# LECTURE 2 - CALCULUS

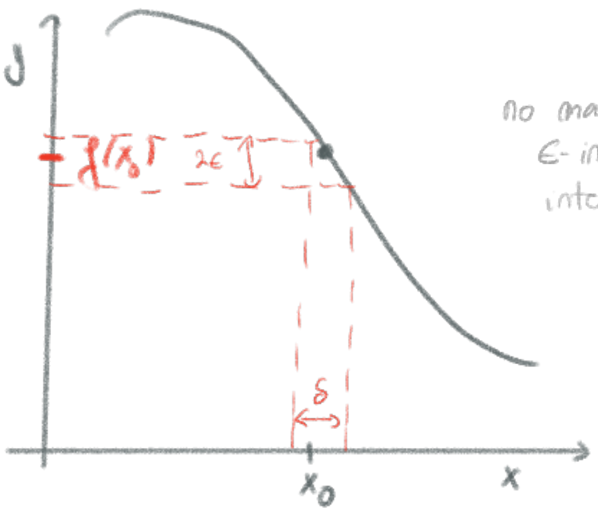
* recap - continuity
* limits (1.2 - 1.3)
* asymptotes (4.6)

## I. Continuity

A function $f(x)$ is continuous at a point $x_0$ of it's domain if, for all points of the domain $x$,

$$\forall \epsilon > 0, \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

* this is a very technical definition, meaning that, as $x$ approaches $x_0$, $f(x)$ approaches $f(x_0)$
* In practice, the function "does not jump", "we do not need to lift the pen"

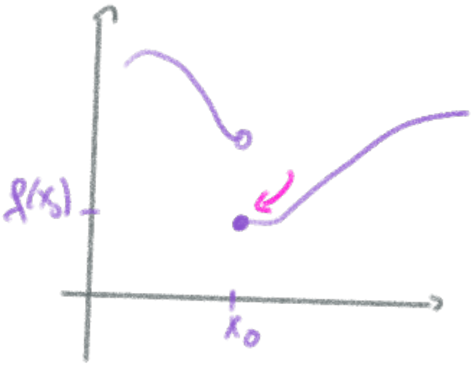


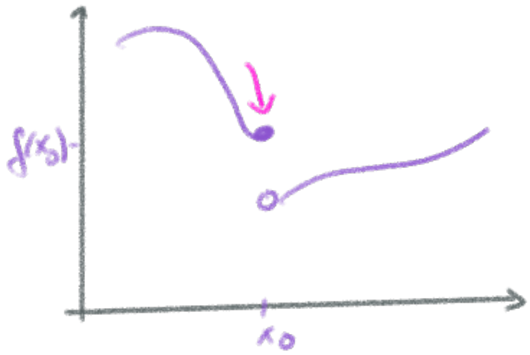

no matter how small we choose $\epsilon > 0$, the $\epsilon$-interval around $f(x_0)$ is mapped onto an interval around $x_0$

* A function is continuous on it's domain if it is continuous on all points of its domain
* All typical functions (polynomials, sin/cos, exp/ln, $\sqrt{\ }$) are continuous on their domain.
* Sums, multiplications, ratios, .., composition of continuous functions are continuous.

↳ note : ON THEIR DOMAIN. For example $f(x) = \frac{1}{x}$ is UNDEFINED at $x = 0$ ($x = 0$ is not in the domain). (this may differ in other textbooks)

* DISCONTINUITIES




$f(x)$ is discontinuous / right continuous at $x_0$

$f(x)$ is discontinuous / left continuous at $x_0$

* left / right continuous : $f(x)$ approaches $f(x_0)$ when $x$ approaches $x_0$ from the left / right

continuous = left and right continuous

* a function is continuous on $[a,b]$ if it is continuous on $(a,b)$, left continuous at $b$ and right continuous at $a$.

* a function is piecewise continuous if it has a finite number of discontinuities.

# II LIMITS

↳ how to describe a function towards the edges of it's domain?

$\lim\limits_{x\to 1} \frac{x-1}{x^2-1}$ ?  $\lim\limits_{x\to 0} x\ln(x^2)$ ?  $\lim\limits_{x\to 0} \frac{\sin(x)}{x}$ ?

