Lecture 7 - Calculus

Overview of the course

- Continuity and limits
- Differentiation
- Integration

END of SECONDARY SCHOOL MATERIAL (for most of you)

- Sequences and series we are here!
- Differential equations
- Partial derivatives and double integrals

Sequences and series

- Sequences
- Infinite Series
- Convergence tests for positive series
- Absolute and conditional convergence

Adams' Ch. 9.1-9.3, Thomas' Ch. 10.1

Sequences

A sequence $\{a_n\}$ is a list of numbers $a_1, a_2, \dots, a_n, \dots$ in a given order

term

index

* a sequence can be seen as a function $f: \mathbb{N} \to \mathbb{R}, \; n \to a_n = f(n)$

Examples: $\sqrt{n}, \; \frac{1}{n}, \; (-1)^n$

$a_1 = 1, \; a_2 = 1, \; \dots \; a_n = a_{n-1} + a_{n-2}$ (recursive formula)

(FIBONNACCI)

$1, \; \frac{-x^2}{2}, \; \frac{x^4}{4!}, \; \frac{-x^6}{6!}, \; \dots$ (a pattern)

$$\to \frac{(-1)^n x^{2n}}{(2n)!}$$

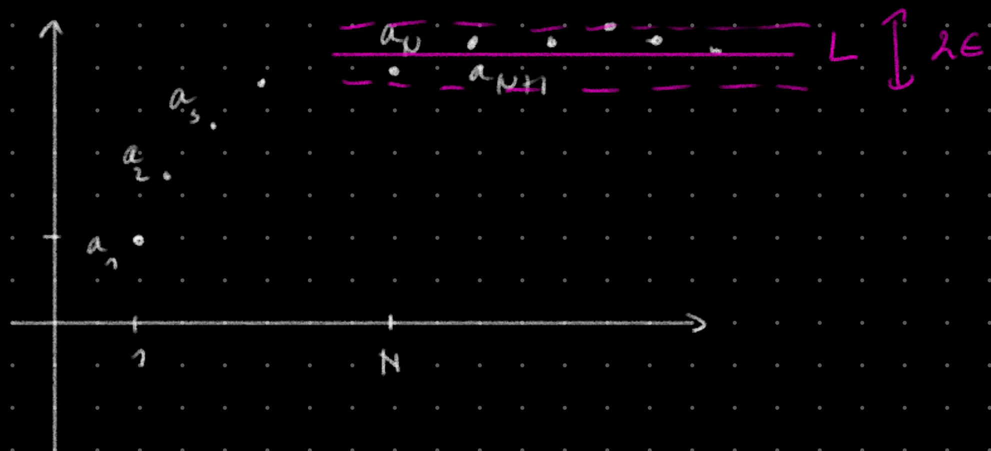
* series and sequences are used to approximate irrational numbers, transcendental functions ($\sin(x)$, $\cos(x)$, e^x, $\ln(x)$, ...) numerically

Convergence of a sequence

A sequence $a_n \rightarrow L$ if $\forall \epsilon > 0, \exists N \in \mathbb{N} : n \geq N \Rightarrow |a_n - L| < \epsilon$

$$\lim_{n \to \infty} a_n = L$$

* this means that, after an index N, all terms a_n are within ϵ-distance from the limit L



* examples of converging sequences

$\frac{1}{n} \rightarrow 0$, $\frac{n}{n+1} \rightarrow 1$, $(0.9)^n \rightarrow 0$

* NOT all sequences converge

→ a sequence diverges to infinity ($a_n \to \pm\infty$) if the terms become arbitrarily large / arbitrarily negative

$$\forall M > 0 \ \exists N \in \mathbb{N} : n \geq N \Rightarrow \begin{matrix} a_n > M \\ a_n < -M \end{matrix}$$

(For any large number M, you can find an index N, such that, past that index, all terms a_n are larger than M)

example : $\sqrt{n} \to +\infty$, $2^n \to +\infty$

→ a sequence diverges if $\lim_{n \to +\infty} a_n$ does not exist

(the sequence does not reach a finite limit or grow arbitrarily large)

example : $(-1)^n$ diverges

* if the sequence can be seen as a real function (i.e. if $f(x), x \in \mathbb{R}$ is defined for $x \geqslant n_0$, and $a_n = f(n)$ for $n \geqslant n_0$, then

$$\lim_{x \to \infty} f(x) = L \implies a_n \to L \qquad (L \text{ can be } \pm\infty)$$

↳ if the function limit exists, then also the sequence limit

↳ not true the other way round, for example $\cos(2\pi n) \to 1$

$\lim_{x \to \infty} \cos(2\pi x)$ DNE

↳ we used this property intuitively

$\frac{1}{n} \to 0 \quad , \sqrt{n} \to \infty$

↳ note: a sequence cannot have vertical asymptotes!

$a_n = \frac{1}{n} \leq 1$, we cannot come arbitrarily close to $n = 0$,

since the domain is $\underline{\underline{\mathbb{N}}}$.

