THEOREM 4

The Squeeze Theorem

Suppose that $f(x) \le g(x) \le h(x)$ holds for all x in some open interval containing a, except possibly at $x = a$ itself. Suppose also that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

Then $\lim_{x\to a} g(x) = L$ also. Similar statements hold for left and right limits.

EXAMPLE 10 Given that $3 - x^2 \le u(x) \le 3 + x^2$ for all $x \ne 0$, find $\lim_{x\to 0} u(x)$.

Solution Since $\lim_{x\to 0}(3 - x^2) = 3$ and $\lim_{x\to 0}(3 + x^2) = 3$, the Squeeze Theorem implies that $\lim_{x\to 0} u(x) = 3$.

EXAMPLE 11 Show that if $\lim_{x\to a} |f(x)| = 0$, then $\lim_{x\to a} f(x) = 0$.

Solution Since $-|f(x)| \le f(x) \le |f(x)|$, and $-|f(x)|$ and $|f(x)|$ both have limit 0 as x approaches a, so does $f(x)$ by the Squeeze Theorem.

EXERCISES 1.2

1. Find: (a) $\lim_{x\to -1} f(x)$, (b) $\lim_{x\to 0} f(x)$, and (c) $\lim_{x\to 1} f(x)$, for the function f whose graph is shown in Figure 1.13.

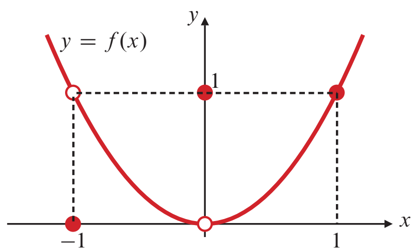

Figure 1.13

2. For the function $y = g(x)$ graphed in Figure 1.14, find each of the following limits or explain why it does not exist.
(a) $\lim_{x\to 1} g(x)$, (b) $\lim_{x\to 2} g(x)$, (c) $\lim_{x\to 3} g(x)$

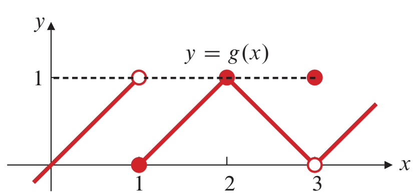

Figure 1.14

In Exercises 3–6, find the indicated one-sided limit of the function g whose graph is given in Figure 1.14.

3. $\lim_{x\to 1-} g(x)$

4. $\lim_{x\to 1+} g(x)$

5. $\lim_{x\to 3+} g(x)$

6. $\lim_{x\to 3-} g(x)$

In Exercises 7–36, evaluate the limit or explain why it does not exist.

7. $\lim_{x\to 4}(x^2 - 4x + 1)$

8. $\lim_{x\to 2} 3(1 - x)(2 - x)$

9. $\lim_{x\to 3} \dfrac{x + 3}{x + 6}$

10. $\lim_{t\to -4} \dfrac{t^2}{4 - t}$

11. $\lim_{x\to 1} \dfrac{x^2 - 1}{x + 1}$

12. $\lim_{x\to -1} \dfrac{x^2 - 1}{x + 1}$

13. $\lim_{x\to 3} \dfrac{x^2 - 6x + 9}{x^2 - 9}$

14. $\lim_{x\to -2} \dfrac{x^2 + 2x}{x^2 - 4}$

15. $\lim_{h\to 2} \dfrac{1}{4 - h^2}$

16. $\lim_{h\to 0} \dfrac{3h + 4h^2}{h^2 - h^3}$

17. $\lim_{x\to 9} \dfrac{\sqrt{x} - 3}{x - 9}$

18. $\lim_{h\to 0} \dfrac{\sqrt{4 + h} - 2}{h}$

19. $\lim_{x\to \pi} \dfrac{(x - \pi)^2}{\pi x}$

20. $\lim_{x\to -2} |x - 2|$

21. $\lim_{x\to 0} \dfrac{|x - 2|}{x - 2}$

22. $\lim_{x\to 2} \dfrac{|x - 2|}{x - 2}$

23. $\lim_{t\to 1} \dfrac{t^2 - 1}{t^2 - 2t + 1}$

24. $\lim_{x\to 2} \dfrac{\sqrt{4 - 4x + x^2}}{x - 2}$

25. $\lim_{t\to 0} \dfrac{t}{\sqrt{4 + t} - \sqrt{4 - t}}$

26. $\lim_{x\to 1} \dfrac{x^2 - 1}{\sqrt{x + 3} - 2}$

27. $\lim_{t\to 0} \dfrac{t^2 + 3t}{(t + 2)^2 - (t - 2)^2}$

28. $\lim_{s\to 0} \dfrac{(s + 1)^2 - (s - 1)^2}{s}$

29. $\lim_{y\to 1} \dfrac{y - 4\sqrt{y} + 3}{y^2 - 1}$

30. $\lim_{x\to -1} \dfrac{x^3 + 1}{x + 1}$

31. $\lim_{x\to 2} \frac{x^4-16}{x^3-8}$

32. $\lim_{x\to 8} \frac{x^{2/3}-4}{x^{1/3}-2}$

33. $\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4}\right)$

34. $\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{1}{x^2-4}\right)$

35. $\lim_{x\to 0} \frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{x^2}$

36. $\lim_{x\to 0} \frac{|3x-1|-|3x+1|}{x}$

The limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ occurs frequently in the study of calculus. (Can you guess why?) Evaluate this limit for the functions f in Exercises 37–42.

37. $f(x) = x^2$

38. $f(x) = x^3$

39. $f(x) = \frac{1}{x}$

40. $f(x) = \frac{1}{x^2}$

41. $f(x) = \sqrt{x}$

42. $f(x) = 1/\sqrt{x}$

Examine the graphs of $\sin x$ and $\cos x$ in Section P.7 to determine the limits in Exercises 43–46.

43. $\lim_{x\to\pi/2} \sin x$

44. $\lim_{x\to\pi/4} \cos x$

45. $\lim_{x\to\pi/3} \cos x$

46. $\lim_{x\to 2\pi/3} \sin x$

47. Make a table of values of $f(x) = (\sin x)/x$ for a sequence of values of x approaching 0, say ± 1.0, ± 0.1, ± 0.01, ± 0.001, ± 0.0001, and ± 0.00001. Make sure your calculator is set in *radian mode* rather than degree mode. Guess the value of $\lim_{x\to 0} f(x)$.

48. Repeat Exercise 47 for $f(x) = \frac{1-\cos x}{x^2}$.

In Exercises 49–60, find the indicated one-sided limit or explain why it does not exist.

49. $\lim_{x\to 2-} \sqrt{2-x}$

50. $\lim_{x\to 2+} \sqrt{2-x}$

51. $\lim_{x\to -2-} \sqrt{2-x}$

52. $\lim_{x\to -2+} \sqrt{2-x}$

53. $\lim_{x\to 0} \sqrt{x^3-x}$

54. $\lim_{x\to 0-} \sqrt{x^3-x}$

55. $\lim_{x\to 0+} \sqrt{x^3-x}$

56. $\lim_{x\to 0+} \sqrt{x^2-x^4}$

57. $\lim_{x\to a-} \frac{|x-a|}{x^2-a^2}$

58. $\lim_{x\to a+} \frac{|x-a|}{x^2-a^2}$

59. $\lim_{x\to 2-} \frac{x^2-4}{|x+2|}$

60. $\lim_{x\to 2+} \frac{x^2-4}{|x+2|}$

Exercises 61–64 refer to the function

$$f(x) = \begin{cases} x-1 & \text{if } x \le -1 \\ x^2+1 & \text{if } -1 < x \le 0 \\ (x+\pi)^2 & \text{if } x > 0. \end{cases}$$

Find the indicated limits.

61. $\lim_{x\to -1-} f(x)$

62. $\lim_{x\to -1+} f(x)$

63. $\lim_{x\to 0+} f(x)$

64. $\lim_{x\to 0-} f(x)$

65. Suppose $\lim_{x\to 4} f(x) = 2$ and $\lim_{x\to 4} g(x) = -3$. Find:

(a) $\lim_{x\to 4} \big(g(x)+3\big)$ (b) $\lim_{x\to 4} x f(x)$

(c) $\lim_{x\to 4} \big(g(x)\big)^2$ (d) $\lim_{x\to 4} \frac{g(x)}{f(x)-1}$.

66. Suppose $\lim_{x\to a} f(x) = 4$ and $\lim_{x\to a} g(x) = -2$. Find:

(a) $\lim_{x\to a} \big(f(x)+g(x)\big)$ (b) $\lim_{x\to a} f(x)\cdot g(x)$

(c) $\lim_{x\to a} 4g(x)$ (d) $\lim_{x\to a} f(x)/g(x)$.

67. If $\lim_{x\to 2} \frac{f(x)-5}{x-2} = 3$, find $\lim_{x\to 2} f(x)$.

68. If $\lim_{x\to 0} \frac{f(x)}{x^2} = -2$, find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} \frac{f(x)}{x}$.

Using Graphing Utilities to Find Limits

Graphing calculators or computer software can be used to evaluate limits at least approximately. Simply "zoom" the plot window to show smaller and smaller parts of the graph near the point where the limit is to be found. Find the following limits by graphical techniques. Where you think it justified, give an exact answer. Otherwise, give the answer correct to 4 decimal places. Remember to ensure that your calculator or software is set for radian mode when using trigonometric functions.

69. $\lim_{x\to 0} \frac{\sin x}{x}$

70. $\lim_{x\to 0} \frac{\sin(2\pi x)}{\sin(3\pi x)}$

71. $\lim_{x\to 1-} \frac{\sin\sqrt{1-x}}{\sqrt{1-x^2}}$

72. $\lim_{x\to 0+} \frac{x-\sqrt{x}}{\sqrt{\sin x}}$

73. On the same graph, plot the three functions $y = x\sin(1/x)$, $y = x$, and $y = -x$ for $-0.2 \le x \le 0.2$, $-0.2 \le y \le 0.2$. Describe the behaviour of $f(x) = x\sin(1/x)$ near $x = 0$. Does $\lim_{x\to 0} f(x)$ exist, and if so, what is its value? Could you have predicted this before drawing the graph? Why?

Using the Squeeze Theorem

74. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.

75. If $2-x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$.

76. (a) Sketch the curves $y = x^2$ and $y = x^4$ on the same graph. Where do they intersect?

(b) The function $f(x)$ satisfies:

$$\begin{cases} x^2 \le f(x) \le x^4 & \text{if } x < -1 \text{ or } x > 1 \\ x^4 \le f(x) \le x^2 & \text{if } -1 \le x \le 1 \end{cases}$$

Find (i) $\lim_{x\to -1} f(x)$, (ii) $\lim_{x\to 0} f(x)$, (iii) $\lim_{x\to 1} f(x)$.

77. On what intervals is $x^{1/3} < x^3$? On what intervals is $x^{1/3} > x^3$? If the graph of $y = h(x)$ always lies between the graphs of $y = x^{1/3}$ and $y = x^3$, for what real numbers a can you determine the value of $\lim_{x\to a} h(x)$? Find the limit for each of these values of a.

78. What is the domain of $x\sin\frac{1}{x}$? Evaluate $\lim_{x\to 0} x\sin\frac{1}{x}$.

79. Suppose $|f(x)| \le g(x)$ for all x. What can you conclude about $\lim_{x\to a} f(x)$ if $\lim_{x\to a} g(x) = 0$? What if $\lim_{x\to a} g(x) = 3$?

