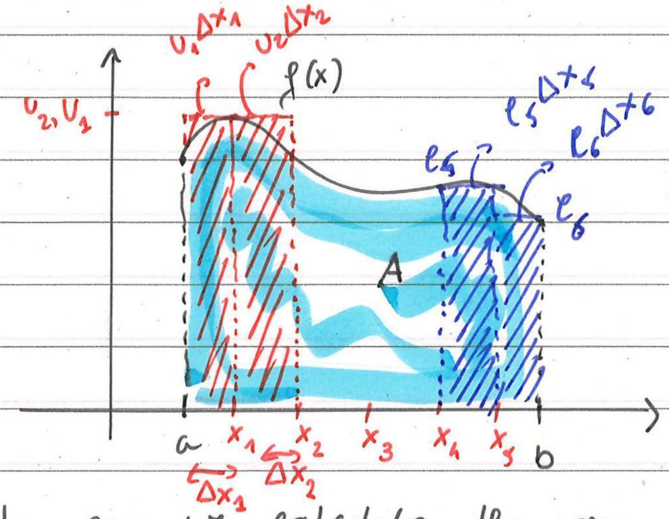
CALCULUS lecture 5 : INTEGRATION

1) Definite integrals - area under a graph
2) Indefinite integrals - antiderivatives
3) Fundamental theorem of Calculus - connection between (anti)derivatives & area.

I Area's as Riemann sums (Adams 5.2 - 5.3)



$f(x)$ is a continuous function on $[a,b]$

- How can we calculate the area A below $f(x)$ and the x-axis?
  ↳ can we find upper and lower boundaries for A?

- Method : we divide $[a,b]$ into sub-intervals $a = x_0 < x_1 < \dots < x_n = b$
  ↳ this is a "partition" of $[a,b]$ $\Delta x_k = x_k - x_{k-1}$

note : this "partition" is not the same as in your Discrete Math.

  - on each sub-interval $[x_{k-1}, x_k]$, f has a maximum u_k / a minimum ℓ_k

  - the sum of the areas of the rectangles $u_k \Delta x_k$ is an upper bound
    $A \leq U(f,P)$. This is an "upper Riemann sum"

    $$\underbrace{U}_{\text{upper Riemann sum}}(\underbrace{f}_{\text{function}}, \underbrace{P}_{\text{partition}}) = \sum_{k=1}^{n} \Delta x_k u_k$$

    upper Riemann sums are always above the curve.

  - the sum of the areas of the rectangles $\ell_k \Delta x_k$ is a lower bound
    $L(f,P) = \sum_{k=1}^{n} \Delta x_k \ell_k \leq A$. This is a "lower Riemann sum"
    lower Riemann sums are always below the curve.

- As we add more points to the partition, i.e. if $\|P\| = \max(|\Delta x_k|) \to 0$, $n \to \infty$
the Riemann sums converge to the area A
(for integrable functions)

- A function f is integrable on $[a,b]$ if there is exactly one A, such that, for every partition P, $L(f,P) \leq A \leq U(f,P)$.
In that case, $A = \int_a^b f(x)\,dx$ (definition of definite integral)

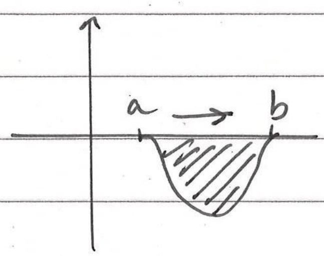
↳ a definite integral is defined as the area under the graph
↳ all Riemann sums (not only upper & lower sums) converge for integrable functions : $\lim\limits_{\substack{\|P\| \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx$
with $c_i \in [x_{i-1}, x_i]$

* all continuous and piecewise continuous (finite number of discontinuities) are integrable.

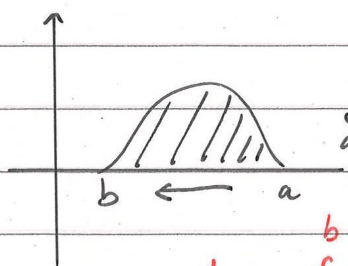
- terminology : $\int_a^b f(x)\,dx$
  - b → upper integration limit b
  - x → integration variable x
  - dx → differential dx
  - f(x) → integrand
  - a → lower integration limit a

II Properties of definite integrals (Adams, 5.4)

* $\int_a^b f(x)\,dx$ is a NUMBER (depends on a and b, not on x)
can be positive or negative (in contrast to area's)



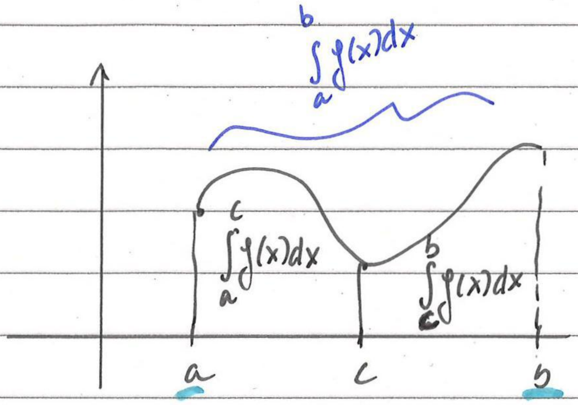
$\int_a^b f(x)\,dx < 0$ (area below x-axis
$f(x) \leq 0$ on $[a,b]$)



$\int_a^b f(x)\,dx < 0$ (dx < 0, negative direction of integration)

↳ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

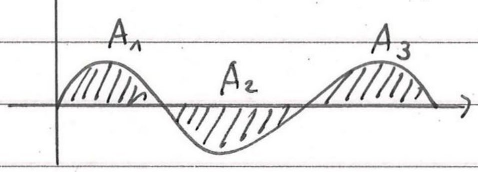
- we can break up an integral : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



this property is mainly useful for piecewise continuous functions and absolute values.

