Logic in Action

Chapter 8: Validity Testing

http://www.logicinaction.org/

For the predicate logic case

The tableau method can be used also to decide the validity of inferences in predicate logic.

We already know how to deal with logical connectives ($\neg, \wedge, \vee, \rightarrow, \leftrightarrow$).

We just need to know how to deal with **quantifiers** ($\exists, \forall$).

Quantifiers (1)

$\exists$	$\exists x\varphi(x) \circ$ $\vert$ $\varphi(a) \overset{+}{\circ}$ **For a new a**	$\circ\ \exists x\varphi(x)$ $\vert$ $\circ\ \varphi(a_1),\ldots,\varphi(a_n)$ **For all existing $a_1,\ldots,a_n$**
$\forall$	$\forall x\varphi(x) \circ$ $\vert$ $\varphi(a_1),\ldots,\varphi(a_n) \circ$ **For all existing $a_1,\ldots,a_n$**	$\circ\ \forall x\varphi(x)$ $\vert$ $\overset{+}{\circ}\ \varphi(a)$ **For a new a**

Existential claims:	$\exists x\varphi(x) \circ$	$\circ\ \forall x\varphi(x)$
Universal claims:	$\circ\ \exists x\varphi(x)$	$\forall x\varphi(x) \circ$

Quantifiers (2)

What if we have a universal claim, but no names?

- We add a new element (because we do not allow empty domains).

$\forall x \varphi(x) \;\circ$	$\circ \; \exists x \varphi(x)$
$\vert$	$\vert$
$\varphi(a) \; \overset{+}{\circ}$	$\overset{+}{\circ} \; \varphi(a)$
For a new $\boldsymbol{a}$	**For a new** $\boldsymbol{a}$

Quantifiers (3)

Important observation.

- Every time a new name is introduced ($\overset{+}{\circ}$), we should **reactivate** every previous universal claim.

Recommendations

When working with predicate tableau, try to follow this order:

1. Work with logical connectives ($\neg, \wedge, \vee, \rightarrow, \leftrightarrow$).
2. Then, when working with existential claims.
3. Finally work with universal claims.

To practice

Which of the following statements are true?

- $\forall x(Px) \models \neg\exists x(\neg Px)$
- $\neg\exists x(Px) \models \forall x(\neg Px)$
- $\forall x\exists yRxy \models \forall xRxx$
- $\forall x\forall yRxy \models \forall xRxx$
- $\forall x\forall yRxy, Rab \models Raa$
- $\forall x(Px \rightarrow Qx) \vee \forall y(Qy \rightarrow Py) \models \forall x\forall y((Px \wedge Qy) \rightarrow (Qx \vee Py))$
- $\forall xPx \rightarrow \forall xQx \models \forall x(Px \rightarrow Qx)$
- $\forall x(Px \rightarrow Qx) \models \forall xPx \rightarrow \forall xQx$
- $\exists y\forall xRxy \models \forall x\exists yRxy$
- $\forall x(Px \rightarrow Qx), \exists x(Px \wedge Rx) \models \exists x(Qx \wedge Rx)$
- $\forall x(Px \rightarrow Qx), \exists x(\neg Px \wedge Rx) \models \exists x(\neg Qx \wedge Rx)$
- $\neg\exists x(Px \wedge Qx), \forall x(Qx \rightarrow Rx) \models \neg\exists x(Px \wedge Rx)$
- $\forall x(Px \rightarrow Qx), \forall x(Qx \rightarrow Rx), \forall x(Rx \rightarrow Px) \models \forall x(Qx \wedge Px)$

Can we always find a counterexample?

Consider the following inference

$$\frac{\forall y \exists x Rxy}{\exists y \forall x Rxy}$$

- What does the inference says?
- Is it valid?
- Can you find a couterexample without using the tableau method?
- Can you find a couterexample with the tableau method?

What can we do?

The problem

- For existential claims, we always introduce a new name.
- But maybe one of the previous names is useful.

The solution

- For existential claims, we will now consider the possibility of a previous name being the adequate one.

Extended rules for existential claims

$\exists x \varphi(x) \ \circ$	$\circ \ \forall x \varphi(x)$
$\varphi(a_i) \ \circ \qquad \varphi(a) \ \overset{+}{\circ}$	$\circ \ \varphi(a_i) \qquad \overset{+}{\circ} \ \varphi(a)$
For an existing a_i and a new a	**For an existing a_i and a new a**

What happen now with $\dfrac{\forall y \exists x Rxy}{\exists y \forall x Rxy}$?

Can we always find a counterexample?

Consider the following inference

$$\frac{\forall y \exists x Rxy, \forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)}{\exists x \exists y (Rxy \wedge Ryx)}$$

- What does the inference says?
- Is it valid?
- Can you find a couterexample without using the tableau method?
- Can you find a couterexample with the tableau method?

What can we do?

The problem

- The tablea method tries to build counterexamples step by step, introducing at most one new name at each step.
- Hence, every model we built is **finite**.
- There are invalid inferences whose counterexamples are **infinite** models.

Important observations

1. The **tableau** method attempts to build a model (domain and relations) with the specified requirements.
2. The presented **tableau** method is **complete** for **proving validity** in **predicate** logic: if an inference with predicate formulas is valid, then its tableau will be closed.
3. The presented **tableau** method is **not complete** for **finding counterexamples** in **predicate** logic: if an inference with predicate formulas is not valid *and its counterexamples is an infinite model*, the tableau will not find it.
4. The presented **tableau** method **cannot** generate **every counterexample** of an invalid inference in **predicate** logic.