section are not surjective; in each case we can find a number in the codomain that is not in the range.

For a function $f : A \to B$ to be surjective it is necessary that $|A| \geq |B|$, since otherwise it would not be possible to assign every element out of B to an element out of A. The following picture shows this:



Surjective function

Again, knowing that $|A| \geq |B|$ is not sufficient for $f : A \to B$ to be surjective.

Example 11 *Of the functions in Example 7, f_1 and f_2 are surjective, whereas f_3, f_5, f_6, f_8, f_{10} and f_{12} are not. This is very easy to find out: just check if the codomain and the range are equal!*

Of course in general it is not that easy to prove that a function is surjective; in fact, sometimes a few little tricks are required before the final conclusion can be made. Obviously a counterexample (in this case an element of the codomain that is not in the range of the function) will always suffice to show that a function is not surjective.

Example 12 *We will prove for the functions in Example 10 whether or not they are surjective.*

(i) $f : \{1, 3, 5, 7, \ldots\} \to \{0, 1, 2, 3, 4, \ldots\}$, *where* $f(x) = \frac{1}{2}x - \frac{1}{2}$ *is surjective.*
Proof: *Let* $y \in \{0, 1, 2, 3, 4, \ldots\}$ *and take* $x = 2y + 1 (\in \{1, 3, 5, 7, \ldots\})$. *Then*

$$\begin{aligned} f(x) &= \tfrac{1}{2}x - \tfrac{1}{2} \\ &= \tfrac{1}{2}(2y + 1) - \tfrac{1}{2} \\ &= y + \tfrac{1}{2} - \tfrac{1}{2} = y, \end{aligned}$$

so f is surjective.

An important question about this proof is how we managed to figure out that we should take $x = 2y+1$. The answer: We did a preliminary calculation. We knew in advance that *we had to come up with an odd-valued x such that $f(x) = y = \frac{1}{2}x - \frac{1}{2}$. Some calculations were then applied to this equation so that it would read $x = g(y)$ (x is some function of y). This was done as follows:*

$$\begin{array}{cll} & y = \frac{1}{2}x - \frac{1}{2} & \textit{(add } \tfrac{1}{2}\textit{)} \\ \Leftrightarrow & y + \frac{1}{2} = \frac{1}{2}x & \textit{(multiply by 2 and put } x \textit{ on the left hand side)} \\ \Leftrightarrow & x = 2y + 1 & \end{array}$$

(ii) *$f : \{2, 4, 6, 8, \ldots\} \to \{\ldots, -4, -3, -2, -1\}$, where $f(x) = -\frac{1}{2}x$ is surjective.*

Proof: *Let $y \in \{\ldots, -4, -3, -2, -1\}$ and take $x = -2y$. Then*

$$f(x) = -\tfrac{1}{2}x = -\tfrac{1}{2} \cdot -2y = y,$$

so f is surjective.

The preliminary calculation: We had to come up with an even-valued x such that $f(x) = y = -\frac{1}{2}x$, which would lead to $x = -2y$. This is the general procedure to prove that a function is surjective; you always have to do some preliminary calculations transform the equation $y = f(x)$ into $x = g(y)$.

(iii) *$f : [1, \infty) \to [-1, \infty)$, where $f(x) = x^2 - 2x$ is surjective.*

Proof: *Let $y \in [-1, \infty)$ and take $x = \sqrt{y+1} + 1 \in [1, \infty)$. Then*

$$\begin{aligned} f(x) &= (\sqrt{y+1} + 1)^2 - 2(\sqrt{y+1} + 1) \\ &= y + 1 + 2\sqrt{y+1} + 1 - 2\sqrt{y+1} - 2 \\ &= y. \end{aligned}$$

The preliminary calculation: Here we use the 'completing the square' technique, discussed at the start of the section. In this exercise we have $y = x^2 - 2x$, which we can rewrite as $y = (x-1)^2 - 1$. This means that

$$\begin{array}{cl} & y + 1 = (x-1)^2 \\ \Rightarrow & \sqrt{y+1} = |x - 1| \textit{ (the absolute value of } x - 1\textit{)} \\ \Rightarrow & \sqrt{y+1} = x - 1 \textit{ , since } x \geq 1 \\ \Rightarrow & x = \sqrt{y+1} + 1 \end{array}$$

(iv&v)[a&b] *For functions that have powers greater than 2 or n^{th} roots it is often very difficult to find the range. In all examples here the functions are not surjective. You can argue that the function values never get bigger than, say, 100 so the range can not possibly be equal to the codomain ($\mathbb{R}$).*

(vi) *$f : \mathbb{N} \to \mathbb{Z}$, where $f(x) = \begin{cases} \frac{1}{2}x - \frac{1}{2} & \textit{if } x \textit{ is odd} \\ -\frac{1}{2}x & \textit{if } x \textit{ is even} \end{cases}$ is surjective.*

Proof: *Let $y \in \mathbb{Z}$. The analysis is split into 2 parts that correspond to how the function*

f is constructed. Notice that for every odd x the corresponding $y = \frac{1}{2}x - \frac{1}{2}$ is nonnegative, whereas the y corresponding to an even x-value is always negative. That's why we split the analysis into negative and nonnegative values for y. So:

