Resit Linear Algebra DSAI.

Stefan: Q1, Q2, Q4, Q5, Q6
Marieke: Q3.

Question 1:

$p(-1)=1 \Rightarrow -a+b-c+d=1$
$p(0)=0 \Rightarrow d=0$
$p(1)=1 \Rightarrow a+b+c+d=1$
$p(2)=-1 \Rightarrow 8a+4b+2c+d=-1$

We need to solve this SLE.

We know $d=0$. So, we need to solve $\begin{cases} -a+b-c=1 \\ a+b+c=1 \\ 8a+4b+2c=-1 \end{cases}$

Adding the first two equations results in $2b=2 \Rightarrow b=1$.
So, we need to solve $\begin{cases} -a-c=0. \\ 8a+2c=-5 \end{cases} \Rightarrow a=-c \Rightarrow -8c+2c=-5$
$\Rightarrow -6c=-5 \Rightarrow c=5/6$ and $a=-5/6$.

So, $a=-5/6$
$b=1$
$c=5/6$
$d=0$.

Alternative sol. to solve the SLE (more time consuming):
Use row operations

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ 8 & 4 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \underset{R_3:\, R_3-8R_1}{\overset{R_2:\, R_2+R_1}{\sim}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & -4 & -6 & -7 & -9 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \overset{R_3:\, R_3+2R_2}{\sim}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & -6 & -3 & -5 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \underset{R_3 * -1/6}{\overset{R_2 * 1/2}{\sim}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1/2 & 5/6 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1: R_1-R_4 \\ R_2: R_2-R_4 \\ \sim \\ R_3: R_3-1/2 R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 5/6 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \overset{R_1:\, R_1-R_3}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1/6 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 5/6 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \overset{R_1:\, R_1-R_2}{\sim}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -5/6 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 5/6 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

So, $a=-5/6$
$b=1$
$c=5/6$
$d=0$.

Question 2:

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 2 & p+3 & 8 & 2 \\ 1 & 2 & p^2 & p \end{array}\right] \underset{R_3:\, R_3-R_1}{\overset{R_2:\, R_2-2R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & p-1 & 0 & -2 \\ 0 & 0 & p^2-4 & p-2 \end{array}\right]$$

There are three pivots if $p-1 \neq 0$ and $p^2-4 \neq 0$.
So, the solution is unique for all values of p apart from 1, 2 and -2.

Question 3:

$$\begin{vmatrix} 5 & 5 & 6 & 6 & 7 \\ 4 & 0 & 4 & 4 & 0 \\ 3 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 5 & 6 & 7 & 1 & 8 \end{vmatrix} = (-1)^{4+4}\cdot 2 \cdot \begin{vmatrix} 5 & 5 & 6 & 7 \\ 4 & 0 & 4 & 0 \\ 3 & 0 & 0 & 0 \\ 5 & 6 & 7 & 8 \end{vmatrix} = 2\cdot(-1)^{3+1}\cdot 3 \cdot \begin{vmatrix} 5 & 6 & 7 \\ 0 & 4 & 0 \\ 6 & 7 & 8 \end{vmatrix}$$

$$= 2\cdot 3\cdot(-1)^{2+2}\cdot 4 \cdot \begin{vmatrix} 5 & 7 \\ 6 & 8 \end{vmatrix} = 2\cdot 3\cdot 4\cdot(5\cdot 8 - 7\cdot 6) = 24\cdot(40-42) = -48.$$

Question 4:

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, as this is always a solution of $A\underline{x} = \underline{0}$.

$\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$, as $A\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = A\left(\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = A\begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} - A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \underline{b} - \underline{b} = \underline{0}$

Similary, $\begin{bmatrix} -5 \\ -3 \\ -1 \end{bmatrix}$ is also a solution.

Question 5:

No, because property (ii) doesn't hold since

$$T\left(2\cdot\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2^2 \\ 2^2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1^2 \\ 1^2 \end{bmatrix} = 2\cdot T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

(Note that property (i) also doesn't hold since

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3^2 \\ 3^2 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \end{bmatrix} \neq \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1^2 \\ 1^2 \end{bmatrix} + \begin{bmatrix} 2^2 \\ 2^2 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)$$)

Question 6:

(a) $|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = (4-\lambda)(4-\lambda) - 1 = \lambda^2 - 8\lambda + 15 = (\lambda-3)(\lambda-5)$

$|A - \lambda I| = 0 \iff (\lambda-3)(\lambda-5) = 0$. So, $\lambda_1 = 3$ and $\lambda_2 = 5$.

