

Lecture 11: Orthogonality and Symmetric Matrices.

(book: 6.1, 6.2, 7.1)

Previous episode: Diagonalization

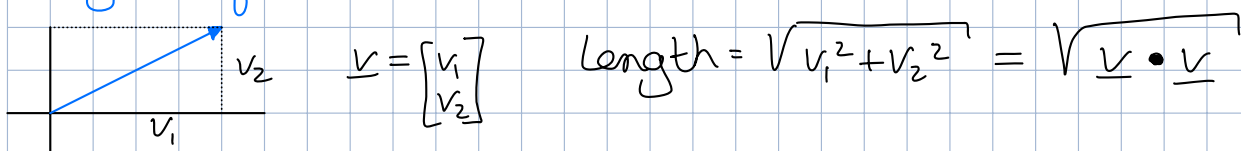
Next episode: Old Exam (Resit 2022-2023)

inner/dot product $\underline{u}, \underline{v} \in \mathbb{R}^n$: $\underline{u} \cdot \underline{v} = \underline{u}^T \cdot \underline{v} = [u_1 \dots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + \dots + u_n v_n$

Properties:

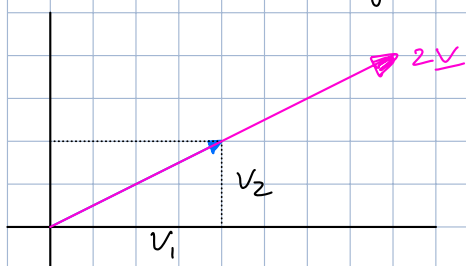
- * $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$ (commutativity)
- * $(\underline{u} + \underline{v}) \cdot \underline{w} = \underline{u} \cdot \underline{w} + \underline{v} \cdot \underline{w}$ (distributivity)
- * $\underline{0} \cdot \underline{u} = 0$
- * $(c \cdot \underline{u}) \cdot \underline{v} = c \cdot (\underline{u} \cdot \underline{v}) = \underline{u} \cdot (c \cdot \underline{v})$
- * $\underline{u} \cdot \underline{u} \geq 0$
- * $\underline{u} \cdot \underline{u} = 0 \Leftrightarrow \underline{u} = \underline{0}$.

Length of a vector:



Similar in \mathbb{R}^3 : $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ length = $\sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{\underline{v} \cdot \underline{v}}$

length/norm of a vector $\underline{v} \in \mathbb{R}^n$: $\|\underline{v}\| = \sqrt{\underline{v} \cdot \underline{v}}$



$$c \cdot \underline{v} = \begin{bmatrix} c \cdot v_1 \\ c \cdot v_2 \end{bmatrix}$$

$$\|c \cdot \underline{v}\| = |c| \cdot \|\underline{v}\|$$

DIY.

unit vector: vector of length 1.

For example, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

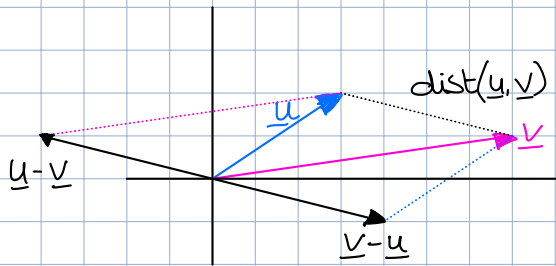
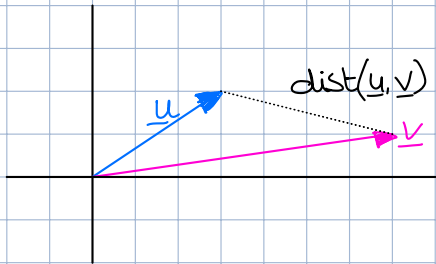
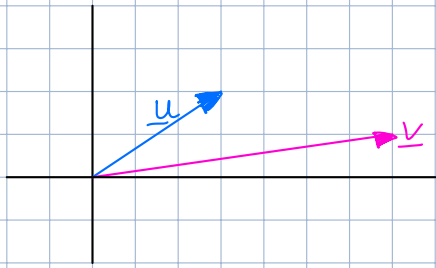
$$\underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \|\underline{v}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$\underline{u} = \frac{1}{\|\underline{v}\|} \underline{v} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

normalizing vector \underline{v} .

