Lecture 3: Solution sets, linear independence
(book: 1.5, 1.7)

Previous lecture: column point of view to an SLE.

Today: homogeneous / nonhomogeneous SLE
+ linear independence.

Homogeneous SLE : $A\underline{x} = \underline{0}$

Is it always consistent? Yes, as there is the trivial solution $\underline{x} = \underline{0}$.

Is there also a nontrivial solution?
No free variables $\to$ No.
At least one free variable $\to$ Yes.

$$\begin{aligned} 2x_1 + 4x_2 &= 0 \\ x_1 + 2x_2 &= 0 \end{aligned}$$

homogeneous SLE.

$$\left[\begin{array}{cc|c} 2 & 4 & 0 \\ 1 & 2 & 0 \end{array}\right] \underset{R_2 : R_2 - \frac{1}{2}R_1}{\sim} \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \underset{R_1 * \frac{1}{2}}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\underline{w}}$$

$\text{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$.

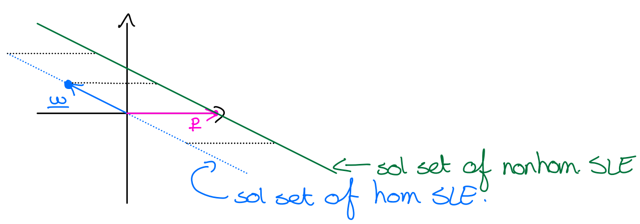
$$\begin{aligned} 2x_1 + 4x_2 &= 6 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

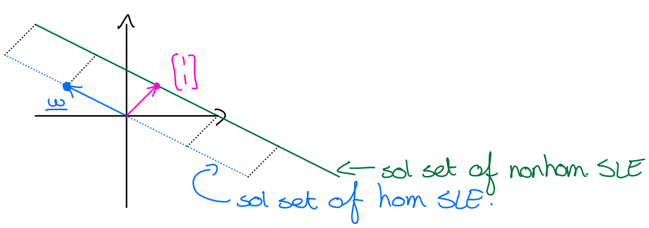
nonhomogeneous SLE.

$$\left[\begin{array}{cc|c} 2 & 4 & 6 \\ 1 & 2 & 3 \end{array}\right] \underset{R_2 : R_2 - \frac{1}{2}R_1}{\sim} \left[\begin{array}{cc|c} 2 & 4 & 6 \\ 0 & 0 & 0 \end{array}\right] \underset{R_1 * \frac{1}{2}}{\sim} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 - 2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

particular sol of the nonhomogeneous SLE.





Observation: the sol. set of $A\underline{x}=\underline{b}$ (when non-empty) is a **translation** of the sol. set of $A\underline{x}=\underline{0}$ for a special vector $\underline{p}$, where $\underline{p}$ is a particular solution of the nonhom. SLE (take $x_2=0$). Any particular solution works.

Theorem: Assume $A\underline{x}=\underline{b}$ is consistent, and let $\underline{p}$ be a particular solution of $A\underline{x}=\underline{b}$. So, $A\underline{p}=\underline{b}$. Then,

Set of all solutions of $A\underline{x}=\underline{b}$

$=$

Set of vectors that can be written as $\underline{q}+\underline{p}$, where $A\underline{q}=\underline{0}$.

Proof:

"$\subseteq$" Let $\underline{v}$ be a solution of $A\underline{x}=\underline{b}$, so $A\underline{v}=\underline{b}$. We need to show that we can write $\underline{v}=\underline{q}+\underline{p}$, where $A\underline{q}=\underline{0}$. So, we need to show that $A\underline{q}=\underline{0}$, where $\underline{q}=\underline{v}-\underline{p}$.
Here we go:
$A\underline{q}=A(\underline{v}-\underline{p})=A\underline{v}-A\underline{p}=\underline{b}-\underline{b}=\underline{0}$ ✓

"$\supseteq$" Let $\underline{v}$ be a vector such that $\underline{v}=\underline{q}+\underline{p}$, where $A\underline{q}=\underline{0}$. We need to show that $\underline{v}$ is a solution of $A\underline{x}=\underline{b}$.
Here we go:
$A\underline{v}=A(\underline{q}+\underline{p})=A\underline{q}+A\underline{p}=\underline{0}+\underline{b}=\underline{b}$ ✓ $\square$

Conclusion:
If we want to solve an SLE $A\underline{x}=\underline{b}$, and we already know the sol. set of the corresponding $A\underline{x}=\underline{0}$, there are three possibilities:

- Row reduce $[A \,\vdots\, \underline{b}]$
- Re-apply the row operations, but now only to $\underline{b}$.
- If we can easily spot a particular solution for $A\underline{x}=\underline{b}$, we add this solution to the sol. set of $A\underline{x}=\underline{0}$.