Case 1: *Suppose that $y \geq 0$ and take $x = 2y+1 \in \mathbb{N}$. Then x is odd, so $f(x) = \frac{1}{2}x - \frac{1}{2} = \frac{1}{2}(2y+1) - \frac{1}{2} = y$.*

Case 2: *Now suppose that $y < 0$ and take $x = -2y \in \mathbb{N}$. Then x is even, so $f(x) = f(-2y) = -\frac{1}{2} \cdot (-2y) = y$, which completes the proof.*

(vii) *$f : \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\} \to \mathbb{N}$, where $f(x) = \frac{1}{x} - 1$ is surjective.*

Proof: *Let $y \in \mathbb{N}$ and take $x = \frac{1}{y+1} \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$. Then*

$$\begin{aligned} f(x) &= \tfrac{1}{x} - 1 \\ &= \frac{1}{\frac{1}{y+1}} - 1 \\ &= y + 1 - 1 = y. \end{aligned}$$

(viii) *$f : (0,1) \to [1, \infty)$, where $f(x) = \begin{cases} \frac{1}{x} - 1 & \text{if } x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\} \\ \frac{1}{x} & \text{if } x \notin \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\} \end{cases}$ is surjective.*

Proof: *Let $y \in [1, \infty)$. We split the analysis into 2 parts.*

Case 1: *If $y \in \mathbb{N}$, then take $x = \frac{1}{y+1} \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$. Then*

$$\begin{aligned} f(x) &= \tfrac{1}{x} - 1 \\ &= \frac{1}{\frac{1}{y+1}} - 1 \\ &= y + 1 - 1 = y. \end{aligned}$$

Case 2: *If $y \notin \mathbb{N}$, then take $x = \frac{1}{y} \in (0,1) \setminus \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$. Then*

$$f(x) = \frac{1}{x} = \frac{1}{\frac{1}{y}} = y,$$

which completes the proof.

The corresponding preliminary calculations: The analysis must be split into 2 parts, because f can take on 2 forms. Notice that if $x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$, then $y = \frac{1}{x} - 1 \in \mathbb{N}$ and if $x \notin \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$, then $y = \frac{1}{x} \notin \mathbb{N}$. Furthermore, $y = \frac{1}{x} - 1 \Leftrightarrow y + 1 = \frac{1}{x} \Leftrightarrow x = \frac{1}{y+1}$ and $y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}$.

Exercise 22 *For the functions in exercise 21 prove whether or not they are surjective.*

a) $g_1 : (0,1] \to [1, \infty)$, *where* $g_1(x) = \frac{1}{x}$.

b) $g_2 : [2,4] \to [5,9]$, *where* $g_2(x) = 2x + 1$.

c) $g_3 : [0,3] \to [0,7]$, *where* $g_3(x) = 2x + 1$.

d) $g_4 : (0,1) \to (1,4)$, *where* $g_4(x) = \frac{8}{6x+2}$.

e) $g_5 : [5,7] \to [1,5]$, *where* $g_5(x) = \frac{x+1}{-2x+16}$.

f) $g_6 : [0,5] \to [-4,\infty)$, *where* $g_6(x) = x^2 + x - 4$.

g) $g_7 : [0,5] \to [-5,\infty)$, *where* $g_7(x) = x^2 - x - 4$.

h) $g_8 : [1,5] \to [1,3]$, *where* $g_8(x) = \sqrt{2x-1}$.

i) $g_9 : [-3,1) \to (0,1]$, *where* $g_9(x) = \frac{1}{x^2+1}$.

j) $g_{10} : (0,2] \to [\frac{1}{5},1)$, *where* $g_{10}(x) = \frac{1}{x^2+1}$.

Exercises 21 k, l, m and n feature functions that have powers greater than 2 or cubic roots. Surjectivity of this type of function will not be asked at the exam. Suffice to say that none of the four given functions are surjective.

Exercise 23 *e') Prove that* $g_{5a} : [5,7] \to [1,4]$, *where* $g_{5a}(x) = \frac{x+1}{-2x+16}$, *is surjective.*

f') Prove that $g_{6a} : [0,5] \to [-4,26]$, *where* $g_{6a}(x) = x^2 + x - 4$, *is surjective.*

g') Prove that $g_{7a} : [0,5] \to [-4\frac{1}{4},16]$, *where* $g_{7a}(x) = x^2 - x - 4$, *is surjective.*

A function that is both injective and surjective, is called **bijective**.

Exercise 24 *Find a bijective function f with domain $(0,1)$ and codomain $[0,1]$.*

Exercise 25 *Find a bijective function f with domain $[0,1]$ and codomain $(0,1)$.*

Exercise 26 *Assume that $m \leq n$. How many of the functions in exercise 18 are injective? (A similar question for surjective functions is very difficult...)*

Section 3.7

A **composite function**, also called a **composition**, notation $f \circ g$ or $f(g(x))$, is a function of a function.