(1) (a) $\lim_{x \to -1} f(x) = 1$

(1b) $\lim_{x \to 0} f(x) = 0$

(1c) $\lim_{x \to 1} f(x) = 1$

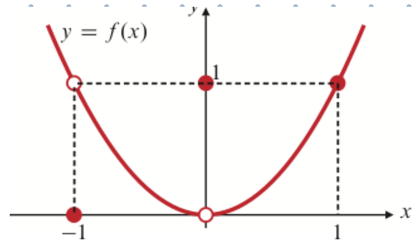

Figure 1.13

(2) (a) $\lim_{x \to 1} g(x)$ doesn't exist

(b) $\lim_{x \to 2} g(x) = 2$

(c) $\lim_{x \to 3} g(x) = 0$

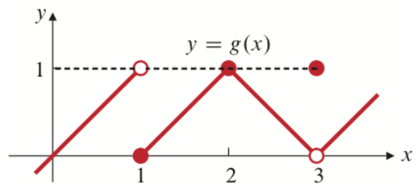

Figure 1.14

(7) $\lim_{x \to 4} (x^2 - 4x + 1) = 1$

$(4)^2 - 4(4) + 1 = 16 - 16 + 1$

(8) $\lim_{x \to 2} 3(1-x)(2-x) = \lim_{x \to 2} 3(2 - x - 2x + x^2) = 0$

$3(2 - 2 - 4 + 4)$

(9) $\lim_{x \to 3} \frac{x+3}{x+6} = \frac{2}{3}$

(10) $\lim_{x \to -4} \frac{x^2}{4-x} = \cancel{\infty}\ 2$ $\quad 16,\ \cancel{0}\ +8$

(11) $\lim_{x \to 1} \frac{x^2-1}{x+1} = 0$

(12) $\lim_{x \to -1} \frac{x^2-1}{x+1} = \lim_{x \to -1} \frac{(x-1)(x+1)}{(x+1)} = \lim_{x \to -1} (x-1) = -2$

(13) $\lim_{x \to 3} \frac{x^2-6x+9}{x^2-9} = \lim_{x \to 3} \frac{(x-3)^2}{(x-3)(x+3)} = \cancel{1} \;//\; \frac{x-3}{x+3} = 0$

(14) $\lim_{x \to -2} \frac{x^2+2x}{x^2-4} = // \frac{x(x+2)}{(x-2)(x+2)} = \lim_{x \to -2} \frac{x}{x-2} = \frac{1}{2}$

(15) $\lim_{x \to 2} \frac{1}{4-x^2} = +\infty \leftarrow \lim^{+} = -\infty, \lim^{-} = +\infty$

lim doesn't exist

(16) $\lim_{x \to 0} \frac{3x+4x^2}{x^2-x^3} = \lim_{x \to 0} \frac{x(3+4x)}{x^2(1-x)}$ $\quad \lim_{x \to 0^+} \frac{3+4x}{x(1-x)} = +\infty$

$= \lim_{x \to 0} \frac{3+4x}{x(1-x)} = \cancel{+\infty}$ $\quad \lim_{x \to 0^-} \frac{3+4x}{x(1-x)} = -\infty$

(17) $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} = // \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} = \lim_{x \to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)} =$

$= \lim_{x \to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{6}$

(18) $\lim\limits_{x\to 0} \dfrac{\sqrt{4+x}-2}{x} = \lim\limits_{x\to 0} \dfrac{(\sqrt{4+x}-2)(\sqrt{4+x}+2)}{x(\sqrt{4+x}+2)}$

$= \lim\limits_{x\to 0} \dfrac{4+x-4}{x(\sqrt{4+x}+2)} =$

$= \lim\limits_{x\to 0} \dfrac{1}{\sqrt{4+x}+2} = \dfrac{1}{4}$

(19) $\lim\limits_{x\to \pi} \dfrac{(x-\pi)^2}{\pi x} = 0$

(20) $\lim\limits_{x\to -2} |x-2| = |-4| = 4$

$\cancel{\lim\limits_{x\to -2^-} -(x-2) = 4}$

$\cancel{\lim\limits_{x\to -2^+} (x-2) = -4}$

(21) $\lim\limits_{x\to 0} \dfrac{|x-2|}{x-2} = -1$

(22) $\lim\limits_{x\to 2} \dfrac{|x-2|}{x-2} \longrightarrow$ limit doesn't exist

$\lim\limits_{x\to 2^+} \dfrac{x-2}{x-2} = 1$ if $x>2$ $\qquad \lim\limits_{x\to 1^+} \dfrac{x+1}{x-1} = +\infty$

$\lim\limits_{x\to 2^-} \dfrac{-(x-2)}{x-2} = -1$ if $x<2$ $\qquad \lim\limits_{x\to 1^-} \dfrac{x+1}{x-1} = -\infty$

(23) $\lim\limits_{x\to 1} \dfrac{x^2-1}{x^2-2x+1} = \lim\limits_{x\to 1} \dfrac{(x+1)(x-1)}{(x-1)^2} = \cancel{\lim\limits_{x\to 1} \dfrac{x+1}{x-1} = +\infty}$

(24) $\lim\limits_{x\to 2} \dfrac{\sqrt{4-4x+x^2}}{x-2} = \lim\limits_{x\to 2} \dfrac{\sqrt{(x-2)^2}}{x-2} =$

$= \lim\limits_{x\to 2} \dfrac{|x-2|}{x-2} = \cancel{\lim\limits_{x\to 2} \dfrac{(x-2)^2}{x-2} = 0}$

$= \lim\limits_{x\to 2^+} \dfrac{x-2}{x-2} = 1$

$\lim\limits_{x\to 2^-} \dfrac{-(x-2)}{x-2} = -1$

the limit doesn't exist

(25) $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{4+x}-\sqrt{4-x}} = \lim\limits_{x\to 0} \dfrac{x(\sqrt{4+x}+\sqrt{4-x})}{(\sqrt{4+x}-\sqrt{4-x})(\sqrt{4+x}+\sqrt{4-x})}$

$= \lim\limits_{x\to 0} \dfrac{x(\sqrt{4+x}+\sqrt{4-x})}{(4+x)-(4-x)} = \lim\limits_{x\to 0} \dfrac{\cancel{x}(\sqrt{4+x}+\sqrt{4-x})}{2\cancel{x}} = \cancel{0}$ 2

(26) $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{\sqrt{x+3} - 2} = \lim\limits_{x \to 1} \dfrac{(x-1)(x+1)(\sqrt{x+3}+2)}{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}$

$= \lim\limits_{x \to 1} \dfrac{\cancel{(x-1)}(x+1)(\sqrt{x+3}+2)}{(x + \cancel{3-4})}$

$= \lim\limits_{x \to 1} (x+1)(\sqrt{x+3}+2) = \cancel{6} \; 8$

$2 \quad 2 \quad 2$

(27) $\lim\limits_{x \to 0} \dfrac{x^2 + 3x}{(x+2)^2 - (x-2)^2} = \dfrac{x(x+3)}{8x}$

$x^2 + 4x + 4 - (x^2 - 4x + 4) = \lim\limits_{x \to 0} \dfrac{x+3}{8} = \dfrac{3}{8}$

$\cancel{x^2} + 4x + \cancel{4} - \cancel{x^2} + 4x - \cancel{4}$

(28) $\lim\limits_{x \to 0} \dfrac{(x+1)^2 - (x-1)^2}{x} = \lim\limits_{x \to 0} \dfrac{4x}{x} = 4$

$\cancel{x^2} + 2x + \cancel{1} - \cancel{x^2} + 2x - \cancel{1}$

(29) $\lim\limits_{x \to 1} \dfrac{x - 4\sqrt{x} + 3}{x^2 - 1} = \lim\limits_{x \to 1} \dfrac{\cancel{(\sqrt{x}-1)}(\sqrt{x}-3)}{(x+1)(\sqrt{x}+1)\cancel{(\sqrt{x}-1)}}$

$(\sqrt{x} - 1)(\sqrt{x} - 3) = \lim\limits_{x \to 1} \dfrac{\sqrt{x} - 3}{(x+1)(\sqrt{x}+1)} \; {-2} \quad 2 = -\dfrac{1}{2}$

(30) $\lim\limits_{x \to -1} \dfrac{x^3 + 1}{x + 1} = \lim\limits_{x \to -1} \dfrac{(x+1)(x^2 - x + 1)}{x+1}$

$(x^3 + 1) = (x+1)(x^2 - x + 1) = \lim\limits_{x \to -1} (x^2 - x + 1) = 3$

(31) $\lim\limits_{x \to 2} \dfrac{x^4 - 16}{x^3 - 8} =\!\!/\!/ \dfrac{(x^2+4)(x^2-4)}{/\!/}$ not a full square

$(x^3 - 2^3) = (x-2)(x^2 + 2x + 4)$

$= \lim\limits_{x \to 2} \dfrac{(x^2+4)(x+2)(x-2)}{(x-2)(x^2+2x+4)}$

$= \lim\limits_{x \to 2} \dfrac{(x^2+4)(x+2)}{x^2 + 2x + 4} = \dfrac{32}{12} = \dfrac{8}{3}$

$8 \quad 4 \quad 32$

(32) $\lim\limits_{x \to 8} \dfrac{\sqrt[3]{x^2} - 4}{\sqrt[3]{x} - 2} = \lim\limits_{x \to 8} \dfrac{(\sqrt[3]{x}+2)\cancel{(\sqrt[3]{x}-2)}}{\cancel{(\sqrt[3]{x}-2)}}$

$= \lim\limits_{x \to 8} (\sqrt[3]{x} + 2) = 4$

(34) $\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{1}{x^2-4}\right)=\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{1}{(x+2)(x-2)}\right)$

$=\lim_{x\to 2}\left(\frac{x+2\quad -1}{(x+2)(x-2)}\right)=\lim_{x\to 2}\frac{x+1}{(x+2)(x-2)}$ $\frac{3}{0}$

$=\cancel{+\infty}$ left limit and right limit are different !!

(35) $\lim_{x\to 0}\frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{x^2}=\lim_{x\to 0}\frac{(\sqrt{2+x^2}-\sqrt{2-x^2})(\sqrt{2+x^2}+\sqrt{2-x^2})}{x^2(\sqrt{2+x^2}+\sqrt{2-x^2})}$

$=\lim_{x\to 0}\frac{(\cancel{2}+x^2-\cancel{2}+x^2)}{\cancel{x^2}(\sqrt{2+x^2}+\sqrt{2-x^2})}=\lim_{x\to 0}\frac{\cancel{x^2}\,2}{\sqrt{2+x^2}+\sqrt{2-x^2}}$

$2\sqrt{2}$

$=\frac{2}{\sqrt{2}}$

✓

(36) $\lim_{x\to 0}\frac{|3x-1|-|3x+1|}{x}=$!

$=\lim_{x\to 0}\frac{(|3x-1|-|3x+1|)(|3x-1|+|3x+1|)}{x(|3x-1|+|3x+1|)}$

$=\lim_{x\to 0}\frac{(3x-1)^2-(3x+1)^2}{x(|3x-1|+|3x+1|)}$

$=\lim_{x\to 0}\frac{\cancel{9x^2}-6x+\cancel{1}-\cancel{9x^2}-6x-\cancel{1}}{//}$

$=\lim_{x\to 0}\frac{-12}{|3x-1|+|3x+1|}=-6$

✓

(33) $\lim_{x\to 0}\frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{x^2}=\lim_{x\to 0}\frac{(\sqrt{2+x^2}-\sqrt{2-x^2})(//+//)}{x^2\quad //}$

$=\lim_{x\to 0}\frac{2+x^2-2+x^2}{x^2\quad //}=\lim_{x\to 0}\frac{2}{\sqrt{2+x^2}+\sqrt{2-x^2}}=\frac{1}{4}$

4

```
> limit((x^2-a^2)/(abs(x-a)),x=a,right);
```

$$2a$$

Finally, we use Maple to confirm the limit discussed in Example 2 in Section 1.2.

```
> limit((1+x^2)^(1/x^2), x=0); evalf(%);
```

$$e$$

$$2.718281828$$

We will learn a great deal about this very important number in Chapter 3.

EXERCISES 1.3

Find the limits in Exercises 1–10.

1. $\lim_{x\to\infty} \dfrac{x}{2x-3}$

2. $\lim_{x\to\infty} \dfrac{x}{x^2-4}$

3. $\lim_{x\to\infty} \dfrac{3x^3-5x^2+7}{8+2x-5x^3}$

4. $\lim_{x\to-\infty} \dfrac{x^2-2}{x-x^2}$

5. $\lim_{x\to-\infty} \dfrac{x^2+3}{x^3+2}$

6. $\lim_{x\to\infty} \dfrac{x^2+\sin x}{x^2+\cos x}$

7. $\lim_{x\to\infty} \dfrac{3x+2\sqrt{x}}{1-x}$

Q **8.** $\lim_{x\to\infty} \dfrac{2x-1}{\sqrt{3x^2+x+1}}$

9. $\lim_{x\to-\infty} \dfrac{2x-1}{\sqrt{3x^2+x+1}}$

10. $\lim_{x\to-\infty} \dfrac{2x-5}{|3x+2|}$

In Exercises 11–32 evaluate the indicated limit. If it does not exist, is the limit ∞, $-\infty$, or neither?

11. $\lim_{x\to3} \dfrac{1}{3-x}$

12. $\lim_{x\to3} \dfrac{1}{(3-x)^2}$

13. $\lim_{x\to3-} \dfrac{1}{3-x}$

14. $\lim_{x\to3+} \dfrac{1}{3-x}$

15. $\lim_{x\to-5/2} \dfrac{2x+5}{5x+2}$

16. $\lim_{x\to-2/5} \dfrac{2x+5}{5x+2}$

17. $\lim_{x\to-(2/5)-} \dfrac{2x+5}{5x+2}$

18. $\lim_{x\to-(2/5)+} \dfrac{2x+5}{5x+2}$

19. $\lim_{x\to2+} \dfrac{x}{(2-x)^3}$

20. $\lim_{x\to1-} \dfrac{x}{\sqrt{1-x^2}}$

21. $\lim_{x\to1+} \dfrac{1}{|x-1|}$

22. $\lim_{x\to1-} \dfrac{1}{|x-1|}$

23. $\lim_{x\to2} \dfrac{x-3}{x^2-4x+4}$

24. $\lim_{x\to1+} \dfrac{\sqrt{x^2-x}}{x-x^2}$

25. $\lim_{x\to\infty} \dfrac{x+x^3+x^5}{1+x^2+x^3}$

26. $\lim_{x\to\infty} \dfrac{x^3+3}{x^2+2}$

! **27.** $\lim_{x\to\infty} \dfrac{x\sqrt{x+1}\,\left(1-\sqrt{2x+3}\right)}{7-6x+4x^2}$

28. $\lim_{x\to\infty} \left(\dfrac{x^2}{x+1}-\dfrac{x^2}{x-1}\right)$

! **29.** $\lim_{x\to-\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$

! **30.** $\lim_{x\to\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$

31. $\lim_{x\to\infty} \dfrac{1}{\sqrt{x^2-2x}-x}$

32. $\lim_{x\to-\infty} \dfrac{1}{\sqrt{x^2+2x}-x}$

33. What are the horizontal asymptotes of $y = \dfrac{1}{\sqrt{x^2-2x}-x}$? What are its vertical asymptotes?

34. What are the horizontal and vertical asymptotes of $y = \dfrac{2x-5}{|3x+2|}$?

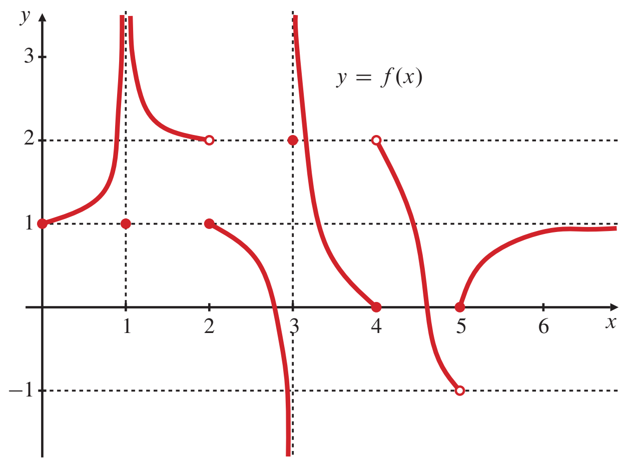

Figure 1.19

The function f whose graph is shown in Figure 1.19 has domain $[0,\infty)$. Find the limits of f indicated in Exercises 35–45.

