Limits

Informal definition

f has limit L at a, if $f(x)$ approaches L when x approaches a

$$\lim_{x \to a} f(x) = L$$

Example:

$$\lim_{x \to 5} \frac{x^2 + x - 2}{x - 4} = 7$$

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1} = \frac{(x-1)(x+2)}{x-1} = \lim_{x \to 1} (x+2) = 3$$

we can cancel out $x-1$ because by definition, $x \neq 1$, so $x - 1 \neq 0$

Formal definition

$$\lim_{x \to x_0} f(x) = L$$

$$\iff \forall \varepsilon > 0 \ \exists \delta > 0 : 0 < |x - x_0| < \delta \implies |f(x) - L| < \varepsilon$$

Left-Right limits

$$\lim_{x \to a^-} f(x) = \lim_{x \uparrow a} f(x)$$

$$\lim_{x \to a^+} f(x) = \lim_{x \downarrow a} f(x)$$

If these do not match, then $\lim_{x \to a} f(x)$ does not exist

Theorems

$\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$

- $\lim_{x \to a} [f(x) \pm g(x)] = L \pm M$
- $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
- $\lim_{x \to a} \sqrt{f(x)} = \sqrt{L}$, $L \geq 0$

Interesting limits

- $$\lim_{x\to1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)}$$
$$= \lim_{x\to1}\frac{x-1}{(x-1)(\sqrt{x}+1)}$$
$$= \lim_{x\to1}\frac{1}{\sqrt{x}+1} = \frac{1}{2}$$

- $\lim_{x\to1}|x-1| = 0$

- $\lim_{x\to1}\frac{|x-1|}{x-1}$

$\hookrightarrow$ $$\left.\begin{aligned}\lim_{x\to1^+}\frac{|x-1|}{x-1} &= \frac{(x-1)}{x-1} = 1\\ \lim_{x\to1^-}\frac{|x-1|}{x-1} &= \frac{-(x-1)}{x-1} = -1\end{aligned}\right]$$ limit doesn't exist

- $\lim_{x\to0} x^2\cos\frac{1}{x} = 0 \leftarrow$ Squeeze Theorem

$f(x) = -x^2 \quad g(x) = x^2\cos(\frac{1}{x})$

$h(x) = x^2 \quad f(x)\le g(x) \le h(x) \Rightarrow \lim_{x\to0} g(x) = 0$

Squeeze Theorem

If $f(x)\le g(x) \le h(x)$ on (a,b), $x_0\in(a,b)$

and $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = L$

$\Rightarrow \lim_{x\to x_0} g(x) = L$

Limits at ∞

$\lim_{x\to\infty} f(x) = L$

$\iff \forall \varepsilon > 0\ \exists M > 0 : x > M$ $x \in \text{dom}$

$\implies |f(x) - L| < \varepsilon$

Examples

$\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}}$

$= \lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2(1+\frac{1}{x^2})}}$

$= \lim_{x\to\pm\infty} \frac{x^{+\infty}}{|x|\sqrt{1+\cancel{\frac{1}{x^2}}^{0}}}$

$\hookrightarrow$ $+\infty$

Note: $|x| = \sqrt{x^2}$

$\sqrt{x^2(a+b)}$

$= \sqrt{x^2}\cdot\sqrt{a+b}$

$= |x|\sqrt{a+b}$

Asymptotes

(1) Vertical
(2) Horizontal
(3) Oblique

Rational functions

$\frac{P(x)}{Q(x)}$ $P(x) \to$ deg. m, $Q(x) \to$ deg. n

if $m \le n$, there always is a H.A. of $y = 0$

Analysing complex polinomials

$f(x) = \frac{x^3+4x-5}{x^2-3x+1} \longrightarrow$

$$\begin{array}{r}
x^2-3x+1 \;/\; x^3+4x-5 \;\backslash\; x+3 \\
x^3-3x^2+x \\ \hline
3x^2+3x-5 \\
3x^2-9x+3 \\ \hline
12x-8
\end{array}$$

$f(x) = x+3 + \frac{12-8}{x^2-3x+1}$

Completing the square

$\sqrt{x^2+2x} = \sqrt{x^2+2x+1-1}$

$= \sqrt{(x+1)^2-1}$

$= \sqrt{(x+1)^2\left(1-\frac{1}{(x+1)^2}\right)} = |x+1|\sqrt{1-\frac{1}{(x+1)^2}}$

