

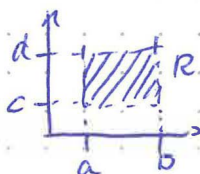
Calculus - lecture 11 Double / iterated integrals

+ Thomas, 15.1-2
(or: Adams, 14.1-2)

I DOUBLE INTEGRALS

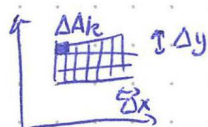
* For a function $f(x,y)$, continuous on a region R

$$R: a \leq x \leq b \\ c \leq y \leq d$$



How to calculate $\iint_R f(x,y) dA$

↳ we make a partition of R into small rectangles ΔA_k , $\Delta A = \Delta x \Delta y$



→ we can take a Riemann sum for points (x_k, y_k) in ΔA_k

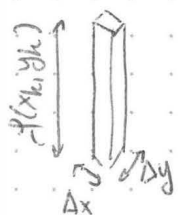
$$S_n = \sum_{k=1}^n f(x_k, y_k) \Delta A_k$$

* the norm $\|P\|$ of a partition is $\max\{\Delta x_k, \Delta y_k\}$ (maximal width (height))

↳ f is integrable over R if this limit exists and is finite

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k) \Delta A_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k, y_k) \Delta A_k = \iint_R f(x,y) dA$$

* an integral is the area under a curve (for $f(x) > 0$).
a double integral is the volume under a surface (for $f(x,y) > 0$)



$f(x_k, y_k) \Delta x \Delta y$ is a volume element of the Riemann sum.

* How to calculate double integrals as iterated integrals?

Example $f(x,y) = 4-x-y$

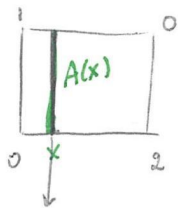
$$R: 0 \leq x \leq 2 \\ 0 \leq y \leq 1$$

$$\iint_R f(x,y) dA = \int_0^2 \int_0^1 (4-x-y) dy dx$$

outer integral (numbered)
inner integral = $A(x)$



$$\iint_R f(x,y) dA = \int_0^2 \int_0^1 (4-x-y) dy dx$$



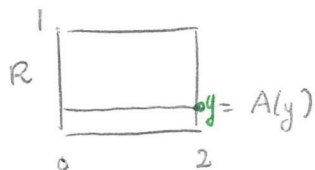
$$A(x) = \int_0^1 (4-x-y) dy = \left[4y - xy - \frac{y^2}{2} \right]_0^1 = 4 - x - \frac{1}{2} = \frac{7}{2} - x$$

the outcome is a function of x
 $\hookrightarrow x$ is treated as a constant in this integration

We first calculate a cross-section along y , for fixed x : $A(x)$

$$\iint_R f(x,y) dA = \int_0^2 \left(\int_0^1 (4-x-y) dy \right) dx = \int_0^2 \left(\frac{7}{2} - x \right) dx = \left[\frac{7}{2}x - \frac{x^2}{2} \right]_0^2 = 7 - 2 = 5$$

* we can also take a cross-section along x



$$\iint_R f(x,y) dA = \int_0^1 A(y) dy = \int_0^1 (6-2y) dy = \left[6y - y^2 \right]_0^1 = 6 - 1 = 5$$

$$A(y) = \int_0^2 (4-x-y) dx = \left[4x - \frac{x^2}{2} - xy \right]_0^2 = (8 - 2 - 2y) = 6 - 2y$$

the outcome is a function of y

\hookrightarrow the result of both calculations is - of course - the same

\rightarrow this is FUBINI's theorem (the order of integration does not matter)

* If $f(x,y)$ is continuous on the rectangular region $R: a \leq x \leq b, c \leq y \leq d$, then

$$\iint_R f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx = \int_c^d \int_a^b f(x,y) dx dy$$

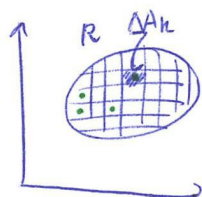
* Example

$$\iint_R (x+y) dA = \int_0^a \int_0^a (x+y) dx dy = \int_0^a \left[\frac{x^2}{2} + xy \right]_0^a dy = \int_0^a \left(\frac{a^2}{2} + ay \right) dy = \frac{a^2}{2} \left[y \right]_0^a + a \left[\frac{y^2}{2} \right]_0^a = \frac{a^3}{2} + \frac{a^3}{2} = a^3$$

