Calculus

Lecture 1: Functions and continuity

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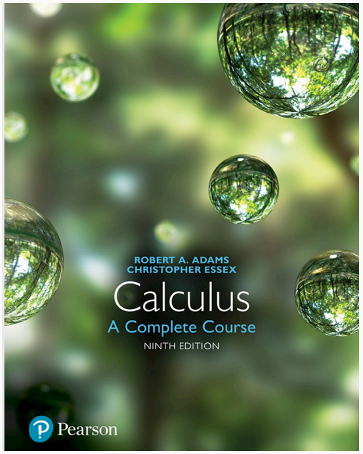
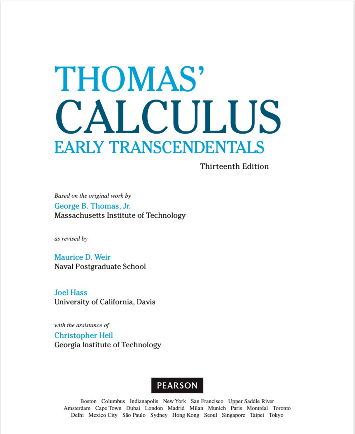
Calculus: Practicalities

- This class has 11 lectures, 8 tutorials, and a Q&A/revision lecture
  - In the tutorials you work on exercises and you have the opportunity to ask questions
- Lecturers: Otti D'Huys (also coordinator), Gijs Schoenmakers
- Teaching Assistants: Fivos Tzavellos, Ankie Fan, Ashkan Saber Karimi, Vitaly Tickovs, Spyridon Giagtzoglou, Riju Mukherjee, Zirui Wang, Jason Tsagkaris

Calculus: Practicalities

- Lecture materials (on Canvas):
  - (sometimes) a summary pdf
  - a scan of our handwritten preparation or lecture material
- Tutorial materials (on Canvas):
  - a list of tutorial exercises
  - pdf with their solutions
  - a checklist per course module, with all relevant concepts and an exhaustive list of useful exercises (no need to try them all...)
- Books: mainly Adams (9^{th} ed) and Thomas (13^{th} ed)





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Calculus: Practicalities

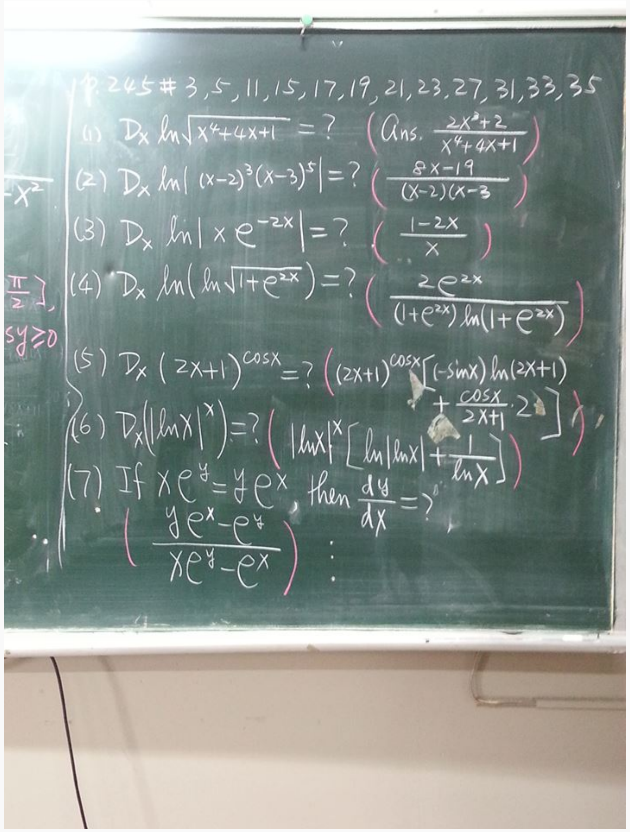
- Evaluation = 100% final exam + 10% bonus quizzes

  Exam:
  - Closed book
  - Formula sheet will be provided
  - Calculators are not allowed

  Quizzes
  - 3 Canvas quizzes, worth 3.33 % each (bonus)
  - Multiple choice and numerical questions
  - The quiz remains open for a week, but once you start it, you have limited time to complete it.
  - Every 2 weeks (more or less)
- How to get help?
  - **Anonymous** discussion boards on Canvas
  - tutorials!
  - No emails

Calculus: Course Contents

- Limits and continuity
- Differentiation
- Integration
- Sequences and series
- Basics of differential equations
- Basics of multivariate calculus



Why Calculus?

