EXERCISES 2.1

In Exercises 1–12, find an equation of the straight line tangent to the given curve at the point indicated.

1. $y = 3x - 1$ at $(1, 2)$
2. $y = x/2$ at $(a, a/2)$
3. $y = 2x^2 - 5$ at $(2, 3)$
4. $y = 6 - x - x^2$ at $x = -2$
5. $y = x^3 + 8$ at $x = -2$
6. $y = \dfrac{1}{x^2+1}$ at $(0, 1)$
7. $y = \sqrt{x+1}$ at $x = 3$
8. $y = \dfrac{1}{\sqrt{x}}$ at $x = 9$
9. $y = \dfrac{2x}{x+2}$ at $x = 2$
10. $y = \sqrt{5 - x^2}$ at $x = 1$
11. $y = x^2$ at $x = x_0$
12. $y = \dfrac{1}{x}$ at $\left(a, \dfrac{1}{a}\right)$

Do the graphs of the functions f in Exercises 13–17 have tangent lines at the given points? If yes, what is the tangent line?

13. $f(x) = \sqrt{|x|}$ at $x = 0$
14. $f(x) = (x-1)^{4/3}$ at $x = 1$
15. $f(x) = (x+2)^{3/5}$ at $x = -2$
16. $f(x) = |x^2 - 1|$ at $x = 1$
17. $f(x) = \begin{cases} \sqrt{x} & \text{if } x \ge 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$ at $x = 0$
18. Find the slope of the curve $y = x^2 - 1$ at the point $x = x_0$. What is the equation of the tangent line to $y = x^2 - 1$ that has slope -3?
19. (a) Find the slope of $y = x^3$ at the point $x = a$.
    (b) Find the equations of the straight lines having slope 3 that are tangent to $y = x^3$.
20. Find all points on the curve $y = x^3 - 3x$ where the tangent line is parallel to the x-axis.
21. Find all points on the curve $y = x^3 - x + 1$ where the tangent line is parallel to the line $y = 2x + 5$.
22. Find all points on the curve $y = 1/x$ where the tangent line is perpendicular to the line $y = 4x - 3$.
23. For what value of the constant k is the line $x + y = k$ normal to the curve $y = x^2$?
24. For what value of the constant k do the curves $y = kx^2$ and $y = k(x-2)^2$ intersect at right angles? *Hint:* Where do the curves intersect? What are their slopes there?

Use a graphics utility to plot the following curves. Where does the curve have a horizontal tangent? Does the curve fail to have a tangent line anywhere?

25. $y = x^3(5-x)^2$
26. $y = 2x^3 - 3x^2 - 12x + 1$
27. $y = |x^2 - 1| - x$
28. $y = |x+1| - |x-1|$
29. $y = (x^2 - 1)^{1/3}$
30. $y = ((x^2 - 1)^2)^{1/3}$
31. If line L is tangent to curve C at point P, then the smaller angle between L and the secant line PQ joining P to another point Q on C approaches 0 as Q approaches P along C. Is the converse true: if the angle between PQ and line L (which passes through P) approaches 0, must L be tangent to C?
32. Let $P(x)$ be a polynomial. If a is a real number, then $P(x)$ can be expressed in the form

$$P(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$$

for some $n \ge 0$. If $\ell(x) = m(x-a) + b$, show that the straight line $y = \ell(x)$ is tangent to the graph of $y = P(x)$ at $x = a$ provided $P(x) - \ell(x) = (x-a)^2 Q(x)$, where $Q(x)$ is a polynomial.

EXERCISES 2.2

Make rough sketches of the graphs of the derivatives of the functions in Exercises 1–4.

1. The function f graphed in Figure 2.18(a).
2. The function g graphed in Figure 2.18(b).
3. The function h graphed in Figure 2.18(c).
4. The function k graphed in Figure 2.18(d).
5. Where is the function f graphed in Figure 2.18(a) differentiable?
6. Where is the function g graphed in Figure 2.18(b) differentiable?

