#### Nico Roos

Data Science and Knowledge Engineering, Maastricht University





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#### Outline

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Hoare Logic and PDL

Inference rules for Hoare Logic

Loop Invariants and Program Correctness

Practical use of Hoare Logic





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Hoare Logic can be viewed as a predecessor of PDL specially aimed at proving program correctness.





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If the pre-condition  $\varphi$  holds before executing  $\alpha$ , then the post-condition  $\psi$  holds after executing  $\alpha$ .



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$$\{x = 1, y = 5\}$$
 x = x + 1  $\{x = 2, y = 5\}$  is a correct statement.





# Hoare Logic and PDL

A triple

$$\{\varphi\} \ \alpha \ \{\psi\}$$

in Hoare Logic is equivalent to

$$\varphi \to [\alpha]\psi$$

in PDL





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Standard Hoare Logic can only provide partial correctness proofs. We cannot say anything about the termination of a block of code  $\alpha$ , while in PDL, we can:

$$\varphi o \langle \alpha \rangle \top$$



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Note that PDL does not and cannot provide a solution for the halting problem. 



#### Partial correctness example

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$$\{x = 1\}$$
 while  $(x > 0)$   $\{x = x + 1\}$   $\{x = 0\}$ 

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$$\{x = 1\}$$
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is a correct statement.

The execution never terminates if the precondition x = 1 holds before the execution of the while-loop.





# Auxiliary variables

We use auxiliary variables to remember the values of variables before the execution of a block of code.

$$\{x = u, y = v\} x = x + 1 \{x = u + 1, y = v\}$$





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auxiliary variables never change their value!





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$$\frac{\varphi[e/x]}{x = e \{\varphi\}}$$

Here e denotes an expression and  $\varphi[e/x]$  denotes that the expression e is substituted for the variable x in  $\varphi$ .

The substitution in the pre-condition guarantees that the expression e is evaluated using the vules of the variables before executing the assignment x = e.





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For example:

$$\{(x+1)=2\}$$
 x = x + 1  $\{x=2\}$ 



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The substitution in the pre-condition guarantees that the expression e is evaluated using the vules of the variables before executing the assignment x = e.

For example:

$$\{(x+1) = 2\} x = x + 1 \{x = 2\}$$
  
 $\Rightarrow \{x = 1\} x = x + 1 \{x = 2\}$ 





#### Another example:

$$\{(x + y) = y + 2\} x = x + y \{x = y + 2\}$$





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$$\{(x + y) = y + 2\} x = x + y \{x = y + 2\}$$
  
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 $\Rightarrow \{x = 2\} x = x + y \{x = y + 2\}$ 

Note that practically, we substitute x = 2 for x on the righthand side of the assignment x = x + y and next interpret the assignment symbol = as the mathematical equal symbol =.





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The rule:

$$\overline{\{\varphi\} \mathbf{x} = \mathbf{e} \{\varphi[\mathbf{e}/\mathbf{x}]\}}$$

can give the following incorrect result:

$$\{x = 0\} x = 1 \{1 = 0\}$$



### Consequence rule

$$\frac{\varphi' \to \varphi, \{\varphi\} \ \alpha \ \{\psi\}, \psi \to \psi'}{\{\varphi'\} \ \alpha \ \{\psi'\}}$$

The rule states that we can strengthen the pre-condition and that we can weaken the post-condition.





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Loop Invariants and Program Correctness

The rule states that we can strengthen the pre-condition and that we can weaken the post-condition.

For example, from:

$$\{x = 1\} x = x + 1 \{x = 2\}$$

we can derive:

$$\{x = 1, y = 5\} x = x + 1 \{x > 1\}$$



## Sequence rule

$$\frac{\{\varphi\} \ \alpha \ \{\psi\}, \{\psi\} \ \beta \ \{\eta\}}{\{\varphi\} \ \alpha; \beta \ \{\eta\}}$$





### Sequence rule

$$\frac{\{\varphi\} \ \alpha \ \{\psi\}, \{\psi\} \ \beta \ \{\eta\}}{\{\varphi\} \ \alpha; \beta \ \{\eta\}}$$

