

Lecture 11 - Calculus

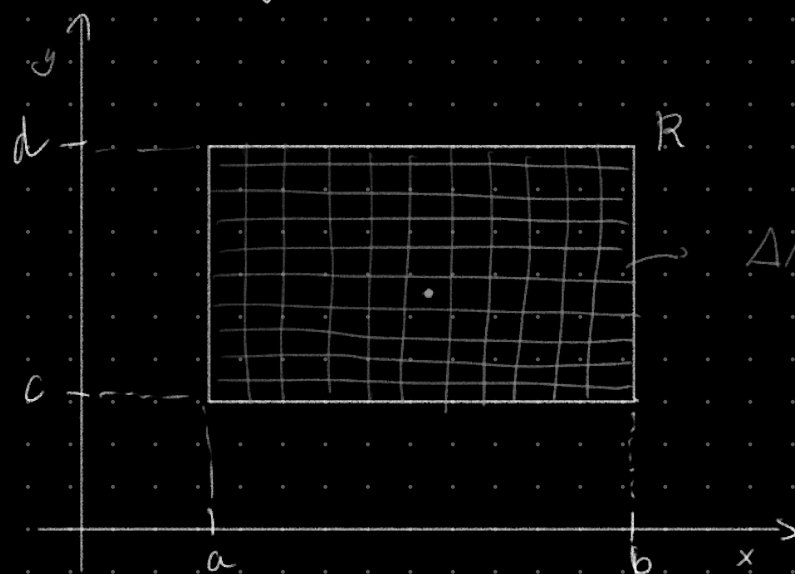
- Limits and continuity
- Differentiation + applications
- Integration
- Sequences and series
- Differential equations
- Introduction to multivariate functions
- Double integrals: today last lecture!

Thomas' Ch. 15.1-2 or Adams' Ch. 14.1-2

Double integrals

Let $f(x,y)$, continuous on a region R .

$$R: \begin{aligned} a &\leq x \leq b \\ c &\leq y \leq d \end{aligned}$$



How to calculate $\iint_R f(x,y) dA$

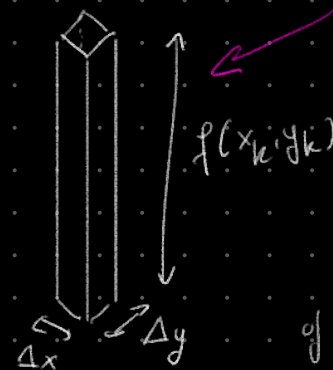
↳ (signed) volume between R and the surface $f(x,y)$

partition of R into small rect. angles ΔA_k , $\Delta A_k = \Delta x \Delta y$

→ Riemann sum

$$S_n = \sum_{k=1}^n \underbrace{\Delta A_k \cdot f(x_k, y_k)}$$

$f(x,y)$ is integrable if $\lim_{n \rightarrow \infty} S_n = I$



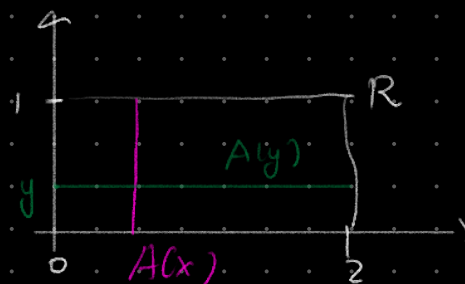
$f(x_k, y_k) \Delta x \Delta y$ is a volume element of the Riemann sum

