FORMULA SHEET

(1) Logic

- **Number sets**

  $\mathbb{Z} \to$ Integers (whole numbers)

  $\mathbb{N} \to$ Natural numbers ($x > 0$)

  $\mathbb{Q} \to$ Rational num: $\frac{a}{b}$, $a, b \in \mathbb{Z}$ $b \neq 0$

  $\mathbb{R} \to$ All numbers

- **Divisibility** $(m = kn)$

  Even numbers $\to 2k$

  Odd numbers $\to 2k+1$

  Prime numbers $\to$ $x \in \mathbb{N}$, $x > 1$ div by only x and 1

- **Logic**

  Tautology $\to$ true for all cases

  Contraddiction $\to$ false for all cases

- **Conditional propositions**

  $p \to q \equiv \neg q \to \neg p \equiv \neg p \vee q$

  $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

- **Quantifiers**

  Universal q. $(\forall)$

  Existential $(\exists)$

  to negate quantif. FLIP the quantif and negate the body

(2) Proof techniques

- **Direct Proof**

  To prove $p \Rightarrow q$: Assume p and prove that q holds

  To disprove a statement

  ↳ prove the negation is true

To prove a $\forall x$ statement
↳ prove for arbitrary element

To prove a $\exists x$ statement
↳ only need to give an example

- **Counterexample**
  To disprove a $\forall x$ statement

- **Contrapositive**
  Quantifiers must NOT change, only the body

- **Mathemathical Induction**
  For statements like $(\forall n \geq N, n \in \mathbb{Z})\, P(n)$
  (1) Base Case: prove $P(n)$
  (2) Induction step: $(\forall n \geq N)(P(n) \rightarrow P(n+1))$

(3) Set Theory

- **Subsets** $B \subseteq A$
  $B \subseteq A \iff \forall x \in B : x \in A$
  $\emptyset \subseteq A$ always true
  $A \subseteq A$ always true
- **Set equality**
  $(A \subseteq B \wedge B \subseteq A) \iff A = B$
- **Union** $A \cup B = \{x : x \in A \vee x \in B\}$
- **Intersection** $A \cap B = \{x : x \in A \wedge x \in B\}$
- **Difference** $A \setminus B = \{x : x \in A \wedge x \notin B\}$

- **Complement** A^c

  $A^c = \{x : x \notin A\}$

  $\emptyset^c = U$ (all elements)

- **Associativity**

  Intersection $A \cap (B \cap C) = (A \cap B) \cap C$

  Union $A \cup (B \cup C) = (A \cup B) \cup C$

- **Distributive laws**

  (1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

  (2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- **De Morgan laws**

  (1) $(A \cup B)^c = A^c \cap B^c$

  (2) $(A \cap B)^c = A^c \cup B^c$

- **Power sets** -> set of all subsets

  $P(B) = \{\emptyset, \{\ell, \emptyset\}, \{\ell\}, \{\emptyset\}\}$

  Cardinality: $|P(A)| = 2^{|A|}$

- **Product sets** $A \times B$

  $A \times B = \{(a,b) : a \in A \wedge b \in B\}$

  Cardinality: $|A \times B| = |A| \times |B|$

  $A \times B = B \times A$ when $A = B$ (or both empty)

- **Set partitions**

  $A_1, \dots, A_k$ are partitions of $A \iff$

  (1) None of the sets is empty

  (2) Union of the sets forms A: $\bigcup_{k=1}^{n} A_k = A$

  (3) No sets intersect: $A_m \cap A_k = \emptyset$, $m \neq k$

(4) Relations

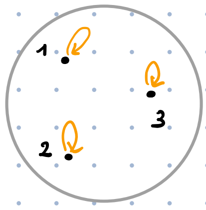
- **Definition** → relationship between el.
  A relation R on set A is a subset of the product set $A \times A$: $x R y$

- **Reflective relation**
  (every element related to itself)

  $\forall x \in A : x R x$

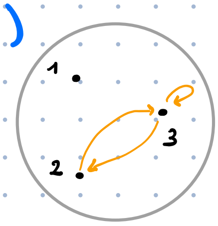
  Neg: $\exists x \in A : x \cancel{R} x$

  e.g. $A = \{1, 2, 3\}$; $x R y$ if $x = y$



- **Symmetric relation**
  (every arrow goes in both directions)

