Overview

- Recap: differential equations
- Functions of multiple variables
- Continuity and limits in 2 dimensions
- Partial derivatives
- Chain rule in multiple dimensions

Adams' Ch. 12.1-5

Differential equations (ODEs)

- An equation that involves derivatives y', y'',… of a function y(x)
- Solution: an explicit formula y(x)
- Often used to model natural phenomena (e.g. Newton's second law), but usually only possible to solve numerically
- We treat two types of first order ODEs that can be solved analytically: linear ODEs and separable ODEs

Solutions are not unique (cfr. Integration constants). In order to have a unique solution, we need n *initial conditions (*n = order of ODE).

- ODE + initial condition = initial value problem (IVP)
- The solution y(x) of an IVP **is** unique. So, if you find a solution (regardless of the method) that satisfies the differential equation and the initial conditions, you have completely solved the problem.

Separable first order differential equations

$y' = \frac{dy}{dx} = f(x) \cdot g(y) \longrightarrow$ solve as $\int \frac{dy}{g(y)} = \int f(x) dx$

(note: the ODE also has constant solutions $y = y_1$, where $g(y_1) = 0$

$\to$ this is relevant for completeness, but not in an initial value problem (unless the initial condition is $y(x_0) = y_1$)

First order linear differential equations

$\frac{dy}{dx} + p(x)\,y = q(x)$ $\longrightarrow$ the general solution $y(x)$ ALWAYS have the structure $y(x) = y_p(x) + K \cdot y_H(x)$

- $y_p(x)$ = 1 (particular) solution of the full non-homogeneous ODE

$$y_p' + p(x)\,y_p = q(x)$$

- $y_H(x)$ = solution of the homogeneous ODE

$$y_H' + p(x)\,y_H = 0$$

Solution strategy

Example: $\frac{dy}{dx} + \frac{y}{x} = 1$

1) Solve homogeneous ODE (always separable)

$\frac{dy}{dx} = -\frac{y}{x} \implies \int \frac{dy}{y} = -\int \frac{dx}{x} \implies \ln|y| = -\ln|x| + C \implies$

$$e^{\ln|y|} = e^{-\ln|x| \cdot c} \Rightarrow y = k \cdot e^{-\ln|x|} = k \frac{1}{e^{\ln|x|}} = \frac{k}{x}$$

$$\Rightarrow y_H(x) = k \cdot \frac{1}{x}$$

2) Find particular solution y_p to non homogeneous ODE.
[If you just "see" a solution, great!)

→ a trick to find this solution: PARAMETER VARIATION

↳ we assume $y(x) = \frac{k(x)}{x}$ (we make the constant k in $y_H(x)$ a function)

→ insert in ODE

$$y(x) = \frac{k(x)}{x} \longrightarrow \frac{dy}{dx} = \frac{k'(x)}{x} - \frac{k(x)}{x^2} \quad \text{(product rule)}$$

$$\underbrace{\frac{dy}{dx} + \frac{y}{x}}_{} = \underbrace{\frac{k'(x)}{x} - \cancel{\frac{k(x)}{x^2}}}_{\frac{dy}{dx}} + \underbrace{\cancel{\frac{k(x)}{x^2}}}_{y/x} = \underbrace{1}_{q(x)}$$

$$\Rightarrow \frac{K'(x)}{x} = 1 \quad \Rightarrow K'(x) = x \quad \Rightarrow K(x) = \frac{x^2}{2} + C$$

(we see that the trick works, since two terms cancel in the ODE, and we find another, easily solvable, ODE for $K(x)$)

$$\longrightarrow \quad y(x) = \frac{K(x)}{x} = \underbrace{\frac{x}{2}}_{\text{particular solution}} + \underbrace{\frac{C}{x}}_{C \cdot y_H(x)} \qquad \longrightarrow \text{ correct structure of solution!}$$

Functions of multiple variables

$f(x_1, x_2, \ldots, x_n)$; in 2D : $z = f(x,y)$

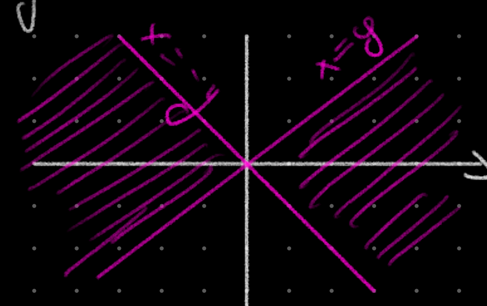
domain : subset of $\mathbb{R}^n$ — domain : subset of $\mathbb{R}^2$

domain convention : domain$(f) = \{(x,y) \in \mathbb{R}^2, f(x,y) \in \mathbb{R}\}$

