Exam Discrete Mathematics.

Question 1:

p	q	$p \Rightarrow q$	$p \wedge \neg q$	$q \wedge \neg q$	$(p \wedge \neg q) \Rightarrow (q \wedge \neg q)$	$(\dots) \Leftrightarrow (\dots)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	F	F	T	T

Question 2:

(a) Base case: $4^{3\cdot 1}+8 = 4^3+8 = 64+8 = 72$, which is divisible by 9. ✓

Induction step: Let $n \in \mathbb{N}$.

Assume $4^{3n}+8$ is divisible by 9. So, $4^{3n}+8 = 9\cdot k$, where $k \in \mathbb{Z}$.

Then,
$$4^{3(n+1)}+8 = 4^{3n+3}+8 = 4^{3n}\cdot 4^3+8 = 64\cdot 4^{3n}+8 = 4^{3n}+8+63\cdot 4^{3n}$$
$$= 9k + 9\cdot 7\cdot 4^{3n} = 9\cdot(k+7\cdot 4^{3n}),$$
which is divisible by 9 because $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. ✓ $\square$

(b) Consider $n=7$.

Then $3^n = 3^7 = 2187 \neq 5040 = 7! = n!$.

Question 3:

(a) False

Consider $A=\{1\}$, $B=\{2\}$ and $C=\{1\}$. Then, $A\setminus C = \emptyset \subseteq \{2\} = B\setminus C$.

However, $A \not\subseteq B$.

So, the implication "$\Leftarrow$" is not true.

(Note that the implication "$\Rightarrow$" is true).

(b) "$\{9^n : n\in\mathbb{N}\} \neq \{3^n : n \in \mathbb{N}\}$"

Consider $x=3$.

Then, $x \in \{3^n : n\in\mathbb{N}\}$ because $x = 3^1$

However, $x \notin \{9^n : n\in\mathbb{N}\}$, because $9^n \geq 9$ for all $n \in \mathbb{N}$. ✓

"$\{9^n : n\in\mathbb{N}\} \subseteq \{3^n : n \in \mathbb{N}\}$"

Let $x \in \{9^n : n \in \mathbb{N}\}$

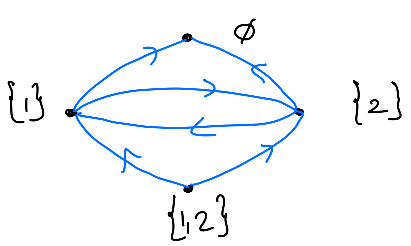
Then, there exists an $n\in\mathbb{N}$ such that $x = 9^n$

Hence, $x = 9^n = (3^2)^n = 3^{2n}$.

Since $2n \in \mathbb{N}$ (because $n\in\mathbb{N}$), we have $x \in \{3^n : n\in\mathbb{N}\}$. ✓ $\square$

Question 4:

(a)



(b). * Reflexive: No, consider $X=\{1\}$. Then $|X\setminus X| = |\emptyset| = 0 \neq 1$. So, $X \not{R} X$.

* Symmetry: No, consider $X=\{1,2\}$ and $Y=\{1\}$. Then, $|X\setminus Y| = |\{2\}| = 1$. So, XRY.
However, $|Y\setminus X| = |\emptyset| = 0 \neq 1$. So, $Y \not{R} X$.

* Transitivity: No, consider $X=\{1,2\}$, $Y=\{1\}$ and $Z=\emptyset$.
Then, $|X\setminus Y| = |\{2\}| = 1$ So, XRY.
And, $|Y\setminus Z| = |\{1\}| = 1$ So, YRZ.
However, $|X\setminus Z| = |\{1,2\}| = 2 \neq 1$. So, $X \not{R} Z$.

* Anti-symmetry: No, consider $X=\{1\}$ and $Y=\{2\}$.
Then, $|X\setminus Y| = |\{1\}| = 1$. So, XRY.
But also $|Y\setminus X| = |\{2\}| = 1$. So, YRX.

(c) Note that: $5x+y$ is even
$\iff$
((x is even) and (y is even)) or ((x is odd) and (y is odd)).

Hence, R is an equivalence relation with two equivalence classes (namely, the even numbers and the odd numbers). So, $\boxed{b}$

Question 5:

(a) We need to choose the positions for the 3 ones.

$n=8$
$k=3$
repetition is not allowed
order is not important

$$\binom{n}{k} = \binom{8}{3} = 56.$$

So, the answer is $\boxed{e}$.