$R: 0 \leq x, y \leq a$

II DOUBLE INTEGRALS OVER GENERAL REGIONS

double integrals can be defined on more general regions than rectangles



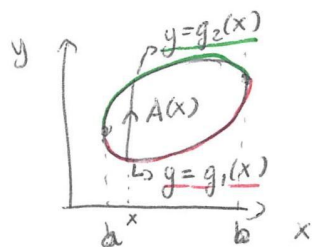
The double integral is the limit of the Riemann sums

$$\iint_R f(x,y) dA = \lim_{\|P\| \rightarrow 0} \sum_k f(x_k, y_k) \Delta A_k = \lim_{n \rightarrow \infty} \sum_k f(x_k, y_k) \Delta A_k$$

* formally, we only take into account rectangles ΔA_k that are fully inside R : as $\|P\| \rightarrow 0$, sufficiently regular areas R are filled

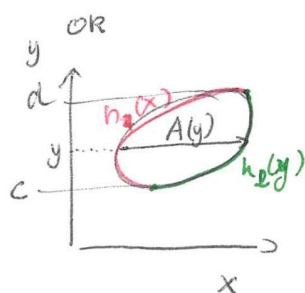
* for $f(x,y) > 0$, continuous on R , the integral $\iint_R f(x,y) dA$ is the VOLUME between R and the surface $f(x,y)$

* to calculate this volume, we can parametrise the region R



$$\text{then } \iint_R f(x,y) dA = \int_a^b A(x) dx$$

$$\text{with the cross-section } A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y) dy$$



$$\text{then } \iint_R f(x,y) dA = \int_c^d A(y) dy$$

$$\text{with the cross-section } A(y) = \int_{h_1(y)}^{h_2(y)} f(x,y) dx$$

→ both results are the same \Rightarrow stronger form of Fubini's theorem.

* If $f(x,y)$ is continuous on a region R

→ if R is defined as $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with $g_1(x), g_2(x)$ continuous on $[a,b]$, then

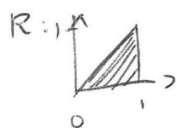
$$\iint_R f(x,y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy dx$$

→ if R is defined as $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with $h_1(y), h_2(y)$ continuous on $[c,d]$, then

$$\iint_R f(x,y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) dx dy$$

Example

$$f(x,y) = 3 - x - y$$



$$1) R: \begin{matrix} x: 0 \rightarrow 1 \\ y: 0 \rightarrow x \end{matrix}$$

$$\begin{aligned} \iint_R f(x,y) dA &= \int_0^1 \int_0^x (3 - x - y) dy dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_0^x dx \\ &= \int_0^1 \left(3x - x^2 - \frac{x^2}{2} \right) dx = \left[\frac{3}{2}x^2 - \frac{1}{2}x^3 \right]_0^1 = 1 \end{aligned}$$

$$2) R: \begin{matrix} y: 0 \rightarrow 1 \\ x: y \rightarrow 1 \end{matrix}$$

$$\begin{aligned} \iint_R f(x,y) dA &= \int_0^1 \int_y^1 (3 - x - y) dx dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_y^1 dy \\ &= \int_0^1 \left(\left(3 - \frac{1}{2} - y \right) - \left(3y - \frac{y^2}{2} - y^2 \right) \right) dy = \left[\frac{5}{2}y - \frac{1}{2}y^2 + \frac{1}{2}y^3 \right]_0^1 \\ &= 1 \end{aligned}$$

* Properties of double integrals

For $f(x,y), g(x,y)$ continuous on R (bounded)

$$* \iint_R c f(x,y) dA = c \iint_R f(x,y) dA \quad \text{for } c \in \mathbb{R}$$

$$* \iint_R (f(x,y) \pm g(x,y)) dA = \iint_R f(x,y) dA \pm \iint_R g(x,y) dA$$

$$* \text{ if } f(x,y) \geq 0 \text{ on } R, \text{ then } \iint_R f(x,y) dA \geq 0$$

$$\text{ if } f(x,y) \geq g(x,y) \text{ on } R, \text{ then } \iint_R f(x,y) dA \geq \iint_R g(x,y) dA$$

* if R is partitioned in R_1, R_2

$$\rightarrow \iint_R f(x,y) dA = \iint_{R_1} f(x,y) dA + \iint_{R_2} f(x,y) dA$$



For * Triangle inequality $\left| \iint_R f(x,y) dA \right| \leq \iint_R |f(x,y)| dA$

* integrals correspond to volumes / negative integrals to volumes below the xy-plane

Examples

$$\bullet \int_0^1 \int_0^x \frac{\sin(x)}{x} dy dx$$

$$\bullet \int_0^1 \int_{y^2}^y dx dy$$

$$\bullet \int_0^2 \int_{-x}^x xy^2 dy dx$$