Systems of linear equations

$$\begin{cases} 2x_1 + x_2 = 5 \\ 3x_1 - 2x_2 = 10 \end{cases} \longrightarrow \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 3 & -2 & 10 \end{array}\right]$$

coef. matrix — augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 3 & -2 & 10 \end{array}\right] \xrightarrow{R_1 = 2R_1} \left[\begin{array}{cc|c} 2 & 2 & 10 \\ 3 & -2 & 10 \end{array}\right] \xrightarrow{R_1 = R_1 + R_2} \left[\begin{array}{cc|c} 5 & 0 & 20 \\ 3 & -2 & 10 \end{array}\right]$$

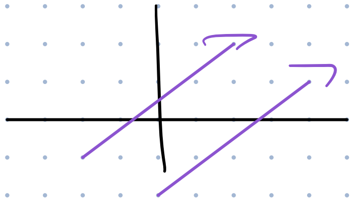
$$\xrightarrow{\frac{1}{5}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 3 & -2 & 10 \end{array}\right] \xrightarrow{R_2 = R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & -2 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 = -\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right] \qquad \begin{array}{l} x_1 = 4 \\ x_2 = 1 \end{array}$$

Types of systems

Inconsistent

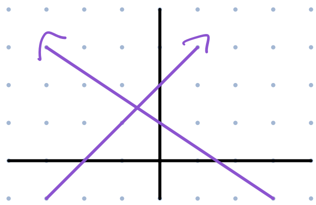
↳ NO SOLUTION

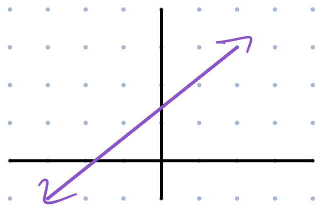


Consistent

↳ 1 SOLUTION

↳ INFINITE SOLUTIONS





Exercise

$$\left[\begin{array}{ccc|c} 1 & -2 & +1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & +5 & +9 & -9 \end{array}\right] \xrightarrow{R_3 = R_3 + 4R_1} \left[\begin{array}{ccc|c} 1 & -2 & +1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{R_2 = R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 28 & -12 \end{array}\right] \xrightarrow{R_1 = R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 11 & -2 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 28 & -12 \end{array}\right]$$

$$\xrightarrow{R_3 = \frac{1}{4}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 11 & -2 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 7 & -3 \end{array}\right]$$

Existence & Uniqueness

- Consistent $\longrightarrow$ solution exists
  - $\hookrightarrow$ is it unique?

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array}\right] \longrightarrow \text{inconsistent system}$$

Row reduction algorithm

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array}\right] \xrightarrow[R_3 = R_3 - 3R_2]{R_2 = R_2 + 2R_1} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 8 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 11 \end{array}\right]$$

inconsistent system

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

- Eigenvectors
- Orthogonality
- Bases
- Inverses
- Nol-Col
- Unit Vector
- Magnitude of matrix

$7 - 4(3) = 7 - 12 = -5$

$$1x_1 + 3x_2 + 0x_3 = -5$$
$$x_3 = 3$$

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

Computatcon of Ax

(1) $A = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(a) $A(\vec{u} + \vec{v}) = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} -2 & +1 \\ 4 & +2 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

(b) $A\vec{u} + A\vec{v} = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} -8 & +2 \\ 16 & +4 \end{bmatrix} + \begin{bmatrix} 6 & -1 \\ -12 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 20 \end{bmatrix} + \begin{bmatrix} 5 \\ -14 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

Homogeneous Systems

$$Ax = 0$$

$$\left[\begin{array}{ccc|c} 4 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 4 & 5 & 9 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 4 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 4 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 3/4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\searrow$ Consistent

$$\vec{x} = \begin{bmatrix} 1/4\, x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1/4 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{cases} x_1 = \frac{1}{4} x_3 \\ x_2 = -2x_3 \end{cases}$$ x_3 is free