Calculation of a double integral

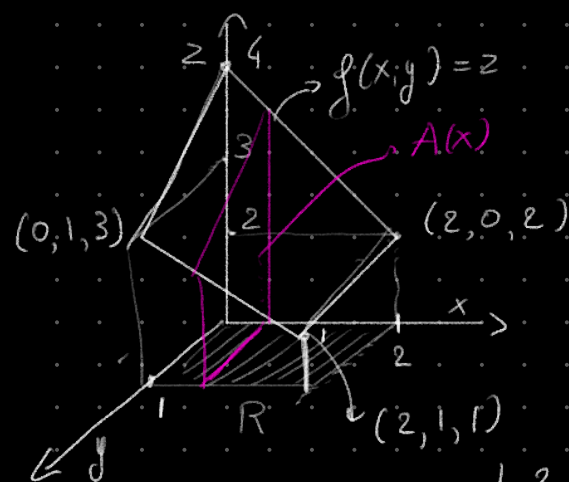
Example : $f(x,y) = 4-x-y$

$R : 0 \leq x \leq 2$

$0 \leq y \leq 1$



$$\iint_R f(x,y) dA = \int_0^2 \underbrace{\int_0^1 (4-x-y) dy}_{A(x)} dx$$



$$A(x) = \int_0^1 (4-x-y) dy = 4-x-\frac{1}{2} = \frac{7}{2}-x$$

$$\int_0^2 A(x) dx = \int_0^2 \left(\frac{7}{2}-x\right) dx = 7-2 = 5$$

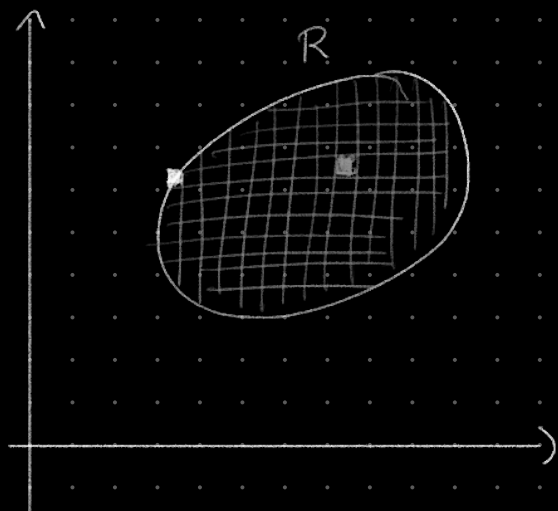
$$\iint_R f(x,y) dA = \int_0^1 \underbrace{\int_0^2 (4-x-y) dx}_{A(y)} dy$$

Fubini's theorem

If $f(x, y)$ is continuous on the rectangular area R : $a \leq x \leq b$
 $c \leq y \leq d$,

$$\text{then } \iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy.$$

Double integrals over general regions

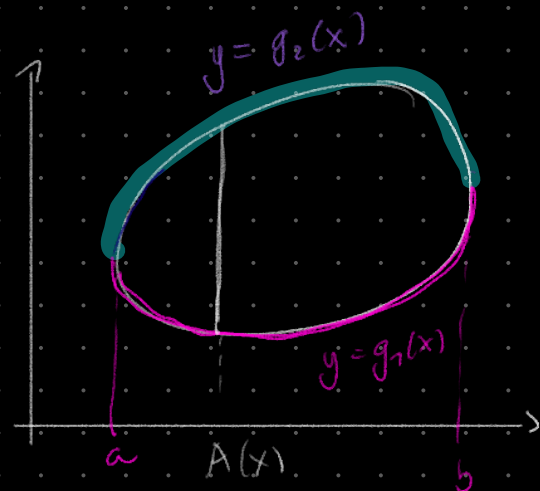


$\iint_R f(x,y) dA$ is the volume between $f(x,y)$ and R

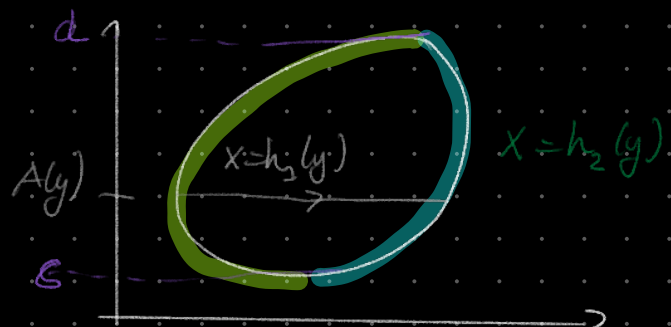
The double integral is the limit of the Riemann sums

$$\iint_R f(x,y) dA = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k, y_k) \Delta A_k$$