An example: Let $f : \mathbb{R}^+ \to \mathbb{R}$, where $f(x) = \sqrt{x}$ (notice that f is injective, but not surjective) and let $g : \mathbb{R} \to \mathbb{R}^+$, where $g(x) = x^2 + 5$ (g is neither injective nor surjective). Now $f \circ g(x)$ is calculated as follows: First we apply the function g to x, resulting in $g(x)$ and then we apply f to that. It is important to realize that this is only possible if $g(x)$ is indeed in the domain of f; otherwise it is impossible to calculate $f(g(x))$ and $f \circ g$ would not be a well-defined function. In our example the codomain of g is $\mathbb{R}^+$, which is also the domain of f, so $f \circ g$ is well-defined. Similarly, since the codomain of f is equal to the domain of g, the composite function $g \circ f$ is well-defined as well. They are not equal however. Let's calculate both:

$f \circ g$: Notice that the domain of the composite function is always equal to the domain of the first function that is applied to x -in this case g-, so the domain of $f \circ g$ is $\mathbb{R}$. Furthermore, the codomain of $f \circ g$ is equal to the codomain of the last function that is applied (f), so it is equal to $\mathbb{R}$ as well. The domain and the codomain of $g \circ f$ are each equal to $\mathbb{R}^+$.

So $f \circ g(x) : \mathbb{R} \to \mathbb{R}$, where $f \circ g(x) = f(g(x)) = f(x^2 + 5) = \sqrt{x^2 + 5}$ and
$g \circ f(x) : \mathbb{R}^+ \to \mathbb{R}^+$, where $g \circ f(x) = g(f(x)) = g(\sqrt{x}) = (\sqrt{x})^2 + 5 = x + 5$.

If $f : A \to B$ and $g : B \to C$, then generally $f \circ g$ is not a well-defined function, whereas $g \circ f : A \to C$ is definitely well-defined.

Exercise 27 *Find a condition on the sets A, B and C that ensures that $f \circ g$ is well-defined.*

Of course there exist composite functions consisting of more than 2 functions simultaneously.

Example 13 *Consider the following sets: $A = \{1, 2, 4, 8\}$, $B = \{0, 6, 9\}$ and $C = \{3, 5, 7\}$ as well as the functions*

$$\begin{aligned}
&f : A \to B, \textit{ where } f(1) = f(2) = 0,\ f(4) = 9 \textit{ and } f(8) = 6 \\
&g : B \to C, \textit{ where } g(0) = g(6) = 3 \textit{ and } g(9) = 5 \\
&h : C \to A, \textit{ where } h(3) = 1,\ h(5) = 2 \textit{ and } h(7) = 8
\end{aligned}$$

and consider the function $h \circ g \circ f$. We have

$$\begin{aligned}
h \circ g \circ f(1) &= h(g(f(1))) = h(g(0)) = h(3) = 1 \\
h \circ g \circ f(2) &= h(g(f(2))) = h(g(0)) = h(3) = 1 \\
h \circ g \circ f(4) &= h(g(f(4))) = h(g(9)) = h(5) = 2 \\
h \circ g \circ f(8) &= h(g(f(8))) = h(g(6)) = h(3) = 1.
\end{aligned}$$

Consider the function $f \circ h \circ g \circ f$. We have

$$\begin{aligned}
f \circ h \circ g \circ f(1) &= f(h \circ g \circ f(1)) = f(1) = 0 \\
f \circ h \circ g \circ f(2) &= f(h \circ g \circ f(2)) = f(1) = 0 \\
f \circ h \circ g \circ f(4) &= f(h \circ g \circ f(4)) = f(2) = 0 \\
f \circ h \circ g \circ f(8) &= f(h \circ g \circ f(8)) = f(1) = 0.
\end{aligned}$$

Sometimes complicated looking functions may have surprisingly easy outcomes.

Section 3.8

Let $f : A \to B$ be a function. Then f performs an operation on $x \in A$, transforming it into $f(x) \in B$. Now consider the operation g that is defined as follows: g undoes the transformantion performed by f, nothing more, nothing less. So if f transforms $x \in A$ into $y \in B$, then g transforms $y \in B$ back into $x \in A$. Formally:

$$\forall x \in A : \quad f(x) = y \ \Rightarrow \ g(y) = x$$

We now discuss the two conditions we have to pose on f in order for g to be a *function*.

1. f has to be surjective: We have defined the operation g to only transform elements $y = f(x)$ back into x, so any y that is not the result of applying f to any $x \in A$, can not be in B.

2. f has to be injective: If there are two values x_1 and x_2 such that $f(x_1) = f(x_2) = y$, then $g(y) = x_1$ and $g(y) = x_2$ simultaneously. Since g is a function, it can assign only one value to y, so apparently $x_1 = x_2$.

So apparently f has to be bijective for g to be a function. If for a function f we can find a function g that undoes f, then f is called **invertible** and g is called the **inverse** of f. Notation: $g = f^{-1}$. Notice that the inverse of a function, if it exists, is unique and that $(f^{-1})^{-1} = f$.

For invertible functions f with a finite domain the inverse can be obtained by assigning to every point y in its codomain the (unique) corresponding point x of its domain for which $f(x) = y$.

