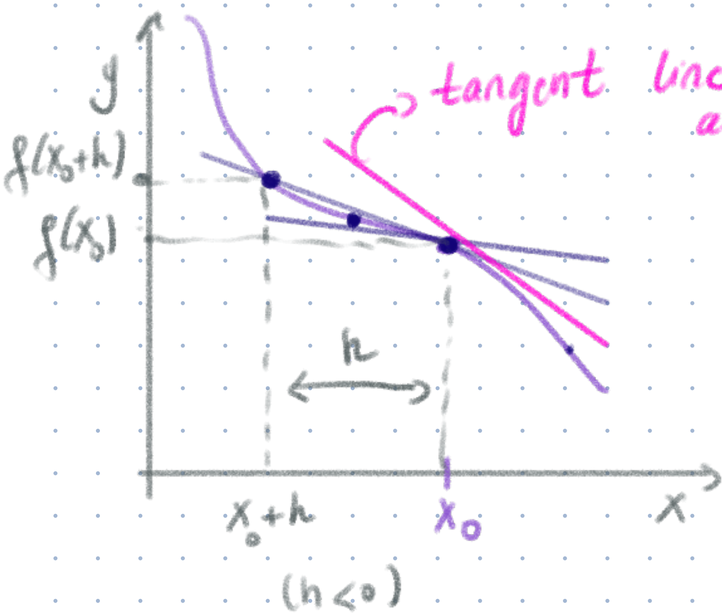
Calculus Lecture 3: Differentiation

- Tangent lines
- Definition of a derivative
- Calculation of derivatives
  - Product rule
  - Chain rule
  - Trigonometric functions
  - Exponential and logarithmic functions
- Higher order derivatives

Adams' Ch. 2.1-6, 3.3

Tangent lines and their slopes



tangent line (touches function at 1 point)

slope: $\frac{\Delta y}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{h}$

slope of tangent line

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

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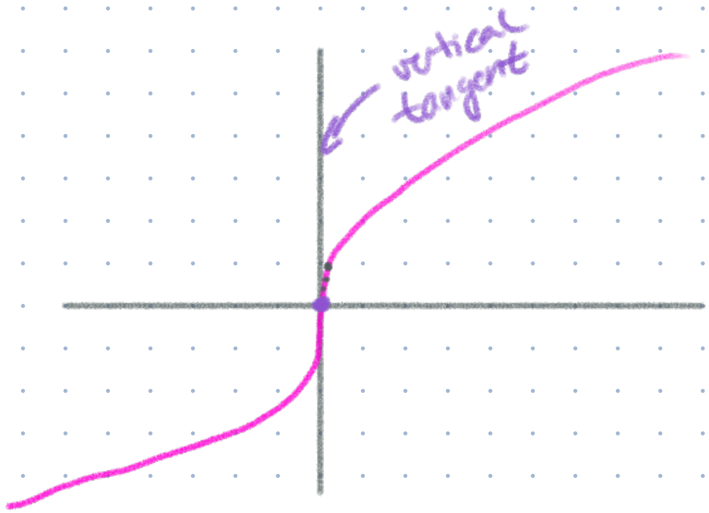
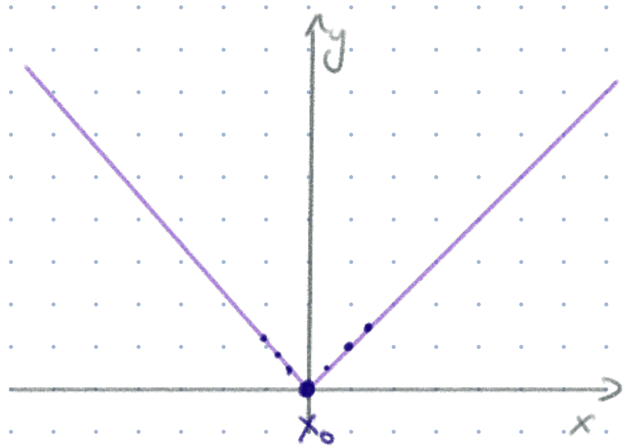
- Can the tangent line be vertical? yes