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Loop Invariants and Program Correctness

For example, from:

$$\{x = 1\} x = x + 1 \{x = 2\}$$
  
 $\{x = 2\} x = x + 1 \{x = 3\}$ 

we can derive:

$$\{x = 1\} x = x + 1; x = x + 1 \{x = 3\}$$



#### Conditional rule

$$\frac{\{\varphi \land \theta\} \ \alpha \ \{\psi\}, \{\varphi \land \neg \theta\} \ \beta \ \{\psi\}}{\{\varphi\} \ \text{if} \ (\theta) \ \{ \ \alpha \ \} \ \text{else} \ \{ \ \beta \ \} \ \{\psi\}}$$





#### Conditional rule

$$\frac{\{\varphi \land \theta\} \ \alpha \ \{\psi\}, \{\varphi \land \neg \theta\} \ \beta \ \{\psi\}}{\{\varphi\} \ \text{if} \ (\theta) \ \{ \ \alpha \ \} \ \text{else} \ \{ \ \beta \ \} \ \{\psi\}}$$

For example, from:

$$\{x \ge y\} z = x \{z = max(x, y)\}$$
  
 $\{x < y\} z = y \{z = max(x, y)\}$ 

we can derive:

$$\{ \} \text{ if } (x \ge y) \ \{ z = x \} \text{ else } \{ z = y \} \{ z = \max(x,y) \}$$





#### While rule

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$$\frac{\{\phi \wedge \psi\} \; \alpha \; \{\psi\}}{\{\psi\} \; \text{while } (\phi) \; \{\; \alpha \; \} \; \{\neg \phi \wedge \psi\}}$$





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$$\frac{\{\phi \wedge \psi\} \ \alpha \ \{\psi\}}{\{\psi\} \ \text{while} \ (\phi) \ \{\ \alpha\ \} \ \{\neg \phi \wedge \psi\}}$$

For example, using the invariant:

$${r = x \times i} r = r + x; i = i + 1 {r = x \times i}$$

we can derive:

$${r = x \times i}$$
 while (i != y)  ${r = r + x; i = i + 1} {r = x \times y}$ 





#### While rule

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we can derive:

$${r = x \times i}$$
 while (i != y)  ${r = r + x; i = i + 1} {r = x \times y}$ 

We can guarantee the pre-condition with the assignments:

$$r = 0; i = 0$$



# Conjunction and Disjunction rule

$$\frac{\{\varphi\} \ \alpha \ \{\psi\}, \{\varphi\} \ \alpha \ \{\psi'\}}{\{\varphi\} \ \alpha \ \{\psi \land \psi'\}}$$





# Conjunction and Disjunction rule

$$\frac{\{\varphi\} \alpha \{\psi\}, \{\varphi\} \alpha \{\psi'\}}{\{\varphi\} \alpha \{\psi \wedge \psi'\}}$$

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Loop Invariants and Program Correctness

$$\frac{\{\varphi\} \alpha \{\psi\}, \{\varphi'\} \alpha \{\psi\}}{\{\varphi \lor \varphi'\} \alpha \{\psi\}}$$



# Conjunction and Disjunction rule

$$\frac{\{\varphi\} \alpha \{\psi\}, \{\varphi\} \alpha \{\psi'\}}{\{\varphi\} \alpha \{\psi \wedge \psi'\}}$$

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Loop Invariants and Program Correctness

$$\frac{\{\varphi\} \ \alpha \ \{\psi\}, \{\varphi'\} \ \alpha \ \{\psi\}}{\{\varphi \lor \varphi'\} \ \alpha \ \{\psi\}}$$

These rules are redundant but convenient to have when proving the correctness of an algorithm.



A vase contains 35 white pebbles and 35 black pebbles. Proceed as follows to draw pebbles from the vase, as long as this is possible.

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Loop Invariants and Program Correctness

- Every round, draw two pebbles from the vase.
- If they have the same colour, then put a black pebble into the vase (you may assume that there are enough additional black pebbles outside of the vase).
- If they have different colours, then put the white pebble back.

In every round one pebble is removed from the vase, so after 69 rounds there is a single pebble left. What is the colour of this pebble?

