

Lecture 2: Vector equations, Matrix equations.

(book: 1.3, 1.4)

Previous lecture: geometric / row point of view to an SLE
+ Gaussian elimination.

Today: column point of view to an SLE.

$$\begin{aligned} x + y &= 30 \\ 2x + 4y &= 74 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 30 \\ 74 \end{bmatrix}$$

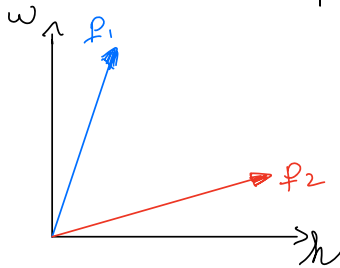
From a machine learning perspective: columns of A are feature vectors (vectors that collect features that we measured for each animal).

For instance, height and weight $\underline{p} = \begin{bmatrix} h \\ w \end{bmatrix}$

Collect data from N animals $\rightarrow N$ feature vectors:

$$\underline{p}_1 = \begin{bmatrix} h_1 \\ w_1 \end{bmatrix}, \underline{p}_2 = \begin{bmatrix} h_2 \\ w_2 \end{bmatrix}, \dots, \underline{p}_N = \begin{bmatrix} h_N \\ w_N \end{bmatrix}.$$

These vectors are points in the feature space (\mathbb{R}^2).

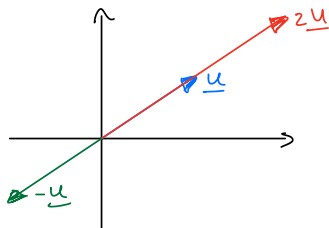


In general: n features $\rightarrow (n \times 1)$ vector $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$

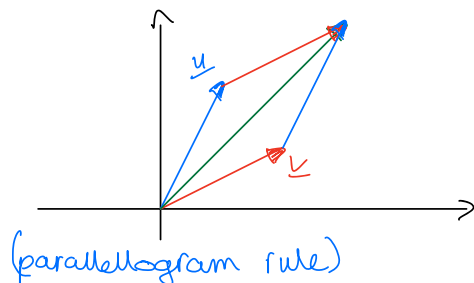
$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

We can perform operations with vectors:

* scaling $\underline{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad 2\underline{u} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \quad -\underline{u} = (-1) \cdot \underline{u} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}.$



* addition $\underline{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\underline{u} + \underline{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$



Algebraic properties of \mathbb{R}^n : book ps3.

Combining these two operations:

Given vectors $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_p\}$ in \mathbb{R}^n and c_1, \dots, c_p the vector

$$\underline{y} = c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + \dots + c_p \cdot \underline{v}_p$$

is called a **linear combination** of $\underline{v}_1, \dots, \underline{v}_p$ with weights c_1, \dots, c_p .

Example: $\underline{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\underline{b} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Q: can \underline{b} be written as a linear combination of \underline{u} and \underline{v} ?
i.e., can we find c_1 and c_2 such that $c_1 \cdot \underline{u} + c_2 \cdot \underline{v} = \underline{b}$?

$$c_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} ? \quad \leftarrow \text{vector equation}$$

$$\begin{bmatrix} c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

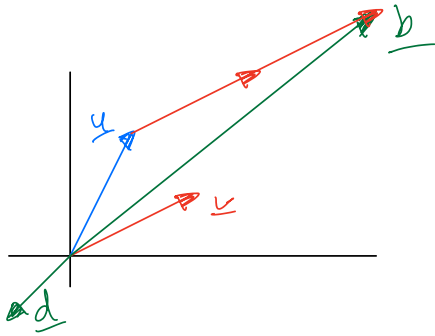
$$\begin{bmatrix} c_1 + 2c_2 \\ 2c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{cases} c_1 + 2c_2 = 5 \\ 2c_1 + c_2 = 4 \end{cases}$$

\leftarrow hey, that is an **SLE**!

* Every SLE can be written as a vector equation, and the other way around.

* Solving an SLE means investigating whether \underline{b} can be written as a linear combination of the columns of A .
(column point of view).



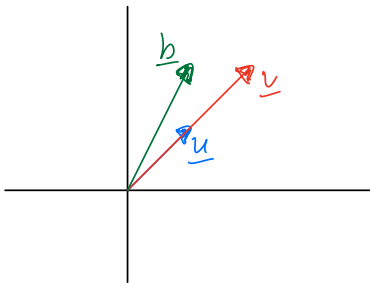
$$\underline{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\underline{b} = \underline{u} + 2 \cdot \underline{v}$$

$$\underline{d} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$-1/3 \underline{u} + (-1/3) \underline{v}$$

Example: (no sol.) $\underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



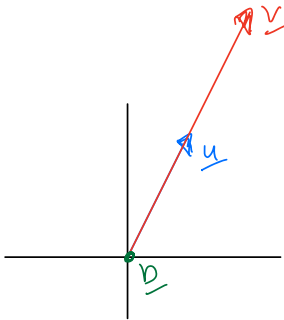
Hence, there is no way to obtain \underline{b} by taking a linear combination of \underline{u} and \underline{v} .