- Calculus was developed to describe motion (mainly by Isaac Newton and Gottfried Leibniz in the 17th century)
- It is a language to describe the world in a numerical way, in terms of functions and their rate of change.
    - Mathematical modelling, control theory, robotics,... all describe systems with differential equations.
    - Probability and statistics: a mathematical description of chance
    - Optimization: finding optimal (extreme) values
    - ...

Functions and continuity - Book chapters

Adams:

- P.1 Real numbers, intervals, absolute value
- P.2 Equation of a line
- P.3 Functions
- P.5. Combining functions
- P.6 Polynomials and Rational functions
- 1.4 Continuity

Real functions

- A function $f : D \to S$ on a set D into a set S is a rule that assigns a **unique** element $f(x) \in S$ to **each** element $x \in D$.
- Domain D $\longrightarrow$ $\mathbb{R}$, or $\mathbb{R}\setminus\{0\}$, $(1,2)$
  - Domain convention:
    $\{x \in \mathbb{R} : f(x) \in \mathbb{R}\}$ $\rightsquigarrow$ if domain not mentioned
  - Open interval: (a,b) $\{x \in \mathbb{R} : a < x < b\}$ 

  - Closed interval: $[a,b]$ $\{x \in \mathbb{R} : a \leq x \leq b\}$ 

- Co-domain S: $\mathbb{R}$
- Range: $\{f(x) \mid x \in D\}$

Domain: check 3 things

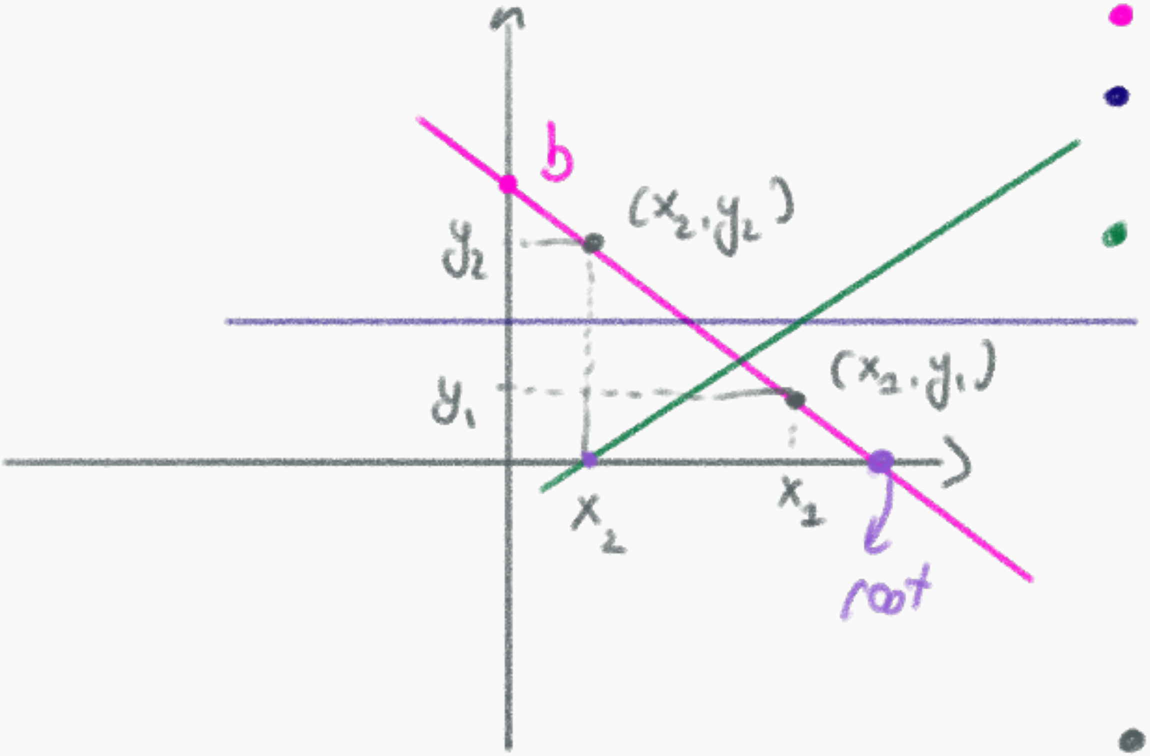
* Domain $(\sqrt{x}) = [0, \infty)$
  No roots of negative numbers
* Domain $(\frac{1}{x}) = \mathbb{R}\setminus\{0\}$
  don't divide by 0
* Domain $(\ln(x)) = (0, \infty)$