$f(x) = \sqrt[3]{x}$

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \pm\infty \quad (\text{here } +\infty)$$

- Does the tangent line always exist?



$$f(x) = |x| \longrightarrow f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} = \operatorname{sgn}(x)$$

no tangent line at $x_0 = 0$

$$f'_{\pm}(0) = \lim_{h \to 0^{\pm}} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^{\pm}} \frac{|0+h| - \cancel{|0|}}{h}$$

$$= \lim_{h \to 0^{\pm}} \frac{|h|}{h} = \lim_{h \to 0^{+}} \frac{-h}{h} = -1$$

The derivative

The derivative of a function at a point c of the domain is de defined as the slope of (the tangent of) the function at c.

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$

The derivative only exists if this limit exists and is finite!

$$\text{domain}(f') \subseteq \text{domain}(f)$$

- *Notations:*

$$y', \quad f'(x), \quad \frac{d}{dx}\big(f(x)\big), \quad D_x f, \quad Df(x), \quad \frac{dy}{dx}$$

$$f'(x_0), \quad \frac{d}{dx}\big(f(x)\big)\Big|_{x=x_0} \quad \frac{dy}{dx}\Big|_{x=x_0}$$

- *Differentiable:*

f is differentiable at x_0 $=$ $f'(x_0)$ exists

- *Singular point:*

x_0 is a singular point $=$ $f'(x_0)$ does not exist

- *Left and right derivatives:* $f'_\pm(x_0) = \lim_{h\to 0^\pm} \frac{f(x_0+h) - f(x_0)}{h}$

Examples

- $f(x) = ax + b \longrightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(a(\cancel{x}+h) + \cancel{b}) - (\cancel{ax}+\cancel{b})}{h}$

$$= \lim_{h \to 0} \frac{ah}{h} = a$$

- $f(x) = c \longrightarrow f'(x) = 0$

- $f(x) = x^2 \Rightarrow f'(x) = 2x \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2\cancel{h}x + \cancel{h^2} - \cancel{x^2}}{\cancel{h}} = 2x$$

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1} \quad (\forall n \in \mathbb{R})$$

Differentiation rules

- Differentiable -> continuous
- For functions f and g differentiable at x, and k constant,
  - $(f+g)'(x) = f'(x) + g'(x)$
  - $(k\ f)' = k \cdot f'(x)$
- (The product rule) For functions f and g differentiable at x
  - $(f(x)g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
  - NOT: $\cancel{f'(x) \cdot g'(x)}$

$$\frac{d}{dx}(f(x) \cdot g(x)) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(f(x+h)g(x+h) - f(x)g(x+h)\right) + \left(f(x)g(x+h) - f(x)g(x)\right)}{h}$$

$$= \lim_{h \to 0} \left(g(x+h)\left(\underbrace{\frac{f(x+h) - f(x)}{h}}_{f'(x)}\right) + f(x)\left(\underbrace{\frac{g(x+h) - g(x)}{h}}_{g'(x)}\right)\right)$$

$$= g(x) f'(x) + f(x) \cdot g'(x)$$

Examples

$f(x) = (x^2+1)(x^3+4)$

$f'(x) = (2x)(x^3+4) + (x^2+1)(3x^2) = 2x^4 + 8x + 3x^4 + 3x^2$
$= 5x^4 + 3x^2 + 8x$

$f(x) = x^5 + x^3 + 4x^2 + 4 \rightarrow f'(x) = 5x^4 + 3x^2 + 8x$

- "power rule" $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$ (valid for $n \in \mathbb{R}$)

proof for $n \in \mathbb{N}$: proof by induction :)

- Base case $n=1$ $\frac{d}{dx}(x^1) = 1$ ✓

- Induction step $P(n) \rightarrow P(n+1)$

$P(n)$: $\underline{\frac{d}{dx}(x^n) = n \cdot x^{n-1}}$ $P(n+1)$: $\underline{\frac{d}{dx}(x^{n+1})} = (n+1) \cdot x^n$

$$\frac{d}{dx}(x^{n+1}) = \frac{d}{dx}(x^n \cdot x) = \frac{d}{dx}(x^n) \cdot x + x^n \cdot \frac{d}{dx}(x)$$

$$= n \cdot \underbrace{x^{n-1} \cdot x}_{x^n} + x^n = (n+1)\,x^n \checkmark \quad \blacksquare$$

