Calculus lecture 5

Recap

- **Continuity** (describes whether a function has gaps)
- **Limits** (to describe how a function behaves when it approaches the edges of it's domain)
- **Derivatives** (slope of the tangent line, describes the rate of change of a function)

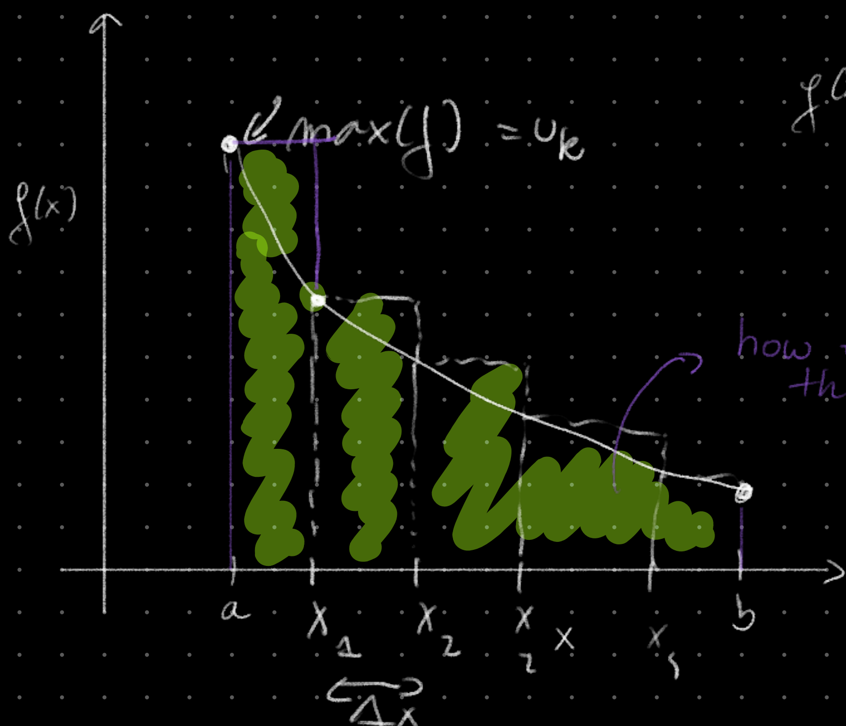
Today: Integration

- **Definite integrals**
  - Areas as Riemann sums
  - Properties of definite integrals
- **Anti-derivatives**
- **Fundamental theorem of Calculus**

Adams' Ch. 5.2-4, Ch. 2.10

The area below a graph

$f(x)$ is a continuous function on $[a,b]$



how to calculate this area A?

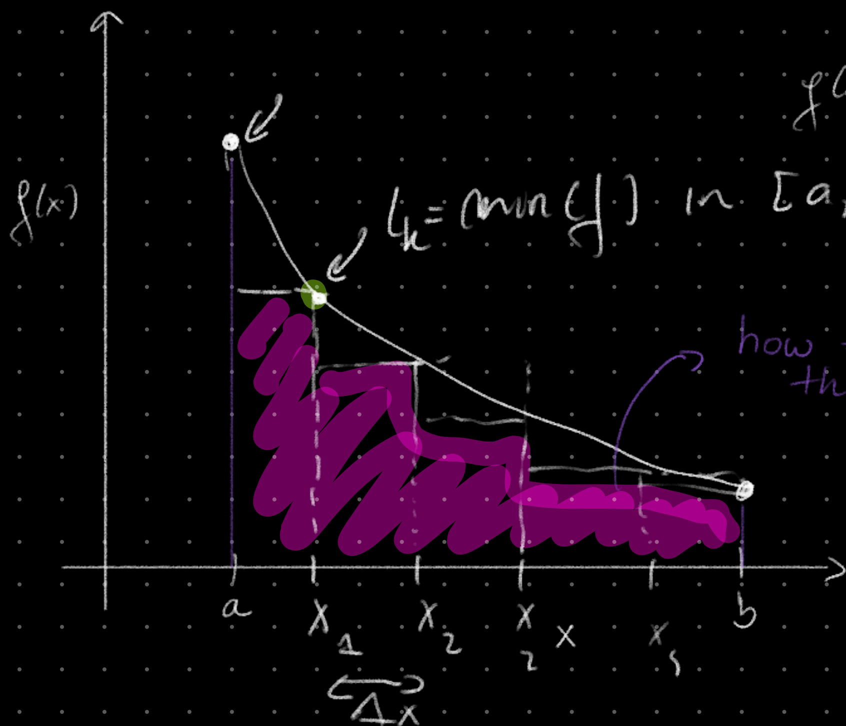
$a < x_1 < x_2 \; \dots \; < b \;\rightarrow$ "partition" of $[a,b]$

