Practice Exam Questions Week 3, **Linear Algebra**

SOLUTIONS.

1. Consider the following matrix A:

$$A = \begin{bmatrix} 3 & -6 & 2 \\ 1 & -2 & 1 \\ -2 & 4 & -2 \end{bmatrix} = [\underline{a}_1 \ \underline{a}_2 \ \underline{a}_3]$$

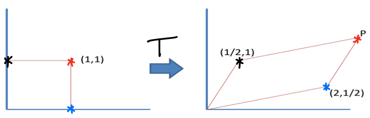
   a. Are the columns of A linearly independent?

No, $\underline{a}_2 = -2 \cdot \underline{a}_1$ and thus $2 \cdot \underline{a}_1 + 1 \cdot \underline{a}_2 + 0 \cdot \underline{a}_3 = \underline{0}$. So, the columns of A are not linearly independent.

   b. Is A invertible? (Hint: use the answer from a.)

No, from a. we know that the columns of A do not form a linearly independent set. Hence, by the Invertible Matrix Theorem, A is not invertible.

2. Consider the following transformation



   a. Construct the matrix which would give this transformation.

$$T(\underline{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$$

$$T(\underline{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

So, the standard matrix for the linear transformation T is $\begin{bmatrix} 2 & 1/2 \\ 1/2 & 1 \end{bmatrix}$.

   b. Give the coordinates of point P.

$$\begin{bmatrix} 2 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 3/2 \end{bmatrix}.$$

3. Consider the following matrix A:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}.$$

   a. Compute the inverse of A.

$$\left[\begin{array}{ccc|ccc} 3 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & -1 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_2 : R_2 - 3R_1 \\ \sim \\ R_3 : R_3 + 2R_1 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right] \underset{\sim}{R_2 : R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right] \underset{\sim}{R_1 : R_1 + R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right] \underset{\sim}{R_2 : R_2 * -1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 2 & 1 \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$

b. Let $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ and find a solution to the linear system of equations $A\mathbf{x} = \mathbf{b}$.

$$\underline{x} = A^{-1}\underline{b} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 8 \end{bmatrix}$$

c. Give an example of an alternative bottom row for A which would make it singular.

$\begin{bmatrix} 3 & 2 & -1 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}$ is, for example, not invertible, because in this matrix there are less than 3 pivot positions.

4. Compute the determinant of the following matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = (-1) * \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{vmatrix} = (-1)*(-1)* \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

$$= (-1)*(-1)*2 = 2.$$

5. True or false? If the given statement is true, briefly explain why. If it is false, give a counterexample.

a. If S is a 3×3 matrix such that $S^2 = 0$, then S^{-1} does not exist.

True. let's prove it by contradiction. Suppose for the sake of contradiction that S^{-1} exists.
Then, $S^2 = 0 \Rightarrow S \cdot S = 0 \Rightarrow S^{-1}S \cdot S = S^{-1} \cdot 0 \Rightarrow S = 0$.
But, if $S = 0$, then S is not invertible (because it doesn't have any pivot positions). Hence, we have a contradiction with our assumption. As a result, S^{-1} does not exist.

b. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent, and the variable x_3 is a free variable in the reduced echelon form of A, then there is a solution with $x_3 = 4$.

True. If x_3 is a free variable, then x_3 can take any value in $\mathbb{R}$. Therefore $x_3 = 4$ is allowed as well. And the system is consistent, so there is a solution for $x_3 = 4$.

c. If you take two vectors in $\mathbb{R}^3$ they will never be linearly dependent.

False, take for example $\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\underline{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$. Then, $\underline{v}_2 = 2\underline{v}_1$ and thus $\{\underline{v}_1, \underline{v}_2\}$ is linearly dependent.

d. If F is (2×2) with $\det(F) = 0$ and g is a (2×1) vector, then the matrix equation $F\mathbf{x} = \mathbf{g}$ is always inconsistent.

False. Take for example $F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\underline{g} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then, $\det(F) = 0$, but $[F \mid g] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$ and thus $F\underline{x} = \underline{g}$ has infinitely many solutions. Hence, $F\underline{x} = \underline{g}$ is consistent.

e. If G is a (3×3) matrix for which $G^2 = I$, then $\det(G) = 1$.

False. Take for example $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Then, $G^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$, but $\det(G) = -1$.

f. If $\det(B) \neq 0$, then B^T is invertible.

True. $\det(B^T) = \det(B)$ and thus $\det(B^T) \neq 0$. Hence, B^T is invertible.

g. If A is a (3×4) matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^3$ onto $\mathbb{R}^4$.

False. In order to perform $A\underline{x}$, $\underline{x}$ needs to be in $\mathbb{R}^4$ and the resulting vector $A\underline{x}$ is in $\mathbb{R}^3$. Therefore, A is a transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$.

h. An elementary row operation on A does not change the determinant of A.

False. Consider for example $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, but $\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2$ and $\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1$.