

Exercises

1	2	3	4	5	6	7	8	9	10
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Surname, First name

Calculus Test

Example exam

a	<input checked="" type="radio"/>	c	d	e	f	→ b
a	b	<input checked="" type="radio"/>	d	e	f	→ c
<input checked="" type="radio"/>	b	c	<input checked="" type="radio"/>	e	f	→ a

1: 18
2: 16
3: 10
4: 10

5: 6

6: 10

7: 10

8: 10

9: 10

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
7	7	7	7	7	7	7
8	8	8	8	8	8	8
9	9	9	9	9	9	9
0	0	0	0	0	0	0

Fill in your answer(s) to the multiple-choice questions as shown above (circles = one correct answer).

Program:

Course code: KEN/BCS 14460

Examiners: Otti D'Huys, Gijs Schoenmakers

Date/time:

Format: Written, closed book

Allowed aids: A formula sheet is attached to the exam.

Instructions to students:

- The exam consists of 9 questions on 12 pages.
- Fill in your name and student ID number on the cover page and tick the corresponding numerals of your student number in the table (top right cover page).
- Answer every question in the reserved space below the question. **Do not write outside the reserved space or on the back of pages, this will not be scanned and will NOT be graded!** As a last resort if you run out of space, use the extra answer space at the end of the exam.
- *In no circumstance write on or near the QR code at the bottom of the page!*
- Ensure that you properly motivate your answers.
- Only use black or dark blue pens, and write in a readable way. Do not use pencils.
- Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in the space reserved for the answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- You have to return all pages of the exam. You are not allowed to take any sheets, even blank, home.
- Good luck!

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Question 1

- 6p 1a For $f(x)$ and $g(x)$ differentiable functions, let $f(1) = 1$, $f(-2) = -2$, $f'(1) = -3$, $f'(-2) = 3$, $g(1) = -2$ and $g'(1) = 1$.

Find $\lim_{x \rightarrow 1} \frac{f(g(2-x)) - g(f(x^2))}{x-1}$

- (a) $-\infty$
 (b) -1
 (c) 0
 (d) 3
 (e) -6
 (f) $+\infty$
 (g) -9
 (h) The limit does not exist
 (i) None of the above

$$\lim_{x \rightarrow 1} \frac{f(g(2-x)) - g(f(x^2))}{x-1} \quad \frac{0}{0}$$

$$= \lim_{x \rightarrow 1} \frac{f'(g(2-x)) \cdot g'(2-x) - g'(f(x^2)) \cdot f'(x^2)}{1} \quad \text{H}$$

$$= f'(-2) - g'(1)(-3)$$

$$= -3 - (-3) = 0$$

- 6p 1b Calculate $\lim_{x \rightarrow 0^-} \frac{x}{\sqrt{x^2-x}}$

- (a) 4
 (b) $-\frac{1}{2}$
 (c) $\frac{1}{2}$
 (d) -1
 (e) $-\infty$
 (f) 0
 (g) $+\infty$
 (h) 1
 (i) The limit does not exist
 (j) None of the above

$$\lim_{x \rightarrow 0^-} \frac{x}{\sqrt{x^2-x}} = 0$$

$$\frac{d}{dx} \sqrt{x^2-x} = \frac{2x-1}{2\sqrt{x^2-x}}$$

- 6p 1c Calculate $\int_0^1 x \ln(x) dx$

- (a) $-\frac{1}{2}$
 (b) This integral diverges to $-\infty$
 (c) 0
 (d) This integral diverges to $+\infty$
 (e) None of the above

$$\int_0^1 x \ln(x) dx =$$

$$f(x) = \ln x \quad f'(x) = \frac{1}{x}$$

$$g(x) = x \quad g'(x) = 1$$

$$= \frac{x^2}{2} \ln x - \int \frac{x}{2} dx$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4}$$

$$\left(\frac{1}{2} \ln 1 - \frac{1}{4} \right) - (0 - 0)$$

$$= -\frac{1}{4}$$

Question 2

- 16p 2 Find all the asymptotes of the function $f(x) = \frac{1}{\sqrt{x^2+3x}+x}$. For the horizontal and oblique asymptotes (if any), specify whether they are one-sided or two-sided. For the vertical asymptotes (if any), specify how the function approaches the asymptotes (i.e. whether the function goes to $+\infty$ or $-\infty$ on the left or right side of the asymptotes.)

