Solutions

Exercise 1: Let A be a set of size n. How many relations exist on A?

$A \times A$ has n^2 elements. Each of them corresponds to a proposition that can be either true or false. So the number of relations on A is 2^{n^2}.

Exercises 2, 3 and 4 concern the following relations:

(i) $x\mathbf{R}_1 y \Leftrightarrow y$ is a multiple of x on $\mathbb{N}$

(ii) $x\mathbf{R}_2 y \Leftrightarrow x < y$ on $S = \{0, 1, 2, 3\}$

(iii) $x\mathbf{R}_3 y \Leftrightarrow xy = y$ on $[0, 1]$

(iv) $x\mathbf{R}_4 y \Leftrightarrow x - y$ is even on $\mathbb{N}$

(v) $x\mathbf{R}_5 y \Leftrightarrow y > x - 1$ on $\mathbb{R}$

(vi) $(x, y)\mathbf{R}_6(a, b) \Leftrightarrow x^2 + y^2 \leq a^2 + b^2$ on $\mathbb{R}^2$

(vii) $(x, y)\mathbf{R}_7(a, b) \Leftrightarrow x + y = a + b$ on $\mathbb{R}^2$

(viii) $X\mathbf{R}_8 Y \Leftrightarrow X \subseteq Y$ on $\mathcal{P}(A)$ for some nonempty set A.

Exercise 2: Show that relations $\mathbf{R}_2$, $\mathbf{R}_3$, $\mathbf{R}_5$, $\mathbf{R}_6$ and $\mathbf{R}_8$ are not symmetric and that $\mathbf{R}_7$ is symmetric.

$\mathbf{R}_2$: Take $x = 1$ and $y = 3$. Then $x < y$, so $x\mathbf{R}_2 y$ is true, but $y \not< x$, so $y\mathbf{R}_2 x$ is false.

$\mathbf{R}_3$: Take $x = 1$ and $y = \frac{1}{\pi}$. Then $xy = y$, so $x\mathbf{R}_3 y$ is true, but $yx \neq x$, so $y\mathbf{R}_3 x$ is false.

$\mathbf{R}_5$: Take $x = 1$ and $y = \pi$. Then $y > x - 1$, so $x\mathbf{R}_5 y$ is true, but $x \not> y - 1$, so $y\mathbf{R}_5 x$ is false.

$\mathbf{R}_6$: Take $(x, y) = (3, 4)$ and $(a, b) = (1, 5)$. Then $x^2 + y^2 \leq a^2 + b^2$, so $(x, y)\,\mathbf{R}_6(a, b)$ is true, but $a^2 + b^2 \not\leq x^2 + y^2$, so $y\mathbf{R}_2 x$ is false.

$\mathbf{R}_7$: Let $(x, y) \in \mathbb{R}^2, (a, b) \in \mathbb{R}^2$ and suppose that $(x, y)\mathbf{R}_7(a, b)$ is true

$$\begin{aligned}
&\Rightarrow x + y = a + b \\
&\Rightarrow a + b = x + y \\
&\Rightarrow (a, b)\mathbf{R}_7(x, y) \text{ is true,}
\end{aligned}$$

which completes the proof.

$\mathbf{R}_8$: Take $X = \emptyset$ and $Y = \{1\}$. Then $X \subseteq Y$, so $X\mathbf{R}_8 Y$ is true, but $Y \not\subseteq X$, so $Y\mathbf{R}_8 X$ is false.

Exercise 3: For relations $\mathbf{R}_1$, $\mathbf{R}_4$, $\mathbf{R}_5$, $\mathbf{R}_6$ and $\mathbf{R}_8$ prove whether or not they are transitive.

$\mathbf{R}_1$ is transitive. **Proof:** Let $x, y, z \in \mathbb{N}$ and suppose that $x\mathbf{R}_1 y$ is true and that $y\mathbf{R}_1 z$ is true. Then y is a multiple of x and z is a multiple of y. But then $y = m \cdot x$ for some natural number m and $z = n \cdot y$ for some natural number n. This means that $z = n \cdot y = n \cdot m \cdot x$ and $n \cdot m$ is a natural number. Therefore z is a multiple of x and $x\mathbf{R}_1 z$ is true.

$\mathbf{R}_4$ is transitive. **Proof:** Let $x, y, z \in \mathbb{N}$ and suppose that $x\mathbf{R}_4 y$ is true and that $y\mathbf{R}_4 z$ is true. Then $x-y$ is even and $y-z$ is even. But then $x-z = x-y+y-z = (x-y)+(y-z)$ is the sum of 2 numbers making it an even number itself. Hence $x\mathbf{R}_4 z$ is true.

$\mathbf{R}_5$ is not transitive. **Proof** (Counterexample): Take $x = \frac{3}{2} \in \mathbb{R}$, $y = \frac{3}{4} \in \mathbb{R}$ and $z = 0 \in \mathbb{R}$. Then $y > x-1$ and $z > y-1$, so $x\mathbf{R}_3 y$ and $y\mathbf{R}_3 z$ are both true. However, $z \not> x-1$, so $x\mathbf{R}_3 z$ is false.

$\mathbf{R}_6$ is transitive. **Proof:** Let $(x,y), (a,b), (c,d) \in \mathbb{R}^2$ and suppose that $(x,y)\, \mathbf{R}_6\, (a,b)$ is true and that $(a,b)\, \mathbf{R}_6\, (c,d)$ is true. Then $x^2+y^2 \leq a^2+b^2$ and $a^2+b^2 \leq c^2+d^2$. But then $x^2+y^2 \leq a^2+b^2 \leq c^2+d^2$ so $(x,y)\, \mathbf{R}_1\, (c,d)$ is true.

$\mathbf{R}_8$ is transitive. **Proof:** Let $X, Y, Z \in \mathcal{P}(A)$ and assume that $X\mathbf{R}_8 Y$ and $Y\mathbf{R}_8 Z$ are true. Then $X \subseteq Y$ and $Y \subseteq Z$. Now, let $x \in X$. Then $x \in Y$ (since $X \subseteq Y$). But then $x \in Z$ (since $Y \subseteq Z$). Hence $X \subseteq Z$ and $X\mathbf{R}_8 Z$ is true.

