Practice Exam Questions Week 5 **Linear Algebra**

SOLUTIONS.

1. Consider the following matrix A and vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 3 & -1 & -2 & -1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

a. Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are both eigenvectors of A. What are the corresponding eigenvalues?

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 3 & -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

So, $\underline{v}_1$ is an eigenvector of A with eigenvalue -1.

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 3 & -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

So, $\underline{v}_2$ is an eigenvector of A with eigenvalue 1.

b. Show that 0 is an eigenvalue of A.

Row 1 and row 2 of A are the same.
Hence, A is not invertible and thus 0 is an eigenvalue of A.

c. Compute a basis for the eigenspace of A for the eigenvalue 0.

$$A - \lambda I = A - 0 \cdot I = A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 3 & -1 & -2 & -1 \end{bmatrix} \begin{matrix} R_2 : R_2 - R_1 \\ R_3 : R_3 + R_1 \\ \sim \\ R_4 : R_4 - 3R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 2 \\ 0 & 2 & -2 & -4 \end{bmatrix} \begin{matrix} R_2 \longleftrightarrow R_4 \\ \sim \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & -2 & -4 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 : \tfrac{1}{2} * R_2 \\ \sim \\ R_3 : -1 * R_3 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 : R_3 - R_2 \\ \sim \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 : R_1 + R_2 \\ \sim \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{cases} x_1 = x_3 + x_4 \\ x_2 = x_3 + 2x_4 \end{cases} \quad \text{So, } \underline{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Hence, a basis for the eigenspace of $\lambda = 0$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

d. Compute a basis for Nul A.

Nul A = Nul $(A - 0 \cdot I)$ = eigenspace of $\lambda = 0$.
Hence, a basis for Nul A is also $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

e. Is the matrix A diagonalizable? If it is, determine a matrix M such that $M^{-1}AM$ is a diagonal matrix. If it is not, explain why not.

We found three distinct eigenvalues: $-1, 1, 0$. Moreover, eigenvalue 0 has multiplicity two. Therefore, the sum of the dimensions of the eigenspaces equals $n=4$.
Hence, A is diagonalizable.

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, $A = MDM^{-1}$ and thus $D = M^{-1}AM$.

f. Compute the matrix A^9.

$$A^9 = (MDM^{-1})^9 = MD^9M^{-1} = M \begin{bmatrix} (1)^9 & 0 & 0 & 0 \\ 0 & (-1)^9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} M^{-1}$$

$$= M \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} M^{-1} = MDM^{-1} = A.$$

So, $A^9 = A$.

2. True or false? If the given statement is true, give a brief explanation. If it is false, give a counterexample.

a. If $A = QR$ and Q is invertible, then A is similar to $B = RQ$;

True. $A = QR \Rightarrow Q^{-1}A = Q^{-1}QR \Rightarrow Q^{-1}A = R \Rightarrow Q^{-1}AQ = \underbrace{RQ}_{B}$
Hence, A is similar to $B = RQ$.

b. An elementary row operation on A does not change the determinant of A.

False. Consider $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ with $\det A = 2$

Moreover, $A \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. and $\det I = 1$.

So, the elementary row operation did change the determinant of A.

c. If λ is an eigenvalue of A, then it is also an eigenvalue of A^T

True. For finding eigenvalues of A, we need to solve $\det(A^T - \lambda I) = 0$.

Note that $\det(A^T - \lambda I) = \det(A^T - \lambda I^T) = \det((A - \lambda I)^T) = \det(A - \lambda I)$.

Since $\det(A-\lambda I)$ gives the eigenvalues of A, and because $\det(A^T-\lambda I) = \det(A-\lambda I)$, the eigenvalues of A and A^T are the same

d. Each eigenvalue of A is also an eigenvalue of A^2.

False. Consider $A=\begin{bmatrix}2&0\\0&2\end{bmatrix}$ with eigenvalues 2 and 2.

$A^2=\begin{bmatrix}4&0\\0&4\end{bmatrix}$ with eigenvalues 4 and 4.

e. If M is a (2×2) matrix such that dim Nul A equals 1, then M has one eigenvalue equal to 0.

