Exercises

1	2	3	4	5	6	7	8	9	10

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
7	7	7	7	7	7	7
8	8	8	8	8	8	8
9	9	9	9	9	9	9
0	0	0	0	0	0	0

Surname, First name

Calculus Test

Example exam

1: 12
2: 16
3: 10
4: 10
5: 6
6: 10
7: 10
8: 10
9: 10

a ~~b~~ c d e f → b
a b ~~c~~ d e f → c
a b c ~~d~~ e f → a

Fill in your answer(s) to the multiple-choice questions as shown above (circles = one correct answer).

Program:
Course code: KEN/BCS 14460
Examiners: Otti D'Huys, Gijs Schoenmakers
Date/time:
Format: Written, closed book
Allowed aids: A formula sheet is attached to the exam.
Instructions to students:

- The exam consists of 9 questions on 12 pages.
- Fill in your name and student ID number on the cover page and tick the corresponding numerals of your student number in the table (top right cover page).
- Answer every question in the reserved space below the question. **Do <u>not</u> write outside the reserved space or on the back of pages, this will not be scanned and will NOT be graded!** As a last resort if you run out of space, use the extra answer space at the end of the exam.
- *In no circumstance write on or near the QR code at the bottom of the page!*
- Ensure that you properly motivate your answers.
- Only use black or dark blue pens, and write in a readable way. Do not use pencils.
- Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in the space reserved for the answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- You have to return all pages of the exam. You are not allowed to take any sheets, even blank, home.
- Good luck!

©copyright 2023 – [dr. Gijs Schoenmakers, dr. Otti D'Huys] - you are not allowed to redistribute this exam, nor any part thereof, without prior written permission of the authors

Question 1

6p **1a** For $f(x)$ and $g(x)$ differentiable functions, let $f(1) = 1$, $f(-2) = -2$, $f'(1) = -3$, $f'(-2) = 3$, $g(1) = -2$ and $g'(1) = 1$.

Find $\lim_{x\to 1} \frac{f(g(2-x))-g(f(x^2))}{x-1}$

$$\lim_{x\to 1} \frac{f[g(2-x)] - g[f(x^2)]}{x-1}$$

$f(1) = 1$
$f(-2) = -2$
$f'(1) = -3$
$f'(-2) = 3$
$g(1) = -2$
$g'(1) = 1$

$$H = \lim_{x\to 1} \frac{f'(g(2-x))\cdot g'(2-x) - g'(f(x^2))\cdot f'(x^2)}{1}$$

$$= f'(-2)\cdot - g'(1)(-3)$$

$$= -3 - (-3)1 = 0$$

- (a) $-\infty$
- (b) -1
- (c) 0
- (d) 3
- (e) -6
- (f) $+\infty$
- (g) -9
- (h) The limit does not exist
- (i) None of the above

6p **1b** Calculate $\lim_{x\to 0^-} \frac{x}{\sqrt{x^2-x}}$ $\overset{H}{=}$

$$\lim_{x\to 0^-} \frac{2\sqrt{x^2-x}}{2x-1} = 0$$

$$\frac{d}{dy}\sqrt{x^2-x} = \frac{2x-1}{2\sqrt{x^2-x}}$$

- (a) 4
- (b) $-\frac{1}{2}$
- (c) $\frac{1}{2}$
- (d) -1
- (e) $-\infty$
- (f) 0
- (g) $+\infty$
- (h) 1
- (i) The limit does not exist
- (j) None of the above

6p **1c** Calculate $\int_0^1 x\ln(x)dx$

$$\int_0^1 x \ln(x)\,dx =$$

$f(x) = \ln x \qquad f'(x) = \frac{1}{x}$
$g'(x) = x \qquad g(x) = \frac{x^2}{2} =$

$$= \frac{x^2}{2}\ln x - \int \frac{x}{2}dx$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4}$$

$$\left(\frac{1}{2}\ln 1 - \frac{1}{4}\right) - (0 - 0)$$

$$= \frac{e}{2} - \frac{1}{4}$$

- (a) $-\frac{1}{2}$
- (b) This integral diverges to $-\infty$
- (c) 0
- (d) This integral diverges to $+\infty$
- (e) None of the above

Question 2

16p **2** Find all the asymptotes of the function $f(x) = \frac{1}{\sqrt{x^2+3x}+x}$. For the horizontal and oblique asymptotes (if any), specify whether they are one-sided or two-sided. For the vertical asymptotes (if any), specify how the function approaches the asymptotes (i.e. whether the function goes to $+\infty$ or $-\infty$ on the left or right side of the asymptotes.)