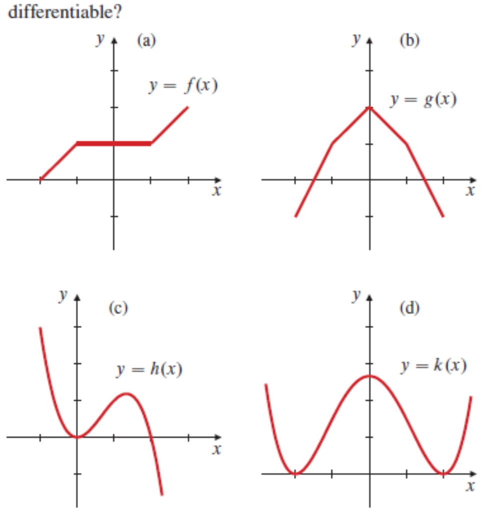


Figure 2.18

Use a graphics utility with differentiation capabilities to plot the graphs of the following functions and their derivatives. Observe the relationships between the graph of y and that of y' in each case. What features of the graph of y can you infer from the graph of y'?

7. $y = 3x - x^2 - 1$
8. $y = x^3 - 3x^2 + 2x + 1$
9. $y = |x^3 - x|$
10. $y = |x^2 - 1| - |x^2 - 4|$

In Exercises 11–24, (a) calculate the derivative of the given function directly from the definition of derivative, and (b) express the result of (a) using differentials.

11. $y = x^2 - 3x$
12. $f(x) = 1 + 4x - 5x^2$
13. $f(x) = x^3$
14. $s = \dfrac{1}{3+4t}$
15. $g(x) = \dfrac{2-x}{2+x}$
16. $y = \dfrac{1}{3}x^3 - x$
17. $F(t) = \sqrt{2t+1}$
18. $f(x) = \dfrac{3}{4}\sqrt{2-x}$
19. $y = x + \dfrac{1}{x}$
20. $z = \dfrac{s}{1+s}$
21. $F(x) = \dfrac{1}{\sqrt{1+x^2}}$
22. $y = \dfrac{1}{x^2}$
23. $y = \dfrac{1}{\sqrt{1+x}}$
24. $f(t) = \dfrac{t^2-3}{t^2+3}$
25. How should the function $f(x) = x \operatorname{sgn} x$ be defined at $x = 0$ so that it is continuous there? Is it then differentiable there?
26. How should the function $g(x) = x^2 \operatorname{sgn} x$ be defined at $x = 0$ so that it is continuous there? Is it then differentiable there?
27. Where does $h(x) = |x^2 + 3x + 2|$ fail to be differentiable?
28. Using a calculator, find the slope of the secant line to $y = x^3 - 2x$ passing through the points corresponding to $x = 1$ and $x = 1 + \Delta x$, for several values of Δx of decreasing size, say $\Delta x = \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$. (Make a table.) Also, calculate $\left.\dfrac{d}{dx}(x^3 - 2x)\right|_{x=1}$ using the definition of derivative.
29. Repeat Exercise 28 for the function $f(x) = \dfrac{1}{x}$ and the points $x = 2$ and $x = 2 + \Delta x$.

Using the definition of derivative, find equations for the tangent lines to the curves in Exercises 30–33 at the points indicated.

(12) $y = \frac{1}{x}$, find tangent line at $(a, \frac{1}{a})$

$$y - \frac{1}{a} = \frac{d}{dx}\left(\frac{1}{x}\right)\Big|_{x=a} \cdot (x - a)$$

$$y = -\frac{1}{x^2}\Big|_{x=a} \cdot (x - a) + \frac{1}{a}$$

$$y = -\frac{1}{a^2}(x - a) + \frac{1}{a}$$

$$y = -\frac{x}{a^2} + \frac{1}{a} + \frac{1}{a}$$

$$y = -\frac{x}{a^2} + \frac{2}{a}$$

(13) $f(x) = \sqrt{|x|}$ at $x = 0$

$$\frac{d}{dx}(|x|) = \frac{x}{|x|} = \mathrm{sgn}(x)$$

$$\lim_{h \to 0} \frac{\sqrt{|x+h|} - \sqrt{|x|}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{|x+0|} - 0}{0} \leftarrow$$ limit does not exist