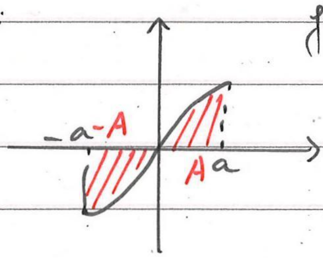
- integration is linear : $\int_a^b (A f(x) + B g(x))\,dx = A\int_a^b f(x)\,dx + B\int_a^b g(x)\,dx$
- triangle inequality : $\int_a^b |f(x)|\,dx \geqslant \left|\int_a^b f(x)\,dx\right|$

$A_1 + A_2 + A_3$ $\qquad$ $|A_1 - A_2 + A_3|$



- for an odd function :

$$\int_{-a}^{a} f(x)\,dx = 0$$



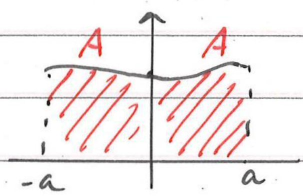
$f(x) = -f(-x)$

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$

$$= -A + A$$

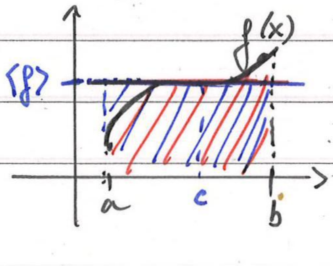
for an even function

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$



- average value of a function (definition) : $\langle f\rangle = \bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$
over $[a,b]$

↳ note : at some $c \in [a,b]$, $f(c) = \langle f\rangle$ (mean value theorem for integrals)



the average $\langle f\rangle$ is the value, such that the blue rectangle area $\langle f\rangle \cdot (b-a)$ is equal to $\int_a^b f(x)\,dx$

III Anti-derivatives (Adams' 2.10) - indefinite integrals

$$\int f(x)\,dx = F(x) + C \iff \frac{d}{dx}(F(x)) = f(x)$$

C: integration constant

$\int f(x)\,dx$: indefinite integral

↳ the indefinite integral is defined as the reverse operation of derivation.

↳ since $\frac{d}{dx}(c) = 0$, it is only defined up to an integration constant.

* Definite integrals are numbers, indefinite integrals are functions

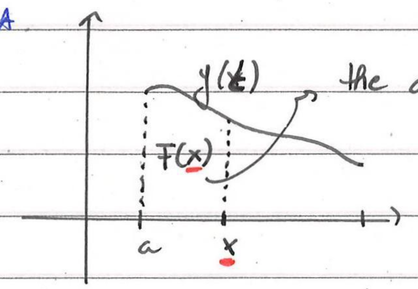
Examples: $\int \sin(x)dx = -\cos(x) + C$, $\int e^x dx = e^x + C$, $\int \frac{dx}{x} = \ln|x| + C$
(see lecture 6 for more)

IV Fundamental theorem of Calculus

For a continuous function $f(x)$ on an interval I, $a \in I$

- let $F(x) = \int_a^x f(t)\,dt$, $x \in I$
  then $F(x)$ is differentiable, and $F'(x) = f(x)$
- if $G'(x) = f(x)$, for a function $G(x)$ on I, then
  $\forall b \in I$: $\int_a^b f(x)\,dx = G(b) - G(a)$

↳ the fundamental theorem of Calculus relates definite integrals (area's below the graph) with indefinite integrals (anti-derivatives)



the area is a function of the integration limits

* sketch of the proof (not exam material)

1) $F'(x) = \lim_{h \to 0} \frac{1}{h}(F(x+h) - F(x))$ (definition of derivative)

$= \lim_{h \to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right)$ (definition of $F(x)$)

$$\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt \qquad \left(\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt\right)$$

$$= \int_x^{x+h} f(t)\,dt \qquad \left(\int_a^c f(t)\,dt + \int_c^b f(t)\,dt = \int_a^b f(t)\,dt\right)$$

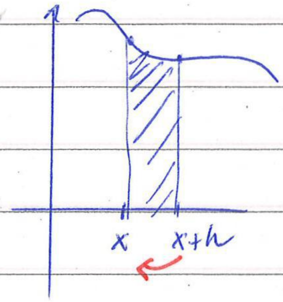
$$\Rightarrow F'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}} (\cancel{h} \cdot f(c)) \quad \text{with } c \in [x, x+h]$$

$f(c) = \langle f \rangle$

(average over $[x, x+h]$)

$$= f(x) \qquad (c \in [x, x+h] \to \text{as } h \to 0,\ c \to x)$$



as $h \to 0$, the area goes to $f(x) \cdot h$

2) $G'(x) = f(x) = F'(x)$

$\Rightarrow G'(x) - F'(x) = (G-F)'(x) = 0$ on I

$\Rightarrow (G-F)(x)$ is constant on I

$\Rightarrow G(x) = F(x) + C$

$\hookrightarrow G(a) = F(a) + C = \underbrace{\cancel{\int_a^a f(t)\,dt}}_{0} + C = C$

$\hookrightarrow G(b) = F(b) + C$

$= \int_a^b f(t)\,dt + G(a)$

$$\Rightarrow \int_a^b f(t)\,dt = G(b) - G(a) \quad \blacksquare$$