

Overview of the course

- Continuity and limits
- Differentiation
- Integration
- Sequences and series
- Introduction to differential equations
- Introduction to multivariate calculus

Today

- Ordinary Differential Equations (ODE)
- Separable ODEs (solution method)
- Linear ODEs
- Parameter variation (solution method)
- Integral equations

Adams' 2.10, 18.1, 18.2, 7.9

Differential equations

- What: An equation that involves derivatives of an unknown function $y(x)$

↳ solution: explicit form $y(x)$

- ODE = Ordinary Differential equation

(↳ stochastic DE's, partial DE's, delay DE's also exist.)

- Examples:

$$F = m a \rightarrow m \frac{d^2}{dx^2} x = F(x, t) \rightarrow \text{solution } x(t)$$

$$y' = 2y(3-y) \rightarrow \text{solution } y(x)$$

$$y'' + 2y' + 3y = \cos(x)$$

- General form: $F(y^{(n)}, y^{(n-1)}, \dots, y', y, x) = 0$
↳ solution $y(x)$
order of ODE
dependent variable
independent variable

Initial value problems (IVP)

- The solution $y(x)$ of a differential equation is not unique (cf. integration constant)
- You need "initial conditions" $y(x_0) = y_0$ to find a unique solution $y(x)$
 - ↳ you need as many initial conditions as the order of the ODE

Example:

$$\int y' = \sin(x) \rightarrow y(x) = -\cos(x) + C$$

$$\begin{cases} y(0) = 1 \end{cases} \rightarrow y(0) = -\cos(0) + C = 1 \Rightarrow C = 2$$

$$y(x) = 2 - \cos(x)$$

$$\frac{d^2 y}{dx^2} = 1 \rightarrow \frac{dy}{dx} = x + C_1 \rightarrow y(x) = \frac{1}{2}x^2 + C_1 x + C_2$$

↳ 2 initial values needed for a unique solution:

$$y'(0) = C_1, \quad y(0) = C_2$$

First order separable ODEs

A separable ODE has the form $\frac{dy}{dx} = f(x) \cdot g(y)$ (typically first order)
→ easily solved by $\int \frac{dy}{g(y)} = \int f(x) dx$

Example: $\frac{dy}{dx} = y \cdot x \Rightarrow \int \frac{dy}{y} = \int x \cdot dx \Rightarrow \ln|y| = \frac{1}{2} x^2 + C$

$$\Rightarrow y(x) = \pm e^{x^2/2} \cdot e^C = k \cdot e^{x^2/2} \quad (k \in \mathbb{R})$$

check: $y' = k \cdot \frac{dx}{dx} e^{x^2/2} = x \cdot k e^{x^2/2} = x \cdot y$

Example 2: $y' = \frac{x}{y}, y(1) = 2$

$$\frac{dy}{dx} = \frac{x}{y} \Leftrightarrow \int y dy = \int x dx \Leftrightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$

$$y(x) = \pm \sqrt{x^2 + k} = \pm \sqrt{3 + x^2} \quad (k = 2C)$$

$$y(1) = \pm \sqrt{1 + k} = 2 \Rightarrow k = 3$$

Linear ODEs

- Linear: linear in $y(x)$ and all derivatives $y'(x), y''(x), \dots, y^{(n)}(x)$

$$a_n(x) \cdot y^{(n)} + \dots + a_1(x) \cdot y' + a_0(x) \cdot y = \underbrace{f(x)}$$

- Homogeneous: $f(x) = 0$

$\rightarrow y(x) = 0$ is a solution

Examples :

linear: $y'' - \sin(x) \cdot y' + 2x \cdot y = \cos(x)$

$$y' = 0$$

$$y' = y x^2$$

nonlinear $y' = \sin(y)$

$$y' = y^2 x$$

$$y'' = y \cdot y' + x$$

Linear homogeneous ODEs

Example: $\frac{d^2 y}{dx^2} = -y(x)$

$$y_1(x) = -\cos(x) \quad \sin(x)$$

$$y_2(x) = \sin(x)$$

$$-y_1 + y_2 = \sin(x) + \cos(x)$$

$$-y_1' + y_2' = +\cos(x) - \sin(x) \quad \downarrow \frac{d}{dx}$$

$$-y_1'' + y_2'' = -\sin(x) - \cos(x) \quad \downarrow \frac{d}{dx} = +y_1 - y_2$$

* If $y_1(x)$ and $y_2(x)$ are solutions of a linear homogeneous ODE, then $a y_1(x) + b y_2(x)$ is a solution. ($a, b \in \mathbb{R}$)

* number of independent solutions = order of ODE

* we can find all solutions by making linear combinations of independent solutions

Non-homogeneous linear ODEs

Form: $\frac{dy}{dx} + p(x)y = q(x)$

* assume a solution $y_H(x)$ of the homogeneous equation

$$\rightarrow \frac{dy_H}{dx} + p(x)y_H = 0$$

$\hookrightarrow C \cdot y_H$ is also a solution, since $\frac{d(C \cdot y_H)}{dx} + p(x)(C \cdot y_H) = 0$
 $C \in \mathbb{R}$ *

* assume a solution $y_P(x)$ of the full ODE (particular solution)

$$\rightarrow \frac{dy_P}{dx} + p(x)y_P = q(x) \quad **$$

(add up * and **)

$$\frac{d}{dx}(y_P + C \cdot y_H) + p(x)(y_P + C \cdot y_H) = q(x)$$

$\Rightarrow y_P(x) + C y_H(x)$ is a solution of the full ODE !!