35. $\lim_{x\to0+} f(x)$

36. $\lim_{x\to1} f(x)$

37. $\lim_{x\to2+} f(x)$

38. $\lim_{x\to2-} f(x)$

39. $\lim_{x\to3-} f(x)$

40. $\lim_{x\to3+} f(x)$

41. $\lim_{x\to4+} f(x)$

42. $\lim_{x\to4-} f(x)$

43. $\lim_{x\to5-} f(x)$

44. $\lim_{x\to5+} f(x)$

45. $\lim_{x\to\infty} f(x)$

46. What asymptotes does the graph in Figure 1.19 have?

Exercises 47–52 refer to the **greatest integer function** $\lfloor x\rfloor$ graphed in Figure 1.20. Find the indicated limit or explain why it does not exist.

(1) $\lim_{x\to\infty} \frac{x}{2x-3} = \lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}(2-\cancel{\frac{3}{x}})} = \frac{1}{2}$

(3) $\lim_{x\to\infty} \frac{3x^3-5x^2+7}{8+2x-5x^3} = \lim_{x\to\infty} \frac{\cancel{x^3}(3-\cancel{\frac{5}{x}}+\cancel{\frac{7}{x^3}})}{\cancel{x^3}(-5+\cancel{\frac{2}{x^2}}+\cancel{\frac{8}{x^3}})} = -\frac{3}{5}$

(2) $\lim_{x\to\infty} \frac{x}{x^2-4} = \lim_{x\to\pm\infty} \frac{x}{x^2} = 0$

(4) $\lim_{x\to-\infty} \frac{x^2-2}{x-x^2} = \lim_{x\to-\infty} \frac{x^2}{-x^2} = -1$

(5) $\lim_{x\to-\infty} \frac{x^2+3}{x^3+2} = \lim_{x\to-\infty} \frac{x^2}{x^3} = 0$

(6) $\lim_{x\to\infty} \frac{x^2+\sin x}{x^2+\cos x}$

$= \lim_{x\to\infty} \frac{\cancel{x^2}(1+\cancel{\frac{\sin x}{x^2}})}{\cancel{x^2}(1+\cancel{\frac{\cos x}{x^2}})}$

$= 1$

- $\lim_{x\to\infty} \frac{\sin x}{x^2} = 0$
- $\lim_{x\to\infty} \frac{\cos x}{x^2} = 0$

(7) $\lim_{x\to\infty} \frac{3x+2\sqrt{x}}{1-x} = \cancel{\lim_{x\to\infty} \frac{(3x+2\sqrt{x})(3x-2\sqrt{x})}{(1-x)(3x-2\sqrt{x})}}$

$= \cancel{\lim_{x\to\infty} \frac{9x^2-4x}{\ }}$

$= \lim_{x\to\infty} \frac{x(3+\frac{2\sqrt{x}}{x})}{x(-1+\frac{1}{x})} = -3$

$\frac{2\sqrt{x}}{x} = \frac{2x}{x\sqrt{x}} = \frac{2}{\sqrt{x}} \to 0$

(8) $\lim_{x\to\infty} \frac{2x-1}{\sqrt{3x^2+x+1}}$

$\sqrt{x^2(3+\frac{1}{x}+\frac{1}{x^2})}$

$= |x|\sqrt{\ }$ //

$\lim_{x\to\infty} \frac{x(2-\frac{1}{x})}{|x|\sqrt{3}}$ //

$\lim_{x\to+\infty} \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}$

Q: does $x\to\infty$ imply $x\to+\infty$

(9) $\lim_{x\to-\infty} \frac{2x-1}{\sqrt{3x^2+x+1}} = -\frac{2}{\sqrt{3}}$

(10) $\lim_{x\to-\infty} \frac{2x-5}{|3x+2|} = \frac{x(2-\frac{5}{x})}{|x(3+\frac{2}{x})|} = -\frac{2}{3}$

(11) $$\lim_{x\to 3} \frac{1}{3-x_0} \quad \text{does not exist}$$

$$\lim_{x\to 3^+} \frac{1}{3-x} = -\infty$$

$$\lim_{x\to 3+} \frac{1}{3-x} = +\infty$$

(12) $$\lim_{x\to 3} \frac{1}{(3-x)^2} = +\infty$$

(13) $$\lim_{x\to 3^-} \frac{1}{3-x} = +\infty$$

$$(3-3^-) \to 0^+$$

$$\frac{-4+25}{5} = \frac{21}{5}$$

(14) $$\lim_{x\to 3^+} \frac{1}{3-x} = -\infty$$

(15) $$\lim_{x\to -\frac{5}{2}} \frac{2x+5}{5x+2} = \frac{-5+5}{-\frac{25}{2}+2} = 0$$

(16) $$\lim_{x\to -\frac{2}{5}} \frac{2x+5}{5x+2} = \frac{-\frac{4}{5}+5}{-2+2} \quad \text{limit does not exist}$$

(17) $$\lim_{x\to -\frac{2}{5}^-} \frac{2x+5}{5x+2} = \frac{\frac{21}{5}}{-2^-+2)0^-} = -\infty$$

$$(1^+ \cdot 1 > 0$$

(18) $$\lim_{x\to -\frac{2}{5}^+} \frac{2x+5}{5x+2} = +\infty$$

(21) $$\lim_{x\to 1^+} \frac{1}{|x-1|} = +\infty$$

(19) $$\lim_{x\to 2^+} \frac{x}{(2-x)^2} = -\infty$$

(22) $$\lim_{x\to 1^-} \frac{1}{|x-1|} = +\infty$$

$$|1^- -1 < 0| > 0$$

$$-1$$

(20) $$\lim_{x\to 1^-} \frac{x}{\sqrt{1-x^2}} = +\infty$$

(23) $$\lim_{x\to 2} \frac{x-3}{x^2-4x+4} = \lim_{x\to 2} \frac{x-3}{(x-2)^2} \; 0^+ = -\infty$$

(24) $$\lim_{x\to 1^+} \frac{\sqrt{x^2-x}}{x-x^2} = \lim_{x\to 1^+} \frac{\sqrt{x(x-1)}}{-x(x-1)} = \lim_{x\to 1^+} \frac{\cancel{x(x-1)}}{-\cancel{x(x-1)}\sqrt{x(x-1)}}$$

$$= \lim_{x\to 1^+} \frac{-1}{\sqrt{x(x-1)}} = -\infty$$

(25) $$\lim_{x\to \infty} \frac{x^5}{x^3} = +\infty$$

(26) $$\lim_{x\to \infty} \frac{x^3+3}{x^2+2} = \lim_{x\to \pm\infty} \frac{x^3}{x^2} = \pm\infty$$?

(28) $\lim_{x \to \infty} \left(\frac{x^2}{x+1} - \frac{x^2}{x-1} \right) =$

$= \lim_{x \to \infty} \frac{x^2(x-1) - x^2(x+1)}{(x+1)(x-1)} =$

$= \lim_{x \to \infty} \frac{x^3 - x^2 - x^3 - x^2}{x^2 - x + x - 1} =$

$= \lim_{x \to \infty} \frac{-2x^2}{x^2 - 1} = \lim_{x \to \infty} -2 = -2$

(33) $f(x) = \frac{1}{\sqrt{x^2 - 2x} - x}$

horizontal asymptotes

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{\sqrt{x^2 - 2x} - x}$

$\sqrt{x^2\left(1 - \frac{2}{x}\right)} - x$

$|x| \sqrt{1 - \frac{2}{x}} - x$

$x > 0: \; x\sqrt{1 - \frac{2}{x}} - x = x\left(\sqrt{1 - \frac{2}{x}} - 1\right)$

$x < 0: \; -x\sqrt{1 - \frac{2}{x}} - x = x\left(-\sqrt{1 - \frac{2}{x}} - 1\right)$

$\left.\begin{array}{l} \lim_{x \to +\infty} \frac{1}{x\left(\sqrt{1 - \cancel{\frac{2}{x}}} - 1\right)} = 0 \\ \lim_{x \to -\infty} \frac{1}{-x\left(\sqrt{1 - \frac{2}{x}} - 1\right)} = 0 \end{array}\right] \lim_{x \to \infty} f(x) = 0$

$f(x)$ has a horizontal asymptote at $y = 0$

(54) $\lim_{x \to 0^+} f(x) = L$

find $\lim_{x \to 0^-} f(x)$

(a) $f(x)$ is even $\to \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

(b) $f(x)$ is odd $\to \lim_{x \to 0^-} = -\lim_{x \to 0^+} f(x)$

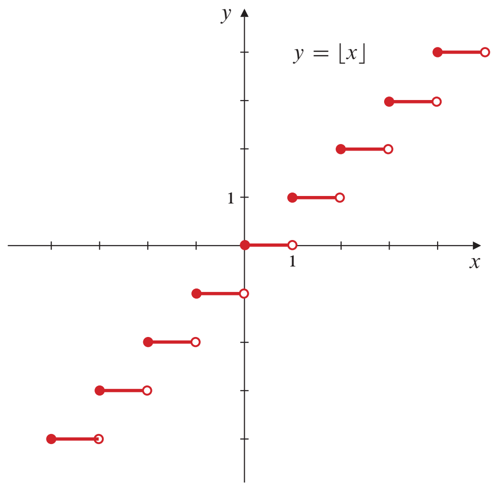

Figure 1.20

47. $\lim_{x\to 3+} \lfloor x \rfloor$

48. $\lim_{x\to 3-} \lfloor x \rfloor$

49. $\lim_{x\to 3} \lfloor x \rfloor$

50. $\lim_{x\to 2.5} \lfloor x \rfloor$

51. $\lim_{x\to 0+} \lfloor 2 - x \rfloor$

52. $\lim_{x\to -3-} \lfloor x \rfloor$

53. Parking in a certain parking lot costs \$1.50 for each hour or part of an hour. Sketch the graph of the function $C(t)$ representing the cost of parking for t hours. At what values of t does $C(t)$ have a limit? Evaluate $\lim_{t\to t_0-} C(t)$ and $\lim_{t\to t_0+} C(t)$ for an arbitrary number $t_0 > 0$.

54. If $\lim_{x\to 0+} f(x) = L$, find $\lim_{x\to 0-} f(x)$ if (a) f is even, (b) f is odd.

55. If $\lim_{x\to 0+} f(x) = A$ and $\lim_{x\to 0-} f(x) = B$, find

(a) $\lim_{x\to 0+} f(x^3 - x)$ (b) $\lim_{x\to 0-} f(x^3 - x)$

(c) $\lim_{x\to 0-} f(x^2 - x^4)$ (d) $\lim_{x\to 0+} f(x^2 - x^4)$.

1.4 Continuity

When a car is driven along a highway, its distance from its starting point depends on time in a *continuous* way, changing by small amounts over short intervals of time. But not all quantities change in this way. When the car is parked in a parking lot where the rate is quoted as "\$2.00 per hour or portion," the parking charges remain at \$2.00 for the first hour and then suddenly jump to \$4.00 as soon as the first hour has passed. The function relating parking charges to parking time will be called *discontinuous* at each hour. In this section we will define continuity and show how to tell whether a function is continuous. We will also examine some important properties possessed by continuous functions.

Continuity at a Point

Most functions that we encounter have domains that are intervals, or unions of separate intervals. A point P in the domain of such a function is called an **interior point** of the domain if it belongs to some *open* interval contained in the domain. If it is not an interior point, then P is called an **endpoint** of the domain. For example, the domain of the function $f(x) = \sqrt{4 - x^2}$ is the closed interval $[-2, 2]$, which consists of interior points in the interval $(-2, 2)$, a left endpoint -2, and a right endpoint 2. The domain of the function $g(x) = 1/x$ is the union of open intervals $(-\infty, 0) \cup (0, \infty)$ and consists entirely of interior points. Note that although 0 is an endpoint of each of those intervals, it does not belong to the domain of g and so is not an endpoint of that domain.

DEFINITION 4

Continuity at an interior point

We say that a function f is **continuous** at an interior point c of its domain if

$$\lim_{x\to c} f(x) = f(c).$$

If either $\lim_{x\to c} f(x)$ fails to exist or it exists but is not equal to $f(c)$, then we will say that f is **discontinuous** at c.