Example 14 *In function f_1 in example 7 - the only invertible function in example 7 - we find: $f^{-1} : \{3, 7, 25\} \to \{1, 3, 5\}$ where $f^{-1}(3) = 3$, $f^{-1}(7) = 1$ and $f^{-1}(25) = 5$.*

Calculating the inverse of a function f with an infinite domain boils down to writing down the equation $y = f(x)$ as the equation $x = g(y)$ that was used in the proof of f being surjective. So, if f turns out to be invertible, then during the proof of f being surjective, we already calculated the inverse function.

Example 15 *We will determine the inverses of functions (iii),(vi) and (viii) of example 10 and we will show that indeed $f^{-1}(f(x)) = x$ for all $x \in A$:*

(iii) Let $f : [1, \infty) \to [-1, \infty)$, where $f(x) = x^2 - 2x$. Then $f^{-1} : [-1, \infty) \to [1, \infty)$, where

$$f^{-1}(y) = \sqrt{y+1} + 1$$

Proof: *Let $x \in [1, \infty)$. Then*

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(x^2 - 2x) \\ &= (\sqrt{(x^2 - 2x) + 1} + 1) \\ &= \sqrt{(x-1)^2} + 1 \\ &= |\, x - 1 \,| + 1 \\ &= x - 1 + 1 \text{ (since } x \geq -1) \\ &= x. \end{aligned}$$

(vi) Let $f : \mathbb{N} \to \mathbb{Z}$, where $f(x) = \begin{cases} \frac{1}{2}x - \frac{1}{2} & \text{if } x \text{ is odd} \\ -\frac{1}{2}x & \text{if } x \text{ is even} \end{cases}$. Then $f^{-1} : \mathbb{Z} \to \mathbb{N}$, where

$$f^{-1}(y) = \begin{cases} 2y + 1 & \text{if } y \geq 0 \\ -2y & \text{if } y < 0 \end{cases}$$

Proof: *Let $x \in \mathbb{N}$. The analysis is split into 2 parts.*

Case 1: *If x is odd, then $f(x) = \frac{1}{2}x - \frac{1}{2} \geq 0$. Consequently*

$$\begin{aligned} f^{-1}(f(x)) &= 2 \cdot f(x) + 1 \\ &= 2 \cdot (\frac{1}{2}x - \frac{1}{2}) + 1 \\ &= x \end{aligned}$$

or

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(\frac{1}{2}x - \frac{1}{2}) \\ &= 2 \cdot (\frac{1}{2}x - \frac{1}{2}) + 1 \\ &= x \end{aligned}$$

(These are the two most common ways to write down this proof.)
Case 2: *If x is even, then $f(x) = -\frac{1}{2}x < 0$. But then*

$$\begin{aligned} f^{-1}(f(x)) &= -2 \cdot f(x) \\ &= -2 \cdot -\frac{1}{2}x \\ &= x. \end{aligned}$$

This completes the proof.

(viii) *Let $f : (0,1) \to [1,\infty)$, where $f(x) = \begin{cases} \frac{1}{x} - 1 & \text{if } x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\} \\ \frac{1}{x} & \text{if } x \notin \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\} \end{cases}$. Then $f^{-1} : [1,\infty) \to (0,1)$, where*

$$f^{-1}(y) = \begin{cases} \frac{1}{y+1} & \text{if } y \in \mathbb{N} \\ \frac{1}{y} & \text{if } y \notin \mathbb{N} \end{cases}$$

Proof: *Let $x \in (0,1)$. The analysis is split into 2 parts.*
Case 1: *If $x \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$, then $f(x) = \frac{1}{x} - 1 \in \mathbb{N}$. Consequently*

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(\frac{1}{x} - 1) \\ &= \frac{1}{\frac{1}{x} - 1 + 1} \\ &= x \end{aligned}$$

or

$$\begin{aligned} f^{-1}(f(x)) &= \frac{1}{f(x) + 1} \\ &= \frac{1}{\frac{1}{x} - 1 + 1} \\ &= x \end{aligned}$$

Case 2: *If* $x \notin \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$*, then* $f(x) = \frac{1}{x} \notin \mathbb{N}$*, so*

$$f^{-1}(f(x)) = \frac{1}{f(x)} = \frac{1}{\frac{1}{x}} = x$$

which completes the proof.

Exercise 28 *For the functions in exercises 21 (and 22) that are both injective and surjective, determine their inverses. I.e. Calculate the inverses of functions* g_1*,* g_2*,* g_4*,* g_8 *and* g_{10}*. Afterwards, prove that your inverse function is correct by showing that for all* x *in the domain of the function we have:* $f^{-1}(f(x)) = x$*.*

Exercise 29 *For the functions* g_{5a}*,* g_{6a} *and* g_{7a} *in exercise 23, determine whether or not they are invertible. If so, then find the inverse.*

Set of practice exercise:

$$\begin{aligned}
g &: [2,4) \to (-12, -\tfrac{8}{3}], \text{ where } g(x) = \frac{2x+4}{x-5} \\
h &: (0,8) \to (-\tfrac{7}{23}, -\tfrac{1}{55}), \text{ where } h(x) = \frac{x-7}{4x+23} \\
k &: (0,\infty) \to (-\tfrac{7}{23}, \tfrac{1}{4}), \text{ where } k(x) = \frac{x-7}{4x+23} \\
l &: (-\infty,-6] \to (\tfrac{1}{4}, 13], \text{ where } l(x) = \frac{x-7}{4x+23} \\
m &: [-2,3) \to [-11,19), \text{ where } m(x) = x^2+5x-5 \\
n &: (0,3] \to [-11,1], \text{ where } n(x) = 3x^2-6x-8 \\
p &: (0,1] \to [-11,-8), \text{ where } p(x) = 3x^2-6x-8
\end{aligned}$$

Show that all functions except n are invertible and calculate their inverses. Show that n is surjective but not injective.

