Surname, First name

Linear Algebra (KEN1410)
Resit

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
7	7	7	7	7	7	7
8	8	8	8	8	8	8
9	9	9	9	9	9	9
0	0	0	0	0	0	0

a b c d e f → b
a b c d e f → c
a b c d e f → a

Fill in your answer(s) to the multiple-choice questions as shown above (circles = one correct answer).

Program: Data Science and Artificial Intelligence
Course code: KEN1410
Examiners: Dr. Marieke Musegaas and Dr. Stefan Maubach
Date/time: Wednesday 5 July 2023, 09:00 - 11:00
Format: Closed book exam
Allowed aids: Pens, simple (non-programmable) calculator from the DACS-list of allowed calculators.
Instructions to students:

- The exam consists of 7 questions on 20 pages.
- Fill in your name and student ID number on the cover page and tick the corresponding numerals of your student number in the table (top right cover page).
- Answer every question in the reserved space below the question. **Do not write outside the reserved space or on the back of pages, this will not be scanned and will NOT be graded!** As a last resort if you run out of space, use the extra answer space at the end of the exam.
- *In no circumstance write on or near the QR code at the bottom of the page!*
- Ensure that you properly motivate your answers.
- Only use black or dark blue pens, and write in a readable way. Do not use pencils.
- Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in the space reserved for the answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- You have to return all pages of the exam. You are not allowed to take any sheets, even blank, home.
- Good luck!

Question 1

10p **1** Consider the polynomial $p(t) = at^3 + bt^2 + ct + d$, where a, b, c and d are real numbers.
It is given that $p(-1) = 1$, $p(0) = 0$, $p(1) = 1$ and $p(2) = -1$.
Compute a, b, c and d.

Question 2

10p **2** Consider the following system of linear equations depending on a parameter p:

$$\begin{aligned} x_1 + \quad 2x_2 + \; 4x_3 &= 2 \\ 2x_1 + (p+3)x_2 + \; 8x_3 &= 2 \\ x_1 + \quad 2x_2 + p^2 x_3 &= p. \end{aligned}$$

Determine the values of p for which this system has exactly one solution.

Question 3

10p

3 Let $A = \begin{bmatrix} 5 & 5 & 6 & 6 & 7 \\ 4 & 0 & 4 & 4 & 0 \\ 3 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 5 & 6 & 7 & 1 & 8 \end{bmatrix}$.

Compute $\det A$.

Question 4

5p

4 Assume that $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$ are solutions of the equation $A\mathbf{x} = \mathbf{b}$, where A is a 3×3 matrix and $\mathbf{b} \neq \mathbf{0}$.

Determine three different solutions of the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Question 5

5p

5 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation such that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}$.

Is T a linear transformation? If yes, prove it. If not, explain why not.

Question 6

Let $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

5p **6a** Show that the matrix A has eigenvalues 3 and 5.

10p **6b** Orthogonally diagonalize matrix A, i.e. find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

Question 7
Answer the following multiple-choice questions.
An explanation is not required.
For every multiple-choice question, there is only **one correct answer**.
Please read the multiple choice instructions on the cover page!

2p **7a** Let A be a 6×4 matrix. Is the following statement true or false?
"The column space of A and the row space of A are orthogonal to each other."

(a) True (b) False

2p **7b** Let A be a 6×4 matrix. Is the following statement true or false?
"The row space of A and the nullspace of A are orthogonal to each other."

(a) True (b) False

2p **7c** Let A be a 6×4 matrix. Is the following statement true or false?
"The column space of A and the nullspace of A are orthogonal to each other."

(a) True (b) False

2p **7d** Let A be a 6×4 matrix. What is the smallest possible dimension of Nul A?

(a) 0
(b) 2
(c) 4
(d) 6
(e) None of the above.

2p **7e** Let A be a 6×4 matrix. What is the largest possible dimension of Col A?

(a) 0
(b) 2
(c) 4
(d) 6
(e) None of the above.