* we can apply a continuous function on a sequence:

if $a_n \to L$, then $f(a_n) \to f(L)$

Example $a_n = n^{\frac{1}{n}}$

$\rightarrow$ let's consider $f(a_n) = \ln(a_n) = \frac{1}{n}\ln(n)$, $a_n = f^{-1}(f(n))$

then $\lim_{x\to\infty} \frac{1}{x}\ln(x) \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$. $= e^{\ln(a_n)}$

$f(a_n) \rightarrow 0$, so $a_n = e^{f(n)} \rightarrow e^0 = 1$

* we can add, subtract, multiply converging sequences:

for $\{a_n\}$, $\{b_n\}$ sequences, $a_n \rightarrow A$, $b_n \rightarrow B$

then $(a_n \pm b_n) \rightarrow A \pm B$, $(a_n \cdot b_n) \rightarrow A \cdot B$

$k a_n \rightarrow kA$ $(k \in \mathbb{R})$

* squeeze theorem for sequences: for $\{a_n\}, \{b_n\}, \{c_n\}$ sequences,

$a_n \leq b_n \leq c_n$ $\forall n$

if $a_n \rightarrow L$, $c_n \rightarrow L$, then $b_n \rightarrow L$

example: since $\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$, and $\pm\frac{1}{n} \rightarrow 0$, $\frac{\sin(n)}{n} \rightarrow 0$

Terminology

A sequence $\{a_n\}$ is

- Bounded above if $\exists M \in \mathbb{R} : \forall n \in \mathbb{N} \quad a_n \leq M$
- Bounded below if $\exists L \in \mathbb{R} : \forall n \in \mathbb{N} \quad a_n \geq L$
- Bounded if bounded above and below

- Increasing: $\forall n \quad a_{n+1} > a_n$
- Decreasing: $\forall n \quad a_{n+1} < a_n$

$\Big\}$ monotonous

- Alternating: $\forall n \quad a_{n+1} \cdot a_n < 0$

- Positive/negative: $\forall n \quad a_n \begin{array}{l} > 0 \\ < 0 \end{array}$

- Every converging sequence is bounded
- A bounded monotonous sequence converges

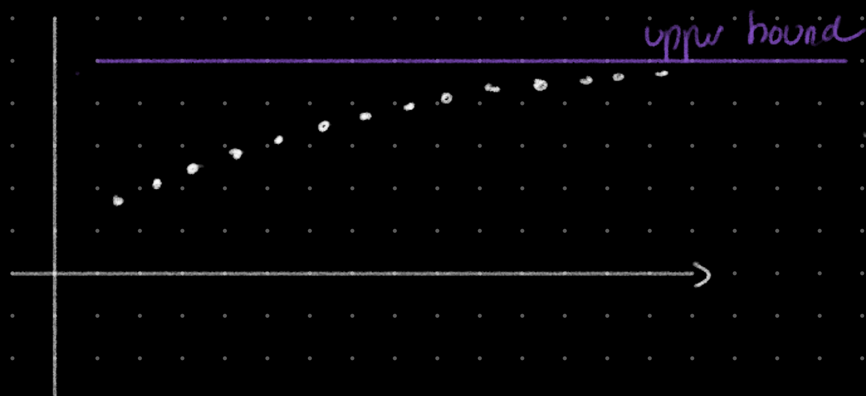
* Intuitive explanation:

- if a sequence converges, $a_n \to L$, then $L+\epsilon$ is an upper bound / $L-\epsilon$ is a lower bound for the terms $a_N, a_{N+1}, \ldots$ (infinitely many)

  → for the first $N-1$ terms there is a minimum term a_{Min}, maximum term a_{Max}

  → As upper bound, take $\max(L+\epsilon, a_{Max})$
  lower bound $\min(L-\epsilon, a_{min})$

- if a monotonous sequence is bounded, than it converges



the sequence cannot increase towards ∞

Infinite series

(Infinite) series = formal sum of infinitely many terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 , \ldots + a_n + \ldots$$

summation index can change

+ a series can be seen as a sequence of partial sum $\{S_n\}$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_n = \sum_{k=1}^{n} s_k$

the series converges to s if $S_n \to s$

$$\sum_{n=1}^{\infty} a_n = s$$

* SERIES ARE AN INDETERMINATE FORM (usually)
  - ↳ we sum up infinitely many terms (that are infinitely small)
  - ↳ usually, we cannot calculate the sum. We can only conclude whether they converge (the sum exists)

Geometric series

$a_n = a r^{n-1}$, $r = \frac{a_{n+1}}{a_n}$ $(a \neq 0,\ r \neq 1)$

↳ constant ratio between terms

* the geometric series is one of the few series where we can calculate the sum

→ $S_1 = a$, $S_2 = a + ar$, $S_3 = a + ar + ar^2$

$$
\begin{aligned}
S_n &= a + ar + ar^2 + \dots + ar^{n-1} \\
-\,r \cdot S_n &= \phantom{a + {}} ar + ar^2 + \dots + ar^{n-1} + ar^n \\
\hline
(1-r) \cdot S_n &= a - ar^n
\end{aligned}
$$