In graphical terms, f is continuous at an interior point c of its domain if its graph has no break in it at the point $(c, f(c))$; in other words, if you can draw the graph through that point without lifting your pen from the paper. Consider Figure 1.21. In (a), f is continuous at c. In (b), f is discontinuous at c because $\lim_{x\to c} f(x) \neq f(c)$. In (c), The only error we have made here is in the assumption (in the first line) that the problem has a solution. It is partly to avoid logical pitfalls like this that mathematicians prove existence theorems.

EXERCISES 1.4

Exercises 1–3 refer to the function g defined on $[-2, 2]$, whose graph is shown in Figure 1.33.

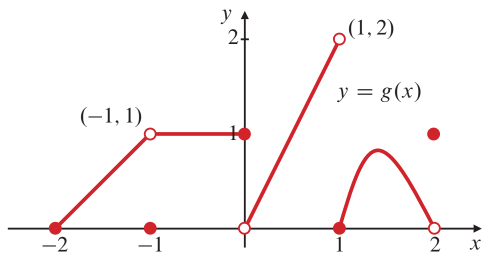

Figure 1.33

1. State whether g is (a) continuous, (b) left continuous, (c) right continuous, and (d) discontinuous at each of the points -2, -1, 0, 1, and 2.

2. At what points in its domain does g have a removable discontinuity, and how should g be redefined at each of those points so as to be continuous there?

3. Does g have an absolute maximum value on $[-2, 2]$? an absolute minimum value?

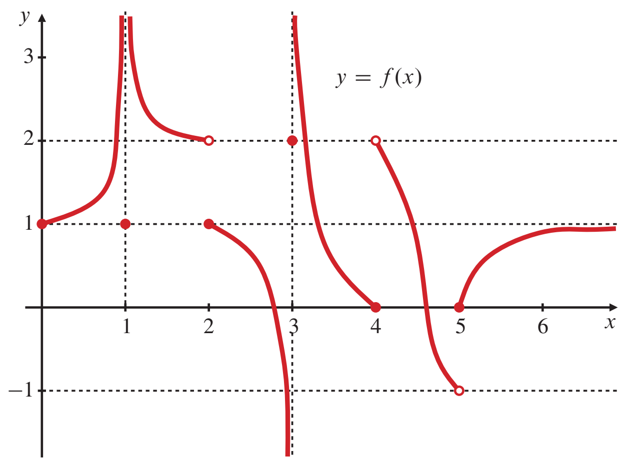

Figure 1.34

4. At what points is the function f, whose graph is shown in Figure 1.34, discontinuous? At which of those points is it left continuous? right continuous?

5. Can the function f graphed in Figure 1.34 be redefined at the single point $x = 1$ so that it becomes continuous there?

6. The function $\operatorname{sgn}(x) = x/|x|$ is neither continuous nor discontinuous at $x = 0$. How is this possible?

In Exercises 7–12, state where in its domain the given function is continuous, where it is left or right continuous, and where it is just discontinuous.

7. $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$

8. $f(x) = \begin{cases} x & \text{if } x < -1 \\ x^2 & \text{if } x \ge -1 \end{cases}$

9. $f(x) = \begin{cases} 1/x^2 & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$

10. $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 0.987 & \text{if } x > 1 \end{cases}$

11. The least integer function $\lceil x \rceil$ of Example 11 in Section P.5.

12. The cost function $C(t)$ of Exercise 53 in Section 1.3.

In Exercises 13–16, how should the given function be defined at the given point to be continuous there? Give a formula for the continuous extension to that point.

13. $\dfrac{x^2 - 4}{x - 2}$ at $x = 2$

14. $\dfrac{1 + t^3}{1 - t^2}$ at $t = -1$

15. $\dfrac{t^2 - 5t + 6}{t^2 - t - 6}$ at 3

16. $\dfrac{x^2 - 2}{x^4 - 4}$ at $\sqrt{2}$

17. Find k so that $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ k - x^2 & \text{if } x > 2 \end{cases}$ is a continuous function.

18. Find m so that $g(x) = \begin{cases} x - m & \text{if } x < 3 \\ 1 - mx & \text{if } x \ge 3 \end{cases}$ is continuous for all x.

19. Does the function x^2 have a maximum value on the open interval $-1 < x < 1$? a minimum value? Explain.

20. The Heaviside function of Example 1 has both absolute maximum and minimum values on the interval $[-1, 1]$, but it is not continuous on that interval. Does this violate the Max-Min Theorem? Why?

Exercises 21–24 ask for maximum and minimum values of functions. They can all be done by the method of Example 9.

21. The sum of two nonnegative numbers is 8. What is the largest possible value of their product?

22. The sum of two nonnegative numbers is 8. What is (a) the smallest and (b) the largest possible value for the sum of their squares?

23. A software company estimates that if it assigns x programmers to work on the project, it can develop a new product in T days, where

$$T = 100 - 30x + 3x^2.$$

How many programmers should the company assign in order to complete the development as quickly as possible?

24. It costs a desk manufacturer $\$(245x - 30x^2 + x^3)$ to send a shipment of x desks to its warehouse. How many desks should it include in each shipment to minimize the average shipping cost per desk?

Find the intervals on which the functions $f(x)$ in Exercises 25–28 are positive and negative.

25. $f(x) = \dfrac{x^2 - 1}{x}$

26. $f(x) = x^2 + 4x + 3$

27. $f(x) = \dfrac{x^2 - 1}{x^2 - 4}$

28. $f(x) = \dfrac{x^2 + x - 2}{x^3}$

29. Show that $f(x) = x^3 + x - 1$ has a zero between $x = 0$ and $x = 1$.

30. Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

31. Show that the function $F(x) = (x-a)^2(x-b)^2 + x$ has the value $(a+b)/2$ at some point x.

32. **(A fixed-point theorem)** Suppose that f is continuous on the closed interval $[0, 1]$ and that $0 \le f(x) \le 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$. (c is called a fixed point of the function f.) *Hint:* If $f(0) = 0$ or $f(1) = 1$, you are done. If not, apply the Intermediate-Value Theorem to $g(x) = f(x) - x$.

33. If an even function f is right continuous at $x = 0$, show that it is continuous at $x = 0$.

34. If an odd function f is right continuous at $x = 0$, show that it is continuous at $x = 0$ and that it satisfies $f(0) = 0$.

Use a graphing utility to find maximum and minimum values of the functions in Exercises 35–38 and the points x where they occur. Obtain 3-decimal-place accuracy for all answers.

35. $f(x) = \dfrac{x^2 - 2x}{x^4 + 1}$ on $[-5, 5]$

36. $f(x) = \dfrac{\sin x}{6 + x}$ on $[-\pi, \pi]$

37. $f(x) = x^2 + \dfrac{4}{x}$ on $[1, 3]$

38. $f(x) = \sin(\pi x) + x(\cos(\pi x) + 1)$ on $[0, 1]$

Use a graphing utility or a programmable calculator and the Bisection Method to solve the equations in Exercises 39–40 to 3 decimal places. As a first step, try to guess a small interval that you can be sure contains a root.

39. $x^3 + x - 1 = 0$

40. $\cos x - x = 0$

Use Maple's `fsolve` routine to solve the equations in Exercises 41–42.

41. $\sin x + 1 - x^2 = 0$ (two roots)

42. $x^4 - x - 1 = 0$ (two roots)

43. Investigate the difference between the Maple routines `fsolve(f,x)`, `solve(f,x)`, and `evalf(solve(f,x))`, where `f := x^3-x-1=0`. Note that no interval is specified for x here.

1.5 The Formal Definition of Limit

The material in this section is optional.

The *informal* definition of limit given in Section 1.2 is not precise enough to enable us to prove results about limits such as those given in Theorems 2–4 of Section 1.2. A more precise *formal* definition is based on the idea of controlling the input x of a function f so that the output $f(x)$ will lie in a specific interval.

EXAMPLE 1 The area of a circular disk of radius r cm is $A = \pi r^2$ cm². A machinist is required to manufacture a circular metal disk having area 400π cm² within an error tolerance of ± 5 cm². How close to 20 cm must the machinist control the radius of the disk to achieve this?

Solution The machinist wants $|\pi r^2 - 400\pi| < 5$, that is,

$$400\pi - 5 < \pi r^2 < 400\pi + 5,$$

or, equivalently,

$$\sqrt{400 - (5/\pi)} < r < \sqrt{400 + (5/\pi)}$$
$$19.96017 < r < 20.03975.$$

Thus, the machinist needs $|r - 20| < 0.03975$; she must ensure that the radius of the disk differs from 20 cm by less than 0.4 mm so that the area of the disk will lie within the required error tolerance.

When we say that $f(x)$ has limit L as x approaches a, we are really saying that we can ensure that the *error* $|f(x) - L|$ will be less than *any* allowed tolerance, no matter how small, by taking x *close enough* to a (but not equal to a). It is traditional to use ϵ, the Greek letter "epsilon," for the size of the allowable *error* and δ, the Greek letter "delta," for the *difference* $x - a$ that measures how close x must be to a to ensure that the error is within that tolerance. These are the letters that Cauchy and Weierstrass used in their pioneering work on limits and continuity in the nineteenth century.

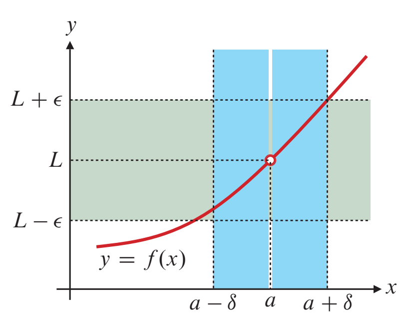

Figure 1.35 If $x \ne a$ and $|x - a| < \delta$, then $|f(x) - L| < \epsilon$

(1)(a) $g(x)$ is not continuous at points $x=-1$, $x=1$, $x=2$, $x=0$

(2) (c) $g(x)$ has removable discontinuities at $x=-1$, $x=2$
they can be removed by redefining $g(x)$
at those points $g(-1)=1$, $g(2)=0$

✓

(17) find k so that $f(x)=\begin{cases} x^2 & x\le 2 \\ k-x^2 & x>2 \end{cases}$ is continuous

dom f : $\mathbb{R}$

Check if function is continuous at $x=2$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$$

$$\lim_{x\to 2^+} (k-x^2) = \lim_{x\to 2^-} x^2$$

$$k-4 = 4 \Rightarrow k=8$$

✓

(18) find m so that $g(x)=\begin{cases} x-m & x<3 \\ 1-mx & x\ge 3 \end{cases}$ is continuous

$$\lim_{x\to 3^+} (1-mx) = \lim_{x\to 3^-} (x-m)$$

$$1-3m = 3-m$$

$$-2m = 2$$

$$m=-1$$

(33) f even and right cont. at $x=0$, show it is continuous at $x=0$

f is even so

- domain is symmetric about y-axis

f is continuous on the right of $x=0$, it must be defined on an interval $[0,h]$, $h>0$.
Being even, f must also be defined for $[-h,h]$

if $x=-y$: $\lim_{x\to 0^-} f(x) = \lim_{y\to 0^+} f(-y) = \lim_{x\to 0^+} f(y) = f(0)$

thus f is continuous on both sides

27. $\lim_{x\to 1+} \dfrac{1}{x-1} = \infty$

28. $\lim_{x\to 1-} \dfrac{1}{x-1} = -\infty$

29. $\lim_{x\to\infty} \dfrac{1}{\sqrt{x^2+1}} = 0$

30. $\lim_{x\to\infty} \sqrt{x} = \infty$

Proving Theorems with the Definition of Limit

31. Prove that limits are unique; that is, if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, prove that $L = M$. *Hint:* Suppose $L \ne M$ and let $\epsilon = |L - M|/3$.

32. If $\lim_{x\to a} g(x) = M$, show that there exists a number $\delta > 0$ such that

$$0 < |x-a| < \delta \quad\Rightarrow\quad |g(x)| < 1 + |M|.$$

(*Hint:* Take $\epsilon = 1$ in the definition of limit.) This says that the values of $g(x)$ are **bounded** near a point where g has a limit.

33. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, prove that $\lim_{x\to a} f(x)g(x) = LM$ (the Product Rule part of Theorem 2). *Hint:* Reread Example 4. Let $\epsilon > 0$ and write

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= |(f(x) - L)g(x) + L(g(x) - M)| \\ &\le |(f(x) - L)g(x)| + |L(g(x) - M)| \\ &= |g(x)||f(x) - L| + |L||g(x) - M| \end{aligned}$$

Now try to make each term in the last line less than $\epsilon/2$ by taking x close enough to a. You will need the result of Exercise 32.

34. If $\lim_{x\to a} g(x) = M$, where $M \ne 0$, show that there exists a number $\delta > 0$ such that

$$0 < |x-a| < \delta \quad\Rightarrow\quad |g(x)| > |M|/2.$$

35. If $\lim_{x\to a} g(x) = M$, where $M \ne 0$, show that

$$\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{M}.$$

Hint: You will need the result of Exercise 34.