How to answer this? Do we even have enough information?



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The effect of drawing two pebbles:

Pebbles Drawn	White Pebbles	Black Pebbles
white white	-2	+1
white black	0	-1
black white	0	-1
black black	0	-1

Notice that the following properties hold:

- Initially, the number of white pebbles is odd.
- If the number of white pebbles is odd before drawing, then the number of white pebbles is odd after drawing.

Hence: if there is one pebble left, it must be white, for if it were black, there would have been an even number of white pebbles.





### **Loop Invariants**

#### The property $\psi$ :

 $\psi = \mathit{The}\ \mathit{number}\ \mathit{of}\ \mathit{white}\ \mathit{pebbles}\ \mathit{is}\ \mathit{odd}$ 

is called a *loop invariant* for this scenario.

#### Characteristics of $\psi$ :

- $\blacktriangleright$   $\psi$  holds initially (before executing the loop)
- $\blacktriangleright$  if  $\psi$  holds before executing the loop body then  $\psi$  holds after executing the loop body
- $\blacktriangleright$   $\psi$ , together with the fact that the loop has ended, implies what we want to prove (i.e., the last pebble is white)





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Consider the following function that computes the square of n:

```
int square (int n) {
  int x = 0;
  int k = 0;
  while (k < n) {
    x = x + 2*k + 1;
    k = k + 1;
  return x;
```

How do we prove that it indeed returns  $n^2$ ?





We need to find the loop invariant  $\psi$  such that:

- $\blacktriangleright$   $\psi$  is true before the *while* loop starts.
- The following Hoare triple is valid:

$$\{(k < n) \land \psi\} \ x = x + 2*k + 1; \ k = k + 1; \ \{\psi\}$$
 (1)

 $\neg (k < n) \land \psi$  implies  $x = n^2$ , so that we can use (1) together with the rule

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$$\frac{\{\phi \wedge \psi\} \ \alpha \ \{\psi\}}{\{\psi\} \ \text{while} \ (\phi) \ \{\ \alpha\ \} \ \{\neg \phi \wedge \psi\}}$$

to derive

$$\{\psi\}$$
 while (k < n) do x = x + 2\*k + 1; k = k + 1;  $\{\neg(k < n) \land \psi\}$ 

which then implies that  $x = n^2$  after executing the loop.



Invariant:  $\psi = (x = k^2) \land (k < n)$ 

Justification:

- $\blacktriangleright$   $\psi$  is true after int x = 0; int k = 0;
- The following Hoare triple is true

$$\{k < n \land \psi\} \ x = x + 2*k + 1; \ k = k + 1; \ \{\psi\}$$

We can derive:

$$\{\psi\}$$
 while (k < n) do  $\{ x = x + 2*k + 1; k = k + 1; \} \{(k = n) \land \psi\}$ 

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Consider the following function that computes the sum of the elements in an array:

```
int sum(int[] a) {
  int s = 0;
  int i = 0;
 while (i != a.length) {
    s = s + a[i];
    i = i + 1;
  return s;
```

Prove that the algorithm returns sum of the elements in the array.



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Loop invariant:  $\psi = (s = \sum_{n \in \{0,...,i-1\}} a[n])$ .

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$$\{(i < a.length) \land \psi\}$$
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From this we can derive:

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$$s = \sum_{n \in \{0, \dots, a.length-1\}} a[n].$$



Consider the following function that returns the factorial of *n*:

```
int Factorial (int n) {
  int f = 1;
  int i = 1;
  while (i < n) {
    i = i + 1;
    f = f * i;
  return f;
```

Prove that the algorithm returns *n*!.



Loop invariant:  $\psi = (f = i!) \land i \leq n$ 

Justification:

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Loop Invariants and Program Correctness





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From this we can derive:

$$\{\psi\}$$
 while (i < n) do  $\{$  i = i + 1; f = f \* i;  $\}$   $\{\neg(i < n) \land \psi\}$ 

- $\neg (i < n) \land \psi$  implies f = n!.
- Note that here we need  $i \leq n$  as a conjunct of  $\psi$ , since  $i \leq n$ together with  $\neg (i < n)$  implies i = n.