Extra Material: Sequences and Series

Part of this section comes from the book 'Introduction to mathematical structures and proofs' by Gerstein [1].

For any natural number n we write $\mathbb{N}_n = \{1, 2, \ldots, n\}$.

Definition 2 *Let A be a set. A sequence of length n in A is a function $f : \mathbb{N}_n \to A$. An infinite sequence is a function $f : \mathbb{N} \to A$.*

Notation: $\{a_i\}_{1 \leq i \leq n}$ or $\{a_i\}_{i=1}^{n}$ for a finite sequence and $\{a_n\}_{n \geq 1}$ or $\{a_n\}_{n=1}^{\infty}$ for an infinite sequence. The number a_n is called the n^{th} **term** of the sequence. Some examples of sequences:

1. Let $A = \{1, 2, 3, 4, 5\}$ and let $f : \mathbb{N}_3 \to A$ where $f(1) = 2$, $f(2) = 4$ and $f(3) = 1$. We then obtain the sequence $\{2, 4, 1\}$, which is different from the sequence $\{2, 1, 4\}$.
2. $A = \{-1, 1\}$ and $f : \mathbb{N} \to A$, where $f(n) = 1$ if n is odd and $f(n) = -1$ if n is even. This sequence can also be described in the following ways:
    - $a_n = 1$ if n is odd and $a_n = -1$ if n is even;
    - $\{1, -1, 1, -1, 1, -1, \ldots\}$;
    - $\{(-1)^{n+1}\}_{n \geq 1}$
3. The famous Fibonacci-sequence $\{1, 1, 2, 3, 5, 8, \ldots\}$

For finite sequences: We have $|\mathbb{N}_n| = n$ and let A be finite as well, say $|A| = m$. Under what condition on m and n does an injective sequence exist? And a surjective one? And how many injective/surjective sequences exist then? (The surjective one is difficult).

In the second example above we have $a_n = (-1)^{n+1}$, so an explicit description of each term in the sequence, i.e. a formula of a_n in terms of n, exists.

Exercise 30 *Give a formula for a_n in terms of n for each of the following sequences:*

- $\{1, 3, 5, 7, 9, 11, \ldots\}$
- $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$
- $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$
- $\{1, 1, 3, 3, 5, 5, 7, 7, 9, \ldots\}$
- $\{2, 3, 5, 8, 12, 17, 23, \ldots\}$
- $\{2, 11, 101, 1001, 10001, 100001, \ldots\}$

Recursively defined sequences

Sometimes terms in a sequence are defined in terms of previous terms, particularly if a pattern can be found. For instance in the sequence $\{2, 3, 4, \ldots\} = \{n+1\}_{n=1}^{\infty}$ each number is 1 bigger than its predecessor. So the sequence can be defined as follows: $a_1 = 2$ and $a_{n+1} = a_n + 1$ for all $n \geq 1$. This is a so-called recursive definition of the sequence.

Exercise 31 *Give a recursive definition of the Fibonacci sequence.*

Exercise 32 *Give a recursive definition for each of the following sequences:*

- $\{1, 3, 5, 7, 9, 11, \ldots\}$
- $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$
- $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$
- $\{1, 1, 3, 3, 5, 5, 7, 7, 9, \ldots\}$
- $\{2, 3, 5, 8, 12, 17, 23, \ldots\}$
- $\{2, 11, 101, 1001, 10001, 100001, \ldots\}$

A recursive definition is an implicit way to describe the elements in a sequence. Instead of giving an (explicit) expression of each element, it gives a relation between two or more elements in the sequence.

Series

Given a infinite sequence $\{a_n\}_{n=1}^{\infty}$ a series is the following sequence: $\{b_n\}_{n=1}^{\infty}$, where $b_n = a_1 + a_2 + a_3 + \ldots + a_n$ (or $b_n = \sum_{i=1}^{n} a_i$). If b_n converges (i.e. it approaches some number as n tends to ∞), we say that the sequence $\{a_n\}_{n=1}^{\infty}$ is summable.

Example 16 *The sequence $\{1, -1, 1, -1, 1, -1, \ldots\}$ is not summable: $b_n = 1$ is n is odd and $b_n = 0$ if n is even, so it does not converge.*

Notice that a sequence $\{a_n\}_{n=1}^{\infty}$ can only be summable if the terms approach 0 and even that is not enough.

Exercise 33 *Two sequences:*

- *Show that the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$ is not summable.*
- *Show that the sequence $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$ is summable. What number does the sum approach?*

Advanced Material: Limits and Continuity of Functions

In this section we discuss two Calculus topics, limits and continuity of functions, in an analytical, discrete mathematics-style way. The functions f that we discuss throughout this section will have domain A and codomain B that are subsets of $\mathbb{R}$.

