

Lecture 8: Sequences and series 2

- Recap: Sequences and Series
- Convergence tests for positive series
- Absolute and conditional convergence

Adams' Ch. 9.1-9.4, Thomas' Ch. 10.1

Sequences

A sequence $\{a_n\}$ is a list of numbers $a_1, a_2, \dots, a_n, \dots$ in a given order

a_n term
 n index

* a sequence can be seen as a function $f: \mathbb{N} \rightarrow \mathbb{R}, n \rightarrow a_n = f(n)$

* a sequence can converge to a limit L , $a_n \rightarrow L$ if the terms approach a constant value L

examples $\frac{1}{n} \rightarrow 0$ $n^{\frac{1}{n}} \rightarrow 1$

$\frac{n}{n+1} \rightarrow 1$ $\frac{1}{2^n} \rightarrow 0$

* otherwise, the sequence can diverge to $\pm \infty$, if the terms become arbitrarily large. $\sqrt[n]{n} \rightarrow \infty$, $(1.5)^n \rightarrow \infty$

* the sequence a_n diverges if $\lim_{n \rightarrow \infty} a_n$ DOES NOT EXIST

example: $(-1)^n$, $\cos(\pi n)$, $\sin(n)$

Infinite series

(Infinite) series = formal sum of infinitely many terms

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$$

└ summation index can change.

+ a series can be seen as a sequence of partial sum $\{s_n\}$

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_n = \sum_{k=1}^n s_k$$

the series converges to s if $s_n \rightarrow s$

$$\sum_{n=1}^{\infty} a_n = s$$

* SERIES ARE AN INDETERMINATE FORM (usually)

↳ we sum up infinitely many terms (that are infinitely small)

↳ usually, we cannot calculate the sum. we can only conclude whether they converge (the sum exists.)

Important series

1. Geometric series

$$a_n = ar^{n-1}$$

$$r = \frac{a_{n+1}}{a_n}$$

↳ constant ratio between terms

→ convergence if $|r| < 1$, $a \neq 0$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

→ divergence to $+\infty$ if $|r| \geq 1$, $a > 0$
to $-\infty$ if $|r| \geq 1$, $a < 0$

→ divergence if $a \neq 0$, $|r| > 1$

2. p-series

$$a_n = \frac{1}{n^p}$$

for $p=1$, $a_n = \frac{1}{n}$ harmonic series

→ convergence for $p > 1$

divergence to $+\infty$ for $p \leq 1$

(shown by integral test.)

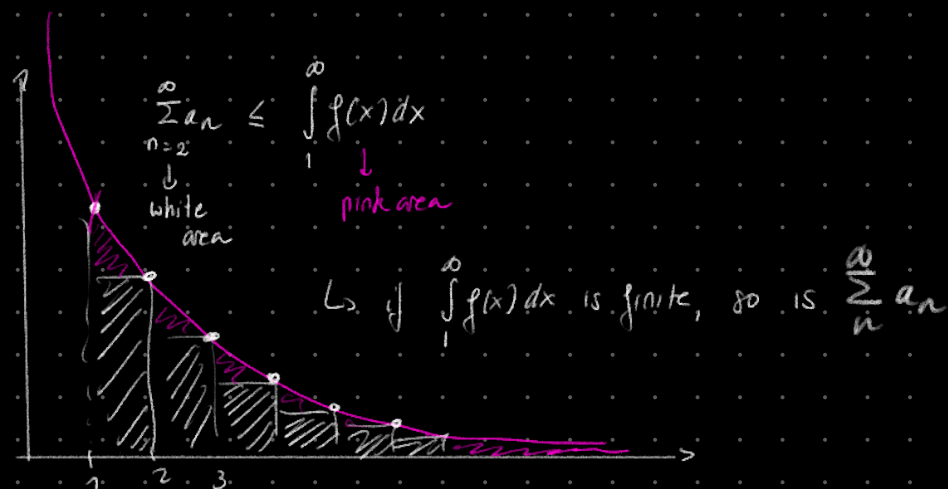
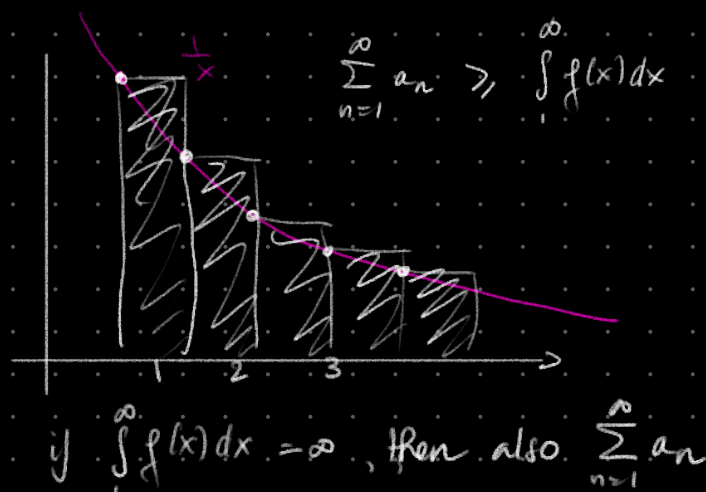
Integral test for positive series

