Exercises

1	2	3	4	5	6	7	8	9	10

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
7	7	7	7	7	7	7
8	8	8	8	8	8	8
9	9	9	9	9	9	9
0	0	0	0	0	0	0

Surname, First name

____

Calculus Test
Example exam

a ~~b~~ c d e f → b
a b ~~c~~ d e f → c
a b c ~~d~~ e f → a

Fill in your answer(s) to the multiple-choice questions as shown above (circles = one correct answer).

Program:
Course code: KEN/BCS 14460
Examiners: Otti D'Huys, Gijs Schoenmakers
Date/time:
Format: Written, closed book
Allowed aids: A formula sheet is attached to the exam.
Instructions to students:

- The exam consists of 9 questions on 12 pages.
- Fill in your name and student ID number on the cover page and tick the corresponding numerals of your student number in the table (top right cover page).
- Answer every question in the reserved space below the question. **Do <u>not</u> write outside the reserved space or on the back of pages, this will not be scanned and will NOT be graded!** As a last resort if you run out of space, use the extra answer space at the end of the exam.
- *In no circumstance write on or near the QR code at the bottom of the page!*
- Ensure that you properly motivate your answers.
- Only use black or dark blue pens, and write in a readable way. Do not use pencils.
- Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in the space reserved for the answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- You have to return all pages of the exam. You are not allowed to take any sheets, even blank, home.
- Good luck!

Question 1

6p **1a** For $f(x)$ and $g(x)$ differentiable functions, let $f(1) = 1$, $f(-2) = -2$, $f'(1) = -3$, $f'(-2) = 3$, $g(1) = -2$ and $g'(1) = 1$.
Find $\lim_{x\to 1} \frac{f(g(2-x))-g(f(x^2))}{x-1}$

- (a) $-\infty$
- (b) -1
- (c) 0
- (d) 3
- (e) -6
- (f) $+\infty$
- (g) ☒ -9
- (h) The limit does not exist
- (i) None of the above

$$\lim_{x\to 1} \frac{f(g(2-x))-g(f(x^2))}{x-1} \stackrel{H}{=}$$

$$\lim_{x\to 1} \frac{f'(g(1))\,g'(1)\cdot(-1) - g'(f(1))\cdot f'(1)\cdot 2}{1} =$$

$$3\cdot 1\cdot(-1) - 1\cdot(-3)\cdot 2 = -3-6 = -9$$

6p **1b** Calculate $\lim_{x\to 0^-} \frac{x}{\sqrt{x^2-x}}$

- (a) 4
- (b) $-\frac{1}{2}$
- (c) $\frac{1}{2}$
- (d) -1
- (e) $-\infty$
- (f) ☒ 0
- (g) $+\infty$
- (h) 1
- (i) The limit does not exist
- (j) None of the above

$$\lim_{x\to 0^-} \frac{x}{\sqrt{x^2-x}} = \lim_{x\to 0^-} \frac{x}{-x\sqrt{1-\frac{1}{x}}} = 0$$

6p **1c** Calculate $\int_0^1 x\ln(x)dx$

- (a) $-\frac{1}{2}$
- (b) This integral diverges to $-\infty$
- (c) 0
- (d) This integral diverges to $+\infty$
- (e) ☒ None of the above

$$\int_0^1 x\ln(x)dx = \lim_{a\to 0^+} \int_a^1 x\ln(x)\,dx$$

$$= \lim_{a\to 0^+}\left(\left[\frac{x^2}{2}\ln(x)\right]_a^1 - \int_a^1 \frac{x\,dx}{2}\right)$$

$$= \lim_{a\to 0^+} \frac{a^2}{2}\ln(a) - \left|\frac{x^2}{4}\right|_a^1$$

$$= 0 - \frac{1}{4}$$

$\ln(x) = u$
$dv = x\,dx$
$du = \frac{dx}{x}$
$v = \frac{x^2}{2}$

$$\lim_{a\to 0^+} \frac{a^2}{2}\ln(a) = \lim_{a\to 0^+} \frac{1}{2}\,\frac{\ln(a)}{\frac{1}{a^2}}$$

$$\stackrel{H}{=} \lim_{a\to 0^+} \frac{1}{2}\,\frac{\frac{1}{a}}{-\frac{2}{a^3}} = 0$$

Question 2

16p **2** Find all the asymptotes of the function $f(x) = \frac{1}{\sqrt{x^2+3x}+x}$. For the horizontal and oblique asymptotes (if any), specify whether they are one-sided or two-sided. For the vertical asymptotes (if any), specify how the function approaches the asymptotes (i.e. whether the function goes to $+\infty$ or $-\infty$ on the left or right side of the asymptotes.)

