Propositional Dynamic Logic

Actions

Can be seen as transitions between states

On a set of states $S = \{s_1, s_2, s_3\}$, actions are binary relations on S.

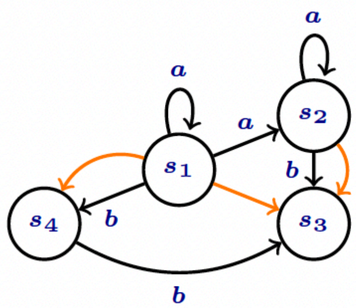
$$R_a := \{(s_1, s_1), (s_1, s_2), (s_2, s_2)\}$$

Identity Relation

$$I := \{(s, s) \mid s \in S\}$$

Composition

$$R_a \circ R_b := \{(s, s') \mid \text{there is } s'' \in S \text{ such that } R_a s s'' \text{ and } R_b s'' s'\}$$



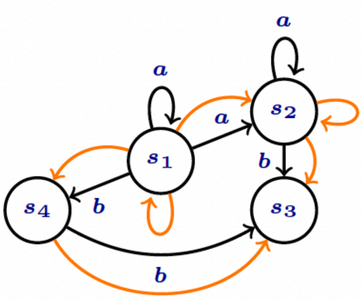
$$R_a := \{(s_1, s_1), (s_1, s_2), (s_2, s_2)\}$$
$$R_b := \{(s_1, s_4), (s_2, s_3), (s_4, s_3)\}$$
$$R_a \circ R_b = \{(s_1, s_4), (s_1, s_3), (s_2, s_3)\}$$

In particular:

$$R_a^0 := I; \quad R_a^1 := R_a \circ R_a^0;$$
$$R_a^2 := R_a \circ R_a^1; \quad R_a^3 := R_a \circ R_a^2$$

Union Relation

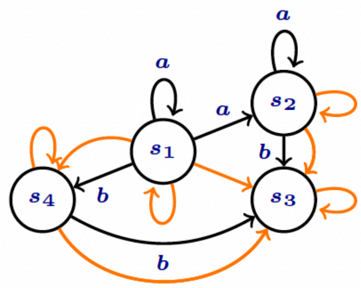
$$R_a \cup R_b := \{(s, s') \mid R_a s s' \text{ or } R_b s s'\}$$



$$R_a := \{(s_1, s_1), (s_1, s_2), (s_2, s_2)\}$$
$$R_b := \{(s_1, s_4), (s_2, s_3), (s_4, s_3)\}$$
$$R_a \cup R_b = \{(s_1, s_1), (s_1, s_2), (s_2, s_2), (s_1, s_4), (s_2, s_3), (s_4, s_3)\}$$

Repetition, zero or more times

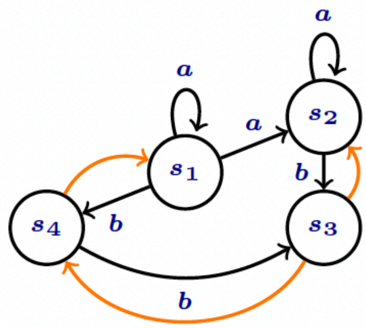
$R_a^* := \{(s,s') \mid R_a^n ss' \text{ for some } n \in \mathbb{N}\}$



$$
\begin{aligned}
R_b &:= \{(s_1,s_4),(s_2,s_3),(s_4,s_3)\}\\
R_b^0 &= \{(s_1,s_1),(s_2,s_2),(s_3,s_3),(s_4,s_4)\}\\
R_b^1 &= \{(s_1,s_4),(s_2,s_3),(s_4,s_3)\}\\
R_b^2 &= \{(s_1,s_4),(s_2,s_3),(s_4,s_3),(s_1,s_3)\}\\
R_b^3 &= \{(s_1,s_4),(s_2,s_3),(s_4,s_3),(s_1,s_3)\}\\
&\vdots\\
R_b^* &= \{(s_1,s_1),(s_2,s_2),(s_3,s_3),(s_4,s_4),\\
&\qquad (s_1,s_4),(s_2,s_3),(s_4,s_3),(s_1,s_3)\}
\end{aligned}
$$

Converse action

$R_a^{-} := \{(s',s) \mid R_a ss'\}$



$$
\begin{aligned}
R_b &:= \{(s_1,s_4),(s_2,s_3),(s_4,s_3)\}\\
R_b^{\smile} &= \{(s_4,s_1),(s_3,s_2),(s_3,s_4)\}
\end{aligned}
$$

Language of PDL

- Formulas →
  - (1) every basic proposition
  - (2) formulas with logical connectives
  - (3) $\langle\alpha\rangle\varphi$, where α is action and φ formula
- Actions →
  - (1) every basic action
  - (2) $\alpha;\beta$, $\alpha\cup\beta$, α^*
  - (3) $?\varphi$, where φ is a formula

Intuitions and abbreviations

$\alpha;\beta$	**sequential composition**: execute α and then β.
$\alpha \cup \beta$	**non-deterministic choice**: execute α or β.
α^*	**repetition**: execute α zero, one, or any *finite* number of times.
$?\varphi$	**test**: check whether φ is true or not.
$\langle\alpha\rangle\,\varphi$	α can be executed in such a way that, after doing it, φ is the case.

We abbreviate $p \vee \neg p$ as $\top$.

We abbreviate $\neg\top$ as $\bot$.

We abbreviate $\neg\langle\alpha\rangle\,\neg\varphi$ as $[\alpha]\,\varphi$.

$[\alpha]\,\varphi$	After any execution of α, φ is the case.

$\langle\alpha\rangle\,\top$	α can be executed.
$[\alpha]\,\bot$	α cannot be executed.
$\langle\alpha\rangle\,\varphi \wedge \neg[\alpha]\,\varphi$	α can be executed it at least two different ways.

Models

↳ Labelled transition systems

(1) S, non-empty set of states

(2) Valuation function → indicates which atomic propositions are true in each state

(3) Binary relation R_a for each action a

$M = \langle S, R_a, V \rangle$

Pointed labelled transition system = Process graph

↳ has a root state