- (The chain rule): If $f(u)$ is differentiable at $u=g(x)$, and $g(x)$ is differentiable at x, then $f(g(x))$ is differentiable and

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

$g(x)=u \qquad \frac{d}{dx}(f(u(x))) = \frac{df}{du} \cdot \frac{du}{dx}$

- (reciprocal rule)

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-1}{(f(x))^2} \cdot f'(x)$$

- (quotient rule) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x) \cdot \frac{1}{g(x)}\right) = f'(x) \cdot \frac{1}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2}$

$$= \frac{f'(x) \cdot g(x) - f(x)g'(x)}{(g(x))^2}$$

Examples:

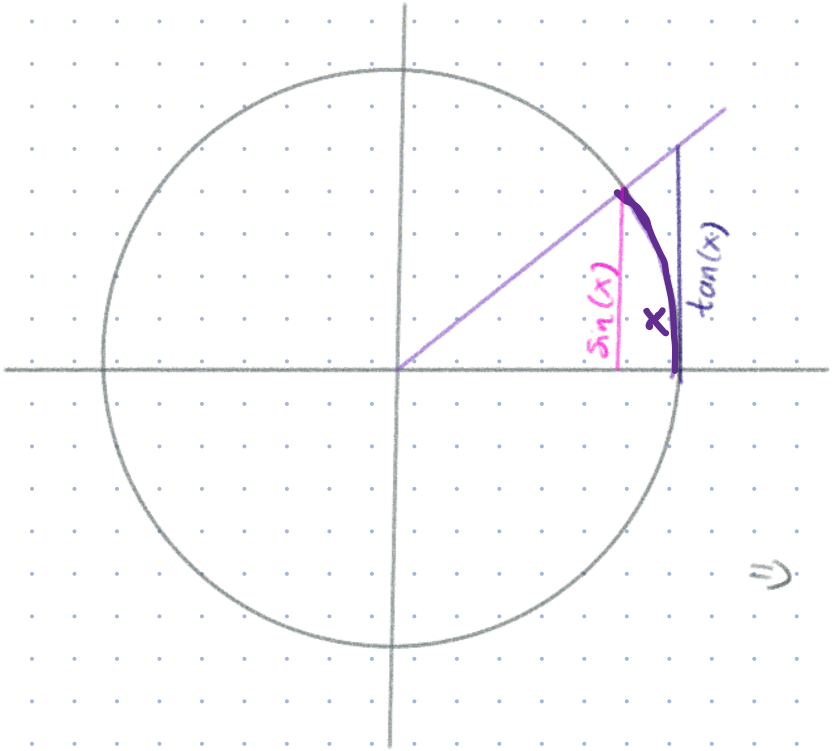
- $\dfrac{d}{dx}\left(\dfrac{1}{1+x^2}\right) = \dfrac{-1}{(1+x^2)^2}\,\dfrac{d}{dx}(1+x^2) = \dfrac{-2x}{(1+x^2)^2}$

- $\dfrac{d}{dx}\left(\dfrac{x^2+1}{\sqrt{x}}\right) = \dfrac{2x\cdot\sqrt{x} - \frac{1}{2\sqrt{x}}(x^2+1)}{x} = \dfrac{2(2x)x - (x^2+1)}{2x\sqrt{x}}$

  $= \dfrac{3x^2 - 1}{2x\sqrt{x}}$

Derivatives of trigonometric functions

- $\lim\limits_{x \to 0} \frac{\sin(x)}{x}$



$x > 0$ $\quad \sin(x) < x < \tan(x)$

$\Leftrightarrow \quad \boxed{\frac{\sin(x)}{x} < 1} < \frac{\tan(x)}{x}$

$$\frac{\tan(x)}{x} = \frac{\sin(x)}{\cos(x) \cdot x}$$

$$\Rightarrow \cos(x) < \frac{\sin(x)}{x}$$

$$\Rightarrow \underset{\downarrow\, 1 \text{ as } x \to 0}{\cos(x)} < \frac{\sin(x)}{x} < \underset{\downarrow\, 1 \text{ as } x \to 0}{1}$$

