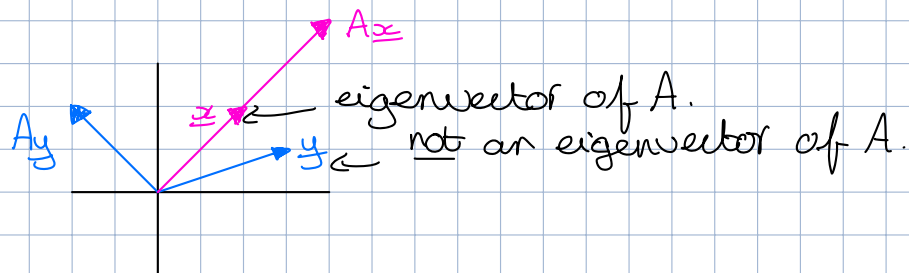


Lecture 9: Eigenvalues and Eigenvectors.

(book: 5.1, 5.2)

Previous episode: Vector Spaces.
Next episode: Diagonalization.

$n \times n$ matrix A $T: \underline{x} \mapsto A\underline{x}$

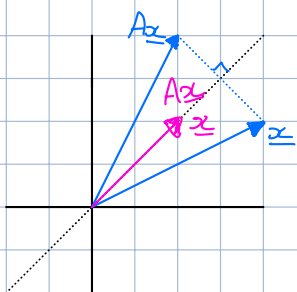


A non-zero $\underline{x} \in \mathbb{R}^n$ is an **eigenvector** of A when $A\underline{x} = \lambda \underline{x}$ for some scalar $\lambda \rightarrow$ **eigenvalue**.

in words: A produces a **scalar multiple** of \underline{x} (the **direction** does not change).
(apart from a minus sign) \neq

Example: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$A\underline{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

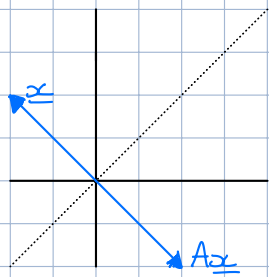


A doesn't change the direction of the vectors on the $x_2 = x_1$

$$A \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

So, $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is an **eigenvector** with **eigenvalue 1**.
 $\lambda_1 = 1$.

\hookrightarrow any vector of the form $\begin{bmatrix} t \\ t \end{bmatrix}$ with $t \neq 0$.



Each vector perpendicular to the line $x_2 = x_1$ is also an eigenvector.

$$A \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = -1 \cdot \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

So, $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$ is an **eigenvector** with **eigenvalue -1**.
 $\lambda_2 = -1$.

\hookrightarrow any vector of the form $\begin{bmatrix} -t \\ t \end{bmatrix}$ with $t \neq 0$.

Suppose $A\underline{x} = \underline{0}$ has a non-trivial solution.
 $\Rightarrow \exists \underline{x} \neq \underline{0} : A\underline{x} = \underline{0} \cdot \underline{x}$

So, each non-trivial solution is an eigenvector with eigenvalue 0.

Recall: A is invertible $\Leftrightarrow A\underline{x} = \underline{0}$ has only the trivial solution.
 $\Leftrightarrow 0$ is not an eigenvalue of A .

Example: Is $\underline{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$? No.

$$A\underline{u} = \lambda \underline{u}?$$

$$A\underline{u} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Is a scalar p an eigenvalue of A ?

$$A\underline{x} = p \cdot \underline{x} \text{ for some vector } \underline{x} \neq \underline{0}?$$

$$\Leftrightarrow A\underline{x} - p\underline{x} = \underline{0} \Leftrightarrow (A - pI)\underline{x} = \underline{0} \text{ has a nontrivial sol?}$$

$$\Leftrightarrow A - pI \text{ has a free variable?}$$

Example: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Is 1 an eigenvalue?

$$A - 1 \cdot I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

x_2 is a free var

So, 1 is an eigenvalue.

What are the corresponding eigenvectors?

$$\begin{bmatrix} 1 & -1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenspace of 1 is $\text{Nul}(A - 1I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

\hookrightarrow a vector space that consists of $\underline{0}$ and all eigenvectors corresponding to the eigenvalue 1.

How to find the eigenvalues?

λ is an eigenvalue $\Leftrightarrow (A - \lambda I)\underline{x} = \underline{0}$ has nontrivial sols.

$\Leftrightarrow A - \lambda I$ is not invertible.

$$\Leftrightarrow \det(A - \lambda I) = 0.$$

So, solve $\det(A - \lambda I) = 0$ for λ .
 \hookrightarrow polynomial of degree n . (characteristic equation polynomial).