(b) So, we have 2 bars and 10 stars.
For example, the solution $x=3$, $y=4$, $z=3$ corresponds with $***|****|***$

$n=3$
$k=10$
repetition is allowed
order is not important

$$\binom{(n-1)+k}{k} = \binom{2+10}{10} = \binom{12}{10} = 66$$

So, the answer is $\boxed{c}$

(c) Denote U: set of passwords made from capital letters and lower case letters.
X: set of passwords made from lower case letters.
We need to calculate $|U\setminus X|$.
$|U| = (26+26)^5 = 380204032$.
$|X| = 26^5 = 11881376$.
$|U\setminus X| = |U| - |X| = 368322656$.
So, the answer is $\boxed{f}$

Question 6:

(a). True.
Proof: Take $x=-1$. Let $y\in\mathbb{Z}$ and let $z\in\mathbb{Z}$.
Assume $x=yz$. So, assume $yz=-1$.
Then, since y and z are both integers, we know $y=1$ and $z=-1$, or the other way around.

*Case 1: $y=1$ and $z=-1$.
So, $y=-z$ ✓
*Case 2: $y=-1$ and $z=1$.
So, $y=-z$ ✓ $\square$

(b) True.
Proof: Let $n \in \mathbb{N}$.
Consider $X=\emptyset$.
Note that $X \in \mathcal{P}(\mathbb{N})$ because $\emptyset \subseteq \mathbb{N}$.
Then, $|X| = 0 < n$, because $n \in \mathbb{N}$ and thus $n \geq 1$. $\square$

(c) The statement $(a \text{ odd}) \wedge (b \text{ odd}) \Rightarrow (ab^2 \text{ odd})$ is being proved.
Hence, its contrapositive $(ab^2 \text{ even}) \Rightarrow (a \text{ even}) \vee (b \text{ even})$ is also being proved.
So, the answer is $\boxed{d}$.

Question 7:

(a) f is not a bijection, because f is not surjective.
Consider $y=0 \in \mathbb{Z}$
We will show that $(\forall x \in \mathbb{Z})\ (f(x) \neq y)$.
Suppose there is an $x \in \mathbb{Z}$ such that $f(x)=0$.
So, $3x^2+2x+1=0$
However, since $2^2-4\cdot 3\cdot 1 = -8 < 0$, there is no solution.
Hence, there is no $x \in \mathbb{Z}$, such that $f(x)=0$.
As a result f is not surjective.
(Note that f is injective).

(b) Consider $f^{-1}: \mathbb{R}\setminus\{2\} \to \mathbb{R}\setminus\{3\}$ defined by $f^{-1}(x) = \dfrac{6x-7}{2x-4}$

Let $x \in \mathbb{R}\setminus\{3\}$, then

$$f^{-1}(f(x)) = \frac{6\cdot\frac{4x-7}{2x-6} - 7}{2\cdot\frac{4x-7}{2x-6} - 4} = \frac{24x-42-14x+42}{8x-14-8x+24} = \frac{10x}{10} = x \checkmark$$

Let $x \in \mathbb{R}\setminus\{2\}$, then

$$f(f^{-1}(x)) = \frac{4\cdot\frac{6x-7}{2x-4} - 7}{2\cdot\frac{6x-7}{2x-4} - 6} = \frac{24x-28-14x+28}{12x-14-12x+24} = \frac{10x}{10} = x \checkmark$$

So, the answer is $\boxed{d}$.

(c). Note that $f(1) = \dfrac{1}{2} + \dfrac{1-(-1)^1}{4} = \tfrac{1}{2} + \tfrac{1}{2} = 1$ and $f(2) = \dfrac{2}{2} + \dfrac{1-(-1)^2}{4} = 1+0 = 1$.
So, f is not injective.

f is surjective. Let $y \in \mathbb{Z}$. Take $x=2y$ (Note that $2y \in \mathbb{Z}$, because $y \in \mathbb{Z}$).
Then $f(x) = f(2y) = \dfrac{2y}{2} + \dfrac{1-(-1)^{2y}}{4} = y + \dfrac{1-((-1)^2)^y}{4} = y + \dfrac{1-1^y}{4}$

$$= y + \frac{1-1}{4} = y+0 = y \checkmark$$

So, the answer is $\boxed{c}$.

Question 8:

ⓐ. The answer is $\boxed{b}$.

Counterexample: $A=\{\{2,3,4\}\}$ and $B=\{\{2,3\}\}$. Then, $A \setminus B = \emptyset$.

The following statement would be correct: $(\{2,3,4\} \subseteq A$ and $\{2,3\} \subseteq B) \Rightarrow (\{4\} \subseteq A \setminus B)$.

ⓑ. All four statements are true, so the answer is $\boxed{e}$.