Calculus 2021
Final Exam ALL MATERIAL

— Do not turn this page before the official start of the exam! —

First Name, Surname: _____

Student ID: _____

Program: Bachelor's DKE)

Course code: KEN1440

Examiner: Alexia Briassouli and Otti D'Huys

Date/time: Monday, March 29$^{\text{th}}$, 2021, 8.30-10.30h

Format: Closed book exam

Allowed aides: Pens. Calculators are not allowed, and are not needed.

Instructions to students:

- The exam consists of 8 questions on 15 pages (excluding the 1 cover page(s)).
- Fill in your name and student ID number on each page, including the cover page.
- Answer every question at the reserved space below the questions. If you run out of space, continue on the back side, and if needed, use the extra blank page.
- Ensure that you properly motivate your answers.
- Do not use red pens, and write in a readable way. Answers that cannot be read easily cannot be graded and may therefore lower your grade.
- You are not allowed to have a communication device within your reach, nor to wear or use a watch.
- You have to return all pages of the exam. You are not allowed to take any sheets, even blank, home.
- If you think a question is ambiguous, or even erroneous, and you cannot ask during the exam to clarify this, explain this in detail in the space reserved for the answer to the question.
- If you have not registered for the exam, your answers will not be graded, and thus handled as invalid.
- ***For all the questions below, the lecture notes and your own critical judgement are sufficient.*** **Good luck!**

The following table will be filled by the examiner:

Question:	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Total
Points:	11	11	20	12	12	13	11	10	100
Score:									

Question 1 (Q1) (11 points)

Limits

(a) (5 points) Evaluate the following limit or explain why it does not exist:

$$\lim_{x\to 0^+} \frac{\ln(x)}{x} \quad \frac{-\infty}{0}$$

$\lim_{x\to 0^+} \frac{\ln x}{x} = -\infty$

(b) (6 points) Evaluate the following limit or explain why it does not exist:

$$\lim_{x\to -\infty} \frac{1}{\sqrt{x^2+2x}-x} \qquad \sqrt{x^2\left(1+\frac{2}{x}\right)} - x$$

$\lim_{x\to -\infty} \frac{-1}{x\left(\sqrt{1+\frac{2}{x}}+1\right)} = 0$ $\qquad = |x|\sqrt{1+\frac{2}{x}} - x$

$= x\left(\sqrt{1+\frac{2}{x}}+1\right)$

Question 2 (Q2) (11 points)
Derivatives

(a) (5 points) Find the derivative of $f(x)$:

$$f(x) = \frac{\sin(x)}{\sqrt{x}}$$

$h(x) = \sin(x)$ $\quad h'(x) = \cos x$

$g(x) = \sqrt{x}$ $\quad g'(x) = \frac{1}{2\sqrt{x}}$

$$f'(x) = \frac{h'(x)g(x) - g'(x)h(x)}{x}$$

$$= \left(\cos x \sqrt{x} - \frac{\sin x}{2\sqrt{x}}\right)\frac{1}{x}$$

$$= \frac{2\cos x - \sin x}{2\sqrt{x}}\,\frac{1}{x}$$

$$= \frac{2\cos x - \sin x}{2x\sqrt{x}}$$

(b) (6 points) Let:

$$f(x) = e^x - 1 \text{ if } x \geq 0 \quad (1)$$
$$= ax + b \text{ if } x < 0 \quad (2)$$

Can you give values to the real parameters a and b such that $f(x)$ is continuous, but NOT differentiable at $x = 0$. Explain all your reasoning.

Continuity

$$\lim_{x \to 0^+} (e^x - 1) = \lim_{x \to 0^-} (ax + b)$$

$$0 = b \quad \Rightarrow \quad b = 0$$

Differentiability

$$f'(x) = \begin{cases} e^x & x > 0 \\ a & x < 0 \end{cases}$$

$\lim_{x \to 0^+} e^x \neq \lim_{x \to 0^-} a$

$a \neq 0$

$\Rightarrow b = 0 \quad \wedge \quad a \in \mathbb{R} - \{0\}$

Question 3 (Q3) (20 points)

Sketching the graph of a function

(a) (20 points) Let :

$$f(x) = \tan x = \frac{\sin(x)}{\cos(x)}$$

1. Determine the domain of f. Is f continuous on its domain? Why?
2. Compute the first derivative of f. Determine from this derivative for what values of x the function f is increasing or decreasing. Does it have local minimum(s), maximum(s)? If yes, at which values of x?
3. Compute the second derivative of f. Determine from this derivative for what values of x the function f is convex (concave up) or concave (concave down). Does it have inflection points? If yes, at which values of x?
4. Find if f has vertical, horizontal or oblique asymptotes.
5. Is f even? Is f odd? Why?
6. Sketch the graph of f based on your previous answers, and show the properties found in your previous answers in the graph.