$$f(x) = |x|^{1/2}$$

$x > 0$: $f'(x) = \frac{1}{2\sqrt{x}}$

$x < 0$: $f'(x) = \frac{1}{2\sqrt{-x}}$

derivative of f does not exist at $x = 0$

(16) $f(x) = |x^2 - 1|$ at $x = 1$

$$f'(x) = \frac{x^2 - 1}{|x^2 - 1|}(2x) = \frac{(2x)(x^2 - 1)}{|x^2 - 1|}$$

with $x^2 - 1 \neq 0 \Rightarrow x^2 \neq 1 \Rightarrow x \neq \pm 1$

the derivative does not exist at $x = 1$, hence no tangent line is present at $x = 1$

(19) (a) $y = x^3$ | ? find slope at $x = a$

$f'(a) = 3a^2$

(b) $g(x)$, $m_g = 3$, $g(x)$ tangent to $f(x)$

$g: y = 3x + c$ $\qquad$ $f'(x) = 3$

$3x^2 = 3$

$$\begin{cases} y = 3x + c \\ y = x^3 \\ 3x^2 = 3 \end{cases} \qquad \begin{cases} \_\_\_ \\ \_\_\_ \\ x = \pm 1 \end{cases}$$

$x = 1$ $\begin{cases} c = -2 \\ y = 1 \\ x = 1 \end{cases}$ $\qquad$ $x = -1$ $\begin{cases} c = -1 + 3 = 2 \\ y = -1 \\ x = -1 \end{cases}$

$y_1 = 3x - 2$ $\qquad$ $y_2 = 3x + 2$

(21) $f(x) = x^3 - x + 1$ $\qquad$ $g(x) = 2x + 5$

$f'(x) = 2 \Rightarrow 3x^2 - 1 = 2 \Rightarrow x^2 = \pm 1$

$y = 2x + c$ $\qquad$ $c = 5$

$x = 1$ $\begin{cases} c = 5 \\ x = 1 \\ y = 1 - 1 + 1 \end{cases}$ $\qquad$ $x = -1$ $\begin{cases} c = 5 \\ x = -1 \\ y = -1 + 1 + 1 \end{cases}$

$(1, 1)$ $\qquad$ $(-1, 1)$

(22) $f(x) = \frac{1}{x}$ $\quad g(x) = 4x - 3$

find all P where $g(x) \perp f(x)$

$$-\frac{1}{f'(x)} = 4 \Rightarrow 4 = +\frac{1}{\frac{1}{x^2}} \Rightarrow 4 = x^2$$

$$\Rightarrow x_{1,2} = \pm 2 \qquad y = mx + c$$

$$\begin{cases} c = -3 \\ x = 2 \\ y = 1/2 \end{cases} \qquad \begin{cases} c = -3 \\ x = -2 \\ y = -1/2 \end{cases}$$

$$(2, \tfrac{1}{2}) \qquad (-2, -\tfrac{1}{2})$$

(23) $x + y = k \perp y = x^2$

g $\quad f$

$y = -x + k$

$$-\frac{1}{f'(x)} = -1 \Rightarrow \frac{1}{2x} = 1 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$$

$$\left[+\frac{1}{2}, \left(+\frac{1}{2}\right)^2\right] \Rightarrow \left(+\frac{1}{2}, \frac{1}{4}\right)$$