The set $\{\underline{v}_1, \dots, \underline{v}_p\}$ is **linearly independent** if

$$c_1\underline{v}_1+c_2\underline{v}_2+\dots+c_p\underline{v}_p=\underline{0} \quad \text{implies} \quad c_1=c_2=\dots=c_p=0.$$

(it has **only** the **trivial solution**).

Otherwise: it's called **linearly dependent**.

Examples: $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$ lin indep. ?

For example, $5\cdot\begin{bmatrix}1\\1\end{bmatrix} + 1\cdot\begin{bmatrix}1\\-2\end{bmatrix} + (-3)\cdot\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ So, lin dep.

$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ lin indep?

$c_1\cdot\begin{bmatrix}1\\0\end{bmatrix} + c_2\cdot\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}c_1+0\\0+c_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\Rightarrow c_1=0$
$\Rightarrow c_2=0.$

Only the trivial solution. So, lin indep.

How can we answer this question in general?
Consider the corresponding homogeneous SLE and reduce it to REF.
* no free variables $\rightarrow$ unique sol (only the trivial sol) $\rightarrow$ lin indep.
* some free variables $\rightarrow$ infinitely many sols $\rightarrow$ lin dep.

If a set contains more vectors than there are entries in each vector.

$$4\Big[\overbrace{\qquad\qquad\qquad\qquad\qquad}^{10}\Big]$$

$\rightarrow$ more columns than rows.
$\rightarrow$ there must be a column without a pivot.
$\rightarrow$ some free variables.
$\rightarrow$ lin dep.

What about a set containg only one vector? is $\{\underline{v}\}$ lin indep?

$\begin{bmatrix}0\\0\end{bmatrix}$ lin dep.

$\begin{bmatrix}1\\1\end{bmatrix}$ lin indep

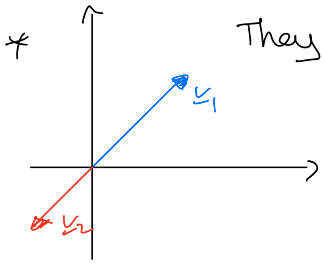
$\begin{bmatrix}1\\0\end{bmatrix}$ lin indep.

* if $\underline{v} \neq \underline{0}$, then we need $c=0$ (only trivial sol). So, $\{\underline{v}\}$ is lin indep.
* if $\underline{v} = \underline{0}$, then c can be anything (also nontrivial sol).
So, $\{\underline{v}\}$ is lin dep.

What about a set containing the zero vector?
Is $\{\underline{v}_1, \dots, \underline{v}_p, \underline{0}\}$ lin indep? $c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_p\underline{v}_p + c_{p+1}\underline{0} = \underline{0}$.
$c_1 = \dots = c_p = 0$, $c_{p+1} = 8$ is for example a nontrivial sol.
So, a set containing the zero vector is always lin dep.

What about a set with two vectors? is $\{\underline{v}_1, \underline{v}_2\}$ lin dep?
Assume $\underline{v}_1 \neq \underline{0}$ and $\underline{v}_2 \neq \underline{0}$.

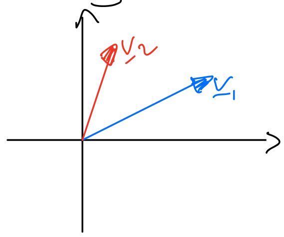
* They lie on the same line.



$$\underline{v}_2 = -2/3\, \underline{v}_1$$
$$\underline{v}_2 + 2/3\, \underline{v}_1 = \underline{0}$$
$$2/3\, \underline{v}_1 + \underline{v}_2 = \underline{0}$$

→ we found a nontriv. sol.
→ $\{\underline{v}_1, \underline{v}_2\}$ is lin dep.

* They do not lie on the same line.



→ lin indep.

Proof: by contradiction :)

Suppose $\{\underline{v}_1, \underline{v}_2\}$ is lin dep.

$\Longrightarrow$ $c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 = \underline{0}$ non triv. sol.

