$p \to q \circ$

$\circ\, p \qquad q \circ$

$\forall x\, Px \circ \neg \exists x \neg Px$

$\forall x\, Px, \exists x \neg Px \circ$

$\forall x\, Px, \neg Pc \circ$

$\forall x\, Px \circ Pc$

$Pc \circ Pc$

$\times$

So inference is valid

~~$p \to q \circ$~~

~~$\circ\, p \qquad p, q \circ$~~

$\forall x\, (Px \to Qx) \models \exists x\, (Px \wedge \neg Qx)$

$\forall x\, (Px \to Qx) \models \exists x\, (Px \land \neg Qx)$

$\forall x\, (Px \to Qx) \circ \exists x\, (Px \land \neg Qx)$

first time, introduce constant;

$Pc \to Qc \circ \exists x\, (Px \land \neg Qx)$

already a constant, so no introducing new ones!

$Pc \to Qc \circ Pc \land \neg Qc$

$\circ Pc, Pc \land \neg Qc$ | $Qc \circ Pc \land \neg Qc$

$\circ Pc, Pc$ | $\circ Pc, \neg Qc$

open

counterexample:
Pc does not hold
(Qc doesn't matter)
$M = (D, I, g)$ $D = \{c\}$

$$\underset{1}{\cdot} \rightarrow \underset{2}{\cdot} \rightarrow \underset{3}{\cdot} \rightarrow \underset{4}{\cdot} \rightarrow \cdot \; \cdots$$

$R \quad x(x+1)$

$\forall y \exists x Rxy \models \exists x \forall y Rxy$

$\forall y \exists x Rxy \circ \exists x \forall y Rxy$) universal step

$\exists x Rxc \overset{+}{\circ} \exists x \forall y Rxy$) existential step

$\forall y \exists x Rxy, Rdc \overset{+}{\circ} \exists x \forall y Rxy$) universal step

" " $\circ$ $\forall y Rcy, \forall y Rdy$) existential

$\forall y \exists x Rxy, Rdc \overset{+}{\circ} Rce, \forall y Rdy, \exists x \forall y Rxy$) exist.

$\forall y \exists x Rxy, Rdc \overset{+}{\circ} Ree, Rdf, \exists x \forall y Rxy$) univ.

$\exists x Rxc, \exists x Rxd, \exists x Rxe, \exists x Rxf, Rdc \circ Rce, Rdf, \exists x \forall y Rxy$

You need extended rules to solve this.

$\exists x \forall y Rxy \models \forall y \exists x Rxy$

$\exists x \forall y Rxy \models \forall y \exists x Rxy$

So inference is valid.

$\exists x \forall y Rxy \quad \circ \quad \forall y \exists x Rxy$

) exist.

$\forall y Rcy \quad \overset{+}{\circ} \quad \forall y \exists x Rxy$

) exist.

$\forall y Rcy \quad \overset{+}{\circ} \quad \exists x Rxd$

) univ

$Rcc, Rcd \quad \circ \quad \exists x Rxd$

) univ

$Rcc, Rcd \quad \circ \quad Rcd, Rdd$

$\times$

$\exists x \forall y (P_x \to Q_y)$ o $\forall x \exists y (P_y \to Q_x)$

) exist.

$\forall y (P_c \to Q_y)$ $\overset{+}{o}$ "

) exist

$\forall y (P_c \to Q_y)$ $\overset{+}{o}$ $\exists y (P_y \to Q_d)$

) un

$P_c \to Q_c , P_c \to Q_d$ o "

) un

" $, P_c \to Q_d$ o $P_c \to Q_d , P_d \to Q_d$

X

$\forall x(Px \to \exists y Qy) \circ \exists x Qx \vee \neg\exists x Px$

$'' \circ \exists x Qx , \neg\exists x Px$

$\forall x(Px \to \exists y Qy), \exists x Px \circ \exists x Qx$ (exist.)

$'' , Pc \circ ''$) univ.

$'' , Pc \circ Qc$ univ

$Pc \to \exists y Qy , Pc \circ Qc$

$Pc \circ Pc, Qc$ ×

$Pc, \exists y Qy \circ Qc$ (exist.

$\forall x(Px \to \exists y Qy), Pc, Qd \circ Qc , \exists x Qx$

$'' , Pc , Qd \circ Qc , Qd$ ×

So: valid

$\forall x Px \to \forall x Qx \circ \underline{\forall x (Px \to Qx)}$ (exist)

$\circ\ Pc \to Qc$

"

$\forall x Qx \circ \underline{Pc \to Qc}$

$\underline{\forall x Qx}, Pc \circ Qc$ (univ)

$\underline{Qc}, Pc \circ \underline{Qc}$

X

$\circ\ \forall x Px, \underline{Pc \to Qc}$

$Pc \circ \underline{\forall x Px}, Qc$ (exist)

$Pc \circ Pd, Qc$

open.

Counterexample: $D = \{c, d\}$

$P = \{c\}$

$Q = \emptyset$

Thus: Not Valid