$$+\frac{1}{2} + \frac{1}{4} = k \Rightarrow k = \frac{3}{4}$$

(24) $f(x) = kx^2$ $\quad g(x) = k(x-2)^2$ $\quad 2k(x-2)$

find k so that $f(x)$ and $g(x)$ intersect at $\perp$

$$kx^2 = k(x-2)^2$$

$$x^2 = x^2 - 4x + 4 \qquad k \neq 0$$

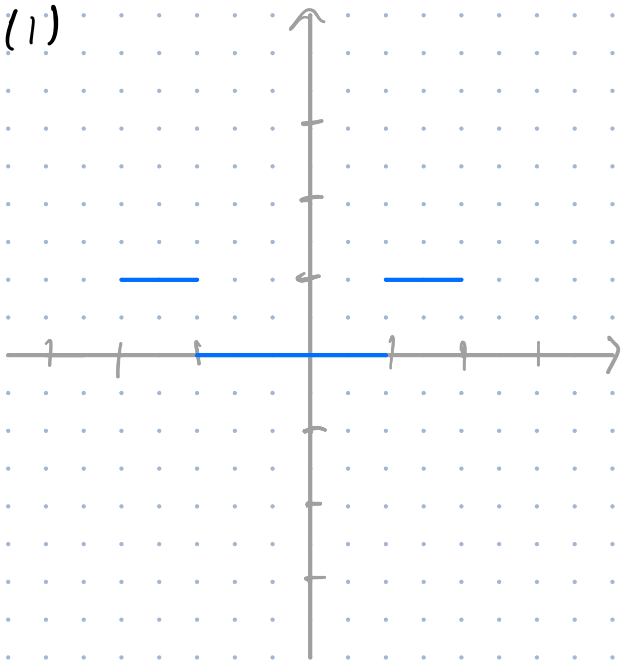
$$-4x + 4 = 0 \Rightarrow x = 1 \quad \therefore \text{intersection at } (1, k)$$

$$f'(1) = -\frac{1}{g'(1)} \Rightarrow 2k = -\frac{1}{(-2k)}$$

$$\Rightarrow 2k = \frac{1}{2k} \Rightarrow 4k^2 = 1$$

$$\Rightarrow k^2 = \frac{1}{4} \Rightarrow k = \pm\frac{1}{2}$$

(1)



(25) $f(x) = x \frac{x}{|x|}$ at $x=0$

dom : $x \neq 0$

$f(x)$ is continuous if $f(0) = 0$

(26) $f(x) = x^2 \frac{x}{|x|}$ at $x = 0$

dom : $x \neq 0$

$\lim_{x \to 0^+} \frac{x^3}{x} = \lim_{x \to 0^-} \frac{x}{-x}$

$x^2 = -1 \Rightarrow x \notin \mathbb{R}$

$\Rightarrow f(x)$ cannot be continuous on $\mathbb{R}$

(27) $f(x) = |x^2 + 3x + 2|$

$f'(x) = \frac{x^2 + 3x + 2}{|x^2 + 3x + 2|} (2x + 3)$

dom f': $x^2 + 3x + 2 \neq 0$

$(x + 2)(x + 1) \neq 0$

$x \neq -2 \quad x \neq -1$

EXERCISES 2.3

In Exercises 1–32, calculate the derivatives of the given functions. Simplify your answers whenever possible.

