

CHAPTER 1. LIMITS AND CONTINUITY

Section 1.1 Examples of Velocity, Growth Rate, and Area (page 63)

1. Average velocity = $\frac{\Delta x}{\Delta t} = \frac{(t+h)^2 - t^2}{h}$ m/s.

2.

h	Avg. vel. over $[2, 2+h]$
1	5.0000
0.1	4.1000
0.01	4.0100
0.001	4.0010
0.0001	4.0001

3. Guess velocity is $v = 4$ m/s at $t = 2$ s.

4. Average velocity on $[2, 2+h]$ is

$$\frac{(2+h)^2 - 4}{(2+h) - 2} = \frac{4 + 4h + h^2 - 4}{h} = \frac{4h + h^2}{h} = 4 + h.$$

As h approaches 0 this average velocity approaches 4 m/s

5. $x = 3t^2 - 12t + 1$ m at time t s.

Average velocity over interval $[1, 2]$ is

$$\frac{(3 \times 2^2 - 12 \times 2 + 1) - (3 \times 1^2 - 12 \times 1 + 1)}{2 - 1} = -3 \text{ m/s.}$$

Average velocity over interval $[2, 3]$ is

$$\frac{(3 \times 3^2 - 12 \times 3 + 1) - (3 \times 2^2 - 12 \times 2 + 1)}{3 - 2} = 3 \text{ m/s.}$$

Average velocity over interval $[1, 3]$ is

$$\frac{(3 \times 3^2 - 12 \times 3 + 1) - (3 \times 1^2 - 12 \times 1 + 1)}{3 - 1} = 0 \text{ m/s.}$$

6. Average velocity over $[t, t+h]$ is

$$\begin{aligned} & \frac{3(t+h)^2 - 12(t+h) + 1 - (3t^2 - 12t + 1)}{(t+h) - t} \\ &= \frac{6th + 3h^2 - 12h}{h} = 6t + 3h - 12 \text{ m/s.} \end{aligned}$$

This average velocity approaches $6t - 12$ m/s as h approaches 0.

At $t = 1$ the velocity is $6 \times 1 - 12 = -6$ m/s.

At $t = 2$ the velocity is $6 \times 2 - 12 = 0$ m/s.

At $t = 3$ the velocity is $6 \times 3 - 12 = 6$ m/s.

7. At $t = 1$ the velocity is $v = -6 < 0$ so the particle is moving to the left.

At $t = 2$ the velocity is $v = 0$ so the particle is stationary.

At $t = 3$ the velocity is $v = 6 > 0$ so the particle is moving to the right.

8. Average velocity over $[t-k, t+k]$ is

$$\begin{aligned} & \frac{3(t+k)^2 - 12(t+k) + 1 - [3(t-k)^2 - 12(t-k) + 1]}{(t+k) - (t-k)} \\ &= \frac{1}{2k} (3t^2 + 6tk + 3k^2 - 12t - 12k + 1 - 3t^2 + 6tk - 3k^2 \\ & \quad + 12t - 12k + 1) \\ &= \frac{12tk - 24k}{2k} = 6t - 12 \text{ m/s,} \end{aligned}$$

which is the velocity at time t from Exercise 7.

9.

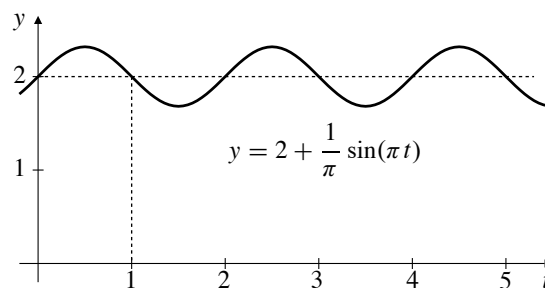


Fig. 1.1.9

At $t = 1$ the height is $y = 2$ ft and the weight is moving downward.

10. Average velocity over $[1, 1+h]$ is

$$\begin{aligned} & \frac{2 + \frac{1}{\pi} \sin \pi(1+h) - \left(2 + \frac{1}{\pi} \sin \pi\right)}{h} \\ &= \frac{\frac{\sin(\pi + \pi h)}{\pi h}}{\pi h} = \frac{\sin \pi \cos(\pi h) + \cos \pi \sin(\pi h)}{\pi h} \\ &= -\frac{\sin(\pi h)}{\pi h}. \end{aligned}$$

h	Avg. vel. on $[1, 1+h]$
1.0000	0
0.1000	-0.983631643
0.0100	-0.999835515
0.0010	-0.999998355

11. The velocity at $t = 1$ is about $v = -1$ ft/s. The “-” indicates that the weight is moving downward.

12. We sketched a tangent line to the graph on page 55 in the text at $t = 20$. The line appeared to pass through the points $(10, 0)$ and $(50, 1)$. On day 20 the biomass is growing at about $(1 - 0)/(50 - 10) = 0.025$ mm²/d.

13. The curve is steepest, and therefore the biomass is growing most rapidly, at about day 45.

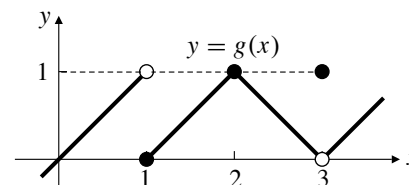


Fig. 1.2.2

we see that

$$\lim_{x \rightarrow 1} g(x) \text{ does not exist}$$

(left limit is 1, right limit is 0)

$$\lim_{x \rightarrow 2} g(x) = 1, \quad \lim_{x \rightarrow 3} g(x) = 0.$$

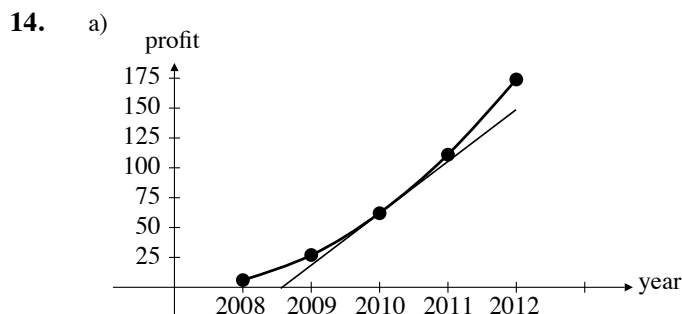


Fig. 1.1.14

- b) Average rate of increase in profits between 2010 and 2012 is

$$\frac{174 - 62}{2012 - 2010} = \frac{112}{2} = 56 \text{ (thousand$/yr).}$$
- c) Drawing a tangent line to the graph in (a) at $t = 2010$ and measuring its slope, we find that the rate of increase of profits in 2010 is about 43 thousand\$/year.

Section 1.2 Limits of Functions (page 71)

1. From inspecting the graph

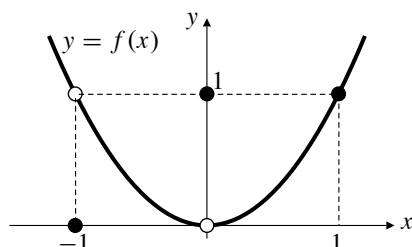


Fig. 1.2.1

we see that

$$\lim_{x \rightarrow -1} f(x) = 1, \quad \lim_{x \rightarrow 0} f(x) = 0, \quad \lim_{x \rightarrow 1} f(x) = 1.$$

2. From inspecting the graph

3. $\lim_{x \rightarrow 1^-} g(x) = 1$

4. $\lim_{x \rightarrow 1^+} g(x) = 0$

5. $\lim_{x \rightarrow 3^+} g(x) = 0$

6. $\lim_{x \rightarrow 3^-} g(x) = 0$

7. $\lim_{x \rightarrow 4} (x^2 - 4x + 1) = 4^2 - 4(4) + 1 = 1$

8. $\lim_{x \rightarrow 2} 3(1 - x)(2 - x) = 3(-1)(2 - 2) = 0$

9. $\lim_{x \rightarrow 3} \frac{x + 3}{x + 6} = \frac{3 + 3}{3 + 6} = \frac{2}{3}$

10. $\lim_{t \rightarrow -4} \frac{t^2}{4 - t} = \frac{(-4)^2}{4 + 4} = 2$

11. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x + 1} = \frac{1^2 - 1}{1 + 1} = \frac{0}{2} = 0$

12. $\lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1} = \lim_{x \rightarrow -1} (x - 1) = -2$

13. $\lim_{x \rightarrow 3} \frac{x^2 - 6x + 9}{x^2 - 9} = \lim_{x \rightarrow 3} \frac{(x - 3)^2}{(x - 3)(x + 3)}$
 $= \lim_{x \rightarrow 3} \frac{x - 3}{x + 3} = \frac{0}{6} = 0$

14. $\lim_{x \rightarrow -2} \frac{x^2 + 2x}{x^2 - 4} = \lim_{x \rightarrow -2} \frac{x}{x - 2} = \frac{-2}{-4} = \frac{1}{2}$

15. $\lim_{h \rightarrow 2} \frac{1}{4 - h^2}$ does not exist; denominator approaches 0 but numerator does not approach 0.

16. $\lim_{h \rightarrow 0} \frac{3h + 4h^2}{h^2 - h^3} = \lim_{h \rightarrow 0} \frac{3 + 4h}{h - h^2}$ does not exist; denominator approaches 0 but numerator does not approach 0.

$$\begin{aligned}
 17. \quad \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} &= \lim_{x \rightarrow 9} \frac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(x - 9)(\sqrt{x} + 3)} \\
 &= \lim_{x \rightarrow 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} = \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 18. \quad \lim_{h \rightarrow 0} \frac{\sqrt{4+h} - 2}{h} &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{4+h} + 2)} \\
 &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4}
 \end{aligned}$$

$$19. \quad \lim_{x \rightarrow \pi} \frac{(x - \pi)^2}{\pi x} = \frac{0^2}{\pi^2} = 0$$

$$20. \quad \lim_{x \rightarrow -2} |x - 2| = |-4| = 4$$

$$21. \quad \lim_{x \rightarrow 0} \frac{|x - 2|}{x - 2} = \frac{|-2|}{-2} = -1$$

$$\begin{aligned}
 22. \quad \lim_{x \rightarrow 2} \frac{|x - 2|}{x - 2} &= \lim_{x \rightarrow 2} \begin{cases} 1, & \text{if } x > 2 \\ -1, & \text{if } x < 2. \end{cases} \\
 \text{Hence, } \lim_{x \rightarrow 2} \frac{|x - 2|}{x - 2} &\text{ does not exist.}
 \end{aligned}$$

$$\begin{aligned}
 23. \quad \lim_{t \rightarrow 1} \frac{t^2 - 1}{t^2 - 2t + 1} &= \lim_{t \rightarrow 1} \frac{(t - 1)(t + 1)}{(t - 1)^2} = \lim_{t \rightarrow 1} \frac{t + 1}{t - 1} \text{ does not exist} \\
 (\text{denominator} \rightarrow 0, \text{ numerator} \rightarrow 2.)
 \end{aligned}$$

$$\begin{aligned}
 24. \quad \lim_{x \rightarrow 2} \frac{\sqrt{4 - 4x + x^2}}{x - 2} &= \lim_{x \rightarrow 2} \frac{|x - 2|}{x - 2} \text{ does not exist.}
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \lim_{t \rightarrow 0} \frac{t}{\sqrt{4+t} - \sqrt{4-t}} &= \lim_{t \rightarrow 0} \frac{t(\sqrt{4+t} + \sqrt{4-t})}{(4+t) - (4-t)} \\
 &= \lim_{t \rightarrow 0} \frac{\sqrt{4+t} + \sqrt{4-t}}{2} = 2
 \end{aligned}$$

