Resit Discrete Mathematics DSAI.

Question 1:

ⓐ

p	q	r	$p \Rightarrow q$	$(p \Rightarrow q) \wedge \neg r$	$p \vee r$	$q \Rightarrow r$	$(p \vee r) \wedge (q \Rightarrow r)$	$(\dots) \Leftrightarrow (\dots)$
T	T	T	T	F	T	T	T	F
T	T	F	T	T	T	F	F	F
T	F	T	F	F	T	T	T	F
T	F	F	F	F	T	T	T	F
F	T	T	T	F	T	T	T	F
F	T	F	T	T	F	F	F	F
F	F	T	T	F	T	T	T	F
F	F	F	T	T	F	T	F	F

Question 2:

ⓐ $P(1)$: $LHS = 1$, $RHS = 1^2 + 10$. $\}$ So, $P(1)$ is not true.

Let $n \in \mathbb{N}$. Assume $P(n)$ is true. So, assume $1+3+5+\dots+(2n-1) = n^2+10$.
Then, $1+3+5+\dots+(2n-1)+(2(n+1)-1) = n^2+10+(2n+1) = n^2+2n+11 = (n+1)^2+10$
Hence, $P(n) \Rightarrow P(n+1)$ for all $n \in \mathbb{N}$.

So, the answer is $\boxed{b}$ $P(n) \Rightarrow P(n+1)$ is true for all $n \in \mathbb{N}$.

ⓑ $P(1)$: $LHS = 1 + \frac{3}{1} = 1+3 = 4$

RHS is $\frac{(1+1)^2}{2} = \frac{2^2}{2} = \frac{4}{2} = 2$ ✗

is $\frac{(1+1)^3}{3} = \frac{2^3}{3} = \frac{8}{3}$ ✗

is $(1+1)^2 = 2^2 = 4$ ✓

Let $n \in \mathbb{N}$. Assume $P(n)$ is true. So, assume $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\dots\left(1+\left(\frac{2n+1}{n^2}\right)\right) = (n+1)^2$

Then, $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\dots\left(1+\left(\frac{2n+1}{n^2}\right)\right)\left(1+\left(\frac{2(n+1)+1}{(n+1)^2}\right)\right) = (n+1)^2\left(1+\left(\frac{2n+3}{(n+1)^2}\right)\right)$

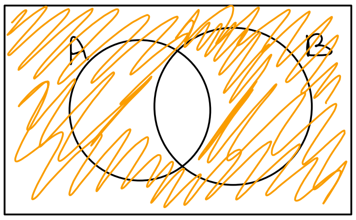
$= (n+1)^2\left(\frac{(n+1)^2+2n+3}{(n+1)^2}\right) = (n+1)^2+2n+3 = n^2+2n+1+2n+3 = n^2+4n+4$

$= (n+2)^2 = ((n+1)+1)^2.$ $\square$

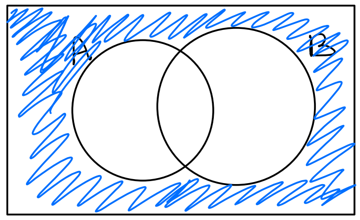
So, the answer is $\boxed{c}$ $(n+1)^2$.

Question 3:

ⓐ $x \notin A \cap B \implies x \notin A$ or $x \notin B$. So, or and thus not and.
So, the proof is incorrect.



$(A \cap B)^c$



$A^c \cap B^c$

So, $(A \cap B)^c \not\subseteq A^c \cap B^c$.
So, the theorem is also incorrect.

So, the answer is $\boxed{d}$ The theorem and the proof are both incorrect.

ⓑ False. Consider the following counterexample:
$A = \{1, 2\}$ $B = \emptyset$ $C = \{3\}$ $D = \{4\}$.
Then, $A \times B = \emptyset \subseteq \{(3,4)\} = C \times D$.
However, $A = \{1,2\} \not\subseteq \{3\} = C$.

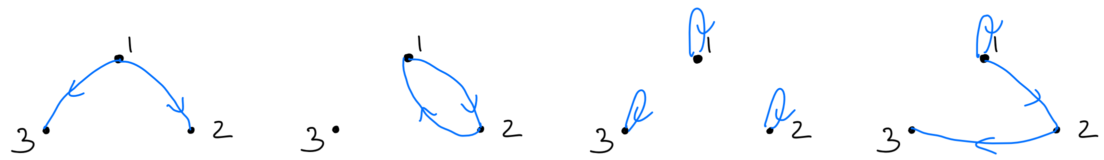
ⓒ True. Proof: Let A, B, C and D be some arbitrary sets.
Distinguish between two cases.

* <u>Case 1:</u> $A = \emptyset$ or $B = \emptyset$. Then $A \times B = \emptyset$ and thus immediately $A \times B \subseteq C \times D$.

