Calculus - lecture 4 : Differentiation (2)

* l'Hôpital
* function sketching

I Indeterminate forms

indeterminate forms : $[\frac{0}{0}]$, $[\frac{\infty}{\infty}]$, $[\infty - \infty]$, $[1^{\infty}]$, $[0^0]$, $[0 \cdot \infty]$

$\hookrightarrow$ l'Hôpital

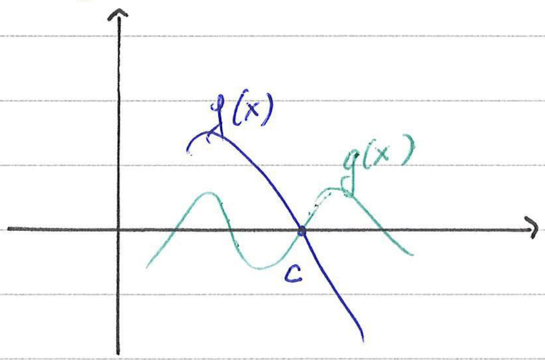
* 1st l'Hôpital rule ($[\frac{0}{0}]$) : for 2 differentiable functions $f(x)$ and $g(x)$ on (a,b) , $g'(x) \neq 0$
  - if $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$
  - if $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$ (where L can be ∞)

  then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$

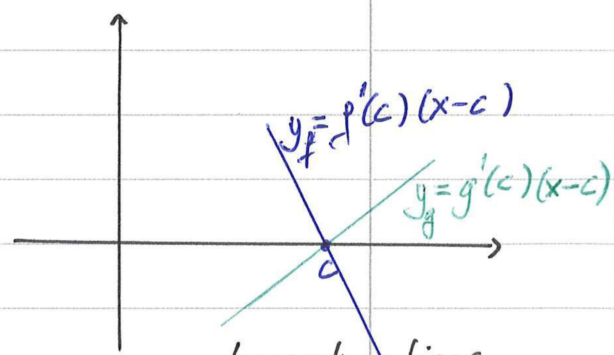
  same applies for $\lim_{x \to b^-} \frac{f(x)}{g(x)}$, or for $\lim_{x \to c} \frac{f(x)}{g(x)}$, $c \in (a,b)$ and for $a, b = \pm \infty$

* Example : $\lim_{x \to 1} \frac{\ln(x)}{x^2 - 1} \overset{H}{=} \lim_{x \to 1} \frac{\frac{1}{x}}{2x} = \frac{1}{2}$

* Intuitive explanation : we replace / approximate the functions by their tangent lines



$\longrightarrow$



tangent lines

$$\lim_{x \to c} \frac{y_f}{y_g} = \frac{f'(c)\,\cancel{(x-c)}}{g'(c)\,\cancel{(x-c)}}$$

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* 2^{nd} l'Hôpital rule ($[\frac{\infty}{\infty}]$) : for 2 differentiable functions $f(x), g(x)$ on (a,b), $g'(x) \neq 0$
   - if $\lim_{x \to a^+} f(x) = \pm\infty$, $\lim_{x \to a^+} g(x) = \pm\infty$
   - if $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$ (where L can be $\pm\infty$)

then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$ (also for $\lim_{x \to b^-} \frac{f}{g}$, $\lim_{x \to c} \frac{f}{g}$ $c \in (a,b)$
$a = \mp\infty$, $b = +\infty$)

* note: the condition $\lim_{x \to c} g(x) = \pm\infty$ is actually sufficient, but if $\lim_{x \to c} f(x) \neq \pm\infty$, there is no point in applying the l'Hôpital rule.

Ex. $\lim_{x \to +\infty} x^2 e^{-x} = \lim_{x \to +\infty} \frac{x^2}{e^x} \overset{H}{=} \lim_{x \to +\infty} \frac{2x}{e^x} \overset{H}{=} \lim_{x \to +\infty} \frac{2}{e^x} = 0$

↳ other indeterminate forms: rewrite until it is in a shape that you can apply l'Hôpital

* 2x oparative

Ex. $\lim_{x \to 0^+} x^x \rightarrow \lim_{x \to 0^+} \ln(x^x) = \lim_{x \to 0^+} x\ln(x) \overset{*}{=} \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x}}$

$\overset{H}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} (-x) = 0$

↳ so, $\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x\ln(x)} = e^0 = 1$

II Function sketching

1) increasing / decreasing function

* a function $f(x)$ on domain I is

monotonous injective {
- increasing if $\forall x_1, x_2 \in I$: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- decreasing if $\forall x_1, x_2 \in I$: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