* limits describe a function $\begin{cases} \text{AROUND} \\ \text{close to} \\ \text{as } x \text{ approaches} \end{cases}$ $x_0$

↳ typically useful if $f(x)$ is undefined / discontinuous at $x_0$

DEFINITION OF LIMIT : $\lim\limits_{x\to x_0} f(x) = L$ if for all points $x$ in the domain of $f$ ; $\forall \epsilon > 0 \ \exists \delta > 0 : |x-x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$

* meaning : as $x$ approaches $x_0$, $f(x)$ approaches $L$

* Connection with continuity : if $x_0$ is in the domain, and $f$ is continuous at $x_0$ $\Leftrightarrow$ $\lim_{x \to x_0} f(x) = f(x_0)$

↳ but $x_0$ is typically not / does not need to be in the domain of $f$.

* examples (how to calculate limits)

1) $\lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{x-1}{(x-1)(x+1)} = \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{2}$

↳ it is allowed to divide numerator and denomitor by $(x-1)$, since $x \neq 1$, $x$ only comes CLOSE to 1, but does not reach it.

2) $\lim_{x \to 2} \left( \frac{4}{x^2-4} - \frac{1}{x-2} \right) = \lim_{x \to 2} \left( \frac{4-(x+2)}{(x+2)(x-2)} \right) = \lim_{x \to 2} \frac{2-x}{(x+2)(x-2)} = \frac{-1}{4}$

* Left / Right limits : $\lim_{x \to x_0^{\mp}} f(x) = L$

↳ we approach $x_0$ from left$^-$ $(x < x_0)$ or right$^+$ $(x > x_0)$

* example : $\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$

$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$

↳ if $\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x)$, then $\lim_{x \to x_0} f(x)$ DOES NOT EXIST

* this happens if $f(x)$ is discontinuous at $x_0$

* if left and right limit are equal, and $x_0$ is not in the domain of $f$, we can define a continuous extension of $f$!

→ $F(x)$ is a continuous extension of $f$ if

$F(x) = f(x)$ for $x \in$ domain $(f)$

$F(x_0) = \lim_{x \to x_0} f(x_0) = L$ if $x = x_0$

* $F(x) = x+1$ is a continuous extension of $f(x) = \frac{x^2-1}{x-1}$

* examples

$$\lim_{x \to 5} \frac{|x-5|}{x^2-25}$$

- $\lim_{x \to 5^+} \frac{|x-5|}{x^2-25} = \lim_{x \to 5^+} \frac{x-5}{(x-5)(x+5)} = \frac{1}{10}$
- $\lim_{x \to 5^-} \frac{|x-5|}{x^2-25} = \lim_{x \to 5^-} \frac{5-x}{(x-5)(x+5)} = \frac{-1}{10}$

The limit does not exist!

$$\lim_{x \to 0} \frac{|x+3| - |3x-3|}{x} = \lim_{x \to 0} \frac{(\cancel{3}+x) - (\cancel{3}-3x)}{x} = 4$$

$$\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \lim_{x \to 0} \frac{(1+x)-(1-x)}{x(\sqrt{1+x} + \sqrt{1-x})} =$$

$$\lim_{x \to 0} \frac{2\cancel{x}}{\cancel{x}} \frac{1}{\sqrt{1+x} + \sqrt{1-x}} = 1$$

* limits at infinity

↳ how does a function behave towards $\pm\infty$ ?

$$\lim_{x \to \pm\infty} f(x) = L \iff \forall \epsilon > 0 \ \exists M > 0 : \begin{matrix} \overset{\to +\infty}{x > M} \\ \underset{\to -\infty}{x < -M} \end{matrix} \Rightarrow |f(x) - L| < \epsilon$$

↳ IF this limit exists, the function approaches a constant value $L$ as $x \to \pm\infty$