$$\begin{aligned}
 26. \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{\sqrt{x+3} - 2} &= \lim_{x \rightarrow 1} \frac{(x-1)(x+1)(\sqrt{x+3} + 2)}{(x+3) - 4} \\
 &= \lim_{x \rightarrow 1} (x+1)(\sqrt{x+3} + 2) = (2)(\sqrt{4} + 2) = 8
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \lim_{t \rightarrow 0} \frac{t^2 + 3t}{(t+2)^2 - (t-2)^2} &= \lim_{t \rightarrow 0} \frac{t(t+3)}{t(t+3)} \\
 &= \lim_{t \rightarrow 0} \frac{t^2 + 4t + 4 - (t^2 - 4t + 4)}{t(t+3)} \\
 &= \lim_{t \rightarrow 0} \frac{t+3}{8} = \frac{3}{8}
 \end{aligned}$$

$$28. \quad \lim_{s \rightarrow 0} \frac{(s+1)^2 - (s-1)^2}{s} = \lim_{s \rightarrow 0} \frac{4s}{s} = 4$$

$$\begin{aligned}
 29. \quad \lim_{y \rightarrow 1} \frac{y - 4\sqrt{y} + 3}{y^2 - 1} &= \lim_{y \rightarrow 1} \frac{(\sqrt{y} - 1)(\sqrt{y} - 3)}{(\sqrt{y} - 1)(\sqrt{y} + 1)(y + 1)} = \frac{-2}{4} = -\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 30. \quad \lim_{x \rightarrow -1} \frac{x^3 + 1}{x + 1} &= \lim_{x \rightarrow -1} \frac{(x+1)(x^2 - x + 1)}{x + 1} = 3
 \end{aligned}$$

$$\begin{aligned}
 31. \quad \lim_{x \rightarrow 2} \frac{x^4 - 16}{x^3 - 8} &= \lim_{x \rightarrow 2} \frac{(x-2)(x+2)(x^2+4)}{(x-2)(x^2+2x+4)} \\
 &= \frac{(4)(8)}{4+4+4} = \frac{8}{3}
 \end{aligned}$$

$$\begin{aligned}
 32. \quad \lim_{x \rightarrow 8} \frac{x^{2/3} - 4}{x^{1/3} - 2} &= \lim_{x \rightarrow 8} \frac{(x^{1/3} - 2)(x^{1/3} + 2)}{(x^{1/3} - 2)} \\
 &= \lim_{x \rightarrow 8} (x^{1/3} + 2) = 4
 \end{aligned}$$

$$\begin{aligned}
 33. \quad \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right) &= \lim_{x \rightarrow 2} \frac{x+2-4}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 34. \quad \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{1}{x^2-4} \right) &= \lim_{x \rightarrow 2} \frac{x+2-1}{(x-2)(x+2)} \\
 &= \lim_{x \rightarrow 2} \frac{x+1}{(x-2)(x+2)} \text{ does not exist.}
 \end{aligned}$$

$$\begin{aligned}
 35. \quad \lim_{x \rightarrow 0} \frac{\sqrt{2+x^2} - \sqrt{2-x^2}}{x^2} &= \lim_{x \rightarrow 0} \frac{(2+x^2) - (2-x^2)}{x^2(\sqrt{2+x^2} + \sqrt{2-x^2})} \\
 &= \lim_{x \rightarrow 0} \frac{2x^2}{2x^2(\sqrt{2+x^2} + \sqrt{2-x^2})} \\
 &= \frac{2}{\sqrt{2} + \sqrt{2}} = \frac{1}{\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 36. \quad \lim_{x \rightarrow 0} \frac{|3x-1| - |3x+1|}{x} &= \lim_{x \rightarrow 0} \frac{(3x-1)^2 - (3x+1)^2}{x(|3x-1| + |3x+1|)} \\
 &= \lim_{x \rightarrow 0} \frac{-12x}{x(|3x-1| + |3x+1|)} = \frac{-12}{1+1} = -6
 \end{aligned}$$

$$\begin{aligned}
 37. \quad f(x) &= x^2 \\
 \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2hx + h^2}{h} = \lim_{h \rightarrow 0} 2x + h = 2x
 \end{aligned}$$

$$\begin{aligned}
 38. \quad f(x) &= x^3 \\
 \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 = 3x^2
 \end{aligned}$$

$$\begin{aligned}
 39. \quad f(x) &= 1/x \\
 \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x - (x+h)}{h(x+h)x} \\
 &= \lim_{h \rightarrow 0} -\frac{1}{(x+h)x} = -\frac{1}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 40. \quad f(x) &= 1/x^2 \\
 \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^2 - (x^2 + 2xh + h^2)}{h(x+h)^2x^2} \\
 &= \lim_{h \rightarrow 0} -\frac{2x+h}{(x+h)^2x^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}
 \end{aligned}$$

$$\begin{aligned}
 41. \quad f(x) &= \sqrt{x} \\
 \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\
 &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
 \end{aligned}$$

$$\begin{aligned}
 42. \quad f(x) &= 1/\sqrt{x} \\
 \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \\
 &= \lim_{h \rightarrow 0} \frac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} \\
 &= -\frac{1}{2x^{3/2}}
 \end{aligned}$$

$$43. \quad \lim_{x \rightarrow \pi/2} \sin x = \sin \pi/2 = 1$$

$$44. \quad \lim_{x \rightarrow \pi/4} \cos x = \cos \pi/4 = 1/\sqrt{2}$$

$$45. \quad \lim_{x \rightarrow \pi/3} \cos x = \cos \pi/3 = 1/2$$

$$46. \quad \lim_{x \rightarrow 2\pi/3} \sin x = \sin 2\pi/3 = \sqrt{3}/2$$

$$47.$$

x	$(\sin x)/x$
± 1.0	0.84147098
± 0.1	0.99833417
± 0.01	0.99998333
± 0.001	0.99999983
0.0001	1.00000000

It appears that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.

$$48.$$

x	$(1 - \cos x)/x^2$
± 1.0	0.45969769
± 0.1	0.49958347
± 0.01	0.49999583
± 0.001	0.49999996
0.0001	0.50000000

It appears that $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$.

$$49. \quad \lim_{x \rightarrow 2^-} \sqrt{2-x} = 0$$

$$50. \quad \lim_{x \rightarrow 2^+} \sqrt{2-x} \text{ does not exist.}$$

$$51. \quad \lim_{x \rightarrow -2^-} \sqrt{2-x} = 2$$

$$52. \quad \lim_{x \rightarrow -2^+} \sqrt{2-x} = 2$$

$$53. \quad \lim_{x \rightarrow 0} \sqrt{x^3 - x} \text{ does not exist.}$$

($x^3 - x < 0$ if $0 < x < 1$)

$$54. \quad \lim_{x \rightarrow 0^-} \sqrt{x^3 - x} = 0$$

$$55. \quad \lim_{x \rightarrow 0^+} \sqrt{x^3 - x} \text{ does not exist. (See # 9.)}$$

$$56. \quad \lim_{x \rightarrow 0^+} \sqrt{x^2 - x^4} = 0$$

$$\begin{aligned}
 57. \quad \lim_{x \rightarrow a^-} \frac{|x-a|}{x^2 - a^2} \\
 = \lim_{x \rightarrow a^-} \frac{|x-a|}{(x-a)(x+a)} = -\frac{1}{2a} \quad (a \neq 0)
 \end{aligned}$$

$$58. \quad \lim_{x \rightarrow a^+} \frac{|x-a|}{x^2 - a^2} = \lim_{x \rightarrow a^+} \frac{x-a}{x^2 - a^2} = \frac{1}{2a}$$

$$59. \quad \lim_{x \rightarrow 2^-} \frac{x^2 - 4}{|x+2|} = \frac{0}{4} = 0$$

$$60. \quad \lim_{x \rightarrow 2^+} \frac{x^2 - 4}{|x+2|} = \frac{0}{4} = 0$$

$$61. f(x) = \begin{cases} x-1 & \text{if } x \leq -1 \\ x^2+1 & \text{if } -1 < x \leq 0 \\ (x+\pi)^2 & \text{if } x > 0 \end{cases}$$

$$\lim_{x \rightarrow -1-} f(x) = \lim_{x \rightarrow -1-} x-1 = -1-1 = -2$$

$$62. \lim_{x \rightarrow -1+} f(x) = \lim_{x \rightarrow -1+} x^2 + 1 = 1 + 1 = 2$$

$$63. \lim_{x \rightarrow 0+} f(x) = \lim_{x \rightarrow 0+} (x + \pi)^2 = \pi^2$$

$$64. \lim_{x \rightarrow 0-} f(x) = \lim_{x \rightarrow 0-} x^2 + 1 = 1$$

$$65. \text{ If } \lim_{x \rightarrow 4} f(x) = 2 \text{ and } \lim_{x \rightarrow 4} g(x) = -3, \text{ then}$$

$$a) \lim_{x \rightarrow 4} (g(x) + 3) = -3 + 3 = 0$$

$$b) \lim_{x \rightarrow 4} x f(x) = 4 \times 2 = 8$$

$$c) \lim_{x \rightarrow 4} (g(x))^2 = (-3)^2 = 9$$

$$d) \lim_{x \rightarrow 4} \frac{g(x)}{f(x) - 1} = \frac{-3}{2 - 1} = -3$$

$$66. \text{ If } \lim_{x \rightarrow a} x f(x) = 4 \text{ and } \lim_{x \rightarrow a} g(x) = -2, \text{ then}$$

$$a) \lim_{x \rightarrow a} (f(x) + g(x)) = 4 + (-2) = 2$$

$$b) \lim_{x \rightarrow a} f(x) \cdot g(x) = 4 \times (-2) = -8$$

$$c) \lim_{x \rightarrow a} 4g(x) = 4(-2) = -8$$

$$d) \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{4}{-2} = -2$$

$$67. \text{ If } \lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} = 3, \text{ then}$$

$$\lim_{x \rightarrow 2} (f(x) - 5) = \lim_{x \rightarrow 2} \frac{f(x) - 5}{x - 2} (x - 2) = 3(2 - 2) = 0.$$

$$\text{Thus } \lim_{x \rightarrow 2} f(x) = 5.$$

$$68. \text{ If } \lim_{x \rightarrow 0} \frac{f(x)}{x^2} = -2 \text{ then}$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^2 \frac{f(x)}{x^2} = 0 \times (-2) = 0,$$

and similarly,

$$\lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} x \frac{f(x)}{x^2} = 0 \times (-2) = 0.$$

69.

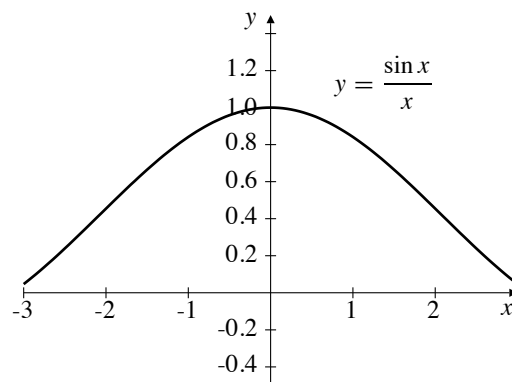


Fig. 1.2.69

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

70.