Exercise 4: For relations $\mathbf{R}_2$, $\mathbf{R}_5$, $\mathbf{R}_6$, $\mathbf{R}_7$ and $\mathbf{R}_8$ prove whether or not they are reflexive.

$\mathbf{R}_2$ is not reflexive. **Proof** (Counterexample): Take $x = 0 \in S$. Then $x \not< x$, so $x\mathbf{R}_2 x$ is false and $\mathbf{R}_2$ is not reflexive.

$\mathbf{R}_5$ is reflexive. **Proof:** Let $x \in \mathbb{R}$. Then $x > x-1$, so $x\mathbf{R}_5 x$ is true and $\mathbf{R}_5$ is reflexive.

$\mathbf{R}_6$ is reflexive. **Proof:** Let $(x,y) \in \mathbb{R}^2$. Then $x^2+y^2 \leq x^2+y^2$, so $(x,y)\, \mathbf{R}_6\, (x,y)$ is true.

$\mathbf{R}_7$ is reflexive. **Proof:** Let $(x,y) \in \mathbb{R}^2$. Then $x+y = x+y$, so $(x,y)\, \mathbf{R}_7\, (x,y)$ is true and $\mathbf{R}_7$ is reflexive.

$\mathbf{R}_8$ is reflexive. **Proof:** Let $X \in \mathcal{P}(A)$. Then $X \subseteq X$, so $X\mathbf{R}_8 X$ is true and $\mathbf{R}_8$ is reflexive.

Exercise 5: The relation in example 2 is reflexive: $1\mathbf{R}1$, $2\mathbf{R}2$ and $3\mathbf{R}3$ are all true. It is also symmetric: For $x \neq y$ we have $x\mathbf{R}y$ and $y\mathbf{R}x$ either both true or both false. And it is also transitive: We have to check all possibilities of x, y and z such that $x\mathbf{R}y$ and $y\mathbf{R}z$ are both true. It turns out that in all of these cases $x\mathbf{R}z$ is true as well. Check this out for yourself.

Exercise 6: $\mathbf{R}_1$ is not reflexive, since $1\mathbf{R}_1 1$ is false; it is symmetric if $1\mathbf{R}_1 3$ and $3\mathbf{R}_1 2$ are both false and it is not transitive, since $1\mathbf{R}_1 2$ and $2\mathbf{R}_1 1$ are both true and $1\mathbf{R}_1 1$ is false.

$\mathbf{R}_2$ is reflexive if $2\mathbf{R}_2 2$ is true; $\mathbf{R}_2$ is not symmetric; For $\mathbf{R}_2$ to be transitive we need that 1) $1\mathbf{R}_2 3$ is true, since $1\mathbf{R}_2 2$ and $2\mathbf{R}_2 3$ are both true; 2) if $3\mathbf{R}_1 2$ is true, then, since $2\mathbf{R}_1 3$ is true as well, also $2\mathbf{R}_1 2$ must be true.

$\mathbf{R}_3$ is reflexive if $3\mathbf{R}_3 3$ is true; $\mathbf{R}_3$ is symmetric if $2\mathbf{R}_3 1$ is false and $3\mathbf{R}_3 2$ is true. For $\mathbf{R}_3$ to be transitive we need that 1) $3\mathbf{R}_3 2$ and $2\mathbf{R}_3 1$ are not both true, as this would mean that $3\mathbf{R}_3 1$ would have to be true as well; 2) if $3\mathbf{R}_3 2$ is true, then, since $2\mathbf{R}_3 3$ is true as well, also $3\mathbf{R}_3 3$ must be true.
smallskip $\mathbf{R}_4$ is not reflexive, since $1\mathbf{R}_4 1$ and $3\mathbf{R}_4 3$ are false; it is symmetric if $1\mathbf{R}_4 3$ and $3\mathbf{R}_4 2$ are both false; For $\mathbf{R}_4$ to be transitive we need that $1\mathbf{R}_4 3$ and $3\mathbf{R}_4 2$ are not both true, since $1\mathbf{R}_4 2$ is false.

Exercise 7: For the relation $x\mathbf{R}y \Leftrightarrow$ '$x - 4y$ is divisible by 3' on $\mathbb{Z}$ we have:
$\mathbf{R}$ is reflexive: Let $x \in \mathbb{Z}$. Then $x - 4x = -3x$, which is divisible by 3.

$\mathbf{R}$ is symmetric: Let $x, y \in \mathbb{Z}$ and assume that $x - 4y$ is divisible by 3. Then $x - 4y = 3k$ for some integer k. But then $4y - x = -3k$, which is divisible by 3. This means that $y - 4x = (4y - x) - 3y + 3x$ is also divisible by 3.
$\mathbf{R}$ is transitive: Let $x, y, z \in \mathbb{Z}$ and assume that $x - 4y$ is divisible by 3 and that $y - 4z$ is divisible by 3. Then $x - 4z = x - 4y + 4y - 4z = (x - 4y) + 3y + (y - 4z)$ is also divisible by 3.

Exercise 8: How many of the relations in exercise 1 are reflexive? And symmetric?

Reflexive: Since $x\mathbf{R}x$ must be true for all x we have $n \cdot (n-1)$ 'free' choices. So: $2^{n(n-1)}$.
Symmetric: For $x \neq y$ we require that $x\mathbf{R}y$ has the same truth value as $y\mathbf{R}x$. This means that we have 'only' $\frac{1}{2}n(n+1)$ free choices. So: $2^{1/2 \cdot n(n+1)}$.