  $\forall a, b \in A : a R b \rightarrow b R a$



  e.g. $A = \{1, 2, 3\}$, $x R y$ if $x + y \geq 5$

  $x R y \iff x + y \geq 5$

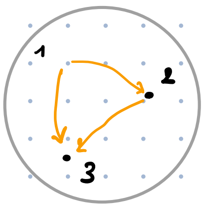
  $\iff y + x \geq 5 \quad y R x$

- **Transitive relation**
  ($a \longrightarrow b \longrightarrow c$)

  $\forall a, b, c \in A : (a R b \wedge b R c) \longrightarrow a R c$

  Neg: $\exists a, b, c \in A : (a R b \wedge b R c) \wedge a \cancel{R} c$

  e.g. if $x < y$



- **Equivalence relation**

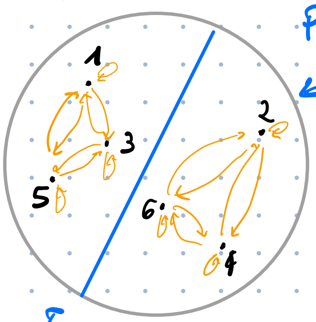
Relation that is reflexive, symmetric and transitive.

Corresponds to a partition of the set

e.g. if $x-y$ is even

Partition of $\mathbb{N}$

← even numbers



odd numbers

- **Antisymmetric relation**

(relation cannot go in both directions)

$\forall x, y \in A: (x R y \wedge y R x) \rightarrow x = y$

$\forall x, y \in A: (x R y \wedge x \neq y) \rightarrow y \not R x$

- **Partial order**

(a relation that is anti-symmetric, reflexive and transitive

e.g. a subset relation

15) Functions

- **Introduction**

  A → Domain

  B → Co-domain

  Range: $\{ f(x), x \in A \}$, range $(f) \subseteq B$

- **Composite functions**

  $f \circ g(x) = f(g(x))$ co-dom (g) = dom(f)

  range $(g) \subseteq$ domain (f)

- **Injective functions**

  (element of domain mapped to different values of co-domain)

  $f: A \to B$ is inj.

  $\iff (\forall x, y \in A)(x \neq y \Rightarrow f(x) \neq f(y))$

  $\iff (\forall x, y \in A)(f(x) = f(y) \Rightarrow x = y)$

- **Surjective functions** ← Range = Co-dom

  each element of co-domain has been mapped from at least 1 element of dom.

  $f: A \to B$ is surjective

  $\iff (\forall y \in B)(\exists x \in A)(y = f(x))$

- **Bijective functions**

  each element of domain mapped to exactly 1 element of co-domain

  $f: A \to B$ is bijective if injecti. and surject

  $\Rightarrow (\forall x \in A)(x \neq y \Rightarrow f(x) \neq f(y))$

  $\wedge (\forall y \in B)(\exists x \in A)(y = f(x))$

- **Inverse functions**

  Need to be bijective

  $f(f^{-1}(y))$, $y \in A$ (adjusted)

  $f^{-1}(f(x))$, $x \in B$ (adjusted)

16) Combinatorics

- **Inclusion - exclusion**

  (2 sets) $|M \cup P| = |M| + |P| - |P \cap M|$

  (3 sets) $|A \cup B \cup C|$

  $= |A| + |B| + |C| - |A \cap B| - |B \cap C| -$

  $- |A \cap C| + |A \cap B \cap C|$

- **Rule of sum**

  $|A| = |A_1| + |A_2| + \dots$

  where $A_1, A_2, \dots,$ are disjoint subsets of A

- **Rule of product**

  n = product of the number of options for each part

- **Disposition with repetitions**

  selection of k objects/positions, out of n objects

  $D^*_{n,k} = (n)^k$ Order ✓ Repetition ✓

- **Simple disposition**

  $D_{n,k} = \dfrac{n!}{(n-k)!}$ Order ✓ Repetition ✗

- **Combination**

  $C_{n,k} = \binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ Order ✗ Repetition ✗

- **Combination with rep.**

  $C^*_{n,k} = \binom{n+k-1}{k}$

  $= \dfrac{(n+k-1)!}{k!(n-1)!}$ Order ✗ Repetition ✓