Suppose $c_1 \neq 0$.
→ $c_1 \underline{v}_1 + c_2 \cdot \underline{v}_2 = \underline{0}$
→ $c_1 \underline{v}_1 = -c_2 \cdot \underline{v}_2$
→ $\underline{v}_1 = -\frac{c_2}{c_1} \cdot \underline{v}_2$ ↯

So, $c_1 = 0$. So, $c_2 \neq 0$.

$$\underset{c_2 \neq 0}{c_2} \cdot \underset{\underline{v}_2 \neq \underline{0}}{\underline{v}_2} = \underline{0} \quad ↯$$

So, $\{\underline{v}_1, \underline{v}_2\}$ is lin indep.

$\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4, \underline{v}_5\}$ lin indep?

$\underline{v}_4$ is a lin comb of $\underline{v}_1, \underline{v}_2, \underline{v}_3$.

$$\underline{v}_4 = 2 \cdot \underline{v}_1 + (-8) \cdot \underline{v}_2 + 3.5 \cdot \underline{v}_3.$$

$$\underset{2}{c_1} \cdot \underline{v}_1 + \underset{-8}{c_2} \cdot \underline{v}_2 + \underset{3.5}{c_3} \cdot \underline{v}_3 + \underset{-1}{c_4} \cdot \underline{v}_4 + \underset{0}{c_5} \cdot \underline{v}_5 = \underline{0} \; ?$$

non triv sol
lin dep.

Theorem: $\{\underline{v}_1, \dots, \underline{v}_p\}$ is lin dep $\Leftrightarrow$ at least one of the vectors is a linear combination of the others.

Proof: "$\Leftarrow$" Assume $\underline{v}_j = c_1 \cdot \underline{v}_1 + \dots + c_{j-1} \underline{v}_{j-1} + c_{j+1} \underline{v}_{j+1} + \dots + c_p \cdot \underline{v}_p$

Then, $c_1 \cdot \underline{v}_1 + \dots + c_{j-1} \underline{v}_{j-1} + (-1) \cdot \underline{v}_j + c_{j+1} \underline{v}_{j+1} + \dots + c_p \underline{v}_p = \underline{0}$.

The weight of $\underline{v}_j$ is nonzero.

So, we found a nontrivial sol.

So, $\{\underline{v}_1, \dots, \underline{v}_p\}$ is lin dep

"$\Rightarrow$" Assume $\{\underline{v}_1, \dots, \underline{v}_p\}$ is lin dep.

Distinguish between two cases:

Case 1: $\underline{v}_1 = \underline{0}$.

Then $\underline{v}_1 = 0 \cdot \underline{v}_2 + \dots + 0 \cdot \underline{v}_p$

So, $\underline{v}_1$ is a lin comb. of the others.

Case 2: $\underline{v}_1 \neq \underline{0}$.

Since $\{\underline{v}_1, \dots, \underline{v}_p\}$ is lin dep, there is a nontrivial sol $c_1 \cdot \underline{v}_1 + \dots + c_p \cdot \underline{v}_p = \underline{0}$.

Let j be the largest subscript for which $c_j \neq 0$.

Note: this subscript exists because it is a nontrivial sol.

Moreover, note that $j=1$ would imply $c_1 \cdot \underline{v}_1 = \underline{0}$, which is not possible because $c_1 \neq 0$ and $\underline{v}_1 \neq \underline{0}$.

Hence, $j > 1$ and $c_1 \underline{v}_1 + \dots + c_j \underline{v}_j + 0 \cdot \underline{v}_j + \dots + 0 \cdot \underline{v}_p = \underline{0}$

$\Rightarrow c_j \underline{v}_j = -c_1 \underline{v}_1 - \dots - c_{j-1} \underline{v}_{j-1} + 0 \cdot \underline{v}_{j+1} + \dots + 0 \cdot \underline{v}_p$

$\Rightarrow \underline{v}_j = \frac{-c_1}{c_j} \underline{v}_1 + \dots + \frac{-c_{j-1}}{c_j} \underline{v}_{j-1} + 0 \cdot \underline{v}_{j+1} + \dots + 0 \cdot \underline{v}_p$

So, $\underline{v}_j$ is a lin comb. of the others. $\square$

So, we actually also already proved:

If $\{\underline{v}_1, \dots, \underline{v}_p\}$ is lin dep. and $\underline{v}_1 \neq \underline{0}$, then there is a $j \in \{2, \dots, p\}$ such that $\underline{v}_j$ is a lin comb. of $\{\underline{v}_1, \dots, \underline{v}_{j-1}\}$.