

1.3 Examples

$$(5) S = [(p \wedge ((q \vee \neg r) \rightarrow \neg q)) \rightarrow (\neg q \vee r)]$$

			A		B		C			
p	q	r	$q \vee \neg r$	$\neg q$	$\neg r$	$A \rightarrow B$	$p \wedge C$	$C \rightarrow B$		
0	0	0	1	1	1					
0	0	1	0	1	0					
0	1	0	1	0	1					
0	1	1	0	0	0					
1	0	0	1	1	1					
1	0	1	0	1	0					
1	1	0	1	0	1					
1	1	1	0	0	0					

1.5 Theory (Direct proofs)

- (a) To prove something for all x , we just need to prove it for an arbitrary x .
- (b) If we choose a number y after an x , we can let y depend on x .
- (c) Assume the antecedent is true, and prove the consequent.

1.5 Direct proof exercises

(1) $\forall a \in \mathbb{Z}$, $2a$ is divisible by 5 $\rightarrow a$ is div. by 5

Assume $5 \nmid 2a \quad \forall a \in \mathbb{Z}$

$$\Rightarrow 2a = 5j \quad j \in \mathbb{Z}$$

\uparrow even \uparrow odd \searrow has to be even

$$2a = 5(2k) \quad k \in \mathbb{Z} \quad j = 2k$$

$$a = 5k$$

\hookrightarrow so a is divisible by 5 \square

(2) $7 \nmid 4a \rightarrow 7 \nmid a$

Assume $7 \nmid 4a$

$$4a = 7j \quad j \in \mathbb{Z}$$

\swarrow even \swarrow odd \searrow has to be even (7 is odd. odd \times even = odd)

$$4a = 7(2k) \quad j = 2k$$

$$2a = 7k$$

$$a = 7\left(\frac{k}{2}\right)$$

↳ a is divisible by 7

(3) Every odd int is a difference of 2 squares

$$2k+1 = a^2 - b^2$$

Review of Combinatorics

(1) Rule of sum & Rule of product

(1) 8 women, 5 men

(i) men president $\rightarrow 5 \times 8$
 female $\quad \quad \quad 8 \times 7$

$$= 5 \cdot 8 + 8 \cdot 7 = 8(5+7) = 12 \cdot 8$$

(2) 3 bread, 5 meat, 6 cheese

Past exam paper

$$(1) \quad (p \rightarrow q) \leftrightarrow ((p \wedge \neg q) \rightarrow (q \wedge \neg q))$$

0	1	0	1	0	0	1	1	0
1	0	0	1	1	1	1	0	0
0	1	1	1	0	0	0	1	0
1	1	1	1	1	0	0	1	0

↳ always true

$$(2)a \quad \forall n \in \mathbb{N}, \quad 4^{3n} + 8 \quad \text{div by } 9$$

Base case $n=1$

$$4^3 + 8 = 64 + 8 = 72 \rightarrow \text{div by } 9 \quad \checkmark$$

Induction step

Assume P holds for n

Prove it holds for $n+1$

$$\begin{aligned} 4^{3(n+1)} + 8 &= 4^{3n+3} + 8 \\ &= 4^{3n} \cdot 4^3 + 8 \\ &= 64 \cdot 4^{3n} + 8 \\ &= \underbrace{4^{3n} + 8}_{\text{divisible by } 9} + 63 \cdot 4^{3n} \end{aligned}$$

divisible by 9
as per our assumption

divisible by 9

so it is divisible by 9

$$(2)b \quad \forall n \in \mathbb{N} \quad 3^n > n!$$

I need calculator to find counter example

$$(3) A \subseteq B \iff A \setminus C \subseteq B \setminus C$$

(5) (a) 8 digit, binary (0,1)
contains exactly 3 ones

— — — 1 — — — — —

$n = 8$ (number of spots)

$k = 3$ (objects to select)

$$\binom{n}{k} = \binom{8}{3} = 56$$

(b) $x + y + z = 10$

10 ones need to be distributed

Repetition

Order is not important

$$\binom{(n-1) + k}{k}$$

(c) 26 letters $\times 2$, 5 characters

$$\underline{26} \quad \underline{52} \quad \underline{52} \quad \underline{52} \quad \underline{52} \quad \times 5$$

$$(6)(a) (\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})(\forall z \in \mathbb{Z}) : (x = yz \rightarrow y = -z)$$

$$y = -z$$

$$z = 1$$

$$x = (-z)(z) = -z^2$$

$$x = -1$$

Take $x = -1$, let $y \in \mathbb{Z}$ and $z \in \mathbb{Z}$

Assume $x = yz$, hence $yz = -1$

then, either $y = -1 \wedge z = 1$ or $y = 1 \wedge z = -1$

$$\text{Case 1: } y = -1 \wedge z = 1$$

$$\text{so, } y = -z \quad \checkmark$$

$$\text{Case 2: } y = 1 \wedge z = -1$$

$$\text{so, } y = -z \quad \checkmark$$

$$(6)(b) (\forall n \in \mathbb{N})(\exists X \in \mathcal{P}(\mathbb{N})) : (|X| < n)$$

Let $n \in \mathbb{N}$

consider $X = \emptyset$

$X \in \mathcal{P}(\mathbb{N})$ because $\emptyset \subseteq \mathbb{N}$

then $|\emptyset| = 0 < n$, since $n \in \mathbb{N} (> 0)$

$$(6)(c) \quad a \text{ and } b \text{ are odd} \rightarrow ab^2 \text{ is odd}$$

Contrapositive: $ab^2 \text{ is even} \rightarrow a \text{ or } b \text{ are even}$

(7) $f: \mathbb{Z} \rightarrow \mathbb{Z}$, $f(x) = 3x^2 + 2x + 1$
? is it bijective?

check if injective

$f: \mathbb{Z} \rightarrow \mathbb{Z}$ is injective

$$\Leftrightarrow (\forall x, y \in \mathbb{Z}) : (x \neq y \rightarrow f(x) \neq f(y))$$

$$\Leftrightarrow (\forall x, y \in \mathbb{Z}) : (f(x) = f(y) \rightarrow x = y)$$

Let $x, y \in \mathbb{Z}$

Assume $x \neq y$