Solutions - Practice Exam Questions - Tutorial 5

1. See picture below. We know from the information in the question that, for all three bands, the number of students that like *only that* band is the same; call this value x. Now, we want to know $p+q+r$. We are told $|A| = 179, |B| = 179, |C| = 17$, so by first principles we know:

$$x + p + q + 5 = 179$$
$$x + p + r + 5 = 179$$
$$x + q + r + 5 = 17$$

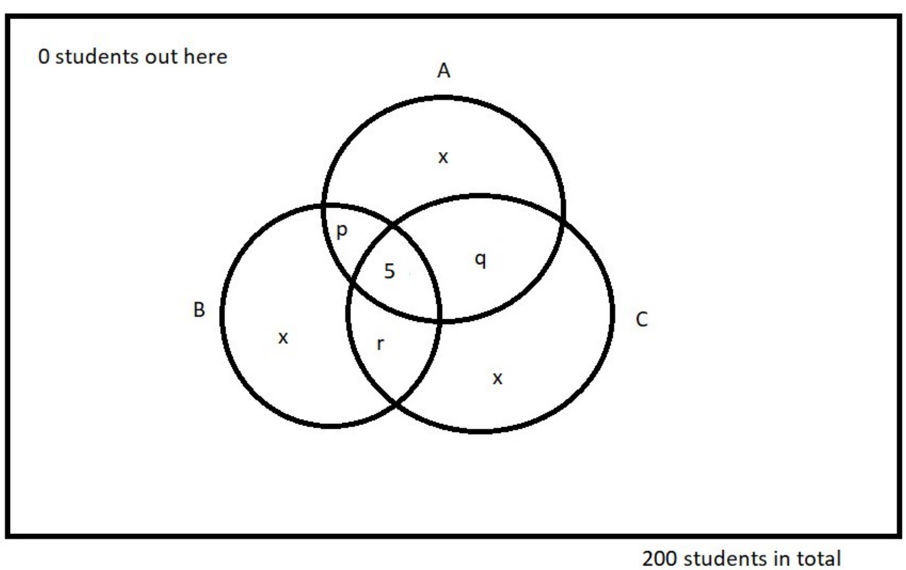
If you add these three equations together and tidy up you get

$$3x + 2p + 2q + 2r = 360 \tag{1}$$

We need one more additional piece of information. Fortunately, we have it: we know that every student likes at least one band (and that there are 200 students in total). So we also know

$$3x + p + q + r + 5 = 200 \tag{2}$$
$$\Rightarrow 3x + p + q + r = 195 \tag{3}$$

If we subtract (3) from (1) we obtain $p + q + r = 165$. And we are done: 165 students like exactly two bands. Magic! (You can now fill in the rest of the numbers if you wish, although that is not necessary to answer the question. You will see that $x = 10$, $p = 163$, $q = 1$, $r = 1$ and that everything works out). $\square$



2.

(a). This is like rolling 10 4-sided dice, and counting the patterns (see Theorem 4 in the book/slides). So the answer is,

$$\binom{(4-1)+10}{10} = \binom{13}{10} = \frac{13!}{10!\,3!} = \frac{13\times 12\times 11}{6}$$

$$= 286.$$

column dividers

number of "crosses"

In an exam we do want to see the final number because it shows you understand what expressions like $\binom{13}{10}$ actually mean.

safe	low-risk	medium	high
x	xxx	xxxx	xx

(b)

$$\frac{7!}{4!} = 7\times 6\times 5 = 210.$$

This is like the chairman, administrator, treasurer example in the notes.

That is: line up the elements of A (in some arbitrary order). Each element of A picks an element of B, but cannot pick elements of B that were picked earlier.

(c) This is $\binom{10}{4}$ ie. Select a subset of 4 positions from $\{1,2,3,4,5,6,7,8,9,10\}$ — this is your choice where to put the "0" symbols.

$= \frac{10!}{6!4!} = 210.$

(d) Nasty!

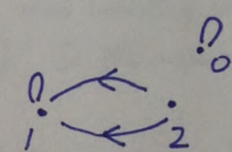
The answer is 2^{n^2}.

Why? Think of an $n \times n$ grid, representing all possible places you can put an arrow or loop. Each box in the grid has two choices: use that arrow/loop, or do not. The grid contains n^2 boxes. So 2^{n^2}

eg. $A = \{0,1,2\}$

	0	1	2
0	✓	×	×
1	×	✓	✓
2	×	✓	×

$\Longleftrightarrow$



3.

(a) $\binom{9}{3} = \frac{9!}{3!\,6!} = 84$

(b) This is like throwing 15 4-sided dice; equivalently, asking how many length 18 strings consist of 15 "x" and 3 "1".

This is $\binom{18}{3} = \binom{18}{15} = 816$.

(c) $4! \times \binom{10}{5}$

permutations of {a,b,c,d} prefix

rule of product

number of ways to select 5 positions (in a length-10 string) where we will put the "e" letters

$= 24 \times 252$

$= 6048$

4.

(a) Same idea as throwing 9 3-sided dice, which corresponds to placing 9 crosses in a table with $3-1=2$ column dividers, which corresponds to strings with 9 crosses and 2 lines (so length 11 in total):

$$\binom{11}{2} = \frac{11!}{2!\,9!} = \underline{55}$$

(b) $$\binom{8}{6} + \binom{8}{7} + \binom{8}{8}$$

$$= \frac{8!}{6!\,2!} + 8 + 1$$

$$= 28 + 8 + 1$$

$$= \underline{37}.$$

(c) $\overset{5}{\cancel{26}} \times$ ways to make a length-5 string with at least one number

number of ways to pick the letter at the beginning

$$= \overset{5}{\cancel{26}} \times \left((5+3)^5 - 5^5 \right)$$

strings made from letters & numbers

strings made from just letters

$$= \overset{5}{\cancel{26}} \times \left[32768 - 3125 \right]$$

$$= \overset{5}{\cancel{26}} \times \cancel{27}\ 29643$$

$$= \underline{\cancel{770718}}. = 148215$$

5.

Answer: Let $n = |S|$. The answer is $n^n - n!$, which in Steven's case is $4^4 - 4! = 256 - 24 = 232$. Where does the expression come from? There are n^n functions from S to S, because each element of the domain S has a free and independent choice of which element in the co-domain S it maps to. There are $n!$ bijective functions, because each element of the domain S has a free choice of which element of the domain S it wants to map to, *but after making its choice that element is no longer available* - recall the Chairperson, Administrator, Treasurer example in class. If we subtract the number of bijective functions from the number of total functions, we get the number of non-bijective functions.

6.

(a)

$$3^5 - 2^5 - 2^5 - 2^5 + 3 = 150$$

3^5: strings with no restriction

first 2^5: strings just using (at most) 0s and 1s

second 2^5: " " 0s and 2s

third 2^5: " " 1s and 2s

3: to compensate for double deletion of all-0, all-1 and all-2 string.

(b) Due to the fact that $|A| = |B|$, a function $f: A \to B$ is an injection if and only if it is a surjection. Hence, to count bijections we can just count <u>injections</u>.

There are 7^7 functions in total, and $7!$ injections,

so number of imperfect functions $= 7^7 - 7! = 818503$

c) $\binom{8+4-1}{8} = 165$

d) $\binom{15}{11} = 1365$