* many convergence tests are for positive series only — adding up positive terms is conceptually easier than adding up positive and negative terms)

* if sequences compare to functions, series compare to improper integrals

↳ if $a_n = f(n)$ for f non-increasing on $[N, \infty)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_N^{\infty} f(x) dx$ both converge or diverge to ∞

PROOF: The series $\sum a_n$ can be seen as both upper and lower Riemann sum.



Comparison test (positive series)

for positive series $\sum a_n$ and $\sum b_n$

$$0 \leq a_n \leq b_n \cdot K \quad (K > 0)$$

• if $\sum a_n$ diverges, then $\sum b_n$ diverges

• if $\sum b_n$ converges, then $\sum a_n$ converges

Examples: $\sum \frac{\ln(n)}{n}$, $\sum \frac{3n+1}{n^3+n}$

for $n \geq 3$, $\frac{\ln(n)}{n} > \frac{1}{n}$
 \downarrow \downarrow
diverges diverges

$$\frac{3n+1}{n^3+n} \leq \frac{3n+1}{n^3} \leq \frac{3n+1}{n^3} = \frac{4}{n^2}$$

\downarrow \downarrow
converges converges

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$$

* we compare to $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ \rightarrow this series DIVERGES, it is a p-series with $p = \frac{1}{2} \rightarrow p < 1$.

* $\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}} \quad \forall n \geq 2$, so, since the smaller series $\sum \frac{1}{\sqrt{n}}$ diverges, the larger series $\sum \frac{1}{\sqrt{n}-1}$ also diverges.

Limit comparison test (positive series)

for $\{a_n\}, \{b_n\}$ positive sequences

- if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$ (exists, can be ∞)
 - if $0 < L < \infty$, $\sum a_n$ and $\sum b_n$ both converge or both diverge
 - if $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ also converges
 - if $L = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ also diverges
- if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$, $0 < L < \infty$, it means that a_n behaves like a multiple of b_n .
↳ they do the same
- if $L = 0$, $\sum a_n$ is a lot smaller than $\sum b_n$. If the larger series converges, so does the smaller series.
- if $L = \infty$, $\sum a_n$ is a lot larger than $\sum b_n$. If the smaller series diverges, so does the larger one.

Example: $\sum_{n=5}^{\infty} \frac{n+5}{n^3-2n+3}$

compare with $\sum \frac{1}{n^2}$ (converging p-series with $p=2$)

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n+5}{n^3-2n+3} \cdot n^2 = \lim_{n \rightarrow \infty} \frac{n^3+5n^2}{n^3-2n+3} = 1$$

\hookrightarrow both series behave the same: they both converge.