$$\text{dom} f: \sqrt{x^2+3x} + x \neq 0$$
$$\sqrt{x^2+3x} \neq (-)x$$
$$x^2+3x \neq x^2$$
$$x \neq 0$$

Vertical Asymptotes

$$\lim_{x\to 0^+} \frac{1}{\sqrt{x^2+3x}+x} = +\infty$$
$$\lim_{x\to 0^-} \frac{1}{-x\sqrt{\left(1+\frac{3}{x}\right)}+x} = -\infty$$

V. A. at $x=0$

$x=0$

Horizontal Asymptotes

$$\lim_{x\to+\infty} // = 0 \leftarrow \text{one-sided}$$

$$\lim_{x\to-\infty} \frac{1}{-x\sqrt{1+\frac{3}{x}}+x} = \lim_{x\to-\infty} \frac{1}{x\left(-\sqrt{1+\frac{3}{x}}+1\right)}$$
$$= 0$$

Oblique Asymptotes

$$\lim_{x\to\infty} \frac{f(x)}{x}$$

$\cancel{0 \cdot \infty = 0}$

$$\lim_{x\to-\infty} x \ln\left(1+\frac{1}{x}\right)$$

$$\lim_{x \to \infty} x \ln\left(1 + \frac{1}{x}\right)$$

Question 3

10p **3** Does the function $f(x) = x\ln(x)$ have a global (absolute) minimum and maximum on $(0, \infty)$?

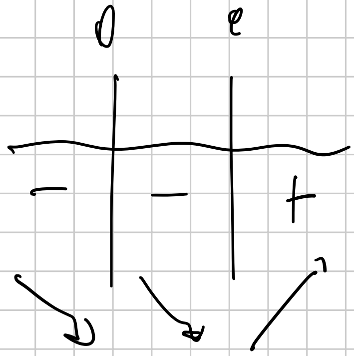
- If so, for which value(s) of x?
- If not, motivate why you can exclude the existence of a global extremum.

$$f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$$

$f'(x) = 0$ — dom f: $x > 0$

$$\ln(x) + 1 = 0$$
$$\ln(x) = 1$$
$$x = e^{-1}$$

$$f'(x) > 0$$
$$\ln(x) > 1$$
$$x > e$$



f(x) has a global minimum at x = e, but has no global maximum as it grows to infinity

Question 4

10p **4** Evaluate the following integral: $\int_0^{\frac{\pi}{2}} (\sin(x))^3 dx$

$u = \cos x$

$du = -\sin x \, dx$

x	u
0	1
$\pi/2$	0

$$\int_0^{\pi/2} \sin^3 x \, dx = \int_0^{\pi/2} (1-\cos^2) \sin x \, dx$$

$$= \int_1^0 (-(1-u^2)) \, du$$

$$= \left[u - \frac{u^3}{3} \right]_1^0$$

$$= \left[\cos x - \frac{(\cos x)^3}{3} \right]_1^0$$

$$= 1 - \frac{1}{3} - (0) = \frac{2}{3}$$

Question 5

← Study series

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{e^n + n}{e^{2n} - n^2}$

1.5p **5a** The series $\sum_{n=1}^{\infty} a_n$ diverges. We can conclude this by comparing to the series $\sum_{n=1}^{\infty} b_n$ with $b_n = \frac{1}{n}$ and $|a_n| \leq b_n$ for all $n \in \mathbb{N}$

(a) True (b) False

1.5p **5b** The series $\sum_{n=1}^{\infty} a_n$ converges. We can conclude this by comparing to the series $\sum_{n=1}^{\infty} b_n$ with $b_n = e^{-n}$ and $a_n \leq b_n$ for all $n \in \mathbb{N}$

(a) True (b) False

1.5p **5c** The series $\sum_{n=1}^{\infty} a_n$ converges. We can conclude this because $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = e^{-1}$

(a) True (b) False

1.5p **5d** The series $\sum_{n=1}^{\infty} a_n$ converges. We can conclude this because $\lim_{n\to\infty} a_n = 0$

