Lecture 8 : Vector Spaces (part 2)
(book: 4.2, 4.3, 4.4, 4.5)

Today: basis, dimension, Nul, Col, Row.

A basis of a vector space V is a set $\{\underline{v}_1, \dots, \underline{v}_p\}$ of vectors in V such that:
* $\{\underline{v}_1, \dots, \underline{v}_p\}$ is lin. indep.
* it spans the whole vector space.

Example: A basis for $\mathbb{R}^3$, standard basis $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$.

But also $\left\{ \begin{bmatrix}3\\0\\-6\end{bmatrix}, \begin{bmatrix}-4\\1\\7\end{bmatrix}, \begin{bmatrix}-2\\1\\5\end{bmatrix} \right\}$. Why?

$$\begin{bmatrix}3 & -4 & -2\\0 & 1 & 1\\-6 & 7 & 5\end{bmatrix} \sim \dots \sim \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

* pivot in every column $\Rightarrow$ lin. indep.
* pivot in every row $\Rightarrow$ spans $\mathbb{R}^3$.

Example: The standard basis for $\mathbb{P}_2$: $\{1, x, x^2\}$

But a basis is also: $\{1+x^2,\ x-3x^2,\ 1+x-3x^2\}$ (later).

Two views of a basis $\mathcal{B}$:
* it is maximal: adding any vector $\underline{v} \in V$ to $\mathcal{B}$, makes it lin dep.
* it is minimal: removing any vector $\underline{v}$ from $\mathcal{B}$, causes $\mathcal{B}$ to no longer span V.

dim (V): # vectors in a basis of V.

$\dim(\mathbb{R}^3) = 3$ $\qquad$ $\dim(\mathbb{R}^n) = n$

$\dim(\mathbb{P}_n) = n+1$.

Let $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ be a basis for V.
For every $\underline{x} \in V$ we can find weights $\underbrace{c_1, \dots, c_n}$ such that
$\underline{x} = c_1 \cdot \underline{b}_1 + \dots + c_n \cdot \underline{b}_n$

coordinates of $\underline{x}$ wrt basis $\mathcal{B}$.

$$[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Note: this set of coordinates is unique.

Example: $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} \right\}$ basis for $\mathbb{R}^3$.

Consider $\underline{x} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$.

What are the coordinates of $\underline{x}$ wrt the basis $\mathcal{B}$?

$$\left[\begin{array}{ccc|c} 3 & -4 & -2 & 5 \\ 0 & 1 & 1 & 2 \\ -6 & 7 & 5 & 4 \end{array}\right] \sim \dots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

Matlab: rref

$$[\underline{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ 5 \\ -2 \end{bmatrix}$$

Example: $\mathcal{B} = \{1+x^2, x-3x^2, 1+x-3x^2\}$ is a basis for $\mathbb{P}_2$. (later).

Consider $p(x) = 6+3x-x^2$.
Find $[p]_{\mathcal{B}}$.

$$6 + 3x - x^2 = c_1 \cdot (1 + x^2) + c_2 \cdot (x - 3x^2) + c_3 \cdot (1 + x - 3x^2).$$

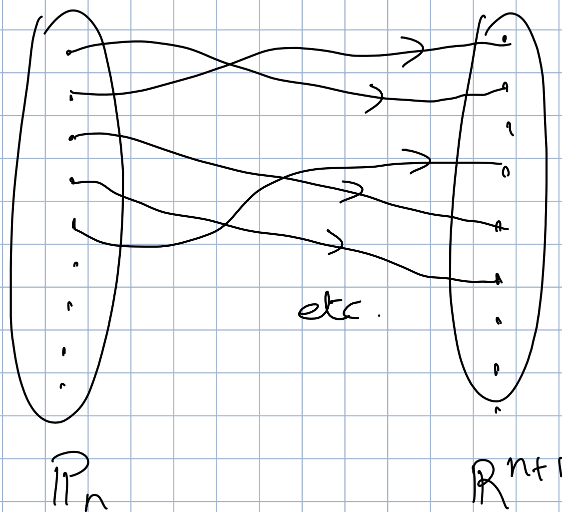
SLE
$$\begin{cases} c_1 + c_3 = 6 \\ c_2 + c_3 = 3 \\ c_1 - 3c_2 - 3c_3 = -1 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 1 & -3 & -3 & -1 \end{array}\right] \sim \dots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

So, $[p]_{\mathcal{B}} = \begin{bmatrix} 8 \\ 5 \\ -2 \end{bmatrix}$. Check: $p(x) = 8(1+x^2) + 5\cdot(x-3x^2) + (-2)\cdot(1+x-3x^2)$. ✓

Theorem: Let $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ be a basis for V.
Then, the linear transformation $T: V \to \mathbb{R}^n$ defined by $T(\underline{x}) = [\underline{x}]_{\mathcal{B}}$ is a bijection.