Non-homogeneous systems $Ax \neq 0$

$$\left[\begin{array}{ccc|c}1&4&-5&0\\2&-1&8&9\end{array}\right] \to \left[\begin{array}{ccc|c}1&4&-5&0\\0&-9&18&9\end{array}\right] \to \left[\begin{array}{ccc|c}1&4&-5&0\\0&1&-2&-1\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1&0&3&4\\0&1&-2&-1\end{array}\right]$$

$$\begin{cases} x_1 = -3x_3 + 4 \\ x_2 = 2x_3 - 1 \\ x_3 \text{ is free} \end{cases} = \begin{bmatrix} -3x_3+4 \\ 2x_3-1 \\ x_3 \end{bmatrix}$$

translation p

$$= \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

Linear Independence

Homogeneous + NON-Trivial sols → DEPENDENT

Homogeneous + NO Non-Trivial sols → INDEPENDENT

↳ free variables = Non-Trivial = DEPENDENT

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&4&2&0\\2&5&1&0\\3&6&0&0\end{array}\right] \leftarrow \text{homogenous system } Ax = 0$$

$$\begin{bmatrix}1&4&2\\2&5&1\\3&6&0\end{bmatrix} = \begin{bmatrix}1&4&2\\0&-3&-3\\0&-6&-6\end{bmatrix} = \begin{bmatrix}1&4&2\\0&-3&-3\\0&0&0\end{bmatrix}$$

$$= \begin{bmatrix}1&4&2\\0&1&1\\0&0&0\end{bmatrix} = \begin{bmatrix}1&0&-2\\0&1&1\\0&0&0\end{bmatrix}$$

$$\begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

there are infinite possible linear dependency relationships

linearly dependent

Relationship: let $x_3 = 2$

↳ $x_1 = 4$; $x_2 = -2$; $x_3 = 2 \Rightarrow 4v_1 - 2v_2 + 2v_3 = 0$

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 5 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous, trivial solution

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

$A x = 0$, where $\vec{x} = 0$

No free variables $\to$ linearly independent

Determining Linear Independence

$u = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} \quad v = \begin{bmatrix} -6 \\ 2 \\ 3 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix} \quad z = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$

$\{u, v\}$
$$\left[\begin{array}{cc|c} 1 & -6 & 0 \\ 2 & 2 & 0 \\ -4 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -6 & 0 \\ 0 & 14 & 0 \\ 0 & 21 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -6 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Infinite solutions

↳ $\vec{u} \neq c\vec{v} \to$ linearly independent

$\{u, v, w, z\} \to$ linearly dependent by Theorem 8 (n. cols > n. rows)

$\{u, v, w\} \to \begin{bmatrix} 1 & -6 & 0 \\ 2 & 2 & -4 \\ -4 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 0 \\ 0 & 14 & -4 \\ -4 & 3 & 0 \end{bmatrix}$

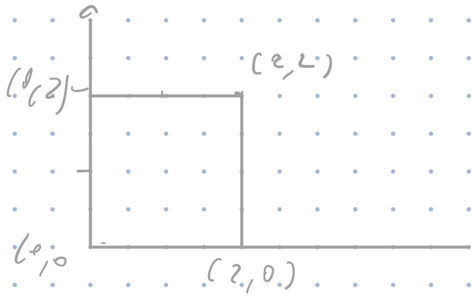
$$\to \begin{bmatrix} 1 & -6 & 0 \\ 0 & 14 & -4 \\ 0 & 21 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 0 \\ 0 & 21 & 0 \\ 0 & 14 & -4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -6 & 0 \\ 0 & 1 & 0 \\ 0 & 14 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \to \text{linearly independent}$$

Matrix Transformations

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x) = Ax$



$T(x) = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$=$

Linear Transformation

Matrix Operations

Multiplication

row 1 of B

$A = m \times n$ ✓ $B = n \times p$ | $AB = m \times p$

$AB = A[b_1, b_2, b_3 \ldots]$

multiplying A by the colums of B

Example

$$A = \overbrace{\begin{bmatrix} 4 & 1 \\ 5 & 2 \end{bmatrix}}^{2} \quad 2\times 2 \qquad B = \begin{bmatrix} 0 & 1 & 3 \\ -2 & 2 & 0 \end{bmatrix} 2 \quad 2\times 3$$

$$AB = A\begin{bmatrix} b_1 & b_2 & b_3 \\ 0 & 1 & 3 \\ -2 & 2 & 0 \end{bmatrix}$$

