# Solutions - Book - Chapter 3:

3.1.1. (a) and (c) are symmetric.

3.1.2. (a) Not symmetric.
$1<2$, but $2 \not< 1$.

(b) Symmetric.

(c) Symmetric.

(d) Not symmetric.
3 is a factor of 12, but 12 is not a factor of 3.

(e) Symmetric.

(f) Symmetric

(g) Symmetric.

3.1.3. (a) Transitive.

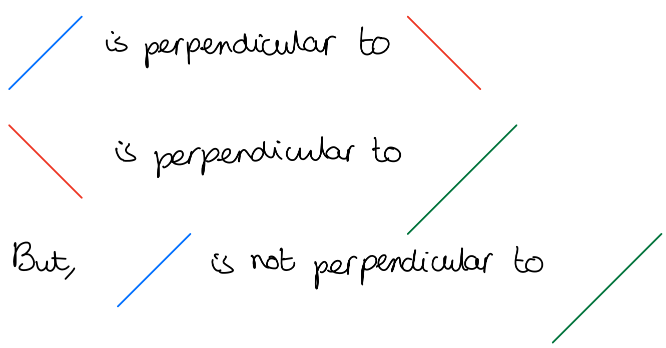
(b) Not transitive.
big is the opposite of small.
small is the opposite of big.
But, big is not the opposite of big.

(c) Transitive.

(d) Transitive.

(e) Transitive.

(f) Not transitive.




(g) Transitive.

3.1.4. (a) Not reflexive
$1 \not< 1$.

(b) Not reflexive.
Big is not the opposite of big.

(c) Reflexive.

(d) Reflexive.

(e) Reflexive.

(f) Not reflexive
A line is not perpendicular to itself.

(g) it depends .... on your definition of parallel.

Answers to some exercises from the book: 3.2.

1(a) yes (ie. is an equiv. relation)

(b) no (not symmetric)(and also not reflexive).

(c) no (not symmetric)(and also not reflexive).

(d) no (a line is not perpendicular to itself)(and also not transitive)

(e) ~~yes~~ (it depends on your definition of parallel whether this relation is reflexive).

2(a) reflexive? yes, $m^2 - m^2 = 0$ and that is divisible by 3. ✓

symmetric? yes, if $m^2 - n^2 = 3k$ (for some integer $k$)

then $n^2 - m^2 = -3k$

$= 3(-k)$ so also divisible by 3. ✓

transitive?

Suppose $mRn$ and $nRp$. So $m^2 - n^2$ is divisible by 3 and $n^2 - p^2$ is divisible by 3.

So $m^2 - n^2 = 3k_1$

and $n^2 - p^2 = 3\cancel{p}k_2$ (where $k_1$ and $k_2$ are integers)

So $m^2 - p^2 = 3(k_1 + k_2)$. ✓ So it is an equivalence relation.

~~By experimenting you should soon see that there are 3 equiv. classes~~

~~$\ldots -9, -6, -3, 0, 3, 6, 9 \ldots$~~

~~$\ldots -8, -5, -2, 1, 4, 7, 10 \ldots$~~ (check if you like)

~~$\ldots -7, \cancel{-4}, -1, 2, 5, 8, 11 \ldots$~~

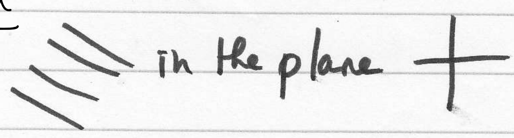
No! This answer is wrong. There are only two equivalence classes: (1) multiples of 3 (2) all the rest!

(b) Reflexive? Yes, because $x + y = x + y$.

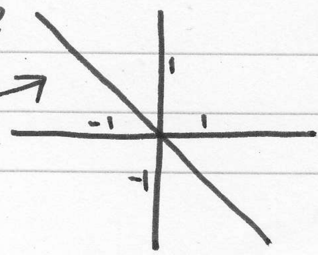
Symmetric? Yes, because $x + y = a + b \Rightarrow a + b = x + y$

Transitive? Yes because $x + y = a + b$ and $a + b = c + d \Rightarrow x + y = c + d$.