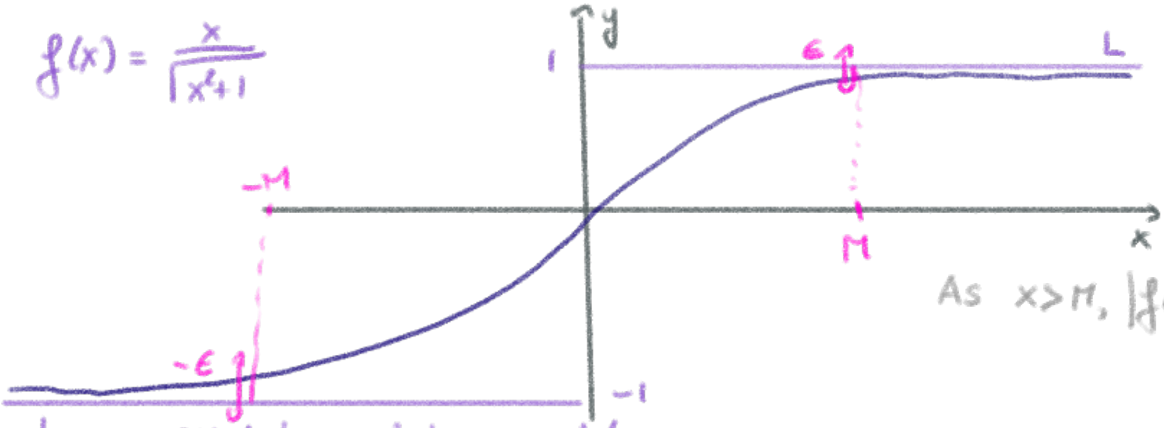
→ in this case, $y = L$ is a horizontal asymptote of $f$

Examples: $\lim_{x \to +\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \to +\infty} \frac{x}{x\sqrt{1+\frac{1}{x^2}}} = 1$

$\lim_{x \to -\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \to -\infty} \frac{x}{-x\sqrt{1+\frac{1}{x^2}}} = -1$

you always need to take the POSITIVE root out of the $\sqrt{\ }$ since $x \to -\infty$, $-x > 0$

$f(x) = \frac{x}{\sqrt{x^2+1}}$




As $x > M$, $|f(x) - L| < \epsilon$

↳ one-sided horizontal asymptote

- other examples : $\lim_{x \to \pm\infty} \frac{1}{1+x^2} = 0$

  $\lim_{x \to -\infty} e^x = 0$

  $\lim_{x \to \pm\infty} \frac{x}{x+1} = 1$

* does $\lim_{x \to \pm\infty} f(x)$ always exist ? NO !!

$\lim_{x \to \pm\infty} \sin(x)$ does not exist, since $f(x)$ does not approach a constant value

↳ we cannot find any large enough $M$, such that $f(x)$ stays within $\epsilon$-distance of a constant $L$

* Infinite limits

↳ some functions become arbitrary large when approaching a finite $x_0 \in \mathbb{R}$, e.g. $\tan(x)$, $\frac{1}{x}$, $\ln(x)$

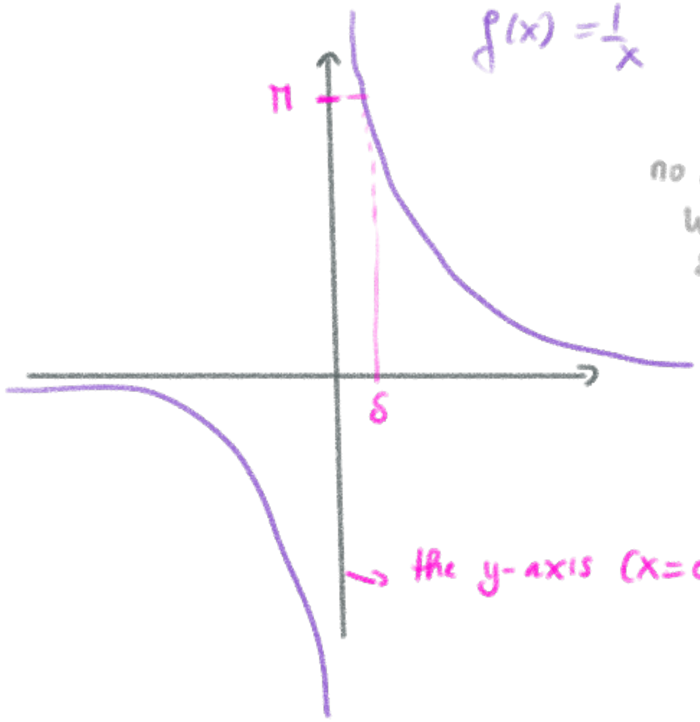
$$\lim_{x \to x_0} f(x) = \pm\infty \iff \forall M > 0 \ \exists \delta > 0 : |x - x_0| < \delta \Rightarrow \begin{array}{l} f(x) > M \ (+\infty) \\ f(x) < -M \ (-\infty) \end{array}$$

