Intro to differential equations

Introduction

- Equation that involves a function and its derivat.
- Solution → explicit form $g(x)$

Ordinary Differential Equations

Only 1 independent variable!

$$F(y^{(n)}, y^{(n-1)}, \dots, y', y, x) = 0$$

order of ODE

independent variable

dependent variable

Solution → $y(x)$

Examples

- $F = m \cdot a$ → $F(x, t) = m \frac{d^2}{dx^2} x$
  ↳ solution is $x(t)$

- $y'' + 2y' + 3y = \cos(x)$

Initial Value Problems (IVP)

$$\begin{cases} y' = \operatorname{sen}(x) \longrightarrow y(x) = -\cos(x) + C \\ y(0) = 1 \longrightarrow y(0) = -\cos(0) + C = 1 \Longrightarrow C = 2 \end{cases}$$

$\Rightarrow y(x) = 2 - \cos(x)$

↳ Initial Value Problem

- Diff. Equations don't have a unique solution, we need to set some conditions

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$$\frac{d^2y}{dx^2} = 1 \longrightarrow \frac{dy}{dx} = x + C_1$$

$$\longrightarrow y(x) = \frac{1}{2}x^2 + C_1 x + C_2$$

We need 2 initial values to find a unique solution

$$y'(0) = C_1, \quad y(0) = C_2$$

↓ You need as many initial conditions as the order of the ODE

First order separable ODEs

- $\frac{dy}{dx} = y \cdot x$

$\Rightarrow \int \frac{dy}{y} = \int x \cdot dx$

$\Rightarrow \ln|y| = \frac{1}{2}x^2 + c$

$\Rightarrow y(x) = \pm e^{x^2/2} \cdot e^c = K e^{\frac{x^2}{2}} \quad (K \in \mathbb{R})$

with an initial value, we could find K

check: $y' = K \cdot \frac{2}{2} x e^{\frac{x^2}{2}} = x \cdot K e^{\frac{x^2}{2}} = x \cdot y$

↳ Definition of Separable ODE

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\, dx$$

More examples

(1) $y' = \frac{x}{y}$, $\quad y(1) = 2$

$\frac{dy}{dx} = \frac{x}{y} \Rightarrow \int y\, dy = \int x\, dx$

$\Leftrightarrow \frac{y^2}{2} = \frac{x^2}{2} + c$

↳ Solved, now write y as function of x

$y(x) = \pm\sqrt{x^2 + K} \quad (K = 2c)$

↳ Now put in initial condition

$y(1) = \pm\sqrt{1 + K} = 2 \Rightarrow K = 3$

$y(x) = +\sqrt{3 + x^2}$

Linear ODEs

↳ linear in $y(x)$ and all derivatives $y'(x)$, $y''(x)$...
↳ only ones solvable → they are separable
↳ functions such as $\sin(y)$ are not allowed

$$a_n(x) \cdot y^{(n)} + \dots + a_1(x) \cdot y' + a_0(x) y = f(x)$$

general form for linear ODE

Linear homogenous ODEs

$$\frac{dy^2}{dx^2} = -y(x)$$

$$y_1(x) = -\cos(x)$$
$$y_2(x) = \sin(x)$$

$$-y_1 + y_2 = \sin(x) + \cos(x) \quad \frac{d}{dx}$$
$$-y_1' + y_2' = \cos(x) - \sin(x) \quad \frac{d}{dx}$$
$$-y_1'' + y_2'' = -\sin(x) - \cos(x)$$
$$= y_1 - y_2$$

↳ the sum of 2 solutions is also a solution

If $y_1(x)$ and $y_2(x)$ are solutions to linear homogenous ODE, then $ay_1(x) + by_2(x)$ is also a solution $(a, b \in \mathbb{R})$

↳ With 2 solutions, we can find all possible solutions by making linear combinations of independent solutions

↳ Number of independent solutions = order of ODE

Non-homogenous linear ODEs

$$\frac{dy}{dx} + p(x)\,y = q(x)$$

(1) Assume a solution $y_H(x)$ to the homogenous eq. $\frac{dy_H}{dx} + p(x)\cdot y_H(x) = 0$