$\Rightarrow S_n = a \dfrac{1-r^n}{1-r}$

→ convergence if $r^n \to 0$, $-1 < r < 1$ or $a = 0$

$S = \dfrac{a}{1-r}$

→ divergence to $+\infty$ if $r \geq 1$, $a > 0$

$-\infty$ $r \geq 1$, $a < 0$

→ divergence if $a \neq 0$, $r < -1$

- (n-th term test for divergence)

  If $\sum_{n=1}^{\infty} a_n \to S$ converges, then $(S_n - S_{n-1}) \to 0 \iff a_n \to 0$

  $S_n \downarrow S$, $S_{n-1} \downarrow S$

(contrapositive): if $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

- Examples. $\sum_{n=1}^{\infty} \frac{n}{n+1} \to \infty$, as $S_n \approx n \cdot 1$ (we add up n terms that approach 1)

  $\sum (-1)^n$ diverges, since $S_1 = -1$, $S_2 = -1+1 = 0$, $S_3 = -1$

  $S_n = 0$ for n even

  $= -1$ for n odd

- only the tail matters: if $\sum_{n=N}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

- sums, differences, multiples of converging series converge

  if $\sum a_n = A$, $\sum b_n = B$, then $\sum (a_n \pm b_n) = A \pm B$, $\sum k a_n = kA$

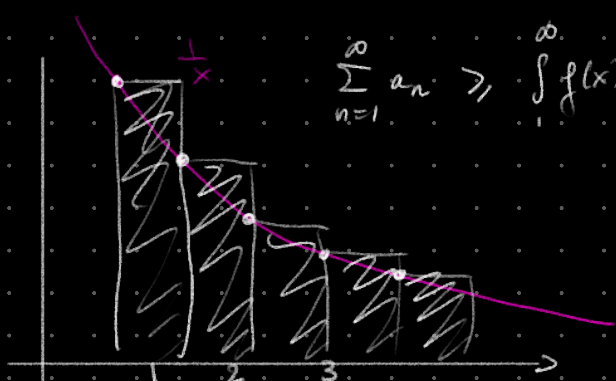
Integral test for positive series

* many convergence tests are for positive series only — adding up positive terms is conceptually easier than adding up positive and negative terms)

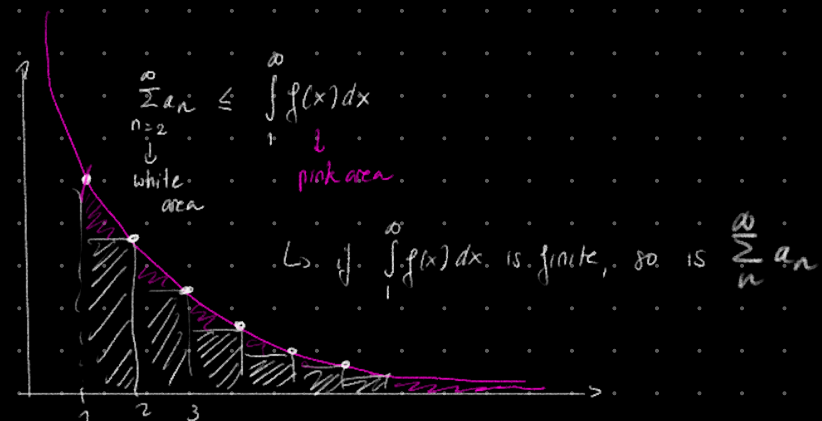
* if sequences compare to functions, series compare to improper integrals

↳ if $a_n = f(n)$, for f non-increasing on $[N, \infty)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(x)\,dx$ both converge or diverge to ∞

PROOF : the series $\sum a_n$ can be seen as both upper and lower Riemann sum.



if $\int_1^{\infty} f(x)\,dx = \infty$, then also $\sum_{n=1}^{\infty} a_n$



p-series = the series $\sum \frac{1}{n^p}$

* $p=1$ $a_n = \frac{1}{n}$, this is the HARMONIC series

→ this series DIVERGES, since $\int_1^\infty \frac{dx}{x} = \lim_{R\to\infty} \int_1^R \frac{dx}{x}$

$$= \lim_{R\to\infty} [\ln(R) - \ln(1)] = +\infty$$

* $p \neq 1$ $a_n = \frac{1}{n^p}$

$$\int_1^\infty \frac{dx}{x^p} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^p} = \lim_{R\to\infty} \left[\frac{1}{1-p} R^{1-p} - \frac{1}{1-p} \right]$$

* for $p<1$, this integral diverges to $+\infty$ → $\sum \frac{1}{n^p} = +\infty$
* for $p>1$ converges to $\frac{1}{p-1}$ → $\sum \frac{1}{n^p}$ converges

* you may know and use the p-series in exercises / the exam without carrying out the integration each time