Natural Deduction

Conditional Proof

$A \to C$

① Assume A

② Reasoning

③ Infer C

④ Conclude $A \to C$ → cancel assumption

Examples

Prove $\vdash A \to (B \to A)$

| A
| ---
| | B
| | ---
| | A ← we assumed A beforehand
| $B \to A$

$A \to (B \to A)$

Rules

Ⓘ Introduction Rule → introduce connectives into a proof

Ⓔ Elimination rule → take connective out

$\wedge$

| A
| B
| $A \wedge B$ $\quad \wedge I$

| $A \wedge B$
| A

| $A \wedge B$
| B

$\wedge E$

$\vee$

A
$A \vee B$

A
$B \vee A$

$\vee I$

$A \vee B$

$\vdash A$
C

$\vdash B$
C

Assume A or B

C $\vee E$

Conditional Proof

$\rightarrow$

$\vdash A$
B

$A \rightarrow B$ $\rightarrow I$

Modus Ponens

$A \rightarrow B$
A
B

$\rightarrow E$

$\leftrightarrow$

$A \rightarrow B$
$B \rightarrow A$
$A \leftrightarrow B$ $\leftrightarrow I$

$A \leftrightarrow B$
$(A \rightarrow B) \wedge (B \rightarrow A)$
$\leftrightarrow E$

$\perp$ -> Arbitrary contradiction (False)

Reductio ad Absurdum

$\neg$

$\vdash A$
$\perp$

$\neg A$ $\neg A$

$\neg A$
A
$\perp$ $\neg E$ $(\perp I)$

Indirect Proof

$\perp$

$\vdash \neg A$
$\perp$

A

$\perp$
A

can be anything

$\perp E$ (X)

Exemples

(1) $p \to (q \to r) \vdash q \to (p \to r)$

$$
\begin{array}{l|l|l|l}
p \to (q \to r) & & & \\
\hline
 & q & & \\
\cline{2-2}
 & & p & \\
\cline{3-3}
 & & q \to r & \to E \\
 & & r & \to E \\
 & p \to r & & \to I \\
q \to (p \to r) & & & \to I
\end{array}
$$

assume q

assume p to get p → r

← prove this using the premise

$$
\begin{array}{|l}
A \to B \\
A \\
B
\end{array}
$$

(2) $p \to (q \to r) \vdash (p \wedge q) \to r$ ↖ Assume this one

$$
\begin{array}{l|l|l}
p \to (q \to r) & & \\
\hline
 & p \wedge q & \\
\cline{2-2}
 & p & \wedge E \\
 & q & \wedge E \\
 & q \to r & \to E \text{ (Modus Ponens)} \\
 & r & \to E \text{ (Modus Ponens)} \\
(p \wedge q) \to r & & \to I \text{ Conditional Proof}
\end{array}
$$

Strategies

(1) Look at the target

$A \vdash B \to C$

(1) A as premise
(2) Assume B
(3) Use premises (1) + derived to infer C
(4) Infer $B \to C$

More Examples

(1) $p \to (q \to r) \vdash (p \to q) \to (p \to r)$

0	$p \to (q \to r)$	← Distribution Axiom
1	$\quad p \to q$	
2	$\quad\quad p$	
3	$\quad\quad q \to r$	MP (2, 0)
4	$\quad\quad q$	MP (2, 1)
5	$\quad\quad r$	MP (4, 3)
6	$\quad p \to r$	$\to$I (2, 5)
7	$(p \to q) \to (p \to r)$	$\to$I (1, 6)

Equivalence

$A \dashv\vdash B$ (Proof both ways)

Examples

$(A \to B) \dashv\vdash \neg(A \wedge \neg B)$

0 $(A \to B)$
- 1 $A \wedge \neg B$
  - 2 A
  - 3 B — MP (0,2)
  - $\neg B$ — (1)
  - $\bot$ — $\bot$I
- $\neg(A \wedge \neg B)$

$\neg(A \wedge \neg B)$
- A
  - $\neg B$
    - $A \wedge \neg B$
    - $\bot$
  - B
- $A \to B$

(2) $\neg A \wedge \neg B \dashv\vdash \neg(A \vee B)$

0 $\neg A \wedge \neg B$
- 1 $A \vee B$
  - 2 A
    - 3 $\neg A$ — 0
    - 4 $\bot$ — RAA
  - 5 B
    - 6 $\neg B$
    - 7 $\bot$
  - 8 $\bot$
- 9 $\neg(A \vee B)$

$\neg(A \vee B)$
- A
  - $A \vee B$ — $\vee$I
  - $\bot$
- $\neg A$
- B
  - $A \vee B$
  - $\bot$
- $\neg B$
- $\neg A \wedge \neg B$

Quantifier Rules

Universal Elimination

$\forall y\ Fx$

Fa

$\forall x\ A(\cdots x \cdots x \cdots)$

$A(\cdots c \cdots c \cdots)$ $\quad \forall E$

Existential Introduction

Fa

$\exists x\ Fx$

$A(\cdots c \cdots c \cdots)$

$\exists x\ A(\cdots c \cdots x \cdots)$ $\quad$ x must not be in A

Universal Introduction

Fa ← a is a new name

$Fa \vee Ga$

$Fa \to (Fa \vee Ga)$

$\forall x\,(Fx \to (Fx \vee Gx))$

$A(\cdots c \cdots c \cdots)$

$\forall x\ A(\cdots x \cdots x \cdots)$

Existential Elimination

$\exists x\ A(\cdots x \cdots x \cdots)$

$A(\cdots c \cdots c \cdots)$

B

B

Examples

(1) $\exists x\, Fx \to Ga \vdash \forall x (Fx \to Ga)$

$$
\begin{array}{ll|l}
0 & \exists x\, Fx \to Ga & \\
\hline
1 & \quad Fb & \leftarrow \text{assume F for arbitrary value} \\
2 & \quad \exists x\, Fx & \exists \text{ intro} \\
3 & \quad Ga & \to E \quad (\text{Modus Pons}) \ (0,2) \\
4 & Fb \to Ga & \to I \quad (1,3) \\
 & \forall x (Fx \to Ga) & \text{arbitrary instance of target} \\
 & & \forall I
\end{array}
$$

(2) $\exists x (Fx \wedge Gx) \vdash \exists x\, Fx \wedge \exists x\, Gx$

$$
\begin{array}{l|l}
\exists x (Fx \wedge Gx) & \text{take arbitrary} \\
\hline
\quad Fa \wedge Ga & \\
\quad Fa & \wedge E \\
\quad \exists x\, Fx & \exists \text{ Intro} \\
\quad Ga & \wedge E \\
\quad \exists x\, Gx & \exists I \\
\quad \exists x\, Fx \wedge \exists x\, Gx & \wedge I \\
\exists x\, Fx \wedge \exists x\, Gx &
\end{array}
$$

More Examples

(1) $q \to (p \to r) \vdash (s \to p) \to (q \to (s \to r))$

0	$q \to (p \to r)$	
1	$s \to p$	
2	q	
3	s	
⋮	⋮	
	r	
7	$s \to r$	
8	$q \to (s \to r)$	
9	$(s \to p) \to (q \to (s \to r))$	

3	s	
4	$p \to r$	MP (0, 2)
5	p	MP (1, 3)
6	r	MP (4, 5)

assume this

if, then

get that