$$\underline{w} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

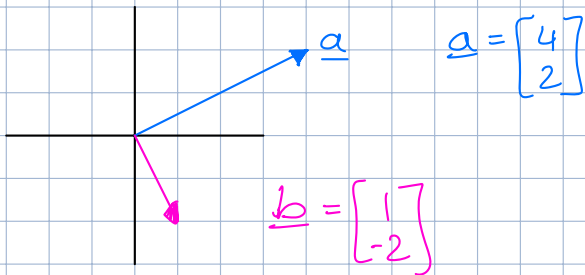
Distance between two vectors:



$$\text{Hence, } \text{dist}(u, v) = \|v - u\| = \|u - v\|.$$

Two vectors are orthogonal ($u \perp v$) if $u \cdot v = 0$.
"perpendicular"

$$\Leftrightarrow \|u + v\|^2 = \|u\|^2 + \|v\|^2.$$



$$a \cdot b = 4 \cdot 1 + 2 \cdot (-2) = 4 - 4 = 0$$

a is perpendicular to b .

Applications:

- * it relates to the correlation coefficient in statistics
- * it is important for matching & feature detection in signal and image processing.

Recall the **Null space** of a matrix A .

$$\text{Nul}(A) = \{ \underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{0} \}$$

$$\begin{bmatrix} \underline{r}_1 \\ \underline{r}_2 \\ \vdots \\ \underline{r}_m \end{bmatrix} \begin{bmatrix} \underline{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{cases} \underline{r}_1^T \cdot \underline{x} = 0 \\ \underline{r}_2^T \cdot \underline{x} = 0 \\ \vdots \\ \underline{r}_m^T \cdot \underline{x} = 0 \end{cases}$$

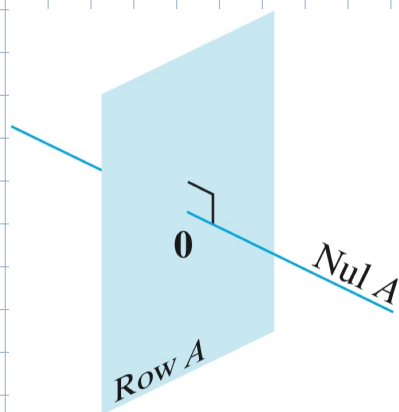
So, every $\underline{x} \in \text{Nul}(A)$ is **orthogonal** to each of the rows of A .

$$(\underline{c}_1 \cdot \underline{r}_1^T + \underline{c}_2 \cdot \underline{r}_2^T + \dots + \underline{c}_m \cdot \underline{r}_m^T) \cdot \underline{x} = \underline{c}_1 (\underline{r}_1^T \cdot \underline{x}) + \underline{c}_2 (\underline{r}_2^T \cdot \underline{x}) + \dots + \underline{c}_m (\underline{r}_m^T \cdot \underline{x}) \\ = \underline{c}_1 \cdot 0 + \underline{c}_2 \cdot 0 + \dots + \underline{c}_m \cdot 0 = 0.$$

So, every vector in $\text{Nul}(A)$ is **orthogonal** to every vector in $\text{Row}(A)$.

$$\Rightarrow \text{Nul}(A) \perp \text{Row}(A).$$

Recall: \downarrow in \mathbb{R}^n \downarrow in \mathbb{R}^n
 $\dim = \#$ free vars $\dim = \#$ pivot rows $\sum \dim = n$.