The equivalence classes are parallel lines in the plane



eg. $E_0 = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x + y = 0\}$



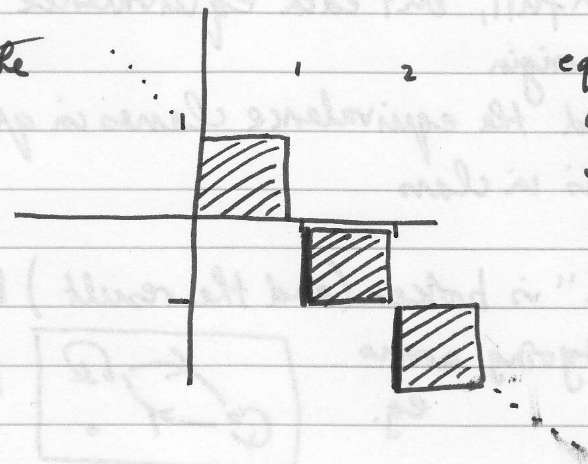

There is an equiv. class for every real

2(c) Reflexive because $\lfloor x \rfloor + \lfloor y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$
Symmetric because $\lfloor x \rfloor + \lfloor y \rfloor = \lfloor a \rfloor + \lfloor b \rfloor \implies \lfloor a \rfloor + \lfloor b \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$

Transitive: suppose $(x,y) R (a,b)$ and $(a,b) R (c,d)$.
Then $\lfloor x \rfloor + \lfloor y \rfloor = \lfloor a \rfloor + \lfloor b \rfloor$ and $\lfloor a \rfloor + \lfloor b \rfloor = \lfloor c \rfloor + \lfloor d \rfloor$

$$\implies \lfloor x \rfloor + \lfloor y \rfloor = \lfloor c \rfloor + \lfloor d \rfloor \implies (x,y) R (c,d) \checkmark$$

~~Equiv classes will again be squares in the plane~~

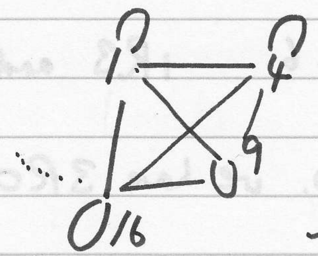



equivalence classes are quite odd. Here, for example, are all the points $(x,y)$ with the property ~~(x,y)~~ $\lfloor x \rfloor + \lfloor y \rfloor = 0$.

There will be similar patterns for all integers $k$ in the expression $\lfloor x \rfloor + \lfloor y \rfloor = k$.

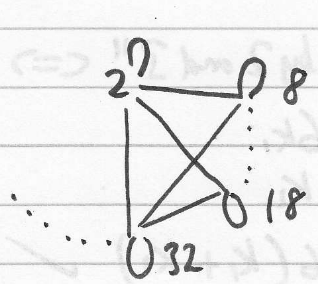
2(d) this is not an equivalence relation! It is not transitive: $(1,2) R (0,0)$
and $(0,0) R (3,4)$ but $(1,2) \not R (3,4)$!

3(a) Equivalence class of 1 is everything related to 1.
That is, all $n$ such that $1 \times n$ is a square

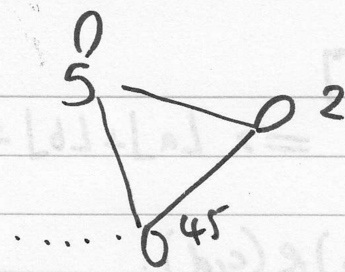
So: 

So $E_1$ contains all the squares.

(b) equiv class of 2 is all $n$ such that $2n$ is a square.



← all the squares divisible by 2, after being divided by 2

(c) 


all the squares divisible by 5, after being divided by 5.

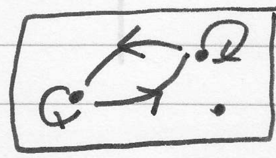
(d) all the squares divisible by $p$, after being divided by $p$.

4(a) won't do it in full, but each equivalence class is a circle around the origin

(b) we computed the equivalence classes in question 3

(c) we did this in class. The plane is partitioned in squares. For example, the equivalence class of $(1\frac{1}{4}, 2\frac{1}{2})$ is $\{(x,y) \in \mathbb{R}^2 : 1 \le x < 2, 2 \le y < 3\}$

5. The "proof" is broken (and the result) because not all elements have outgoing arrows

eg. 


This is ~~reflex~~ symmetric, transitive but not reflexive!