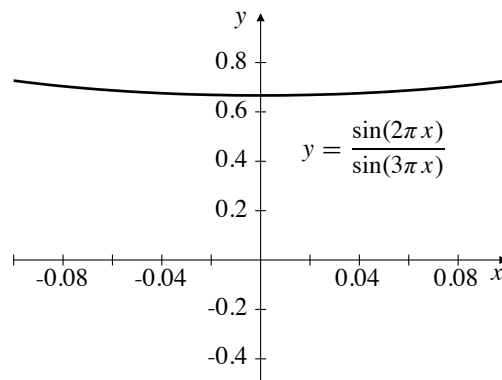


Fig. 1.2.70

$$\lim_{x \rightarrow 0} \sin(2\pi x) / \sin(3\pi x) = 2/3$$

71.

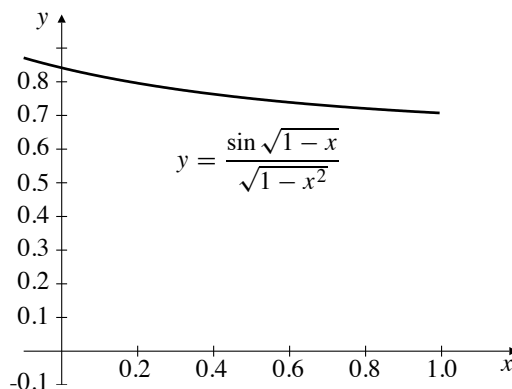


Fig. 1.2.71

$$\lim_{x \rightarrow 1-} \frac{\sin \sqrt{1-x}}{\sqrt{1-x^2}} \approx 0.7071$$

72.

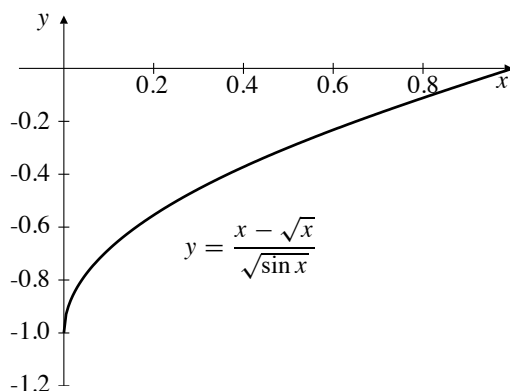


Fig. 1.2.72

$$\lim_{x \rightarrow 0^+} \frac{x - \sqrt{x}}{\sqrt{\sin x}} = -1$$

73.

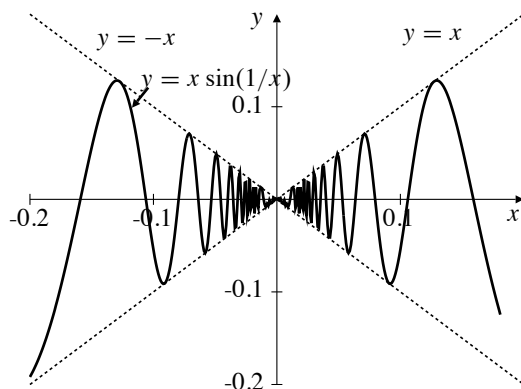


Fig. 1.2.73

$f(x) = x \sin(1/x)$ oscillates infinitely often as x approaches 0, but the amplitude of the oscillations decreases and, in fact, $\lim_{x \rightarrow 0} f(x) = 0$. This is predictable because $|x \sin(1/x)| \leq |x|$. (See Exercise 95 below.)

74. Since $\sqrt{5-2x^2} \leq f(x) \leq \sqrt{5-x^2}$ for $-1 \leq x \leq 1$, and $\lim_{x \rightarrow 0} \sqrt{5-2x^2} = \lim_{x \rightarrow 0} \sqrt{5-x^2} = \sqrt{5}$, we have $\lim_{x \rightarrow 0} f(x) = \sqrt{5}$ by the squeeze theorem.

75. Since $2 - x^2 \leq g(x) \leq 2 \cos x$ for all x , and since $\lim_{x \rightarrow 0} (2 - x^2) = \lim_{x \rightarrow 0} 2 \cos x = 2$, we have $\lim_{x \rightarrow 0} g(x) = 2$ by the squeeze theorem.

76. a)

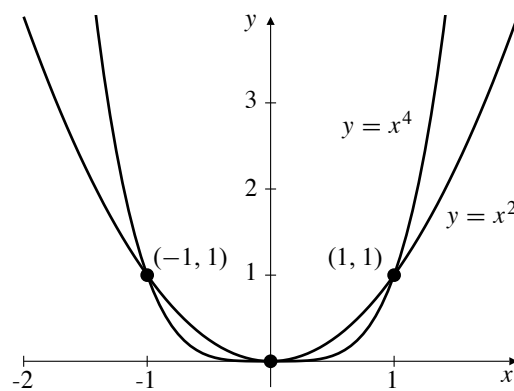


Fig. 1.2.76

- b) Since the graph of f lies between those of x^2 and x^4 , and since these latter graphs come together at $(\pm 1, 1)$ and at $(0, 0)$, we have $\lim_{x \rightarrow \pm 1} f(x) = 1$ and $\lim_{x \rightarrow 0} f(x) = 0$ by the squeeze theorem.

77. $x^{1/3} < x^3$ on $(-1, 0)$ and $(1, \infty)$. $x^{1/3} > x^3$ on $(-\infty, -1)$ and $(0, 1)$. The graphs of $x^{1/3}$ and x^3 intersect at $(-1, -1)$, $(0, 0)$, and $(1, 1)$. If the graph of $h(x)$ lies between those of $x^{1/3}$ and x^3 , then we can determine $\lim_{x \rightarrow a} h(x)$ for $a = -1$, $a = 0$, and $a = 1$ by the squeeze theorem. In fact

$$\lim_{x \rightarrow -1} h(x) = -1, \quad \lim_{x \rightarrow 0} h(x) = 0, \quad \lim_{x \rightarrow 1} h(x) = 1.$$

78. $f(x) = x \sin \frac{1}{x}$ is defined for all $x \neq 0$; its domain is $(-\infty, 0) \cup (0, \infty)$. Since $|\sin t| \leq 1$ for all t , we have $|f(x)| \leq |x|$ and $-|x| \leq f(x) \leq |x|$ for all $x \neq 0$. Since $\lim_{x \rightarrow 0} (-|x|) = 0 = \lim_{x \rightarrow 0} |x|$, we have $\lim_{x \rightarrow 0} f(x) = 0$ by the squeeze theorem.

79. $|f(x)| \leq g(x) \Rightarrow -g(x) \leq f(x) \leq g(x)$
 Since $\lim_{x \rightarrow a} g(x) = 0$, therefore $0 \leq \lim_{x \rightarrow a} f(x) \leq 0$.
 Hence, $\lim_{x \rightarrow a} f(x) = 0$.
 If $\lim_{x \rightarrow a} g(x) = 3$, then either $-3 \leq \lim_{x \rightarrow a} f(x) \leq 3$ or $\lim_{x \rightarrow a} f(x)$ does not exist.

Section 1.3 Limits at Infinity and Infinite Limits (page 78)

- $\lim_{x \rightarrow \infty} \frac{x}{2x-3} = \lim_{x \rightarrow \infty} \frac{1}{2-(3/x)} = \frac{1}{2}$
- $\lim_{x \rightarrow \infty} \frac{x}{x^2-4} = \lim_{x \rightarrow \infty} \frac{1/x}{1-(4/x^2)} = \frac{0}{1} = 0$

$$3. \lim_{x \rightarrow \infty} \frac{3x^3 - 5x^2 + 7}{8 + 2x - 5x^3} = \lim_{x \rightarrow \infty} \frac{3 - \frac{5}{x} + \frac{7}{x^3}}{\frac{8}{x^3} + \frac{2}{x^2} - 5} = -\frac{3}{5}$$

$$4. \lim_{x \rightarrow -\infty} \frac{x^2 - 2}{x - x^2} = \lim_{x \rightarrow -\infty} \frac{1 - \frac{2}{x^2}}{\frac{1}{x} - 1} = \frac{1}{-1} = -1$$

$$5. \lim_{x \rightarrow -\infty} \frac{x^2 + 3}{x^3 + 2} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x} + \frac{3}{x^3}}{1 + \frac{2}{x^3}} = 0$$

$$6. \lim_{x \rightarrow \infty} \frac{x^2 + \sin x}{x^2 + \cos x} = \lim_{x \rightarrow \infty} \frac{1 + \frac{\sin x}{x^2}}{1 + \frac{\cos x}{x^2}} = \frac{1}{1} = 1$$

We have used the fact that $\lim_{x \rightarrow \infty} \frac{\sin x}{x^2} = 0$ (and similarly for cosine) because the numerator is bounded while the denominator grows large.

$$7. \lim_{x \rightarrow \infty} \frac{3x + 2\sqrt{x}}{1 - x} = \lim_{x \rightarrow \infty} \frac{3 + \frac{2}{\sqrt{x}}}{\frac{1}{x} - 1} = -3$$

$$8. \lim_{x \rightarrow \infty} \frac{2x - 1}{\sqrt{3x^2 + x + 1}} = \lim_{x \rightarrow \infty} \frac{x \left(2 - \frac{1}{x}\right)}{|x| \sqrt{3 + \frac{1}{x} + \frac{1}{x^2}}} \quad \text{(but } |x| = x \text{ as } x \rightarrow \infty)$$

? why shouldn't this be true if $x \rightarrow +\infty$? is $+\infty$ implied

$$= \lim_{x \rightarrow \infty} \frac{2 - \frac{1}{x}}{\sqrt{3 + \frac{1}{x} + \frac{1}{x^2}}} = \frac{2}{\sqrt{3}}$$

$$9. \lim_{x \rightarrow -\infty} \frac{2x - 1}{\sqrt{3x^2 + x + 1}} = \lim_{x \rightarrow -\infty} \frac{2 - \frac{1}{x}}{-\sqrt{3 + \frac{1}{x} + \frac{1}{x^2}}} = -\frac{2}{\sqrt{3}},$$

because $x \rightarrow -\infty$ implies that $x < 0$ and so $\sqrt{x^2} = -x$.