$$\begin{aligned} \text{dom } f: \sqrt{x^2+3x} + x &\neq 0 \\ \sqrt{x^2+3x} &\neq (-)x \\ x^2+3x &\neq x^2 \\ x &\neq 0 \end{aligned}$$

Vertical Asymptotes

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{x^2+3x}+x} &= +\infty \\ \lim_{x \rightarrow 0^-} \frac{1}{-x\sqrt{1+\frac{3}{x}}+x} &= -\infty \end{aligned} \left. \vphantom{\lim_{x \rightarrow 0^+}} \right\} \begin{array}{l} \text{V. A. at } x=0 \\ x=0 \end{array}$$

Horizontal Asymptotes

$$\lim_{x \rightarrow +\infty} // = 0 \leftarrow \text{one-sided}$$

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{1}{-x\sqrt{1+\frac{3}{x}}+x} &= \lim_{x \rightarrow -\infty} \frac{1}{x(-\sqrt{1+\frac{3}{x}}+1)} \\ &= 0 \end{aligned}$$

Oblique Asymptotes

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x}$$

$$\cancel{0 \cdot \infty = 0}$$

$$\lim_{x \rightarrow -\infty} x \ln\left(1+\frac{1}{x}\right)$$

$$\lim_{x \rightarrow -\infty} x \ln \left(1 + \frac{1}{x} \right)$$

Question 3

- 10p 3 Does the function $f(x) = x \ln(x)$ have a global (absolute) minimum and maximum on $(0, \infty)$?
- If so, for which value(s) of x ?
 - If not, motivate why you can exclude the existence of a global extremum.

$$f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$$

$$f'(x) = 0 \quad \text{dom } f: x > 0$$

$$\ln(x) + 1 = 0$$

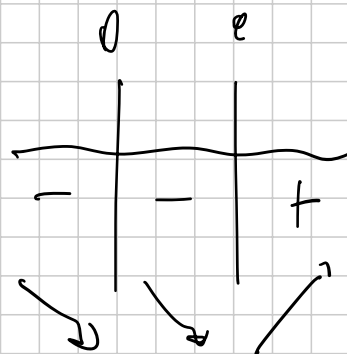
$$\ln(x) = -1$$

$$x = e^{-1}$$

$$f'(x) > 0$$

$$\ln(x) > -1$$

$$x > e^{-1}$$



$f(x)$ has a global minimum at $x = e^{-1}$,
but has no global maximum as it grows
to infinity

Question 4

10p 4 Evaluate the following integral: $\int_0^{\frac{\pi}{2}} (\sin(x))^3 dx$

$$\begin{aligned}
 u &= \cos x \\
 du &= -\sin x \, dx \\
 \begin{array}{c|c} x & u \\ \hline 0 & 1 \\ \pi/2 & 0 \end{array} & \quad \int_0^{\pi/2} \sin^3 x \, dx = \int_0^{\pi/2} (1 - \cos^2 x) \sin x \, dx \\
 &= \int_1^0 (1 - u^2) (-du) \\
 &= \left[u - \frac{u^3}{3} \right]_1^0 \\
 &= \left[\cancel{\cos x} - \frac{(\cancel{\cos x})^3}{3} \right]_1^0 \\
 &= 1 - \frac{1}{3} - (0) = \frac{2}{3}
 \end{aligned}$$

Question 5

Study series

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{e^n + n}{e^{2n} - n^2}$

1.5p 5a The series $\sum_{n=1}^{\infty} a_n$ diverges. We can conclude this by comparing to the series $\sum_{n=1}^{\infty} b_n$ with $b_n = \frac{1}{n}$ and $|a_n| \leq b_n$ for all $n \in \mathbb{N}$

(a) True (b) False

1.5p 5b The series $\sum_{n=1}^{\infty} a_n$ converges. We can conclude this by comparing to the series $\sum_{n=1}^{\infty} b_n$ with $b_n = e^{-n}$ and $a_n \leq b_n$ for all $n \in \mathbb{N}$

(a) True (b) False

1.5p 5c The series $\sum_{n=1}^{\infty} a_n$ converges. We can conclude this because $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = e^{-1}$