Equation of a line

$f(x) = ax + b$

$a \to$ slope

$b \to$ y-intercept



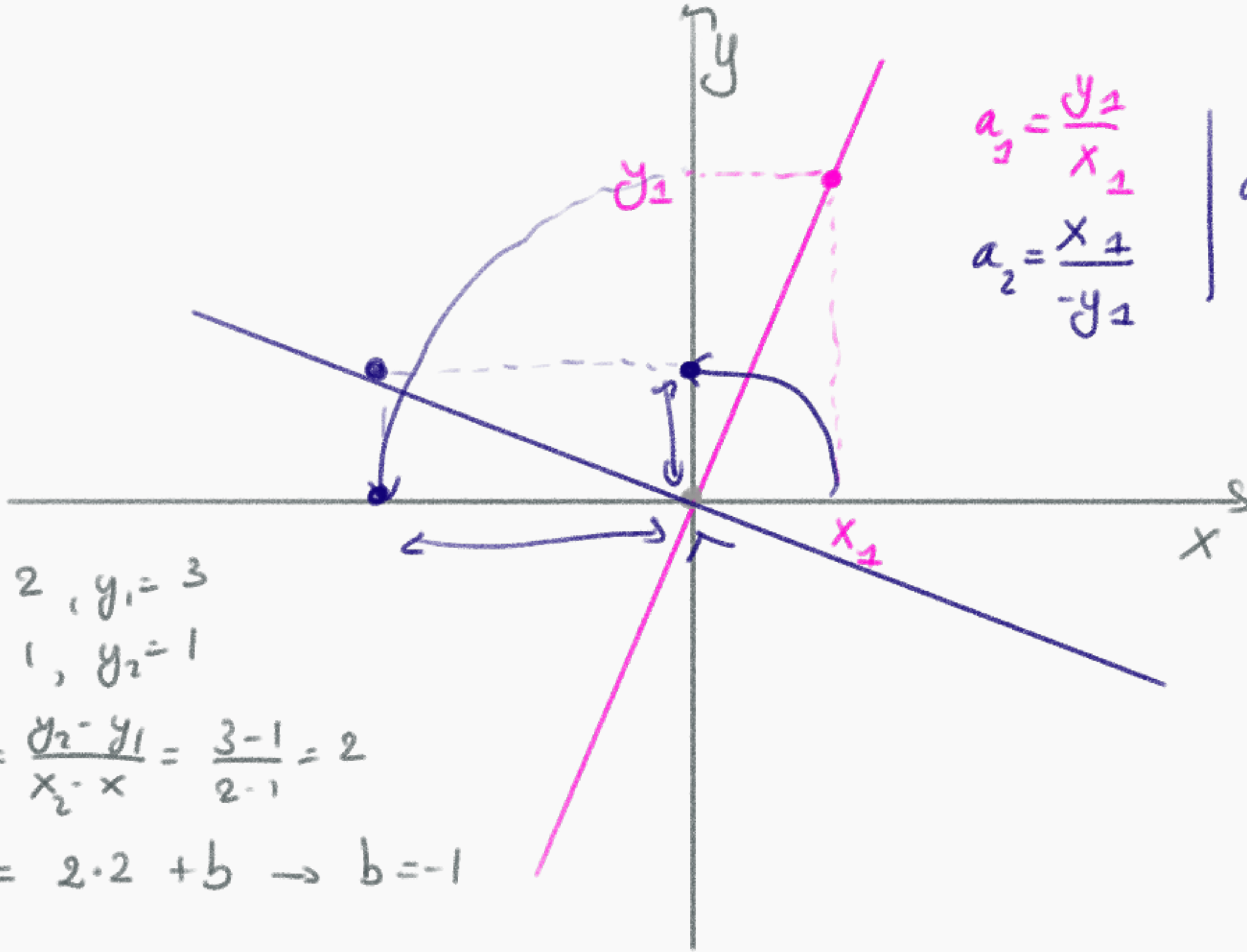
- negative slope $a < 0$
- $a = 0$ (zero slope constant function
- $a > 0$

$$a = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \mapsto$$

$$-b = a x_1 - y_1$$

- lines with the same slope are parallel
- lines with $a_1 \cdot a_2 = -1$ are perpendicular