Limits using the definition

Prove that $\lim_{x \to 3} (x^2 + x - 6) = 6$

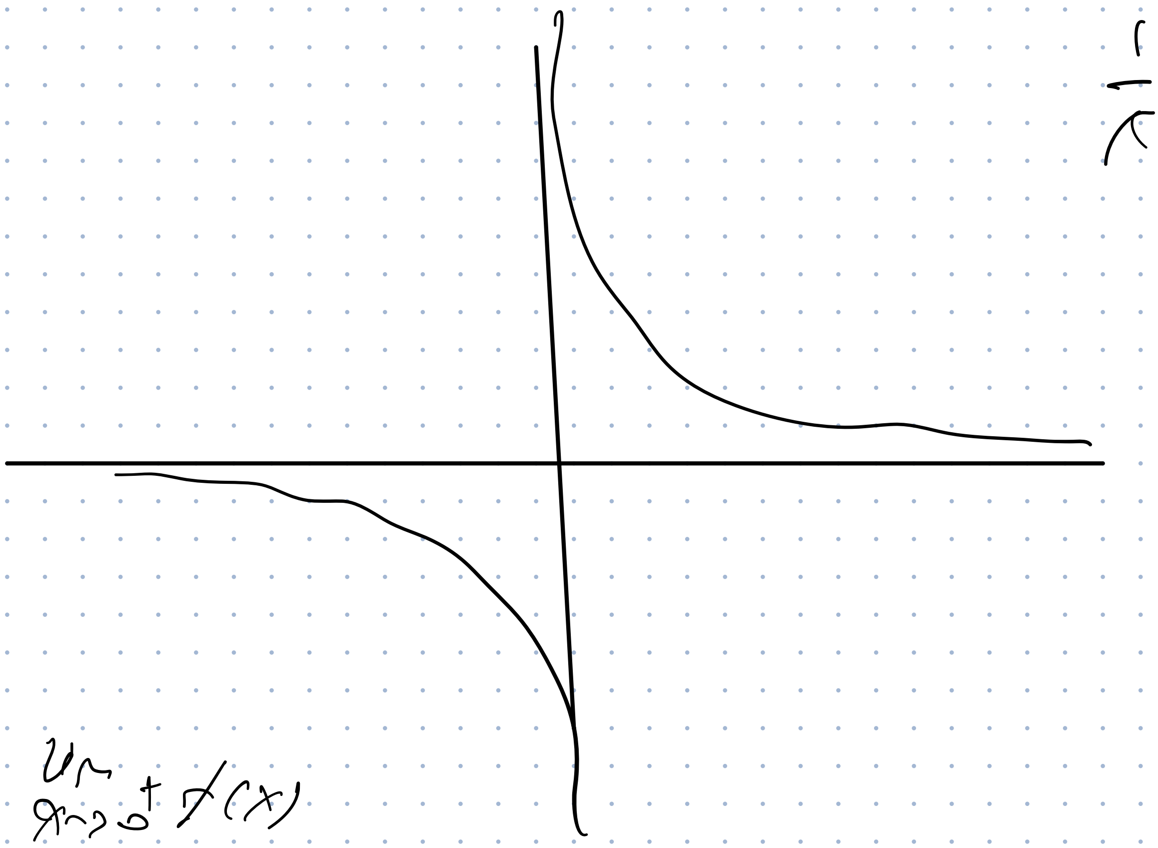
Proof:

- Let $\varepsilon > 0$
- take $\delta = \min(1,$
- Assume that $0 < |x - \varepsilon| < d$

  ⇐ distance between x and ε is less tan δ
- then $|x^2 + x - 6 - 6| = |x^2 + x - 12|$

$$= |(x-3)(x+4)| = |x-3||x+4|$$

$$< \delta |x+4| < 8\delta \leq 8 \frac{\varepsilon}{8} = \varepsilon \quad \square$$



