Calculus Lecture 6: Integration techniques

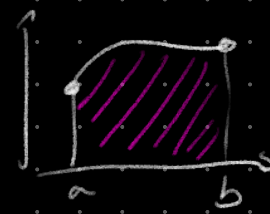
- **Recap:** definite and indefinite integrals
- **Substitution** (inverse chain rule)
- **Integration by parts** (inverse product rule)
- **Partial fraction decomposition** (rational functions)
- **Improper integrals**

Adams' Ch. 5.6, 6.1, 6.2, 6.5

Recap

- Definite integral
  - Area below a graph
  - Limit of a Riemann sum of rectangular areas

$$\int_a^b f(x)\,dx =$$



NUMBER

- Indefinite integral = anti-derivative

$$\int f(x)\,dx = F(x) + C \quad \text{if} \quad F'(x) = f(x)$$

FUNCTION of x

- The fundamental theorem of Calculus connects definite and indefinite integrals

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{, with} \quad F'(x) = f(x)$$

This lecture: how to calculate integrals

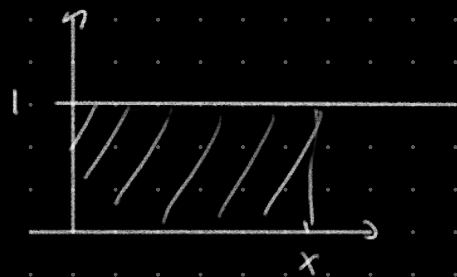
Simple integrals (that are on your formula sheet)

$\int e^{-x^2}\,dx$ no analytical form

$\int dx = \int 1\cdot dx$

$= x + C$

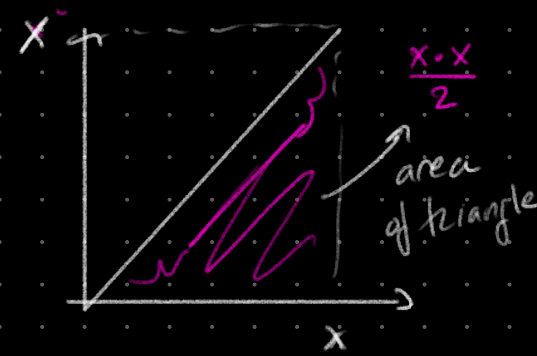
$\int_0^x f(t)\,dt$



↳ area $= 1\cdot x = x$

$= \int_0^x f(t)\,dt$

$\int_0^x t\,dt$



$\int \frac{dx}{x} = \ln|x| + C$

$\frac{d}{dx}(\ln|x|) \rightarrow$ for $x>0$ $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

for $x<0$ $\frac{d}{dx}(\ln(-x)) = \frac{1}{(-x)}\cdot -1 = \frac{1}{x}$

Substitution

chain rule: $\int \frac{d}{dx}(f(g(x)))dx = \int (\underbrace{f'(g(x))}_{f'(u)} \cdot \underbrace{g'(x)}_{du})dx$

$\downarrow$

$f(g(x)) + C$

$\left[\begin{array}{l} u = g(x) \\ du = g'(x)dx \end{array} \right.$

$\int \sin(3x)\,dx = \int \sin(u) \cdot \frac{1}{3} du = \frac{1}{3} \int \sin(u) du = -\frac{1}{3}\cos(u) + C = -\frac{1}{3}\cos(3x) + C$

$u = 3x$
$du = 3 \cdot dx$

$\int \frac{dx}{x+1} = \int \frac{du}{u} = \ln|u| + C = \ln|x+1| + C$

$u = x+1$
$du = dx$

$u = \cos(x)$
$du = -\sin(x)\,dx$

$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)} dx = \int \frac{-du}{u} = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos(x)| + C$

$\int \frac{\overbrace{x\,dx}^{du}}{x^2+1} = \int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C + \frac{1}{2}\ln(1+x^2) + C$

$u = x^2+1 \quad du = 2x\,dx$

Substitution - definite integrals

$$\int_0^{\pi/4} \tan(x)\cdot dx = \int_0^{\pi/4} \frac{\sin(x)}{\cos(x)}\, dx = \int_1^{\sqrt{2}/2} \frac{-du}{u} = \int_{\sqrt{2}/2}^{1} \frac{du}{u} = \Big[\ln(u)\Big]_{\frac{\sqrt{2}}{2}}^{1} = \ln(1) - \ln\left(\frac{\sqrt{2}}{2}\right)$$

$$= 0 + \ln(\sqrt{2}) = \frac{1}{2}\ln(2)$$

$u = \cos(x)$

$du = -\sin(x)dx$

$u(0) = \cos(0) = 1 \longrightarrow$ lower integration limit

$u\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \longrightarrow$ upper integration limit