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Hoare logic can be used to prove partial program correctness.





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Practical use of Hoare Logic

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Note that full correctness requires addressing the termination of a program, which is not address in standard Hoare Logic.

There are tools that support checking the correctness of a Java program. See for instance the Java Modeling Language (JML).





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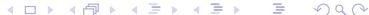
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Programming language such as Eiffel support design by contract.

The Java Modeling Language (JML) enable the use of design by contract in Java programming.





## Loop invariants

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Specifying loop invariants for (every) loop in your program is a pragmatic way to improve correctness of your program





Practical use of Hoare Logic

## Loop invariants

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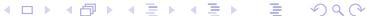


Practical use of Hoare Logic

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```
{n > 1}
                           ← pre-condition
int Factorial (int n) {
  int f = 1;
  int i = 1;
  \{1 < i \land i \le n \land f = i!\} \leftarrow \text{loop invariant}
  while (i < n) {
    i = i + 1;
    f = f * i;
  return f;
\{f = n!\}
                            ← post-condidtion
Note that i! == (i-1)! * i.
```





```
Formulated in JML:
/*0 requires 1 <= n;
ensures \result == (\product int i; 1<=i && i<=n; i);</pre>
0*/
int Factorial (int n) {    int f = 1;
  int i = 1;
  /*@ loop_invariant i<=n &&
  f==(\product int j; 1<=j && j<=i; j); @*/
  while (i < n) {
    i = i + 1;
    f = f * i;
    return f;
```

# Sorting example

```
public class Sort {
  public int[] a;
  /*@ public normal_behavior
    @ requires a.length > 0 && 0<= start && start < a.length;</pre>
    @ ensures (\forall int i; start<=i && i<a.length;a[\result] >= a[i]);
    @ ensures start <= \result && \result < a.length;</pre>
    0*/
  int /*@ strictly_pure @*/ max(int start) {
    int counter = start;
    int idx = start:
    /*@ loop_invariant start<=counter && counter<=a.length &&
      @ start<=idx && idx<a.length && start<a.length &&
      @ (\forall int x; x>=start && x<counter; a[idx]>=a[x]);
      @ assignable \strictly_nothing;
      @ decreases a.length - counter;
      0*/
    while (counter < a.length) {</pre>
      if (a[counter] > a[idx])
      idx = counter;
      counter = counter+1;
    return idx;
```



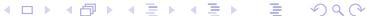


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```
/*@ public normal_behavior
  @ requires a.length > 0;
  \mathbb{Q} ensures (\forall int i; 0 \le i \&\& i \le a.length-1; a[i] >= a[i+1]);
  0*/
void sort() {
int pos = 0;
int idx = 0;
/*@ loop_invariant 0<=pos && pos<=a.length && 0<=idx && idx<a.length
  0 \&\& (\forall x = 0 \&\& x = 1; a[x] = a[x+1]) \&\&
  0 \pmod{0} == (\text{y}); y = pos && y < a.length; <math>a[pos-1] >= a[y]);
  @ assignable a[*];
  @ decreases a.length - pos;
  0*/
while (pos < a.length-1) {
  idx = max(pos);
  int tmp = a[idx];
  a[idx] = a[pos];
  a[pos] = tmp;
  pos = pos+1;
```





#### Tools

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The loop invariants, as well as pre- and post conditions, and class invariants can be checked:

- during the execution of a program; i.e. runtime testing,
- before execution of a program; i.e., automated program verification.





#### **Tools**

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The loop invariants, as well as pre- and post conditions, and class invariants can be checked:

- during the execution of a program; i.e. runtime testing,
- before execution of a program; i.e., automated program verification.

Loop invariants, pre- and post conditions, and class invariants can be specified for Java programs as special comments in JML.





Practical use of Hoare Logic

#### **Tools**

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A program with loop invariants, pre- and post conditions, and class invariants:

- can run as a normal Java program; i.e., the special comment are ignored,
- can run as a Java program that performs runtime testing, for instance: jmlrac,
- can be verified at compile time, for instance: ESC/Java2.