1. $y = 3x^2 - 5x - 7$

2. $y = 4x^{1/2} - \dfrac{5}{x}$

3. $f(x) = Ax^2 + Bx + C$

4. $f(x) = \dfrac{6}{x^3} + \dfrac{2}{x^2} - 2$

5. $z = \dfrac{s^5 - s^3}{15}$

6. $y = x^{45} - x^{-45}$

7. $g(t) = t^{1/3} + 2t^{1/4} + 3t^{1/5}$

8. $y = 3\sqrt[3]{t^2} - \dfrac{2}{\sqrt{t^3}}$

9. $u = \dfrac{3}{5}x^{5/3} - \dfrac{5}{3}x^{-3/5}$

10. $F(x) = (3x - 2)(1 - 5x)$

11. $y = \sqrt{x}\left(5 - x - \dfrac{x^2}{3}\right)$

12. $g(t) = \dfrac{1}{2t - 3}$

13. $y = \dfrac{1}{x^2 + 5x}$

14. $y = \dfrac{4}{3 - x}$

15. $f(t) = \dfrac{\pi}{2 - \pi t}$

16. $g(y) = \dfrac{2}{1 - y^2}$

17. $f(x) = \dfrac{1 - 4x^2}{x^3}$

18. $g(u) = \dfrac{u\sqrt{u} - 3}{u^2}$

19. $y = \dfrac{2 + t + t^2}{\sqrt{t}}$

20. $z = \dfrac{x - 1}{x^{2/3}}$

21. $f(x) = \dfrac{3 - 4x}{3 + 4x}$

22. $z = \dfrac{t^2 + 2t}{t^2 - 1}$

23. $s = \dfrac{1 + \sqrt{t}}{1 - \sqrt{t}}$

24. $f(x) = \dfrac{x^3 - 4}{x + 1}$

25. $f(x) = \dfrac{ax + b}{cx + d}$

26. $F(t) = \dfrac{t^2 + 7t - 8}{t^2 - t + 1}$

27. $f(x) = (1 + x)(1 + 2x)(1 + 3x)(1 + 4x)$

28. $f(r) = (r^{-2} + r^{-3} - 4)(r^2 + r^3 + 1)$

29. $y = (x^2 + 4)(\sqrt{x} + 1)(5x^{2/3} - 2)$

30. $y = \dfrac{(x^2 + 1)(x^3 + 2)}{(x^2 + 2)(x^3 + 1)}$

31. $y = \dfrac{x}{2x + \dfrac{1}{3x + 1}}$

32. $f(x) = \dfrac{(\sqrt{x} - 1)(2 - x)(1 - x^2)}{\sqrt{x}(3 + 2x)}$

Calculate the derivatives in Exercises 33–36, given that $f(2) = 2$ and $f'(2) = 3$.

33. $\dfrac{d}{dx}\left(\dfrac{x^2}{f(x)}\right)\bigg|_{x=2}$

34. $\dfrac{d}{dx}\left(\dfrac{f(x)}{x^2}\right)\bigg|_{x=2}$

35. $\dfrac{d}{dx}\left(x^2 f(x)\right)\bigg|_{x=2}$

36. $\dfrac{d}{dx}\left(\dfrac{f(x)}{x^2 + f(x)}\right)\bigg|_{x=2}$

37. Find $\dfrac{d}{dx}\left(\dfrac{x^2 - 4}{x^2 + 4}\right)\bigg|_{x=-2}$.

38. Find $\dfrac{d}{dt}\left(\dfrac{t(1 + \sqrt{t})}{5 - t}\right)\bigg|_{t=4}$.

39. If $f(x) = \dfrac{\sqrt{x}}{x + 1}$, find $f'(2)$.

40. Find $\dfrac{d}{dt}\Big((1 + t)(1 + 2t)(1 + 3t)(1 + 4t)\Big)\Big|_{t=0}$.

41. Find an equation of the tangent line to $y = \dfrac{2}{3 - 4\sqrt{x}}$ at the point $(1, -2)$.

42. Find equations of the tangent and normal to $y = \dfrac{x + 1}{x - 1}$ at $x = 2$.

43. Find the points on the curve $y = x + 1/x$ where the tangent line is horizontal.

44. Find the equations of all horizontal lines that are tangent to the curve $y = x^2(4 - x^2)$.

2.4

In Exercises 22–29, express the derivative of the given function in terms of the derivative f' of the differentiable function f.