In general :
$$\int_a^b f'(g(x))\cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f'(u)\, du$$

$u = g(x)$

$du = g'(x)dx$

Integration by parts (inverse product rule)

$$\int \frac{d}{dx}(u \cdot v)\,dx = \int u' \cdot v\,dx + \int v' \cdot u\,dx \quad \text{product rule}$$

$$\downarrow$$

$$u \cdot M = \int M \cdot du + \int u \cdot dM \Rightarrow \int u \cdot dM = u \cdot M - \int M \cdot du$$

$$\int xe^x\,dx = xe^x - \int e^x \cdot dx = xe^x - e^x + C$$

$u = x \longrightarrow du = dx$

$dM = e^x dx \longrightarrow M = e^x$

$$\int \ln(x)\,dx$$

$u = \ln(x) \longrightarrow du = \frac{dx}{x}$

$dM = dx \longrightarrow M = x$

DIY

$$\int x\sin(x)\,dx = -x \cdot \cos(x) - \int -\cos(x) \cdot dx = -x \cdot \cos(x) + \sin(x) + C$$

$u = x \longrightarrow du = dx$

$dM = \sin(x)dx \longrightarrow M = -\cos(x)$

Rational functions - partial fraction decomposition

Idea: we write a rational function $\frac{P(x)}{Q(x)}$ as a sum of a polynomial and "simple" fractions $\frac{A_k}{x - x_k}$ that we can easily integrate.

$\longrightarrow$ poles

$$\frac{P(x)}{Q(x)} = P_1(x) + \frac{A_1}{x - x_1} + \dots + \frac{A_n}{x - x_n}$$

How? 1) factorize $Q(x) = (x - x_1)(x - x_2)$

2) we assume that $P(x)$ has a lower degree than $Q(x)$

↳ otherwise, you can find $P_1(x)$ by long division of polynomes

$\longrightarrow$ if $Q(x) = (x - x_1)(x - x_2)$ is of degree 2,

$P(x)$ has maximally degree 1

$\longrightarrow$ $P(x) = ax + b$ (with $a = 0$ possible)

$$\Rightarrow \frac{P(x)}{Q(x)} = \frac{ax+b}{(x-x_1)(x-x_2)} = \frac{A_1}{x-x_1} + \frac{A_2}{x-x_2}$$

$$\Rightarrow \frac{A_1(x-x_2) + A_2(x-x_1)}{(x-x_1)(x-x_2)} = \frac{ax+b}{(x-x_1)(x-x_2)} \Rightarrow \begin{cases} (A_1 + A_2)\cancel{x} = a\cancel{x} \\ A_1 x_2 + A_2 x_1 = -b \end{cases}$$

2 equations
2 unknowns
$\longrightarrow$ you can solve this!

Example: $\int \frac{dx}{x^2-4} = \int \left(\frac{1}{4} \frac{1}{x-2} - \frac{1}{4} \frac{1}{x+2} \right) dx = \frac{1}{4} \int \frac{dx}{x-2} - \frac{1}{4} \int \frac{dx}{x+2}$

$$= \frac{1}{4} \ln|x-2| - \frac{1}{4} \ln|x+2| + C$$

$$= \frac{1}{4} \ln\left|\frac{x-2}{x+2}\right| + C$$

$$\frac{1}{x^2-4} = \frac{A_1}{x-2} + \frac{A_2}{x+2}$$

$$\Rightarrow 1 = A_1(x+2) + A_2(x-2)$$

$x = 2 \rightarrow 1 = A_1(2+2) + \cancel{A_2(2-2)}$

$$1 = 4A_1 \Rightarrow A_1 = \frac{1}{4}$$

$x = -2 \rightarrow 1 = A_1\cancel{(-2+2)} + A_2(-2-2)$

$$1 = -4 \cdot A_2 \Rightarrow A_2 = -\frac{1}{4}$$

the procedure is similar for higher degree polynomials

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-x_1} + \cdots + \frac{A_n}{x-x_n}, \quad A_k = \lim_{x \to x_k} \frac{P(x)}{Q(x)} \cdot (x-x_k)$$

What if $Q(x) = (x-x_0)^2$?