$$10. \lim_{x \rightarrow -\infty} \frac{2x - 5}{|3x + 2|} = \lim_{x \rightarrow -\infty} \frac{2x - 5}{-(3x + 2)} = -\frac{2}{3}$$

$$11. \lim_{x \rightarrow 3} \frac{1}{3 - x} \text{ does not exist.}$$

$$12. \lim_{x \rightarrow 3} \frac{1}{(3 - x)^2} = \infty$$

$$13. \lim_{x \rightarrow 3^-} \frac{1}{3 - x} = \infty$$

$$14. \lim_{x \rightarrow 3^+} \frac{1}{3 - x} = -\infty$$

$$15. \lim_{x \rightarrow -5/2} \frac{2x + 5}{5x + 2} = \frac{0}{\frac{-25}{2} + 2} = 0$$

$$16. \lim_{x \rightarrow -2/5} \frac{2x + 5}{5x + 2} \text{ does not exist.}$$

$$17. \lim_{x \rightarrow -(2/5)^-} \frac{2x + 5}{5x + 2} = -\infty$$

$$18. \lim_{x \rightarrow -2/5^+} \frac{2x + 5}{5x + 2} = \infty$$

$$19. \lim_{x \rightarrow 2^+} \frac{x}{(2 - x)^3} = -\infty$$

$$20. \lim_{x \rightarrow 1^-} \frac{x}{\sqrt{1 - x^2}} = \infty$$

$$21. \lim_{x \rightarrow 1^+} \frac{1}{|x - 1|} = \infty$$

$$22. \lim_{x \rightarrow 1^-} \frac{1}{|x - 1|} = \infty$$

$$23. \lim_{x \rightarrow 2} \frac{x - 3}{x^2 - 4x + 4} = \lim_{x \rightarrow 2} \frac{x - 3}{(x - 2)^2} = -\infty$$

$$24. \lim_{x \rightarrow 1^+} \frac{\sqrt{x^2 - x}}{x - x^2} = \lim_{x \rightarrow 1^+} \frac{-1}{\sqrt{x^2 - x}} = -\infty$$

$$25. \lim_{x \rightarrow \infty} \frac{x + x^3 + x^5}{1 + x^2 + x^3} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2} + 1 + x^2}{\frac{1}{x^3} + \frac{1}{x} + 1} = \infty$$

$$26. \lim_{x \rightarrow \infty} \frac{x^3 + 3}{x^2 + 2} = \lim_{x \rightarrow \infty} \frac{x + \frac{3}{x^2}}{1 + \frac{2}{x^2}} = \infty \quad \text{not } \pm \infty?$$

$$27. \lim_{x \rightarrow \infty} \frac{x\sqrt{x+1}(1 - \sqrt{2x+3})}{7 - 6x + 4x^2} = \lim_{x \rightarrow \infty} \frac{x^2 \left(\sqrt{1 + \frac{1}{x}}\right) \left(\frac{1}{\sqrt{x}} - \sqrt{2 + \frac{3}{x}}\right)}{x^2 \left(\frac{7}{x^2} - \frac{6}{x} + 4\right)}$$

$$= \frac{1(-\sqrt{2})}{4} = -\frac{1}{4}\sqrt{2}$$

$$28. \lim_{x \rightarrow \infty} \left(\frac{x^2}{x+1} - \frac{x^2}{x-1} \right) = \lim_{x \rightarrow \infty} \frac{-2x^2}{x^2 - 1} = -2$$

$$\begin{aligned}
 29. \quad \lim_{x \rightarrow -\infty} (\sqrt{x^2 + 2x} - \sqrt{x^2 - 2x}) &= \lim_{x \rightarrow -\infty} \frac{(x^2 + 2x) - (x^2 - 2x)}{\sqrt{x^2 + 2x} + \sqrt{x^2 - 2x}} \\
 &= \lim_{x \rightarrow -\infty} \frac{4x}{(-x) \left(\sqrt{1 + \frac{2}{x}} + \sqrt{1 - \frac{2}{x}} \right)} \\
 &= -\frac{4}{1+1} = -2
 \end{aligned}$$

$$\begin{aligned}
 30. \quad \lim_{x \rightarrow \infty} (\sqrt{x^2 + 2x} - \sqrt{x^2 - 2x}) &= \lim_{x \rightarrow \infty} \frac{x^2 + 2x - x^2 + 2x}{\sqrt{x^2 + 2x} + \sqrt{x^2 - 2x}} \\
 &= \lim_{x \rightarrow \infty} \frac{4x}{x \sqrt{1 + \frac{2}{x}} + x \sqrt{1 - \frac{2}{x}}} \\
 &= \lim_{x \rightarrow \infty} \frac{4}{\sqrt{1 + \frac{2}{x}} + \sqrt{1 - \frac{2}{x}}} = \frac{4}{2} = 2
 \end{aligned}$$

$$\begin{aligned}
 31. \quad \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2 - 2x} - x} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 2x} + x}{(\sqrt{x^2 - 2x} + x)(\sqrt{x^2 - 2x} - x)} \\
 &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 2x} + x}{x^2 - 2x - x^2} \\
 &= \lim_{x \rightarrow \infty} \frac{x(\sqrt{1 - (2/x)} + 1)}{-2x} = \frac{2}{-2} = -1
 \end{aligned}$$

$$32. \quad \lim_{x \rightarrow -\infty} \frac{1}{\sqrt{x^2 + 2x} - x} = \lim_{x \rightarrow -\infty} \frac{1}{|x|(\sqrt{1 + (2/x)} + 1)} = 0$$

33. By Exercise 35, $y = -1$ is a horizontal asymptote (at the right) of $y = \frac{1}{\sqrt{x^2 - 2x} - x}$. Since

$$\lim_{x \rightarrow -\infty} \frac{1}{\sqrt{x^2 - 2x} - x} = \lim_{x \rightarrow -\infty} \frac{1}{|x|(\sqrt{1 - (2/x)} + 1)} = 0,$$

$y = 0$ is also a horizontal asymptote (at the left).

Now $\sqrt{x^2 - 2x} - x = 0$ if and only if $x^2 - 2x = x^2$, that is, if and only if $x = 0$. The given function is undefined at $x = 0$, and where $x^2 - 2x < 0$, that is, on the interval $[0, 2]$. Its only vertical asymptote is at $x = 0$, where

$$\lim_{x \rightarrow 0^-} \frac{1}{\sqrt{x^2 - 2x} - x} = \infty.$$

34. Since $\lim_{x \rightarrow \infty} \frac{2x - 5}{|3x + 2|} = \frac{2}{3}$ and $\lim_{x \rightarrow -\infty} \frac{2x - 5}{|3x + 2|} = -\frac{2}{3}$, $y = \pm(2/3)$ are horizontal asymptotes of $y = (2x - 5)/|3x + 2|$. The only vertical asymptote is $x = -2/3$, which makes the denominator zero.

$$35. \quad \lim_{x \rightarrow 0^+} f(x) = 1$$

$$36. \quad \lim_{x \rightarrow 1} f(x) = \infty$$

37.

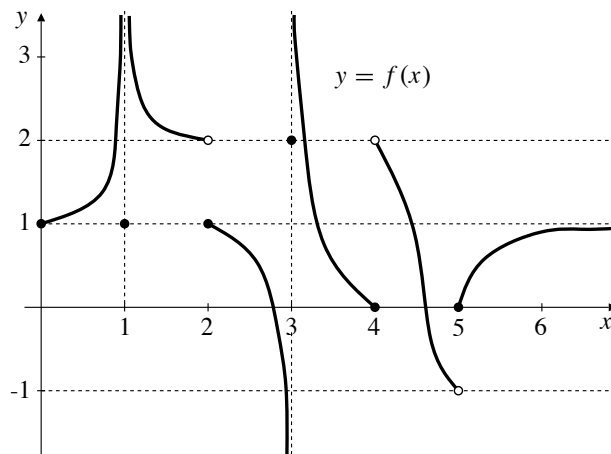


Fig. 1.3.37

$$\lim_{x \rightarrow 2^+} f(x) = 1$$

$$38. \quad \lim_{x \rightarrow 2^-} f(x) = 2$$

$$39. \quad \lim_{x \rightarrow 3^-} f(x) = -\infty$$

$$40. \quad \lim_{x \rightarrow 3^+} f(x) = \infty$$

$$41. \quad \lim_{x \rightarrow 4^+} f(x) = 2$$

$$42. \quad \lim_{x \rightarrow 4^-} f(x) = 0$$

$$43. \quad \lim_{x \rightarrow 5^-} f(x) = -1$$

$$44. \quad \lim_{x \rightarrow 5^+} f(x) = 0$$

$$45. \quad \lim_{x \rightarrow \infty} f(x) = 1$$

$$46. \quad \text{horizontal: } y = 1; \text{ vertical: } x = 1, x = 3.$$

$$47. \quad \lim_{x \rightarrow 3^+} \lfloor x \rfloor = 3$$

$$48. \quad \lim_{x \rightarrow 3^-} \lfloor x \rfloor = 2$$

$$49. \quad \lim_{x \rightarrow 3} \lfloor x \rfloor \text{ does not exist}$$

$$50. \quad \lim_{x \rightarrow 2.5} \lfloor x \rfloor = 2$$

$$51. \quad \lim_{x \rightarrow 0^+} \lfloor 2 - x \rfloor = \lim_{x \rightarrow 2^-} \lfloor x \rfloor = 1$$

$$52. \quad \lim_{x \rightarrow -3^-} \lfloor x \rfloor = -4$$

$$\begin{aligned}
 53. \quad &\lim_{t \rightarrow t_0} C(t) = C(t_0) \text{ except at integers } t_0 \\
 &\lim_{t \rightarrow t_0^-} C(t) = C(t_0) \text{ everywhere} \\
 &\lim_{t \rightarrow t_0^+} C(t) = C(t_0) \text{ if } t_0 \neq \text{an integer} \\
 &\lim_{t \rightarrow t_0^+} C(t) = C(t_0) + 1.5 \text{ if } t_0 \text{ is an integer}
 \end{aligned}$$

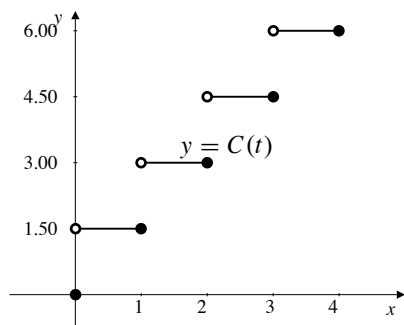


Fig. 1.3.53

54. $\lim_{x \rightarrow 0^+} f(x) = L$

(a) If f is even, then $f(-x) = f(x)$.
Hence, $\lim_{x \rightarrow 0^-} f(x) = L$.

(b) If f is odd, then $f(-x) = -f(x)$.
Therefore, $\lim_{x \rightarrow 0^-} f(x) = -L$.

55. $\lim_{x \rightarrow 0^+} f(x) = A, \quad \lim_{x \rightarrow 0^-} f(x) = B$

a) $\lim_{x \rightarrow 0^+} f(x^3 - x) = B$ (since $x^3 - x < 0$ if $0 < x < 1$)

b) $\lim_{x \rightarrow 0^-} f(x^3 - x) = A$ (because $x^3 - x > 0$ if $-1 < x < 0$)

c) $\lim_{x \rightarrow 0^-} f(x^2 - x^4) = A$

d) $\lim_{x \rightarrow 0^+} f(x^2 - x^4) = A$ (since $x^2 - x^4 > 0$ for $0 < |x| < 1$)

Section 1.4 Continuity (page 87)

1. g is continuous at $x = -2$, discontinuous at $x = -1, 0, 1$, and 2 . It is left continuous at $x = 0$ and right continuous at $x = 1$.

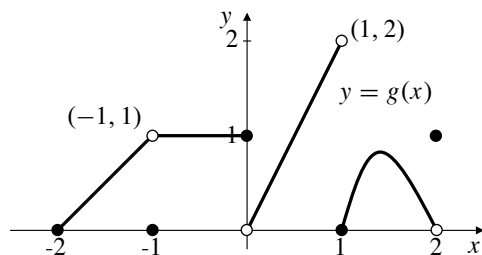


Fig. 1.4.1

2. g has removable discontinuities at $x = -1$ and $x = 2$. Redefine $g(-1) = 1$ and $g(2) = 0$ to make g continuous at those points.

3. g has no absolute maximum value on $[-2, 2]$. It takes on every positive real value less than 2, but does not take the value 2. It has absolute minimum value 0 on that interval, assuming this value at the three points $x = -2$, $x = -1$, and $x = 1$.
4. Function f is discontinuous at $x = 1, 2, 3, 4$, and 5 . f is left continuous at $x = 4$ and right continuous at $x = 2$ and $x = 5$.