Examples



$$a_1 = \frac{y_1}{x_1} \qquad a_2 = \frac{x_1}{-y_1}$$

$$a_1 \cdot a_2 = -1$$

$x_1 = 2$, $y_1 = 3$

$x_2 = 1$, $y_2 = 1$

$$a = \frac{y_2 - y_1}{x_2 - x} = \frac{3-1}{2-1} = 2$$

$$3 = 2 \cdot 2 + b \rightarrow b = -1$$

Absolute value

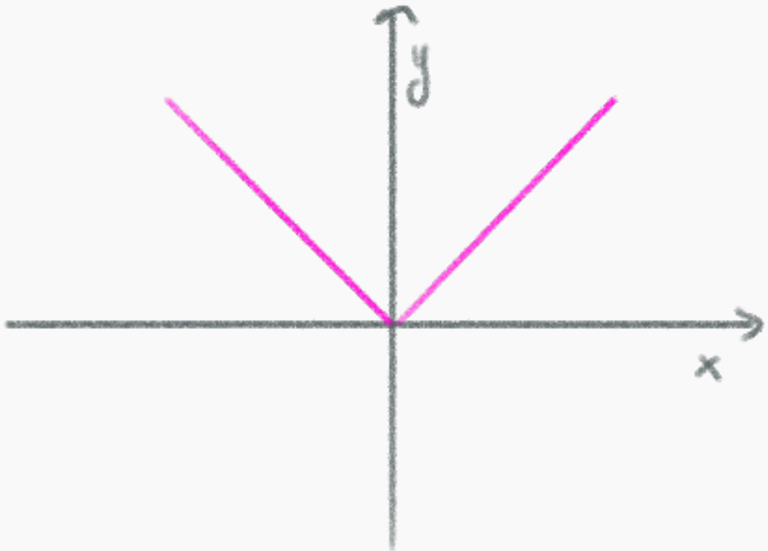
$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

↳ distance from 0

→ domain $\mathbb{R}$

$|a \cdot b| = |a|\,|b|$

$|a+b| \leq |a| + |b|$ (triangle inequality)



Polynomials

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

$P(x)$ ↳ $a_n \neq 0$

- Degree of a polynomial: n (highest degree)
- Root of a polynomial: $P(r) = 0$ $\quad P(x) = (x-r)Q(x)$ ↳ degree $n-1$
- Number of (complex) roots of a polynomial: $\longrightarrow n$ (degree)

  ↳ a root with multiplicity m appears m times

$n=1$ linear function

$n=0$ constant function

$n=2$ $\quad a_2x^2 + a_1x + a_0 \longrightarrow$ parabola

$x^3 - 2x^2 + 2x - 1$

$= (x-1)Q(x)$

$P(x) = x^2 - 3x + 2 \quad r_1 = +2, \; r_2 = +1$

$P(r_1) = P(+2) = 0 \longrightarrow P(x) = (x-r_1)(x-r_2) = \underbrace{(x-2)}(x-1)$