36. Use the facts proved in Exercises 33 and 35 to prove the Quotient Rule (part 5 of Theorem 2): if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, where $M \ne 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

37. Use the definition of limit twice to prove Theorem 7 of Section 1.4; that is, if f is continuous at L and if $\lim_{x\to c} g(x) = L$, then

$$\lim_{x\to c} f(g(x)) = f(L) = f\left(\lim_{x\to c} g(x)\right).$$

38. Prove the Squeeze Theorem (Theorem 4 in Section 1.2). *Hint:* If $f(x) \le g(x) \le h(x)$, then

$$\begin{aligned} |g(x) - L| &= |g(x) - f(x) + f(x) - L| \\ &\le |g(x) - f(x)| + |f(x) - L| \\ &\le |h(x) - f(x)| + |f(x) - L| \\ &= |h(x) - L - (f(x) - L)| + |f(x) - L| \\ &\le |h(x) - L| + |f(x) - L| + |f(x) - L| \end{aligned}$$

Now you can make each term in the last expression less than $\epsilon/3$ and so complete the proof.

CHAPTER REVIEW

Key Ideas

- **What do the following statements and phrases mean?**
  - the average rate of change of $f(x)$ on $[a, b]$
  - the instantaneous rate of change of $f(x)$ at $x = a$
  - $\lim_{x\to a} f(x) = L$
  - $\lim_{x\to a+} f(x) = L$, $\lim_{x\to a-} f(x) = L$
  - $\lim_{x\to\infty} f(x) = L$, $\lim_{x\to-\infty} f(x) = L$
  - $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a+} f(x) = -\infty$
  - f is continuous at c.
  - f is left (or right) continuous at c.
  - f has a continuous extension to c.
  - f is a continuous function.
  - f takes on maximum and minimum values on interval I.
  - f is bounded on interval I.
  - f has the intermediate-value property on interval I.
- **State as many "laws of limits" as you can.**
- **What properties must a function have if it is continuous and its domain is a closed, finite interval?**
- **How can you find zeros (roots) of a continuous function?**

Review Exercises

1. Find the average rate of change of x^3 over $[1, 3]$.

2. Find the average rate of change of $1/x$ over $[-2, -1]$.

3. Find the rate of change of x^3 at $x = 2$.

4. Find the rate of change of $1/x$ at $x = -3/2$.

Evaluate the limits in Exercises 5–30 or explain why they do not exist.

5. $\lim_{x\to 1} (x^2 - 4x + 7)$

6. $\lim_{x\to 2} \dfrac{x^2}{1 - x^2}$

7. $\lim_{x\to 1} \dfrac{x^2}{1 - x^2}$

8. $\lim_{x\to 2} \dfrac{x^2 - 4}{x^2 - 5x + 6}$

9. $\lim_{x\to 2} \dfrac{x^2 - 4}{x^2 - 4x + 4}$

10. $\lim_{x\to 2-} \dfrac{x^2 - 4}{x^2 - 4x + 4}$

11. $\lim_{x\to -2+} \dfrac{x^2-4}{x^2+4x+4}$

12. $\lim_{x\to 4} \dfrac{2-\sqrt{x}}{x-4}$

13. $\lim_{x\to 3} \dfrac{x^2-9}{\sqrt{x}-\sqrt{3}}$

14. $\lim_{h\to 0} \dfrac{h}{\sqrt{x+3h}-\sqrt{x}}$

15. $\lim_{x\to 0+} \sqrt{x-x^2}$

16. $\lim_{x\to 0} \sqrt{x-x^2}$

17. $\lim_{x\to 1} \sqrt{x-x^2}$

18. $\lim_{x\to 1-} \sqrt{x-x^2}$

19. $\lim_{x\to\infty} \dfrac{1-x^2}{3x^2-x-1}$

20. $\lim_{x\to -\infty} \dfrac{2x+100}{x^2+3}$

21. $\lim_{x\to -\infty} \dfrac{x^3-1}{x^2+4}$

22. $\lim_{x\to\infty} \dfrac{x^4}{x^2-4}$

23. $\lim_{x\to 0+} \dfrac{1}{\sqrt{x-x^2}}$

24. $\lim_{x\to 1/2} \dfrac{1}{\sqrt{x-x^2}}$

25. $\lim_{x\to\infty} \sin x$

26. $\lim_{x\to\infty} \dfrac{\cos x}{x}$

27. $\lim_{x\to 0} x \sin \dfrac{1}{x}$

28. $\lim_{x\to 0} \sin \dfrac{1}{x^2}$

29. $\lim_{x\to -\infty} [x+\sqrt{x^2-4x+1}]$

30. $\lim_{x\to\infty} [x+\sqrt{x^2-4x+1}]$

At what, if any, points in its domain is the function f in Exercises 31–38 discontinuous? Is f left or right continuous at these points? In Exercises 35 and 36, H refers to the Heaviside function: $H(x) = 1$ if $x \ge 0$ and $H(x) = 0$ if $x < 0$.

31. $f(x) = x^3 - 4x^2 + 1$

32. $f(x) = \dfrac{x}{x+1}$

33. $f(x) = \begin{cases} x^2 & \text{if } x > 2 \\ x & \text{if } x \le 2 \end{cases}$

34. $f(x) = \begin{cases} x^2 & \text{if } x > 1 \\ x & \text{if } x \le 1 \end{cases}$

35. $f(x) = H(x-1)$

36. $f(x) = H(9-x^2)$

37. $f(x) = |x| + |x+1|$

38. $f(x) = \begin{cases} |x|/|x+1| & \text{if } x \ne -1 \\ 1 & \text{if } x = -1 \end{cases}$

Challenging Problems

1. Show that the average rate of change of the function x^3 over the interval $[a, b]$, where $0 < a < b$, is equal to the instantaneous rate of change of x^3 at $x = \sqrt{(a^2+ab+b^2)/3}$. Is this point to the left or to the right of the midpoint $(a+b)/2$ of the interval $[a, b]$?

2. Evaluate $\lim_{x\to 0} \dfrac{x}{|x-1|-|x+1|}$.

3. Evaluate $\lim_{x\to 3} \dfrac{|5-2x|-|x-2|}{|x-5|-|3x-7|}$.

4. Evaluate $\lim_{x\to 64} \dfrac{x^{1/3}-4}{x^{1/2}-8}$.

5. Evaluate $\lim_{x\to 1} \dfrac{\sqrt{3+x}-2}{\sqrt[3]{7+x}-2}$.

6. The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \ne 0$:

$$r_+(a) = \frac{-1+\sqrt{1+a}}{a} \quad \text{and} \quad r_-(a) = \frac{-1-\sqrt{1+a}}{a}.$$

(a) What happens to the root $r_-(a)$ when $a \to 0$?

(b) Investigate numerically what happens to the root $r_+(a)$ when $a \to 0$ by trying the values $a = 1, \pm 0.1, \pm 0.01, \ldots$. For values such as $a = 10^{-8}$, the limited precision of your calculator may produce some interesting results. What happens, and why?

(c) Evaluate $\lim_{a\to 0} r_+(a)$ mathematically by using the identity

$$\sqrt{A}-\sqrt{B} = \frac{A-B}{\sqrt{A}+\sqrt{B}}.$$

7. TRUE or FALSE? If TRUE, give reasons; if FALSE, give a counterexample.

(a) If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} (f(x)+g(x))$ does not exist.

(b) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a} (f(x)+g(x))$ does not exist.

(c) If f is continuous at a, then so is $|f|$.

(d) If $|f|$ is continuous at a, then so is f.

(e) If $f(x) < g(x)$ for all x in an interval around a, and if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$.

8. (a) If f is a continuous function defined on a closed interval $[a, b]$, show that $R(f)$ is a closed interval.

(b) What are the possibilities for $R(f)$ if $D(f)$ is an open interval (a, b)?

9. Consider the function $f(x) = \dfrac{x^2-1}{|x^2-1|}$. Find all points where f is not continuous. Does f have one-sided limits at those points, and if so, what are they?

10. Find the minimum value of $f(x) = 1/(x-x^2)$ on the interval $(0, 1)$. Explain how you know such a minimum value must exist.

11. (a) Suppose f is a continuous function on the interval $[0, 1]$, and $f(0) = f(1)$. Show that $f(a) = f\left(a+\dfrac{1}{2}\right)$ for some $a \in \left[0, \dfrac{1}{2}\right]$.

Hint: Let $g(x) = f\left(x+\dfrac{1}{2}\right) - f(x)$, and use the Intermediate-Value Theorem.

(b) If n is an integer larger than 2, show that $f(a) = f\left(a+\dfrac{1}{n}\right)$ for some $a \in \left[0, 1-\dfrac{1}{n}\right]$.

Chapter review

Challenging problems

(2) $$\lim_{x\to 0} \frac{x}{|x-1| - |x+1|}$$

$$\lim_{x\to 0} \frac{x(|x-1|+|x+1|)}{(|x-1|-|x+1|)(|x-1|+|x+1|)}$$

$$\lim_{x\to 0} \frac{x(|x-1|+|x+1|)}{(x-1)^2-(x+1)^2}$$

$x^2-2x+1-(x^2+2x+1)$

$\cancel{x^2}-2x+\cancel{1}-\cancel{x^2}-2x\cancel{-1}$

$$\lim_{x\to 0} \frac{|x-1|+|x+1|}{-4} = -\frac{1}{2}$$

(3) $$\lim_{x\to 3} \frac{|5-2x|-|x-2|}{|x-5|-|3x-7|} =$$

$\lim_{x\to 3} 5-2x < 0 \Rightarrow |5-2x| = -5+2x$

$\lim_{x\to 3} x-2 > 0 \Rightarrow |x-2| = x-2$

$\lim_{x\to 3} x-5 < 0 \Rightarrow |x-5| = -x+5$

$\lim_{x\to 3} 3x-7 > 0 \Rightarrow |3x-7| = 3x-7$

$$= \lim_{x\to 3} \frac{-5+2x-(x-2)}{-x+5-(3x-7)}$$

$-5+2x-x+2 = x-3$

$-x+5-3x+7 = -4x+12 = -4(x-3)$

$$= \lim_{x\to 3} \frac{x-3}{-4(x-3)} = \lim_{x\to 3} -\frac{1}{4} = -\frac{1}{4}$$

(4) $$\lim_{x\to 64} \frac{\sqrt[3]{x}-4}{\sqrt{x}-8}$$ $\Rightarrow$ Apply de L'Hôpital rule

Thus,

$$\begin{aligned} 2s\frac{ds}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - \sqrt{2}\frac{dx}{dt}y - \sqrt{2}x\frac{dy}{dt} \\ &= 400(2x - \sqrt{2}y) + 100(2y - \sqrt{2}x), \end{aligned}$$

since $dx/dt = 400$ and $dy/dt = 100$. When $t = 1/100$ (i.e., 36 s after $t = 0$), we have $x = 4$ and $y = 1$. Hence,

$$\begin{aligned} s^2 &= 1 + 16 + 1 - 4\sqrt{2} = 18 - 4\sqrt{2} \\ s &\approx 3.5133. \\ \frac{ds}{dt} &= \frac{1}{2s}\left(400(8 - \sqrt{2}) + 100(2 - 4\sqrt{2})\right) \approx 322.86. \end{aligned}$$

The aircraft and the car are separating at a rate of about 323 km/h after 36 s. (Note that it was necessary to convert 36 s to hours in the solution. In general, all measurements should be in compatible units.)

EXERCISES 4.1

1. Find the rate of change of the area of a square whose side is 8 cm long, if the side length is increasing at 2 cm/min.
2. The area of a square is decreasing at 2 ft^2/s. How fast is the side length changing when it is 8 ft?
3. A pebble dropped into a pond causes a circular ripple to expand outward from the point of impact. How fast is the area enclosed by the ripple increasing when the radius is 20 cm and is increasing at a rate of 4 cm/s?
4. The area of a circle is decreasing at a rate of 2 cm^2/min. How fast is the radius of the circle changing when the area is 100 cm^2?
5. The area of a circle is increasing at $1/3$ km^2/h. Express the rate of change of the radius of the circle as a function of (a) the radius r and (b) the area A of the circle.
6. At a certain instant the length of a rectangle is 16 m and the width is 12 m. The width is increasing at 3 m/s. How fast is the length changing if the area of the rectangle is not changing?
7. Air is being pumped into a spherical balloon. The volume of the balloon is increasing at a rate of 20 cm^3/s when the radius is 30 cm. How fast is the radius increasing at that time? (The volume of a ball of radius r units is $V = \frac{4}{3}\pi r^3$ cubic units.)
8. When the diameter of a ball of ice is 6 cm, it is decreasing at a rate of 0.5 cm/h due to melting of the ice. How fast is the volume of the ice ball decreasing at that time?
9. How fast is the surface area of a cube changing when the volume of the cube is 64 cm^3 and is increasing at 2 cm^3/s?
10. The volume of a right circular cylinder is 60 cm^3 and is increasing at 2 cm^3/min at a time when the radius is 5 cm and is increasing at 1 cm/min. How fast is the height of the cylinder changing at that time?
11. How fast is the volume of a rectangular box changing when the length is 6 cm, the width is 5 cm, and the depth is 4 cm, if the length and depth are both increasing at a rate of 1 cm/s and the width is decreasing at a rate of 2 cm/s?
12. The area of a rectangle is increasing at a rate of 5 m^2/s while the length is increasing at a rate of 10 m/s. If the length is 20 m and the width is 16 m, how fast is the width changing?
13. A point moves on the curve $y = x^2$. How fast is y changing when $x = -2$ and x is decreasing at a rate of 3?
14. A point is moving to the right along the first-quadrant portion of the curve $x^2y^3 = 72$. When the point has coordinates $(3, 2)$, its horizontal velocity is 2 units/s. What is its vertical velocity?
15. The point P moves so that at time t it is at the intersection of the curves $xy = t$ and $y = tx^2$. How fast is the distance of P from the origin changing at time $t = 2$?
16. **(Radar guns)** A police officer is standing near a highway using a radar gun to catch speeders. (See Figure 4.6.) He aims the gun at a car that has just passed his position and, when the gun is pointing at an angle of 45° to the direction of the highway, notes that the distance between the car and the gun is increasing at a rate of 100 km/h. How fast is the car travelling?