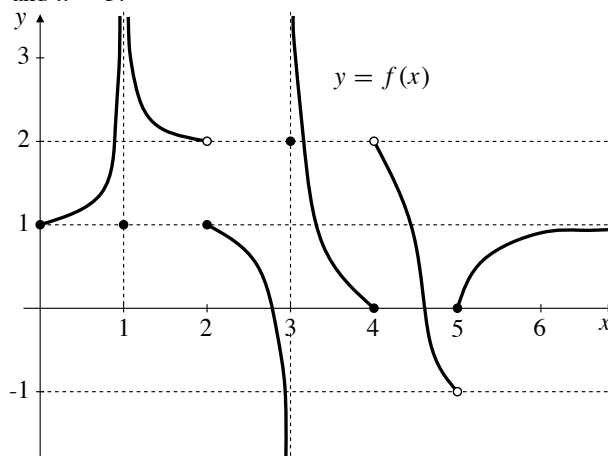


Fig. 1.4.4

5. f cannot be redefined at $x = 1$ to become continuous there because $\lim_{x \rightarrow 1} f(x) (= \infty)$ does not exist. (∞ is not a real number.)
6. $\operatorname{sgn} x$ is not defined at $x = 0$, so cannot be either continuous or discontinuous there. (Functions can be continuous or discontinuous only at points in their domains!)
7. $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$ is continuous everywhere on the real line, even at $x = 0$ where its left and right limits are both 0, which is $f(0)$.
8. $f(x) = \begin{cases} x & \text{if } x < -1 \\ x^2 & \text{if } x \geq -1 \end{cases}$ is continuous everywhere on the real line except at $x = -1$ where it is right continuous, but not left continuous.

$$\begin{aligned} \lim_{x \rightarrow -1^-} f(x) &= \lim_{x \rightarrow -1^-} x = -1 \neq 1 \\ &= f(-1) = \lim_{x \rightarrow -1^+} x^2 = \lim_{x \rightarrow -1^+} f(x). \end{aligned}$$

9. $f(x) = \begin{cases} 1/x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous everywhere except at $x = 0$, where it is neither left nor right continuous since it does not have a real limit there.
10. $f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 0.987 & \text{if } x > 1 \end{cases}$ is continuous everywhere except at $x = 1$, where it is left continuous but not right continuous because $0.987 \neq 1$. Close, as they say, but no cigar.

11. The least integer function $\lceil x \rceil$ is continuous everywhere on \mathbb{R} except at the integers, where it is left continuous but not right continuous.
12. $C(t)$ is discontinuous only at the integers. It is continuous on the left at the integers, but not on the right.
13. Since $\frac{x^2 - 4}{x - 2} = x + 2$ for $x \neq 2$, we can define the function to be $2 + 2 = 4$ at $x = 2$ to make it continuous there. The continuous extension is $x + 2$.
14. Since $\frac{1 + t^3}{1 - t^2} = \frac{(1 + t)(1 - t + t^2)}{(1 + t)(1 - t)} = \frac{1 - t + t^2}{1 - t}$ for $t \neq -1$, we can define the function to be $3/2$ at $t = -1$ to make it continuous there. The continuous extension is $\frac{1 - t + t^2}{1 - t}$.
15. Since $\frac{t^2 - 5t + 6}{t^2 - t - 6} = \frac{(t - 2)(t - 3)}{(t + 2)(t - 3)} = \frac{t - 2}{t + 2}$ for $t \neq 3$, we can define the function to be $1/5$ at $t = 3$ to make it continuous there. The continuous extension is $\frac{t - 2}{t + 2}$.
16. Since $\frac{x^2 - 2}{x^4 - 4} = \frac{(x - \sqrt{2})(x + \sqrt{2})}{(x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)} = \frac{x + \sqrt{2}}{(x + \sqrt{2})(x^2 + 2)}$ for $x \neq \pm\sqrt{2}$, we can define the function to be $1/4$ at $x = \sqrt{2}$ to make it continuous there. The continuous extension is $\frac{x + \sqrt{2}}{(x + \sqrt{2})(x^2 + 2)}$. (Note: cancelling the $x + \sqrt{2}$ factors provides a further continuous extension to $x = -\sqrt{2}$.)
17. $\lim_{x \rightarrow 2^+} f(x) = k - 4$ and $\lim_{x \rightarrow 2^-} f(x) = 4 = f(2)$. Thus f will be continuous at $x = 2$ if $k - 4 = 4$, that is, if $k = 8$.
18. $\lim_{x \rightarrow 3^-} g(x) = 3 - m$ and $\lim_{x \rightarrow 3^+} g(x) = 1 - 3m = g(3)$. Thus g will be continuous at $x = 3$ if $3 - m = 1 - 3m$, that is, if $m = -1$.
19. x^2 has no maximum value on $-1 < x < 1$; it takes all positive real values less than 1, but it does not take the value 1. It does have a minimum value, namely 0 taken on at $x = 0$.
20. The Max-Min Theorem says that a continuous function defined on a closed, finite interval must have maximum and minimum values. It does not say that other functions cannot have such values. The Heaviside function is not continuous on $[-1, 1]$ (because it is discontinuous at $x = 0$), but it still has maximum and minimum values. Do not confuse a theorem with its converse.
21. Let the numbers be x and y , where $x \geq 0$, $y \geq 0$, and $x + y = 8$. If P is the product of the numbers, then

$$P = xy = x(8 - x) = 8x - x^2 = 16 - (x - 4)^2.$$

Therefore $P \leq 16$, so P is bounded. Clearly $P = 16$ if $x = y = 4$, so the largest value of P is 16.

22. Let the numbers be x and y , where $x \geq 0$, $y \geq 0$, and $x + y = 8$. If S is the sum of their squares then

$$\begin{aligned} S &= x^2 + y^2 = x^2 + (8 - x)^2 \\ &= 2x^2 - 16x + 64 = 2(x - 4)^2 + 32. \end{aligned}$$

Since $0 \leq x \leq 8$, the maximum value of S occurs at $x = 0$ or $x = 8$, and is 64. The minimum value occurs at $x = 4$ and is 32.

23. Since $T = 100 - 30x + 3x^2 = 3(x - 5)^2 + 25$, T will be minimum when $x = 5$. Five programmers should be assigned, and the project will be completed in 25 days.

24. If x desks are shipped, the shipping cost per desk is

$$\begin{aligned} C &= \frac{245x - 30x^2 + x^3}{x} = x^2 - 30x + 245 \\ &= (x - 15)^2 + 20. \end{aligned}$$

This cost is minimized if $x = 15$. The manufacturer should send 15 desks in each shipment, and the shipping cost will then be \$20 per desk.

25. $f(x) = \frac{x^2 - 1}{x} = \frac{(x - 1)(x + 1)}{x}$
 $f = 0$ at $x = \pm 1$. f is not defined at 0.
 $f(x) > 0$ on $(-1, 0)$ and $(1, \infty)$.
 $f(x) < 0$ on $(-\infty, -1)$ and $(0, 1)$.
26. $f(x) = x^2 + 4x + 3 = (x + 1)(x + 3)$
 $f(x) > 0$ on $(-\infty, -3)$ and $(-1, \infty)$
 $f(x) < 0$ on $(-3, -1)$.
27. $f(x) = \frac{x^2 - 1}{x^2 - 4} = \frac{(x - 1)(x + 1)}{(x - 2)(x + 2)}$
 $f = 0$ at $x = \pm 1$.
 f is not defined at $x = \pm 2$.
 $f(x) > 0$ on $(-\infty, -2)$, $(-1, 1)$, and $(2, \infty)$.
 $f(x) < 0$ on $(-2, -1)$ and $(1, 2)$.
28. $f(x) = \frac{x^2 + x - 2}{x^3} = \frac{(x + 2)(x - 1)}{x^3}$
 $f(x) > 0$ on $(-2, 0)$ and $(1, \infty)$
 $f(x) < 0$ on $(-\infty, -2)$ and $(0, 1)$.
29. $f(x) = x^3 + x - 1$, $f(0) = -1$, $f(1) = 1$.
Since f is continuous and changes sign between 0 and 1, it must be zero at some point between 0 and 1 by IVT.
30. $f(x) = x^3 - 15x + 1$ is continuous everywhere.
 $f(-4) = -3$, $f(-3) = 19$, $f(1) = -13$, $f(4) = 5$.
Because of the sign changes f has a zero between -4 and -3 , another zero between -3 and 1, and another between 1 and 4.

31. $F(x) = (x - a)^2(x - b)^2 + x$. Without loss of generality, we can assume that $a < b$. Being a polynomial, F is continuous on $[a, b]$. Also $F(a) = a$ and $F(b) = b$. Since $a < \frac{1}{2}(a + b) < b$, the Intermediate-Value Theorem guarantees that there is an x in (a, b) such that $F(x) = (a + b)/2$.

32. Let $g(x) = f(x) - x$. Since $0 \leq f(x) \leq 1$ if $0 \leq x \leq 1$, therefore, $g(0) \geq 0$ and $g(1) \leq 0$. If $g(0) = 0$ let $c = 0$, or if $g(1) = 0$ let $c = 1$. (In either case $f(c) = c$.) Otherwise, $g(0) > 0$ and $g(1) < 0$, and, by IVT, there exists c in $(0, 1)$ such that $g(c) = 0$, i.e., $f(c) = c$.

33. The domain of an even function is symmetric about the y -axis. Since f is continuous on the right at $x = 0$, therefore it must be defined on an interval $[0, h]$ for some $h > 0$. Being even, f must therefore be defined on $[-h, h]$. If $x = -y$, then

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{y \rightarrow 0^+} f(-y) = \lim_{y \rightarrow 0^+} f(y) = f(0).$$

Thus, f is continuous on the left at $x = 0$. Being continuous on both sides, it is therefore continuous.

34. f odd $\Leftrightarrow f(-x) = -f(x)$
 f continuous on the right $\Leftrightarrow \lim_{x \rightarrow 0^+} f(x) = f(0)$
 Therefore, letting $t = -x$, we obtain

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{t \rightarrow 0^+} f(-t) = \lim_{t \rightarrow 0^+} -f(t) \\ &= -f(0) = f(-0) = f(0). \end{aligned}$$

Therefore f is continuous at 0 and $f(0) = 0$.

35. max 1.593 at -0.831 , min -0.756 at 0.629
 36. max 0.133 at $x = 1.437$; min -0.232 at $x = -1.805$
 37. max 10.333 at $x = 3$; min 4.762 at $x = 1.260$
 38. max 1.510 at $x = 0.465$; min 0 at $x = 0$ and $x = 1$
 39. root $x = 0.682$
 40. root $x = 0.739$
 41. roots $x = -0.637$ and $x = 1.410$
 42. roots $x = -0.7244919590$ and $x = 1.220744085$
 43. fsolve gives an approximation to the single real root to 10 significant figures; solve gives the three roots (including a complex conjugate pair) in exact form involving the quantity $(108 + 12\sqrt{69})^{1/3}$; evalf(solve) gives approximations to the three roots using 10 significant figures for the real and imaginary parts.

Section 1.5 The Formal Definition of Limit (page 92)

1. We require $39.9 \leq L \leq 40.1$. Thus

$$\begin{aligned} 39.9 &\leq 39.6 + 0.025T \leq 40.1 \\ 0.3 &\leq 0.025T \leq 0.5 \\ 12 &\leq T \leq 20. \end{aligned}$$

The temperature should be kept between 12°C and 20°C .