$$\Delta x_n = x_{n+1} - x_n$$

$$U = \sum_k f(u_k) \cdot \Delta x \; > \; A$$

$\rightarrow$ as $\Delta x \rightarrow 0$, $\; U(f,P) \rightarrow A$

$f(x)$ is a continuous function on $[a,b]$



$a < x_1 < x_2 \quad \dots \quad < b \longrightarrow$ "partition" of $[a,b]$
P

$\Delta x_n = x_{n+1} - x_n$

$$A \geqslant L(f,P) = \sum_k f(l_k)\,\Delta x_k$$

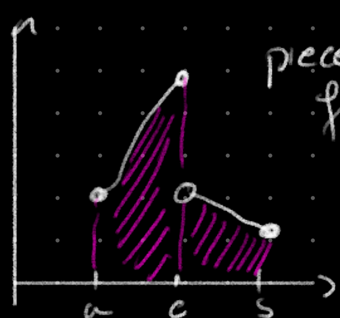
↳ as $\Delta x \to 0$, $L(f,P)$ converges to A

Definition of a definite integral :
A function f is integrable on $[a, b]$ if there is exactly one A, such that, for every partition P, $L(f, P) \leq A \leq U(f, P)$.
In that case, $A = \int_a^b f(x)dx$

- Definite integral: area between the graph and the x-axis
  (Note: a definite integral can be positive or negative!)

- For integrable functions, all Riemann sums converge (not only upper and lower sums

- Which functions are integrable?



piecewise continuous functions are integrable

- Terminology:

upper integration limit

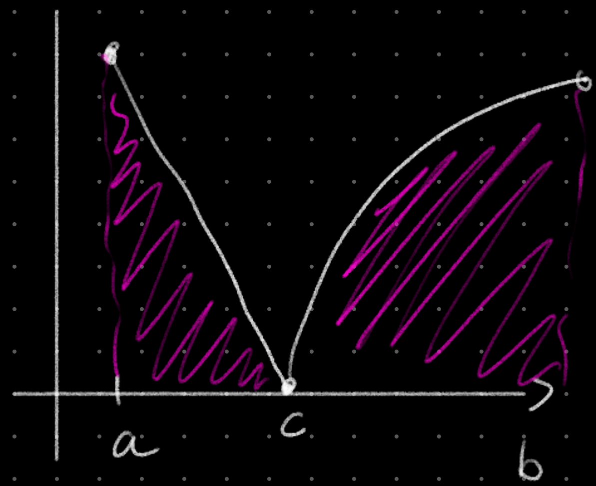
differential dx

$$\int_a^b f(x)\,dx$$

integrand

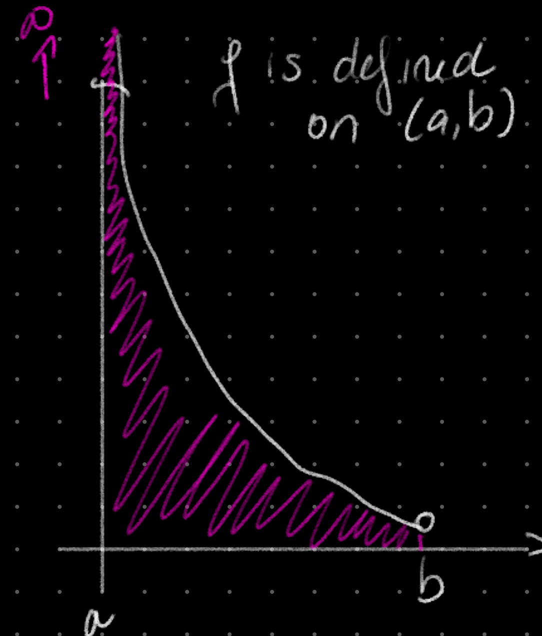
x = integration variable

lower limit



$f(x)$ is not differentiable at c

f is integrable on $[a,b]$



f is defined on (a,b)

improper integral

= INDETERMINATE FORM

may or may not exist!

