1.1.2 m = anne is a mathematician
r = anne won the race

(a) $m \wedge r$

(b) $\neg m$

(c) $m \vee r$

(d) $m \wedge \neg r$

(e) $\neg m \wedge \neg r$ (you could also say $\neg(m \vee r)$ for example).

1.1.3 d = too difficult
g = good presentation.
e = enjoyed.

(a) $d \wedge g$
(b) $e \wedge \neg g$
(c) $(d \vee g) \wedge \neg(d \wedge g)$
(d) $\neg e \wedge \cancel{\neg d}\, d$
(e) $\cancel{d \wedge g \wedge \neg e}$ $d \wedge \neg g \wedge \neg e$ (could do $(d \wedge \neg g) \wedge \neg e$
or: $d \wedge (\neg g \wedge \neg e)$ etc.
but brackets don't matter)

1.1.4
(a) It's always FALSE. You can see this in different ways eg. truth table

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

or you can say: is it ever true? To be true $p \wedge q$ needs to be true, so p and q both need to be true, but then $\neg(p \vee q)$ is false, so the whole expression is false. So it can't ever be true.

1.1.5.

(a) It's always true. More formally the expression is $(p \vee q) \vee \neg(p \wedge q)$

We can show this with a truth table:

p	q	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \vee \neg(p \wedge q)$
T	T	T	T	F	T
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	T

Or you could say: let's make this false. Then $(p \vee q)$ has to be false, so p and q are both false. But then $p \wedge q$ is false, so $\neg(p \wedge q)$ is true. So the expression is true i.e. it can't be made false!

1.2.2 (a)

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

(b)

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

(c)

p	q	$p \vee q$	$p \wedge \neg(p \vee q)$
F	F	F	F
F	T	T	F
T	F	T	F
T	T	T	F

1.3.1

(a)

p	q	$q \vee p$	$p \Leftrightarrow (q \vee p)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

(b)

p	q	$q \Rightarrow p$	$p \Leftrightarrow (q \Rightarrow p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	F

c)

p	q	r	$q \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

(we could have seen beforehand there was only one way of making this false:- make p true and $(q \Rightarrow r)$ false, so make p true and make q true and r false.)

p	q	r	$q \Rightarrow r$	$\neg(q \Rightarrow r)$	$p \wedge \neg(q \Rightarrow r)$
T	T	T	T	F	F
T	T	F	F	T	T
T	F	T	T	F	F
T	F	F	T	F	F
F	T	T	T	F	F
F	T	F	F	T	F
F	F	T	T	F	F
F	F	F	T	F	F

(again, we could have seen there was only one way to make it true:- p true and $\neg(q \Rightarrow r)$ true, so p true and $(q \Rightarrow r)$ false, so p true and q true and r false.)

1.3.2

p	q	$p \Rightarrow q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T.

1.3.3. $\underbrace{(p \wedge q)}_{\text{both true}} \vee \underbrace{(\neg p \wedge \neg q)}_{\text{both false}}$.

1.4.2. (a) There exists a fish that does not swim. → original statement.

$\forall$ fish f : f can swim.

$\neg(\forall$ fish f : f can swim$)$

$= \exists$ fish f : f cannot swim

(b) No newspaper exaggerates the truth

original statement:

$\exists$ newspaper n : n exaggerates the truth.

Now, $\neg(\exists$ newspaper n : n exaggerates the truth$)$

$= \forall$ newspapers n : n DOES NOT exaggerate the truth. = no newspaper exaggerates the truth.

(c) original statement:

$\neg(\exists$ student s : s cannot ~~use~~ use a computer$)$.

So negation is just: $\exists$ student s: s cannot use a computer

= Some student in mathematics is unable to use a computer

(d) $\neg(\exists$ president p : p is economical with the truth$)$

So negation is just $\exists$ president p : p is economical with the truth.

$\approx$ Some president is economical with the truth.

1.4.3. (a) For every number $x^{>0}$ there exists a number $y>0$ such that

$$x^2 = y.$$

(I think this is true: take $y = x^2$).

eg. $x=2, y=4$.

(b) For every x and for every y there exists a z such that

$$x + z = y.$$

(Again I think this is true:- just take $z = y - x$.)

(c) $\neg(\exists x)(\forall y)(y > x)$

$= (\forall x)\neg(\forall y)(y > x)$

$= (\forall x)(\exists y)\neg(y > x)$

$= (\forall x)(\exists y)(y \leq x)$.

For every x there exists a y such that $y \leq x$.

(true for the real numbers).

1.4.4 (a) Some good students do not study hard. (You tell me it this is true! ☺)

i.e. "at least one"

(b) Some males ~~do~~ do give ~~bot~~ birth to their young.

i.e. "at least one"

true: Seahorse

(c) At least one of Shakespeare's plays is not a comedy.