22. $f(2t + 3)$

23. $f(5x - x^2)$

24. $\left[f\left(\dfrac{2}{x}\right)\right]^3$

25. $\sqrt{3 + 2f(x)}$

26. $f\left(\sqrt{3 + 2t}\right)$

27. $f\left(3 + 2\sqrt{x}\right)$

28. $f\Big(2f\big(3f(x)\big)\Big)$

29. $f\Big(2 - 3f\big(4 - 5t\big)\Big)$

EXERCISES 2.5

1. Verify the formula for the derivative of $\csc x = 1/(\sin x)$.

2. Verify the formula for the derivative of $\cot x = (\cos x)/(\sin x)$.

Find the derivatives of the functions in Exercises 3–36. Simplify your answers whenever possible. Also be on the lookout for ways you might simplify the given expression before differentiating it.

3. $y = \cos 3x$

4. $y = \sin \dfrac{x}{5}$

5. $y = \tan \pi x$

6. $y = \sec ax$

7. $y = \cot(4 - 3x)$

8. $y = \sin((\pi - x)/3)$

9. $f(x) = \cos(s - rx)$

10. $y = \sin(Ax + B)$

11. $\sin(\pi x^2)$

12. $\cos(\sqrt{x})$

13. $y = \sqrt{1 + \cos x}$

14. $\sin(2\cos x)$

15. $f(x) = \cos(x + \sin x)$

16. $g(\theta) = \tan(\theta \sin\theta)$

17. $u = \sin^3(\pi x/2)$

18. $y = \sec(1/x)$

19. $F(t) = \sin at \cos at$

20. $G(\theta) = \dfrac{\sin a\theta}{\cos b\theta}$

21. $\sin(2x) - \cos(2x)$

22. $\cos^2 x - \sin^2 x$

23. $\tan x + \cot x$

24. $\sec x - \csc x$

25. $\tan x - x$

26. $\tan(3x)\cot(3x)$

27. $t\cos t - \sin t$

28. $t\sin t + \cos t$

57. Use the method of Example 1 to evaluate $\lim\limits_{h\to 0} \dfrac{1 - \cos h}{h^2}$.

58. Find values of a and b that make

$$f(x) = \begin{cases} ax + b, & x < 0 \\ 2\sin x + 3\cos x, & x \ge 0 \end{cases}$$

differentiable at $x = 0$.

(33) $\frac{d}{dx}\left(\frac{x^2}{f(x)}\right)\Big|_{x=2}$ $\quad f(2)=2$

$f'(2)=3$

$$= \frac{2x\,f(x) - x^2 f'(x)}{f(x)^2}\Big|_{x=2}$$

$$= \frac{\underbrace{4\cdot 2 - 4\cdot 3}}{4} = 2-3 = -1$$

(34) $\frac{d}{dx}\left(\frac{f(x)}{x^2}\right)\Big|_{x=2}$

$$= \frac{f'(x)x^2 - 2x\,f(x)}{x^4}\Big|_{x=2}$$

$$= \frac{3\cdot 4 - 4\cdot 2}{16} = \frac{3\cdot 2}{4} = \frac{6}{4} = \frac{3}{2}$$

(35) $\frac{d}{dx}\left(x^2 f(x)\right)\Big|_{x=2}$

$$= 2x\,f(x) + x^2 f'(x)\Big|_{x=2}$$

$$= 4(2) + 4(3) = 8 + 12 = 20$$

(36) $\frac{d}{dx}\left(\frac{f(x)}{x^2+f(x)}\right)\Big|_{x=2}$ $\quad g(x) = x^2 + f(x)$

$g'(x) = 2x + f'(x)$

$$= \frac{f'(x)\left[x^2+f(x)\right] - f(x)\left[2x+f'(x)\right]}{(x^2+f(x))^2}\Big|_{x=2}$$

$=$

Solve the equations in Exercises 11–14 for x.

11. $2^{x+1} = 3^x$

12. $3^x = 9^{1-x}$

13. $\dfrac{1}{2^x} = \dfrac{5}{8^{x+3}}$

14. $2^{x^2-3} = 4^x$

Find the domains of the functions in Exercises 15–16.

15. $\ln \dfrac{x}{2-x}$

16. $\ln(x^2 - x - 2)$

Solve the inequalities in Exercises 17–18.