$$\int_a^b f(x)\,dx = \lim_{R\to b} \int_a^R f(x)\,dx$$

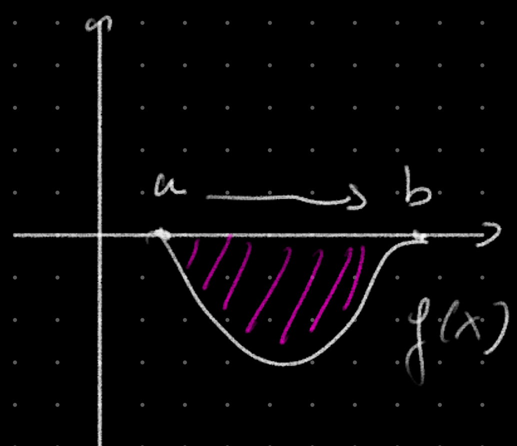
$f(x) = 0$ for $x \in \mathbb{Q}$

$f(x) = 1$ for $x \in \mathbb{R} \setminus \mathbb{Q}$

Properties of definite integrals

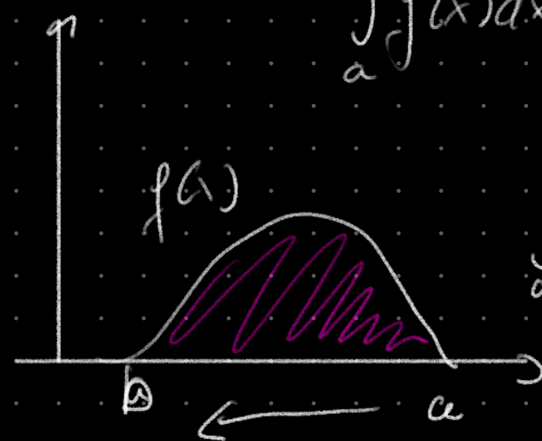
- $\int_a^b f(x)dx$ is a NUMBER $= \int_a^b f(t)dt$

↳ when is $\int_a^b f(x)\,dx$ negative?



$\int_a^b f(x)dx < 0$ if $f(x) < 0$ (and $a<b$)