(d) There is an integer x such that for all integers y, $x^2 \neq y$.

FALSE:- because if x is an integer then x^2 is also an integer, so taking $y = x^2$ makes this bit false

1.4.4 (e) ~~z~~ Negation:- $(\forall \text{ integer } x)(\exists \text{ integer } y)(x^2 \neq y)$.

<u>true</u>:- we can always take y equal to x^2+1

(ie. $x^2 \neq x^2+1$).

1.5.1. Suppose $\sqrt{5}$ is rational. So $\sqrt{5} = \frac{a}{b}$ where $a \in \cancel{\mathbb{Q}}\ \mathbb{Z}$, $b \in \cancel{\mathbb{Q}}\ \mathbb{Z}$, $b \neq 0$.

Assume that common factors of a and b have been cancelled out.

Now, $5 = \frac{a^2}{b^2}$ so $5b^2 = a^2$.

So a^2 is divisible by 5.

I claim that this means a is also divisible by 5.

if not $a = 5k+1$

or $a = 5k+2$

or $a = 5k+3$

or $a = 5k+4$ for some integer k.

but $(5k+1)^2 = 25k^2 + 10k + 1 = a^2$ and this is <u>not</u> divisible by 5 because of the +1.

$(5k+2)^2 = 25k^2 + 20k + 4 = a^2$ " " because of the +4.

$(5k+3)^2 = 25k^2 + 30k + 9$ .. " " because of the 9

$(5k+4)^2 = 25k^2 + 40k + 16$ " " because of the 16.

So a <u>is</u> also divisible by 5.

So we can write $a = 5k$ for some k.

So $5b^2 = (5k)^2$

$5b^2 = 25k^2$.

$b^2 = 5k^2$. So b^2 is divisible by 5

so b is divisible by 5 (using argument)

So a and b have a common factor of 5, CONTRADICTION!

□

1.5.2 (a). $p=3.$ $q=3.$

$3^2+3^2=18$, 18 is not prime.

(b). $a=2$, $b=-10$.

$a^2=4$. $b^2=100$.

(c). $x=-1$ because $(-1)(-1)(-1)(-1)=1$.

1.5.3.

If x is an odd integer then x can be written $(2a+1)$ for some integer a.

If y is an odd integer then y can be written $(2b+1)$ for some integer b.

So $x+y=(2a+1)+(2b+1)=2(a+b)+2$

and this value is divisible by 2 $\Rightarrow$ so $x+y$ is even.

1.5.4. If x is odd then . . . . . . $(2a+1)$

If ~~x~~ y is odd then . . . . $(2b+1)$

$xy=(2a+1)(2b+1)=4ab+2a+2b+1$ this means it is not divisible by 2, so odd.

1.5.5. Assume not $(-2<x<2)$.

So $x\geq 2$ or $x\leq -2$.

If $x\geq 2$ then $x^2-4\geq 0$ ie. $x^2-4<0$ is FALSE.

If $x\leq -2$ then $x^2-4\geq 0$ ie. $x^2-4<0$ is FALSE.

So we are done.

1.5.6. (a) If I don't get a job then I don't study math.

(b) If ~~x≥2~~ $x^2<0$ then x is not an integer greater than zero

(c) If the moon does not go round the earth then the earth does not go round the sun.

(d) if the product of all the primes plus 1 is prime then there is an infinite number of primes.

1.6.1 Can we make 8p with 5p and 3p stamps?

Yes: $8 = 5 + 3$. This is the base case $P(8)$.

Assume $P(n)$ is true for some $n \geq 8$.

Consider $P(n+1)$.

(i) If you can make n with ~~the~~ a 5p coin, then replace it with $2 \times 3_p$ coins.

(ii) if you can make n with three 3p coins, replace it with $2 \times 5p$ coins.

So if n can be made with either (i) or (ii) we are done.

if neither (i) nor (ii) holds then ~~n~~ all ways of making n involve no 5p coins and at most 2 ~~3~~ 3p coins. So $n \leq 6$.

But $n \geq 8$ by assumption. So the two cases are sufficient, and we are done.

1.6.2. $\forall n \geq 4 : 2^n \geq n^2$. (Notice that the proposition $P(n+1)$ in this case states that $2^{n+1} \geq (n+1)^2$, where $(n+1)^2 = n^2 + 2n + 1$).

Proof:

Basic step: The proposition $P(4)$ states that $2^4 \geq 4^2$, which is true.