Horizontal asymptotes

1) $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{\sqrt{x^2+3x}+x} = 0$

$\to$ one-sided horizontal asymptote $y=0$

2) $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{\sqrt{x^2+3x}+x} = \lim_{x\to-\infty} \frac{\sqrt{x^2+3x}-x}{(x^2+3x)-x^2}$

$= \lim_{x\to-\infty} \frac{-x\sqrt{1+3/x}-x}{3x} = \lim_{x\to-\infty} \frac{-\sqrt{1+3/x}-1}{3} = -\frac{2}{3}$

$\to$ one-sided horizontal asymptote $y = -\frac{2}{3}$

(so there are no oblique asymptotes)

Vertical asymptotes

$\to$ poles? $\sqrt{x^2+3x}+x = 0 \Leftrightarrow -x = \sqrt{x^2+3x}$

$\Rightarrow x^2 = x^2+3x \Rightarrow x=0$

$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{1}{\sqrt{x^2+3x}+x} = +\infty$

$\lim_{x\to0^-} f(x)$ does not exist, since $f(x)$ is not defined for

$x^2+3x<0 \Leftrightarrow -3<x<0$

$\to$ vertical asymptote $x=0$

Question 3

10p **3** Does the function $f(x) = x\ln(x)$ have a global (absolute) minimum and maximum on $(0,\infty)$?
- If so, for which value(s) of x?
- If not, motivate why you can exclude the existence of a global extremum.

- Domain $(0,\infty)$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = 0$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} x\cdot\ln(x) = +\infty$$

- Critical points? $f'(x) = \ln(x) + 1$

$\Rightarrow f'(x) < 0$ for $\ln(x) < -1 \iff 0 < x < e^{-1}$

($f(x)$ decreases)

$f'(x) > 0$ for $\ln(x) > -1 \iff x > e^{-1}$

($f'(x)$ increases)

$\rightarrow f(x)$ has a local minimum at $x = e^{-1}$, $f(e^{-1}) = -e^{-1}$

$\hookrightarrow$ this is a global minimum, since $f(e^{-1}) < 0$

$f(e^{-1}) < +\infty$

$\rightarrow f(x)$ does not have a global maximum, since

$\lim_{x\to\infty} f(x) = +\infty$

Question 4

10p **4** Evaluate the following integral: $\int_0^{\frac{\pi}{2}} (\sin(x))^3 dx$

$$\int_0^{\pi/2} \sin^3(x) = \int_0^{\pi/2} \sin^2(x)\,(\sin(x)\,dx) = -\int_1^0 (1-u^2)\,du = \int_0^1 (1-u^2)\,du$$

$u = \cos(x)$ $\quad u(0) = 1$

$du = -\sin(x)\,dx$ $\quad u(\frac{\pi}{2}) = 0$

$$= \left[u - \frac{u^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

$=$

Question 5

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{e^n+n}{e^{2n}-n^2}$

$$\frac{e^n+n}{e^{2n}-n^2} = \frac{e^n+n}{(e^n-n)(e^n+n)} = \frac{1}{e^n-n}$$

1.5p **5a** The series $\sum_{n=1}^{\infty} a_n$ diverges. We can conclude this by comparing to the series $\sum_{n=1}^{\infty} b_n$ with $b_n = \frac{1}{n}$ and $|a_n| \leq b_n$ for all $n \in \mathbb{N}$

- (a) True
- (b) False ☒

you can only prove divergence if $a_n \geq K \cdot b_n$ and $\sum b_n$ diverges

1.5p **5b** The series $\sum_{n=1}^{\infty} a_n$ converges. We can conclude this by comparing to the series $\sum_{n=1}^{\infty} b_n$ with $b_n = e^{-n}$ and $a_n \leq b_n$ for all $n \in \mathbb{N}$

- (a) True
- (b) False ☒

$a_n > b_n$

1.5p **5c** The series $\sum_{n=1}^{\infty} a_n$ converges. We can conclude this because $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = e^{-1}$

- (a) True ☒
- (b) False

1.5p **5d** The series $\sum_{n=1}^{\infty} a_n$ converges. We can conclude this because $\lim_{n\to\infty} a_n = 0$

- (a) True
- (b) False ☒

you cannot conclude anything about $\sum a_n$ if $a_n \to 0$

Question 6

10p **6** Determine which values of $x \in \mathbb{R}\backslash\{0\}$, the given series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} n\left(\frac{2}{x}\right)^n$$

1) Check for absolute convergence, consider $\sum n \left|\frac{2}{x}\right|^n$

2) Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \left|\frac{2}{x}\right|^{n+1-n} = \lim_{n\to\infty} \frac{n+1}{n} \left|\frac{2}{x}\right| = \left|\frac{2}{x}\right|$$

$\Rightarrow$ if $\left|\frac{2}{x}\right| < 1$, the series is absolutely convergent

$\Leftrightarrow$ $x < -2$ or $x > 2$

$\Rightarrow$ if $\left|\frac{2}{x}\right| > 1$, the series (and sequence) diverge.