Exercise 9: What elements of $\mathbb{Z}$ are in the same equivalence class as 0? We will call this equivalence class E_0. We have:

$$\begin{aligned} E_0 &= \{m \in S : m\mathbf{R}0 \text{ is true}\} \\ &= \{m \in S : m - 0 \text{ is divisible by } 3\} \\ &= \{m \in S : m \text{ is divisible by } 3\} \\ &= \{\ldots, -6, -3, 0, 3, 6, \ldots\} \end{aligned}$$

Now what elements of $\mathbb{Z}$ are in the same equivalence class as 1? We will call this equivalence class E_1. We have:

$$\begin{aligned} E_1 &= \{m \in S : m\mathbf{R}1 \text{ is true}\} \\ &= \{m \in S : m - 4 \cdot 1 \text{ is divisible by } 3\} \\ &= \{m \in S : m - 4 \text{ is divisible by } 3\} \\ &= \{\ldots, -5, -2, 1, 4, 7, \ldots\} \end{aligned}$$

Now what elements of S are in the same equivalence class as 2? We will call this equivalence class E_2. We have:

$$\begin{aligned} E_2 &= \{m \in S : m\mathbf{R}2 \text{ is true}\} \\ &= \{m \in S : m - 4 \cdot 2 \text{ is divisible by } 3\} \\ &= \{m \in S : m - 8 \text{ is divisible by } 3\} \\ &= \{\ldots, -4, -1, 2, 5, 8, \ldots\} \end{aligned}$$

Exercise 10: First we prove that the relations are reflexive, symmetric and transitive.

Relation $\mathbf{R}_9$:

$\mathbf{R}_9$ is reflexive. **Proof:** Let $x \in \{0, 1, 2, \ldots, 20\}$. Then x has the same remainder on division by 5 as x, so $x\mathbf{R}_9x$ is true.

$\mathbf{R}_9$ is symmetric. **Proof:** Let $x, y \in \{0, 1, 2, \ldots, 20\}$ and assume that $x\mathbf{R}_9y$ is true. Then x has the same remainder on division by 5 as y. But then y has the same remainder on division by 5 as x, so $y\mathbf{R}_9x$ is true.

$\mathbf{R}_9$ is transitive. **Proof:** Let $x, y \in \{0, 1, 2, \ldots, 20\}$ and assume that both $x\mathbf{R}_9y$ and $y\mathbf{R}_9z$ are true. Then x has the same remainder on division by 5 as y and y has the same remainder on division by 5 as z. But then x has the same remainder on division by 5 as z, so $x\mathbf{R}_9 z$ is true.

The construction of the equivalence classes of $\mathbf{R}_9$

What elements of $S = \{0, 1, 2, \ldots, 20\}$ are in the same equivalence class as 0? We will call this equivalence class E_0. We have:

$$\begin{aligned} E_0 &= \{m \in S : 0\mathbf{R}_9 m \text{ is true}\} \\ &= \{m \in S : 0 \equiv m \pmod 5\} \\ &= \{m \in S : m = 5 \cdot k \text{ for some integer } k\} \\ &= \{m \in S : m \text{ is divisible by } 5\} \\ &= \{0, 5, 10, 15, 20\} \end{aligned}$$

Now what elements of S are in the same equivalence class as 1? We will call this equivalence class E_1. We have:

$$\begin{aligned} E_1 &= \{m \in S : 1\mathbf{R}_9 m \text{ is true}\} \\ &= \{m \in S : 1 \equiv m \pmod 5\} \\ &= \{m \in S : m = 5 \cdot k + 1 \text{ for some integer } k\} \\ &= \{1, 6, 11, 16\} \end{aligned}$$

Now what elements of S are in the same equivalence class as 2? We will call this equivalence class E_2. We have:

$$\begin{aligned} E_2 &= \{m \in S : 2\mathbf{R}_9 m \text{ is true}\} \\ &= \{m \in S : 2 \equiv m \pmod 5\} \\ &= \{m \in S : m = 5 \cdot k + 2 \text{ for some integer } k\} \\ &= \{2, 7, 12, 17\} \end{aligned}$$

Continuing this analysis we find two more equivalence classes: $E_3 = \{3, 8, 13, 18\}$ and $E_4 = \{4, 9, 14, 19\}$.

Relation $\mathbf{R}_{10}$:

$\mathbf{R}_{10}$ is reflexive. **Proof:** Let $x \in \{1, 2, \ldots, 20\}$. Then $x \cdot x$ is a square, so $x\mathbf{R}_{10}x$ is true.

$\mathbf{R}_{10}$ is symmetric. **Proof:** Let $x, y \in \{1, 2, \ldots, 20\}$ and assume that $x\mathbf{R}_{10}y$ is true. Then $x \cdot y$ is a square. But then $y \cdot x = x \cdot y$ is a square as well, so $y\mathbf{R}_{10}x$ is true.

$\mathbf{R}_{10}$ is transitive. **Proof:** Let $x, y, z \in \{1, 2, \ldots, 20\}$ and assume that $x\mathbf{R}_{10}y$ is true and that $y\mathbf{R}_{10}z$ is true. Then $x \cdot y$ is a (perfect) square and $y \cdot z$ is a (perfect) square. This means that there exist natural numbers m and n such that $x \cdot y = m^2$ and $y \cdot z = n^2$. But then

$$x \cdot z = \tfrac{x \cdot y}{y} \cdot \tfrac{y \cdot z}{y} = \tfrac{m^2 \cdot n^2}{y^2} = (\tfrac{mn}{y})^2.$$

Remains to prove that $\frac{mn}{y}$ is an integer. The following argument suffices: Notice that $\frac{m^2}{y}$ is an integer (it is equal to x). This means that every prime factor in y is also in m^2. But all prime factors in m^2 are also in m, but then only half as many times. This means that every prime factor in y is also in m, and if a prime factor appears in y more than once, say q times, then it appears at least $\frac{q}{2}$ times in m. The exact same argument holds for n, because $\frac{n^2}{y}$ is equal to z, which is an integer. But that means that $m \cdot n$ contains all prime factors of y at least as many times as y, which means that $\frac{mn}{y}$ is an integer.

