

Lecture 7: Vector Spaces (part 1)

(book: 4.1, 4.2).

Previous lecture: Determinants

Applications of Vector Spaces:

- * Vector Space Search Engines. (see the YouTube video on Canvas)
- * Digital Signal Processing (Section 4.7)
- * Fourier Series (follow-up course Numerical Mathematics)

Intro to Vector Space: 3Blue1Brown - Abstract vector spaces (see Canvas)

So far: $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n, \dots$ were our vector spaces.

- * adding vectors in $\mathbb{R}^n \rightarrow$ another vector in \mathbb{R}^n .
- * scaling vectors in $\mathbb{R}^n \rightarrow$ another vector in \mathbb{R}^n .

Definition of a vector space V :

A non-empty set of objects (vectors) with the following 10 rules: (axioms)

- ① $\underline{u}, \underline{v} \in V \Rightarrow \underline{u} + \underline{v}$ is another vector in V (closed under addition)
- ② $\underline{u} \in V, c \in \mathbb{R} \Rightarrow c \cdot \underline{u}$ is another vector in V (closed under scalar mult.)
- ③ $\underline{u} + \underline{v} = \underline{v} + \underline{u}$ (commutativity)
- ④ $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$ (associativity)
- ⑤ + ⑥ $\exists \underline{0} \in V: \underline{u} + \underline{0} = \underline{u}$ and $\underline{u} + (-\underline{u}) = \underline{0}$ (zero vector)
- ⑦ $1 \cdot \underline{u} = \underline{u}$
- ⑧ - ⑩ see the book.

Example: Is \mathbb{R}_+^2 a vector space? **No.**

$$\hookrightarrow \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0 \right\}$$

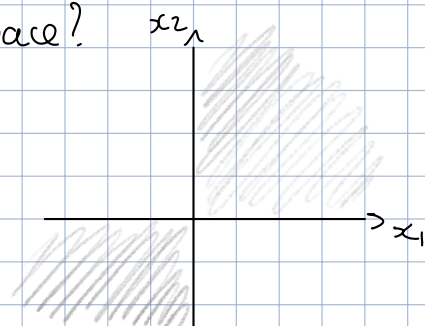


② **X** e.g. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbb{R}_+^2$, but $-1 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \notin \mathbb{R}_+^2$

① **✓** Let $\underline{u}, \underline{v} \in \mathbb{R}_+^2$ Then, $\underline{u} + \underline{v} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$ (Note: $u_1 + v_1 \geq 0$ and $u_2 + v_2 \geq 0$ are circled in blue)

Example: Is $\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_1 \cdot x_2 \geq 0 \right\}$ a vector space?

① **X** e.g. $\underline{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$.
Then $\underline{u} + \underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $1 \cdot (-1) < 0$.



Example: Is P_n a vector space?

↳ set of polynomials of degree at most n .
i.e., all polynomials of the form
 $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n$.

$n=3 \quad P_3$

$$\begin{aligned} p(x) &= x^3 \in P_3 \\ 8.5x^4 &\notin P_3 \\ 9x^2 + x^3 &\in P_3 \\ 7.5x^3 + x^4 &\notin P_3 \end{aligned}$$

①. Let $p, q \in P_n$ $p+q \in P_n$? $p(x) = a_0 + a_1x + \dots + a_nx^n$
 $q(x) = b_0 + b_1x + \dots + b_nx^n$.

$$\begin{aligned} (p+q)(x) &= p(x) + q(x) \\ &= (a_0+b_0) + (a_1+b_1)x + \dots + (a_n+b_n)x^n \end{aligned}$$

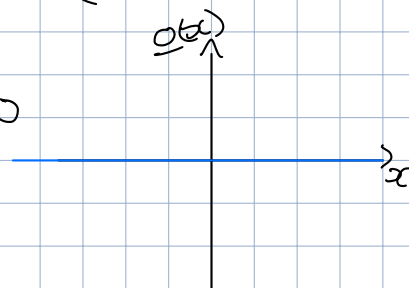
So, $p+q \in P_n$ ✓

② Let $p \in P_n$ and $c \in \mathbb{R}$. $p(x) = a_0 + a_1x + \dots + a_nx^n$

$$\begin{aligned} (c \cdot p)(x) &= c \cdot p(x) = c \cdot (a_0 + a_1x + \dots + a_nx^n) \\ &= c \cdot a_0 + (c \cdot a_1)x + \dots + (c \cdot a_n)x^n \end{aligned}$$

So, $c \cdot p \in P_n$ ✓

③-⑤ DIY ✓ But what is $\underline{0}$? $\underline{0}(x) = 0$



Similarly: P is also a vector space.
↳ set of all polynomials

Example: Set of polynomials of the form

$$p(x) = 0 + a_1x + a_2x^2 + \dots + a_nx^n$$

① ✗ → So, **not** a vector space.