(a) True (b) False

1.5p 5d The series $\sum_{n=1}^{\infty} a_n$ converges. We can conclude this because $\lim_{n \rightarrow \infty} a_n = 0$

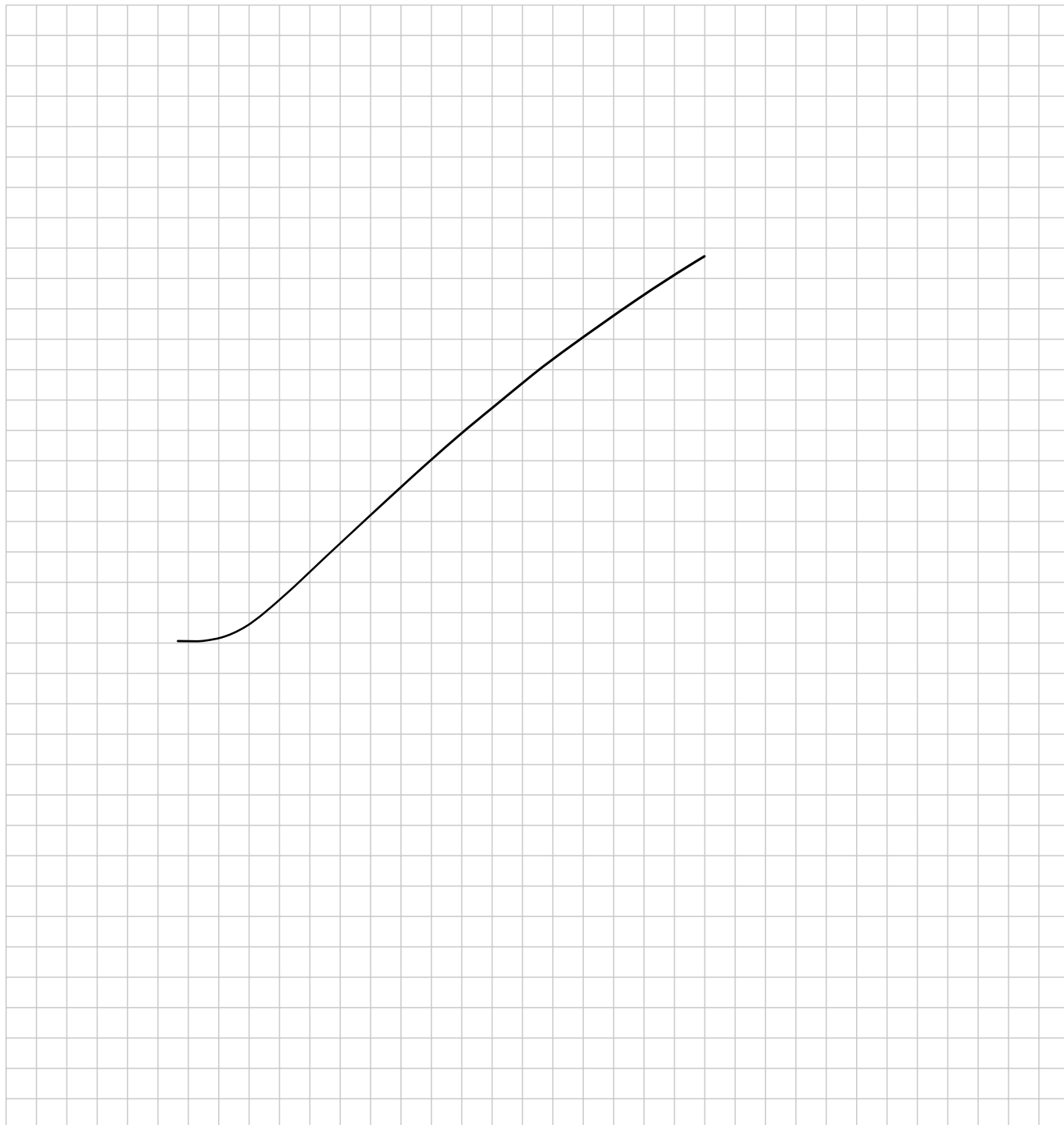
(a) True (b) False

Question 6

- 10p **6** Determine which values of $x \in \mathbb{R} \setminus \{0\}$, the given series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} n \left(\frac{2}{x} \right)^n$$

?