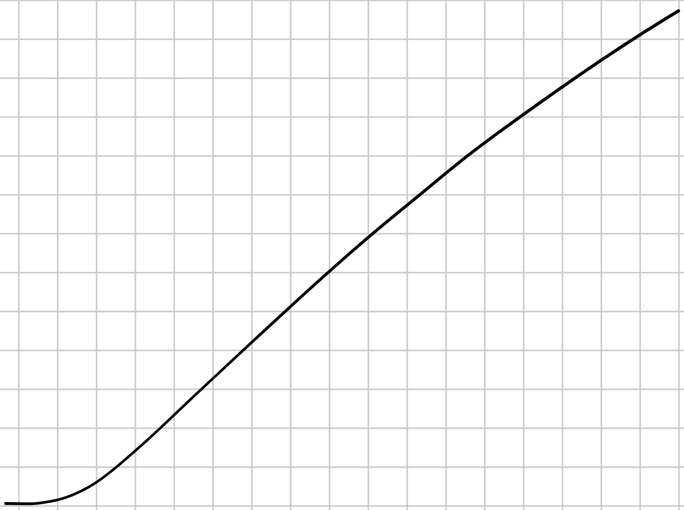
(a) True (b) False

Question 6

10p **6** Determine which values of $x \in \mathbb{R}\backslash\{0\}$, the given series is absolutely convergent, conditionally convergent or divergent.

?

$$\sum_{n=1}^{\infty} n\left(\frac{2}{x}\right)^n$$



Question 7

10p **7** Find the solution $y = y(x)$ to the given initial value problem.

$$\begin{cases} y' + y = x \\ y(0) = 3 \end{cases} \qquad \left\{ y' - \frac{1}{y} = 0 \right.$$

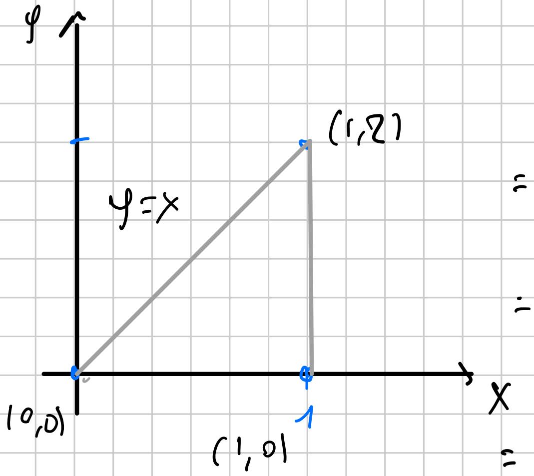
Question 8

10p **8** For the function $f(x, y) = \sin(2x - y)$, give the equation of the tangent plane at $(0, 0, f(0, 0))$

Question 9

10p **9** Evaluate the double integral over a triangle T with vertices $(0,0),(1,0),(1,1)$:

$$\iint_T e^{2x+y}\,dA$$



$$\iint_T (e^{2x+y})\,dA$$
$$= \int_0^1 \int_0^x e^{2x+y}\,dy\,dx$$
$$= \int_0^1 \left[e^{2x+y} \right]_0^x dx$$
$$= \int_0^1 \left(e^{3x} - e^{2x} \right) dx$$
$$= \left[\frac{1}{3} e^{3x} \right]_0^1 - \left[\frac{1}{2} e^{2x} \right]_0^1$$
$$= \frac{1}{3}e^3 - \frac{1}{3} - \left(\frac{1}{2}e^2 - \frac{1}{2} \right)$$
$$= \frac{1}{3}e^3 + \frac{1}{2}e^2 + \frac{1}{6}$$

$$-\frac{1}{3} + \frac{1}{2} = \frac{3-2}{6} = \frac{1}{6} \qquad \frac{1}{12}$$

$$\int e^{2x+y}\,dy$$
$$= e^{2x+y}$$

Extra space

10a

$$\int_0^{\frac{\pi}{2}} (\operatorname{sen}(x))^3 \, dx = \int_0^{\frac{\pi}{2}} \sin^3(x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\operatorname{sen}^2(x)\, dx \qquad u = \cos(x)$$

$$= \int_0^{\pi/2} \left[\sin^2(x) - \cos^2(x)\sin^2(x)\right] dx \qquad \frac{du}{dx} = -\sin(x)$$

$$du = -\sin(x)\, dx$$

$$dx = \left[-\frac{1}{\sin(x)}\right] du$$

$$= \int_0^{\pi/2} \left[1 - u^2\right] \operatorname{sen}^2(x) \left(-\frac{1}{\sin(x)}\right) du$$

$$= \int_0^{\pi/2} (-1 + u^2)\, du \Rightarrow \left[-u + \frac{u^3}{3}\right] \Big|_0^{\frac{\pi}{2}} =$$

10b

10c

10d

This page is left blank intentionally