↳ in this case, $f(x)$ has a vertical asymptote $x = x_0$

* watch out : left and right limits are often different.

example : $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ , $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\lim_{x \to 0} \frac{1}{x}$ DOES NOT EXIST




no matter how large we choose $M>0$, we can always find $\delta < \frac{1}{M}$ such that if $0<x<\delta$, $f(x)>M$.

the y-axis ($x=0$) is a vertical asymptote

## III ASYMPTOTES

Asymptote = the function approaches a straight line.

→ horizontal asymptote ($y=a$) $\Leftrightarrow \lim\limits_{x\to\pm\infty} f(x)=a$

→ vertical asymptote ($x=b$) $\Leftrightarrow \lim\limits_{x\to b^{\pm}} f(x)=\pm\infty$
($b \neq \pm\infty$)

→ oblique asymptote ($y=ax+b$) ($a\neq 0$)

$\Leftrightarrow \lim\limits_{x\to\pm\infty} (f(x)-(ax+b))=0$

$\Leftrightarrow \lim\limits_{x\to\pm\infty} \frac{f(x)}{x}=a$ and $\lim\limits_{x\to\pm\infty} (f(x)-ax)=b$

↳ horizontal and oblique asymptotes can one-sided (only at $+\infty$ or only at $-\infty$) or two-sided

↳ (at one side) horizontal and oblique asymptotes exclude each other

Examples:

$$f(x) = \frac{x^2}{x^2-1} \quad \text{has}$$

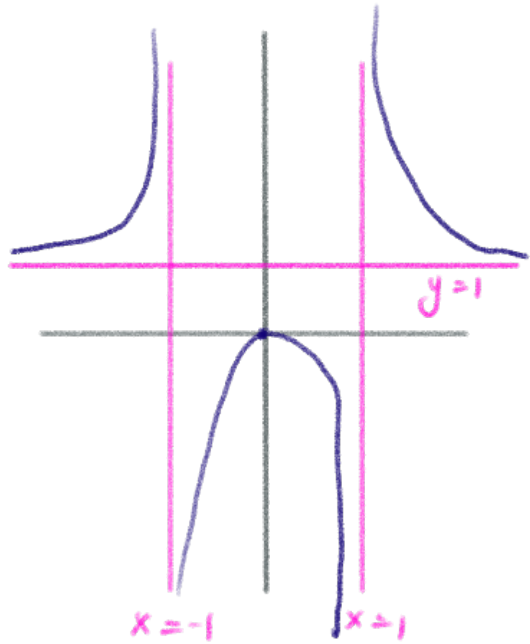
- vertical asymptotes $x=1$ and $x=-1$

$$\lim_{x \to 1^+} \frac{x^2}{x^2-1} = +\infty \;, \quad \lim_{x \to 1^-} \frac{x^2}{x^2-1} = -\infty$$

$$\lim_{x \to -1^+} \frac{x^2}{x^2-1} = -\infty \;, \quad \lim_{x \to -1^-} \frac{x^2}{x^2-1} = +\infty$$

- a 2-sided horizontal asymptote $y=1$

$$\lim_{x \to \pm\infty} \frac{x^2}{x^2-1} = 1$$




$$f(x) = \sqrt{x^2+1} \quad \text{has}$$

- a 1-sided oblique asymptote $y=x$

$$\begin{cases} \lim_{x \to +\infty} \frac{\sqrt{x^2+1}}{x} = 1 = a \\ \lim_{x \to +\infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x \to +\infty} \frac{\cancel{x^2}+1 - \cancel{x^2}}{\sqrt{x^2+1}+x} = 0 = b \end{cases}$$

- a 1-sided oblique asymptote $y=-x$

$$\lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{x} = -1 = a$$

$$\lim_{x \to -\infty} \left(\sqrt{x^2+1} + x\right) = \lim_{x \to -\infty} \frac{\cancel{x^2}+1 - \cancel{x^2}}{\sqrt{x^2+1}-x} = 0 = b$$