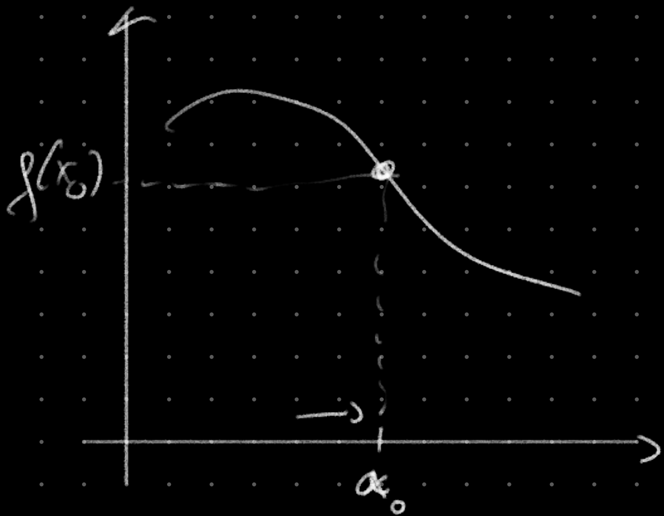
domain of $\frac{1}{x^2+y^2} = \mathbb{R}^2 \setminus \{(0,0)\}$

domain of $\sqrt{9-x^2-y^2}$ $\quad x^2+y^2 \leq 9$ (circle with $r=3$)

domain of $\sqrt{x^2-y^2}$ $\quad x^2 \geq y^2$



Continuity



for 1D (univariate functions)

$f(x)$ is continuous at a iff, for $x \in \text{domain}(f)$

$$\forall \epsilon > 0 \;\; \exists \delta > 0 : |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

(if x approaches a, then $f(x)$ approaches $f(a)$)

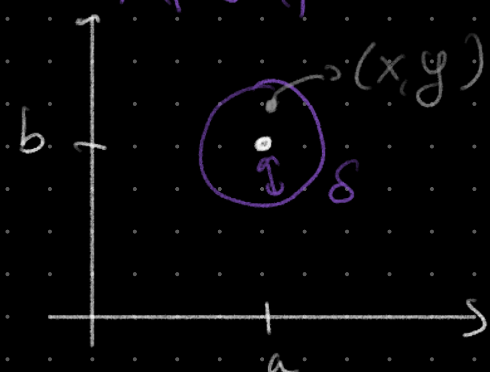
Continuity for multivariate functions)

$f(x,y)$ is continuous at (a,b) iff, for $(x,y) \in$ domain ( )

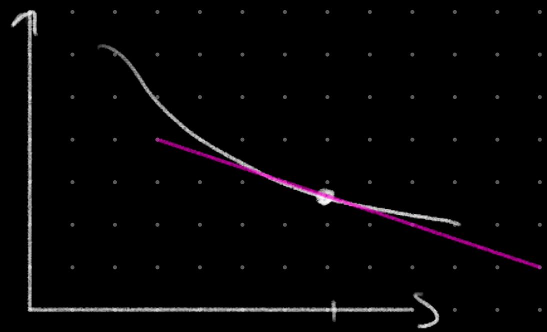
$$\forall \epsilon > 0 \;\; \exists \delta > 0 : \underbrace{\sqrt{(x-a)^2 + (y-b)^2}}_{\text{Euclidean distance}} < \delta \Rightarrow |f(x,y) - f(a,b)| < \epsilon$$

(if (x,y) approaches (a,b), $f(x,y)$ approaches $f(a,b)$)

when (x,y) is in the circle, $|f(x,y) - f(a,b)| < \epsilon$



Partial derivatives



$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

PARTIAL DERIVATIVE

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$\frac{\partial f(x,y)}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

other notations:

$$\frac{\partial}{\partial x} f(x,y) = f_1(x,y) = f_x(x,y)$$

$$= D_x(x,y) = D_1(x,y)$$

$$\left.\frac{\partial f}{\partial x}\right|_a^b = \frac{\partial}{\partial x} f(a,b)$$

$$\frac{\partial}{\partial x}(xy) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{x}+\cancel{h}) \cdot y - \cancel{xy}}{\cancel{h}} = y$$

$$\frac{\partial}{\partial x}(x+y) = 1$$

* how to calculate : like "normal derivatives". If you calculate $\frac{\partial f}{\partial x}$, you treat y like a constant. If you calculate $\frac{\partial f}{\partial y}$, you treat x like a constant.