Rational functions

$$f(x) = \frac{P(x)}{Q(x)}$$

$P(x)$, $Q(x)$ polynomials

domain $= \mathbb{R} \setminus \{$roots of $Q(x)\}$

$\hookrightarrow$ poles

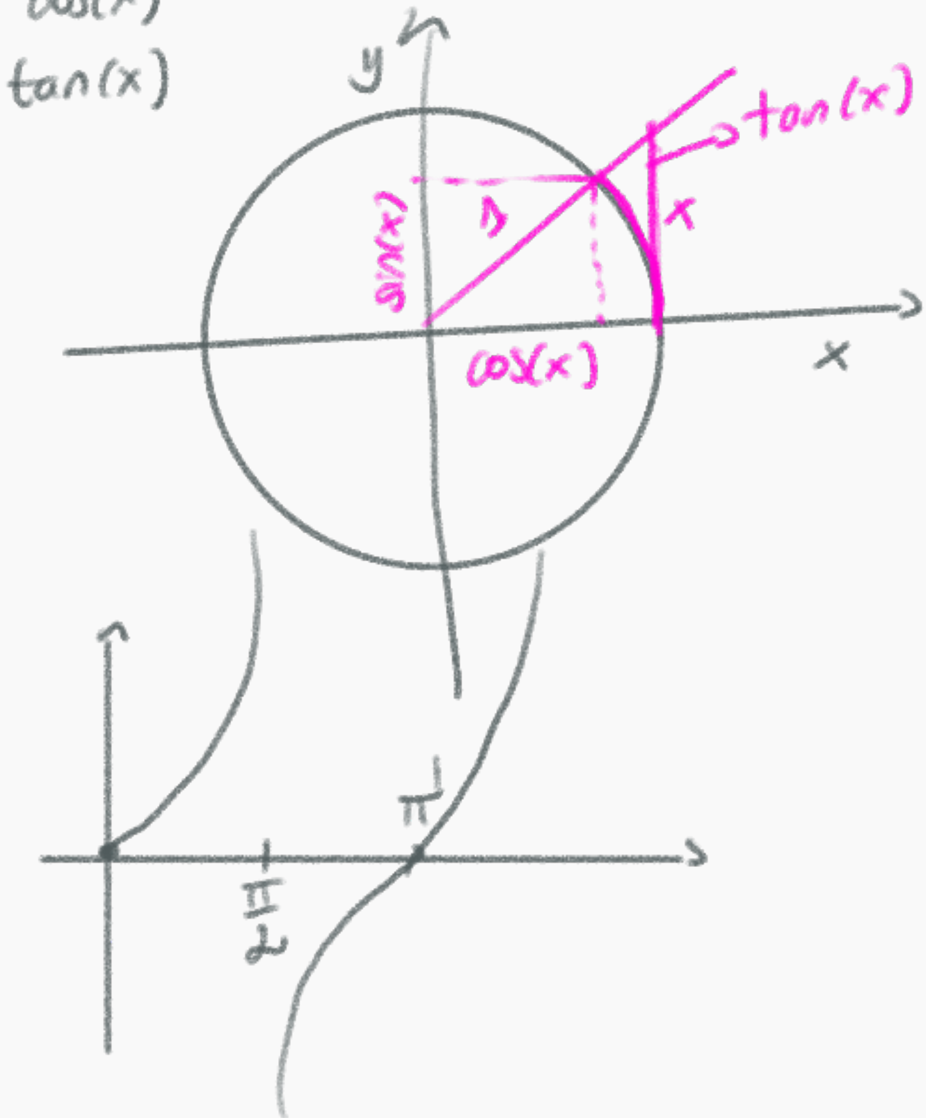
Trigonometric functions

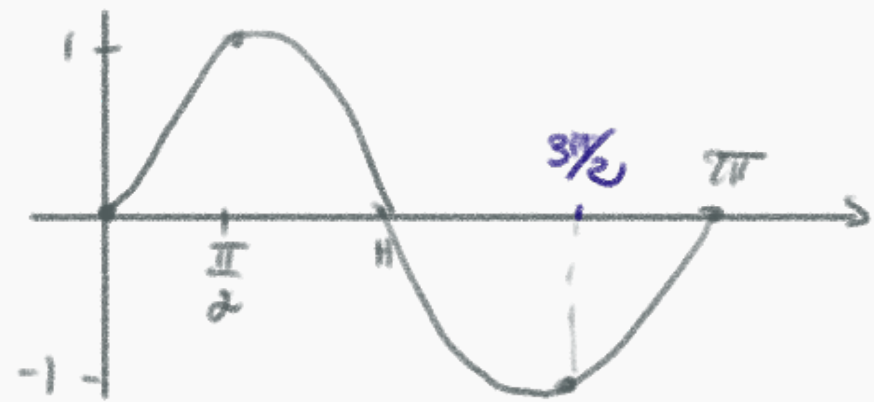
sin(x)

cos(x)

tan(x)

$\tan(x) = \dfrac{\sin(x)}{\cos(x)}$





Even and odd functions

- Even functions → mirror around y-axis

  $f(x) = f(-x)$

  $|x|$, $\cos(x)$, x^2

- Odd functions → mirror around origin

  $f(x) = -f(-x)$

  $\sin(x)$, x^3, $\sqrt[3]{x}$

Continuity

A function $f(x)$ is **continuous** at an interior point x_0 of its domain if, for all points x in the domain,

$$\forall \epsilon > 0,\ \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

- (there are no jumps)
- if we x approaches x_0, then $f(x)$ approaches $f(x_0)$

- Every "typical" function is continuous
- Every sum / difference / product / composition of continuous functions is continuous.