Proof: See the book (Theorem 9).

Hence, the vectors can be abstract objects, but using the coordinates we can translate them to real numbers.



bijection.

Example: Show that $\{1+x^2, x-3x^2, 1+x-3x^2\}$ forms a basis for $\mathbb{P}_2$.

The coordinate mapping of the standard basis $\{1, x, x^2\}$ produces the coordinate vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-3\end{bmatrix}, \begin{bmatrix}1\\1\\-3\end{bmatrix}$.

Do those vectors form a basis in $\mathbb{R}^3$? Yes.

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & -3 & -3\end{bmatrix} \sim \cdots \sim \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

So, $\{1+x^2, x-3x^2, 1+x-3x^2\}$ forms a basis for $\mathbb{P}_2$.

Three important subspaces associated with an $m \times n$ matrix A.

* $Nul(A)$
* $Col(A)$
* $Row(A)$

Recall from Monday: $Nul(A) = \{ \underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{0} \}$ is a subspace of $\mathbb{R}^n$.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -6 & 4 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 2 & -6 & 4 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underline{x} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \qquad Nul(A) = Span \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Note: * # vectors in the spanning set = # free variables.
* the vectors in the spanning set are always lin indep.

Basis for $Nul(A)$: $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. $\quad \dim(Nul(A)) = 2$

So, in general:
* $\dim(Nul(A))$ = # free vars.
* the vectors from parametric vector form form a basis for $Nul(A)$.

$Col(A) = Span\{\underline{a}_1, \dots, \underline{a}_n\}$ is a subspace of $\mathbb{R}^m$

How to find a basis for Col A?

Intermezzo.

The Spanning Set Theorem:

Let $\underline{v}_1, \dots, \underline{v}_p \in V$ and let $H = \text{Span}\{\underline{v}_1, \dots, \underline{v}_p\}$.

* If $\underline{v}_k$ is a lin comb. of the other vectors, then $\{\underline{v}_1, \dots, \underline{v}_p\} \setminus \{\underline{v}_k\}$ still spans H.
* If $H \neq \{\underline{0}\}$, some subset of $\{\underline{v}_1, \dots, \underline{v}_p\}$ spans H.

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* 1st strategy: keep deleting columns if they are dependent on others. The columns that you end up with form a basis for $\text{Col}(A)$.

Easy when in reduced echelon form.

$$B = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\left.\begin{aligned} \underline{b}_2 &= 3 \cdot \underline{b}_1 \\ \underline{b}_4 &= -1 \cdot \underline{b}_1 - 2 \cdot \underline{b}_3 \end{aligned}\right\} \text{Remove } \underline{b}_2 \text{ and } \underline{b}_4.$$

So, $\underbrace{\{\underline{b}_1, \underline{b}_3\}}_{\hookrightarrow \text{ pivot columns.}}$ forms a basis and $\dim(\text{Col}(B)) = 2$. True in general.

Intermezzo.

Linear dependence relations between colums do not change under row operations.

If $A \sim B$, then $A\underline{x} = \underline{0}$ and $B\underline{x} = \underline{0}$ have the same set of solutions.

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 2 & 6 & -1 & 0 \\ 1 & 3 & 0 & -1 \end{bmatrix} \sim \dots \sim B$$

So, we also have
$$\underline{a}_2 = 3 \cdot \underline{a}_1$$
$$\underline{a}_4 = -1 \cdot \underline{a}_1 - 2 \cdot \underline{a}_3$$

So, $\{\underline{a}_1, \underline{a}_3\}$ forms a basis for $\text{Col}(A)$. and $\dim(\text{Col}(A)) = 2$.

be careful! of A! Not of REF!

* 2nd strategy: the pivot columns of A form a basis for $\text{Col}(A)$.
Hence, $\dim(\text{Col}(A)) = \#\text{pivot cols}$.

Row(A) = Span$\{\underline{r}_1, \dots, \underline{r}_m\}$ is a subspace of $\mathbb{R}^n$. Row(A) = Col(A^T).

Note: $\boxed{A \sim B \Rightarrow \text{Row}(A) = \text{Row}(B)}$. (Warning Col(A) $\neq$ Col(B)).

idea: $A \sim B$

$\Rightarrow$ rows of B are a lin comb. of the rows of A.

$\Rightarrow$ any lin comb of the rows of B are also a lin comb of the rows of A.

$\Rightarrow$ Row(B) $\subseteq$ Row(A).

$A \sim B \Rightarrow B \sim A \Rightarrow \dots \Rightarrow$ Row(A) $\subseteq$ Row B.

So, Row(A) = Row(B).

