Practice Exam Questions - Tutorial 2

1. Use induction to prove the following statement

   (a) For all integers $n \geq 1$,
   $$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

   (b) For all integers $n \geq 1$, $7^n - 4^n$ is divisible by 3.

2. Use induction to prove the following statements.

   (a) For all integers $n \geq 1$,
   $$\sum_{i=1}^{n} (i \times (i!)) \quad = \quad (n+1)! - 1$$
   where as usual $n!$ refers to "n factorial" i.e. $n \times (n-1) \times ... \times 1$.

   (b) For all integers $n \geq 1$, $2^{3n} - 3^n$ is divisible by 5.

3. Use induction to prove the following statement.

   - For all integers $n \geq 1$,
   $$\sum_{i=1}^{n} i(i+2) = \frac{n(n+1)(2n+7)}{6}$$

4. Use induction to prove the following statement.

   - For all integers $n \geq 1$,
   $$\sum_{i=1}^{n} (2i-1)(2i) = \frac{n(n+1)(4n-1)}{3}$$

5. Let $A = \{\{7\}, 2, \{4, \{5, 6\}\}, 4\}$. Are the following statements true or false? Briefly motivate your answer.

   ✓ (a) $7 \in A$ False, $\{7\} \in A$

   ✓ (b) $\{2, 4\} \subseteq A$ True: $\{2,4\} \subseteq A \iff \forall x \in \{2,4\} : x \in A$

   ✓ (c) $\{5, 6\} \subseteq A$ False : 5 and 6 are not elements of A

   ✓ (d) $\{7\} \in A$ True : $\{7\}$ is an elemento of A

   ✓ (e) $\emptyset \subseteq A$ True, this is always true

   ✓ (f) $\{4, \emptyset\} \subseteq A$ False, $\emptyset \notin A$

   ✓ (g) $|A| = 5$ False, $|A| = 4$

6. Let $A = \{2, 3, 4, 5\}$, $B = \{4, 5, 6, 7\}$, $C = \{4, 5\}$. Which of the following statements are true? Briefly motivate your answer.

   ✓ T (a) $(A \setminus C) \cup B = A \cup B$

   $\{2,3\} \cup \{4,5,6,7\} = \{2,3,4,5,6,7\}$

   $A \cup B = \{2,3,4,5,6,7\}$

✓ F (b) $(C \setminus B) \cup \{6,7\} = \emptyset$ $\{\}$

✓ T (c) $B \cap A = C$ $\{4,5\} = \{4,5\}$

✓ T (d) $A \cap (B \setminus C) \subseteq \emptyset$ $\emptyset \subseteq \emptyset$

✗ T (e) $\emptyset \in A \cap \{7,8\}$ $\emptyset$ always a subset of set
but we are checking if it is an element

✓ F (f) $\{\{2\}\} \subseteq A$ $\{2\} \subseteq A$

7. Prove or disprove the following statement.
   - For all sets A, B and C, $(B \cap (A^c \cup C)^c = \emptyset) \Leftrightarrow (A \subseteq B^c \cup C)$.

8. Prove or disprove the following statement.
   - For all sets A, B and C, $(C \subseteq B \setminus A) \Leftrightarrow ((A \cap C = \emptyset) \wedge (B^c \subseteq C^c))$.

9. Prove or disprove the following statement.
   - For all sets A, B, and C, $(B \subseteq A^c \cup C) \Leftrightarrow ((A \cap B) \setminus (A \cap C) = \emptyset)$.

10. Prove or disprove the following statement.
    - For all sets A, B, and C, $(A \cup (C^c \setminus B)) = ((A \cup C^c) \setminus B)$.

(1) 1. Use induction to prove the following statement

(a) For all integers $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

(b) For all integers $n \geq 1$, $7^n - 4^n$ is divisible by 3.

(a) $\forall x \in \mathbb{Z}, n \geq 1 : \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

Base case: $P(1)$

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{2}$$

$$LS = \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+2)} = \frac{1}{2}$$

$$RS = \frac{1}{2}$$

$LS = RS \quad \square$

Induction step

Assume the claim holds for n

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

Show that it holds for $n+1$

algebra

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+1+1)}$$

Use assumpt. $\longrightarrow$

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n(n+2) + 1}{(n+1)(n+2)} = \frac{n^2 + 2n + 1}{(n+1)(n+2)}$$

$$= \frac{(n+1)^{\cancel{2}}}{(\cancel{n+1})(n+2)} = \frac{n+1}{n+2} = \frac{(n+1)}{(n+1)+1}$$