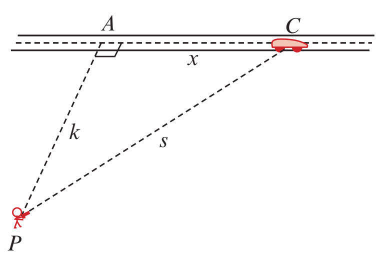

Figure 4.6

17. If the radar gun of Exercise 16 is aimed at a car travelling at 90 km/h along a straight road, what will its reading be when it is aimed making an angle of 30° with the road?

18. The top of a ladder 5 m long rests against a vertical wall. If the base of the ladder is being pulled away from the base of the wall at a rate of 1/3 m/s, how fast is the top of the ladder slipping down the wall when it is 3 m above the base of the wall?

19. A man 2 m tall walks toward a lamppost on level ground at a rate of 0.5 m/s. If the lamp is 5 m high on the post, how fast is the length of the man's shadow decreasing when he is 3 m from the post? How fast is the shadow of his head moving at that time?

20. A woman 6 ft tall is walking at 2 ft/s along a straight path on level ground. There is a lamppost 5 ft to the side of the path. A light 15 ft high on the lamppost casts the woman's shadow on the ground. How fast is the length of her shadow changing when the woman is 12 feet from the point on the path closest to the lamppost?

21. **(Cost of production)** It costs a coal mine owner $\$C$ each day to maintain a production of x tonnes of coal, where $C = 10{,}000 + 3x + x^2/8{,}000$. At what rate is the production increasing when it is 12,000 tonnes and the daily cost is increasing at \$600 per day?

22. **(Distance between ships)** At 1:00 p.m. ship A is 25 km due north of ship B. If ship A is sailing west at a rate of 16 km/h and ship B is sailing south at 20 km/h, at what rate is the distance between the two ships changing at 1:30 p.m?

23. What is the first time after 3:00 p.m. that the hands of a clock are together?

24. **(Tracking a balloon)** A balloon released at point A rises vertically with a constant speed of 5 m/s. Point B is level with and 100 m distant from point A. How fast is the angle of elevation of the balloon at B changing when the balloon is 200 m above A?

25. Sawdust is falling onto a pile at a rate of 1/2 m^3/min. If the pile maintains the shape of a right circular cone with height equal to half the diameter of its base, how fast is the height of the pile increasing when the pile is 3 m high?

26. **(Conical tank)** A water tank is in the shape of an inverted right circular cone with top radius 10 m and depth 8 m. Water is flowing in at a rate of 1/10 m^3/min. How fast is the depth of water in the tank increasing when the water is 4 m deep?

27. **(Leaky tank)** Repeat Exercise 26 with the added assumption that water is leaking out of the bottom of the tank at a rate of $h^3/1{,}000$ m^3/min when the depth of water in the tank is h m. How full can the tank get in this case?

28. **(Another leaky tank)** Water is pouring into a leaky tank at a rate of 10 m^3/h. The tank is a cone with vertex down, 9 m in depth and 6 m in diameter at the top. The surface of water in the tank is rising at a rate of 20 cm/h when the depth is 6 m. How fast is the water leaking out at that time?

29. **(Kite flying)** How fast must you let out line if the kite you are flying is 30 m high, 40 m horizontally away from you, and moving horizontally away from you at a rate of 10 m/min?

30. **(Ferris wheel)** You are on a Ferris wheel of diameter 20 m. It is rotating at 1 revolution per minute. How fast are you rising or falling when you are 6 m horizontally away from the vertical line passing through the centre of the wheel?

31. **(Distance between aircraft)** An aircraft is 144 km east of an airport and is travelling west at 200 km/h. At the same time, a second aircraft at the same altitude is 60 km north of the airport and travelling north at 150 km/h. How fast is the distance between the two aircraft changing?

32. **(Production rate)** If a truck factory employs x workers and has daily operating expenses of $\$y$, it can produce $P = (1/3)x^{0.6}y^{0.4}$ trucks per year. How fast are the daily expenses decreasing when they are \$10,000 and the number of workers is 40, if the number of workers is increasing at 1 per day and production is remaining constant?

33. A lamp is located at point $(3, 0)$ in the xy-plane. An ant is crawling in the first quadrant of the plane and the lamp casts its shadow onto the y-axis. How fast is the ant's shadow moving along the y-axis when the ant is at position $(1, 2)$ and moving so that its x-coordinate is increasing at rate 1/3 units/s and its y-coordinate is decreasing at 1/4 units/s?

34. A straight highway and a straight canal intersect at right angles, the highway crossing over the canal on a bridge 20 m above the water. A boat travelling at 20 km/h passes under the bridge just as a car travelling at 80 km/h passes over it. How fast are the boat and car separating after one minute?

35. **(Filling a trough)** The cross section of a water trough is an equilateral triangle with top edge horizontal. If the trough is 10 m long and 30 cm deep, and if water is flowing in at a rate of 1/4 m^3/min, how fast is the water level rising when the water is 20 cm deep at the deepest?

36. **(Draining a pool)** A rectangular swimming pool is 8 m wide and 20 m long. (See Figure 4.7.) Its bottom is a sloping plane, the depth increasing from 1 m at the shallow end to 3 m at the deep end. Water is draining out of the pool at a rate of 1 m^3/min. How fast is the surface of the water falling when the depth of water at the deep end is (a) 2.5 m? (b) 1 m?

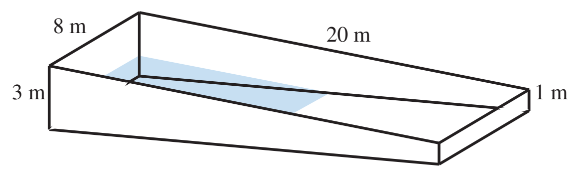

Figure 4.7

! **37.** One end of a 10 m long ladder is on the ground. The ladder is supported partway along its length by resting on top of a 3 m high fence. (See Figure 4.8.) If the bottom of the ladder is 4 m from the base of the fence and is being dragged along the ground away from the fence at a rate of 1/5 m/s, how fast is the free top end of the ladder moving (a) vertically and (b) horizontally?

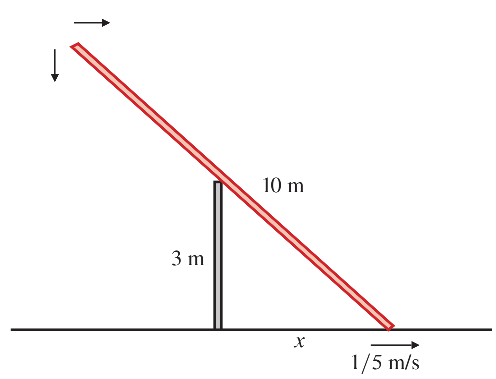

Figure 4.8

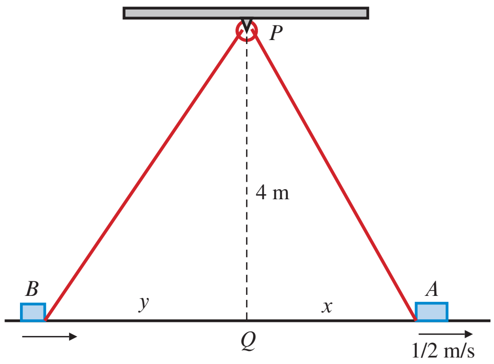

Figure 4.9

38. Two crates, A and B, are on the floor of a warehouse. The crates are joined by a rope 15 m long, each crate being hooked at floor level to an end of the rope. The rope is stretched tight and pulled over a pulley P that is attached to a rafter 4 m above a point Q on the floor directly between the two crates. (See Figure 4.9.) If crate A is 3 m from Q and is being pulled directly away from Q at a rate of 1/2 m/s, how fast is crate B moving toward Q?

39. (Tracking a rocket) Shortly after launch, a rocket is 100 km high and 50 km downrange. If it is travelling at 4 km/s at an angle of 30° above the horizontal, how fast is its angle of elevation, as measured at the launch site, changing?

40. (Shadow of a falling ball) A lamp is 20 m high on a pole. At time $t = 0$ a ball is dropped from a point level with the lamp and 10 m away from it. The ball falls under gravity (its acceleration is 9.8 m/s^2) until it hits the ground. How fast is the shadow of the ball moving along the ground (a) 1 s after the ball is dropped? (b) just as the ball hits the ground?

41. (Tracking a rocket) A rocket blasts off at time $t = 0$ and climbs vertically with acceleration 10 m/s^2. The progress of the rocket is monitored by a tracking station located 2 km horizontally away from the launch pad. How fast is the tracking station antenna rotating upward 10 s after launch?

4.2 Finding Roots of Equations

Finding solutions (roots) of equations is an important mathematical problem to which calculus can make significant contributions. There are only a few general classes of equations of the form $f(x) = 0$ that we can solve exactly. These include **linear equations**:

$$ax + b = 0, \quad (a \neq 0) \qquad \Rightarrow \qquad x = -\frac{b}{a}$$

and **quadratic equations**:

$$ax^2 + bx + c = 0, \quad (a \neq 0) \qquad \Rightarrow \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Cubic and quartic (3rd- and 4th-degree polynomial) equations can also be solved, but the formulas are very complicated. We usually solve these and most other equations approximately by using numerical methods, often with the aid of a calculator or computer.

In Section 1.4 we discussed the Bisection Method for approximating a root of an equation $f(x) = 0$. That method uses the Intermediate-Value Theorem and depends only on the continuity of f and our ability to find an interval $[x_1, x_2]$ that must contain the root because $f(x_1)$ and $f(x_2)$ have opposite signs. The method is rather slow; it requires between three and four iterations to gain one significant figure of precision in the root being approximated.

If we know that f is more than just continuous, we can devise better (i.e., faster) methods for finding roots of $f(x) = 0$. We study two such methods in this section:

(a) **Fixed-Point Iteration**, which looks for solutions of an equation of the form $x = f(x)$. Such solutions are called **fixed points** of the function f.

(b) **Newton's Method**, which looks for solutions of the equation $f(x) = 0$ as fixed points of the function $g(x) = x - \dfrac{f(x)}{f'(x)}$, that is, points x such that $x = g(x)$. This method is usually very efficient, but it requires that f be differentiable.

taking ln of both sides,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(1+\sin\frac{3}{x}\right) \qquad [\infty\cdot 0]$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\sin\dfrac{3}{x}\right)}{\dfrac{1}{x}} \qquad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to\infty} \frac{\dfrac{1}{1+\sin\dfrac{3}{x}}\left(\cos\dfrac{3}{x}\right)\left(-\dfrac{3}{x^2}\right)}{-\dfrac{1}{x^2}} = \lim_{x\to\infty} \frac{3\cos\dfrac{3}{x}}{1+\sin\dfrac{3}{x}} = 3.$$

Hence, $\displaystyle\lim_{x\to\infty}\left(1+\sin\frac{3}{x}\right)^x = e^3$.

EXERCISES 4.3

Evaluate the limits in Exercises 1–32.

1. $\displaystyle\lim_{x\to 0} \frac{3x}{\tan 4x}$

2. $\displaystyle\lim_{x\to 2} \frac{\ln(2x-3)}{x^2-4}$

3. $\displaystyle\lim_{x\to 0} \frac{\sin ax}{\sin bx}$

4. $\displaystyle\lim_{x\to 0} \frac{1-\cos ax}{1-\cos bx}$

5. $\displaystyle\lim_{x\to 0} \frac{\sin^{-1} x}{\tan^{-1} x}$

6. $\displaystyle\lim_{x\to 1} \frac{x^{1/3}-1}{x^{2/3}-1}$

7. $\displaystyle\lim_{x\to 0} x\cot x$

8. $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{\ln(1+x^2)}$

9. $\displaystyle\lim_{t\to \pi} \frac{\sin^2 t}{t-\pi}$

10. $\displaystyle\lim_{x\to 0} \frac{10^x-e^x}{x}$

11. $\displaystyle\lim_{x\to \pi/2} \frac{\cos 3x}{\pi-2x}$

12. $\displaystyle\lim_{x\to 1} \frac{\ln(ex)-1}{\sin \pi x}$

13. $\displaystyle\lim_{x\to \infty} x\sin\frac{1}{x}$

14. $\displaystyle\lim_{x\to 0} \frac{x-\sin x}{x^3}$

15. $\displaystyle\lim_{x\to 0} \frac{x-\sin x}{x-\tan x}$

16. $\displaystyle\lim_{x\to 0} \frac{2-x^2-2\cos x}{x^4}$

17. $\displaystyle\lim_{x\to 0+} \frac{\sin^2 x}{\tan x - x}$

18. $\displaystyle\lim_{r\to \pi/2} \frac{\ln\sin r}{\cos r}$

19. $\displaystyle\lim_{t\to \pi/2} \frac{\sin t}{t}$

20. $\displaystyle\lim_{x\to 1-} \frac{\arccos x}{x-1}$

21. $\displaystyle\lim_{x\to \infty} x(2\tan^{-1} x - \pi)$

22. $\displaystyle\lim_{t\to (\pi/2)-} (\sec t - \tan t)$

23. $\displaystyle\lim_{t\to 0} \left(\frac{1}{t}-\frac{1}{te^{at}}\right)$

24. $\displaystyle\lim_{x\to 0+} x^{\sqrt{x}}$

25. $\displaystyle\lim_{x\to 0+} (\csc x)^{\sin^2 x}$

26. $\displaystyle\lim_{x\to 1+} \left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$

27. $\displaystyle\lim_{t\to 0} \frac{3\sin t - \sin 3t}{3\tan t - \tan 3t}$

28. $\displaystyle\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

29. $\displaystyle\lim_{t\to 0} (\cos 2t)^{1/t^2}$

30. $\displaystyle\lim_{x\to 0+} \frac{\csc x}{\ln x}$

31. $\displaystyle\lim_{x\to 1-} \frac{\ln\sin \pi x}{\csc \pi x}$

32. $\displaystyle\lim_{x\to 0} (1+\tan x)^{1/x}$

33. **(A Newton quotient for the second derivative)** Evaluate $\lim_{h\to 0} \dfrac{f(x+h)-2f(x)+f(x-h)}{h^2}$ if f is a twice differentiable function.

