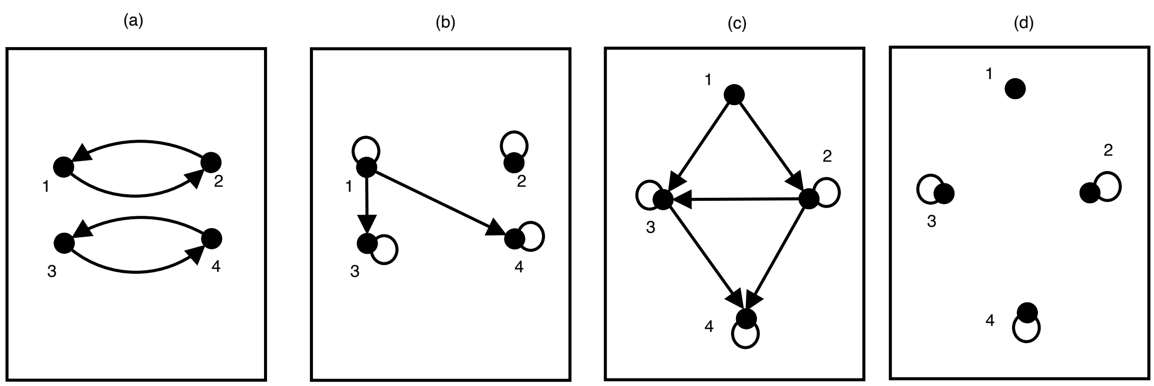
Practice Exam Questions - Tutorial 4

1. (a) Let R be the relation on $\mathbb{Z}$ defined as follows: xRy means "$2x - 2y \leq 3$". Is R anti-symmetric? Prove or disprove that it has that property.
   (b) Let R be the relation on $\mathbb{R}$ defined as follows: xRy means "$x \geq |y|$" where "$|\cdot|$" means "absolute value". Is R anti-symmetric? Prove or disprove that it has that property.
   (c) Let R be the relation on $\mathbb{Z}$ defined as follows: xRy means "$(x - y) + 1$ is natural". Is R anti-symmetric? Prove or disprove that it has that property.

2. Let $A = \{1, 2, 3, 4\}$. For each of the four relations (a)-(d) shown below on A, state whether the relation is anti-symmetric. (No motivation is required).



3. Let $A = \{0, 1\}$. Prove the following statement. For all relations R on A, if R is transitive but not anti-symmetric, then R is reflexive. *Note: the proof here can be very short!*

4. Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be the function defined as

$$f(x) = \begin{cases} 3x & \text{if } 0 \leq x \leq 6 \\ 12 - x & \text{if } x > 6 \end{cases}$$

   What is the range of f? Is f injective? Is f surjective? Is f bijective? Motivate your answer rigorously.

5. Let $A = \{x \in \mathbb{R} : x \neq 1\}$. Consider the function $f : A \to A$ where

$$f(x) = \frac{x+1}{x-1}.$$

   Is this function injective? Surjective? Bijective? For each of these three properties, prove or disprove that it has this property.

6. (a) Suppose $A = \{0, 1, 2, 3\}$. Give a function $f : A \to A$ such that $(f \circ f)(0) = 2$, $(f \circ f)(1) = 3$, $(f \circ f)(2) = 0$ and $(f \circ f)(3) = 1$. Recall that $(f \circ f)(x)$ is alternative notation for $f(f(x))$. *The function f does not have to have a "real-world" or algebraic meaning: just drawing a function diagram is fine, or writing down the values of $f(0)$, $f(1)$, $f(2)$ and $f(3)$!*

(b) Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and let $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the function defined as follows, where c is a non-negative constant.

$$f(x) = \begin{cases} 6x & \text{if } 0 \leq x \leq c \\ (x+1)^2 - 1 & \text{if } x > c \end{cases}$$

There exists only one strictly positive value for the constant c which can make f invertible. What is it? **Hint:** $c \in \{2, 3, 4, 5\}$. Motivate this by proving that the function, for your choice of c, is invertible and give also f^{-1}.

7. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ and $g : \mathbb{N} \rightarrow \mathbb{N}$ be functions defined as follows.

$$f(x) = \begin{cases} x^2 & \text{if } 1 \leq x \leq 10 \\ 2x + 1 & \text{if } x > 10 \end{cases}$$

$$g(x) = \begin{cases} x + 2 & \text{if } 1 \leq x \leq 50 \\ 3x & \text{if } x > 50 \end{cases}$$

Give, in the same style as the definitions for $f(x)$ and $g(x)$, the definition of the function $(g \circ f)(x)$.

8. (a) Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and let $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the function defined as follows, where ${}^+\sqrt{x}$ denotes the positive square root of x:

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 6 \\ {}^+\sqrt{(x+3)} + c & \text{if } x > 6 \end{cases}$$

There exists only one non-negative value for the constant c which can make f invertible. What is it? **Hint:** $c \in \{31, 32, 33\}$! Motivate this by proving that the function, for your choice of c, is invertible and give also f^{-1}.

(b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined as follows: $f(x) = x^2 - 10$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined as $g(x) = 4x + 7$ (if $x < 0$) and $g(x) = 5x^3$ (if $x \geq 0$). Write down the values of $(g \circ f)(3)$ and $(g \circ f)(4)$. Recall that $(g \circ f)(x)$ is alternative notation for $g(f(x))$.

9. (a) Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and let $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the function defined as follows:

$$f(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } 0 \leq x < 6 \\ 2x + c & \text{if } x \geq 6 \end{cases}$$

There exists only one strictly positive value for the constant c which can make f invertible. What is it? (**Hint:** $c \in \{3, 4, 5, 6, 7, 8\}$!) Motivate this by proving that the function, for your choice of c, is invertible and give also f^{-1}.

(b) Suppose $A = \{0, 1, 2, 3\}$ and $B = \{0, 1, 2, 3, 4\}$. Let $g : B \rightarrow B$ be defined as $g(x) = 4 - x$. Write down a function $f : A \rightarrow B$ such that $(g \circ f)(0) = 4$, $(g \circ f)(1) = 2$, $(g \circ f)(2) = 0$ and $(g \circ f)(3) = 3$. Recall that $(g \circ f)(x)$ is alternative notation for $g(f(x))$.