Definition 3 *Let f be a function with domain A and let c be a real number such that A and c have the following property: There exist $a < c$ and $b > c$ such that $(a, b)\backslash\{c\} \subseteq A$, i.e. the domain of f includes the interval (a, b) except for, perhaps, the point c. (Notice that if $A = (a, b)$ or $A = [a, b]$, then any value of c strictly between a and b satisfies this condition.) A real number l is called the **limit** of the function f at c if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x) - l| < \varepsilon \text{ for every } x \in A \text{ for which } 0 < |x - c| < \delta.$$

The inequalities $0 < |x - c| < \delta$ say that $c - \delta < x < c + \delta$ and $x \neq c$.

Interpretation: We want $f(x)$ to be arbitrarily close to some number l (because we want $|f(x)-l| < \varepsilon$ and ε can be arbitrarily close to 0) and apparently we can achieve this by choosing our x close enough (but not equal!) to c, namely within distance δ from c, where we can choose the value of δ.

If l is the limit of a function f at c, then we say that f approaches l as x approaches c or that f converges to l at c. This is written as

$$\lim_{x \to c} f(x) = l \quad \text{or} \quad f(x) \to l \text{ as } x \to c.$$

Let's formalize the statement in the definition a little bit so that we can use it to prove things. We say that f (with domain A) has limit l at c if

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall\, x \in A: \quad 0 < |x - c| < \delta \;\; \Rightarrow \;\; |f(x) - l| < \varepsilon.$$

Example 17 *Let $f : \mathbb{R}\backslash\{3\} \to \mathbb{R}$ where $f(x) = 5x+1$. Notice that 3 is not in the domain of f, but for the existence of a limit that is not important; we just want to know what happens with $f(x)$ is x gets closer and closer to 3, not what happens if x is equal to 3. We will prove that $\lim_{x\to 3} f(x) = 16$. It is pretty obvious that this is the case: If we plug in values closer and closer to 3, we find function values closer and closer to 16. For example, $f(2) = 11$, $f(2\frac{1}{2}) = 13\frac{1}{2}$, $f(2\frac{2}{3}) = 14\frac{1}{3}$, $f(2\frac{3}{4}) = 14\frac{3}{4}$, $f(2\frac{4}{5}) = 15$, $f(2\frac{5}{6}) = 15\frac{1}{6}$, $f(2\frac{6}{7}) = 15\frac{2}{7}$, etc. In fact, if we create any sequence $\{x_n\}_{n=1}^{\infty}$ such that x_n approaches 3 as n tends to ∞ (without x_n ever being equal to 3), then $\{f(x_n)\}_{n=1}^{\infty}$ is a sequence approaching 16. But that's not a proof. So let's prove that $\lim_{x\to 3} f(x) = 16$ using the definition.*

Proof. *Let $\varepsilon > 0$. Take $\delta = \frac{1}{5}\varepsilon$ and assume that $0 < |x - 3| < \delta$. Then*

$$|f(x) - 16| = |5x + 1 - 16| = |5x - 15| = |5(x - 3)| = 5 \cdot |x - 3| < 5 \cdot \delta = 5 \cdot \tfrac{1}{5}\varepsilon = \varepsilon,$$

which completes the proof. ■

This example is fairly straightforward. The main question is how we came up with the idea to *choose $\delta = \frac{1}{5}\varepsilon$. (Notice that the δ is chosen after the ε.) This required a preliminary calculation similar to proofs of a function being surjective. Here we know that $|x-3| < \delta$ and we want to know for what values of δ this implies that $|f(x) - 16| < \varepsilon$. So, we do a calculation.*

$$|f(x) - 16| = |5x + 1 - 16| = |5x - 15| = |5(x - 3)| = 5 \cdot |x - 3|$$

Apparently $|f(x)-16| < \varepsilon \quad 5\cdot|x-3| < \varepsilon$! But we already know that $|x-3| < \delta$, so $5\cdot|x-3| < 5\delta$. As long as $5\delta \leq \varepsilon$ we are done! So we take $\delta = \frac{1}{5}\varepsilon$.

Notice that in example 17 the domain of the function f does not include 3. This shows that, in order for $\lim_{x\to c} f(x) = l$ to exist, it is not required that c is in the domain of f. This is why in the definition of a limit we need this rather elaborate description of the number c and the domain A of f. Now let's see what happens if we make example 17 a little different.

Example 18 *Let $g : \mathbb{R} \to \mathbb{R}$ where*

$$g(x) = \begin{cases} 5x + 1 & x \in \mathbb{R}\backslash\{3\} \\ \pi & x = 3 \end{cases}$$

Notice that the only difference between the function g in this example and the function f in example 17 is that the number 3 is in the domain of g and not in the domain of f and that we have $g(3) = \pi$. We can prove that $\lim_{x\to 3} g(x) = 16$. In fact, the whole proof that $\lim_{x\to 3} g(x) = 16$ is exactly the same as it was for f. Why is that the case? Notice that in the proof it says that $0 < |x - 3| < \delta$, so in particular it says that $0 < |x - 3|$. This means that, even though 3 is in the domain of g, the values of x that we consider, are not equal to 3. The analysis only concerns values of x that are very close to 3 (arbitrarily close in fact), and for all those values of x the functions f and g have the same function values. The fact that $g(3)$ (which is π) is not equal to $\lim_{x\to 3} g(x)$ (which is 16), does not make a difference.

