Calculus - ODE summary

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1 Ordinary Differential Equations (ODEs), order, linearity

- **Ordinary Differential Equations (ODEs)** involve derivatives of only **one** variable [1]
- **Order** of a differential equation = order of the highest order derivative appearing in it.
- **Linear ODEs** contain derivatives of y of any order, but in a **linear** combination, with coefficients that are functions of x:

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0 y(x) = f(x)$$

  * Homogeneous linear ODEs if $f(x) = 0$
  * Nonhomogeneous linear ODEs if $f(x) \neq 0$

- Do **not** confuse the order of a differential equation with linearity! (see examples below)
- Remember notations for derivatives e.g. $y^{(2)}(x) = d^2y/dx^2$, $y^{(n)}(x) = d^ny/dx^n$ etc

1.1 Examples of linear/nonlinear ODEs

- Example 1:

$$\frac{d^2y}{dx^2} + x^3 y = \sin(x)$$

- This is a **second order** ODE because of the second order derivative $\frac{d^2y}{dx^2}$
- It is **linear** because none of the derivatives appear in a power higher than 1.
- Example 2:

$$\frac{d^3y}{dx^3} + 4x\left(\frac{dy}{dx}\right)^2 = \frac{d^2y}{dx^2}$$

- This is a **third order** ODE because of the third order derivative $\frac{d^3y}{dx^3}$.
- It is **non-linear** because of the derivative that appears squared in $\left(\frac{dy}{dx}\right)^2$.
- Example 3:

$$\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^3 = \left(\frac{dy}{dx}\right)^2$$

- This is a **first order** ODE because only first order derivatives appear $\frac{dy}{dx}$
- It is **non-linear** because the derivative appears cubed in $\left(\frac{dy}{dx}\right)^3$.

[1] Partial Differential Equations contain *partial* derivatives with respect to more than one variable (*not part of this class*).

2 Types of ODEs

This is a summary of ODEs you will see in this class. Specific methods for separable and non-separable ODEs follow.

- General form of first order linear ODE:

$$\frac{dy}{dx} = f(x, y)$$

- Simplest form of first order linear ODE has:

$$f(x, y) = f(x)$$

  leading to:

$$\frac{dy}{dx} = f(x)$$

  Can be solved by simple integration. Its solution is the antiderivative:

$$y = \int f(x)dx$$

- **Separable ODEs** (next simplest to solve):

$$\frac{dy}{dx} = f(x)g(y)$$

- **First order linear ODEs (non-separable)**:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Next pages: HOW TO SOLVE Separable/Non-separable ODEs! (section 7.9, Adams)

3 How to solve Separable ODEs

$$\frac{dy}{dx} = f(x)g(y)$$

3.1 Separation of variables for Separable ODEs

- **Separate** the *separable* ODE with the dependent variable (e.g. y) on one side and the independent (e.g. x) on the other side:

$$\frac{dy}{g(y)} = f(x)dx$$

- Then **integrate** both sides to solve.
- Note: don't forget the constant C in the indefinite integrals! If initial values are given, you can determine C based on them.

3.2 Example for Separable ODEs

$$\frac{dy}{dx} = \frac{x}{y}$$

- Separate dependent/independent variables:

$$ydy = xdx$$

- Integrate both sides to solve:

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1$$

  or

$$y^2 - x^2 = C,\ C = 2C_1$$

  where C is an arbitrary constant (it does not matter if it is written as $2C_1$ or C. Its value is determined by Initial Values if/when these are given, otherwise the constant can remain general!)

Next page: HOW TO SOLVE Non-separable ODEs! (section 7.9, Adams)

4 How to solve Non-Separable ODEs (1st order linear ODEs)

$$\frac{dy}{dx} + p(x)y = q(x)$$

4.1 Parameter Variation

- First solve the homogeneous equation with a constant parameter K: $y = Ke^{-\mu(x)}$.
- Then solve the non-homogeneous equation by varying the parameter K as a function of x $K(x)$: $y = K(x)e^{-\mu(x)}$
- Replace $y = K(x)e^{-\mu(x)}$ in the nonhomogeneous ODE. This gives:

$$K'(x) = e^{\mu(x)}q(x)$$

- Integrate the resulting $K'(x)$ to get $K(x)$ and $y = K(x)e^{-\mu(x)}$

  See example 7-part II in section 7.9 Adams