The construction of the equivalence classes of $\mathbf{R}_{10}$

What elements of $S = \{1, 2, \ldots, 20\}$ are in the same equivalence class as 1? We will call this equivalence class E_1. We have:

$$\begin{aligned} E_1 &= \{m \in S : 1\mathbf{R}_{10}m \text{ is true}\} \\ &= \{m \in S : 1 \cdot m \text{ is a square}\} \\ &= \{m \in S : m \text{ is a square}\} \\ &= \{1, 4, 9, 16\} \end{aligned}$$

Now what elements of S are in the same equivalence class as 2? We will call this equivalence class E_2. We have:

$$\begin{aligned} E_2 &= \{m \in S : 2\mathbf{R}_{10}m \text{ is true}\} \\ &= \{m \in S : 2 \cdot m \text{ is a square}\} \\ &= \{2, 8, 18\} \end{aligned}$$

Now what elements of S are in the same equivalence class as 3? We will call this equivalence class E_3. We have:

$$\begin{aligned} E_3 &= \{m \in S : 3\mathbf{R}_{10}m \text{ is true}\} \\ &= \{m \in S : 3 \cdot m \text{ is a square}\} \\ &= \{3, 12\} \end{aligned}$$

The other equivalence classes are $E_5 = \{5, 20\}$ (you may call this class E_4 if you want to!), $E_6 = \{6\}$, $E_7 = \{7\}$, $E_{10} = \{10\}$, $E_{11} = \{11\}$, $E_{13} = \{13\}$, $E_{14} = \{14\}$, $E_{15} = \{15\}$, $E_{17} = \{17\}$ and $E_{19} = \{19\}$.

Exercise 11: The 'equals'-relation "=" is a typical example of a relation that is both symmetric and antisymmetric.

Exercise 12: For relations $\mathbf{R}_2$, $\mathbf{R}_5$, $\mathbf{R}_7$ and $\mathbf{R}_8$ in example 1 as well as relations $\mathbf{R}_9$ and $\mathbf{R}_{10}$ in exercise 10, prove whether or not they are antisymmetric.

$\mathbf{R}_2$ is antisymmetric. **Proof:** Let $x, y \in \{0, 1, 2, 3\}$ and assume that $x\mathbf{R}_2y$ is true and that $x \neq y$. Then $x < y$ and hence $y \not< x$.

$\mathbf{R}_5$ is not antisymmetric. **Proof:** Take $x = 0$ and $y = \frac{1}{2}$. Then $y > x - 1$ and $x > y - 1$, so $x\mathbf{R}_5y$ and $y\mathbf{R}_5x$ are both true. However, $x \neq y$.

$\mathbf{R}_7$ is not antisymmetric. **Proof:** Take $(x, y) = (0, 0)$ and $(a, b) = (1, -1)$. Then $x + y = a + b$

and $a+b=x+y$, so $(x,y)\mathbf{R}_7(a,b)$ and $(a,b)\mathbf{R}_7(x,y)$ are both true. However, $(x,y)\neq(a,b)$.

$\mathbf{R}_8$ is antisymmetric. **Proof:** Let $X,Y\in\mathcal{P}(A)$ and assume that $X\subseteq Y$ and $Y\subseteq X$. Then $X=Y$.

$\mathbf{R}_9$ is not antisymmetric. **Proof:** Take $x=0$ and $y=5$. Then x has the same remainder on division by 5 as y and y has the same remainder on division by 5 as x, so $x\mathbf{R}_9y$ and $y\mathbf{R}_9x$ are both true. However, $x\neq y$.

$\mathbf{R}_{10}$ is not antisymmetric. **Proof:** Take $x=2$ and $y=18$. Then $x\cdot y$ is a square and $y\cdot x$ is a square, so $x\mathbf{R}_{10}y$ and $y\mathbf{R}_{10}x$ are both true. However, $x\neq y$.

Exercise 13: The relation in example 2 is not antisymmetric. **Proof:** Take $x=1$ and $y=3$. Then $x\mathbf{R}y$ and $y\mathbf{R}x$ are both true. However, $x\neq y$.

Exercise 14: Concerning the relations in exercise 6 we have: $\mathbf{R}_1$ is not antisymmetric, since $1\mathbf{R}_12$ and $2\mathbf{R}_11$ are both true; $\mathbf{R}_2$ is antisymmetric as long as $3\mathbf{R}_22$ is false; $\mathbf{R}_3$ is antisymmetric if $3\mathbf{R}_32$ is false and $\mathbf{R}_4$ is always antisymmetric.

Exercise 15: The relation in exercise 7 is not antisymmetric. **Proof:** Take $x=0$ and $y=3$. Then $x-4y=-12$, which is divisible by 3. Furthermore, $y-4x=3$ is also divisible by 3. Therefore both $x\mathbf{R}y$ and $y\mathbf{R}x$ are true. However, $x\neq y$.

Exercise 16: How many of the relations in exercise 1 are antisymmetric?

Antisymmetry requires that for any tuple (x,y) with $x\neq y$ we have that $x\mathbf{R}y$ and $y\mathbf{R}x$ are not both true. So, for all $x\mathbf{R}x$ we have two possibilities (and there are n of those) and for all couples $x\mathbf{R}y$ and $y\mathbf{R}x$ where $x\neq y$ we have three possibilities in total. Hence, the total number of antisymmetric relations is $2^n\cdot 3^{\,1/2\cdot n(n-1)}$.

Exercise 17: In relation-form this function looks as follows:

$$\begin{aligned}\mathbf{R}=&\{(\emptyset,0),(\{1\},1),(\{2\},2),(\{3\},3),(\{1,2\},4),(\{1,3\},5),(\{2,3\},6),(\{1,2,3\},10)\}\\ &\text{on }\mathcal{P}(\{1,2,3\})\times\mathbb{R}\end{aligned}$$

Exercise 18: For every $a\in A$ we need one outgoing arrow. This arrow can point to any of the n elements of B, so n possibilities per element of A. There are m elements in A, so in total we find n^m.

Exercise 20: For the functions f_3, f_5, f_6, f_8, f_{10} and f_{12} in Example 7, determine whether or not they are injective.