And, since $\text{Col}(A) = \text{Row}(A^T)$
we also have $\text{Col}(A) \perp \text{Nul}(A^T)$.

W : subspace of \mathbb{R}^n .

W^\perp ("W perpendicular"): orthogonal complement of W .
 \hookrightarrow all vectors in \mathbb{R}^n that are orthogonal to W .

$$(\text{Row}(A))^\perp = \text{Nul}(A) \quad . \quad (\text{Col}(A))^\perp = \text{Nul}(A^T).$$

W^\perp is also a subspace of \mathbb{R}^n (exc. 3d Ch 6.1).
 $(W^\perp)^\perp = W$.

In general, $\dim(W) + \dim(W^\perp) = n$.

$\{v_1, \dots, v_k\}$ is an orthogonal set if $v_i \cdot v_j = 0$ for all $i \neq j$.

Theorem: If $S = \{v_1, \dots, v_k\}$ is an orthogonal set and $0 \notin S$, then S is linearly independent and thus S forms a basis for $\text{Span}\{v_1, \dots, v_k\}$.

Proof: Book Thm 4.

$\{v_1, \dots, v_k\}$ is an orthonormal set if it is an orthogonal set of unit vectors.
 \hookrightarrow vectors of length 1.

How to test whether $\{v_1, \dots, v_k\}$ is orthogonal/orthonormal?

Create $A = \begin{bmatrix} | & & | \\ v_1 & \dots & v_k \\ | & & | \end{bmatrix}$

Compute $A^T A = \begin{bmatrix} -v_1^T - \\ \vdots \\ -v_k^T - \end{bmatrix} \begin{bmatrix} | & \dots & | \\ v_1 & \dots & v_k \\ | & & | \end{bmatrix} = \begin{bmatrix} & & j \\ i & & \\ & & v_i \cdot v_j \end{bmatrix}$

$\{v_1, \dots, v_k\}$ is orthogonal $\Leftrightarrow A^T A$ is diagonal.

$\{v_1, \dots, v_k\}$ is orthonormal $\Leftrightarrow A^T A$ is identity matrix.

A square matrix A is an orthogonal matrix $\Leftrightarrow A^T A = I_n \Leftrightarrow A^{-1} = A^T$.

Watch out the terminology: an orthogonal matrix has orthonormal columns

Orthogonal basis for a subspace W of \mathbb{R}^n : it is a basis of W , where the vectors form an orthogonal set.

Let $\{\underline{u}_1, \dots, \underline{u}_k\}$ be an orthogonal basis of W .

Let $\underline{y} \in W$.

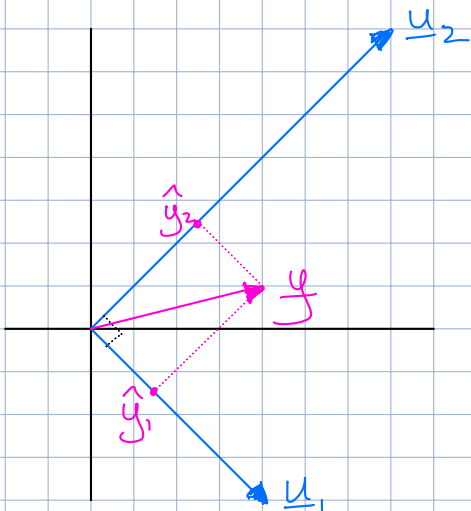
Then, $\underline{y} = c_1 \underline{u}_1 + \dots + c_k \underline{u}_k$.