2. Since 1.2% of 8,000 is 96, we require the edge length x of the cube to satisfy $7904 \leq x^3 \leq 8096$. It is sufficient that $19.920 \leq x \leq 20.079$. The edge of the cube must be within 0.079 cm of 20 cm.

3. $3 - 0.02 \leq 2x - 1 \leq 3 + 0.02$
 $3.98 \leq 2x \leq 4.02$
 $1.99 \leq x \leq 2.01$

4. $4 - 0.1 \leq x^2 \leq 4 + 0.1$
 $1.9749 \leq x \leq 2.0024$

5. $1 - 0.1 \leq \sqrt{x} \leq 1.1$
 $0.81 \leq x \leq 1.21$

6. $-2 - 0.01 \leq \frac{1}{x} \leq -2 + 0.01$
 $-\frac{1}{2.01} \geq x \geq -\frac{1}{1.99}$
 $-0.5025 \leq x \leq -0.4975$

7. We need $-0.03 \leq (3x + 1) - 7 \leq 0.03$, which is equivalent to $-0.01 \leq x - 2 \leq 0.01$. Thus $\delta = 0.01$ will do.

8. We need $-0.01 \leq \sqrt{2x + 3} - 3 \leq 0.01$. Thus

$$\begin{aligned} 2.99 &\leq \sqrt{2x + 3} \leq 3.01 \\ 8.9401 &\leq 2x + 3 \leq 9.0601 \\ 2.97005 &\leq x \leq 3.03005 \\ 3 - 0.02995 &\leq x - 3 \leq 0.03005. \end{aligned}$$

Here $\delta = 0.02995$ will do.

9. We need $8 - 0.2 \leq x^3 \leq 8.2$, or $1.9832 \leq x \leq 2.0165$. Thus, we need $-0.0168 \leq x - 2 \leq 0.0165$. Here $\delta = 0.0165$ will do.

- 10.** We need $1 - 0.05 \leq 1/(x+1) \leq 1 + 0.05$, or $1.0526 \geq x+1 \geq 0.9524$. This will occur if $-0.0476 \leq x \leq 0.0526$. In this case we can take $\delta = 0.0476$.

- 11.** To be proved: $\lim_{x \rightarrow 1} (3x+1) = 4$.

Proof: Let $\epsilon > 0$ be given. Then $|(3x+1)-4| < \epsilon$ holds if $3|x-1| < \epsilon$, and so if $|x-1| < \delta = \epsilon/3$. This confirms the limit.

- 12.** To be proved: $\lim_{x \rightarrow 2} (5-2x) = 1$.

Proof: Let $\epsilon > 0$ be given. Then $|(5-2x)-1| < \epsilon$ holds if $|2x-4| < \epsilon$, and so if $|x-2| < \delta = \epsilon/2$. This confirms the limit.

- 13.** To be proved: $\lim_{x \rightarrow 0} x^2 = 0$.

Let $\epsilon > 0$ be given. Then $|x^2-0| < \epsilon$ holds if $|x-0| = |x| < \delta = \sqrt{\epsilon}$.

- 14.** To be proved: $\lim_{x \rightarrow 2} \frac{x-2}{1+x^2} = 0$.

Proof: Let $\epsilon > 0$ be given. Then

$$\left| \frac{x-2}{1+x^2} - 0 \right| = \frac{|x-2|}{1+x^2} \leq |x-2| < \epsilon$$

provided $|x-2| < \delta = \epsilon$.

- 15.** To be proved: $\lim_{x \rightarrow 1/2} \frac{1-4x^2}{1-2x} = 2$.

Proof: Let $\epsilon > 0$ be given. Then if $x \neq 1/2$ we have

$$\left| \frac{1-4x^2}{1-2x} - 2 \right| = |(1+2x)-2| = |2x-1| = 2 \left| x - \frac{1}{2} \right| < \epsilon$$

provided $|x - \frac{1}{2}| < \delta = \epsilon/2$.

- 16.** To be proved: $\lim_{x \rightarrow -2} \frac{x^2+2x}{x+2} = -2$.

Proof: Let $\epsilon > 0$ be given. For $x \neq -2$ we have

$$\left| \frac{x^2+2x}{x+2} - (-2) \right| = |x+2| < \epsilon$$

provided $|x+2| < \delta = \epsilon$. This completes the proof.

- 17.** To be proved: $\lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{2}$.

Proof: Let $\epsilon > 0$ be given. We have

$$\left| \frac{1}{x+1} - \frac{1}{2} \right| = \left| \frac{1-x}{2(x+1)} \right| = \frac{|x-1|}{2|x+1|}.$$

If $|x-1| < 1$, then $0 < x < 2$ and $1 < x+1 < 3$, so that $|x+1| > 1$. Let $\delta = \min(1, 2\epsilon)$. If $|x-1| < \delta$, then

$$\left| \frac{1}{x+1} - \frac{1}{2} \right| = \frac{|x-1|}{2|x+1|} < \frac{2\epsilon}{2} = \epsilon.$$

This establishes the required limit.

- 18.** To be proved: $\lim_{x \rightarrow -1} \frac{x+1}{x^2-1} = -\frac{1}{2}$.

Proof: Let $\epsilon > 0$ be given. If $x \neq -1$, we have

$$\left| \frac{x+1}{x^2-1} - \left(-\frac{1}{2}\right) \right| = \left| \frac{1}{x-1} - \left(-\frac{1}{2}\right) \right| = \frac{|x+1|}{2|x-1|}.$$

If $|x+1| < 1$, then $-2 < x < 0$, so $-3 < x-1 < -1$ and $|x-1| > 1$. Let $\delta = \min(1, 2\epsilon)$. If $0 < |x-(-1)| < \delta$ then $|x-1| > 1$ and $|x+1| < 2\epsilon$. Thus

$$\left| \frac{x+1}{x^2-1} - \left(-\frac{1}{2}\right) \right| = \frac{|x+1|}{2|x-1|} < \frac{2\epsilon}{2} = \epsilon.$$

This completes the required proof.

- 19.** To be proved: $\lim_{x \rightarrow 1} \sqrt{x} = 1$.

Proof: Let $\epsilon > 0$ be given. We have

$$|\sqrt{x} - 1| = \left| \frac{x-1}{\sqrt{x}+1} \right| \leq |x-1| < \epsilon$$

provided $|x-1| < \delta = \epsilon$. This completes the proof.

- 20.** To be proved: $\lim_{x \rightarrow 2} x^3 = 8$.

Proof: Let $\epsilon > 0$ be given. We have

$|x^3-8| = |x-2||x^2+2x+4|$. If $|x-2| < 1$, then $1 < x < 3$ and $x^2 < 9$. Therefore $|x^2+2x+4| \leq 9+2 \times 3+4 = 19$. If $|x-2| < \delta = \min(1, \epsilon/19)$, then

$$|x^3-8| = |x-2||x^2+2x+4| < \frac{\epsilon}{19} \times 19 = \epsilon.$$

This completes the proof.

- 21.** We say that $\lim_{x \rightarrow a} f(x) = L$ if the following condition holds: for every number $\epsilon > 0$ there exists a number $\delta > 0$, depending on ϵ , such that

$$a - \delta < x < a \quad \text{implies} \quad |f(x) - L| < \epsilon.$$

- 22.** We say that $\lim_{x \rightarrow -\infty} f(x) = L$ if the following condition holds: for every number $\epsilon > 0$ there exists a number $R > 0$, depending on ϵ , such that

$$x < -R \quad \text{implies} \quad |f(x) - L| < \epsilon.$$

- 23.** We say that $\lim_{x \rightarrow a} f(x) = -\infty$ if the following condition holds: for every number $B > 0$ there exists a number $\delta > 0$, depending on B , such that

$$0 < |x-a| < \delta \quad \text{implies} \quad f(x) < -B.$$

24. We say that $\lim_{x \rightarrow \infty} f(x) = \infty$ if the following condition holds: for every number $B > 0$ there exists a number $R > 0$, depending on B , such that

$$x > R \text{ implies } f(x) > B.$$

25. We say that $\lim_{x \rightarrow a+} f(x) = -\infty$ if the following condition holds: for every number $B > 0$ there exists a number $\delta > 0$, depending on B , such that

$$a < x < a + \delta \text{ implies } f(x) < -B.$$

26. We say that $\lim_{x \rightarrow a-} f(x) = \infty$ if the following condition holds: for every number $B > 0$ there exists a number $\delta > 0$, depending on B , such that

$$a - \delta < x < a \text{ implies } f(x) > B.$$

27. To be proved: $\lim_{x \rightarrow 1+} \frac{1}{x-1} = \infty$. Proof: Let $B > 0$ be given. We have $\frac{1}{x-1} > B$ if $0 < x-1 < 1/B$, that is, if $1 < x < 1 + \delta$, where $\delta = 1/B$. This completes the proof.

28. To be proved: $\lim_{x \rightarrow 1-} \frac{1}{x-1} = -\infty$. Proof: Let $B > 0$ be given. We have $\frac{1}{x-1} < -B$ if $0 > x-1 > -1/B$, that is, if $1 - \delta < x < 1$, where $\delta = 1/B$. This completes the proof.

29. To be proved: $\lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^2+1}} = 0$. Proof: Let $\epsilon > 0$ be given. We have

$$\left| \frac{1}{\sqrt{x^2+1}} \right| = \frac{1}{\sqrt{x^2+1}} < \frac{1}{x} < \epsilon$$

provided $x > R$, where $R = 1/\epsilon$. This completes the proof.

30. To be proved: $\lim_{x \rightarrow \infty} \sqrt{x} = \infty$. Proof: Let $B > 0$ be given. We have $\sqrt{x} > B$ if $x > R$ where $R = B^2$. This completes the proof.

31. To be proved: if $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} f(x) = M$, then $L = M$.

Proof: Suppose $L \neq M$. Let $\epsilon = |L - M|/3$. Then $\epsilon > 0$. Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_1 > 0$ such that $|f(x) - L| < \epsilon$ if $|x - a| < \delta_1$. Since $\lim_{x \rightarrow a} f(x) = M$, there exists $\delta_2 > 0$ such that $|f(x) - M| < \epsilon$ if $|x - a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. If $|x - a| < \delta$, then

$$\begin{aligned} 3\epsilon &= |L - M| = |(f(x) - M) + (L - f(x))| \\ &\leq |f(x) - M| + |f(x) - L| < \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

This implies that $3 < 2$, a contradiction. Thus the original assumption that $L \neq M$ must be incorrect. Therefore $L = M$.

32. To be proved: if $\lim_{x \rightarrow a} g(x) = M$, then there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|g(x)| < 1 + |M|$. Proof: Taking $\epsilon = 1$ in the definition of limit, we obtain a number $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|g(x) - M| < 1$. It follows from this latter inequality that

$$|g(x)| = |(g(x) - M) + M| \leq |g(x) - M| + |M| < 1 + |M|.$$

33. To be proved: if $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then $\lim_{x \rightarrow a} f(x)g(x) = LM$.

Proof: Let $\epsilon > 0$ be given. Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_1 > 0$ such that $|f(x) - L| < \epsilon/(2(1 + |M|))$ if $0 < |x - a| < \delta_1$. Since $\lim_{x \rightarrow a} g(x) = M$, there exists $\delta_2 > 0$ such that $|g(x) - M| < \epsilon/(2(1 + |L|))$ if $0 < |x - a| < \delta_2$. By Exercise 32, there exists $\delta_3 > 0$ such that $|g(x)| < 1 + |M|$ if $0 < |x - a| < \delta_3$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $|x - a| < \delta$, then

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= |(f(x) - L)g(x) + L(g(x) - M)| \\ &\leq |(f(x) - L)g(x)| + |L(g(x) - M)| \\ &= |f(x) - L||g(x)| + |L||g(x) - M| \\ &< \frac{\epsilon}{2(1 + |M|)}(1 + |M|) + |L|\frac{\epsilon}{2(1 + |L|)} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $\lim_{x \rightarrow a} f(x)g(x) = LM$.