Inductive step: Let $n \geq 4$ and suppose that $P(n)$ is true (so $2^n \geq n^2$). Then

$$\begin{aligned}
2^{n+1} &= 2 \cdot 2^n \\
&\geq 2 \cdot n^2 \text{ (by the induction hypothesis)} \\
&= n^2 + n^2 \\
&= n^2 + n \cdot n \\
&\geq n^2 + 4n \text{ (since } n \geq 4) \\
&= n^2 + 2n + 2n \\
&\geq n^2 + 2n + 2 \cdot 4 \text{ (since } n \geq 4) \\
&> n^2 + 2n + 1 \\
&= (n+1)^2
\end{aligned}$$

1.6.3 (a) Check base case $P(1)$ holds:

$$1^3 = \left(\left(\tfrac{1}{2}\right)(1)(2)\right)^2 = 1 \checkmark$$

Assume $P(n)$ holds.

$$P(n+1) = \underbrace{1^3 + 2^3 + \cdots + n^3}_{\text{by induction we can use the other expression}} + (n+1)^3.$$

$$= \left\{\left(\tfrac{1}{2}\right)n(n+1)\right\}^2 + (n+1)^3.$$

$$= \left(\tfrac{1}{2}\right)^2 n^2(n+1)^2 + (n+1)^3.$$

$$= (n+1)^2\left[\tfrac{1}{4}n^2 + (n+1)\right]$$

$$= (n+1)^2\left[\tfrac{1}{4}n^2 + n + 1\right]$$

$$= (n+1)^2\left[\tfrac{1}{4}(n^2+4n+4)\right] = (n+1)^2\left[\tfrac{1}{4}(n+2)^2\right]$$

$$= (n+1)^2\left[\left(\tfrac{1}{2}\right)^2(n+2)^2\right]$$

$$= \left\{\left(\tfrac{1}{2}\right)(n+1)(n+2)\right\}^2 \quad \square.$$

(b)

$\forall n \in \mathbb{N} : 6^n - 5n + 4$ is divisible by 5. (Notice that the proposition $P(n+1)$ in this case states that $6^{n+1} - 5(n+1) + 4$ is divisible by 5).

Proof:

Basic step: $P(1)$ states that $6^1 - 5 \cdot 1 + 4$ is divisible by 5, which is true.

Inductive step: Let $n \geq 1$ and suppose that $P(n)$ is true (so $6^n - 5n + 4$ is divisible by 5). Then

$$
\begin{aligned}
6^{n+1} - 5(n+1) + 4 &= 6 \cdot 6^n - 5(n+1) + 4 \\
&= (5+1) \cdot 6^n - 5(n+1) + 4 \\
&= 5 \cdot 6^n - 5 + 6^n - 5n + 4.
\end{aligned}
$$

Now $6^n - 5n + 4$ is divisible by 5 by the induction hypothesis and $5 \cdot 6^n - 5$ is also divisible by 5, so $6^{n+1} - 5(n+1) + 4$ is divisible by 5 and $P(n+1)$ is true.

1.6.3 (c). Check base case $P(1)$ i.e. $n = 1$.

left hand side is $1 + a^1 = 1 + a$.

right hand side is $\dfrac{a^2 - 1}{a - 1} = \dfrac{(a-1)(a+1)}{a-1}$

$= a + 1$. ✓

Assume $P(n)$ holds.

$$\underbrace{1 + a + a^2 + \cdots a^n}_{} + a^{n+1} \overset{\text{by induction}}{=} \left(\frac{a^{n+1} - 1}{a - 1}\right) + a^{n+1}$$

$$= \left(\frac{a^{n+1} - 1}{a - 1}\right) + \frac{a^{n+1}(a-1)}{(a-1)}$$

$$= \frac{a^{n+1} - 1 + a^{n+2} - a^{n+1}}{a - 1}.$$

$$= \frac{a^{n+2} - 1}{a - 1}$$

$\square$.

1.6.3 (d). Check $P(1)$:

$1^2 = 1^3$ ✓.

Assume $P(n)$ holds.

~~"$P(n+1)$" too~~

$$(1+2+\cdots+n+(n+1))^2 = (A+(n+1))^2.$$

where $A = 1+2+\cdots+n$.

$$(A+(n+1))^2 = A^2 + 2A(n+1) + (n+1)^2.$$

By induction we can replace A^2 by $1^3+2^3+\cdots+n^3$

So $= (1^3+2^3+\cdots+n^3) + 2(1+2+\cdots+n)(n+1) + (n+1)^2$

$= 1^3+2^3+\cdots+n^3 + 2\frac{n}{2}(n+1)(n+1) + (n+1)^2$

$= 1^3+2^3+\cdots+n^3 + n(n+1)^2 + (n+1)^2$

$= 1^3+2^3+\cdots+n^3 + (n+1)^3$ $\square$.

because sum of first n natural numbers is $\frac{n}{2}(n+1)$