* <u>Case 2:</u> $A \neq \emptyset$ and $B \neq \emptyset$. So, $A \times B \neq \emptyset$.
Assume $A \subseteq C$ and $B \subseteq D$.
Let $(a,b) \in A \times B$. Then $a \in A$ and $b \in B$.
Since $A \subseteq C$, we know $a \in C$
and since $B \subseteq D$, we know $b \in D$.
So, $(a,b) \in C \times D$. $\square$

Question 4:

(a)



Symm	✗	✓	✓	✗
Refl	✗	✗	✓	✗
Trans	✓	✗	✓	✗

So, the answer is $\boxed{b}$

(b) R is an equivalence relation. Proof:

* Reflexive: let $x \in \mathbb{N}$. $3x-5x=-2x$, which is even. So, xRx ✓

* Symmetry: let $x \in \mathbb{N}$ and let $y \in \mathbb{N}$.
Assume xRy. So, $3x-5y$ is even.
$\Rightarrow 3x-5y = 2\cdot k$ for some $k \in \mathbb{Z}$.
$\Rightarrow 3y = 3x-2y-2k$.
As a result, $3y-5x = (3x-2y-2k)-5x$
$= -2x-2y-2k$
$= 2(-x-y-k)$, which is even because $-x-y-k \in \mathbb{Z}$.
So, yRx ✓

* Transitivity: let $x \in \mathbb{N}$, $y \in \mathbb{N}$ and $z \in \mathbb{N}$.
Assume xRy. So, $3x-5y=2k$ for some $k \in \mathbb{Z}$.
and assume yRz. So, $3y-5z=2\ell$ for some $\ell \in \mathbb{Z}$.
Then, $3x-5z = (2k+5y)-(3y-2\ell)$
$= 2k+2\ell+2y$
$= 2(k+\ell+y)$, which is even because $k+\ell+y \in \mathbb{Z}$.
So, xRz ✓

$\square$

Question 5:

(a) We need to choose the positions for the 8 zeroes.

$n = 12$
$k = 8$
without repetion.
order is not important.

$$\binom{n}{k} = \binom{12}{8} = 495.$$

So, the answer is $\boxed{b}$

(b) $n = 8$
$k = 3$
without repetition
order is important.

$$\frac{n!}{(n-k)!} = \frac{8!}{5!} = 8 \cdot 7 \cdot 6 = 336.$$

So, the answer is $\boxed{d}$

(c) $n = 3$
$k = 8$
with repetition
order is not important

$$\binom{(n-1)+k}{k} = \binom{2+8}{8} = \binom{10}{8} = 45.$$

So, the answer is $\boxed{a}$

Question 6:

(a) $\neg\big((\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(y^2 > x)\big)$

$(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})(y^2 \leq x)$

So, the answer is $\boxed{d}$

(b) We need to find a counterexample for the statement $(\forall x \in \mathbb{N})(\forall y \in \mathbb{N})(xy > 1)$.

The only counterexample is $x = 1$ and $y = 1$.

(c) True. Proof: let $x \in \mathbb{N}$.
Take $y = x + 1$. Since $x \in \mathbb{N}$, we also have $x + 1 \in \mathbb{N}$, and thus $y \in \mathbb{N}$.
Note that $x < x + 1$ and thus $x < y$.
So $\frac{x}{y} < 1$.

Hence, $\frac{x}{y} \notin \mathbb{N}$. $\square$

Question 7:

(a) No, because, for example, $f(1) = -\left(\frac{1-1}{2}\right) = 0 \notin \mathbb{N}$.

(b). f is injective. Proof: Let $x \in \mathbb{N}$ and let $y \in \mathbb{N}$ and assume $f(x) = f(y)$.

Case 1: x and y are both even.
$f(x) = f(y) \Rightarrow \frac{x}{2} = \frac{y}{2} \Rightarrow x = y$ ✓

Case 2: x and y are both odd.
$f(x) = f(y) \Rightarrow -\left(\frac{x-1}{2}\right) = -\left(\frac{y-1}{2}\right) \Rightarrow \frac{x-1}{2} = \frac{y-1}{2}$
$\Rightarrow x-1 = y-1 \Rightarrow x = y$. ✓

Case 3: x is even, y is odd.
$f(x) = \frac{x}{2} > 0$ and $f(y) = -\left(\frac{y-1}{2}\right) \leq -\left(\frac{1-1}{2}\right) = 0$.
So, $f(x) \neq f(y)$. ✓

Case 4: x is odd, y is even.
Similar to Case 3 we can conclude $f(x) \leq 0$ and $f(y) > 0$.
So, $f(x) \neq f(y)$. ✓
□

(c) Let $x \in \mathbb{N}$, x even. Then, $\frac{x}{2} \geq \frac{2}{2} = 1$.
Let $x \in \mathbb{N}$, x odd. Then, $-\left(\frac{x+1}{2}\right) \leq -\left(\frac{1+1}{2}\right) = -1$.
Hence, there is no $x \in \mathbb{N}$ such that $g(x) = 0$.
So, g is not surjective.

Question 8:

(a) The answer is $\boxed{c}$ because $\{a\} \subseteq A$ and not $\{a\} \in A$.

(b) The answer is $\boxed{c}$ because in this case $A \cup B \cup C = C$.

(c) The answer is $\boxed{e}$. If $\{2\} \subseteq A$ and $\{3\} \subseteq A$, then $\{2,3\} \subseteq A$.