Surname, First name

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Linear Algebra (KEN1410)
Exam

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
7	7	7	7	7	7	7
8	8	8	8	8	8	8
9	9	9	9	9	9	9
0	0	0	0	0	0	0

a b c d e f → b
a b c d e f → c
a b c d e f → a

Fill in your answer(s) to the multiple-choice questions as shown above (circles = one correct answer).

Program: Data Science and Artificial Intelligence
Course code: KEN1410
Examiners: dr. Marieke Musegaas and dr. Stefan Maubach
Date/time: Tuesday 29 March 2022, 09:00 - 11:00
Format: Closed book exam
Allowed aids: Pens, simple (non-programmable) calculator from the DKE-list of allowed calculators.
Instructions to students:

- The exam consists of 8 questions on 18 pages.
- Fill in your name and student ID number on the cover page.
- Answer every question in the reserved space below the question. **Do not write outside the reserved space or on the back of pages, this will not be scanned and will NOT be graded!** As a last resort if you run out of space, use the extra answer space at the end of the exam.
- *In no circumstance write on the QR code at the bottom of the page!*
- Ensure that you properly motivate your answers.
- Before answering the questions, please first read all the exam questions, and then make a plan to spend the two hours.
- Only use black or dark blue pens, and write in a readable way. No Pencils. Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in the space reserved for the answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- You have to return all pages of the exam. You are not allowed to take any sheets, even blank, home.
- Good luck!

Exercise 1

Consider the following matrix A:

$$A = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 1 & 2 & 1 & -2 \\ 2 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

Define $T : \mathbb{R}^4 \to \mathbb{R}^4$ by $T(\mathbf{x}) = A\mathbf{x}$.

15p

1 Determine all $\mathbf{x} \in \mathbb{R}^4$ such that $T(\mathbf{x}) = \begin{bmatrix} 8 \\ 2 \\ 5 \\ 4 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 2 & 0 & 2 & 4 & 8 \\ 1 & 2 & 1 & -2 & 2 \\ 2 & 0 & 1 & 2 & 5 \\ 1 & 0 & 1 & 2 & 4 \end{array}\right] \quad R_1 \longleftrightarrow R_2 \; \sim \; \left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 2 \\ 2 & 0 & 2 & 4 & 8 \\ 2 & 0 & 1 & 2 & 5 \\ 1 & 0 & 1 & 2 & 4 \end{array}\right] \quad \begin{array}{l} R_2 : R_2 - 2 \cdot R_1 \\ \sim \; R_3 : R_3 - 2 \cdot R_1 \\ R_4 : R_4 - R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 2 \\ 0 & -4 & 0 & 8 & 4 \\ 0 & -4 & -1 & 6 & 1 \\ 0 & -2 & 0 & 4 & 2 \end{array}\right] \quad R_2 : R_2 * -1/4 \; \sim \; \left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & -4 & -1 & 6 & 1 \\ 0 & -2 & 0 & 4 & 2 \end{array}\right] \quad \begin{array}{l} \sim \\ R_3 : R_3 + 4 \cdot R_2 \\ R_4 : R_4 + 2 \cdot R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 : R_3 * -1 \; \sim \; \left[\begin{array}{cccc|c} 1 & 2 & 1 & -2 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 : R_1 - R_3 \; \sim$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -4 & -1 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 : R_1 - 2 \cdot R_2 \; \sim \; \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underline{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

Exercise 2

Let $a, b, c, d \in \mathbb{R}$ and consider the following system of linear equations

$$\begin{aligned} x_1 + ax_2 + bx_3 &= 14 \\ cx_2 + dx_3 &= -40 \end{aligned}$$

This system of linear equations has a solution set that looks like

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix},$$

where $\lambda \in \mathbb{R}$.

15p **2** Compute a, b, c and d.

$$\underline{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \end{array}\right] \underset{R_1 : R_1 + 3 \cdot R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & 1 & 2 & 5 \end{array}\right] \underset{R_2 : R_2 * -8}{\sim}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -8 & -16 & -40 \end{array}\right]$$

So, $a = 3$
$b = 4$
$c = -8$
$d = -16$.

Exercise 3

5p **3** Provide, **IF POSSIBLE**, an example of a subset H of $\mathbb{R}^2$ that has the following three properties:

- the zero vector is in H,
- H is **NOT** closed under vector addition,
- H is closed under multiplication by scalars.

(Note: only provide your answer. An explanation is not required.)

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 x_2 \geq 0 \right\}$$

Exercise 4

A matrix A has after a couple of row operations the following form

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & -2 \\ 0 & -6 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

5p **4a** Provide the dimensions of the following four vector spaces: Nul A, Col A, Row A and Nul A^T. (Note: only provide your answer. An explanation is not required.)