5p

7f Let $A = \begin{bmatrix} 1 & 1 & -2 & -1 \\ -1 & 4 & -3 & 1 \\ 0 & 7 & 1 & -8 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$.

Does $\mathbf{u}$ belong to Nul A? Does $\mathbf{v}$ belong to Col A?

- (a) $\mathbf{u}$ belongs to Nul A. $\mathbf{v}$ belongs to Col A.
- (b) $\mathbf{u}$ belongs to Nul A. $\mathbf{v}$ does not belong to Col A.
- (c) $\mathbf{u}$ does not belong to Nul A. $\mathbf{v}$ belongs to Col A.
- (d) $\mathbf{u}$ does not belong to Nul A. $\mathbf{v}$ does not belong to Col A.

5p

7g What is the dot product (inner product) of $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$?

- (a) $\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$
- (b) $\begin{bmatrix} 4 \\ 3 \\ -4 \end{bmatrix}$
- (c) 0
- (d) 5
- (e) The dot product cannot be computed for these vectors.
- (f) None of the above.

5p

7h If $\mathbf{y} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then the orthogonal projection of $\mathbf{y}$ onto Span $\{\mathbf{u}\}$ is

- (a) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- (b) $\begin{bmatrix} \frac{3}{2} \\ \frac{-1}{2} \end{bmatrix}$
- (c) $\begin{bmatrix} 10 \\ 5 \end{bmatrix}$
- (d) $\begin{bmatrix} \frac{1}{10} \\ \frac{3}{10} \end{bmatrix}$
- (e) None of the above.

5p

7i If A is a 3×3 matrix such that $A\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $A\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, then the product $A\begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$ is

- (a) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- (b) $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$
- (c) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$
- (d) $\begin{bmatrix} 9 \\ 10 \\ 11 \end{bmatrix}$
- (e) Not uniquely determined by the information given.

5p

7j For what value (or values) of p is the vector $\begin{bmatrix} 1 \\ 2 \\ p \\ 5 \end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$?

- (a) -1 only
- (b) 1 only
- (c) 3 only
- (d) for no value of p
- (e) for infinitely many values of p
- (f) None of the above.

5p **7k** Recall that $\mathbb{P}_2$ denotes the set of polynomials of degree at most 2. In other words, $\mathbb{P}_2$ consists of all polynomials of the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2,$$

where the coefficients a_0, a_1, a_2 and the variable t are real numbers.

Let V be the subset of $\mathbb{P}_2$ made up of only those polynomials $\mathbf{p}(t)$ such that $\mathbf{p}(1) = 0$, i.e.

$$V = \{\mathbf{p} \in \mathbb{P}_2 : \mathbf{p}(1) = 0\}.$$

Similarly, let W be the subset of $\mathbb{P}_2$ made up of only those polynomials $\mathbf{p}(t)$ such that $\mathbf{p}(0) = 1$, i.e.

$$W = \{\mathbf{p} \in \mathbb{P}_2 : \mathbf{p}(0) = 1\}.$$

Is V is a subspace of $\mathbb{P}_2$? Is W is a subspace of $\mathbb{P}_2$?

- (a) V is a subspace of $\mathbb{P}_2$. W is a subspace of $\mathbb{P}_2$.
- (b) V is a subspace of $\mathbb{P}_2$. W is not a subspace of $\mathbb{P}_2$.
- (c) V is not a subspace of $\mathbb{P}_2$. W is a subspace of $\mathbb{P}_2$.
- (d) V is not a subspace of $\mathbb{P}_2$. W is not a subspace of $\mathbb{P}_2$.

5p **7l** Let $A = \begin{bmatrix} a & -3 \\ 2 & b \end{bmatrix}$, where a, b are real numbers. Given is that the characteristic polynomial of A is $\lambda^2 - 19$.
How many different values can a have?

- (a) There are at least three different values a can have.
- (b) There are exactly two values a can have.
- (c) There is only one value a can have.
- (d) There are no posisble values for a.

Extra space

If you use these extra answer boxes, **please mention clearly in your main answer that part of your answer can be found here!**

8a

8b

8c

8d