Exam Disrete Mathematics.

## Question 1:

| $p$ | $q$ | $p \Rightarrow q$ | $p \wedge \neg q$ | $q \wedge \neg q$ | $(p \wedge \neg q) \Rightarrow (q \wedge \neg q)$ | $(\dots) \Leftrightarrow (\dots)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | F | T | T |
| F | F | T | F | F | T | T |

## Question 2:

ⓐ Base case: $4^{3\cdot 1}+8 = 4^3+8 = 64+8 = 72$, which is divisible by $9$. ✓

Induction step: Let $n \in \mathbb{N}$.

Assume $4^{3n}+8$ is divisible by $9$. So, $4^{3n}+8 = 9\cdot k$, where $k \in \mathbb{Z}$

Then,

$$4^{3(n+1)}+8 = 4^{3n+3}+8 = 4^{3n}\cdot 4^3+8 = 64\cdot 4^{3n}+8 = 4^{3n}+8+63\cdot 4^{3n}$$

$$= 9k + 9\cdot 7\cdot 4^{3n} = 9\cdot\left(k+7\cdot 4^{3n}\right),$$ which is divisible by $9$ because $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. ✓ $\square$

ⓑ Consider $n=7$.

Then $3^n = 3^7 = 2187 \neq 5040 = 7! = n!$.

## Question 3:

ⓐ False

Consider $A=\{1\}$, $B=\{2\}$ and $C=\{1\}$. Then $A\setminus C = \emptyset \subseteq \{2\} = B\setminus C$.

However, $A \not\subseteq B$.

So, the implication "$\Leftarrow$" is not true.

(Note that the implication "$\Rightarrow$" is true).

ⓑ "$\{9^n : n\in\mathbb{N}\} \neq \{3^n : n\in\mathbb{N}\}$"

Consider $x=3$.

Then, $x \in \{3^n : n\in\mathbb{N}\}$ because $x=3^1$

However, $x \notin \{9^n : n\in\mathbb{N}\}$, because $9^n \geq 9$ for all $n \in \mathbb{N}$. ✓

"$\{9^n : n\in\mathbb{N}\} \subseteq \{3^n : n\in\mathbb{N}\}$"

Let $x \in \{9^n : n\in\mathbb{N}\}$

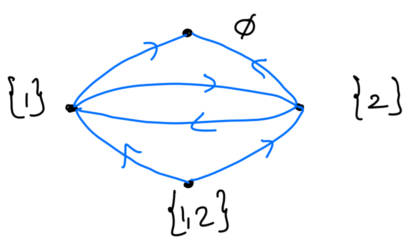
Then, there exists an $n\in\mathbb{N}$ such that $x = 9^n$

Hence, $x = 9^n = (3^2)^n = 3^{2n}$.

Since $2n \in \mathbb{N}$ (because $n\in\mathbb{N}$), we have $x \in \{3^n : n\in\mathbb{N}\}$. ✓ $\square$

## Question 4:

ⓐ



(b). * Reflexive: No, consider $X=\{1\}$. Then $|X\setminus X| = |\emptyset| = 0 \neq 1$. So, $X \not R X$.

* Symmetry: No, consider $X=\{1,2\}$ and $Y=\{1\}$. Then, $|X\setminus Y| = |\{2\}| = 1$. So, $XRY$.
However, $|Y\setminus X| = |\emptyset| = 0 \neq 1$. So, $Y \not R X$.

* Transitivity: No, consider $X=\{1,2\}$, $Y=\{1\}$ and $Z=\emptyset$.
Then, $|X\setminus Y| = |\{2\}| = 1$ So, $XRY$.
And, $|Y\setminus Z| = |\{1\}| = 1$ So, $YRZ$.
However, $|X\setminus Z| = |\{1,2\}| = 2 \neq 1$. So, $X \not R Z$.

* Anti-symmetry: No, consider $X=\{1\}$ and $Y=\{2\}$.
Then, $|X\setminus Y| = |\{1\}| = 1$. So, $XRY$.
But also $|Y\setminus X| = |\{2\}| = 1$. So, $YRX$.

(c) Note that: $5x+y$ is even
$\iff$
$((x$ is even$)$ and $(y$ is even$))$ or $((x$ is odd$)$ and $(y$ is odd$))$.

Hence, R is an equivalence relation with two equivalence classes (namely, the even numbers and the odd numbers). So, $\boxed{b}$

## Question 5:

(a) We need to choose the positions for the 3 ones.

$n=8$
$k=3$
repetition is not allowed
order is not important

$$\binom{n}{k} = \binom{8}{3} = 56.$$

So, the answer is $\boxed{e}$.