17. $\ln(2x-5) > \ln(7-2x)$

18. $\ln(x^2-2) \le \ln x$

In Exercises 19–48, differentiate the given functions. If possible, simplify your answers.

19. $y = e^{5x}$

20. $y = xe^x - x$

21. $y = \dfrac{x}{e^{2x}}$

22. $y = x^2 e^{x/2}$

23. $y = \ln(3x-2)$

24. $y = \ln|3x-2|$

25. $y = \ln(1+e^x)$

26. $f(x) = e^{(x^2)}$

27. $y = \dfrac{e^x + e^{-x}}{2}$

28. $x = e^{3t} \ln t$

29. $y = e^{(e^x)}$

30. $y = \dfrac{e^x}{1+e^x}$

31. $y = e^x \sin x$

32. $y = e^{-x} \cos x$

33. $y = \ln \ln x$

34. $y = x \ln x - x$

35. $y = x^2 \ln x - \dfrac{x^2}{2}$

36. $y = \ln|\sin x|$

37. $y = 5^{2x+1}$

38. $y = 2^{(x^2-3x+8)}$

39. $g(x) = t^x x^t$

40. $h(t) = t^x - x^t$

41. $f(s) = \log_a(bs+c)$

42. $g(x) = \log_x(2x+3)$

43. $y = x^{\sqrt{x}}$

44. $y = (1/x)^{\ln x}$

45. $y = \ln|\sec x + \tan x|$

46. $y = \ln|x + \sqrt{x^2 - a^2}|$

47. $y = \ln\left(\sqrt{x^2+a^2} - x\right)$

48. $y = (\cos x)^x - x^{\cos x}$

49. Find the nth derivative of $f(x) = xe^{ax}$.

50. Show that the nth derivative of $(ax^2 + bx + c)e^x$ is a function of the same form but with different constants.

51. Find the first four derivatives of e^{x^2}.

52. Find the nth derivative of $\ln(2x+1)$.

53. Differentiate (a) $f(x) = (x^x)^x$ and (b) $g(x) = x^{(x^x)}$. Which function grows more rapidly as x grows large?

! **54.** Solve the equation $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = a$, where $a > 0$. The exponent tower goes on forever.

Use logarithmic differentiation to find the required derivatives in Exercises 55–57.

55. $f(x) = (x-1)(x-2)(x-3)(x-4)$. Find $f'(x)$.

56. $F(x) = \dfrac{\sqrt{1+x}\,(1-x)^{1/3}}{(1+5x)^{4/5}}$. Find $F'(0)$.

57. $f(x) = \dfrac{(x^2-1)(x^2-2)(x^2-3)}{(x^2+1)(x^2+2)(x^2+3)}$. Find $f'(2)$. Also find $f'(1)$.

58. At what points does the graph $y = x^2 e^{-x^2}$ have a horizontal tangent line?

59. Let $f(x) = xe^{-x}$. Determine where f is increasing and where it is decreasing. Sketch the graph of f.

60. Find the equation of a straight line of slope 4 that is tangent to the graph of $y = \ln x$.

61. Find an equation of the straight line tangent to the curve $y = e^x$ and passing through the origin.

62. Find an equation of the straight line tangent to the curve $y = \ln x$ and passing through the origin.

63. Find an equation of the straight line that is tangent to $y = 2^x$ and that passes through the point $(1, 0)$.

64. For what values of $a > 0$ does the curve $y = a^x$ intersect the straight line $y = x$?

65. Find the slope of the curve $e^{xy} \ln \dfrac{x}{y} = x + \dfrac{1}{y}$ at $(e, 1/e)$.

66. Find an equation of the straight line tangent to the curve $xe^y + y - 2x = \ln 2$ at the point $(1, \ln 2)$.

67. Find the derivative of $f(x) = Ax \cos \ln x + Bx \sin \ln x$. Use the result to help you find the indefinite integrals

$$\int \cos \ln x\, dx \text{ and } \int \sin \ln x\, dx.$$

! **68.** Let $F_{A,B}(x) = Ae^x \cos x + Be^x \sin x$. Show that $(d/dx)F_{A,B}(x) = F_{A+B,B-A}(x)$.