- f_3: Take $x=1$ and $y=5$. Then $f_3(x)=25=f_3(y)$, but $x\neq y$, so f_3 is not injective.
- f_5: Let $x,y\in\{1,2,3,4,5\}$ and assume that $f_5(x)=f_5(y)$. Then $3x=3y$, so $x=y$ and f_5 is injective.
- f_6: Let $x,y\in\{1,2,3,4,5\}$ and assume that $f_6(x)=f_6(y)$. Then $3x=3y$ (notice that $f_6(1)=3=3\cdot 1$), so $x=y$ and f_6 is injective.

- f_8: Let $x, y \in \{1,2,3,4,5\}$ and assume that $f_8(x) = f_8(y)$. Then $x^2 = y^2$, so $x^2 - y^2 = 0$. But then $(x-y)(x+y) = 0$, so $x - y = 0 \vee x + y = 0$. Since $x + y$ can not be equal to 0, we conclude that $x - y = 0$. Hence $x = y$ and f_8 is injective.

- f_{10}: Take $x = -5$ and $y = 7$. Then $f_{10}(x) = 36 = f_{10}(y)$, but $x \neq y$, so f_{10} is not injective.

- f_{12}: Take $x = 1$ and $y = 3$. Then $f_{12}(x) = 3 = f_{12}(y)$, but $x \neq y$, so f_{12} is not injective.

Exercises 21, 22 & 28:

- g_1: We prove that $g_1 : (0,1] \to [1,\infty)$, where $g_1(x) = \frac{1}{x}$ is injective and surjective, we find its inverse and calculate $g_1^{-1}(g_1)$.

  g_1 is injective. **Proof:** Let $x \in (0,1]$, $y \in (0,1]$. Then $g_1(x) = g_1(y) \Rightarrow \frac{1}{x} = \frac{1}{y} \Rightarrow x = y$.

  g_1 is surjective. **Proof:** Let $y \in [1,\infty)$ and take $x = \frac{1}{y} \in (0,1]$. Then $g_1(x) = \frac{1}{\frac{1}{y}} = y$. The corresponding preliminary calculation: $y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}$.

  We now know that the inverse function of g_1 will be $g_1^{-1} : [1,\infty) \to (0,1]$, where $g_1^{-1}(y) = \frac{1}{y}$. To show that this is correct, let $x \in (0,1]$. Then $g_1^{-1}(g_1(x)) = g_1^{-1}(\frac{1}{x}) = \frac{1}{\frac{1}{x}} = x$. Notice that for this function g_1^{-1} and g_1 differ only in the domain and the codomain!

- g_2: We prove that $g_2 : [2,4] \to [5,9]$, where $g_2(x) = 2x+1$ is invertible, we find its inverse and calculate $g_2^{-1}(g_2)$.

  g_2 is injective. **Proof:** Let $x \in [2,4]$ and $y \in [2,4]$. Then $g_2(x) = g_2(y) \Rightarrow 2x + 1 = 2y + 1 \Rightarrow x = y$.

  g_2 is surjective. **Proof:** Let $y \in [5,9]$ and take $x = \frac{y-1}{2} \in [2,4]$. Then $g_2(x) = 2\left(\frac{y-1}{2}\right) + 1 = y$. The corresponding preliminary calculation: $y = 2x + 1 \Leftrightarrow 2x = y - 1 \Leftrightarrow x = \frac{y-1}{2}$.

  The inverse function: $g_2^{-1} : [5,9] \to [2,4]$, where $g_2^{-1}(y) = \frac{y-1}{2}$. **Proof:** Let $x \in [2,4]$. Then $g_2^{-1}(g_2(x)) = g_2^{-1}(2x+1) = \frac{2x+1-1}{2} = x$.

- g_3: We prove that $g_3 : [0,3] \to [0,7]$, where $g_3(x) = 2x + 1$ is injective but not surjective.

  g_3 is injective. **Proof:** Let $x \in [0,3]$ and $y \in [0,3]$. Then $g_3(x) = g_3(y) \Rightarrow 2x + 1 = 2y + 1 \Rightarrow x = y$.

  g_3 is not surjective. **Proof:** Take $y = 0$ ($\in [0,7]$). Then in order for $g_3(x)$ to be equal to y we need that $2x + 1 = 0$, so $x = -\frac{1}{2} \notin [0,3]$. Apparently there is no $x \in [0,3]$ with $g_3(x) = 0$, so g_3 is not surjective.

- g_4: We prove that $g_4 : (0,1) \to (1,4)$, where $g_4(x) = \frac{8}{6x+2}$ is invertible, we find its inverse and calculate $g_4^{-1}(g_4)$.

  g_4 is injective. **Proof:** Let $x \in (0,1)$, $y \in (0,1)$ and suppose that $g_4(x) = g_4(y)$. Then $\frac{8}{6x+2} = \frac{8}{6y+2} \Rightarrow 48y + 16 = 48x + 16 \Rightarrow x = y$.

  g_4 is surjective. **Proof:** Let $y \in (1,4)$ and take $x = \frac{4-y}{3y} \in (0,1)$. Then $g_4(x) = \frac{8}{6x+2} = \frac{8}{6 \cdot \frac{4-y}{3y} + 2} = \frac{24y}{24-6y+6y} = y$.

The inverse function: $g_4^{-1} : (1,4) \to (0,1)$, where $g_4^{-1}(y) = \frac{4-y}{3y}$. **Proof:** Let $x \in (0,1)$. Then $g_4^{-1}(g_4(x)) = g_4^{-1}(\frac{8}{6x+2}) = \frac{4-\frac{8}{6x+2}}{3\cdot\frac{8}{6x+2}} = \frac{24x+8-8}{24} = x$.

- g_5: We prove that $g_5 : [5,7] \to [1,5]$, where $g_5(x) = \frac{x+1}{-2x+16}$ is injective but not surjective.

  g_5 is injective. **Proof:** Let $x, y \in [5,7]$. Then

$$\begin{aligned} g_5(x) = g_5(y) &\Rightarrow \tfrac{x+1}{-2x+16} = \tfrac{y+1}{-2y+16} \\ &\Rightarrow (-2x+16)(y+1) = (-2y+16)(x+1) \\ &\Rightarrow \ldots \Rightarrow x = y. \end{aligned}$$

  g_5 is not surjective. **Proof:** Take $y = 5$ $(\in [1,5])$. Then in order for $g_5(x)$ to be equal to y we need that $\frac{x+1}{-2x+16} = 5 \Rightarrow -10x + 80 = x + 1 \Rightarrow x = \frac{79}{11} \notin [5,7]$. Apparently there is no $x \in [5,7]$ with $g_5(x) = 5$, so g_5 is not surjective.