Example 19 *Let $f : \mathbb{R}\backslash\{-1\}$, where $f(x) = \frac{x^2-1}{x+1}$. Prove that $\lim_{x\to -1} f(x) = -2$.*

Notice that this example is a bit more tricky, as the expression $\frac{x^2-1}{x+1}$ is not well-defined for $x = -1$ (it says $\frac{0}{0}$ then).

Proof. *Let $\varepsilon > 0$, take $\delta = \varepsilon$ and assume that $0 < |x + 1| < \delta$. Then*

$$\begin{aligned} |f(x) - (-2)| &= \left|\tfrac{x^2-1}{x+1} + 2\right| = \left|\tfrac{x^2-1+2x+2}{x+1}\right| \\ &= \left|\tfrac{x^2+2x+1}{x+1}\right| = \left|\tfrac{(x+1)^2}{x+1}\right| \\ &= |x + 1| < \delta = \varepsilon \end{aligned}$$

which completes the proof. ■

The preliminary calculation for this example goes as follows: $|f(x) - (-2)| = \left|\frac{x^2-1}{x+1} + 2\right| = \left|\frac{x^2-1+2x+2}{x+1}\right| = \left|\frac{x^2+2x+1}{x+1}\right| = \left|\frac{(x+1)^2}{x+1}\right| = |x + 1| < \delta$ so we choose $\delta = \varepsilon$.

Example 20 *Prove that* $\lim_{x\to -2} x^3 - 5x - 1 = 1$.

Proof. *Let* $\varepsilon > 0$. *Take* $\delta = \min\{1, \frac{1}{16}\varepsilon\}$ *and assume that* $0 < |x+2| < \delta$. *Then*

$$\begin{aligned} |x^3 - 5x - 1 - 1| &= |x^3 - 5x - 2| = |(x+2)(x^2 - 2x - 1)| \\ &= |x+2| \cdot |x^2 - 2x - 1| < \delta \cdot |x^2 - 2x - 1| \leq 16\delta \leq \varepsilon \end{aligned}$$

which completes the proof. ■

The preliminary calculation:

$$|x^3 - 5x - 1 - 1| = |x^3 - 5x - 2| = |(x+2)(x^2 - 2x - 1)| = |x+2| \cdot |x^2 - 2x - 1| < \delta \cdot |x^2 - 2x - 1|$$

since $|x+2| < \delta$. *After this we need to estimate the value of* $|x^2 - 2x - 1|$. *In order to find an estimate, we start by assuming that* $\delta \leq 1$. *This is an important 'trick': We can always assume that our choice for* δ *is going to be smaller than some prefixed number and we generally choose 1 because that makes the calculations relatively easy. Now, given that* $\delta \leq 1$ *and that* $0 < |x+2| < \delta$, *we know that* $-3 < x < -1$. *But then*

$$|x^2 - 2x - 1| \leq |x^2| + |-2x| + |-1| \leq |(-3)^2| + |-2 \cdot -3| + |-1| = 9 + 6 + 1 = 16,$$

so $\delta \cdot |x^2 - 2x - 1| \leq 16\delta$. *Then, by having* $\delta \leq \frac{1}{16}\varepsilon$, *we make sure that the resulting expression is smaller than* ε. *All in all we need that* $\delta \leq 1$ *and* $\delta \leq \frac{1}{16}\varepsilon$, *so we take* $\delta = \min\{1, \frac{1}{16}\varepsilon\}$.

Now for a few more complicated examples. In each case filling in the corresponding value of x would lead to the expression $\frac{0}{0}$. Some notation: There is no function mentioned in any of the following. Including a function f in these exercises boils down to assuming it to be the expression to the left hand side of the =-symbol and then to let the domain consist of all real numbers where this expression is well-defined. In example 21 below that means that the domain of f is $\mathbb{R}\backslash\{-3, 3\}$ and in example 22 the domain of f is $[-4, \infty)\backslash\{0\}$.

Example 21 *Prove that* $\lim_{x\to 3} \frac{x^2-6x+9}{x^2-9} = 0$.