② ✗

③ ✗

Let V be a vector space and let $W \subseteq V$.

When is W also a vector space?

- ① $\underline{w}_1, \underline{w}_2 \in W \Rightarrow \underline{w}_1 + \underline{w}_2 \in W$. (closed under addition)
- ② $\underline{w} \in W, c \in \mathbb{R} \Rightarrow c \cdot \underline{w} \in W$ (closed under scalar mult.)
- ③ $\underline{0} \in W$ (if $W \neq \emptyset$, then ③ follows from ②)

(All the other axioms are fulfilled because $W \subseteq V$ and V is a vector space).

W is called a subspace of V .

Example: Vector space V . $W = \emptyset$

$$W \subseteq V.$$

Is W a vector space?

Is W a subspace of V ?

- ① ✓
- ② ✓
- ③ ✗

(because $W = \emptyset$ and thus $\underline{0} \notin W$) \rightarrow So W is **not** a subspace of V .

Example: \mathbb{R}^3 is a vector space.

$$W = \left\{ \begin{bmatrix} a \\ b \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

$$W \subseteq \mathbb{R}^3.$$

- ① ✗
- ② ✗
- ③ ✗

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W.$$

\rightarrow So, W is **not** a subspace of \mathbb{R}^3 .

Example: $W = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of \mathbb{R}^3

Example: $W = \text{Span} \left\{ \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \right\}$ Subspace of \mathbb{R}^3 ? Yes.
(W is a line in \mathbb{R}^3). ✓

Theorem: Let V be a vector space.

If $\underline{v}_1, \dots, \underline{v}_p \in V$, then $W = \text{Span} \{ \underline{v}_1, \dots, \underline{v}_p \}$ is a subspace of V .

Proof:

① We need to show $W \subseteq V$.

Since V is a vector space:

So, $c_1 \underline{v}_1 \in V, \dots, c_p \underline{v}_p \in V$. (because of property ② of V).

So, $c_1 \underline{v}_1 + \dots + c_p \underline{v}_p \in V$ (because of prop ① of V)

So, $W \subseteq V$. ✓

①. If $\underline{x}, \underline{y} \in W$, then $\underline{x} + \underline{y} = c_1 \underline{v}_1 + \dots + c_p \underline{v}_p + d_1 \underline{v}_1 + \dots + d_p \underline{v}_p = (c_1 + d_1) \underline{v}_1 + \dots + (c_p + d_p) \underline{v}_p$.

So, $\underline{x} + \underline{y} \in W$. ✓

②. If $\underline{x} \in W, a \in \mathbb{R}$, then $a \cdot \underline{x} = a \cdot (c_1 \underline{v}_1 + \dots + c_p \underline{v}_p) = (a \cdot c_1) \underline{v}_1 + \dots + (a \cdot c_p) \underline{v}_p$.

So, $a \cdot \underline{x} \in W$. ✓

③. $\underline{0} = 0 \cdot \underline{v}_1 + \dots + 0 \cdot \underline{v}_p$ So, $\underline{0} \in W$. ✓

□

Theorem: Let A be an $m \times n$ matrix.

The set of all solutions of $A\underline{x} = \underline{0}$ is a subspace of \mathbb{R}^n .
 $= \text{Nul}(A)$. W .

Proof:

①. A is $m \times n$, so \underline{x} needs to have n elements. So, $W \subseteq \mathbb{R}^n$. ✓

②. If $\underline{u}, \underline{v} \in W$, then $A(\underline{u} + \underline{v}) = A\underline{u} + A\underline{v} = \underline{0} + \underline{0} = \underline{0}$ ✓
 $\underline{u} \in W \Rightarrow A\underline{u} = \underline{0}$
 $\underline{v} \in W \Rightarrow A\underline{v} = \underline{0}$

③. If $\underline{u} \in W$ and $c \in \mathbb{R}$, then $A(c \cdot \underline{u}) = c \cdot A\underline{u} = c \cdot \underline{0} = \underline{0}$ ✓
 $\underline{u} \in W \Rightarrow A\underline{u} = \underline{0}$

④. $A\underline{0} = \underline{0}$. So, $\underline{0} \in W$. ✓

□