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$$

compare with $\sum \frac{1}{n^{3/2}}$ (converging p-series with $p = \frac{3}{2}$)

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\ln(n)}{n^2} \cdot n^{3/2} = \lim_{n \rightarrow \infty} \frac{\ln(n)}{\sqrt{n}}$$

$$\lim_{x \rightarrow \infty} \frac{\ln(x)}{\sqrt{x}} \stackrel{+}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x} = 0$$

since $\sum b_n$ converges, $\sum a_n$ also converges.

note: the integral test is also possible here.

Ratio test (positive series)

for a positive sequence $\{a_n\}$,

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho \quad (\text{can be } \infty)$$

$$\text{if } \rho < 1 \Rightarrow \sum a_n \text{ converges}$$

$$\text{if } \rho > 1 \Rightarrow \sum a_n \text{ diverges (and } a_n \rightarrow \infty)$$

$$\text{if } \rho = 1 \text{ NO CONCLUSION !!}$$

$$\text{Example } \sum \frac{3^n}{n!} \quad a_n = \frac{3^n}{n!}$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \lim_{n \rightarrow \infty} \frac{3}{n+1} = 0$$

\Rightarrow CONVERGENCE

$$\sum \frac{n^n}{n!}$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n \rightarrow \infty} \frac{1}{n+1} \cdot \left(\frac{n+1}{n}\right)^n \cdot (n+1)$$

$$= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \quad \text{DIVERGENCE!}$$

Absolute and conditional convergence

The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum |a_n|$ is convergent

↳ absolute convergence implies convergence

↳ a series that is convergent, but not absolutely convergent,

is called CONDITIONALLY convergent.

* $\sum \frac{(-1)^n}{n^2}$ is absolutely convergent, since $\sum |a_n| = \sum \frac{1}{n^2}$ converges.

* $\sum \frac{(-1)^n}{n}$ is NOT absolutely convergent, since $\sum |a_n| = \sum \frac{1}{n}$ diverges

Alternating series test

- If
- 1) $\{a_n\}$ is an alternating sequence.
 - 2) $|a_{n+1}| < |a_n|$ for $n > N$ (decreasing in absolute value)
 - 3) $a_n \rightarrow 0$.

Then $\sum a_n$ converges.

Note: first test for absolute convergence with another test. If $\sum a_n$ is not absolutely convergent, $\{a_n\}$ is alternating, use the alternating series test for to check for conditional convergence.

$\sum \frac{(-1)^n}{n}$ is conditionally convergent.

Example: for what values of $x \in \mathbb{R}$ does $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n \cdot 2^n}$ converge absolutely, converge conditionally, or diverge?

$$1) \sum_{n=1}^{\infty} \frac{(x-5)^n}{2^n \cdot n} = \sum_{n=1}^{\infty} \left(\frac{x-5}{2}\right)^n \cdot \frac{1}{n}$$

2) Apply ratio test to $\sum \left|\frac{x-5}{2}\right|^n \cdot \frac{1}{n}$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x-5}{2} \right|^{n+1} \cdot \left| \frac{x-5}{2} \right|^{-n} \cdot \frac{n}{n+1} = \left| \frac{x-5}{2} \right|$$

\Rightarrow absolute convergence for $\left| \frac{x-5}{2} \right| < 1 \Leftrightarrow 3 < x < 7$

\Rightarrow divergence (also for the sequence) for $\left| \frac{x-5}{2} \right| > 1$

\rightarrow if $x > 7$, divergence to ∞

$x < 3$, divergence (alternating sequence)

• if $x = 7$, $\sum \frac{(x-5)^n}{2^n \cdot n} = \sum \frac{1}{n} \rightarrow$ divergence to ∞

• if $x = 3$, $\sum \frac{(x-5)^n}{2^n \cdot n} = \sum \frac{(-1)^n}{n} \rightarrow$ conditional convergence

\hookrightarrow alternating

$\hookrightarrow a_n \rightarrow 0$

$\hookrightarrow |a_{n+1}| < |a_n|$

alternating series test implies convergence.