What are the weights c_1, \dots, c_k ?

$$\begin{aligned} \underline{y} \cdot \underline{u}_1 &= (c_1 \underline{u}_1 + \dots + c_k \underline{u}_k) \cdot \underline{u}_1 = c_1 \underline{u}_1 \cdot \underline{u}_1 + \cancel{c_2 \underline{u}_2 \cdot \underline{u}_1} + \dots + \cancel{c_k \underline{u}_k \cdot \underline{u}_1} \\ &= c_1 \underline{u}_1 \cdot \underline{u}_1 \end{aligned}$$

$$\Rightarrow c_1 = \frac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \quad \dots \quad c_k = \frac{\underline{y} \cdot \underline{u}_k}{\underline{u}_k \cdot \underline{u}_k}$$

So, it's easy to find the weights :)
(non-orthogonal basis: solving an SLE).



$$\underline{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \underline{y} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\frac{3 \cdot 2 + 5 \cdot 1}{2 \cdot 2 + 1 \cdot 1}$$

$$\underline{y} = \hat{\underline{y}}_1 + \hat{\underline{y}}_2 = \underbrace{\frac{\underline{y} \cdot \underline{u}_1}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1}_{\text{orthogonal projection of } \underline{y} \text{ onto } \underline{u}_1} + \underbrace{\frac{\underline{y} \cdot \underline{u}_2}{\underline{u}_2 \cdot \underline{u}_2} \underline{u}_2}_{\text{orthogonal projection of } \underline{y} \text{ onto } \underline{u}_2}$$

orthogonal projection
of \underline{y} onto \underline{u}_1 .

orthogonal projection
of \underline{y} onto \underline{u}_2 .

Recall from a previous episode:

A is diagonalizable \Leftrightarrow the sum of the dimensions of the eigenspaces equals n .

Symmetric matrix: $A = A^T$.

$$\begin{bmatrix} -1 & 6 & -4 \\ 6 & 2 & 0 \\ -4 & 0 & 3 \end{bmatrix}$$

For an $n \times n$ symmetric matrix:

- * All eigenvalues are real numbers.
- * Eigenvectors from different eigenspaces are orthogonal.
- * A is diagonalizable. !

Proof: DIY.

A is called orthogonally diagonalizable if there is orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$ or $A = PDP^T$.

A is orthogonally diagonalizable $\Leftrightarrow A$ is symmetric.

Example: $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ with eigenvalues -2 and 7 .

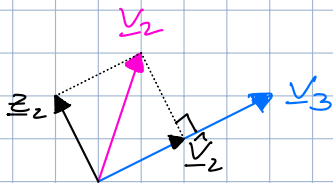
Orthogonally diagonalize A .

$$A - (-2)I = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{x} = x_3 \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix} \quad \underline{v}_1 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

$$A - 7I = \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 1/2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{x} = x_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \quad \underline{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We need to make \underline{v}_2 and \underline{v}_3 orthogonal.



Projection of \underline{v}_2 onto \underline{v}_3 : $\hat{\underline{v}}_2 = \frac{\underline{v}_2 \cdot \underline{v}_3}{\underline{v}_3 \cdot \underline{v}_3} \underline{v}_3 = \frac{-1/2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 0 \\ -1/4 \end{bmatrix}$

Component of \underline{v}_2 orthogonal to \underline{v}_3 :

$$\underline{z}_2 = \underline{v}_2 - \hat{\underline{v}}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -1/4 \\ 0 \\ -1/4 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$

Note: \underline{z}_2 is also an eigenvector because it's a linear combination of \underline{v}_2 and \underline{v}_3 .

Moreover $\underline{z}_2 \perp \underline{v}_3$ 😊

So $\{\underline{z}_2, \underline{v}_3\} = \left\{ \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ forms an **orthogonal basis** for the eigenspace.

Normalize $\underline{v}_1, \underline{z}_2, \underline{v}_3$: $\underline{u}_1 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$, $\underline{u}_2 = \begin{bmatrix} -1/\sqrt{10} \\ 4/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$, $\underline{u}_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$
 ↳ make the length equal to 1.

Then, $P = [\underline{u}_1 \ \underline{u}_2 \ \underline{u}_3]$ and $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$. $A = PDP^{-1}$.
 ↳ orthogonal matrix $A = PDP^T$ $P^T P = I_n$.