Ex 3.3

1. This is not a partial order! It is not anti-symmetric:

$3$ divides $-3$

$-3$ divides $3$ but $3 \neq -3$!

2(a) reflexive (because $m-m=0$ which is divisible by 2 or 3)

symmetric (because $m-n = -(n-m)$ which doesn't change divisibility)

NOT transitive:

$m=1$ $n=3$ $p=6$. $1R3$ and $3R6$ but $1 \not R 6$.

NOT anti-symmetric:

eg. $n=3$, $m=0$. we have $3R0$ and $0R3$ but $0 \neq 3$.

(b) reflexive, symmetric, NOT antisymmetric (same proof as above)

This is transitive because "divisible by 2 and 3" $\iff$ "divisible by 6"

so $mRn \Rightarrow m-n = 6k_1$

$nRp \Rightarrow n-p = 6k_2$

so $m-p = 6(k_1+k_2)$ ✓.

(c) 2 5

(c) this is a well-known partial order: reflexive, ~~sym~~ transitive, anti-symmetric (and not symmetric in this case).

(d) reflexive? yes because $m < m+1$.
symmetric? no because $1 < 10+1$ but it's not true that $10 < 1+1$.

transitive? no, because $2 \cdot 9 < 2+1$
and $2 < 1 \cdot 1 + 1$
but $2 \cdot 9 \not< 1 \cdot 1 + 1$.

anti-symmetric? no: $1 < 1 \cdot 5 + 1$
and $1 \cdot 5 < 1 + 1$
but $1 \neq 1 \cdot 5$!

(e) reflexive? no, because eg. $3 \not< 3-1$.
symmetric? no, because $3 < 10-1$ but $10 \not< 3-1$.

Transitive. suppose $mRn \Rightarrow m < n-1$
suppose $nRp \Rightarrow n < p-1 \Rightarrow n-1 < p-2$.
So $m < p-2 \Rightarrow m < p-1 \Rightarrow mRp$. So yes.

anti-symmetric? yes, if $mRn$ then $m < n-1$. Now, is it possible for $nRm$ to hold too? That requires $n < m-1$.
but $m < n-1$ so $n < (n-1)-1$
$n < n-2$ which is not possible.

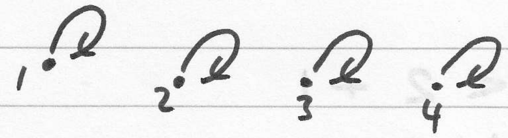
So it is ~~any~~ anti-symmetric.

3. yes it is.

4. no. dog R fog and fog R dog but fog $\neq$ dog (ie. not antisymmetric)

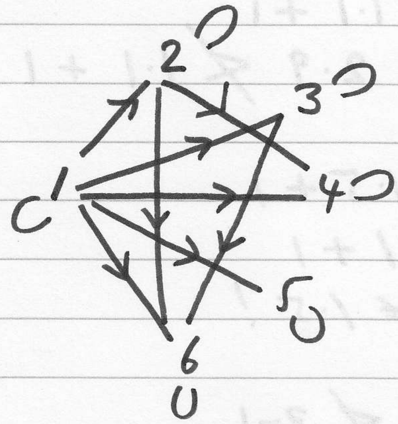
3.4

1 (a)



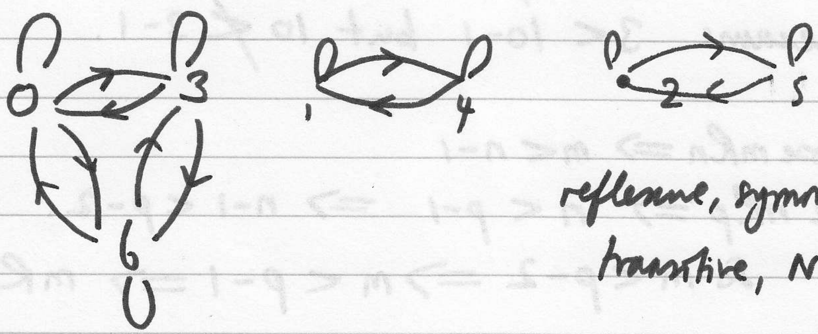

reflexive, symmetric, transitive, anti-symmetric.

(b)




reflexive, NOT symmetric, transitive, anti-symmetric.

(c)




reflexive, symmetric, transitive, NOT anti-symmetric.