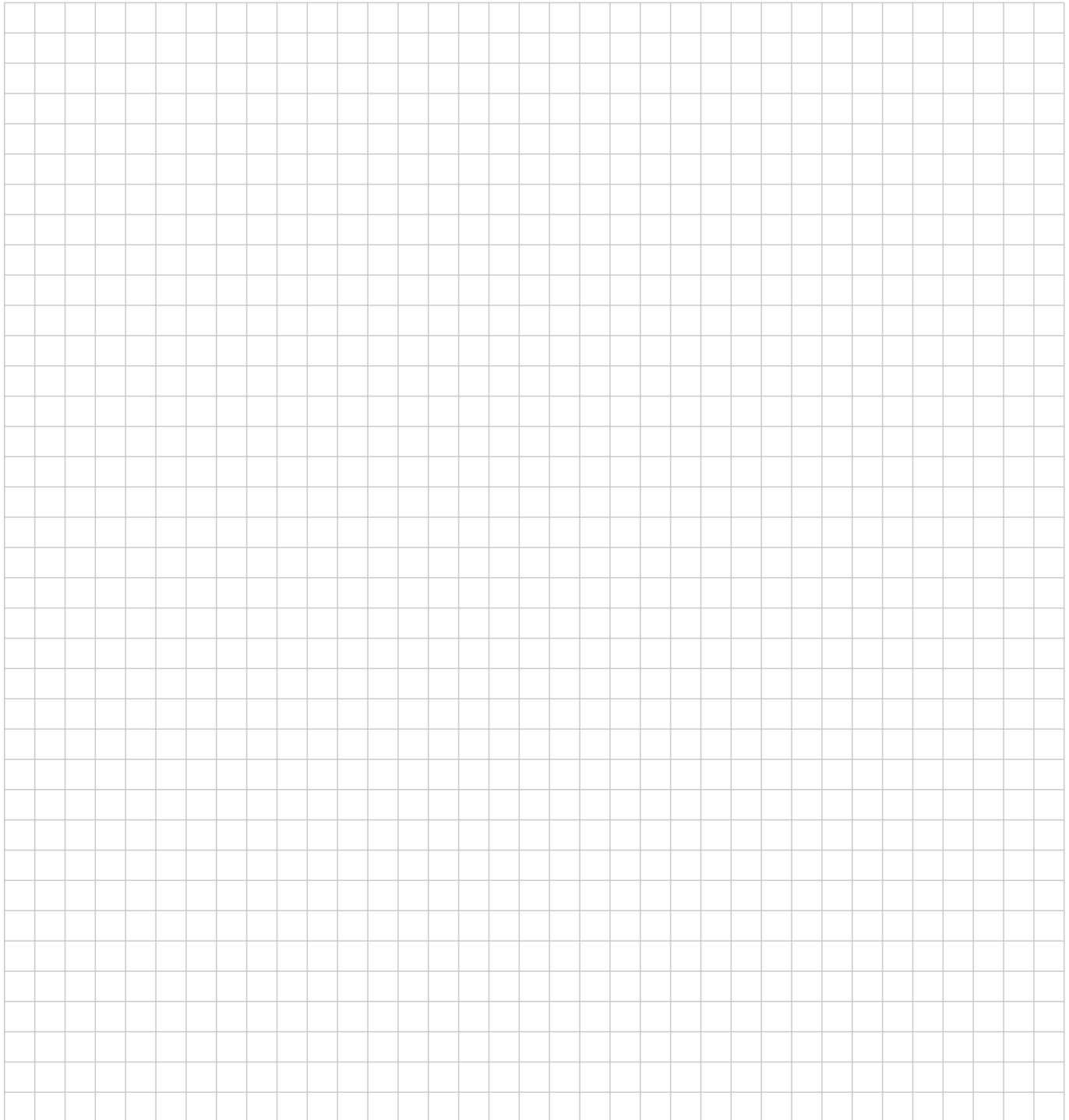
Differential equation

Question 7

10p 7 Find the solution $y = y(x)$ to the given initial value problem.

