Solutions - Practice Exam Questions - Tutorial 3

1. (a) $\{(4,2),(4,3),(5,2),(5,3)\}$

   (b) $\emptyset$

   (c) $\{(\emptyset,2),(\emptyset,3),(\{6\},2),(\{6\},3)\}$

   (d) It's just $\mathbb{P}(B)$ i.e. $\{\emptyset,\{4\},\{5\},\{4,5\}\}$

   (e) $\{\emptyset,\{4\},\{5\},\{6\},\{4,5\},\{4,6\},\{5,6\},\{4,5,6\}\}$

   (f) $\{((2,4),6),((3,4),6),((2,5),6),((3,5),6)\}$

   (g) $\{(\emptyset,\emptyset),(\{4\},\emptyset),(\{5\},\emptyset),(\{4,5\},\emptyset)\}$

   (h) $E\setminus A$ is equal to $\{\{2,3\},\emptyset\}$ so $(E\setminus A)\times A$ is equal to $\{(\{2,3\},2),(\{2,3\},3),(\emptyset,2),(\emptyset,3)\}$

2. (a) $B \setminus C = \{1, \{3\}\}$

   and $A = \{\emptyset\}$

   So $A \times (B\setminus C) = \{(\emptyset, 1), (\emptyset, \{3\})\}$.

   (b) $A \cap D = \emptyset$

   so $\mathbb{P}(A \cap D) = \mathbb{P}(\emptyset) = \{\emptyset\}$

3. $B \setminus A = B = \{\emptyset\}$.

   $\cancel{\mathbb{P}(B) = \mathbb{P}}$

   $\mathbb{P}(B\setminus A) = \mathbb{P}(B) = \{\emptyset, B\}$

   $= \{\emptyset, \{\emptyset\}\}$.

   $\mathbb{P}(B\setminus A) \cup A$

   $= \{\emptyset, \{\emptyset\}\} \cup \emptyset$

   $= \{\emptyset, \{\emptyset\}\}$

   (b) 1/2/4

   12/4

   14/2

   24/1

   124

   5 in total, where "12/4" means $\{\{1,2\}, \{4\}\}$ and so on.

4. (a) $A=\{1\}, B=\{2\}$. Observe that $\{1,2\}\in\mathbb{P}(A\cup B)$ but $\{1,2\}\notin\mathbb{P}(A)\cup\mathbb{P}(B)$.

(b) The property is $(A \subseteq B) \vee (B \subseteq A)$.

To prove this, I need to prove that for all sets A and B, $M \Leftrightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) \cup \mathbb{P}(B)$.

I first prove the $\Rightarrow$ direction. So let A and B be arbitrary sets. Assume that M holds. If $A \subseteq B$, then $\mathbb{P}(A \cup B) = \mathbb{P}(B)$. Now, note that $A \subseteq B \Rightarrow \mathbb{P}(A) \subseteq \mathbb{P}(B)$. (Can you see why?) Hence, $\mathbb{P}(A) \cup \mathbb{P}(B) = \mathbb{P}(B)$. Summarizing, if $A \subseteq B$ then both sides evaluate to $\mathbb{P}(B)$ so the claim holds. The analysis is symmetrical if $B \subseteq A$.

I now prove the $\Leftarrow$ direction. To do this, I will prove the contrapositive: I will show that if M is not true, then $\mathbb{P}(A \cup B) \neq \mathbb{P}(A) \cup \mathbb{P}(B)$. Now, if M does not hold, then $A \not\subseteq B \wedge B \not\subseteq A$. So A contains some element x that B does not, and B contains some element y that A does not (and $x \neq y$). Observe that $\{x, y\} \subseteq A \cup B$, so $\{x, y\} \in \mathbb{P}(A \cup B)$. However, $\{x, y\} \not\subseteq A$, so $\{x, y\} \notin \mathbb{P}(A)$. Similarly, $\{x, y\} \not\subseteq B$, so $\{x, y\} \notin \mathbb{P}(B)$. Hence, $\{x, y\} \notin \mathbb{P}(A) \cup \mathbb{P}(B)$. Hence, $\mathbb{P}(A \cup B) \neq \mathbb{P}(A) \cup \mathbb{P}(B)$ (because one side contains $\{x, y\}$ and the other doesn't). $\square$

5. (a)

Symmetric: no, e.g. $x = 0$, $y = 3$. We have $2(0) - 2(3) = -6 \leq 3$

but $2(3) - 2(0) = 6 \not\leq 3$.