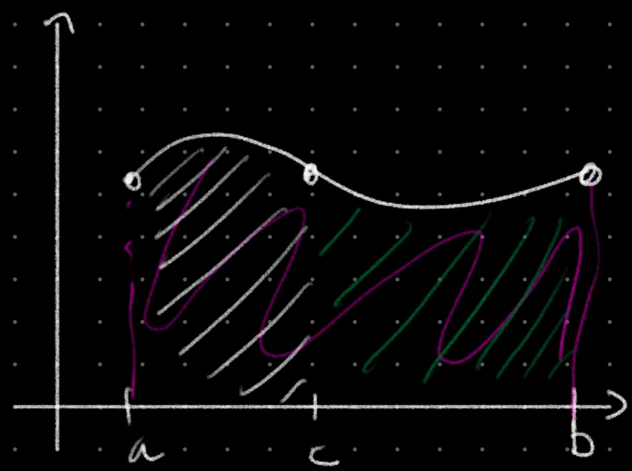
$\int_a^b f(x)dx = -\int_b^a f(x)dx$



$\int_a^b f(x)dx < 0$

$(dx < 0)$

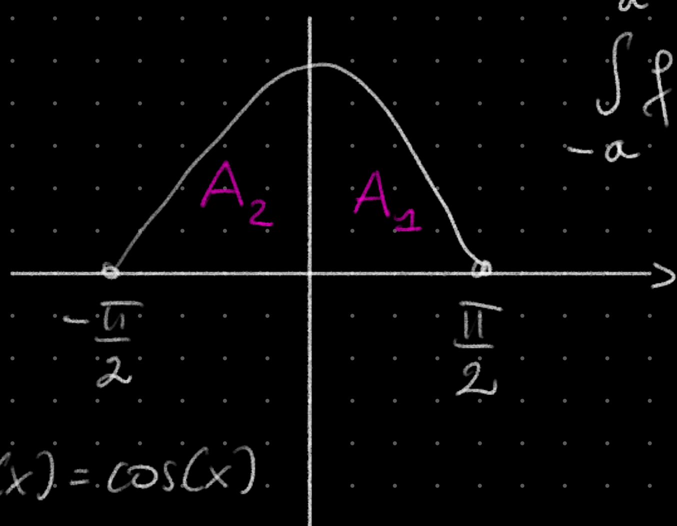
- we can break up the domain



$$\int_a^b f(x)\,dx = \int_c^b f(x)\,dx + \int_a^c f(x)\,dx$$

* this works well for $|x|$ or piecewise defined functions

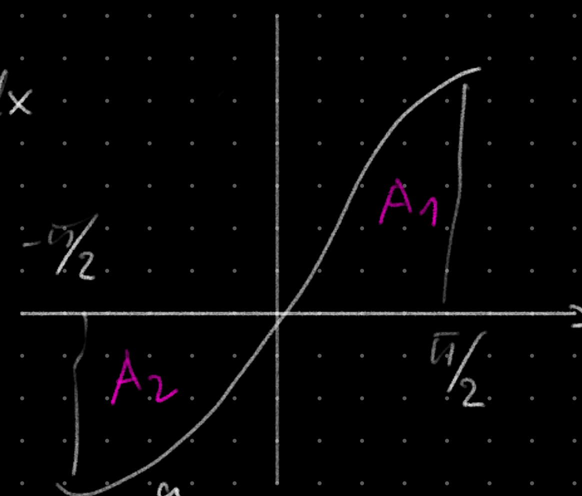
- Integrating even and odd functions



$f(x) = \cos(x)$

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

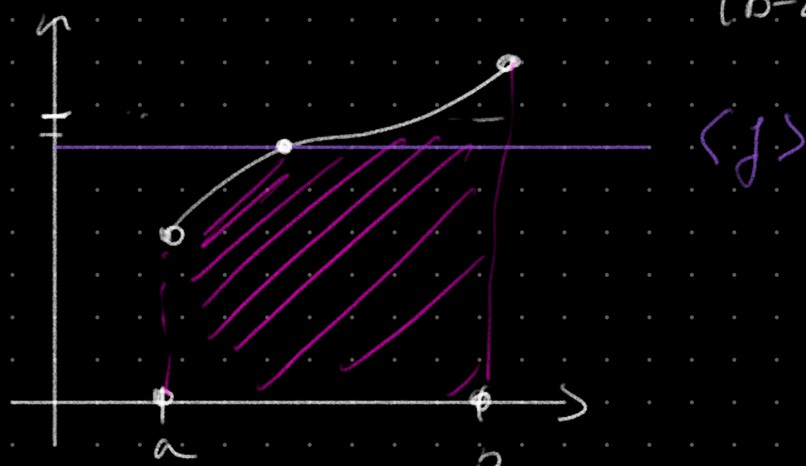
$\sin(x)$



$$\int_{-a}^{a} f(x)\,dx = 0$$

- Average value of a function on $[a, b]$

$$\langle f \rangle = \bar{f} = \frac{1}{b-a} \int_a^b f(x)\,dx$$



$$(b-a)\langle f \rangle = \int_a^b f(x)\,dx$$

there is a $c \in [a, b]$
for which $f(c) = \langle f \rangle$
if f is continuous

Anti-derivatives - indefinite integrals

$$\int f(x)dx = F(x) + c \Leftrightarrow \frac{d}{dx}(F(x)) = f(x)$$

integration constant

* $\int f(x)dx$ is a function of x

  $\int f(x)dx$ is only defined up to a constant

  $\rightarrow$ NOT UNIQUE

* not always possible to calculate.