34. To be proved: if $\lim_{x \rightarrow a} g(x) = M$ where $M \neq 0$, then there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|g(x)| > |M|/2$.

Proof: By the definition of limit, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|g(x) - M| < |M|/2$ (since $|M|/2$ is a positive number). This latter inequality implies that

$$|M| = |g(x) + (M - g(x))| \leq |g(x)| + |g(x) - M| < |g(x)| + \frac{|M|}{2}.$$

It follows that $|g(x)| > |M| - (|M|/2) = |M|/2$, as required.

35. To be proved: if $\lim_{x \rightarrow a} g(x) = M$ where $M \neq 0$, then

$$\lim_{x \rightarrow a} \frac{1}{g(x)} = \frac{1}{M}.$$

Proof: Let $\epsilon > 0$ be given. Since $\lim_{x \rightarrow a} g(x) = M \neq 0$, there exists $\delta_1 > 0$ such that $|g(x) - M| < \epsilon|M|^2/2$ if $0 < |x - a| < \delta_1$. By Exercise 34, there exists $\delta_2 > 0$ such that $|g(x)| > |M|/2$ if $0 < |x - a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. If $0 < |x - a| < \delta$, then

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|M||g(x)|} < \frac{\epsilon|M|^2}{2} \frac{2}{|M|^2} = \epsilon.$$

This completes the proof.

36. To be proved: if $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M \neq 0$,

$$\text{then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Proof: By Exercises 33 and 35 we have

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} f(x) \times \frac{1}{g(x)} = L \times \frac{1}{M} = \frac{L}{M}.$$

37. To be proved: if f is continuous at L and $\lim_{x \rightarrow c} g(x) = L$, then $\lim_{x \rightarrow c} f(g(x)) = f(L)$.

Proof: Let $\epsilon > 0$ be given. Since f is continuous at L , there exists a number $\gamma > 0$ such that if $|y - L| < \gamma$, then $|f(y) - f(L)| < \epsilon$. Since $\lim_{x \rightarrow c} g(x) = L$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|g(x) - L| < \gamma$. Taking $y = g(x)$, it follows that if $0 < |x - c| < \delta$, then $|f(g(x)) - f(L)| < \epsilon$, so that $\lim_{x \rightarrow c} f(g(x)) = f(L)$.

38. To be proved: if $f(x) \leq g(x) \leq h(x)$ in an open interval containing $x = a$ (say, for $a - \delta_1 < x < a + \delta_1$, where $\delta_1 > 0$), and if $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$, then also $\lim_{x \rightarrow a} g(x) = L$.

Proof: Let $\epsilon > 0$ be given. Since $\lim_{x \rightarrow a} f(x) = L$, there exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then $|f(x) - L| < \epsilon/3$. Since $\lim_{x \rightarrow a} h(x) = L$, there exists $\delta_3 > 0$ such that if $0 < |x - a| < \delta_3$, then $|h(x) - L| < \epsilon/3$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $0 < |x - a| < \delta$, then

$$\begin{aligned} |g(x) - L| &= |g(x) - f(x) + f(x) - L| \\ &\leq |g(x) - f(x)| + |f(x) - L| \\ &\leq |h(x) - f(x)| + |f(x) - L| \\ &= |h(x) - L + L - f(x)| + |f(x) - L| \\ &\leq |h(x) - L| + |f(x) - L| + |f(x) - L| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus $\lim_{x \rightarrow a} g(x) = L$.

Review Exercises 1 (page 93)

1. The average rate of change of x^3 over $[1, 3]$ is

$$\frac{3^3 - 1^3}{3 - 1} = \frac{26}{2} = 13.$$

2. The average rate of change of $1/x$ over $[-2, -1]$ is

$$\frac{(1/(-1)) - (1/(-2))}{-1 - (-2)} = \frac{-1/2}{1} = -\frac{1}{2}.$$

3. The rate of change of x^3 at $x = 2$ is

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(2+h)^3 - 2^3}{h} &= \lim_{h \rightarrow 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\ &= \lim_{h \rightarrow 0} (12 + 6h + h^2) = 12. \end{aligned}$$

4. The rate of change of $1/x$ at $x = -3/2$ is

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\frac{1}{-(3/2)+h} - \left(\frac{1}{-3/2}\right)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{2}{2h-3} + \frac{2}{3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{2(3+2h-3)}{3(2h-3)h} \\ &= \lim_{h \rightarrow 0} \frac{4}{3(2h-3)} = -\frac{4}{9}. \end{aligned}$$

5. $\lim_{x \rightarrow 1} (x^2 - 4x + 7) = 1 - 4 + 7 = 4$

$$6. \lim_{x \rightarrow 2} \frac{x^2}{1 - x^2} = \frac{2^2}{1 - 2^2} = -\frac{4}{3}$$

7. $\lim_{x \rightarrow 1} \frac{x^2}{1 - x^2}$ does not exist. The denominator approaches 0 (from both sides) while the numerator does not.

$$8. \lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 - 5x + 6} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)(x-3)} = \lim_{x \rightarrow 2} \frac{x+2}{x-3} = -4$$

9. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 - 4x + 4} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)^2} = \lim_{x \rightarrow 2} \frac{x+2}{x-2}$ does not exist. The denominator approaches 0 (from both sides) while the numerator does not.

10. $\lim_{x \rightarrow 2^-} \frac{x^2 - 4}{x^2 - 4x + 4} = \lim_{x \rightarrow 2^-} \frac{x + 2}{x - 2} = -\infty$
11. $\lim_{x \rightarrow -2+} \frac{x^2 - 4}{x^2 + 4x + 4} = \lim_{x \rightarrow -2+} \frac{x - 2}{x + 2} = -\infty$
12. $\lim_{x \rightarrow 4} \frac{2 - \sqrt{x}}{x - 4} = \lim_{x \rightarrow 4} \frac{4 - x}{(2 + \sqrt{x})(x - 4)} = -\frac{1}{4}$
13. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{\sqrt{x} - \sqrt{3}} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)(\sqrt{x} + \sqrt{3})}{x - 3}$
 $= \lim_{x \rightarrow 3} (x + 3)(\sqrt{x} + \sqrt{3}) = 12\sqrt{3}$
14. $\lim_{h \rightarrow 0} \frac{h}{\sqrt{x + 3h} - \sqrt{x}} = \lim_{h \rightarrow 0} \frac{h(\sqrt{x + 3h} + \sqrt{x})}{(x + 3h) - x}$
 $= \lim_{h \rightarrow 0} \frac{\sqrt{x + 3h} + \sqrt{x}}{3} = \frac{2\sqrt{x}}{3}$
15. $\lim_{x \rightarrow 0+} \sqrt{x - x^2} = 0$
16. $\lim_{x \rightarrow 0} \sqrt{x - x^2}$ does not exist because $\sqrt{x - x^2}$ is not defined for $x < 0$.
17. $\lim_{x \rightarrow 1} \sqrt{x - x^2}$ does not exist because $\sqrt{x - x^2}$ is not defined for $x > 1$.
18. $\lim_{x \rightarrow 1-} \sqrt{x - x^2} = 0$
19. $\lim_{x \rightarrow \infty} \frac{1 - x^2}{3x^2 - x - 1} = \lim_{x \rightarrow \infty} \frac{(1/x^2) - 1}{3 - (1/x) - (1/x^2)} = -\frac{1}{3}$
20. $\lim_{x \rightarrow -\infty} \frac{2x + 100}{x^2 + 3} = \lim_{x \rightarrow -\infty} \frac{(2/x) + (100/x^2)}{1 + (3/x^2)} = 0$
21. $\lim_{x \rightarrow -\infty} \frac{x^3 - 1}{x^2 + 4} = \lim_{x \rightarrow -\infty} \frac{x - (1/x^2)}{1 + (4/x^2)} = -\infty$
22. $\lim_{x \rightarrow \infty} \frac{x^4}{x^2 - 4} = \lim_{x \rightarrow \infty} \frac{x^2}{1 - (4/x^2)} = \infty$
23. $\lim_{x \rightarrow 0+} \frac{1}{\sqrt{x - x^2}} = \infty$
24. $\lim_{x \rightarrow 1/2} \frac{1}{\sqrt{x - x^2}} = \frac{1}{\sqrt{1/4}} = 2$
25. $\lim_{x \rightarrow \infty} \sin x$ does not exist; $\sin x$ takes the values -1 and 1 in any interval (R, ∞) , and limits, if they exist, must be unique.
26. $\lim_{x \rightarrow \infty} \frac{\cos x}{x} = 0$ by the squeeze theorem, since

$$-\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x} \quad \text{for all } x > 0$$

$$\text{and } \lim_{x \rightarrow \infty} (-1/x) = \lim_{x \rightarrow \infty} (1/x) = 0.$$

27. $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$ by the squeeze theorem, since

$$-|x| \leq x \sin \frac{1}{x} \leq |x| \quad \text{for all } x \neq 0$$

$$\text{and } \lim_{x \rightarrow 0} (-|x|) = \lim_{x \rightarrow 0} |x| = 0.$$

28. $\lim_{x \rightarrow 0} \sin \frac{1}{x^2}$ does not exist; $\sin(1/x^2)$ takes the values -1 and 1 in any interval $(-\delta, \delta)$, where $\delta > 0$, and limits, if they exist, must be unique.

29. $\lim_{x \rightarrow -\infty} [x + \sqrt{x^2 - 4x + 1}]$
 $= \lim_{x \rightarrow -\infty} \frac{x^2 - (x^2 - 4x + 1)}{x - \sqrt{x^2 - 4x + 1}}$
 $= \lim_{x \rightarrow -\infty} \frac{4x - 1}{x - |x|\sqrt{1 - (4/x) + (1/x^2)}}$
 $= \lim_{x \rightarrow -\infty} \frac{x[4 - (1/x)]}{x + x\sqrt{1 - (4/x) + (1/x^2)}}$
 $= \lim_{x \rightarrow -\infty} \frac{4 - (1/x)}{1 + \sqrt{1 - (4/x) + (1/x^2)}} = 2.$

Note how we have used $|x| = -x$ (in the second last line), because $x \rightarrow -\infty$.