- non-increasing if $\forall x_1, x_2 \in I$: $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$
- non-decreasing if $\forall x_1, x_2 \in I$: $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

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* connection with 1^{st} derivative.

for $f(x)$ continuous on $[a,b]$, differentiable on (a,b)

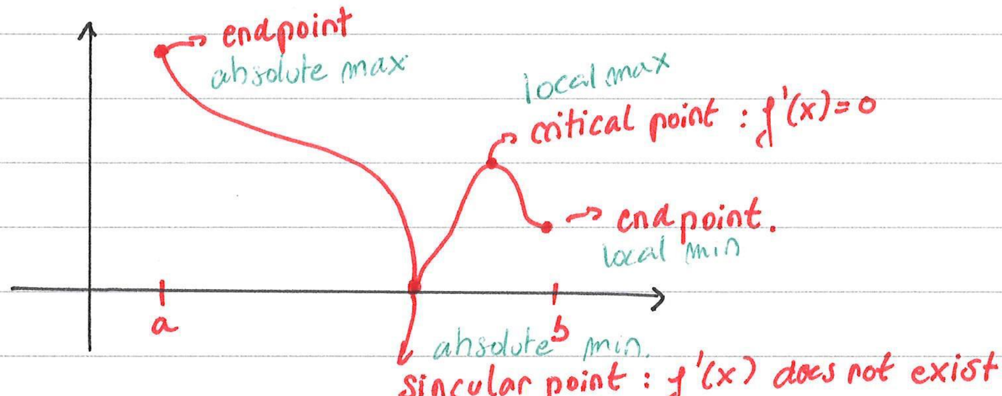
- if $f'(x) > 0$ on (a,b), then f is increasing
- if $f'(x) < 0$ on (a,b), then f is decreasing on $[a,b]$
- if $f'(x) = 0$ on (a,b), then f is constant

Example: $f(x) = x^3 - 12x + 1 \rightarrow f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$

x		-2		2	
$f'(x)$	$+$	0	$-$	0	$+$
	$\nearrow$		$\searrow$		$\nearrow$

$f(x)$ is increasing on $(-\infty,-2) \cup (2,+\infty)$
decreasing on $(-2,2)$

2) Extrema (minima and maxima)



* a continuous function f on a closed and bounded interval $[a,b]$ has an absolute minimum and an absolute maximum (min-max theorem)
* these extreme values can be at: endpoints, singular points critical points.
* local $\begin{cases} \text{minima} \\ \text{maxima} \end{cases}$ at x_0: $\exists h > 0 \ \forall x \in (x_0-h, x_0+h)$ $\begin{matrix} f(x) \geq f(x_0) \\ f(x) \leq f(x_0) \end{matrix}$

* if $f(x)$ has an extremum (possibly local.) at x_0 and $f'(x_0)$ exists and $x_0 \in (a,b)$, then $f'(x_0) = 0$

assume f has a local maximum at x_0

$$\hookrightarrow f'(x_0) = \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h} \leq 0$$
$$= \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} \geq 0$$
$$\Big\} \ f'(x_0) = 0$$

$\hookrightarrow$ socrative: is the opposite true: $f'(x_0) = 0 \Rightarrow f$ has a local extremum at x_0

* $f(x)$ has a minimum and maximum on $[a,b]$
  ↳ what about open intervals? $\mathbb{R}$?

→ for f continuous on (a,b)

$$\lim_{x\to a^+} f(x) = L$$
$$\lim_{x\to b^-} f(x) = M$$

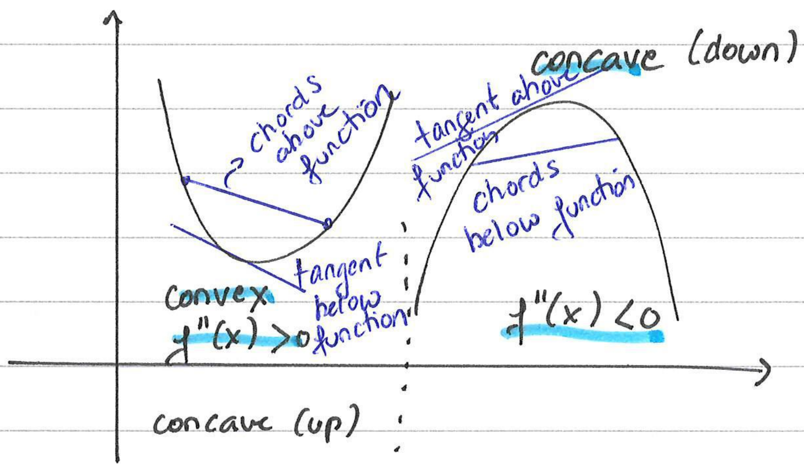
↳ if $\exists x \in (a,b)$: $f(x) > L, M$, then f has a maximum
$f(x) < L, M$, then f has a minimum.
↳ absolute