$$\begin{cases} y' + y = x \\ y(0) = 3 \end{cases}$$

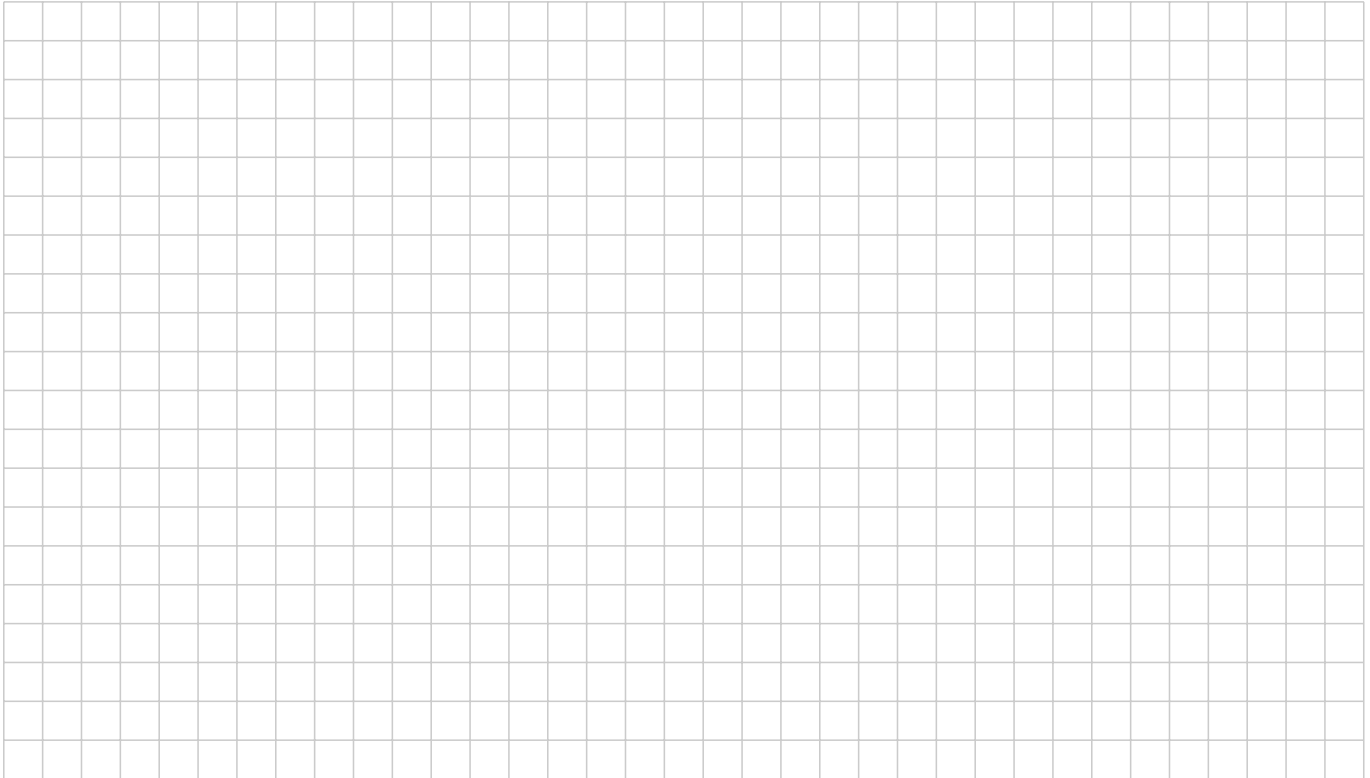
$$\int y' - \frac{1}{y} = 0$$



Multivariate

Question 8

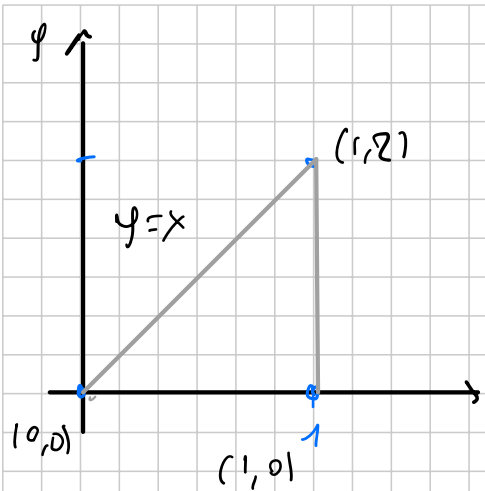
10p **8** For the function $f(x, y) = \sin(2x - y)$, give the equation of the tangent plane at $(0, 0, f(0, 0))$