Proof. *Let* $\varepsilon > 0$, *take* $\delta = \min\{1, 5\varepsilon\}$ *and assume that* $0 < |x-3| < \delta$. *Then*

$$\begin{aligned} \left|\frac{x^2 - 6x + 9}{x^2 - 9} - 0\right| = \left|\frac{(x-3)^2}{(x-3)(x+3)}\right| = \left|\frac{x-3}{x+3}\right| &= \frac{|x-3|}{|x+3|} \\ &< \frac{\delta}{|x+3|} \quad \textit{(because } |x-3| < \delta\textit{)} \\ &\leq \frac{\delta}{5} \quad (\delta \leq 1 \textit{ implies that } 2 < x < 4 \textit{ and therefore } 5 \leq |x+3| \leq 7) \\ &\leq \frac{5\varepsilon}{5} \quad \textit{(because } \delta \leq 5\varepsilon\textit{)} \\ &= \varepsilon, \end{aligned}$$

which completes the proof. ■

The preliminary calculation: $\left|\frac{x^2-6x+9}{x^2-9} - 0\right| = \left|\frac{(x-3)^2}{(x-3)(x+3)}\right| = \left|\frac{x-3}{x+3}\right| = \frac{|x-3|}{|x+3|} < \frac{\delta}{|x+3|}$. *If we take* $\delta \le 1$, *then* $2 \le x \le 4$, *so* $|x+3| \ge 5$ *and* $\frac{\delta}{|x+3|} \le \frac{\delta}{5}$ *and* $\frac{\delta}{5} \le \varepsilon \Leftrightarrow \delta \le 5\varepsilon$

Example 22 *Prove that* $\lim\limits_{x\to 0} \frac{\sqrt{4+x}-2}{x} = \frac{1}{4}$.

Proof. *Let* $\varepsilon > 0$, *take* $\delta = \min\{1, 50\varepsilon\}$ *and assume that* $0 < |x| < \delta$. *Then*

$$
\begin{aligned}
\left|\frac{\sqrt{4+x}-2}{x} - \frac{1}{4}\right| &= \left|\frac{(\sqrt{4+x}-2)(\sqrt{4+x}+2)}{x(\sqrt{4+x}+2)} - \frac{1}{4}\right| \\
&= \left|\frac{4+x-4}{x(\sqrt{4+x}+2)} - \frac{1}{4}\right| = \left|\frac{1}{\sqrt{4+x}+2} - \frac{1}{4}\right| \\
&= \left|\frac{4-\sqrt{4+x}-2}{4(\sqrt{4+x}+2)}\right| = \left|\frac{2-\sqrt{4+x}}{4\sqrt{4+x}+8}\right| \\
&= \left|\frac{(2-\sqrt{4+x})(2+\sqrt{4+x})}{(4\sqrt{4+x}+8)(2+\sqrt{4+x})}\right| \\
&= \left|\frac{-x}{32+4x+16\sqrt{4+x})}\right| = \frac{|x|}{|32+4x+16\sqrt{4+x}|} \\
&< \frac{\delta}{32+4x+16\sqrt{4+x}} \qquad \textit{(*)} \\
&< \frac{\delta}{50} \le \frac{1}{50} \cdot 50\varepsilon = \varepsilon,
\end{aligned}
$$

which completes the proof. ■

Preliminary calculations/arguments corresponding to (*): Notice that, by setting $\delta \le 1$, *we know that* x *will be between* -1 *and* 1, *or* $-1 \le x \le 1$. *This means that we can estimate the value of the ugly-looking expression in the denominator. Namely, we know that:*

$$32 + 4x + 16\sqrt{4+x} \ge 32 + 4 \cdot (-1) + 16\sqrt{4-1} \approx 55.7 > 50.$$

So the expression in the denominator is always bigger than 50 and δ *divided by this number will be smaller than* $\frac{\delta}{50}$. *Now, as long as* $\delta \le 50\varepsilon$, *we then know that* $\frac{\delta}{50}$ *is smaller than* ε. *Hence our choice of* $\delta = \min\{1, 50\varepsilon\}$.

Limits of these kind of expressions do not necessarily exist. You have to be particularly careful when there is an absolute value involved in the expression. The following example shows this nicely.

Example 23 $\lim\limits_{x\to 2} \frac{|x-2|}{x-2}$ *does not exist.*

Proof. Notice first that $\lim\limits_{x\to 2} |x-2|$ does exist (it equals 0). We have:

$$\lim_{x\uparrow 2} \frac{|x-2|}{x-2} = \lim_{x\uparrow 2} \frac{2-x}{x-2} = \lim_{x\uparrow 2} -1 = -1 \quad \text{(left limit)}$$

and

$$\lim_{x\downarrow 2} \frac{|x-2|}{x-2} = \lim_{x\downarrow 2} \frac{x-2}{x-2} = \lim_{x\downarrow 2} 1 = 1 \quad \text{(right limit)}.$$

Since the left limit and the right limit are not equal, THE limit does not exist. This will never happen in any of the exercises. ■

Exercise 34 *Use the formal definition of a limit to prove the following:*

1. $\lim_{x\to -2} \frac{x^2+2x}{x^2-4} = \frac{1}{2}$
2. $\lim_{x\to 0} \frac{x}{\sqrt{1+3x}-1} = \frac{2}{3}$
3. $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \frac{1}{6}$
4. $\lim_{x\to 1} \frac{x^2-1}{\sqrt{x+3}-2} = 6$
5. $\lim_{x\to 0} \frac{x}{\sqrt{4+x}-\sqrt{4-x}} = 2$
6. $\lim_{x\to 2} \frac{x^3-8}{x^2-4} = 3$
7. $\lim_{x\to 0} \frac{2x}{|x-1|-|x+1|} = -1$
8. $\lim_{x\to 0} \frac{|3x-8|-|x^2-4x+2|}{\sqrt{9-x}-2} = 12$