- g_6: We prove that $g_6 : [0,5] \to [-4,\infty)$, where $g(x) = x^2 + x - 4$ is injective but not surjective.

  g_6 is injective. **Proof:** Let $x \in [0,5]$ and $y \in [0,5]$. Then

$$\begin{aligned} g(x) = g(y) &\Rightarrow x^2 + x - 4 = y^2 + y - 4 \\ &\Rightarrow x^2 - y^2 + x - y = 0 \\ &\Rightarrow (x+y)(x-y) + 1\cdot(x-y) = 0 \\ &\Rightarrow (x+y+1)(x-y) = 0 \\ &\Rightarrow x - y = 0 \text{ (since } x+y+1 \geq 1) \end{aligned}$$

  g_6 is not surjective. **Proof:** Take $y = 100 \in [-4,\infty)$. Then in order for $g_6(x)$ to be equal to y we need that

$$\begin{aligned} x^2 + x - 4 = 100 &\Rightarrow x^2 + x - 104 = 0 \\ &\Rightarrow x = \tfrac{-1+\sqrt{1+4\cdot104}}{2} \vee x = \tfrac{-1-\sqrt{1+4\cdot104}}{2}. \end{aligned}$$

  In either case $x \notin [0,5]$. Apparently there is no $x \in [0,5]$ with $g_6(x) = 100$, so g_6 is not surjective.

- g_7: We prove that $g_7 : [0,5] \to [-5,\infty)$, where $g_7(x) = x^2 - x - 4$ is not injective and not surjective.

  g_7 is not injective. **Proof:** Take $x = 0 \in [0,5]$ and $y = 1 \in [0,5]$. Then $g_7(x) = -4$ and $g_7(y) = -4$, but $x \neq y$.

  g_7 is not surjective. **Proof:** Take $y = 237 \in [-5,\infty)$. Then in order for $g_7(x)$ to be equal to y we need that

$$\begin{aligned} x^2 - x - 4 = 237 &\Rightarrow x^2 - x - 241 = 0 \\ &\Rightarrow x = \tfrac{1+\sqrt{1+4\cdot241}}{2} \vee x = \tfrac{1-\sqrt{1+4\cdot241}}{2}. \end{aligned}$$

  In either case $x \notin [0,5]$. Apparently there is no $x \in [0,5]$ with $g_6(x) = 237$, so g_6 is not surjective.

- g_8: We prove that $g_8 : [1,5] \to [1,3]$, where $g_8(x) = \sqrt{2x-1}$ is invertible, we find its inverse and calculate $g_8^{-1}(g_8)$.

  g_8 is injective. **Proof:** Let $x \in [1,5]$, $y \in [1,5]$. Then

  $$\begin{aligned} g_8(x) = g_8(y) \ &\Rightarrow \ \sqrt{x-1} = \sqrt{y-1} \\ &\Rightarrow \ x - 1 = y - 1 \text{ (by taking squares)} \\ &\Rightarrow \ x = y \end{aligned}$$

  g_8 is surjective. **Proof:** Let $y \in [1,3]$ and take $x = \frac{y^2+1}{2} \in [1,5]$. Then

  $$\begin{aligned} g_8(x) &= \sqrt{2x-1} \\ &= \sqrt{2 \cdot \tfrac{y^2+1}{2} - 1} \\ &= \sqrt{y^2} \\ &= |y| \\ &= y \quad \text{since } y \geq 0 \end{aligned}$$

  The inverse function: $g_8^{-1} : [1,3] \to [1,5]$, where $g_8(y) = \frac{y^2+1}{2}$. **Proof:** Let $x \in [1,3]$. Then

  $$\begin{aligned} g_8^{-1}(g_8(x)) &= g_8^{-1}(\sqrt{2x-1}) \\ &= \tfrac{(\sqrt{2x-1})^2+1}{2} \\ &= \tfrac{|2x-1|+1}{2} \\ &= \tfrac{2x-1+1}{2} \ \text{ because } 2x-1 \geq 0 \\ &= x. \end{aligned}$$

- g_9: We prove that $g_9 : [-3,1) \to (0,1]$, where $g_9(x) = \frac{1}{x^2+1}$ is neither injective nor surjective.

  g_9 is not injective. **Proof:** Take $x = -\frac{1}{2}$ and $y = \frac{1}{2}$. Then $g_9(x) = \frac{4}{5}$ and $g_9(y) = \frac{4}{5}$, but $x \neq y$.

  g_9 is not surjective. **Proof:** Take $y = \frac{1}{20}$ $(\in (0,1])$. Then in order for $g_9(x)$ to be equal to y we need that $\frac{1}{x^2+1} = \frac{1}{20} \ \Rightarrow \ x^2 + 1 = 20 \ \Rightarrow \ x = \sqrt{19} \lor x = -\sqrt{19}$. In either case $x \notin [-3,1)$.