Q2 ↑

3 (a) all 4

(b) reflexive, not transitive, not symmetric, anti-symmetric.

(c) reflexive, transitive, not symmetric, anti-symmetric.

(d) not reflexive, not transitive, not symmetric, anti-symmetric.

F



3.5.

1(a) no ($-1$ not in $\mathbb{N}$)

(b) yes.

(c) no, $f(1)$ not in $\mathbb{N}$

(d). no, $f(0)$ is defined in two ways.

(e) yes.

2(a) domain = $\mathbb{Z}$, co-domain = $\mathbb{N}$, range = $\{\}$.

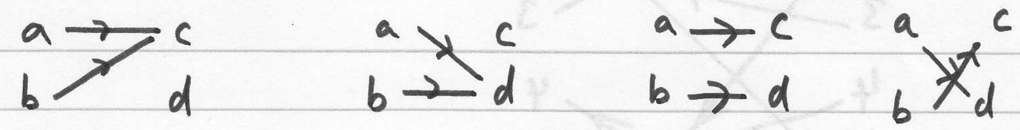
(b) all three are $\mathbb{Z}$

(c). domain = $\mathbb{N}$, co-domain = $\mathbb{Z}$, range = $\{-1, 0, 1, 2, 3, \ldots\}$.

(d). domain = ~~$\mathbb{N}$~~ $\mathbb{Z}$, co-domain = $\mathbb{Z}$, range = $\mathbb{Z}^+ = \{x \in \mathbb{Z} : x \geq 0\}$.

3.
4. } fsck that!
5.

3.6 (a) there are 4 functions



neither injective nor sujective.

injective and sujective

2(a) $f(x) = x$

(b). $f(x) = 2x$

(c). $f(1) = 1$

$f(x) = x-1$ (if $x \geq 2$).

(d) $f(1) = 1$

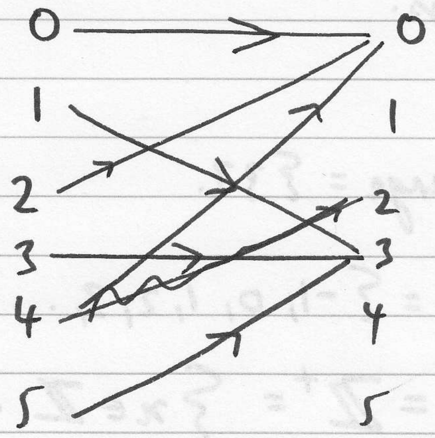
$f(2) = 1$

$f(x) = x$ ($x \geq 3$).

3. It's certainly not surjective because nothing maps to odd numbers!

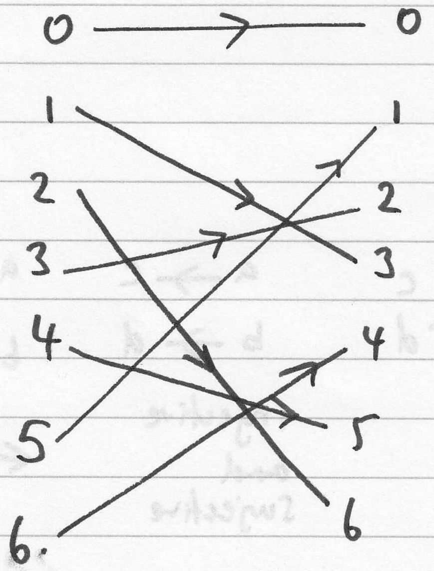
is it injective? no. $f(3) = f(6)$ and $3 \neq 6$.

4.




neither injective nor surjective.

5.




This is injective and ~~bijective~~ surjective.
(ie. it's a bijection)