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-x_0)} + \frac{A_2}{(x-x_0)^2}$$

$$\frac{x+3}{(x-2)^2} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} \Rightarrow x+3 = A_1(x-2) + A_2$$

$$\Rightarrow x = A_1 x \Rightarrow A_1 = 1$$

$$3 = -2A_1 + A_2 \Rightarrow A_2 = 5$$

$$\int \frac{x+3}{(x-2)^2}\,dx = \int \left(\frac{1}{x-2} + \frac{5}{(x-2)^2} \right) dx = \int \frac{dx}{x-2} + 5\int \frac{dx}{(x-2)^2}$$

$$= \ln|x-2| - 5\frac{1}{x-2} + C$$

What if $Q(x) = (x-x_0)(ax^2+bx+c)$?

$\hookrightarrow$ no real roots

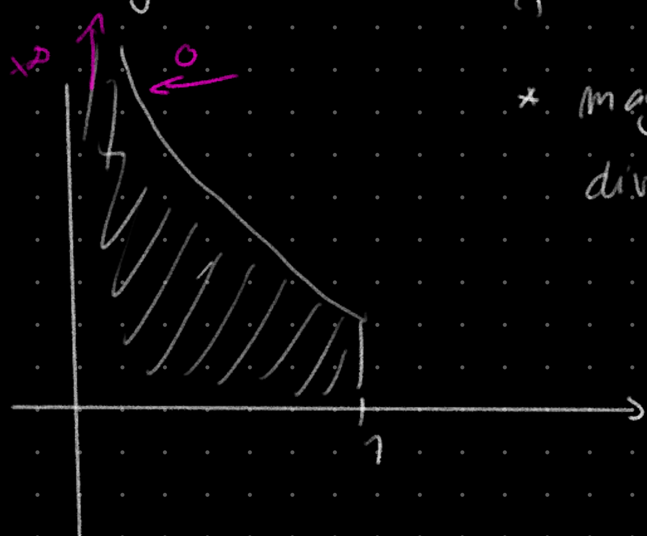
$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-x_0} + \frac{Bx+C}{ax^2+bx+c}$$

Improper integrals

- Type I: integrating next to a vertical asymptote
- $$\int_0^1 \frac{dx}{x} = \lim_{a \to 0^+} \int_a^1 \frac{dx}{x} = \lim_{a \to 0^+} \left(\cancel{\ln(1)} - \ln(a)\right) = +\infty$$

↳ diverges to ∞

↳ always indeterminate forms!



* may converge : Area is finite

diverge : Area is $\pm\infty$

↳ if $\int_a^b f(x)\,dx = +\infty$, then, if $g(x) \geq f(x)$, then $\int_a^b g(x)\,dx = +\infty$

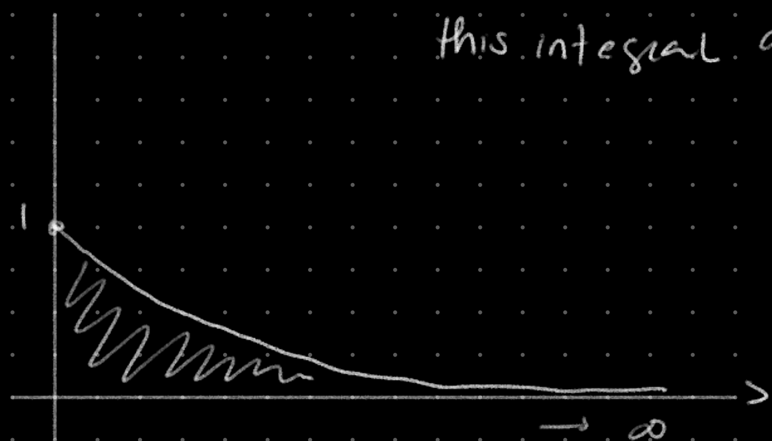
Improper integrals

- Type II: integrating to infinity

$$\int_0^{+\infty} e^{-x}\,dx = \lim_{R\to\infty} \int_0^R e^{-x}\,dx = \lim_{R\to\infty} \left[-e^{-x}\right]_0^R = \lim_{R\to\infty} \left(-e^{-R} - (-e^{0})\right)$$

$$= 0 + 1$$

this integral converges!



- Improper integrals are an INDETERMINATE FORM. They may
  - converge: the area is finite
  - diverge to $\pm\infty$: the area grows arbitrarily large
  - diverge / not exist: it is not possible to calculate the area