(1) The domain is not continuous

domf: $\cos(x) \neq 0$

$x \neq \frac{\pi}{2} + k\pi$

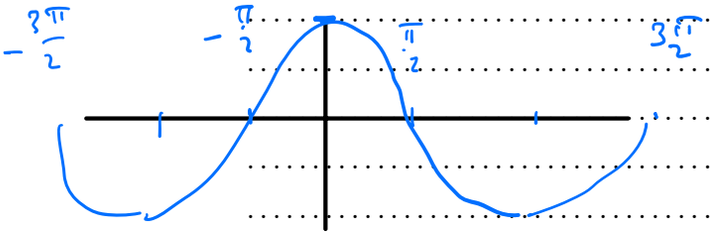
(2) $f'(x) = \dfrac{\cos x \cos x + \sin x \sin x}{\cos x^2} = \dfrac{1}{\cos^2 x}$

$f'(x) = 0 : \quad \dfrac{1}{\cos^2 x} = 0 \quad \nexists x \in \mathbb{R}$

$f'(x) > 0 : \quad$ 1. $1 > 0$ $\quad t = \cos x$

2. $\cos^2 x > 0 \quad t^2 > 0$

$\cos x < 0 \quad \vee \quad \cos x > 0$

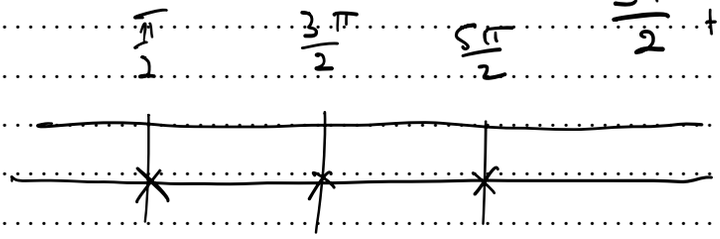


(a) $\cos x < 0$

$\frac{\pi}{2} + 2k\pi < x < \frac{3\pi}{2} + 2k\pi$

(b) $\cos x > 0$

$\frac{3\pi}{2} + 2k\pi < x < \frac{5\pi}{2} + 2k\pi$



Question 4 (Q4) (12 points)

Integrals

(a) (5 points) Evaluate the following integral:

$$\int \frac{x}{x^2-1} dx$$

(b) (7 points) Evaluate the integral of the following piecewise defined function:

$$\int_1^4 |x^2 - 5x + 6| dx$$

Question 5 (Q5) (12 points)
Sequences, Series

(a) (5 points) Find the sum of the following series or show it diverges to infinity:

$$\sum_{k=0}^{\infty} \frac{2^{k+3}}{e^{k-3}}$$

(b) (7 points) Determine if the given series converges or diverges using an appropriate test.

$$\sum_{n=1}^{\infty} \left| \sin \frac{1}{n^2} \right|$$

You can use the p-series or geometric series for comparison.

Question 6 (Q6) (13 points)

Differential Equations

(a) (6 points) Find the solution $y = y(x)$ to the given initial value problem. On what interval is the solution valid?

$$\begin{cases} y' = x^{-2} - x^{-3} \\ y(-1) = 0 \end{cases}$$

(b) (7 points) Find the solution $y = y(x)$ to the given initial value problem. On what interval is the solution valid?

$$\begin{cases} y'' = \cos x \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

Question 7 (Q7) (11 points)

Multivariate Calculus, Partial Derivatives

not in exam

(a) (5 points) Find the limit or explain why it doesn't exist

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$$

(b) (6 points) If $z = f(x, y)$ where $x = 2s + 3t$, $y = 3s - 2t$, find

$$\frac{\partial^2 z}{\partial s \partial t}$$

Question 8 (Q8) (10 points)

Double Integrals

(a) (5 points) Evaluate the double integral by iteration:

$$\int\int_R (x^2 + y^2)dA, \text{ where R is the rectangle } 0 \leq x \leq a, 0 \leq y \leq b$$

(b) (5 points) Evaluate the double integral by iteration:

$$\int_0^2 \int_0^y y^2 e^{xy} dxdy$$

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