34. If f has a continuous third derivative, evaluate

$$\lim_{h\to 0} \frac{f(x+3h)-3f(x+h)+3f(x-h)-f(x-3h)}{h^3}.$$

35. **(Proof of the second l'Hôpital Rule)** Fill in the details of the following outline of a proof of the second l'Hôpital Rule (Theorem 4) for the case where a and L are both finite. Let $a < x < t < b$ and show that there exists c in (x, t) such that

$$\frac{f(x)-f(t)}{g(x)-g(t)} = \frac{f'(c)}{g'(c)}.$$

Now juggle the above equation algebraically into the form

$$\frac{f(x)}{g(x)} - L = \frac{f'(c)}{g'(c)} - L + \frac{1}{g(x)}\left(f(t) - g(t)\frac{f'(c)}{g'(c)}\right).$$

It follows that

$$\left|\frac{f(x)}{g(x)} - L\right|$$
$$\le \left|\frac{f'(c)}{g'(c)} - L\right| + \frac{1}{|g(x)|}\left(|f(t)| + |g(t)|\left|\frac{f'(c)}{g'(c)}\right|\right).$$

Now show that the right side of the above inequality can be made as small as you wish (say, less than a positive number ϵ) by choosing first t and then x close enough to a. Remember, you are given that $\lim_{c\to a+}\big(f'(c)/g'(c)\big) = L$ and $\lim_{x\to a+} |g(x)| = \infty$.

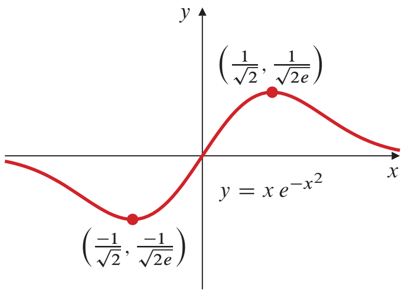

Figure 4.25 The graph for Example 6

Note that $f(0) = 0$ and that f is an odd function ($f(-x) = -f(x)$), so the graph is symmetric about the origin. Also,

$$\lim_{x\to\pm\infty} x e^{-x^2} = \left(\lim_{x\to\pm\infty} \frac{1}{x}\right)\left(\lim_{x\to\pm\infty} \frac{x^2}{e^{x^2}}\right) = 0 \times 0 = 0$$

because $\lim_{x\to\pm\infty} x^2 e^{-x^2} = \lim_{u\to\infty} u e^{-u} = 0$ by Theorem 5 of Section 3.4. Since $f(x)$ is positive at $x = 1/\sqrt{2}$ and is negative at $x = -1/\sqrt{2}$, f must have absolute maximum and minimum values by Theorem 8. These values can only be the values $\pm 1/\sqrt{2e}$ at the two critical points. The graph is shown in Figure 4.25. The x-axis is an asymptote as $x \to \pm\infty$.

EXERCISES 4.4

In Exercises 1–17, determine whether the given function has any local or absolute extreme values, and find those values if possible.

1. $f(x) = x + 2$ on $[-1, 1]$

2. $f(x) = x + 2$ on $(-\infty, 0]$

3. $f(x) = x + 2$ on $[-1, 1)$

4. $f(x) = x^2 - 1$

5. $f(x) = x^2 - 1$ on $[-2, 3]$

6. $f(x) = x^2 - 1$ on $(2, 3)$

7. $f(x) = x^3 + x - 4$ on $[a, b]$

8. $f(x) = x^3 + x - 4$ on (a, b)

9. $f(x) = x^5 + x^3 + 2x$ on $(a, b]$

10. $f(x) = \dfrac{1}{x-1}$

11. $f(x) = \dfrac{1}{x-1}$ on $(0, 1)$

12. $f(x) = \dfrac{1}{x-1}$ on $[2, 3]$

13. $f(x) = |x - 1|$ on $[-2, 2]$

14. $|x^2 - x - 2|$ on $[-3, 3]$

15. $f(x) = \dfrac{1}{x^2+1}$

16. $f(x) = (x+2)^{2/3}$

17. $f(x) = (x-2)^{1/3}$

In Exercises 18–40, locate and classify all local extreme values of the given function. Determine whether any of these extreme values are absolute. Sketch the graph of the function.

18. $f(x) = x^2 + 2x$

19. $f(x) = x^3 - 3x - 2$

20. $f(x) = (x^2 - 4)^2$

21. $f(x) = x^3(x-1)^2$

22. $f(x) = x^2(x-1)^2$

23. $f(x) = x(x^2-1)^2$

24. $f(x) = \dfrac{x}{x^2+1}$

25. $f(x) = \dfrac{x^2}{x^2+1}$

26. $f(x) = \dfrac{x}{\sqrt{x^4+1}}$

27. $f(x) = x\sqrt{2 - x^2}$

28. $f(x) = x + \sin x$

29. $f(x) = x - 2\sin x$

30. $f(x) = x - 2\tan^{-1} x$

31. $f(x) = 2x - \sin^{-1} x$

32. $f(x) = e^{-x^2/2}$

33. $f(x) = x\, 2^{-x}$

34. $f(x) = x^2 e^{-x^2}$

35. $f(x) = \dfrac{\ln x}{x}$

36. $f(x) = |x + 1|$

37. $f(x) = |x^2 - 1|$

38. $f(x) = \sin|x|$

39. $f(x) = |\sin x|$

40. $f(x) = (x-1)^{2/3} - (x+1)^{2/3}$

In Exercises 41–46, determine whether the given function has absolute maximum or absolute minimum values. Justify your answers. Find the extreme values if you can.

41. $\dfrac{x}{\sqrt{x^2+1}}$

42. $\dfrac{x}{\sqrt{x^4+1}}$

43. $x\sqrt{4 - x^2}$

44. $\dfrac{x^2}{\sqrt{4 - x^2}}$

45. $\dfrac{1}{x \sin x}$ on $(0, \pi)$

46. $\dfrac{\sin x}{x}$

47. If a function has an absolute maximum value, must it have any local maximum values? If a function has a local maximum value, must it have an absolute maximum value? Give reasons for your answers.

48. If the function f has an absolute maximum value and $g(x) = |f(x)|$, must g have an absolute maximum value? Justify your answer.

49. **(A function with no max or min at an endpoint)** Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that f is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$ but that it has neither a local maximum nor a local minimum value at the endpoint $x = 0$.

4.5 Concavity and Inflections

Like the first derivative, the second derivative of a function also provides useful information about the behaviour of the function and the shape of its graph: it determines whether the graph is *bending upward* (i.e., has increasing slope) or *bending downward* (i.e., has decreasing slope) as we move along the graph toward the right.

PROOF Suppose that $f'(x_0) = 0$ and $f''(x_0) < 0$. Since

$$\lim_{h \to 0} \frac{f'(x_0 + h)}{h} = \lim_{h \to 0} \frac{f'(x_0 + h) - f'(x_0)}{h} = f''(x_0) < 0,$$

it follows that $f'(x_0 + h) < 0$ for all sufficiently small positive h, and $f'(x_0 + h) > 0$ for all sufficiently small negative h. By the first derivative test (Theorem 7), f must have a local maximum value at x_0. The proof of the local minimum case is similar.

The functions $f(x) = x^4$ (Figure 4.30), $f(x) = -x^4$, and $f(x) = x^3$ (Figure 4.27) all satisfy $f'(0) = 0$ and $f''(0) = 0$. But x^4 has a minimum value at $x = 0$, $-x^4$ has a maximum value at $x = 0$, and x^3 has neither a maximum nor a minimum value at $x = 0$ but has an inflection there. Therefore, we cannot make any conclusion about the nature of a critical point based on knowing that $f''(x) = 0$ there.

EXAMPLE 3 Find and classify the critical points of $f(x) = x^2 e^{-x}$.

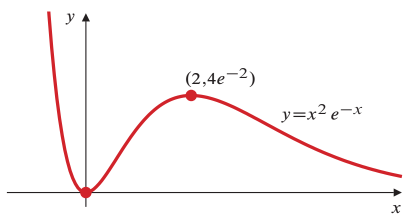

Figure 4.33 The critical points of $f(x) = x^2 e^{-x}$

Solution We begin by calculating the first two derivatives of f:

$$\begin{aligned}
f'(x) &= (2x - x^2)e^{-x} = x(2 - x)e^{-x} = 0 \quad \text{at } x = 0 \text{ and } x = 2,\\
f''(x) &= (2 - 4x + x^2)e^{-x}\\
f''(0) &= 2 > 0, \qquad f''(2) = -2e^{-2} < 0.
\end{aligned}$$

Thus, f has a local minimum value at $x = 0$ and a local maximum value at $x = 2$. See Figure 4.33.

For many functions the second derivative is more complicated to calculate than the first derivative, so the First Derivative Test is likely to be of more use in classifying critical points than is the Second Derivative Test. Also note that the First Derivative Test can classify local extreme values that occur at endpoints and singular points as well as at critical points.

It is possible to generalize the Second Derivative Test to obtain a higher derivative test to deal with some situations where the second derivative is zero at a critical point. (See Exercise 40 at the end of this section.)

EXERCISES 4.5

In Exercises 1–22, determine the intervals of constant concavity of the given function, and locate any inflection points.

1. $f(x) = \sqrt{x}$

2. $f(x) = 2x - x^2$

3. $f(x) = x^2 + 2x + 3$

4. $f(x) = x - x^3$

5. $f(x) = 10x^3 - 3x^5$

6. $f(x) = 10x^3 + 3x^5$

7. $f(x) = (3 - x^2)^2$

8. $f(x) = (2 + 2x - x^2)^2$

9. $f(x) = (x^2 - 4)^3$

10. $f(x) = \dfrac{x}{x^2 + 3}$

11. $f(x) = \sin x$

12. $f(x) = \cos 3x$

13. $f(x) = x + \sin 2x$

14. $f(x) = x - 2\sin x$

15. $f(x) = \tan^{-1} x$

16. $f(x) = x\, e^x$

17. $f(x) = e^{-x^2}$

18. $f(x) = \dfrac{\ln(x^2)}{x}$

19. $f(x) = \ln(1 + x^2)$

20. $f(x) = (\ln x)^2$

21. $f(x) = \dfrac{x^3}{3} - 4x^2 + 12x - \dfrac{25}{3}$

22. $f(x) = (x - 1)^{1/3} + (x + 1)^{1/3}$

23. Discuss the concavity of the linear function $f(x) = ax + b$. Does it have any inflections?

Classify the critical points of the functions in Exercises 24–35 using the Second Derivative Test whenever possible.

24. $f(x) = 3x^3 - 36x - 3$

25. $f(x) = x(x - 2)^2 + 1$

26. $f(x) = x + \dfrac{4}{x}$

27. $f(x) = x^3 + \dfrac{1}{x}$

28. $f(x) = \dfrac{x}{2^x}$

29. $f(x) = \dfrac{x}{1 + x^2}$

30. $f(x) = xe^x$

31. $f(x) = x \ln x$

32. $f(x) = (x^2 - 4)^2$

33. $f(x) = (x^2 - 4)^3$

34. $f(x) = (x^2 - 3)e^x$ **35.** $f(x) = x^2 e^{-2x^2}$

36. Let $f(x) = x^2$ if $x \geq 0$ and $f(x) = -x^2$ if $x < 0$. Is 0 a critical point of f? Does f have an inflection point there? Is $f''(0) = 0$? If a function has a nonvertical tangent line at an inflection point, does the second derivative of the function necessarily vanish at that point?

37. Verify that if f is concave up on an interval, then its graph lies above its tangent lines on that interval. *Hint:* Suppose f is concave up on an open interval containing x_0. Let $h(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$. Show that h has a local minimum value at x_0 and hence that $h(x) \geq 0$ on the interval. Show that $h(x) > 0$ if $x \neq x_0$.

38. Verify that the graph $y = f(x)$ crosses its tangent line at an inflection point. *Hint:* Consider separately the cases where the tangent line is vertical and nonvertical.

39. Let $f_n(x) = x^n$ and $g_n(x) = -x^n$, $(n = 2, 3, 4, \ldots)$. Determine whether each function has a local maximum, a local minimum, or an inflection point at $x = 0$.