Examples

$$\frac{\partial}{\partial x}(xy) = y$$

$$\frac{\partial}{\partial y}(x+y) = 1$$

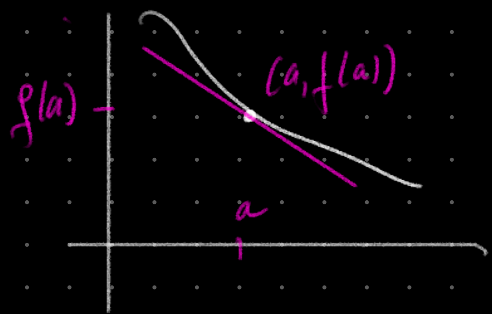
$$\frac{\partial}{\partial x}\left(\ln\left[\frac{x}{y}\right]\right) = \frac{\partial}{\partial x}\left(\ln|x| - \ln|y|\right) = \frac{1}{x}$$

$$\frac{\partial}{\partial x}\left(\ln\left[\frac{x}{y}\right]\right) = \frac{\cancel{y}}{x}\cdot\frac{1}{\cancel{y}}$$

$$\frac{\partial}{\partial y}\left(\sqrt{x^2+y^2}\right) = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

The tangent plane

1D



Equation of tangent line: $y = f(a) + f'(a)(x-a)$

- In 2D, we have a "tangent plane" that touches the surface $f(x,y)$

  Equation of tangent plane at $(a, b, f(a,b))$:

$$z(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\,(x-a) + \frac{\partial f}{\partial y}(a,b)\,(y-b)$$

Example: $f(x,y) = \sin(xy)$. Tangent plane at $\left(-1, \frac{\pi}{3}, f\left(-1, \frac{\pi}{3}\right)\right)$?

- $f\left(-1, \frac{\pi}{3}\right) = \sin\left(\frac{-\pi}{3}\right) = -\frac{\sqrt{3}}{2}$
- $\frac{\partial f}{\partial x} = y\cos(xy) \rightarrow \frac{\partial f}{\partial x}\left(-1, \frac{\pi}{3}\right) = \frac{\pi}{3}\cos\left(\frac{-\pi}{3}\right) = \frac{\pi}{6}$
- $\frac{\partial f}{\partial y} = x\cos(xy) \rightarrow \frac{\partial f}{\partial y}\left(-1, \frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$

Tangent plane: $z = -\frac{\sqrt{3}}{2} + \frac{\pi}{6}(x+1) - \frac{1}{2}\left(y - \frac{\pi}{3}\right)$

Higher order derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = f_{11} = D_{xx} f = \dots$$

↳ there are 4 second order derivatives: $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial y}$

→ if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Example: $\frac{\partial^2}{\partial x \partial y}(\sin(x+y)) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\sin(x+y)\right) = \frac{\partial}{\partial x}\cos(x+y)$

$= -\sin(x+y)$

$$\frac{\partial^2}{\partial y \partial x}(\sin(x+y)) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\sin(x+y)\right) = \frac{\partial}{\partial y}\cos(x+y) =$$

$$-\sin(x+y)$$

Chain rule in multiple dimensions

(coordinate transformations)

1D: $\frac{d}{dx}(f(g(x)) = f'(g(x)) \cdot g'(x)$

2D $f(x,y)$, $x(t)$ and $y(t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$f(x,y)$ $x(r,\theta)$ and $y(r,\theta)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

Example: $z(x,y) = \dfrac{1}{(x+y)^2}$, $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{-2}{(x+y)^3}\left(\frac{\partial x}{\partial r} + \frac{\partial y}{\partial r}\right) = \frac{-2\,(\cos\theta+\sin\theta)}{r^3(\cos\theta+\sin\theta)^3} = \frac{-2}{r^3}\frac{1}{(\cos\theta+\sin\theta)^2}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{-2}{(x+y)^3}\left(\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\right) = \frac{-2}{r^3}\,\frac{-r\sin\theta + r\cos\theta}{(\cos\theta+\sin\theta)^3} = \frac{2}{r^2}\,\frac{\sin\theta-\cos\theta}{(\cos\theta+\sin\theta)^3}$$

- $z(x,y) = \sqrt{x^2+y^2}$, $x(t) = 2t$, $y(t) = 5-t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dt}$$

$$= \frac{2t\cdot 2 - (5-t)}{\sqrt{4t^2+(5-t)^2}} = \frac{5t-5}{\sqrt{5t^2-10t+25}}$$

$\longrightarrow$ if you write $z = \sqrt{x^2+y^2} = \sqrt{5t^2-10t+25}$, and compute $\frac{dz}{dt}$, you get the same result.

$$\frac{\partial^2}{\partial x \partial y}(\sin(xy)) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(\sin(xy))\right) = \frac{\partial}{\partial x}(x\cos(xy))$$

$$= \cos(xy) - xy\sin(xy)$$