Reflexive? Yes, because $2x - 2x = 0$ and $0 \leq 3$.

Transitive? No: take $x = 2, y = 1, z = 0$. We get $2(2) - 2(1) \leq 3$ ✓

$2(1) - 2(0) \leq 3$ ✓

$2(2) - 2(0) \not\leq 3$.

background, not necessary to write this on an exam.

(How I got here? I wasn't sure at first so I

Started by trying to prove it true. I assumed xRy and yRz,

So $2x - 2y \leq 3$ and $2y - 2z \leq 3$.

I tried to simplify this, dividing everything by 2 to get:

$x - y \leq 1.5$ and $y - z \leq 1.5$.

Now, x and y are integers so this implies

$x - y \leq 1$ and $y - z \leq 1$.

Adding these inequalities, I get

$x - z \leq 2$.

Now, xRz means "$2x - 2z \leq 3$" which is equivalent to

"$x - z \leq 1.5$".

We know $x - z \leq 2$, so if the claim is false then ~~the~~ a

counterexample must have the form $x - z = 2$.

Which gave me a clue to find $x = 2, y = 1, z = 0$.)

(b)

It is not reflexive.

Take eg $x = -1$. Is xRx?

That is, is $-1 \geq |-1|$?

$\iff -1 \geq 1$? No!

It is not symmetric.

For example, $3 R -2$ because $3 \geq |-2|$ ✓

but $-2 \not R 3$ because $-2 \not\geq |3|$ ✗.

It is transitive.

Assume $x \geq |y|$ and $y \geq |z|$. We need to show that $x \geq |z|$

Now, $x \geq |y| \implies x \geq 0$.

$y \geq |z| \implies y \geq 0 \implies x \geq y$ (because $|y| = y$).

~~Combined~~

Now, combine $x \geq y$ and $y \geq |z|$

and we get $x \geq |z|$ ✓

(c)

Reflexive?

Let $x \in \mathbb{Z}$. Is it true that $x - x + 1$ is natural?

Yes, because $x - x + 1 = 1$ and $1 \in \mathbb{N}$ ✓

So it is reflexive.

Symmetric?

No. Let $x = 10$ and $y = 4$.

Clearly, $10 - 4 + 1 = 7$ and $7 \in \mathbb{N}$ ✓

but $4 - 10 + 1 = -5$ and $-5 \notin \mathbb{N}$.

So not symmetric.

Transitive?

Let x, y, z be arbitrary integers such that xRy and yRz. Need to show xRz.

Now,

$xRy \Rightarrow x - y + 1$ is natural

$\Rightarrow x - y + 1 \geq 1$

$\Rightarrow x - y \geq 0$

$\Rightarrow x \geq y$.

$yRz \Rightarrow y - z + 1$ is natural

$\Rightarrow y - z + 1 \geq 1$

$\Rightarrow y - z \geq 0$

$\Rightarrow y \geq z$

Combining $x \geq y$ and $y \geq z$ we get $x \geq z$.

Now, $x \geq z$

$\Rightarrow x - z \geq 0$

$\Rightarrow x - z + 1 \geq 1$

$\Rightarrow x - z + 1$ is natural (because x and z are integers, meaning $x - z$ is integer). So yes, transitive!!

6.

	R	S	T
a	✗	✓	✗
b	✓	✗	✓
c	✗	✗	✗
d	✗	✓	✓

7. There are *two* equivalence classes: (1) All integers ≥ 0 and (2) All integers ≤ -1.

   If you want to know *why* these are the equivalence classes, observe firstly that (1) and (2) together partition $\mathbb{Z}$. Next, observe that if you take any two integers x and y, both ≥ 0, then $(x + \frac{1}{2}) > 0$ and $(y + \frac{1}{2}) > 0$, so $(x + \frac{1}{2})(y + \frac{1}{2}) \geq 0$. Similarly, if you take any two integers x and y, both ≤ -1, then $(x + \frac{1}{2}) < 0$ and $(y + \frac{1}{2}) < 0$, so $(x + \frac{1}{2})(y + \frac{1}{2}) \geq 0$. Now, suppose you take (without loss of generality) an integer $x \geq 0$ and an integer $y \leq -1$. We observe that $(x + \frac{1}{2}) > 0$ and $(y + \frac{1}{2}) < 0$, so $(x + \frac{1}{2})(y + \frac{1}{2}) < 0$ i.e. x and y are not related. In other words, everything in (1) is mutually related, everything in (2) is mutually related, but nothing in (1) is related to anything in (2).