Question 1:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \xrightarrow{R_2:\ R_2+2\cdot R_1} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right] \quad R_3:\ R_3-2\cdot R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \underline{x} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix}$$

So, for example $\underline{x} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 0 \cdot \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$

and $\underline{x} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = -3 \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + -1 \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 1 \cdot \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}$

Question 2:

Suppose T is a linear transformation.

Then, $T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) - T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$

$$= \begin{bmatrix} 1 \\ -3 \\ 6 \\ 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 6 \\ 2 \\ 5 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad ↯$$

So, $T(\underline{0}) \neq \underline{0}$. Hence, T cannot be a linear transformation.

Question 3:

$$\det A = \begin{vmatrix} p & 0 & 0 \\ 1 & p & 2 \\ 0 & -1 & 2 \end{vmatrix} = p \cdot \begin{vmatrix} p & 2 \\ -1 & 2 \end{vmatrix} = p \cdot (2p+2)$$

A is not invertible $\Leftrightarrow \det A = 0 \Leftrightarrow p\cdot(2p+2) = 0 \Leftrightarrow p = 0 \vee p = -1.$

Question 4:

$$\left.\begin{array}{l}\text{trace } A = 0 \implies \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \text{every row sum is } 3 \implies \lambda_1 = 3 \\ \det A = 6 \implies \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 6.\end{array}\right\} \left.\begin{array}{l}\lambda_2 + \lambda_3 = -3 \implies \lambda_3 = -3 - \lambda_2 \\ \\ \end{array}\right] 3 \cdot \lambda_2 \cdot (-3 - \lambda_2) = 6$$

$$\implies -3\lambda_2^2 - 9\lambda_2 - 6 = 0 \implies \lambda_2^2 + 3\lambda_2 + 2 = 0 \implies (\lambda_2 + 2)(\lambda_2 + 1) = 0$$

$$\lambda_2 = -2 \lor \lambda_2 = -1$$

$$\Downarrow \qquad \Downarrow$$

$$\lambda_3 = -1 \qquad \lambda_3 = -2.$$

Hence, the eigenvalues of A are: $3, -1, -2$.

Question 5:

$(\text{Row } A)^{\perp} = \text{Nul } A$. So, let's calculate Row A.

$$\begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \underset{R_3 : R_3 - 5 \cdot R_1}{\overset{R_2 : R_2 + R_1}{\sim}} \begin{bmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 14 & -35 & 42 \end{bmatrix} \underset{R_3 : R_3 + 7 \cdot R_2}{\sim}$$

$$\begin{bmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, two lin indep vectors that are orthogonal to Nul A:

$$\begin{bmatrix} 1 \\ -4 \\ 9 \\ -7 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -2 \\ 5 \\ -6 \end{bmatrix}.$$

Question 6:

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \underset{R_2 : R_2 - 2 \cdot R_1}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

So, every vector in Row A is of the form $\alpha \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$.

So, $\underline{u} - \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix} = \begin{bmatrix} \alpha - 3 \\ -\alpha + 3\beta - 2 \\ \beta - 9 \end{bmatrix}$.

This vector needs to be orthogonal to both $\begin{bmatrix}1\\-1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\3\\1\end{bmatrix}$.

So, $\begin{cases}(\alpha-3)-(-\alpha+3\beta-2)=0\\3(-\alpha+3\beta-2)+(\beta-9)=0\end{cases} \iff \begin{cases}2\alpha-3\beta=1\\-3\alpha+10\beta=15\end{cases}$ +

$-\alpha+7\beta=16 \Rightarrow \alpha=7\beta-16$

$2\cdot(7\beta-16)-3\beta=1 \Rightarrow 14\beta-32-3\beta=1 \Rightarrow 11\beta=33 \Rightarrow \beta=3 \Rightarrow \alpha=7\cdot 3-16=5.$

(OR: $\left[\begin{array}{cc|c}2 & -3 & 1\\-3 & 10 & 15\end{array}\right] \underset{R_2:\,R_2+3/2\,R_1}{\sim} \left[\begin{array}{cc|c}2 & -3 & 1\\0 & 11/2 & 33/2\end{array}\right] \underset{R_2:\,R_2 * 2/11}{\overset{R_1:\,R_1 * 1/2}{\sim}} \left[\begin{array}{cc|c}1 & -3/2 & 1/2\\0 & 1 & 3\end{array}\right]$

$\underset{R_1:\,R_1+3/2\,R_2}{\sim} \left[\begin{array}{cc|c}1 & 0 & 5\\0 & 1 & 3\end{array}\right]$)

Hence, $\underline{u} = 5\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix} + 3\cdot\begin{bmatrix}0\\3\\1\end{bmatrix} = \begin{bmatrix}5\\4\\3\end{bmatrix}$.

Question 7:

ⓐ $\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}$. Then we need $\alpha+\gamma=0 \Rightarrow \gamma=-\alpha$ and $\alpha+\beta=0. \Rightarrow \beta=-\alpha$ Hence, $\begin{bmatrix}\alpha\\-\alpha\\-\alpha\end{bmatrix}$.

The vector needs to be a unit vector. So, $\alpha^2+(-\alpha)^2+(-\alpha)^2=1$.

$\Rightarrow 3\alpha^2=1 \Rightarrow \alpha^2=1/3 \Rightarrow \alpha=\frac{1}{\sqrt{3}}$ or $\alpha=\frac{-1}{\sqrt{3}}$.

So, $\begin{bmatrix}\frac{1}{\sqrt{3}}\\\frac{-1}{\sqrt{3}}\\\frac{-1}{\sqrt{3}}\end{bmatrix}$ and $\begin{bmatrix}\frac{-1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}$.

ⓑ $\begin{bmatrix}3 & -2 & 4\\-2 & 6 & 2\\4 & 2 & 3\end{bmatrix}\begin{bmatrix}2\\1\\-2\end{bmatrix} = \begin{bmatrix}-4\\-2\\4\end{bmatrix} = -2\cdot\begin{bmatrix}2\\1\\-2\end{bmatrix}$

$\begin{bmatrix}3 & -2 & 4\\-2 & 6 & 2\\4 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\-2\\0\end{bmatrix} = \begin{bmatrix}7\\-14\\0\end{bmatrix} = 7\cdot\begin{bmatrix}1\\-2\\0\end{bmatrix}$

$\begin{bmatrix}3 & -2 & 4\\-2 & 6 & 2\\4 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}7\\0\\7\end{bmatrix} = 7\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}$

So, $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

(c) For example: $\underline{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\underline{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(d) For example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(e). For example: the first quadrant, namely $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, \ y \geq 0 \right\}$.