2×3

$$= \begin{bmatrix} 0-2 & 4+2 & 12+3 \\ 0-4 & 5+4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 6 & 15 \\ -4 & 9 & 0 \end{bmatrix}$$

Row-Column-Rule

rows

$I_m A = A I_n = A \ (m \times n)$

cols

Transpose of a matrix

$$A_{3\times 4} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$A^{T}_{4\times 3} = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$$

Properties:

Inverse of a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$A' = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$$

$$A' = \frac{1}{8} \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/8 \\ 0 & 1/4 \end{bmatrix}$$

$$AA' = I, \quad \Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & -1/8 \\ 0 & 1/4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solving system with the inverse

$Ax = b$

$\hookrightarrow x = A^{-1} b$

$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$Ax = b \implies \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$x = A^{-1} b$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$$

$18 - 20 = -2$

$$= \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

$$x = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -9 + 14 \\ \text{—} \\ \text{—} \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Elementary matrices

Determinants and Co-factors

$C_{ij} = (-1)^{i+j} \det A_{ij}$

$$A = \begin{bmatrix} 4 & 0 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det A = a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31}$$

$$= 4 \, (-1)^2 \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix}$$

$$= 4 \, (+1) = 4$$

$$\begin{bmatrix} 5 & -1 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & 8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{bmatrix}$$

$$\det A = 2 \cdot C_{1,3}$$

$$= 2 \cdot (-1)^{1+3} \det A_{13}$$

$$\det A_{13} = \begin{vmatrix} 0 & 3 & -4 \\ -5 & 8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$$

$$= -5 \cdot C_{2,1} = -5 \, (-1)^{2+1} \cdot \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix}$$

$$= +5 \, (-18 + 20)$$

$$= 10$$

$$= 2 \cdot 10 = 20$$

$\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ $A = \begin{bmatrix} 2 & -6 & 12 \\ -3 & 9 & 9 \\ 0 & 0 & 1 \end{bmatrix}$

$Ax = 0$ ← Space of x, if $Ax = 0$

← infinite solutions

free variable

$x_1 = 3x_2$

$x_3 = 0$

Column Spaces

Eigenvectors

- Determine if $\vec{x}$ is an eigenvector of A
  $A\vec{x} = \vec{u}$
  if $\lambda \cdot \vec{u} = \vec{x}$, then $\vec{x}$ is eigenvector
- Determine eigenvalue of A for $\vec{x}$ (eigen)
- Show that x is eigenvalue of A, and find eigenvector

$x = 3 \qquad A = \begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix}$

$A\vec{x} = \lambda\vec{x} \Rightarrow A\vec{x} - \lambda\vec{x} = 0 \Rightarrow$

(a) Show 3 is eigenvalue $\Rightarrow \boxed{\vec{x}(A - \lambda I) = 0}$

$$\begin{bmatrix} 2 & 3 \\ -1 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}$$

if we end up with a free variable,

if end up with an I matrix, we only have the trivial solution, so no eigenvector

(b) find eigenvector

$$\left[\begin{array}{cc|c} 0 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

BASIS!

$$x = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

All the vectors in this format are eigenvectors

$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ $\lambda = 2$, find eigenspace

$$Ax = \lambda x$$

$$Ax - \lambda x = 0$$

$$\vec{x}\,(A - \lambda I) = 0$$

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 3 \\ 0 & \underbrace{0 \quad\quad 0}_{\text{2 free variables}} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = \frac{1}{2}x_2 + 3x_3 \\ x_2 \text{ free} \\ x_3 \text{ free} \end{cases} = \begin{bmatrix} 1/2\, x^2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x^2\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$