$$\lim_{x \to 0^+} \frac{\sin(x)}{x} = 1$$

$\rightarrow \lim\limits_{x \to 0} \frac{\sin(x)}{x} = 1$ since $\frac{\sin(x)}{x}$ is even

$$\rightarrow \frac{d}{dx}(\sin(x)) = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left(x+\frac{h}{2}\right)}{h}$$

approaches 1 as $h \to 0$

$$= \underbrace{\lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}_{1} \cdot \underbrace{\lim_{h\to 0} \cos\left(x+\frac{h}{2}\right)}_{\cos(x)}$$

$$= \cos(x)$$

trig. identities:

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$

$$\sin(a-b) = \sin(a)\cos(b) - \sin(b)\cos(a)$$

$$\rightarrow \sin(a+b) - \sin(a-b) = 2\sin(b)\cos a$$

↳ here: $a = x+\frac{h}{2}$

$b = \frac{h}{2}$

$a+b = x+h$

$a-b = x$

(alternative calculation)

$$\frac{d}{dx}(\sin(x)) = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} =$$

$$\lim_{h\to 0} \frac{(\sin(x)\cos(h) + \sin(h)\cos(x)) - \sin(x)}{h} = \lim_{h\to 0}\left(\sin(x)\cdot\frac{\cos(h)-1}{h} + \cos(x)\frac{\sin(h)}{h}\right)$$

to 0, see below

$$= \lim_{h\to 0}\left(\cos(x)\frac{\sin(h)}{h}\right) = \cos(x)$$

$$\cdot \lim_{h\to 0} \frac{\cos(h)-1}{h} = \lim_{h\to 0} \frac{(\cos(h)-1)(\cos(h)+1)}{h(\cos(h)+1)}$$

$$= \lim_{h\to 0} \frac{\cos^2(h)-1}{h(\cos(h)+1)} = \lim_{h\to 0} \frac{-\sin^2(h)}{h(\cos(h)+1)}$$

$$= \lim_{h\to 0}\left(\frac{\sin(h)}{h}\cdot\frac{-\sin(h)}{\cos(h)+1}\right) = 0$$

↳ to 1 ↳ to 0

other trig. limits:

$$\frac{d}{dx}(\cos(x)) = \frac{d}{dx}\left(\sin\left(\frac{\pi}{2}-x\right)\right) = -\cos\left(\frac{\pi}{2}-x\right) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\cdot\cos(x) - (-\sin(x)\cdot\sin(x))}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

reciprocal rule

- $$\frac{d}{dx}(e^x) = e^x$$

$$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = e^x \lim_{h\to 0}\frac{e^h-1}{h}$$

e is chosen / determined such, that this limit is 1

the exponential function $f(x) = e^x$ is the only function that is equal to its derivative!

- $$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

proof: $y = \ln(x) \iff e^y = x$ (def of logarithm)

$\Rightarrow \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$

$\Rightarrow e^y \frac{dy}{dx} = 1$ (chain rule)

$\Rightarrow x\frac{dy}{dx} = 1$

$\Rightarrow dy/dx = 1/x$

Examples

- $\frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) = \frac{-\sin(x)\cdot\sin(x) - \cos(x)\cdot\cos(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)}$

- $\frac{d}{dx}\left(e^{-\sin(3x)}\right) = e^{-\sin(3x)} \frac{d}{dx}\left(-\sin(3x)\right) = e^{-\sin(3x)} \left(-\cos(3x)\right) \frac{d}{dx}(3x)$

chain rule : $\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} \cdot f'(x)$

$\frac{d}{dx}\left(-\sin(f(x))\right) = -\cos(f(x)) \cdot f'(x)$

$= -3\cos(3x)\, e^{-\sin(3x)}$

Higher order derivatives

$$y'' = f''(x) = \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx} f(x)\right) = D_x^2 f(x)$$

$$f(x) = x^3 \longrightarrow f'(x) = 3x^2 \longrightarrow f''(x) = 6 \cdot x \longrightarrow f'''(x) = 6 \longrightarrow f^{(4)} = 0$$