1.3 Examples

(5) $S = [(p \wedge ((q \vee \neg r) \to \neg q)) \to (\neg q \vee r)]$

			A	B	C		
p	q	r	$q \vee \neg r$	$\neg q \vee r$	$A \to \neg q$	$p \wedge C$	$C \to B$
0	0	0	1				
0	0	1					
0	1	0					
1	0	0					
1	0	1					
1	1	0					
0	1	1					
1	1	1					

1.5 Theory (Direct proofs)

(a) To prove something for all x, we just need to prove it for an arbitrary x.

(b) If we choose a number y after an x, we can let y depend on x.

(c) Assume the antecedent is true, and prove the consequence

1.5 Direct proof exercises

(1) $\forall a \in \mathbb{Z}$, $2a$ is divisible by 5 $\to$ a is div. by 5

Assume $5 \mid 2a \quad \forall a \in \mathbb{Z}$

$\Rightarrow \underbrace{2a}_{\text{even}} = \underbrace{5}_{\text{odd}}\, j \qquad j \in \mathbb{Z}$

j has to be even

$2a = 5(2k) \qquad k \in \mathbb{Z} \qquad j = 2k$

$a = 5k$

↳ so a is divisible by 5 $\square$

(2) $7 \mid 4a \to 7 \mid a$

Assume $7 \mid 4a$

$\underbrace{4a}_{\text{even}} = \underbrace{7}_{\text{odd}}\, j \qquad j \in \mathbb{Z}$

j has to be even (7 is odd. odd × even = odd)

$4a = 7(2k)$ $\qquad j = 2k$

$2a = 7k$

$a = 7\left(\frac{k}{2}\right)$

↳ a is divisible by 7

(3) Every odd int is a difference of 2 squares

$2k+1 = a^2 - b^2$

Review of Combinatorics

(1) Rule of sum & Rule of product

(1) 8 women, 5 men

(i) men president → 5×8

female $\qquad + \quad 8 \times 7$

$= 5 \cdot 8 + 8 \cdot 7 = 8(5+7) = 12 \cdot 8$

(2) 3 bread, 5 meat, 6 cheese

Past exam paper

(1) $(p \rightarrow q) \leftrightarrow ((p \wedge \neg q) \rightarrow (q \wedge \neg q))$

F

0	1	0	1	0	0	1	1	0
1	0	0	1	1	1	1	0	0
0	1	1	1	0	0	0	1	0
1	1	1	1	1	0	0	1	0

↳ always true

(2)a $\forall n \in \mathbb{N}$, $4^{3n} + 8$ div by 9

Base case $n=1$

$4^3 + 8 = 64 + 8 = 72 \rightarrow$ div by 9 ✓

Induction step

Assume P holds for n

Prove it holds for n+1

$$4^{3(n+1)} + 8 = 4^{3n+3} + 8$$
$$= 4^{3n} \cdot 4^3 + 8$$
$$= 64 \cdot 4^{3n} + 8$$
$$= \underbrace{4^{3n} + 8}_{\text{divisible by 9 as per our assumption}} + \underbrace{63 \cdot 4^{3n}}_{\text{divisible by 9}}$$

so it is divisible by 9

(2)b $\forall n \in \mathbb{N} : 3^n > n!$

I need calculator to find counter example

(3) $A \subseteq B \leftrightarrow A \setminus C \subseteq B \setminus C$

(5) (a) 8 digit, binary (0, 1)
contains exactly 3 ones

_ _ _ 1 _ _ _ _ _

$n = 8$ (number of spots)

$k = 3$ (objects to select)

$\binom{n}{k} = \binom{8}{3} = 56$

(b) $x + y + z = 10$

10 ones need to be distributed

Repetition

Order is not important

$\binom{(n-1)+k}{k}$

(c) 26 letters $\times 2$, 5 characters

$\underline{26}\ \underline{52}\ \underline{52}\ \underline{52}\ \underline{52} \times 5$

(6)(a) $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})(\forall z \in \mathbb{Z}): (x = yz \rightarrow y = -z)$

$y = -z$ $\qquad$ $z = 1$

$x = (-z)(z) = -z^2$ $\qquad$ $x = -1$

Take $x = -1$, let $y \in \mathbb{Z}$ and $z \in \mathbb{Z}$

Assume $x = yz$, hence $yz = -1$

then, either $y = -1 \wedge z = 1$ or $y = 1 \wedge z = -1$

Case 1: $y = -1 \wedge z = 1$

so, $y = -z$ ✓

Case 2: $y = 1 \wedge z = -1$

so, $y = -z$ ✓

(6)(b) $(\forall n \in \mathbb{N})(\exists X \in P(\mathbb{N})): (|X| < n)$

Let $n \in \mathbb{N}$

consider $X = \emptyset$

$X \in P(\mathbb{N})$ because $\emptyset \subseteq \mathbb{N}$

then $|\emptyset| = 0 < n$, since $n \in \mathbb{N}$ (>0)

(6)(c) a and b are odd $\longrightarrow$ ab^2 is odd

Contropositive: ab^2 is even $\rightarrow$ a or b are even

(7) $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = 3x^2 + 2x + 1$

? is it bijective ?

Check if injective

$f: \mathbb{Z} \to \mathbb{Z}$ is injective

$\Longleftrightarrow (\forall x, y \in \mathbb{Z}) : (x \neq y \to f(x) \neq f(y))$

$\Longleftrightarrow (\forall x, y \in \mathbb{Z}) : (f(x) = f(y) \to x = y)$

Let $x, y \in \mathbb{Z}$

Assume $x \neq y$