(b) $(\forall n \geq 1)(7^n - 4^n$ divisible by 3)

Base case: $P(1)$

$7^{(1)} - 4^{(1)} = 3$

3 is divisible by 3 $\square$

Inductive step

Assume the claim holds for n

$$7^n - 4^n = 3K, \text{ with } K \in \mathbb{Z}$$

Show that it holds for $n+1$

$$7^{(n+1)} - 4^{(n+1)} = 7 \cdot 7^{(n)} - 4 \cdot 4^{(n)} \quad \text{algebra}$$

$$= 7 \cdot 7^{(n)} - 7 \cdot 4^n + 3 \cdot 4^{(n)}$$

$$= \underbrace{7\underbrace{(7^n - 4^n)}_{\text{divisible by 3 per assumption}} + \underbrace{3 \cdot 4^n}_{\text{also div. by 3}}}_{\text{S divisible by 3}}$$

|2|

(a) For all integers $n \geq 1$,

$$\sum_{i=1}^{n} (i \times (i!)) = (n+1)! - 1$$

where as usual $n!$ refers to "n factorial" i.e. $n \times (n-1) \times ... \times 1$.

Base step: $P(1)$

RS: $(1 \times (1!)) = 1$

LS: $(1+1)! - 1 = 2 - 1 = 1$

RS = LS $\square$

Induction step

Assume claim holds for an arbitrary $n >$

Show it holds for $n+1$

$$\sum_{i=1}^{n+1} (i \times (i!)) = \sum_{i=1}^{n} (i \times (i!)) + [(n+1)(n+1)!]$$

Use assumption

$$= (n+1)! - 1 + (n+1)(n+1)!$$

$$= (n+1)!(n+1+1) - 1$$

$$= (n+1)!(n+2) - 1$$

$$= (n+2)! - 1$$

$$= [(n+1)+1]! - 1 \quad \square$$

(2) (b) (b) For all integers $n \geq 1$, $2^{3n} - 3^n$ is divisible by 5.

$(\forall n \in \mathbb{Z})(n \geq 1)$: $2^{3n} - 3^n$ div. by 5

Base step $(n = 1)$

$2^3 - 3^1 = 8 - 3 = 5$

5 is divisible by 5 $\square$

Induction step

Let n be an arbitrary integer ≥ 1

$P(n) = 2^{3n} - 3^n$ is div. by 5

holds for n

Show that P holds for $n+1$

$$2^{3(n+1)} - 3^{(n+1)} = 2^{(3n+3)} - 3^{(n+1)}$$

$$= 8 \cdot 2^{3n} - 3 \cdot 3^n$$

$$= 8 \cdot 2^{3n} - 8 \cdot 3^n + 5 \cdot 3^n$$

$$= 8\,(2^{3n} - 3^n) + 5 \cdot 3^n$$

div. by 5 by assumption

div. by 5

So, $P(n+1)$ holds

(3) • For all integers $n \geq 1$,

$$\sum_{i=1}^{n}(2i-1)(2i) = \frac{n(n+1)(4n-1)}{3}$$

Base step: $n = 1$

LS: $(2 \cdot 1 - 1)(2 \cdot 1) = 2$

RS: $\frac{1(1+1)(4-1)}{3} = \frac{6}{3} = 2$ $\square$

Induction step

Let n be an arbitrary integer, $n \geq 1$

$$P(n) = \sum_{i=1}^{n}(2i-1)(2i) = \frac{n(n+1)(4n-1)}{3}$$

holds for n

$(n+1)(n+2)(4n+3)$

$(n+1)(4n^2+3n+8n+6)$

Show that P holds for $n+1$

$$\sum_{i=1}^{n+1}(2i-1)(2i) = \underbrace{\sum_{i=1}^{n}(2i-1)(2i)}_{\text{assumption}} + [2(n+1)-1](2(n+1))$$

$$= \frac{n(n+1)(4n-1)}{3} + 4(n+1)(n+1) - 2(n+1)$$

$$= \frac{n(n+1)(4n-1) + 12(n+1)(n+1) - 6(n+1)}{3}$$

$$= \frac{(n+1)[n(4n-1) + 12(n+1) - 6]}{3}$$

$$= \frac{(n+1)[4n^2 - n + 12n + 12 - 6]}{3}$$

$$= \frac{(n+1)[4n^2 + 11n + 6]}{3}$$

$$= \frac{(n+1)(n+2)(4n+3)}{3}$$

$$= \frac{(n+1)[(n+1)+1][4(n+1)-1]}{3} \qquad \square$$

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