Question 9

10p 9 Evaluate the double integral over a triangle T with vertices $(0,0)$, $(1,0)$, $(1,1)$:

$$\iint_T e^{2x+y} dA$$



$$\int e^{2x+y} dy$$

$$= e^{2x+y}$$

$$\begin{aligned} & \iint_T (e^{2x+y}) dA \\ &= \int_0^1 \int_0^x e^{2x+y} dy dx \\ &= \int_0^1 \left[e^{2x+y} \right]_0^x dx \\ &= \int_0^1 (e^{3x} - e^{2x}) dx \\ &= \left[\frac{1}{3} e^{3x} \right]_0^1 - \left[\frac{1}{2} e^{2x} \right]_0^1 \\ &= \frac{1}{3} e^3 - \frac{1}{3} - \left(\frac{1}{2} e^2 - \frac{1}{2} \right) \\ &= \frac{1}{3} e^3 + \frac{1}{2} e^2 + \frac{1}{6} \end{aligned}$$

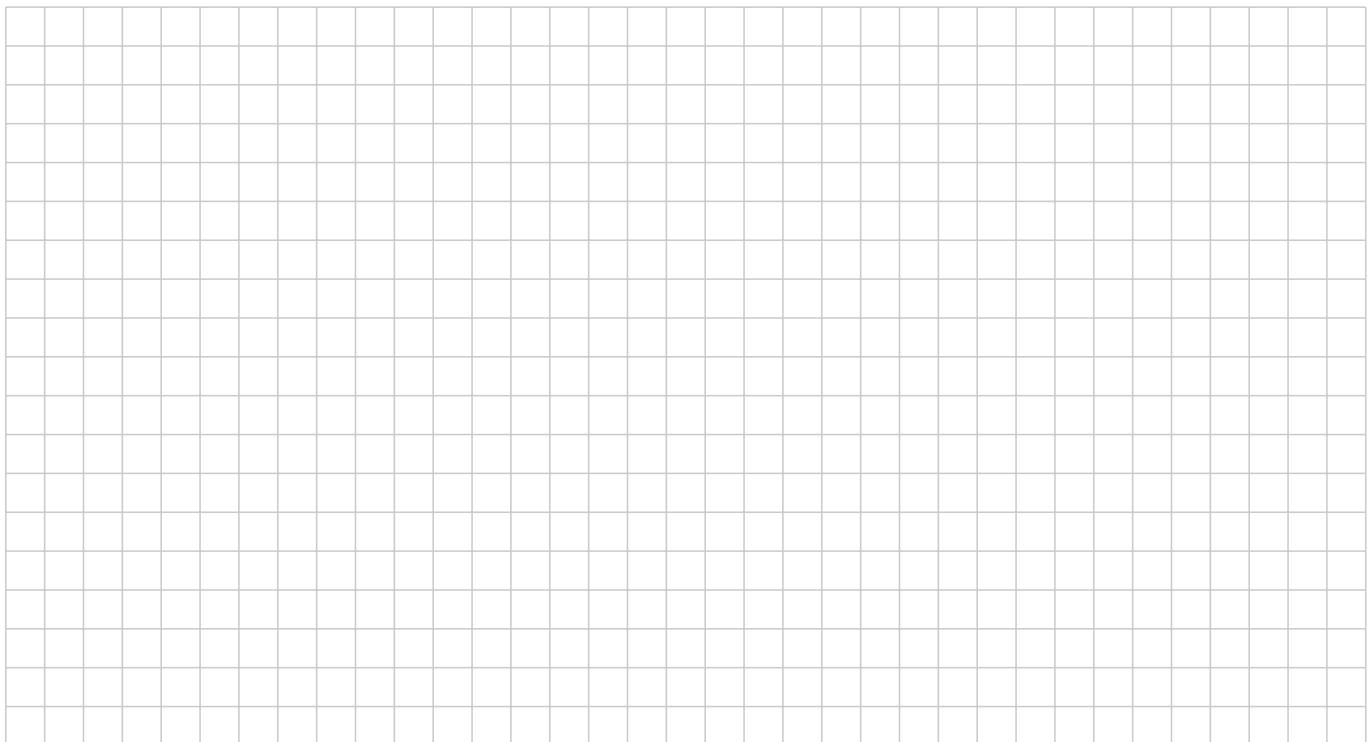
$$-\frac{1}{3} + \frac{1}{2} = \frac{3-2}{6} = \frac{1}{6} \quad \frac{1}{12}$$