(2) Assume a solution $y_p(x)$ to the full eq.

$$\frac{dy_p}{dx} + p(x)\cdot y_p(x) = q(x)$$

(3) $\Rightarrow$ then $\frac{d}{dx}(\underbrace{y_H + y_p}_{\text{particular solution}}) + p(x)(y_H + y_p) = q(x)$

General Solution: $y_p(x) + c\cdot y_H(x)$

Examples

(1) $y' + 2y = 3 \leftarrow$ remove this to make it homogeneous

(1) homogeneous eq: $y' + 2y = 0$

$$\Rightarrow \frac{dy}{dx} = -2y \Leftrightarrow \int \frac{dy}{dx} = \int -2\,dx$$

$$\Rightarrow \ln|y| = -2x + c$$

$$\Rightarrow y_H(x) = K\cdot e^{-2x}$$

↑ rewritten in terms of x

(2) $y' + 2y = 3 \rightarrow y_p = \frac{3}{2} \leftarrow$ find quickest way to find p. solution

↳ we concider it a constant function

↓ constant is usually easy

BUT not always possible

(3) General solution: $y(x) = y_H(x) + y_p(x)$

$$= Ke^{-2x} + \frac{3}{2}$$

Parameter variation: solving linear first order ODE

$$y' + p(x)\,y = q(x)$$

(1) Solve homogeneous eq.

$$y' + p(x)\,y = 0$$

Remove everything that does not multiply y

separable

$$\Rightarrow \int \frac{dy}{y} = -\int p(x)\,dx$$

$$\ln|y| = -\mu(x)$$

$$y_H(x) = K \cdot e^{-\mu(x)}$$

We make the parameter a function of x

(2) Find particular solution

Assume: $y(x) = \underbrace{K(x)} \cdot e^{-\mu(x)}$

product derivative rule

$$y'(x) = \underbrace{K'(x)\,e^{-\mu(x)}}_{q(x)} - \underbrace{K(x)\,p(x)\,e^{-\mu(x)}}_{\substack{p(x)\cdot y(x) \\ \text{by definition}}}$$

$$K'(x)\,e^{-\mu(x)} = q(x)$$

$$\rightarrow K'(x) = e^{\mu(x)}\,q(x)$$

$$K(x) = \int e^{\mu(x)}\,q(x)\,dx$$

Formula

$$\left[\, y(x) = K(x)\,e^{-\mu(x)} = e^{-\mu(x)} \int e^{\mu(x)}\,q(x)\,dx \,\right]$$

Example

$$\begin{cases} y' + \frac{y}{x} = 1 & (x > 0) \\ y(1) = 1 & \leftarrow \text{initial condition} \end{cases}$$

(1) Homogeneous eq.

$$y' = -\frac{y}{x} \iff \int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\Rightarrow \ln|y| = -\ln|x| + C$$

$$\Rightarrow e^{\ln|y|} = e^{-\ln|x| + C}$$

$$\Rightarrow |y| = \frac{e^c}{e^{\ln|x|}}$$

$$\Rightarrow |y| = \frac{e^c}{|x|} \qquad K = e^c$$

$$\Rightarrow y = \frac{K}{x}$$

$$\boxed{y_H = \frac{K}{x}}$$

(2) Parameter for particular solution

$$y' = \frac{K'(x)}{x} - \frac{K(x)}{x^2} \quad \text{(product rule, derivative of)}$$

$$\frac{K'(x)}{x} - \cancel{\frac{K(x)}{x^2}} = 1 - \cancel{\frac{K(x)}{x^2}}$$

homogeneous solution ✓

$$K'(x) = x \Rightarrow K(x) = \frac{x^2}{2} + C$$

$$y(x) = \frac{K(x)}{x} = \frac{1}{x}\left(\frac{x^2}{2} + c\right) = \frac{x}{2} + \frac{c}{x}$$

(3) Consider initial conditions

$$y(1) = 1 \longrightarrow y(1) = \frac{1}{2} + \frac{c}{1} \Rightarrow c = \frac{1}{2}$$

Final solution: $y(x) = \frac{x}{2} + \frac{1}{2x}$