Nul A : 0
Col A : 3
Row A : 3

$\text{Nul}(A^T) = 4 - \text{Col}(A^T) = 4 - \text{Row}(A) = 4-3=1.$

5p **4b** Provide, **IF POSSIBLE**, a basis of the following two vector spaces: Col A and Row A. If it is not possible to provide a basis for a vector space, explain why.

Basis for Col A not possible because we don't know the original matrix.

Basis for Row A : $\{[1 \ \ 3 \ \ 0], [0 \ \ 4 \ \ -2], [0 \ \ 0 \ \ 1]\}$

Exercise 5

Consider the following matrix A:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

5p **5a** Show that 3 is an eigenvalue of A (hint: find an eigenvector).

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So, 3 is an eigenvalue.

10p **5b** Show that 0 is an eigenvalue of A. And find two corresponding linearly independent eigenvectors.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

So, 0 is an eigenvalue.

Two lin indep. eigenvectors: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

5p **5c** Is the matrix A diagonalizable? Briefly explain.
(Note: you do not need to diagonalize A. You only need to state whether it is possible to diagonalize A.)

Yes, A is diagonalizable because we found 3 lin indep eigenvectors.

Exercise 6

Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

10p **6** Write $\mathbf{y}$ as the sum of a vector in $\mathrm{Span}\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

The orthogonal projection of $\underline{y}$ onto $\underline{u}$ is

$$\hat{\underline{y}} = \frac{\underline{y} \cdot \underline{u}}{\underline{u} \cdot \underline{u}} \underline{u} = \frac{20}{50} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$$

and the component of $\underline{y}$ orthogonal to $\underline{u}$ is

$$\underline{y} - \hat{\underline{y}} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$$

Thus, $\underline{y} = \hat{\underline{y}} + (\underline{y} - \hat{\underline{y}}) = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$

a vector in $\mathrm{Span}\{\underline{u}\}$

a vector orthogonal to $\underline{u}$.

Exercise 7

10p **7** Prove or disprove the following statement.

Let $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ be two vectors that have the same length ($||\mathbf{x}|| = ||\mathbf{y}||$) and define $\mathbf{u} = \mathbf{x} + \mathbf{y}$ and $\mathbf{v} = \mathbf{x} - \mathbf{y}$. Then, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to each other.

The statement is true.

Proof: $\underline{u} \cdot \underline{v} = (\underline{x}+\underline{y}) \cdot (\underline{x}-\underline{y}) = \underline{x} \cdot (\underline{x}-\underline{y}) + \underline{y} \cdot (\underline{x}-\underline{y})$

$= \underline{x} \cdot \underline{x} \cancel{- \underline{x} \cdot \underline{y}} + \cancel{\underline{y} \cdot \underline{x}} - \underline{y} \cdot \underline{y}$

$= \underline{x} \cdot \underline{x} - \underline{y} \cdot \underline{y}$

$= ||\underline{x}||^2 - ||\underline{y}||^2$

$= ||\underline{y}||^2 - ||\underline{y}||^2$ (because $\underline{x}$ and $\underline{y}$ have the same length: $||\underline{x}|| = ||\underline{y}||$)

$= 0.$

So, $\underline{u}$ and $\underline{v}$ are orthogonal to each other.

Exercise 8

True or False? For each of the statements below: state whether the statement is true or false. (Note: An explanation is not required. **Please read the multiple choice instructions on the cover page!**)

3p **8a** If two rows of a square matrix A are the same, then $\det A = 0$.

(a) True (b) False

So, the rows are not lin indep $\Rightarrow A^T$ is not invertible $\Rightarrow A$ is not invertible $\Rightarrow \det A = 0$.

3p **8b** If A is a 6×8 matrix, then it is possible that it has a 1-dimensional null space.

(a) True (b) False

the null space has dimension at least 2

3p **8c** Two orthogonal vectors are automatically also linearly independent.

(a) True (b) False

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are orthogonal but not lin indep.

3p **8d** If λ is an eigenvalue of A, then it is also an eigenvalue of A^T.

(a) True (b) False

$\det(A^T - \lambda I) = \det(A^T - (\lambda I)^T) = \det((A - \lambda I)^T) = \det(A - \lambda I)$

3p **8e** Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. If $\mathbf{x} \cdot \mathbf{y} = 0$, then there does not exist a plane in $\mathbb{R}^3$ that contains both $\mathbf{x}$ and $\mathbf{y}$.

(a) True (b) False

Consider $x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Then, $x \cdot y = 0$.

Since x and y are lin. indep., they span a plane in $\mathbb{R}^3$. Hence, they both belong to this plane.

Extra space

If you use these extra answer boxes, **please mention clearly in your main answer that part of your answer can be found here!**

9a

9b

9c

9d

9e