Extra space

10a

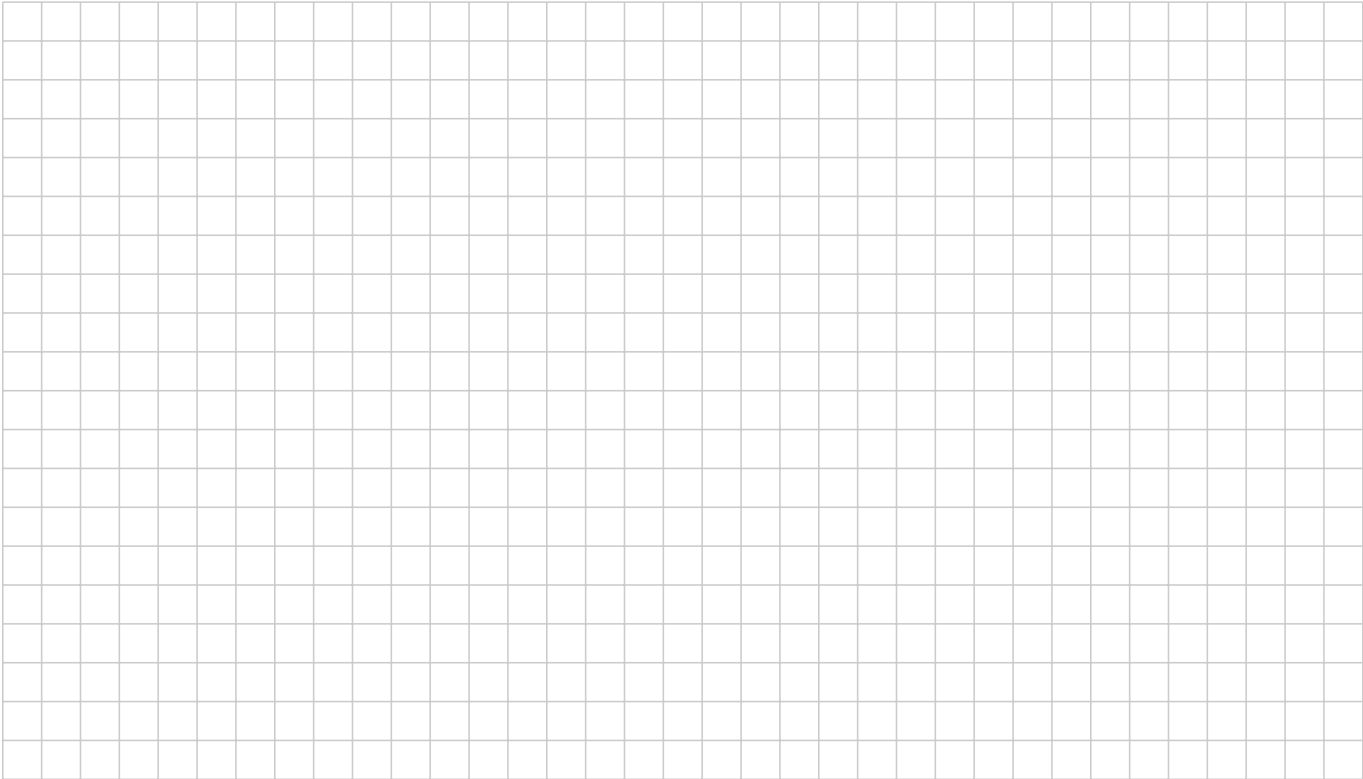
$$\begin{aligned}
 & \int_0^{\frac{\pi}{2}} (\sin(x))^3 dx = \int_0^{\frac{\pi}{2}} \sin^3(x) dx \\
 & = \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \sin^2(x) dx \quad u = \cos(x) \\
 & = \int_0^{\frac{\pi}{2}} [\sin^2(x) - \cos^2(x) \sin^2(x)] dx \quad \frac{du}{dx} = -\sin(x) \\
 & \quad du = -\sin(x) dx \\
 & \quad dx = \left[-\frac{1}{\sin(x)} \right] du \\
 & = \int_0^{\frac{\pi}{2}} \left[1 - u^2 \right] \sin^2(x) \left(-\frac{1}{\sin(x)} \right) du \\
 & = \int_0^{\frac{\pi}{2}} (-1 + u^2) du \Rightarrow \left[-u + \frac{u^3}{3} \right] \Big|_0^{\frac{\pi}{2}} =
 \end{aligned}$$