If B is RREF, then the nonzero rows of B are lin indep and thus form a basis for both Row(B) and Row(A).

$$B = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, dim (Row(A)) = # pivot rows.

$\boxed{\text{Rank}(A)} = \boxed{\dim(\text{Col}(A))}$ = # pivot cols = # pivots = # pivot rows $= \boxed{\dim(\text{Row}(A))}$

$\hookrightarrow$ = n − # nonpivot cols.

= n − # free variables.

$\boxed{= n - \dim(\text{Nul}(A))}$

Recall a linear transformation $T: \underline{x} \longrightarrow A\underline{x}$

A : m×n
domain: $\mathbb{R}^n$
codomain: $\mathbb{R}^m$.

* Range = $\{T(\underline{x}) : \underline{x} \in \mathbb{R}^n\} = \{A\underline{x} : \underline{x} \in \mathbb{R}^n\}$ = Col(A)

So, T is surjective $\Leftrightarrow$ range = codomain $\Leftrightarrow$ Col(A) = $\mathbb{R}^m$
$\Leftrightarrow$ dim(Col(A)) = m $\Leftrightarrow$ Rank(A) = m

* Kernel = $\{\underline{x} \in \mathbb{R}^n : T(\underline{x}) = \underline{0}\} = \{x \in \mathbb{R}^n : A\underline{x} = \underline{0}\}$ = Nul(A).

So T is injective $\Leftrightarrow$ $T(\underline{x}) = \underline{0}$ has only the trivial sol.

$\Leftrightarrow$ Kernel = $\{\underline{0}\}$ $\Leftrightarrow$ Nul(A) = $\{\underline{0}\}$ $\Leftrightarrow$ dim(Nul(A)) = 0.

$\Leftrightarrow$ Rank(A) = n

Summary (so far):

Let A be an $m \times n$ matrix with columns $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$.

$m \geq n$: $\begin{bmatrix} A \end{bmatrix}$

The following statements are equivalent:

1. A has a pivot in every column
2. A has n pivot positions
3. There are no free variables
4. $A\underline{x} = \underline{0}$ has only the trivial sol.
5. $\{\underline{a}_1, \underline{a}_2, \dots \underline{a}_n\}$ is linearly indep.
6. $T: \underline{x} \mapsto A\underline{x}$ is one-to-one/injective
7. $\text{Nul } A = \{\underline{0}\}$
8. $\dim \text{Nul } A = 0$
9. $\text{rank } A = n$

$m \leq n$: $\begin{bmatrix} & A & \end{bmatrix}$

The following statements are equivalent:

a. A has a pivot in every row.
b. A has m pivot positions.
c. The echelon form of A does not contain a row of all zeros.
d. $A\underline{x} = \underline{b}$ is consistent for every $\underline{b}$ in $\mathbb{R}^m$.
e. $\text{Span}\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n\} = \mathbb{R}^m$.
f. $T: \underline{x} \mapsto A\underline{x}$ is onto/surjective.
g. $\text{Col } A = \mathbb{R}^m$
h. $\dim \text{Col } A = m$
i. $\text{rank } A = m$

If A is square ($n=m$), then statements 2 and b are equivalent. Hence, the following statements are equivalent for square matrices.

* 1 – 9, a – i
* A is invertible
* There is a matrix C such that $CA = I_n$ and $AC = I_n$
* A is row equivalent to I_n.
* A^T is invertible.
* $\det A \neq 0$
* The columns of A form a basis for $\mathbb{R}^n$.

A: $m \times n$ matrix.

	definition	subspace of	equal to $\mathbb{R}^n$ / $\mathbb{R}^m$ / $\mathbb{R}^n$ if	equal to $\{\underline{0}\}$ if
Nul A	$\{\underline{x} : A\underline{x} = \underline{0}\}$	$\mathbb{R}^n$	A is the zero matrix	A has a pivot in every column
Col A	Span$\{\underline{a}_1, \dots, \underline{a}_n\}$	$\mathbb{R}^m$	A has a pivot in every row	A is the zero matrix
Row A	Span$\{\underline{r}_1, \dots, \underline{r}_m\}$	$\mathbb{R}^n$	A^T has a pivot in every row	A is the zero matrix

	dimension
Nul A	# free var's in the equation $A\underline{x} = \underline{0}$ = # nonpivot columns in A = n - rank A
Col A	# pivot columns in A. = # pivot columns in A = rank A
Row A	# non-zero rows in the echelon form of A = # pivot columns in A = rank A

	finding a basis
Nul A	Find the general sol. of $A\underline{x} = \underline{0}$. Write the solution in parametric vector form where the weights are the free var's. The corresponding vectors form a basis for Nul A.
Col A	The pivot columns of A (so, of A itself, and thus <u>not</u> the pivot columns of a reduced form of A)
Row A	The nonzero rows of an echelon form of A.