(b) So, we have 2 bars and 10 stars.
For example, the solution $x=3$, $y=4$, $z=3$ corresponds with ***|****|***

$n=3$
$k=10$
repetition is allowed
order is not important

$$\binom{(n-1)+k}{k} = \binom{2+10}{10} = \binom{12}{10} = 66$$

So, the answer is $\boxed{c}$

(c) Denote $U$: set of passwords made from capital letters and lower case letters.
$X$: set of passwords made from lower case letters.
We need to calculate $|U\setminus X|$.
$|U| = (26+26)^5 = 380204032$.
$|X| = 26^5 = 11881376$.
$|U\setminus X| = |U| - |X| = 368322656$.
So, the answer is $\boxed{f}$

## Question 6:

(a). True.
Proof: Take $x=-1$. Let $y \in \mathbb{Z}$ and let $z \in \mathbb{Z}$.
Assume $x=yz$. So, assume $yz=-1$.
Then, since $y$ and $z$ are both integers, we know $y=1$ and $z=-1$, or the other way around.

*Case 1: $y = 1$ and $z = -1$.
So, $y = -z$ ✓

*Case 2: $y = -1$ and $z = 1$.
So, $y = -z$ ✓

□

(b) True.
Proof: Let $n \in \mathbb{N}$.
Consider $X = \phi$.
Note that $X \in \mathcal{P}(\mathbb{N})$ because $\phi \subseteq \mathbb{N}$.
Then, $|X| = 0 < n$, because $n \in \mathbb{N}$ and thus $n \geq 1$. □

(c) The statement $(a \text{ odd}) \wedge (b \text{ odd}) \Rightarrow (ab^2 \text{ odd})$ is being proved.
Hence, its contrapositive $(ab^2 \text{ even}) \Rightarrow (a \text{ even}) \vee (b \text{ even})$ is also being proved.
So, the answer is $\boxed{d}$.

## Question 7:

(a) $f$ is not a bijection, because $f$ is not surjective.
Consider $y = 0 \in \mathbb{Z}$
We will show that $(\forall x \in \mathbb{Z})\ (f(x) \neq y)$.
Suppose there is an $x \in \mathbb{Z}$ such that $f(x) = 0$.
So, $3x^2 + 2x + 1 = 0$
However, since $2^2 - 4 \cdot 3 \cdot 1 = -8 < 0$, there is no solution.
Hence, there is no $x \in \mathbb{Z}$, such that $f(x) = 0$.
As a result $f$ is not surjective.
(Note that $f$ is injective).

(b) Consider $f^{-1}: \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{3\}$ defined by $f^{-1}(x) = \dfrac{6x - 7}{2x - 4}$

Let $x \in \mathbb{R} \setminus \{3\}$, then

$$f^{-1}(f(x)) = \frac{6 \cdot \frac{4x-7}{2x-6} - 7}{2 \cdot \frac{4x-7}{2x-6} - 4} = \frac{24x - 42 - 14x + 42}{8x - 14 - 8x + 24} = \frac{10x}{10} = x \quad ✓$$

Let $x \in \mathbb{R} \setminus \{2\}$, then

$$f(f^{-1}(x)) = \frac{4 \cdot \frac{6x-7}{2x-4} - 7}{2 \cdot \frac{6x-7}{2x-4} - 6} = \frac{24x - 28 - 14x + 28}{12x - 14 - 12x + 24} = \frac{10x}{10} = x \quad ✓$$

So, the answer is $\boxed{d}$.

(c). Note that $f(1) = \dfrac{1}{2} + \dfrac{1 - (-1)^1}{4} = \tfrac{1}{2} + \tfrac{1}{2} = 1$ and $f(2) = \dfrac{2}{2} + \dfrac{1 - (-1)^2}{4} = 1 + 0 = 1$.
So, $f$ is not injective.

$f$ is surjective. Let $y \in \mathbb{Z}$. Take $x = 2y$ (Note that $2y \in \mathbb{Z}$, because $y \in \mathbb{Z}$).
Then $f(x) = f(2y) = \dfrac{2y}{2} + \dfrac{1 - (-1)^{2y}}{4} = y + \dfrac{1 - ((-1)^2)^y}{4} = y + \dfrac{1 - 1^y}{4}$

$$= y + \frac{1-1}{4} = y + 0 = y \quad ✓$$

So, the answer is $\boxed{c}$.

## Question 8:

a. The answer is $\boxed{b}$.
Counterexample: $A=\{\{2,3,4\}\}$ and $B=\{\{2,3\}\}$. Then, $A\setminus B=\emptyset$.
The following statement would be correct: $(\{2,3,4\}\subseteq A$ and $\{2,3\}\subseteq B)\Rightarrow(\{4\}\subseteq A\setminus B)$.

b. All four statements are true, so the answer is $\boxed{e}$.