* example: • x^2 has an absolute minimum (0)
  since $\lim_{x\to+\infty} x^2 = \lim_{x\to-\infty} x^2 = +\infty$

  • $\frac{1}{1+x^2}$ has an absolute maximum (0)
  since $\lim_{x\to\pm\infty} f(x) = 0$, and $\forall x \in \mathbb{R} : f(x) > 0$

(socrative) $2x$

3) Concavity and inflection points



* inflection point: point where concavity changes + tangent exists
* if $f''(x_0)$ exists and x_0 is an inflection point, then $f''(x_0) = 0$

⇒ this comes all together in function sketching!
* domain
* continuous on domain?
* even / odd function
* asymptotes
* first derivative → extreme values, increasing / decreasing intervals
* second derivative → convex / concave intervals, inflection points

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examples

$$f(x) = \frac{x^2 + 2x + 4}{2x}$$

$$f(x) = x^2 e^{-x}$$

* domain: $\mathbb{R}$, continuous on domain
* not even, not odd : $f(-x) = x^2 e^{x} \neq -f(x)$
  $\neq f(x)$
* horizontal asymptote, one-sided.

$$\lim_{x \to +\infty} x^2 e^{-x} = \lim_{x \to +\infty} \frac{x^2}{e^x} \overset{H}{=} \lim_{x \to +\infty} \frac{2x}{e^x} \overset{H}{=} \lim_{x \to +\infty} \frac{2}{e^x} = 0$$

* $f'(x) = 2xe^{-x} - x^2 e^{-x} = x(2-x)e^{-x}$

  $f'(x) = 0$ for $x = 0$ and $x = 2$

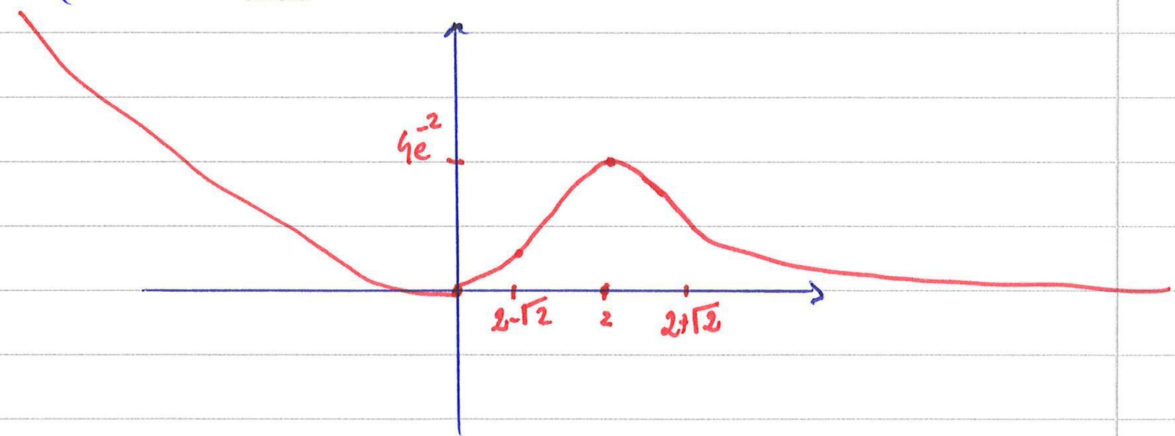
x		0		2	
$f(x)$		0		$4e^{-2}$	
$f'(x)$	$\searrow$	0	$\nearrow$	0	$\searrow$

absolute minimum at $x = 0$

* $f''(x) = (2-2x)e^{-x} - x(2-x)e^{-x} = (x^2 - 4x + 2)e^{-x}$

  $0 = x^2 - 4x + 2 \iff x_{1,2} = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$

x		0		$2-\sqrt{2}$		2		$2+\sqrt{2}$	
$f(x)$	$+\infty \leftarrow$	0		$(6-4\sqrt{2})e^{-(2-\sqrt{2})}$		$4e^{-2}$		$(6+4\sqrt{2})e^{-(2+\sqrt{2})}$	$\rightarrow 0$
$f'(x)$	$\searrow$	0	$\nearrow$			0 max	$\searrow$		
$f''(x)$	$\cup$ min			0		$\cap$		0	$\cup$



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