3.7 1. $(f\circ f)(1) = f(f(1)) = f(3) = 2.$

$(f\circ f)(2) = f(f(2)) = f(1) = 3.$

$(f\circ f)(3) = f(f(3)) = f(2) = 1.$

$(f\circ f\circ f)(1) = f(f(f(1))) = f(2) = 1$

$\ldots(2) = f(f(f(2))) = f(3) = 2.$

$\ldots(3) = f(f(f(3))) = f(1) = 3.$

$(f\circ f\circ f\circ f)(1) = f(f(f(f(1)))) = f(1) = 3.$

$(2) = f(\ldots 2) = f(2) = 1$

$(3) = f(\ldots 3) = f(3) = 2.$

2. $f\circ f = f(f(x)) = f(x^3) = (x^3)^3$

$g\circ g = g(g(x)) = g(x-3) = x-6$

$f\circ g = f(g(x)) = f(x-3) = (x-3)^3.$

$g\circ f = g(f(x)) = g(x^3) = x^3-3.$

3. $f\circ f = f(f(x)) = f(x^2) = (x^2)^2$

$$g\circ g = \begin{cases} x-6 & x<0. \\ 4x & x\geq 0. \end{cases}$$

$$f\circ g = f(g(x)) = \begin{cases} (x-3)^2 & : x<0. \\ (2x)^2 & : x\geq 0. \end{cases}$$

$g\circ f = g(f(x)) = g(x^2) = 2x^2.$

4(a) Proof by contrapositive. Assume $f$ is ~~not~~ <u>not</u> injective. Then there exists $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

But then $(g \circ f)(x) = g(f(x))$

$(g \circ f)(y) = g(f(y))$ ) So $x$ and $y$ map to same element.

So $(g \circ f)$ is not injective.

Hence $g \circ f$ injective $\Rightarrow f$ is injective.

(b) Let $x$ and $y$ be arbitrary elements of $A$ such that $a \neq b$.

Now, $f(x) \neq f(y)$ because $f$ is injective.

and $g(f(x)) \neq g(f(y))$ because $g$ is injective.

Hence $(g \circ f)$ is injective.

(c) Let $c$ be an arbitrary element of $C$. Because $g$ is surjective, there exists a $b \in B$ s.t. $g(b) = c$. Furthermore, because $f$ is surjective, there exists an $a \in A$ such that $f(a) = b$.

Now, $(g \circ f)(a) = g(f(a)) = g(b) = c$. ✓.

5. $(f \circ g)(x) = (3x - 1)^2$.

$(g \circ f)(x) = 3x^2 - 1$.

$9x^2 - 6x + 1 = 3x^2 - 1$

~~$6x^2 - 6x + 1 = 0$~~

$6x^2 - 6x + 2 = 0$.

$3x^2 - 3x + 1 = 0$.

Perhaps I made a mistake but this has no (real) solutions!

6. They fsck'd this question up ...

ignore it.

3.8. #1.

(a) $y = 3x + 2.$

$x = \frac{y-2}{3}.$ so $f^{-1}(x) = \frac{x-2}{3}.$

(it is easy to check that $f$ is injective and surjective.)

Check: is $f^{-1}(f(x)) = x$?

Is $f^{-1}(3x+2) = x$?

is $\frac{(3x+2)-2}{3} = x$? yes ✓.

(b) This is not even a function, because $f(0.5) < 0$!

(c) This is not surjective, so not a bijection, so not invertible.
It is not surjective because (e.g.) there is no $x \in \mathbb{Z}$ such that $f(x) = 3$.

(d) not surjective — nothing maps to 2!

2. Quick check: do we actually believe it is invertible?
is it injective? yes:- suppose $x \neq y$. if $x, y \geq 1$ or $x, y < 1$ then it is not possible for $f(x) = f(y)$.
if $x < 1$ and $y \geq 1$ then $f(x) < 1$ and $f(y) \geq 1$, so then it is also not possible for $f(x) = f(y)$.

is it surjective?
suppose $y \geq 1$: take $x = {}^{+}\sqrt{y}$
suppose $y < 1$: take $x = y$.

So the inverse seems to be: $f^{-1}(x) = \begin{cases} {}^{+}\sqrt{x} & : x \geq 1 \\ x & : x < 1. \end{cases}$

Check, is $f^{-1}(f(x)) = x$?

suppose $x \geq 1$, then $f^{-1}(f(x)) = f^{-1}(x^2) = {}^{+}\sqrt{x^2} = x.$
Suppose $x < 1$, then $f^{-1}(f(x)) = f^{-1}(x) = x.$ ✓

3. There are $b^a$ functions from $A$ to $B$.

$\cancel{\underbrace{b \times (b-1) \times (b-2) \cdots}_{a \text{ times}}}$ ~~of them will be~~

The functions will only be invertible if $|A| = |B|$, and then there will be $n!$ of them where $n = |A| = |B|$. Recall $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$.

4. This was actually in bonus exercises 5. (answers on Eleum)

5. So was this! (answers on Eleum).