Systems of linear equations

$$\begin{cases} 2x_1 + x_2 = 5 \\ 3x_1 - 2x_2 = 10 \end{cases} \longrightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}}_{\text{coef. matrix}} \quad \underbrace{\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 3 & -2 & 10 \end{array}\right]}_{\text{augmented matrix}}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 3 & -2 & 10 \end{array}\right] \xrightarrow{R_1 = 2R_1} \left[\begin{array}{cc|c} 2 & 2 & 10 \\ 3 & -2 & 10 \end{array}\right] \xrightarrow{R_1 = R_1 + R_2} \left[\begin{array}{cc|c} 5 & 0 & 20 \\ 3 & -2 & 10 \end{array}\right]$$

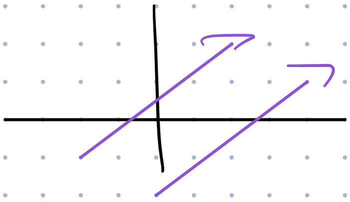
$$\xrightarrow{\frac{1}{5}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 3 & -2 & 10 \end{array}\right] \xrightarrow{R_2 = R_2 - 3R_1} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & -2 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 = -\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right] \qquad \begin{aligned} x_1 &= 4 \\ x_2 &= 1 \end{aligned}$$

Types of systems

Inconsistent

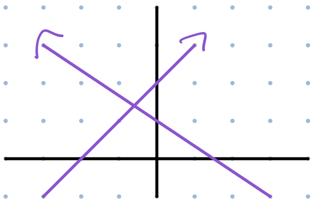
↳ NO SOLUTION

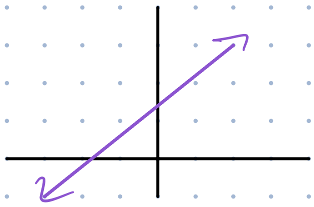


Consistent

↳ 1 SOLUTION

↳ INFINITE SOLUTIONS





Exercise

$$\left[\begin{array}{ccc|c} 1 & -2 & +1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & +5 & +9 & -9 \end{array}\right] \xrightarrow{R_3 = R_3 + 4R_1} \left[\begin{array}{ccc|c} 1 & -2 & +1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \xrightarrow{R_2 = R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 28 & -12 \end{array}\right] \xrightarrow{R_1 = R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 11 & -2 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 28 & -12 \end{array}\right]$$

$$\xrightarrow{R_3 = \frac{1}{4} R_3} \left[\begin{array}{ccc|c} 1 & 0 & 11 & -2 \\ 0 & 1 & 5 & -1 \\ 0 & 0 & 7 & -3 \end{array}\right]$$

Existence & Uniqueness

- Consistent → solution exists
  - ↳ is it unique?

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array}\right] \xrightarrow{R_1 \Leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array}\right] \rightarrow \text{inconsistent system}$$

Row reduction algorithm

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array}\right] \xrightarrow[R_3 = R_3 - 3R_2]{R_2 = R_2 + 2R_1} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 8 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 11 \end{array}\right]$$

↑ inconsistent system

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$7 - 4(3) = 7 - 12 = -5$

- Eigenvectors
- Orthogonality
- Bases
- Inverses
- Nol-Col
- Unit Vector
- Magnitude of matrix

$$1x_1 + 3x_2 + 0x_3 = -5$$
$$x_3 = 3$$

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

Computation of Ax

(1) $A = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(a) $A(\vec{u} + \vec{v}) = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} -2 & +1 \\ 4 & +2 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

(b) $A\vec{u} + A\vec{v} = \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} -8 & +2 \\ 16 & +4 \end{bmatrix} + \begin{bmatrix} 6 & -1 \\ -12 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 20 \end{bmatrix} + \begin{bmatrix} 5 \\ -14 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

Homogeneous Systems

$$Ax = 0$$

$$\left[\begin{array}{ccc|c} 4 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 4 & 5 & 9 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 4 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 4 & 2 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 3/4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Consistent

$$\vec{x} = \begin{bmatrix} 1/4\, x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1/4 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{cases} x_1 = \frac{1}{4} x_3 \\ x_2 = -2x_3 \end{cases} \quad x_3 \text{ is free}$$

Non-homogeneous systems $Ax \neq 0$

$$\left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{array}\right]$$

$$\begin{cases} x_1 = -3x_3 + 4 \\ x_2 = 2x_3 - 1 \\ x_3 \text{ is free} \end{cases} = \begin{bmatrix} -3x_3 + 4 \\ 2x_3 - 1 \\ x_3 \end{bmatrix}$$

translation p

$$= \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

Linear Independence

Homogeneous + NON-Trivial sols $\rightarrow$ DEPENDENT

Homogeneous + NO Non-Trivial sols $\rightarrow$ INDEPENDENT

free variables = Non-Trivial = DEPENDENT

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right] \leftarrow \text{homogenous system } Ax = 0$$

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \\ x_3 \text{ is free} \end{cases} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

there are infinite possible linear dependency relationships

linearly dependent

Relationship: let $x_3 = 2$

$\hookrightarrow x_1 = 4;\ x_2 = -2;\ x_3 = 2 \Rightarrow 4v_1 - 2v_2 + 2v_3 = 0$

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 5 & 8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 13 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Homogeneous, trivial solution

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

$Ax = 0$, where $\vec{x} = 0$

No free variables $\rightarrow$ linearly independent

Determining Linear Independence

$$u = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} \quad v = \begin{bmatrix} -6 \\ 2 \\ 3 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix} \quad z = \begin{bmatrix} 3 \\ 7 \\ -5 \end{bmatrix}$$

$\{u, v\}$

$$\left[\begin{array}{cc|c} 1 & -6 & 0 \\ 2 & 2 & 0 \\ -4 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -6 & 0 \\ 0 & 14 & 0 \\ 0 & 21 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -6 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

↳ inconsistent system

$x_1 = 0$

$x_2 = 0$

$Ax = 0$, where $\vec{x} = 0$

↳ $\vec{u} \neq c\vec{v} \rightarrow$ linearly independent

$\{u, v, w\}$

$\{u, v, w, z\} \rightarrow$ linearly dependent by Theorem 8 (n. cols > n. rows)