Lecture 10: Diagonalization
(book: 5.2, 5.3)

Previous episode: Eigenvalues and Eigenvectors.
Next episode: Orthogonality and Symmetric Matrices.

Let A and B be two $n \times n$ matrices.
A and B are similar $\iff \exists$ invertible matrix P st
$A = PBP^{-1}$ or $B = P^{-1}AP$.

Theorem: If A and B are similar, then they have the same eigenvalues.

Proof: $|B - \lambda I| = |P^{-1}AP - \lambda I| = |P^{-1}AP - \lambda P^{-1}P| = |P^{-1}(AP - \lambda P)|$

$$= |P^{-1}(AP - \lambda IP)| = |P^{-1}(A - \lambda I)P| = |P^{-1}| \cdot |A - \lambda I| \cdot |P|$$

$$= \frac{1}{\cancel{|P|}} \cdot |A - \lambda I| \cdot \cancel{|P|} = |A - \lambda I|$$

So, A and B have the same characteristic equation
$\Rightarrow$ same eigenvalues $\square$

$$A^k = \underbrace{A \cdot A \cdot A \cdot \ldots \cdot A \cdot A}_{k \text{ times}}$$

For a diagonal matrix this is easy.

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad D^2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix}$$

$$D^3 = D^2 \cdot D = \begin{bmatrix} 2^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2^3 & 0 \\ 0 & 3^3 \end{bmatrix} \qquad D^k = \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}$$

If A is similar to a diagonal matrix, then it's also easy.
$\hookrightarrow A = PDP^{-1}$.

$$A^k = \underbrace{A \cdot A \cdot A \cdot \ldots \cdot A \cdot A}_{k \text{ times}} = \underbrace{PD\cancel{P^{-1}} \cdot \cancel{P}D\cancel{P^{-1}} \cdot \cancel{P}D\cancel{P^{-1}} \cdot \ldots \cdot \cancel{P}D\cancel{P^{-1}} \cdot \cancel{P}DP^{-1}}_{k \text{ times}}.$$

$$= PD^kP^{-1}.$$

A is called diagonalizable if A is similar to a diagonal matrix.

* How to build D?

$\begin{cases} A \text{ is similar to } D \Rightarrow A \text{ and } D \text{ have the same eigenvalues } \lambda_1, \dots, \lambda_n. \\ D \text{ is a diagonal matrix} \Rightarrow \text{the eigenvalues are on the diagonal.} \end{cases}$

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}.$$

* How to build P?

$P = [\underline{v}_1 \ \dots \ \underline{v}_n]$ where $\{\underline{v}_1, \dots, \underline{v}_n\}$ are lin indep vectors in $\mathbb{R}^n$.
↳ because P is invertible.

$A = PDP^{-1} \Rightarrow AP = PD$, where

$AP = A[\underline{v}_1 \ \dots \ \underline{v}_n] = [A\underline{v}_1 \ A\underline{v}_2 \ \dots \ A\underline{v}_n]$

and $PD = [\underline{v}_1 \ \dots \ \underline{v}_n] \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = [\lambda_1 \underline{v}_1 \ \lambda_2 \underline{v}_2 \ \dots \ \lambda_n \underline{v}_n]$.

So, for $AP = PD$ we need $A\underline{v}_i = \lambda_i \underline{v}_i \quad \forall i \in \{1, \dots, n\}$.

$\Rightarrow \underline{v}_1, \dots, \underline{v}_n$ are eigenvectors corresponding to the eigenvalues $\lambda_1, \dots, \lambda_n$.

So, A is diagonalizable $\Leftrightarrow$ A has n lin indep eigenvectors.

If there are n distinct eigenvalues $\Rightarrow$ n lin indep. eigenvectors.

Monday: eigenvectors from different eigenspaces are lin indep.

$\Rightarrow$ A is diagonalizable.

Example: $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $\quad \lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_1 \neq \lambda_2$:)

So, A is diagonalizable. and $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ What is P?

eigenspace of $\lambda_1 = 1$: $[A - 1 \cdot I \,|\, 0] = \left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \underline{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So, take for example $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

eigenspace of $\lambda_2 = 2$: $[A-2I \,\vdots\, 0] = \begin{bmatrix} -1 & 2 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$ $\underline{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

So, take for example $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Then, $P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Check: $A = PDP^{-1}$. $AP = PD$.

$$AP = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$$ ✓

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* What if the eigenvalues of A are not all distinct? (some have mult. >1)

$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 2$.
$(\lambda-1)(\lambda-2)^2 = 0$.

Is A diagonalizable? It depends....

\# of lin indep eigenvectors corresponding to λ = dim of the eigenspace of λ = $\dim(\mathrm{Nul}(A-\lambda I)) \leq$ mult. of λ.

if strictly $<$ for some λ, then A is not diagonalizable.

Theorem: an $n \times n$ matrix A is diagonalizable $\iff$ the sum of the dimensions of the eigenspaces equals n.

Example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Is A diagonalizable?

$\lambda = 0$ with mult 2.

$A - 0I = A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ 1 free var. $x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So, $\dim(\mathrm{Nul}(A-\lambda I)) = 1 < 2 =$ multiplicity of λ.
and thus A is not diagonalizable.

Example $A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}$

* Is -5 an eigenvalue?
* Is A diagonalizable?

$A - (-5)I = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

There are free variables, so -5 is indeed an eigenvalue.
$\dim(\mathrm{Nul}(A-(-5)I)) = 2$.

So, the multiplicity of $\lambda=-5$ is either 2 or 3.
yes ← 2 ; 3 → no.

$$\left.\begin{aligned} \text{trace}(A) &= (-4)+(-3)+(-2) = -9 \\ \text{trace}(A) &= \lambda_1+\lambda_2+\lambda_3 = (-5)+(-5)+\lambda_3 = -10+\lambda_3 \end{aligned}\right\} \quad \begin{aligned} -9 &= -10+\lambda_3 \\ \Rightarrow \lambda_3 &= 1. \end{aligned}$$

So, A is diagonalizable because the multiplicity of $\lambda=-5$ is 2.

$$D = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

eigenspace of -5: $\underline{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\quad \underline{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\quad \underline{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

eigenspace of 1: $A - 1\cdot I = \begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix}$

$\underline{x} = x_3 \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$ $\quad$ So, $\underline{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Then $P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

DIY: Verify that $AP = PD$. ✓

Applications of diagonalization:
* Markov Processes.
* Dynamical Systems.
* Difference Equations.

Summary. Is A diag?

Does A have n distinct eigenvalues?

Yes

yes, $A = PDP^{-1}$

No.

Is the sum of the dimensions of the eigenspaces equal to n?

Yes

yes, $A = PDP^{-1}$

No

No, A is not diag.

$\mathbb{C}^2 \to \mathbb{C}^2 \qquad T(\underline{x}) = A\underline{x} \qquad A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$

$\lambda_1 = \cos\varphi + i \cdot \sin\varphi \qquad \underline{x} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda_2 = \cos\varphi - i \cdot \sin\varphi \qquad \underline{x} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$

DIY.

$V = \mathbb{R}^+$.

* addition of vectors u and : $u * v$.
* multiplication of a vector u by a scalar c : u^c.

What is the zero vector in this context? 1

Verify the 10 axioms. ✓