10b

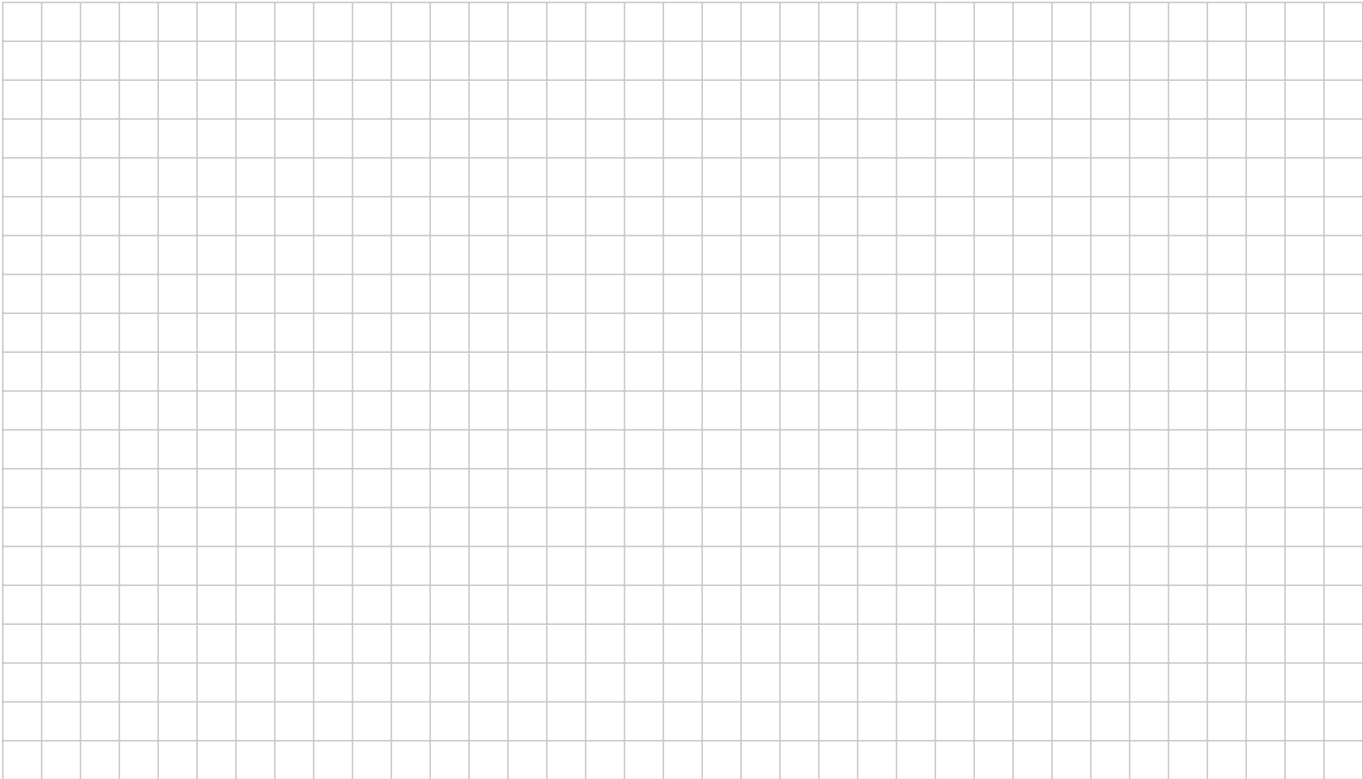




10c



10d



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