\hookrightarrow in this way we can solve for all initial conditions,
(by adapting C) \rightarrow "general solution"

Example $y' + 2y = 3$

Homogeneous equation: $y' + 2y = 0$

$$\Rightarrow \frac{dy}{dx} = -2y \Leftrightarrow \int \frac{dy}{y} = \int -2 dx \Rightarrow \ln |y| = -2x + C$$

$$\Rightarrow y_H(x) = K \cdot e^{-2x}$$

$$y' + 2y = 3 \quad y_p = \frac{3}{2}$$

GENERAL SOLUTION $y(x) = y_H(x) + y_p(x)$

$$= K \cdot e^{-2x} + \frac{3}{2}$$

Solving linear first order ODEs : parameter variation

$$y' + p(x)y = q(x)$$

1) Solve homogeneous equation $\frac{dy}{dx} + p(x)y = 0$

* this is separable $\int \frac{dy}{y} = -\int p(x)dx = -\mu(x) + C$

$$\Leftrightarrow \ln |y| = -\mu(x) + C$$

$$\Leftrightarrow y_H(x) = e^{-\mu(x)} \cdot e^C = h \cdot e^{-\mu(x)}$$

2) We make an assumption about the general solution

\rightarrow we assume $y(x) = k(x)e^{-\mu(x)}$

\hookrightarrow we change the constant in $y_H(x)$ into a function $k(x)$

\rightarrow we insert into the ODE

$$y(x) = k(x)e^{-\mu(x)}$$

$$y'(x) = k'(x)e^{-\mu(x)}$$

$$+ k(x)\left(-\frac{d}{dx}\mu(x)\right)e^{-\mu(x)} \quad \left. \vphantom{\begin{matrix} y(x) = k(x)e^{-\mu(x)} \\ y'(x) = k'(x)e^{-\mu(x)} \end{matrix}} \right\} \text{product rule}$$

$$\rightarrow y'(x) = h'(x) e^{-\mu(x)} - p(x) h(x) e^{-\mu(x)} \quad (\text{def. } \mu(x))$$

insert in ODE:

$$y' + p(x)y = \underbrace{h'(x) e^{-\mu(x)} - p(x) h(x) e^{-\mu(x)}}_{y'} + \underbrace{p(x) h(x) e^{-\mu(x)}}_{p(x) \cdot y} = q(x)$$

$$\Rightarrow h'(x) e^{-\mu(x)} = q(x)$$

$$\Rightarrow h'(x) = q(x) e^{\mu(x)}$$

$$\Rightarrow h(x) = \int q(x) e^{\mu(x)} dx$$

$$\Rightarrow y(x) = h(x) e^{-\mu(x)} = e^{-\mu(x)} \int q(x) e^{\mu(x)} dx$$

\rightarrow this is the integrating factor formula o.o

(do not learn by heart ... often leads to sign errors)

Example
$$\begin{cases} y' + \frac{y}{x} = 1 & (x > 0) \\ y(1) = 1 \end{cases}$$

1) Homogeneous equation $y' + \frac{y}{x} = 0 \Leftrightarrow \frac{dy}{dx} = -\frac{y}{x} \Leftrightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}$

$$\Leftrightarrow \ln|y| = -\ln|x| + C$$

$$\Leftrightarrow y_H(x) = e^{-\ln|x|} \cdot K = \frac{K}{e^{\ln(x)}} = \frac{K}{x}$$

2) Parameter variation $y = \frac{h(x)}{x}$

$$y'(x) = \frac{h'(x)}{x} - \frac{h(x)}{x^2} = 1 - \frac{y}{x} = 1 - \frac{h(x)}{x^2}$$

product rule ODE

$$\Rightarrow \frac{h'(x)}{x} = 1 \Rightarrow h'(x) = x \Rightarrow h(x) = \frac{x^2}{2} + C$$

$$\hookrightarrow y(x) = \frac{h(x)}{x} = \frac{1}{x} \left(\frac{x^2}{2} + C \right) = \frac{x}{2} + \frac{C}{x}$$

3) Initial condition $y(1) = 1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$

$y(x) = \frac{x}{2} + \frac{1}{2x}$
end result.

Integral equations

$$y(x) = a + b \int_c^x F(y(t), t) dt$$

$$\begin{cases} y' = b F(y, x) & \text{ODE} \\ y(c) = a + \int_c^c F(y, t) dt = a & \text{NP} \end{cases}$$

Example: $y(x) = 3 + 2 \int_1^x t y(t) dt$

$$\begin{cases} y' = 2xy \\ y(1) = 3 \end{cases} \rightarrow \frac{dy}{dx} = 2xy \rightarrow \int \frac{dy}{y} = \int 2x dx$$

$$\begin{aligned} \ln|y| &= x^2 + C \\ y(x) &= e^{x^2 + C} \\ &= k \cdot e^{x^2} \end{aligned}$$

$$y(1) = k \cdot e = 3 \rightarrow k = \frac{3}{e}$$