Exercises

1	2	3	4	5	6	7	8

Surname, First name

1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	5	5	5	5	5	5
6	6	6	6	6	6	6
7	7	7	7	7	7	7
8	8	8	8	8	8	8
9	9	9	9	9	9	9
0	0	0	0	0	0	0

Linear Algebra (KEN1410)
Exam

a ~~b~~ c d e f → b
a b ~~c~~ d e f → c
a b c ~~d~~ e f → a

Fill in your answer(s) to the multiple-choice questions as shown above (circles = one correct answer).

Program: Data Science and Artificial Intelligence
Course code: KEN1410
Examiners: Dr. Marieke Musegaas and Dr. Stefan Maubach
Date/time: Monday 3 April 2023, 09:00 - 11:00
Format: Closed book exam
Allowed aids: Pens, simple (non-programmable) calculator from the DACS-list of allowed calculators.
Instructions to students:

- The exam consists of 7 questions on 20 pages.
- Fill in your name and student ID number on the cover page and tick the corresponding numerals of your student number in the table (top right cover page).
- Answer every question in the reserved space below the question. **Do not write outside the reserved space or on the back of pages, this will not be scanned and will NOT be graded!** As a last resort if you run out of space, use the extra answer space at the end of the exam.
- *In no circumstance write on or near the QR code at the bottom of the page!*
- Ensure that you properly motivate your answers.
- Only use black or dark blue pens, and write in a readable way. Do not use pencils.
- Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in the space reserved for the answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- You have to return all pages of the exam. You are not allowed to take any sheets, even blank, home.
- Good luck!

Question 1

10p **1** Consider the following matrix A depending on a parameter p:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & p & 3 \\ 5 & 3 & p \end{bmatrix}.$$

Determine for which value(s) of p the matrix A is **not** invertible.

Question 2

5p **2** Is the following statement true? If yes, provide a proof. If no, provide a counterexample.

"If an $n \times n$ matrix A is symmetric, then A is invertible."

Question 3

5p

3 Let $\mathbf{u} = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \sqrt{3} \\ \sqrt{2} \end{bmatrix}$. Determine whether $\{\mathbf{u}, \mathbf{v}\}$ is a basis for $\mathbb{R}^2$.

Question 4

10p **4** Consider the following matrix A:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Is the matrix A diagonalizable?
(Note: you do not need to diagonalize A. You only need to state if it is possible to diagonalize A.)

Question 5

5p

5 Determine two distinct vectors in Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix} \right\}$ with length 1.

Question 6

Consider the following matrix A:

$$A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}.$$

It is given that the matrix A has eigenvalues 1 and -2.

5p

6a Show that $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ forms a basis for the eigenspace corresponding to the eigenvalue 1.

5p

6b Show that $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ forms a basis for the eigenspace corresponding to the eigenvalue -2.

10p **6c** Orthogonally diagonalize matrix A, i.e. find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

Question 7

Answer the following multiple-choice questions.
An explanation is not required.
For every multiple-choice question, there is only **one correct answer**.
Please read the multiple choice instructions on the cover page!

5p **7a** Consider the following matrices A, B and C:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}.$$

Which of the following operations **can not** be performed?

- a AB
- b AC
- c $2A + 3C$
- d $3B + 5C$
- e All of the above operations can be performed.

5p **7b** Find a value k such that the system of linear equations $A\mathbf{x} = \mathbf{b}$ corresponding to the following augmented matrix has infinitely many solutions.

$$\begin{bmatrix} 2 & 2 & -4 & 3 \\ 1 & 3 & -2 & 4 \\ -4 & k & 8 & -6 \end{bmatrix}.$$

- a $k = -12$
- b $k = -4$
- c $k = 4$
- d $k = 12$
- e None of these k values will give infinitely many solutions.

5p **7c** The following augmented matrix is almost in reduced echelon form. What is the solution of the corresponding system of linear equations $A\mathbf{x} = \mathbf{b}$?

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) $\mathbf{x} = \begin{bmatrix} 2 \\ 5 \\ 0 \\ 7 \end{bmatrix}$

(b) $\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ 7 \\ 0 \end{bmatrix}$

(c) $\mathbf{x} = \begin{bmatrix} 7 \\ 2 \\ 5 \\ 0 \end{bmatrix}$

(d) The system of linear equations does not have a solution.

(e) The system of linear equations has infinitely many solutions.

5p **7d** Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 5 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ a \\ b \end{bmatrix}$. If $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, then we must have

(a) $a = 3$ and $b = 1$

(b) $a = 2$ and $b = 0$

(c) $a = 2$ and $b = -1$

(d) $a = 1$ and $b = 3$

(e) $a = 0$ and $b = 2$

(f) $a = -1$ and $b = 2$

(g) None of the above

5p

7e Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Is T injective (one-to-one)? Is T surjective (onto)?

- (a) T is both injective and surjective
- (b) T is injective, but not surjective
- (c) T is surjective, but not injective
- (d) T is neither injective nor surjective

5p

7f Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Which one of the following subsets of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$?

- (a) $\left\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} + \mathbf{u} = \mathbf{0}\right\}$
- (b) $\left\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \cdot \mathbf{u} = 0\right\}$
- (c) $\left\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \cdot \mathbf{x} = 1\right\}$
- (d) $\left\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \cdot \mathbf{u} = 1\right\}$
- (e) None of the above.

5p **7g** Recall that $\mathbb{P}_3$ denotes the set of polynomials of degree at most 3. In other words, $\mathbb{P}_3$ consists of all polynomials of the form

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n,$$

where the coefficients $a_0, \ldots, a_n$ and the variable t are real numbers.

Let H be the subspace of $\mathbb{P}_3$ made up of only those polynomials $\mathbf{p}(t)$ such that $\mathbf{p}(0) = 0$, i.e.

$$H = \{\mathbf{p} \in \mathbb{P}_3 \mid \mathbf{p}(0) = 0\}.$$

Then,

- (a) $\{t, t^2, t^3\}$ is a basis for H.
- (b) $\{1, t, t^2, t^3\}$ is a basis for H.
- (c) None of the above.

5p **7h** Consider the following matrix A:

$$A = \begin{bmatrix} 1 & 1 & -2 & -1 \\ -1 & 4 & -3 & 1 \\ 0 & 7 & 1 & -8 \end{bmatrix}.$$

A vector in Nul(A) is

a $\begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}$

b $\begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$

c $\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

d $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$

e None of the above.

5p **7i** Consider the following matrix A:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

A vector in Col(A) is

- a $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
- b $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$
- c $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
- d $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$
- e None of the above.

Extra space
If you use these extra answer boxes, **please mention clearly in your main answer that part of your answer can be found here!**

8a

8b

8c

8d

8e

8f