(b) $A - 3I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underset{R_2 : R_2 - R_1}{\sim} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

So, basis for $\lambda = 3$: $\underline{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Normalize $\underline{v}_1$: $\|\underline{v}_1\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$\underline{u}_1 = \frac{1}{\|\underline{v}_1\|}\underline{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

$A - 5I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \underset{R_2 : R_2 + R_1}{\sim} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \underset{R_1 * -1}{\sim} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

So, basis for $\lambda = 5$: $\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Normalize $\underline{v}_2$: $\|\underline{v}_2\| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\underline{u}_2 = \frac{1}{\|\underline{v}_2\|} \underline{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

So, $D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ and $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Question 7:

(a) Col A is a subspace of $\mathbb{R}^6$
Row A is a subspace of $\mathbb{R}^4$.
So, they cannot be orthogonal.
So, $\boxed{b}$

(b) True, because Nul A $\perp$ Row A.
So, $\boxed{a}$

(c) Col A is a subspace of $\mathbb{R}^6$
Nul A is a subspace of $\mathbb{R}^4$.
So, they cannot be orthogonal.
So, $\boxed{b}$

(d) Since A is a 6×4 matrix, A can have 0 free variables (when there is a pivot in every column). So, the smallest possible dimension of Nul A is 0.
So, $\boxed{a}$

(e) Since A is a 6×4 matrix, A has a most 4 pivot columns. So, the largest possible dimension of Col A is 4.
So, $\boxed{c}$

(f) $A\underline{u} = \begin{bmatrix} -2 \\ -13 \\ 11 \end{bmatrix} \neq \underline{0}$, so $\underline{u} \notin$ Nul A.

Since A is a 3×4 matrix, Col A is a subspace of $\mathbb{R}^3$. Hence, as $\underline{v} \in \mathbb{R}^4$, we know $\underline{v} \notin$ Col A.
So, $\boxed{d}$

(g) $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} = 0 \cdot 4 + 1 \cdot 2 + (-1)(-3) = 0 + 2 + 3 = 5.$
So, $\boxed{d}$

(h) $\frac{\underline{y} \cdot \underline{u}}{\underline{u} \cdot \underline{u}} \underline{u} = \frac{3 \cdot 2 + (-1) \cdot 1}{2 \cdot 2 + 1 \cdot 1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{5}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
So, $\boxed{a}$

(i) $A\begin{bmatrix}6\\7\\8\end{bmatrix} = A\left(2\cdot\begin{bmatrix}3\\4\\5\end{bmatrix} + (-1)\cdot\begin{bmatrix}0\\1\\2\end{bmatrix}\right) = 2\cdot A\begin{bmatrix}3\\4\\5\end{bmatrix} + (-1)\cdot A\begin{bmatrix}0\\1\\2\end{bmatrix}$

$= 2\cdot\begin{bmatrix}0\\1\\0\end{bmatrix} + (-1)\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\2\\0\end{bmatrix}$

So, $\boxed{b}$

(j) $1\cdot\begin{bmatrix}0\\1\\1\\1\end{bmatrix} + 4\cdot\begin{bmatrix}0\\0\\0\\1\end{bmatrix} + 1\cdot\begin{bmatrix}1\\1\\2\\0\end{bmatrix} = \begin{bmatrix}1\\2\\p\\5\end{bmatrix}$ So, we need $1\cdot 1 + 4\cdot 0 + 1\cdot 2 = p$.
$\iff p = 3$.

So, $\boxed{c}$

(k) The zero polynomial $p(t)=0$ does not belong to W. So, W is not a subspace of $\mathbb{P}_2$.
As for V:
* the zero polynomial $p(t)=0$ belongs to V.
* if $\underline{u}\in V$ and $\underline{v}\in V$, then $(\underline{u}+\underline{v})(1) = \underline{u}(1)+\underline{v}(1) = 0+0 = 0$.
  So, $\underline{u}+\underline{v}\in V$.
* if $\underline{u}\in V$ and $c\in\mathbb{R}$, then $(c\cdot\underline{u})(1) = c\cdot\underline{u}(1) = c\cdot 0 = 0$.
  So, $c\cdot\underline{u}\in V$.

Hence, V is a subspace of $\mathbb{P}_2$.
So, $\boxed{b}$

(l) $|A-\lambda I| = \begin{vmatrix}a-\lambda & -3\\ 2 & b-\lambda\end{vmatrix} = (a-\lambda)(b-\lambda)+6 = \lambda^2 + (-a-b)\lambda + ab + 6.$
So, $-a-b=0 \Rightarrow b=-a$.
$ab+6=-19 \Rightarrow a(-a)+6=-19 \Rightarrow -a^2+6=-19 \Rightarrow a^2=25 \Rightarrow$
$a=5$ or $a=-5$.
So, $\boxed{b}$.