$\Leftrightarrow$ $-2 < x < 2$

$\Rightarrow$ if $\left|\frac{2}{x}\right| = 1$, no conclusion

$\rightarrow$ we treat $x = 2$ and $x = -2$ separately

3) $x = 2 \rightarrow \sum n$ diverges

$x = -2 \rightarrow \sum n(-1)^n$ diverges

4) Summary $\sum n\left(\frac{2}{x}\right)^n$ is absolutely convergent for

$x < -2$ and $x > 2$

$\sum n\left(\frac{2}{x}\right)^n$ diverges for $-2 \leq x \leq 2$

Question 7

10p **7** Find the solution $y = y(x)$ to the given initial value problem.

$$\begin{cases} y' + y = x \\ y(0) = 3 \end{cases}$$

This is a linear, first order, non homogeneous ODE.

1) Homogeneous equation

$y' + y = 0 \Rightarrow y' = -y \Rightarrow \int \frac{dy}{y} = -\int dx \Rightarrow \ln|y| = -x + C$

$\Rightarrow y_H(x) = k \cdot e^{-x}$

2) Parameter variation

assume $y(x) = k(x) \cdot e^{-x}$

$y' + y = k'(x) e^{-x} - \cancel{k(x) e^{-x}} + \cancel{k(x) e^{-x}} = x$

$\Rightarrow k'(x) = x e^{x}$

$\Rightarrow k(x) = \int x e^{x} dx = x e^{x} - \int e^{x} dx = x e^{x} - e^{x} + C$

$u = x \qquad du = dx$

$dv = e^{x} dx \qquad v = e^{x}$

$\Rightarrow y(x) = k(x) e^{-x} = (x e^{x} - e^{x} + C) \cdot e^{-x} = x - 1 + C \cdot e^{-x}$

$y(0) = -1 + C = 3 \Rightarrow C = 4$

$\boxed{y(x) = x - 1 + 4 \cdot e^{-x}}$

$y' + y = x \qquad y' = 1 - 4e^{-x} \qquad y' + y = (x \cancel{-1} + \cancel{4e^{-x}}) + (\cancel{1} - \cancel{4e^{-x}})$

$= x$

Question 8

10p **8** For the function $f(x, y) = \sin(2x - y)$, give the equation of the tangent plane at $(0, 0, f(0, 0))$

Tangent plane equation.

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

Here : $f(0,0) = 0$ $a = 0$, $b = 0$

$$\frac{\partial f}{\partial x} = 2\cos(2x-y) \longrightarrow \frac{\partial f}{\partial x}(0,0) = 2$$

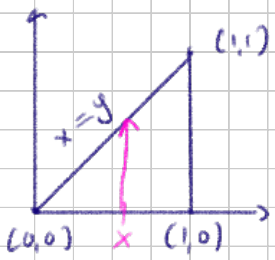
$$\frac{\partial f}{\partial y} = -\cos(2x-y) \longrightarrow \frac{\partial f}{\partial y}(0,0) = -1$$

$$\longrightarrow z = 2x - y$$

Question 9

10p **9** Evaluate the double integral over a triangle T with vertices $(0,0),(1,0),(1,1)$:

$$\iint_T e^{2x+y} dA$$



$x: 0 \to 1$
$y: 0 \to x$

$$\iint_T e^{2x+y}\, dA = \int_0^1 \int_0^x e^{2x+y}\, dy\, dx$$

$$A(x) = \int_0^x e^{2x+y}\, dy = e^{2x}\int_0^x e^y\, dy = e^{2x}\left[e^y\right]_0^x = e^{3x} - e^{2x}$$

$$\int_0^1 A(x)\, dx = \int_0^1 (e^{3x} - e^{2x})\, dx = \frac{1}{3}\left[e^{3x}\right]_0^1 - \frac{1}{2}\left[e^{2x}\right]_0^1$$

$$= \frac{1}{3}(e^3 - 1) - \frac{1}{2}(e^2 - 1) = \frac{e^3}{3} - \frac{e^2}{2} + \frac{1}{6}$$

Extra space

10a

10b

10c

10d

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