Calculating integrals over general regions



$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$



$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

Fubini's theorem (stronger form)

If $f(x, y)$ is continuous on a region R

→ if R is defined as $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$

with $g_1(x)$ and $g_2(x)$ continuous on $[a, b]$, then

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

→ if R is defined as $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$

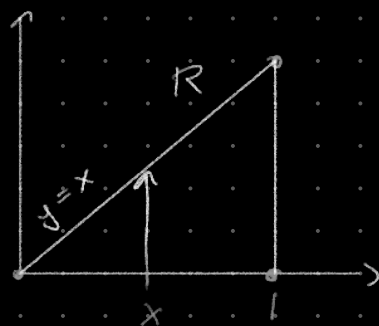
with $h_1(y)$ and $h_2(y)$ continuous on $[c, d]$, then

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

Example

$$f(x, y) = 3 - x - y$$

R = triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$

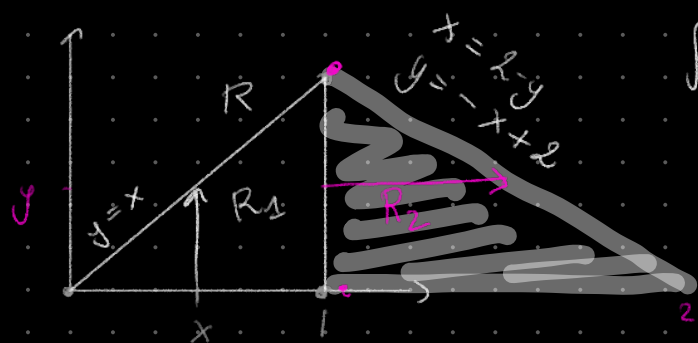


$$\begin{aligned} x &: 0 \rightarrow 1 \\ y &: 0 \rightarrow x \end{aligned}$$

$$\iint_R f(x, y) dA = \int_0^1 \overbrace{\int_0^x (3 - x - y) dy}^{A(x)} dx$$

$$A(x) = \int_0^x (3 - x - y) dy = \left[3y - xy - \frac{y^2}{2} \right]_0^x = 3x - x^2 - \frac{x^2}{2} = 3x - \frac{3x^2}{2}$$

$$\int_0^1 A(x) dx = \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_0^1 = \frac{3}{2} - \frac{1}{2} = 1$$



$$\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA$$

$$\begin{aligned} R_2: & y: 0 \rightarrow 1 \\ & x: 1 \rightarrow 2 - y \end{aligned}$$

$$\iint_R (x+y) \, dA$$



$$x: 0 \rightarrow 1$$

OR

$$y: 0 \rightarrow 1$$

$$y: 0 \rightarrow 1-x$$

$$x: 0 \rightarrow 1-y$$

$$\int_0^1 \int_0^{1-x} (x+y) \, dy \, dx$$

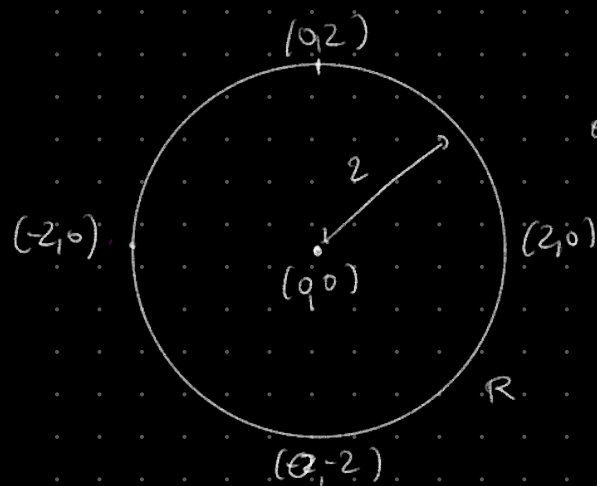
$$A(x) = \int_0^{1-x} (x+y) \, dy = \left[xy + \frac{y^2}{2} \right]_0^{1-x} = x(1-x) + \frac{(1-x)^2}{2} = \frac{1}{2} - \frac{x^2}{2}$$