- g_{10}: We prove that $g_{10} : (0,2] \to [\frac{1}{5}, 1)$, where $g_{10}(x) = \frac{1}{x^2+1}$ is invertible, we find its inverse and calculate $g_{10}^{-1}(g_{10})$.

g_{10} is injective. **Proof:** Let $x, y \in (0, 2]$. Then

$$\begin{aligned} g_{10}(x) = g_{10}(y) \Rightarrow \ & \tfrac{1}{x^2+1} = \tfrac{1}{y^2+1} \\ \Rightarrow \ & x^2 + 1 = y^2 + 1 \\ \Rightarrow \ & x^2 - y^2 = 0 \\ \Rightarrow \ & (x-y)(x+y) = 0 \\ \Rightarrow \ & x = y \quad \text{because } x + y \neq 0 \end{aligned}$$

g_{10} is surjective. **Proof:** Let $y \in [\frac{1}{5}, 1)$ and take $x = \sqrt{\frac{1}{y} - 1}$. Then

$$\begin{aligned} g_{10}(x) &= \tfrac{1}{x^2+1} \\ &= \frac{1}{\left(\sqrt{\frac{1}{y}-1}\right)^2+1} \\ &= \frac{1}{\left|\frac{1}{y}-1\right|+1} \\ &= \frac{1}{\frac{1}{y}-1+1} \quad (\text{since } \tfrac{1}{y} - 1 \geq 0) \\ &= y. \end{aligned}$$

The inverse function: $g_{10}^{-1} : [\frac{1}{5}, 1) \to (0, 2]$, where $g_{10}^{-1}(y) = \sqrt{\frac{1}{y} - 1}$. **Proof:** Let $x \in (0, 2]$. Then

$$\begin{aligned} g_{10}^{-1}(g_{10}(x)) &= g_{10}^{-1}(\tfrac{1}{x^2+1}) \\ &= \sqrt{\frac{1}{\frac{1}{x^2+1}} - 1} \\ &= \sqrt{x^2} \\ &= x \quad (\text{since } x > 0) \end{aligned}$$

- g_{11}: We prove that $g_{11} : [0, 5] \to \mathbb{R}$, where $g_{11}(x) = x^4 - 3x^3 + 1$ is not injective.
  Proof: Take $x = 0$ and $y = 3$. Then $g_{11}(x) = 1$ and $g_{11}(y) = 1$, so $g_{11}(x) = g_{11}(y)$, but $x \neq y$.

- g_{12}: We prove that $g_{12} : [3, 4] \to \mathbb{R}$, where $g_{12}(x) = x^4 - 3x^3 + 1$ is injective.
  Proof: Let $x, y \in [3, 4]$. Then

$$\begin{aligned} g_{12}(x) = g_{12}(y) \ &\Rightarrow \ x^4 - 3x^3 + 1 = y^4 - 3y^3 + 1 \\ &\Rightarrow \ x^4 - y^4 - 3(x^3 - y^3) = 0 \\ &\Rightarrow \ (x-y)(x^3 + x^2y + xy^2 + y^3) - 3(x-y)(x^2 + xy + y^2) = 0 \quad (\text{by } (1)) \\ &\Rightarrow \ (x-y)(x^3 + x^2y + xy^2 + y^3 - 3x^2 - 3xy - 3y^2) = 0 \end{aligned}$$

Now, since $x \geq 3$ and $y \geq 3$, we have that $x^3 \geq 3x^2$, $y^3 \geq 3y^2$ and $x^2y \geq 3xy$. This means that $x^3 + x^2y + xy^2 + y^3 - 3x^2 - 3xy - 3y^2 > 0$. We conclude that apparently $x - y = 0$ and hence $x = y$.

- g_{13} : We prove that $g_{13} : [0, 3] \to \mathbb{R}$, where $g_{13}(x) = \sqrt[3]{4x} - x$ is not injective.

  Proof: Take $x = 0$ and $y = 2$. Then $g_{13}(x) = 0$ and $g_{13}(y) = \sqrt[3]{8} - 2 = 0$, so $g_{13}(x) = g_{13}(y)$, but $x \neq y$.

- g_{14} : We prove that $g_{14} : [1, 2] \to \mathbb{R}$, where $g_{14}(x) = \sqrt[3]{4x} - x$ is injective.

  Proof: Let $x, y \in [1, 2]$. Then

$$\begin{aligned}
g_{14}(x) = g_{14}(y) &\Rightarrow \sqrt[3]{4x} - x = \sqrt[3]{4y} - y \\
&\Rightarrow \sqrt[3]{4x} - \sqrt[3]{4y} - (x - y) = 0 \\
&\Rightarrow \sqrt[3]{4} \cdot (\sqrt[3]{x} - \sqrt[3]{y}) - (x - y) = 0 \\
&\Rightarrow \tfrac{\sqrt[3]{4}(x-y)}{\sqrt[3]{(x)^2} + \sqrt[3]{xy} + \sqrt[3]{(y)^2}} - (x - y) = 0 \qquad \text{(by (1))} \\
&\Rightarrow (x - y)(\tfrac{\sqrt[3]{4}}{\sqrt[3]{(x)^2} + \sqrt[3]{xy} + \sqrt[3]{(y)^2}} - 1) = 0
\end{aligned}$$

  Now, since $x \geq 1$ and $y \geq 1$, we have that $\sqrt[3]{(x)^2} + \sqrt[3]{xy} + \sqrt[3]{(y)^2} \geq 3\sqrt[3]{1} = 3$, which implies that $\frac{\sqrt[3]{4}}{\sqrt[3]{(x)^2} + \sqrt[3]{xy} + \sqrt[3]{(y)^2}} - 1 \leq \frac{\sqrt[3]{4}}{3} - 1 < 0$. We conclude that apparently $x - y = 0$ and hence $x = y$.

Exercise 23:

e') Prove that $g_{5a} : [5, 7] \to [1, 4]$, where $g_{5a}(x) = \frac{x+1}{-2x+16}$, is surjective.

Proof: Let $y \in [1, 4]$ and take $x = \frac{16y-1}{2y+1}$ $(\in [5, 7])$. Then

$$\begin{aligned}
g_{5a}(x) &= \tfrac{x+1}{-2x+16} \\
&= \tfrac{\frac{16y-1}{2y+1} + 1}{-2\frac{16y-1}{2y+1} + 16} \\
&= \tfrac{\frac{16y-1+2y+1}{2y+1}}{\frac{-32y+2+32y+16}{2y+1}} \\
&= \tfrac{18y}{18} = y.
\end{aligned}$$

The corresponding preliminary calculation:

$$\begin{aligned}
\tfrac{x+1}{-2x+16} = y &\Leftrightarrow y(-2x + 16) = x + 1 \\
&\Leftrightarrow -2yx + 16y = x + 1 \\
&\Leftrightarrow x(2y + 1) = 16y - 1 \\
&\Leftrightarrow x = \tfrac{16y-1}{2y+1}
\end{aligned}$$

f') Prove that $g_{6a} : [0,5] \to [-4,26]$, where $g_{6a}(x) = x^2 + x - 4$, is surjective.