False. Consider $M=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ with dim Nul $A=1$, but with two eigenvalues equal to zero.

f. Let B be an $(n \times n)$ matrix. Let $\mathbf{e}_1$ be the first column of the identity matrix I_n. If $\mathbf{e}_1$ is an eigenvector of B with eigenvalue 1, then the first column of B is $\mathbf{e}_1$.

True. $\underline{e}_1$ is an eigenvector of B with eigenvalue 1, so $B\underline{e}_1=\underline{e}_1$. Note that multiplying each row of B with the vector $\underline{e}_1$ has the effect of picking up the first entry in each row. Hence, equivalently, it selects the first column of B. So, $B\underline{e}_1=\underline{b}_1$. Hence, $\underline{b}_1=\underline{e}_1$.

g. Any invertible matrix can also be diagonalized.

False. Consider $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$. $\det(A)=1\neq 0$, so A is invertible.

Eigenvalues of A are: 1 and 1.

$A-1\cdot I=\begin{bmatrix}0&1\\0&0\end{bmatrix}$. Hence, the dimension of the eigenspace of $\lambda=1$ is 1.

Hence, the sum of the dimensions of the eigenspaces equals $1<2=n$. And thus A is not diagonalizable.

h. If A and B are 2×2 matrices which can both be diagonalized, then their sum $C = A + B$ can also be diagonalized.

False. Consider $A=\begin{bmatrix}0&0\\0&1\end{bmatrix}$ and $B=\begin{bmatrix}1&1\\0&0\end{bmatrix}$.

Both A and B have distinct eigenvalues and therefore they are diagonalizable.

However $C=A+B=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ is not diagonalizable. (see g.)

i. If K and L are 3×3 matrices and $\mathbf{v}$ is an eigenvector of K and also of L, then $\mathbf{v}$ is an eigenvector of the matrix product KL.

True. $\underline{v}$ is an eigenvector of K, so $K\underline{v}=\lambda\underline{v}$ for some λ in $\mathbb{R}$.
$\underline{v}$ is an eigenvector of L, so $L\underline{v}=\beta\underline{v}$ for some β in $\mathbb{R}$

Then, $KL\underline{v}=K(\beta\underline{v})=\beta K\underline{v}=\underbrace{\beta\lambda}_{=\gamma}\underline{v}=\gamma\underline{v}$ with γ in $\mathbb{R}$.

Therefore, $\underline{v}$ is also an eigenvector of KL with eigenvalue $\gamma=\beta\lambda$.

j. If $\mathbf{x}$ is an eigenvector of an invertible matrix P, then it is also an eigenvector of P^{-1}

True. $\underline{x}$ is an eigenvector of P, so $P\underline{x} = \lambda \underline{x}$ where $\underline{x} \neq \underline{0}$

$\Rightarrow P^{-1}P\underline{x} = P^{-1}(\lambda \underline{x}) \Rightarrow I\underline{x} = \lambda(P^{-1}\underline{x}) \Rightarrow \underline{x} = \lambda(P^{-1}\underline{x})$.

Since P is invertible, we know $\lambda \neq 0$ and thus we can multiply with $\frac{1}{\lambda}$.

Hence, $P^{-1}\underline{x} = \frac{1}{\lambda}\underline{x}$.

Therefore, $\underline{x}$ is also an eigenvector of P^{-1}, with eigenvalue $\frac{1}{\lambda}$.

3. Consider the following matrix Q:

$$Q = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Find the characteristic polynomial and the eigenvalues of Q.

$$\det(Q - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ -3 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\cdot(-1)^{3+3}\cdot\begin{vmatrix} -1-\lambda & 0 \\ -3 & 4-\lambda \end{vmatrix}$$

$= (2-\lambda)(-1-\lambda)(4-\lambda)$

The characteristic equation is $(2-\lambda)(-1-\lambda)(4-\lambda) = 0$.

Hence, the eigenvalues of Q are 2, -1 and 4.