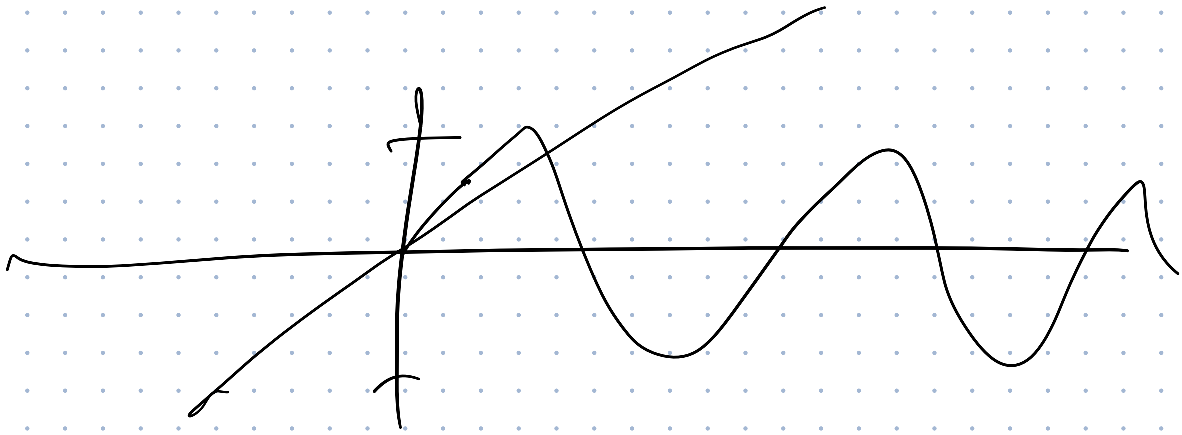
(1) $\lim_{x \to x_0} f(x) = L$ ←

(2) $\lim_{x \to x_0} f(x) = \infty$

(3) $\lim_{x \to \infty} f(x) = L$ ←

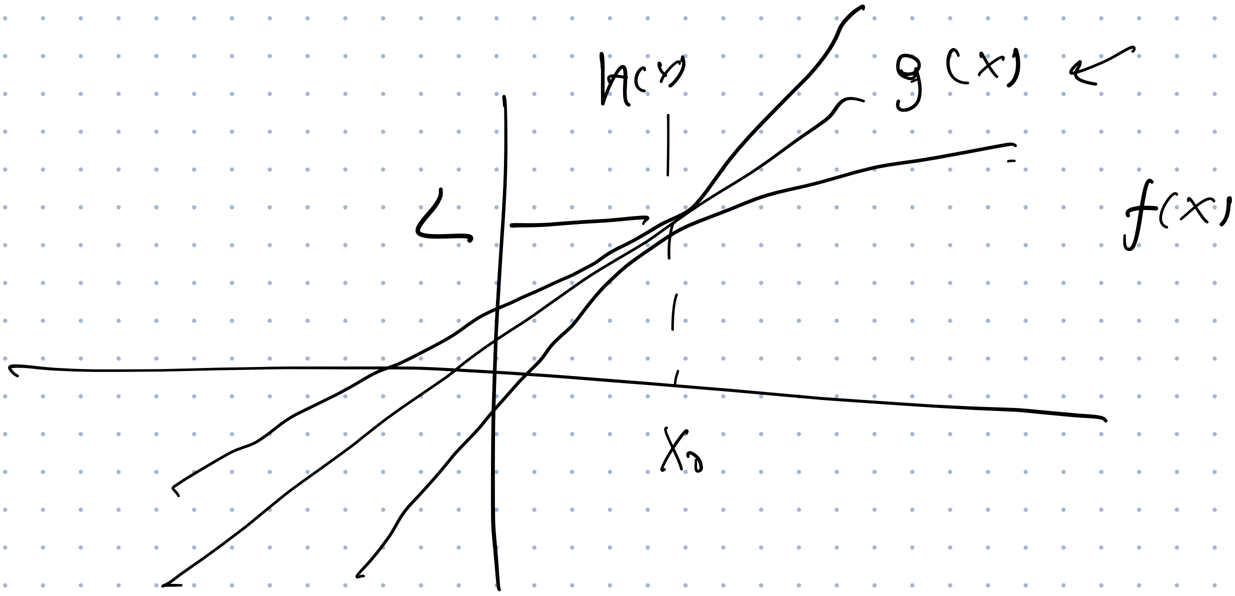
(4) $\lim_{x \to \infty} f(x) = \infty$

$$\lim_{x \to x_0} \sqrt{x^2 \cdot 2} = \lim_{x \to x_0} |x| \sqrt{2}$$



$$\lim_{x \to \infty} x^2 \cdot \sin x$$

$$-x^2 \leq x^2 \sin x \leq x^2$$



$$f(x) \le g(x) \le h(x)$$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L$$

$$\lim_{x \to x_0} g(x) = L$$