Proof: Let $y \in [-4,26]$ and take $x = \sqrt{y + 4\frac{1}{4}} - \frac{1}{2} \in [0,5]$. Then

$$\begin{aligned} g_{6a}(x) &= x^2 + x - 4 \\ &= \left(\sqrt{y+4\tfrac{1}{4}} - \tfrac{1}{2}\right)^2 + \sqrt{y+4\tfrac{1}{4}} - \tfrac{1}{2} - 4 \\ &= y + 4\tfrac{1}{4} - \sqrt{y+4\tfrac{1}{4}} + \tfrac{1}{4} + \sqrt{y+4\tfrac{1}{4}} - \tfrac{1}{2} - 4 \\ &= y. \end{aligned}$$

The corresponding preliminary calculations:

$$\begin{aligned} & y = x^2 + x - 4 = (x + \tfrac{1}{2})^2 - 4\tfrac{1}{4} \ \text{ (completing the square)} \\ \Leftrightarrow\ & (x+\tfrac{1}{2})^2 = y + 4\tfrac{1}{4} \\ \Leftrightarrow\ & x + \tfrac{1}{2} = \sqrt{y+4\tfrac{1}{4}} \ \vee\ -(x+\tfrac{1}{2}) = \sqrt{y+4\tfrac{1}{4}} \end{aligned}$$

Now, $-(x+\frac{1}{2}) = \sqrt{y+4\frac{1}{4}}$ leads to values of x that are not in the domain, so we take $x + \frac{1}{2} = \sqrt{y+4\frac{1}{4}}$, which implies that $x = \sqrt{y+4\frac{1}{4}} - \frac{1}{2}$.

g') Prove that $g_{7a} : [0,5] \to [-4\frac{1}{4}, 16]$, where $g_{7a}(x) = x^2 - x - 4$, is surjective.

Proof: Let $y \in [-4\frac{1}{4}, 16]$ and take $x = \sqrt{y + 4\frac{1}{4}} + \frac{1}{2} \in [0,5]$. Then

$$\begin{aligned} g_{7a}(x) &= x^2 - x - 4 \\ &= \left(\sqrt{y+4\tfrac{1}{4}} + \tfrac{1}{2}\right)^2 - \sqrt{y+4\tfrac{1}{4}} - \tfrac{1}{2} - 4 \\ &= y + 4\tfrac{1}{4} + \sqrt{y+4\tfrac{1}{4}} + \tfrac{1}{4} - \sqrt{y+4\tfrac{1}{4}} - \tfrac{1}{2} - 4 \\ &= y. \end{aligned}$$

The corresponding preliminary calculations:

$$\begin{aligned} & y = x^2 - x - 4 = (x - \tfrac{1}{2})^2 - 4\tfrac{1}{4} \ \text{ (completing the square)} \\ \Leftrightarrow\ & (x-\tfrac{1}{2})^2 = y + 4\tfrac{1}{4} \\ \Leftrightarrow\ & x - \tfrac{1}{2} = \sqrt{y+4\tfrac{1}{4}} \ \vee\ -(x-\tfrac{1}{2}) = \sqrt{y+4\tfrac{1}{4}} \end{aligned}$$

Now, $-(x-\frac{1}{2}) = \sqrt{y+4\frac{1}{4}}$ may lead to values of x that are not in the domain, so we take $x - \frac{1}{2} = \sqrt{y+4\frac{1}{4}}$, which implies that $x = \sqrt{y+4\frac{1}{4}} + \frac{1}{2}$.

Exercise 24: An example of a bijective function $f : (0,1) \to [0,1]$ is

$$f(x) = \begin{cases} 0 & \text{if } x = \frac{1}{2} \\ 2x & \text{if } x \in \left\{\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\right\} \\ 1 & \text{if } x = \frac{3}{4} \\ 2x - 1 & \text{if } x \in \left\{\frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots\right\} \\ x & \text{otherwise} \end{cases}$$

Exercise 25: An example of a bijective function $f : [0,1] \to (0,1)$ is

$$f(x) = \begin{cases} \frac{1}{3} & \text{if } x = 0 \\ \frac{1}{2}x & \text{if } x \in \left\{\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \ldots\right\} \\ \frac{1}{5} & \text{if } x = 1 \\ \frac{1}{2}x & \text{if } x \in \left\{\frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \ldots\right\} \\ x & \text{otherwise} \end{cases}$$

Exercise 26: Assume that $m \leq n$. How many of the functions in exercise 18 are injective?

We have n possibilities for the first outcome, then $n-1$ for the second, $n-2$ for the third etc. In total: $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-m+1) = \frac{n!}{(n-m)!}$.

Exercise 27: Let $f : A \to B$ and $g : B \to C$. In order for $f \circ g$ to be well-defined, we need that $C \subseteq A$.

Exercise 29:

- g_{5a} is injective. The proof is the same as for g_5. It is also surjective, as proved in exercise 10. Therefore it is invertible. Its inverse follows directly from the preliminary calculation of the proof that g_{5a} is surjective. It is $g_{5a}^{-1} : [5,7] \to [1,4]$, where

$$g_{5a}^{-1}(y) = \frac{16y-1}{2y+1}.$$

- g_{6a} is injective. The proof is the same as for g_6. It is also surjective, as proved in exercise 10. Therefore it is invertible. Its inverse follows directly from the preliminary calculation of the proof that g_{6a} is surjective. It is $g_{6a}^{-1} : [-4,26] \to [0,5]$, where

$$g_{6a}^{-1}(y) = \sqrt{y + 4\tfrac{1}{4}} - \frac{1}{2}.$$

- g_{7a} is not injective. The proof is the same as for g_7. This means that it is not invertible either.

References

[1] Gerstein, L.J. (1996): Introduction to Mathematical Structures and Proofs