Practice Exam Questions Week 1, **Linear Algebra**

SOLUTIONS.

1. Consider the following linear system of equations:

$$\begin{aligned} 2x_1 - 3x_2 - 7x_3 + 13x_4 + x_5 &= -3 \\ x_1 + x_2 - x_3 + 4x_4 + 2x_5 &= 4 \\ -x_1 + x_2 + 3x_3 - 6x_4 - x_5 &= 0 \end{aligned}$$

(a) Determine the augmented matrix of this system and compute its reduced row echelon form.

$$\left[\begin{array}{ccccc|c} 2 & -3 & -7 & 13 & 1 & -3 \\ 1 & 1 & -1 & 4 & 2 & 4 \\ -1 & 1 & 3 & -6 & -1 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_2 \sim$$

augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 4 & 2 & 4 \\ 2 & -3 & -7 & 13 & 1 & -3 \\ -1 & 1 & 3 & -6 & -1 & 0 \end{array}\right] \begin{array}{l} R_2 : R_2 - 2R_1 \\ \sim \\ R_3 : R_3 + R_1 \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 4 & 2 & 4 \\ 0 & -5 & -5 & 5 & -3 & -11 \\ 0 & 2 & 2 & -2 & 1 & 4 \end{array}\right] \begin{array}{l} R_3 : R_3 + \tfrac{2}{5}R_2 \\ \sim \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 4 & 2 & 4 \\ 0 & -5 & -5 & 5 & -3 & -11 \\ 0 & 0 & 0 & 0 & -1/5 & -2/5 \end{array}\right] \begin{array}{l} R_3 : -5 * R_3 \\ \sim \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 4 & 2 & 4 \\ 0 & -5 & -5 & 5 & -3 & -11 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 : R_1 - 2R_3 \\ \sim \\ R_2 : R_2 + 3R_3 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 4 & 0 & 0 \\ 0 & -5 & -5 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_2 : -1/5 * R_2 \\ \sim \end{array} \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 4 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 : R_1 - R_2 \\ \sim \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 5 & 0 & -1 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

reduced row echelon form of the augmented matrix.

(b) Compute the whole solution set.

$$\begin{cases} x_1 = -1 + 2x_3 - 5x_4 \\ x_2 = 1 - x_3 + x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 2 \end{cases}$$

2. True or false? If the given statement is true, briefly explain why. If it is false, give a counterexample.

(a) If a system of linear equations has a unique solution, then the number of basic variables is larger than the number of free variables.

True. If the solution is unique, then there are no free variables. Hence, # basic variables > # free variables, because # free variables = 0.

(b) If a linear system of four equations in four variables has a coefficient matrix with a pivot in each column, then the system has a unique solution.

True. Since there is a pivot in each column of the coefficient matrix, we know that every variable is a basic variable. So, there are no free variables. Hence, the system has a unique solution.

(c) A consistent system of linear equations with fewer equations than unknowns (also called underdetermined system) can never have an unique solution.

True. In general, # basic variables $\leq$ # equations.
In this case, we also have # equations $<$ # variables.
So, we have in this case: # basic variables $<$ # variables.
Hence, there must be at least one free variable.
So, it cannot have a unique solution.

(d) Suppose a (3×5) coefficient matrix for a system has three pivot columns. Then the system is consistent.

True. If the (3×5) coefficient matrix has three pivot columns, then the reduced row echelon form of the augmented matrix cannot contain a row of the form $[0 \cdots 0 \; b]$ with b nonzero.