30. $\lim_{x \rightarrow \infty} [x + \sqrt{x^2 - 4x + 1}] = \infty + \infty = \infty$
31. $f(x) = x^3 - 4x^2 + 1$ is continuous on the whole real line and so is discontinuous nowhere.
32. $f(x) = \frac{x}{x + 1}$ is continuous everywhere on its domain, which consists of all real numbers except $x = -1$. It is discontinuous nowhere.
33. $f(x) = \begin{cases} x^2 & \text{if } x > 2 \\ x & \text{if } x \leq 2 \end{cases}$ is defined everywhere and discontinuous at $x = 2$, where it is, however, left continuous since $\lim_{x \rightarrow 2-} f(x) = 2 = f(2)$.
34. $f(x) = \begin{cases} x^2 & \text{if } x > 1 \\ x & \text{if } x \leq 1 \end{cases}$ is defined and continuous everywhere, and so discontinuous nowhere. Observe that $\lim_{x \rightarrow 1-} f(x) = 1 = \lim_{x \rightarrow 1+} f(x)$.
35. $f(x) = H(x - 1) = \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}$ is defined everywhere and discontinuous at $x = 1$ where it is, however, right continuous.
36. $f(x) = H(9 - x^2) = \begin{cases} 1 & \text{if } -3 \leq x \leq 3 \\ 0 & \text{if } x < -3 \text{ or } x > 3 \end{cases}$ is defined everywhere and discontinuous at $x = \pm 3$. It is right continuous at -3 and left continuous at 3 .
37. $f(x) = |x| + |x + 1|$ is defined and continuous everywhere. It is discontinuous nowhere.
38. $f(x) = \begin{cases} |x|/|x + 1| & \text{if } x \neq -1 \\ 1 & \text{if } x = -1 \end{cases}$ is defined everywhere and discontinuous at $x = -1$ where it is neither left nor right continuous since $\lim_{x \rightarrow -1} f(x) = \infty$, while $f(-1) = 1$.

Challenging Problems 1 (page 94)

1. Let $0 < a < b$. The average rate of change of x^3 over $[a, b]$ is

$$\frac{b^3 - a^3}{b - a} = b^2 + ab + a^2.$$

The instantaneous rate of change of x^3 at $x = c$ is

$$\lim_{h \rightarrow 0} \frac{(c+h)^3 - c^3}{h} = \lim_{h \rightarrow 0} \frac{3c^2h + 3ch^2 + h^3}{h} = 3c^2.$$

If $c = \sqrt{(a^2 + ab + b^2)/3}$, then $3c^2 = a^2 + ab + b^2$, so the average rate of change over $[a, b]$ is the instantaneous rate of change at $\sqrt{(a^2 + ab + b^2)/3}$.

Claim: $\sqrt{(a^2 + ab + b^2)/3} > (a+b)/2$.

Proof: Since $a^2 - 2ab + b^2 = (a-b)^2 > 0$, we have

$$\begin{aligned} 4a^2 + 4ab + 4b^2 &> 3a^2 + 6ab + 3b^2 \\ \frac{a^2 + ab + b^2}{3} &> \frac{a^2 + 2ab + b^2}{4} = \left(\frac{a+b}{2}\right)^2 \\ \sqrt{\frac{a^2 + ab + b^2}{3}} &> \frac{a+b}{2}. \end{aligned}$$

2. For x near 0 we have $|x-1| = 1-x$ and $|x+1| = x+1$. Thus

$$\lim_{x \rightarrow 0} \frac{x}{|x-1| - |x+1|} = \lim_{x \rightarrow 0} \frac{x}{(1-x) - (x+1)} = -\frac{1}{2}.$$

3. For x near 3 we have $|5-2x| = 2x-5$, $|x-2| = x-2$, $|x-5| = 5-x$, and $|3x-7| = 3x-7$. Thus

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{|5-2x| - |x-2|}{|x-5| - |3x-7|} &= \lim_{x \rightarrow 3} \frac{2x-5 - (x-2)}{5-x - (3x-7)} \\ &= \lim_{x \rightarrow 3} \frac{x-3}{4(3-x)} = -\frac{1}{4}. \end{aligned}$$

4. Let $y = x^{1/6}$. Then we have

$$\begin{aligned} \lim_{x \rightarrow 64} \frac{x^{1/3} - 4}{x^{1/2} - 8} &= \lim_{y \rightarrow 2} \frac{y^2 - 4}{y^3 - 8} \\ &= \lim_{y \rightarrow 2} \frac{(y-2)(y+2)}{(y-2)(y^2 + 2y + 4)} \\ &= \lim_{y \rightarrow 2} \frac{y+2}{y^2 + 2y + 4} = \frac{4}{12} = \frac{1}{3}. \end{aligned}$$

5. Use $a-b = \frac{a^3 - b^3}{a^2 + ab + b^2}$ to handle the denominator. We have

$$\begin{aligned} &\lim_{x \rightarrow 1} \frac{\sqrt{3+x} - 2}{\sqrt[3]{7+x} - 2} \\ &= \lim_{x \rightarrow 1} \frac{3+x-4}{\sqrt{3+x}+2} \times \frac{(7+x)^{2/3} + 2(7+x)^{1/3} + 4}{(7+x)-8} \\ &= \lim_{x \rightarrow 1} \frac{(7+x)^{2/3} + 2(7+x)^{1/3} + 4}{\sqrt{3+x}+2} = \frac{4+4+4}{2+2} = 3. \end{aligned}$$

6. $r_+(a) = \frac{-1 + \sqrt{1+a}}{a}$, $r_-(a) = \frac{-1 - \sqrt{1+a}}{a}$.

- a) $\lim_{a \rightarrow 0} r_-(a)$ does not exist. Observe that the right limit is $-\infty$ and the left limit is ∞ .
b) From the following table it appears that $\lim_{a \rightarrow 0} r_+(a) = 1/2$, the solution of the linear equation $2x - 1 = 0$ which results from setting $a = 0$ in the quadratic equation $ax^2 + 2x - 1 = 0$.

a	$r_+(a)$
1	0.41421
0.1	0.48810
-0.1	0.51317
0.01	0.49876
-0.01	0.50126
0.001	0.49988
-0.001	0.50013

$$\begin{aligned} \text{c) } \lim_{a \rightarrow 0} r_+(a) &= \lim_{a \rightarrow 0} \frac{\sqrt{1+a} - 1}{a} \\ &= \lim_{a \rightarrow 0} \frac{(1+a) - 1}{a(\sqrt{1+a} + 1)} \\ &= \lim_{a \rightarrow 0} \frac{1}{\sqrt{1+a} + 1} = \frac{1}{2}. \end{aligned}$$

7. TRUE or FALSE

- a) If $\lim_{x \rightarrow a} f(x)$ exists and $\lim_{x \rightarrow a} g(x)$ does not exist, then $\lim_{x \rightarrow a} (f(x) + g(x))$ does not exist.
TRUE, because if $\lim_{x \rightarrow a} (f(x) + g(x))$ were to exist then

$$\begin{aligned} \lim_{x \rightarrow a} g(x) &= \lim_{x \rightarrow a} (f(x) + g(x) - f(x)) \\ &= \lim_{x \rightarrow a} (f(x) + g(x)) - \lim_{x \rightarrow a} f(x) \end{aligned}$$

would also exist.

- b) If neither $\lim_{x \rightarrow a} f(x)$ nor $\lim_{x \rightarrow a} g(x)$ exists, then $\lim_{x \rightarrow a} (f(x) + g(x))$ does not exist.
FALSE. Neither $\lim_{x \rightarrow 0} 1/x$ nor $\lim_{x \rightarrow 0} (-1/x)$ exist, but $\lim_{x \rightarrow 0} ((1/x) + (-1/x)) = \lim_{x \rightarrow 0} 0 = 0$ exists.

- c) If f is continuous at a , then so is $|f|$.
TRUE. For any two real numbers u and v we have

$$||u| - |v|| \leq |u - v|.$$

This follows from

$$\begin{aligned} |u| &= |u - v + v| \leq |u - v| + |v|, \quad \text{and} \\ |v| &= |v - u + u| \leq |v - u| + |u| = |u - v| + |u|. \end{aligned}$$

Now we have

$$||f(x)| - |f(a)|| \leq |f(x) - f(a)|$$

so the left side approaches zero whenever the right side does. This happens when $x \rightarrow a$ by the continuity of f at a .

- d) If $|f|$ is continuous at a , then so is f .
FALSE. The function $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ is discontinuous at $x = 0$, but $|f(x)| = 1$ everywhere, and so is continuous at $x = 0$.

- e) If $f(x) < g(x)$ in an interval around a and if $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$ both exist, then $L < M$.

FALSE. Let $g(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ and let $f(x) = -g(x)$. Then $f(x) < g(x)$ for all x , but $\lim_{x \rightarrow 0} f(x) = 0 = \lim_{x \rightarrow 0} g(x)$. (Note: under the given conditions, it is TRUE that $L \leq M$, but not necessarily true that $L < M$.)

8. a) To be proved: if f is a continuous function defined on a closed interval $[a, b]$, then the range of f is a closed interval.
Proof: By the Max-Min Theorem there exist numbers u and v in $[a, b]$ such that $f(u) \leq f(x) \leq f(v)$ for all x in $[a, b]$. By the Intermediate-Value Theorem, $f(x)$ takes on all values between $f(u)$ and $f(v)$ at values of x between u and v , and hence at points of $[a, b]$. Thus the range of f is $[f(u), f(v)]$, a closed interval.
- b) If the domain of the continuous function f is an open interval, the range of f can be any interval (open, closed, half open, finite, or infinite).

9. $f(x) = \frac{x^2 - 1}{|x^2 - 1|} = \begin{cases} -1 & \text{if } -1 < x < 1 \\ 1 & \text{if } x < -1 \text{ or } x > 1 \end{cases}$
 f is continuous wherever it is defined, that is at all points except $x = \pm 1$. f has left and right limits -1 and 1 , respectively, at $x = 1$, and has left and right limits 1 and -1 , respectively, at $x = -1$. It is not, however, discontinuous at any point, since -1 and 1 are not in its domain.

10. $f(x) = \frac{1}{x - x^2} = \frac{1}{\frac{1}{4} - (\frac{1}{4} - x + x^2)} = \frac{1}{\frac{1}{4} - (x - \frac{1}{2})^2}$.
Observe that $f(x) \geq f(1/2) = 4$ for all x in $(0, 1)$.

11. Suppose f is continuous on $[0, 1]$ and $f(0) = f(1)$.

- a) To be proved: $f(a) = f(a + \frac{1}{2})$ for some a in $[0, \frac{1}{2}]$.
Proof: If $f(1/2) = f(0)$ we can take $a = 0$ and be done. If not, let

$$g(x) = f(x + \frac{1}{2}) - f(x).$$

Then $g(0) \neq 0$ and

$$g(1/2) = f(1) - f(1/2) = f(0) - f(1/2) = -g(0).$$

Since g is continuous and has opposite signs at $x = 0$ and $x = 1/2$, the Intermediate-Value Theorem assures us that there exists a between 0 and $1/2$ such that $g(a) = 0$, that is, $f(a) = f(a + \frac{1}{2})$.

- b) To be proved: if $n > 2$ is an integer, then $f(a) = f(a + \frac{1}{n})$ for some a in $[0, 1 - \frac{1}{n}]$.
Proof: Let $g(x) = f(x + \frac{1}{n}) - f(x)$. Consider the numbers $x = 0, x = 1/n, x = 2/n, \dots, x = (n-1)/n$. If $g(x) = 0$ for any of these numbers, then we can let a be that number. Otherwise, $g(x) \neq 0$ at any of these numbers. Suppose that the values of g at all these numbers has the same sign (say positive). Then we have

$$f(1) > f(\frac{n-1}{n}) > \dots > f(\frac{2}{n}) > \frac{1}{n} > f(0),$$

which is a contradiction, since $f(0) = f(1)$. Therefore there exists j in the set $\{0, 1, 2, \dots, n-1\}$ such that $g(j/n)$ and $g((j+1)/n)$ have opposite sign. By the Intermediate-Value Theorem, $g(a) = 0$ for some a between j/n and $(j+1)/n$, which is what we had to prove.