Example (∞ many solutions in \mathbb{R}^2): $\underline{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$2 \cdot \underline{u} + (-1) \cdot \underline{v} = \underline{b}$$

$$0 \cdot \underline{u} + 0 \cdot \underline{v} = \underline{b}$$

$$\underline{u} + (-1/2) \cdot \underline{v} = -\underline{b}$$



there are ∞ many ways to linearly combine \underline{u} and \underline{v} .

$$\begin{cases} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array} \right] \xrightarrow{R_2: R_2 - 2 \cdot R_1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

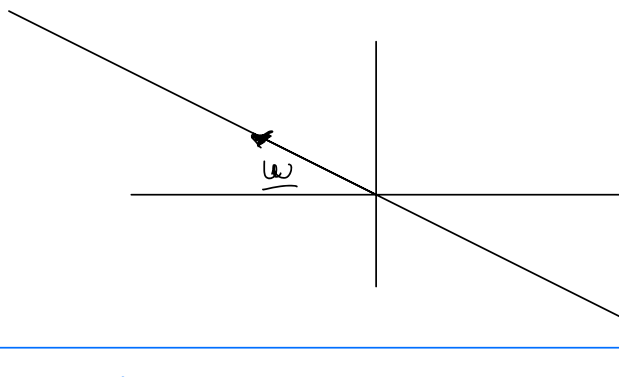
$$\begin{cases} x_1 = -2x_2 \\ x_2 \text{ is free} \end{cases} \quad (\text{parametric form})$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad (\text{parametric vector form})$$

Hence, the solution set is $x_2 \cdot \underline{w}$, where $\underline{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

i.e., $\text{Span}\{\underline{w}\}$

↳ any scalar multiple of \underline{w} .



any solution on this line is a solution to the SUE.

Example (∞ many solutions in \mathbb{R}^3)

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= 0 \\ 2x_1 - 6x_2 + 4x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 2 & -6 & 4 & 0 \end{array} \right] \xrightarrow{R_2: R_2 - 2 \cdot R_1} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{cases} x_1 = 3x_2 - 2x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$$

parametric form

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \cdot \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}_{\underline{w}_1} + x_3 \cdot \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\underline{w}_2}$$

Hence, a solution is any linear combination of \underline{w}_1 and \underline{w}_2 .

$\text{Span}\{\underline{w}_1, \underline{w}_2\}$

So, the solution set is a plane in \mathbb{R}^3 .

Given a set of p vectors $\{\underline{v}_1, \dots, \underline{v}_p\}$ in \mathbb{R}^n , the span of this set of vectors is the set of all possible linear combinations of the vectors in this set.

→ $\text{Span}\{\underline{v}_1, \dots, \underline{v}_p\}$ contains any vector \underline{y} that can be written as

$$\underline{y} = c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + \dots + c_p \cdot \underline{v}_p.$$

Hence, solving an SUE boils down to investigating whether \underline{b} belongs to the Span of the columns of A .

We can see a linear combination of vectors as the **product** of a matrix (A) and a vector (\underline{x}).

Definition of $A\underline{x}$: **linear combination** of the columns of A with the entries of \underline{x} being the weights.

$$A\underline{x} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

More efficient $A\underline{x} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 3 + (-1) \cdot 7 \\ 0 \cdot 4 + (-5) \cdot 3 + 3 \cdot 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

We need: # columns of A = # row/entries of \underline{x} .

Properties of the matrix-vector product: p. 65

Three things with the same solution set:

- * the **SLE** with augmented matrix $[\underline{a}_1 \ \underline{a}_2 \ \dots \ \underline{a}_n \mid \underline{b}]$
- * the **vector equation** $x_1 \cdot \underline{a}_1 + x_2 \cdot \underline{a}_2 + \dots + x_n \cdot \underline{a}_n = \underline{b}$.
- * the **matrix equation** $[\underline{a}_1 \ \underline{a}_2 \ \dots \ \underline{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underline{b} \quad A\underline{x} = \underline{b}$

Example: $A = \begin{bmatrix} 1 & 5 & -3 \\ 0 & 1 & -2 \\ -2 & -10 & 6 \end{bmatrix}$

Q: Is $A\underline{x} = \underline{b}$ **consistent** for every $\underline{b} \in \mathbb{R}^3$? **No!**

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & b_1 \\ 0 & 1 & -2 & b_2 \\ -2 & -10 & 6 & b_3 \end{array} \right] \sim \begin{array}{l} R_3: R_3 + 2 \cdot R_1 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 5 & -3 & b_1 \\ 0 & 1 & -2 & b_2 \\ 0 & 0 & 0 & b_3 + 2b_1 \end{array} \right]$$

The SLE is consistent iff $b_3 + 2b_1 = 0$.

For example, the SLE is inconsistent if $\underline{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
 consistent if $\underline{b} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

4 equivalent statements:

- * The columns of A **span** \mathbb{R}^m .
- * Each $\underline{b} \in \mathbb{R}^m$ is a **linear combination** of the columns of A .
- * For each $\underline{b} \in \mathbb{R}^m$, $A\underline{x} = \underline{b}$ has a **solution**.
- * A has a **pivot** position in every row.
 \hookrightarrow (coefficient matrix!)