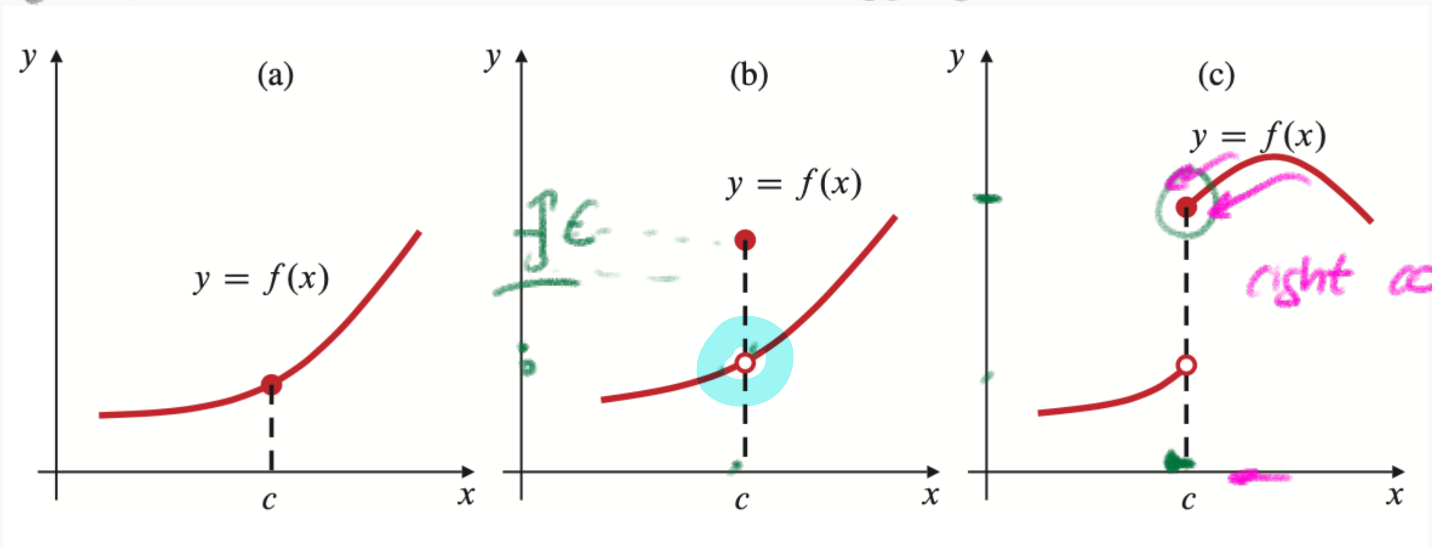
Continuity

- A function $f(x)$ is **discontinuous** at c if there is a "jump" at $x = c$ (and $f(c)$ is defined)
- Examples:

removable discontinuity

continuous

discontinuous at c



Note: (in this course) f can only be discontinous at c if c is in the domain of f, i.e. if $f(c)$ exists (in this course). So $f(x) = \frac{1}{x}$ is not discontinous at $x = 0$, it is undefined.

Left and right continuity

A function $f(x)$ is

- right continuous at c if $f(x)$ approaches $f(x_0)$ if x approaches x_0 from the right
- left continuous at c if " from the left
- continuous at c if it is both right and left continuous at c.

- right continuous
  $\forall \epsilon > 0 \ \exists \delta > 0 : 0 < x - x_0 < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$

- left continuous
  $\forall \epsilon > 0 \ \exists \delta > 0 : 0 < x_0 - x < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$

Continuous functions on an interval

A function $f(x)$ is

- continuous on an interval $[a, b]$ if
  - it is continuous at all interior points
  - left continuous at b
  - right continuous at a
- piecewise continuous on $[a, b]$ if there are a finite number of discontinuities on $[a,b]$