$$= \cancel{x} - x^2 + \frac{1}{2} - \cancel{x} + \frac{x^2}{2}$$

$$\int_0^1 A(x) \, dx = \int_0^1 \left(\frac{1}{2} - \frac{x^2}{2} \right) \, dx = \frac{1}{2} - \left[\frac{x^3}{6} \right]_0^1 = \frac{1}{3}$$

An example with a more complex region R .

(the example in class was too complicated.)



equation of circle $x^2 + y^2 = 2^2 \Rightarrow y = \pm \sqrt{4 - x^2}$

$R: x: -2 \rightarrow 2$
 $y: -\sqrt{4-x^2} \rightarrow +\sqrt{4-x^2}$

$f(x,y) = x + y^2$

$\iint_R f(x,y) dA = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x + 2y^2) dy dx$

$A(x) = \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x + 2y^2) dy = \left[xy + \frac{2}{3} y^3 \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} = 2x\sqrt{4-x^2} + \frac{4}{3} \sqrt{4-x^2}^3$

$\int_{-2}^2 A(x) dx = \int_{-2}^2 2x\sqrt{4-x^2} dx + \frac{4}{3} \int_{-2}^2 \sqrt{4-x^2}^3 dx = \frac{8}{3} \int_{-\pi/2}^{\pi/2} \sqrt{4-\sin^2(u)} \cos(u) du$

$x = 2\sin(u)$
 $dx = 2\cos(u) du$
 $x = \pm 2 \rightarrow u = \pm \frac{\pi}{2}$

odd function, symmetric domain.

$$= \frac{16}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(u) du = \frac{4}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos(2u))^2 du = \frac{4}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + 2\cos(2u) + \left(\frac{1}{2} + \frac{\cos(4u)}{2}\right)\right) du$$

$$\cos^4(u) =$$

$$\cos(2u) = 2\cos^2 u - 1 \rightarrow \cos^2 u = \frac{1}{2} (1 + \cos(2u))$$

$$\cos^2(2u) = \frac{1}{2} (1 + \cos(4u))$$

$$= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du + \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2u) du + \frac{2}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(4u) du = 2\pi + \frac{4}{3} [\sin(2u)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{6} [\sin(4u)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \boxed{2\pi}$$

THIS INTEGRAL IS A DEMONSTRATION THAT DOUBLE INTEGRALS CAN BECOME VERY DIFFICULT QUICKLY

→ such an integral is way too long and complex for the exam!!

Properties of double integrals

For $f(x,y)$, $g(x,y)$ continuous on a bounded region R .

$$* \iint_R C \cdot f(x,y) dA = C \cdot \iint_R f(x,y) dA \quad \text{for } C \in \mathbb{R}$$

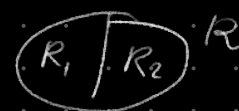
$$* \iint_R (f(x,y) \pm g(x,y)) dA = \iint_R f(x,y) dA \pm \iint_R g(x,y) dA$$

$$* \text{ if } f(x,y) \geq 0 \text{ on } R, \text{ then } \iint_R f(x,y) dA \geq 0.$$

\rightarrow choose parametrization correctly!

$$* \text{ if } f(x,y) \geq g(x,y) \text{ on } R, \text{ then } \iint_R f(x,y) dA \geq \iint_R g(x,y) dA$$

* if R is partitioned into R_1 and R_2



$$\text{then } \iint_R f(x,y) dA = \iint_{R_1} f(x,y) dA + \iint_{R_2} f(x,y) dA$$