Example: Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

$$A - \lambda I = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3.$$

$$\det(A - \lambda I) = 0 \iff \lambda^2 - 4\lambda + 3 = 0 \iff (\lambda - 1)(\lambda - 3) = 0 \\ \iff \lambda_1 = 1, \lambda_2 = 3.$$

And find the corresponding eigenvectors.

$$\lambda_1 = 1: A - \lambda_1 I = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ with } x_2 \neq 0.$$

$$\lambda_2 = 3: A - \lambda_2 I = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ with } x_2 \neq 0.$$

Example: $A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$

$$\underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad A \underline{v} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} = 10 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda_1 = 10.$$

$$\underline{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad A \underline{u} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \lambda_2 = 0.$$

Example: $A = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 2 & 2 \\ 6 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 5 \\ 4 & 2 & 2 \\ 6 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = 8 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Example: $A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 1 & 0 \\ 0 & -3-\lambda & 1 \\ 0 & 0 & -3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (5-\lambda)(-3-\lambda)(-3-\lambda) = 0$$

So, $\lambda_1 = 5$ and $\lambda_2 = -3$ (with multiplicity 2).

So, for triangular or diagonal matrices, the eigenvalues are the entries on the main diagonal.

Example $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \left(= \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix} \text{ with } \varphi = \pi/2 \right)$

Give me the eigenvalues.

$$\mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

$$\mathbb{C}^2 \rightarrow \mathbb{C}^2$$

$$\lambda^2 + 1 = 0 \Leftrightarrow \lambda^2 = -1$$

So, $\lambda_1 = i$ and $\lambda_2 = -i$

So, eigenvalues can also be complex numbers.

$$\lambda_1 = i : A - \lambda_1 I = \begin{bmatrix} -i & -1 \\ -1 & -i \end{bmatrix} \sim \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

$$\underline{x} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\lambda_2 = -i : \dots \quad \underline{x} = x_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

DIY: $A = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}$ Determine the eigenvalues and eigenvectors.

Properties:

* A is invertible $\Leftrightarrow 0$ is not an eigenvalue of A .

* Exc 19, Ch 5.2: $\det(A) = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n$.

* $\sum_{i=1}^n a_{ii} = \lambda_1 + \lambda_2 + \dots + \lambda_n$.
 $\underbrace{\hspace{1cm}} \rightarrow \text{trace}(A)$.

* Thm 2: If $\underline{v}_1, \dots, \underline{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of a matrix A , then

$\{v_1, \dots, v_r\}$ is linearly independent.

Applications to Graph Theory.

Given adjacency matrix A ($n \times n$, symmetric)

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n$.

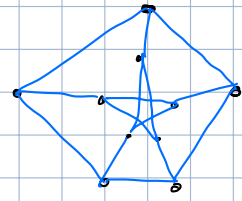
$\chi(G)$ chromatic number.

* Hoffman lower bound

$$\chi(G) \geq 1 + \frac{\lambda_1}{-\lambda_n}$$

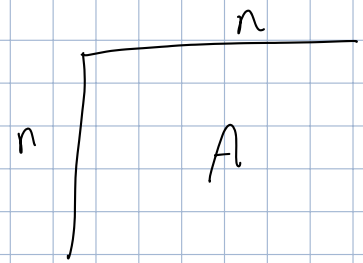
* Wilf upper bound $\chi(G) \leq \lambda_1 + 1$.

More applications? Read the [Google Page Rank algorithm](#) paper.



$$\chi(G) = 3$$

Petersen graph.



Summary (so far):

Let A be an $m \times n$ matrix with columns a_1, a_2, \dots, a_n .

$$m \geq n: \begin{bmatrix} A \end{bmatrix}$$

The following statements are equivalent:

- ① A has a pivot in every column
- ② A has n pivot positions
- ③ There are no free variables
- ④ $A\underline{x} = \underline{0}$ has only the trivial sol.
- ⑤ $\{a_1, a_2, \dots, a_n\}$ is linearly indep.
- ⑥ $T: \underline{x} \mapsto A\underline{x}$ is one-to-one/
injective
- ⑦ $\text{Nul } A = \{\underline{0}\}$
- ⑧ $\dim \text{Nul } A = 0$
- ⑨ $\text{rank } A = n$

$$m \leq n: \begin{bmatrix} A \end{bmatrix}$$

The following statements are equivalent:

- Ⓐ A has a pivot in every row.
- Ⓑ A has m pivot positions.
- Ⓒ The echelon form of A does not contain a row of all zeros.
- Ⓓ $A\underline{x} = \underline{b}$ is consistent for every \underline{b} in \mathbb{R}^m .
- Ⓔ $\text{Span}\{a_1, a_2, \dots, a_n\} = \mathbb{R}^m$.
- Ⓕ $T: \underline{x} \mapsto A\underline{x}$ is onto/surjective.
- Ⓖ $\text{Col } A = \mathbb{R}^m$
- Ⓗ $\dim \text{Col } A = m$
- Ⓘ $\text{rank } A = m$

If A is square ($n=m$), then statements ② and ⑥ are equivalent.
Hence, the following statements are equivalent for square matrices.

* ① - ⑨, Ⓐ - Ⓘ

* A is invertible

* There is a matrix C such that $CA = I_n$ and $AC = I_n$

* A is row equivalent to I_n .

* A^T is invertible.

* $\det A \neq 0$

* The columns of A form a basis for \mathbb{R}^n .

* 0 is not an eigenvalue of A .