Predicate Logic

Introduction

Statement	Translation
John walks	$R\,j$
Bob walks	$R\,b$ ← object

↑ property

Synthax

- Symbols for constants (objects) ⟶ a, b, c
- Symbols vor variables ⟶ x, y, z
- Symbols for predicates ⟶ A, B, C
- Logical operators $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- Quantifiers $\forall x$, $\exists x$

Syllogisms

- All A are B : $\forall x\,(Ax \rightarrow Bx)$
- Some A are B: $\exists x\,(Ax \wedge Bx)$
- All A are not B: $\forall x\,(Ax \rightarrow \neg Bx)$ $(\neg \exists x\,($

Relations

- John sees Mary ⟶ $S\,j\,m$
- John gives Mary the book ⟶ $G\,j\,m\,b$

Complex quantifier patterns

- Everyone sees someone $\forall x\, \exists y\,(S\,x\,y)$
- Everyone is seen by someone $\forall x\, \exists y\,(S\,y\,x)$

Examples

$L\,x\,y$ ⟶ x loves y
$G\,x$ ⟶ x is girl
$B\,x$ ⟶ x is boy

Every boy loves a girl
$\forall x\,(Bx \rightarrow \exists y\,(Gy \wedge Lxy))$

Every girl who loves all boys does not love every girl:

$$\forall x\,((Gx \wedge \varphi(x)) \longrightarrow \psi(x))$$

Intuitive validities

- $\neg \forall x\ \varphi x \equiv \exists x\ \neg \varphi x$

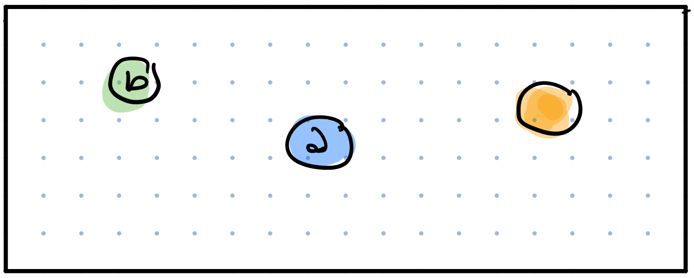
  any predicate

- $\neg \exists x\ \varphi x \equiv \forall x\ \neg \varphi x$
- $\forall x\ \varphi x \equiv \neg \exists x\ \neg \varphi x$
- $\exists x\ \varphi x \equiv \neg \forall x\ \neg \varphi x$
- $\neg \forall x (\varphi x \rightarrow \psi x) \equiv \exists x\ \neg (\varphi x \rightarrow \psi x)$

  $\equiv \exists x (\varphi x \wedge \neg \psi x)$

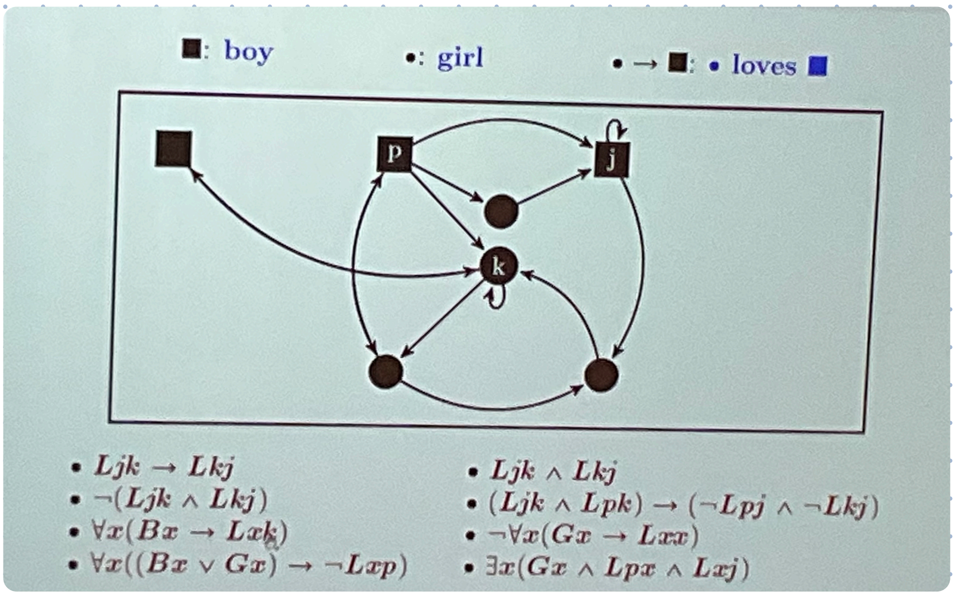
- $\forall x (\varphi x \wedge \psi x) \equiv \forall x\ \varphi x \wedge \forall x\ \psi x$

Evaluating formulas



Color is a property

- Ba → true (a is blue)
- $\exists x\ Sx \vee Cb$ (x is either square or circle)
- $Ra \rightarrow Sb$ true (a is not red)



Language

- Term t is a variable / constant
- Formula

  1. $P\, t_1\, t_2$
  2. $\varphi \wedge \psi$
  3. $\forall x\, \varphi$

Substitution

- Inside a term $\longrightarrow$ replacing the occurrences of the variable y for the term t inside s

  $(s)^y_t$

  For a constant: $(c)^y_t := c$ $\qquad (a)^x_c = a$

  For a variable: $\begin{cases} (x)^y_t := x \\ (y)^y_t := t \end{cases}$

- Inside a formula

$$\begin{aligned}
(Pt_1 \cdots t_n)^y_t &:= P(t_1)^y_t \cdots (t_n)^y_t \\
(\neg\varphi)^y_t &:= \neg(\varphi)^y_t \\
(\varphi \wedge \psi)^y_t &:= (\varphi)^y_t \wedge (\psi)^y_t \\
(\varphi \vee \psi)^y_t &:= (\varphi)^y_t \vee (\psi)^y_t \\
(\varphi \to \psi)^y_t &:= (\varphi)^y_t \to (\psi)^y_t \\
(\varphi \leftrightarrow \psi)^y_t &:= (\varphi)^y_t \leftrightarrow (\psi)^y_t
\end{aligned}$$

$$\begin{cases} (\forall x\varphi)^y_t := \forall x(\varphi)^y_t \\ (\forall y\varphi)^y_t := \forall y\varphi \end{cases} \qquad \begin{cases} (\exists x\varphi)^y_t := \exists x(\varphi)^y_t \\ (\exists y\varphi)^y_t := \exists y\varphi \end{cases}$$

Models

A model is a tuple $\mathcal{M} = \langle D, I, g \rangle$

- D is the domain $\rightarrow$ non-empty collection of obj
- I interpretation function $\rightarrow$ assigns to each symbol
  - a relation