Examples : $\int \sin(x)dx = -\cos(x) + C$

$\int x\,dx = \frac{x^2}{2} + C$

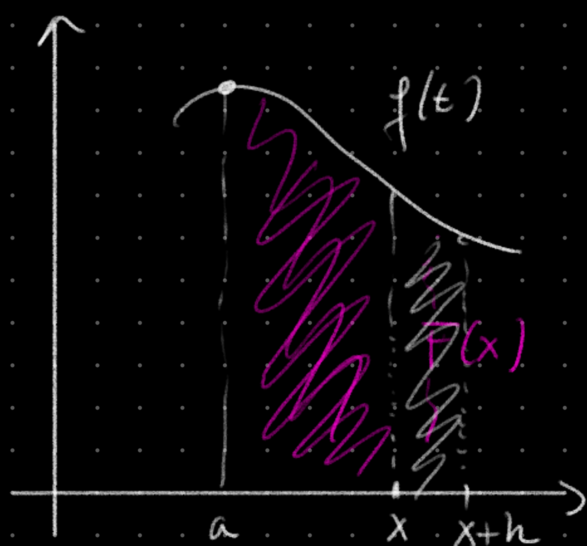
$\int dx = x + C$ $\qquad$ $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$

Fundamental theorem of Calculus

For a continuous function $f(x)$ on an interval I, $a \in I$

1. let $F(x) = \int_a^x f(t)dt, x \in I$, then $F(x)$ is differentiable, and $F'(x) = f(x)$.
2. if $G'(x) = f(x)$ for a function $G(x)$ on I, then, for all $b \in I$, $\int_a^b f(t)dt = G(b) - G(a)$

↳ relates definite integrals (= area below graph) and indefinite integrals (= anti-derivative)



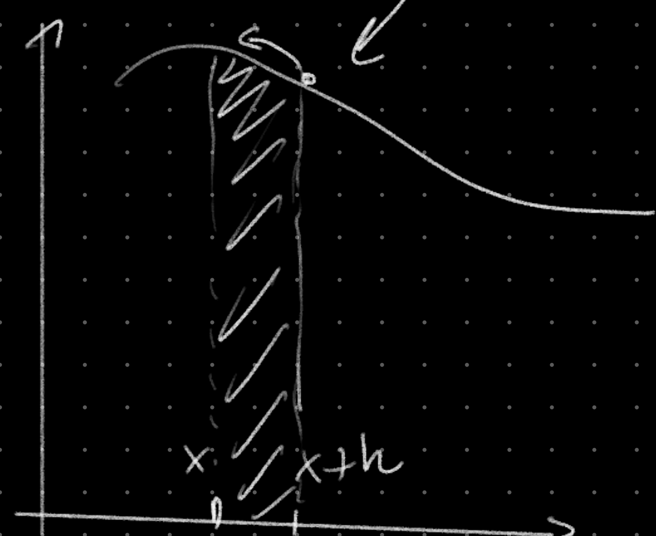
$F(a) = 0$

$F(x)$ = area under the graph from a to x

$$\frac{dF}{dx} = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \left(\frac{1}{h} \int_a^{x+h} f(t)dt - \int_a^{x} f(t)dt \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\cancel{\int_a^x f(t)\,dt} + \int_x^{x+h} f(t)\,dt - \cancel{\int_a^x f(t)\,dt} \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = f(x)$$



$$2) \quad G'(x) = f(x) = F'(x)$$

$$\Rightarrow \quad G'(x) - F'(x) = 0$$

$$\Rightarrow \quad \frac{d}{dx}(G-F)(x) = 0$$

$$\Rightarrow G - F = C \quad \Leftrightarrow \quad G(x) = F(x) + C$$

$$\hookrightarrow G(a) = F(a) + C = C$$

$$G(b) = F(b) + C = F(b) + G(a)$$

$$\Rightarrow F(b) = \int_a^b f(t)\,dt = G(b) - G(a)$$