! **69.** Using the results of Exercise 68, find
(a) $(d^2/dx^2)F_{A,B}(x)$ and (b) $(d^3/dx^3)e^x \cos x$.

! **70.** Find $\dfrac{d}{dx}(Ae^{ax} \cos bx + Be^{ax} \sin bx)$ and use the answer to help you evaluate

(a) $\displaystyle\int e^{ax} \cos bx\, dx$ and (b) $\displaystyle\int e^{ax} \sin bx\, dx$.

? **71.** Prove identity (ii) of Theorem 2 by examining the derivative of the left side minus the right side, as was done in the proof of identity (i).

? **72.** Deduce identity (iii) of Theorem 2 from identities (i) and (ii).

? **73.** Prove identity (iv) of Theorem 2 for rational exponents r by the same method used for Exercise 71.

! **74.** Let $x > 0$, and let $F(x)$ be the area bounded by the curve $y = t^2$, the t-axis, and the vertical lines $t = 0$ and $t = x$. Using the method of the proof of Theorem 1, show that $F'(x) = x^2$. Hence, find an explicit formula for $F(x)$. What is the area of the region bounded by $y = t^2$, $y = 0$, $t = 0$, and $t = 2$?

! **75.** Carry out the following steps to show that $2 < e < 3$. Let $f(t) = 1/t$ for $t > 0$.

(a) Show that the area under $y = f(t)$, above $y = 0$, and between $t = 1$ and $t = 2$ is less than 1 square unit. Deduce that $e > 2$.

(b) Show that all tangent lines to the graph of f lie below the graph. *Hint:* $f''(t) = 2/t^3 > 0$.

(c) Find the lines T_2 and T_3 that are tangent to $y = f(t)$ at $t = 2$ and $t = 3$, respectively.

(d) Find the area A_2 under T_2, above $y = 0$, and between $t = 1$ and $t = 2$. Also find the area A_3 under T_3, above $y = 0$, and between $t = 2$ and $t = 3$.

(e) Show that $A_2 + A_3 > 1$ square unit. Deduce that $e < 3$.

Exam style questions

(1) $f(x)$ and $g(x)$ are differentiable on $\mathbb{R}$

$f(0) = 0$ $\quad g(0) = -1$

$f'(0) = 3$ $\quad g'(0) = 2$

$f'(-1) = 2$

$? : \frac{d}{dx}\left[f(x+g(x))\right]\Big|_{x=0}$

$$\frac{d}{dx}\left[f(x+g(x))\right]\Big|_{x=0}$$

← its a composed function!

$$= f'(0+g(0)) \cdot \frac{d}{dx}\left[x+g(x)\right]\Big|_{x=0}$$

$$= \underbrace{f'(-1)}_{2} \cdot \left[1 + \underbrace{g'(0)}_{2}\right]$$

$$= 2 \cdot 3 = 6$$

(2) $f(x) = \begin{cases} a e^{bx} & x > 0 \\ 2x-1 & x \leq 0 \end{cases}$ $\quad$ | a, b so that $f(x)$ is continuous and diff. at $x=0$

(a) Continuity at $x=0$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$$

$$\lim_{x \to 0^+} (a e^{bx}) = \lim_{x \to 0^-} (2x-1)$$

$$\Rightarrow a = -1$$

(b) Differentiability at $x=0$

$$f'(x) = \begin{cases} a \cdot b \cdot e^{bx} & x > 0 \\ 2 & x < 0 \end{cases}$$

$$\lim_{x \to 0^+} (a \cdot b \cdot e^{bx}) = \lim_{x \to 0^-} (2)$$

$$-b = 2$$

$a = -1$

$b = -2$