40. **(Higher Derivative Test)** Use your conclusions from Exercise 39 to suggest a generalization of the Second Derivative Test that applies when

$$f'(x_0) = f''(x_0) = \ldots = f^{(k-1)}(x_0) = 0, \quad f^{(k)}(x_0) \neq 0,$$

for some $k \geq 2$.

41. This problem shows that no test based solely on the signs of derivatives at x_0 can determine whether every function with a critical point at x_0 has a local maximum or minimum or an inflection point there. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Prove the following:

(a) $\lim_{x\to 0} x^{-n} f(x) = 0$ for $n = 0, 1, 2, 3, \ldots$.

(b) $\lim_{x\to 0} P(1/x) f(x) = 0$ for every polynomial P.

(c) For $x \neq 0$, $f^{(k)}(x) = P_k(1/x) f(x) (k = 1, 2, 3, \ldots)$, where P_k is a polynomial.

(d) $f^{(k)}(0)$ exists and equals 0 for $k = 1, 2, 3, \ldots$.

(e) f has a local minimum at $x = 0$; $-f$ has a local maximum at $x = 0$.

(f) If $g(x) = xf(x)$, then $g^{(k)}(0) = 0$ for every positive integer k and g has an inflection point at $x = 0$.

42. A function may have neither a local maximum nor a local minimum nor an inflection at a critical point. Show this by considering the following function:

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that $f'(0) = f(0) = 0$, so the x-axis is tangent to the graph of f at $x = 0$; but $f'(x)$ is not continuous at $x = 0$, so $f''(0)$ does not exist. Show that the concavity of f is not constant on any interval with endpoint 0.

4.6 Sketching the Graph of a Function

When sketching the graph $y = f(x)$ of a function f, we have three sources of useful information:

(i) **the function f itself**, from which we determine the coordinates of some points on the graph, the symmetry of the graph, and any asymptotes;

(ii) **the first derivative, f'**, from which we determine the intervals of increase and decrease and the location of any local extreme values; and

(iii) **the second derivative, f''**, from which we determine the concavity and inflection points, and sometimes extreme values.

Items (ii) and (iii) were explored in the previous two sections. In this section we consider what we can learn from the function itself about the shape of its graph, and then we illustrate the entire sketching procedure with several examples using all three sources of information.

We could sketch a graph by plotting the coordinates of many points on it and joining them by a suitably smooth curve. This is what computer software and graphics calculators do. When carried out by hand (without a computer or calculator), this simplistic approach is at best tedious and at worst can fail to reveal the most interesting aspects of the graph (singular points, extreme values, and so on). We could also compute the slope at each of the plotted points and, by drawing short line segments through these points with the appropriate slopes, ensure that the sketched graph passes through each plotted point with the correct slope. A more efficient procedure is to obtain the coordinates of only a few points and use qualitative information from the function and its first and second derivatives to determine the *shape* of the graph between these points.

EXERCISES 4.6

1. Figure 4.43 shows the graphs of a function f, its two derivatives f' and f'', and another function g. Which graph corresponds to each function?

2. List, for each function graphed in Figure 4.43, such information that you can determine (approximately) by inspecting the graph (e.g., symmetry, asymptotes, intercepts, intervals of increase and decrease, critical and singular points, local maxima and minima, intervals of constant concavity, inflection points).

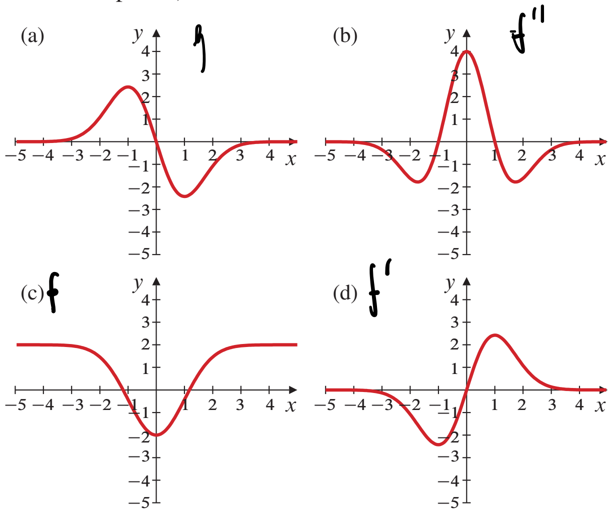

Figure 4.43

3. Figure 4.44 shows the graphs of four functions:

$$f(x) = \frac{x}{1-x^2}, \qquad g(x) = \frac{x^3}{1-x^4},$$

$$h(x) = \frac{x^3 - x}{\sqrt{x^6+1}}, \qquad k(x) = \frac{x^3}{\sqrt{|x^4-1|}}.$$

Which graph corresponds to each function?

4. Repeat Exercise 2 for the graphs in Figure 4.44.

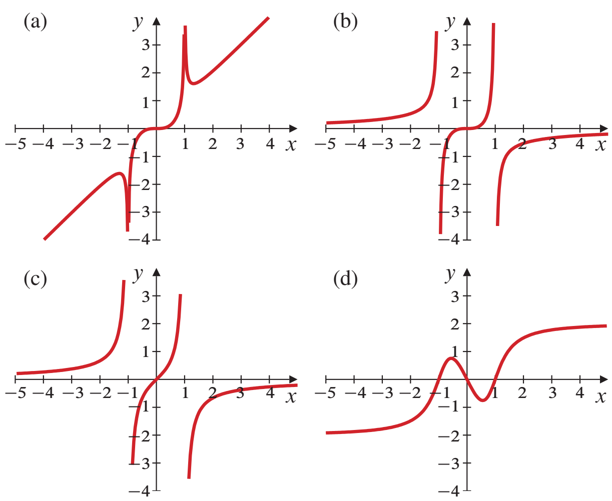

Figure 4.44

In Exercises 5–6, sketch the graph of a function that has the given properties. Identify any critical points, singular points, local maxima and minima, and inflection points. Assume that f is continuous and its derivatives exist everywhere unless the contrary is implied or explicitly stated.

5. $f(0) = 1$, $f(\pm 1) = 0$, $f(2) = 1$, $\lim_{x\to\infty} f(x) = 2$, $\lim_{x\to-\infty} f(x) = -1$, $f'(x) > 0$ on $(-\infty, 0)$ and on $(1, \infty)$, $f'(x) < 0$ on $(0, 1)$, $f''(x) > 0$ on $(-\infty, 0)$ and on $(0, 2)$, and $f''(x) < 0$ on $(2, \infty)$.

6. $f(-1) = 0$, $f(0) = 2$, $f(1) = 1$, $f(2) = 0$, $f(3) = 1$, $\lim_{x\to\pm\infty}(f(x) + 1 - x) = 0$, $f'(x) > 0$ on $(-\infty, -1)$, $(-1, 0)$ and $(2, \infty)$, $f'(x) < 0$ on $(0, 2)$, $\lim_{x\to-1} f'(x) = \infty$, $f''(x) > 0$ on $(-\infty, -1)$ and on $(1, 3)$, and $f''(x) < 0$ on $(-1, 1)$ and on $(3, \infty)$.

In Exercises 7–39, sketch the graphs of the given functions, making use of any suitable information you can obtain from the function and its first and second derivatives.

7. $y = (x^2 - 1)^3$

8. $y = x(x^2 - 1)^2$

9. $y = \dfrac{2-x}{x}$

10. $y = \dfrac{x-1}{x+1}$

11. $y = \dfrac{x^3}{1+x}$

12. $y = \dfrac{1}{4+x^2}$

13. $y = \dfrac{1}{2-x^2}$

14. $y = \dfrac{x}{x^2-1}$

15. $y = \dfrac{x^2}{x^2-1}$

16. $y = \dfrac{x^3}{x^2-1}$

17. $y = \dfrac{x^3}{x^2+1}$

18. $y = \dfrac{x^2}{x^2+1}$

19. $y = \dfrac{x^2-4}{x+1}$

20. $y = \dfrac{x^2-2}{x^2-1}$

21. $y = \dfrac{x^3-4x}{x^2-1}$

22. $y = \dfrac{x^2-1}{x^2}$

23. $y = \dfrac{x^5}{(x^2-1)^2}$

24. $y = \dfrac{(2-x)^2}{x^3}$

25. $y = \dfrac{1}{x^3-4x}$

26. $y = \dfrac{x}{x^2+x-2}$

27. $y = \dfrac{x^3-3x^2+1}{x^3}$

28. $y = x + \sin x$

29. $y = x + 2\sin x$

30. $y = e^{-x^2}$

31. $y = xe^x$

32. $y = e^{-x}\sin x, \ (x \ge 0)$

33. $y = x^2 e^{-x^2}$

34. $y = x^2 e^x$

35. $y = \dfrac{\ln x}{x}, \ (x > 0)$

36. $y = \dfrac{\ln x}{x^2}, \ (x > 0)$

37. $y = \dfrac{1}{\sqrt{4-x^2}}$

38. $y = \dfrac{x}{\sqrt{x^2+1}}$

39. $y = (x^2 - 1)^{1/3}$

40. What is $\lim_{x\to0+} x\ln x$? $\lim_{x\to0} x\ln|x|$? If $f(x) = x\ln|x|$ for $x \ne 0$, is it possible to define $f(0)$ in such a way that f is continuous on the whole real line? Sketch the graph of f.

41. What straight line is an asymptote of the curve $y = \dfrac{\sin x}{1+x^2}$? At what points does the curve cross this asymptote?

Exercises 4,6

4. Exam level questions

Calculate the following limits:

- (1) $\lim_{x\to 0} \frac{|x-1|-|2x+1|}{x}$
- (2) $\lim_{x\to \infty} \sqrt{x^2-x}-x$
- (3) $\lim_{x\to -\infty} \sqrt{x^2-x}-x$
- (4) $\lim_{x\to 3^-} (\frac{1}{x-3}+\frac{6}{9-x^2})$
- (5) $\lim_{x\to 1} \frac{x^2-1}{|x-1|}$

(1) $\lim_{x\to 0} \frac{|x-1|-|2x+1|}{x} = \lim_{x\to 0} \frac{(|x-1|-|2x+1|)(|x-1|+|2x+1|)}{x(|x-1|+|2x+1|)}$

$= \lim_{x\to 0} \frac{(x-1)^2-(2x+1)^2}{x(|x-1|+|2x+1|)}$

x^2-2x+1

$-(4x^2+4x+1)$

$= \lim_{x\to 0} \frac{-3x(x+2)}{x(|x-1|+|2x+1|)}$

$x^2-2x\cancel{+1}-4x^2-4x\cancel{-1}$

$= \lim_{x\to 0} \frac{-3(x+2)}{|x-1|+|2x+1|} = -3$

$-3x^2-6x$

$-3x(x+2)$

(2) $\lim_{x\to \infty} (\sqrt{x^2-x}-x) = \lim_{x\to \infty} (\sqrt{x^2(1-\frac{1}{x})}-x)$

$= \lim_{x\to \infty} (|x|\sqrt{1-\frac{1}{x}}-x)$

$x \to +\infty$: $\lim_{x\to +\infty} x(\sqrt{1-\frac{1}{x}}-1) \longrightarrow \infty \cdot 0$

$\lim_{x\to \infty} \frac{(\sqrt{x^2-x}-x)(\sqrt{x^2-x}+x)}{\sqrt{x^2-x}+x} = \lim_{x\to \infty} \frac{x^2-x-x^2}{\sqrt{x^2-x}+x}$

$= \lim_{x\to +\infty} \frac{-x}{x(\sqrt{1-\frac{1}{x}}+1)} = \lim_{x\to +\infty} \frac{-1}{\sqrt{1-\frac{1}{x}}+1} = -\frac{1}{2}$

$1-1=0$

(3) $\lim\limits_{x\to-\infty}\left(\sqrt{x^2-x}-x\right)$

$$\cancel{=\lim_{x\to-\infty}\frac{-x}{-x\left(\sqrt{1-\frac{1}{x}}+1\right)}=\lim_{x\to-\infty}\frac{+1}{\sqrt{1-\frac{1}{x}}+1}=+\frac{1}{2}}$$

$1^+ - 1 \quad 0^+$

$$\lim_{x\to-\infty}\sqrt{x^2-x}=+\infty$$

$$\lim_{x\to-\infty}\left(\sqrt{x^2-x}-x\right)=+\infty$$

(4) $\lim\limits_{x\to3^-}\left(\frac{1}{x-3}+\frac{6}{9-x^2}\right)=\lim\limits_{x\to3^-}\frac{1}{x-3}-\frac{6}{(x-3)(x+3)}$

$$=\lim_{x\to3^-}\frac{x+3-6}{(x-3)(x+3)}=\lim_{x\to3^-}\frac{1}{x+3}=\frac{1}{6}$$

?

(5) $\lim\limits_{x\to1}\frac{x^2-1}{|x-1|}=\lim\limits_{x\to1}\frac{(x-1)(x+2)}{|x-1|}$

$\lim\limits_{x\to1^+}(x+2)=3$ (should be 2)

$\lim\limits_{x\to1^-}\frac{(x-1)(x+2)}{-(x+1)}=\lim\limits_{x\to1^-}-(x+2)=-3$

(should be -2)

$\Rightarrow \lim\limits_{x\to1} f(x)$ does not exist

Solution Let $f(t) = \cos t$, so that $f'(t) = -\sin t$ and $f''(t) = -\cos t$. The value of a nearest to 36° for which we know $\cos a$ is $a = 30° = \pi/6$, so we use the linearization about that point:

$$L(x) = \cos\frac{\pi}{6} - \sin\frac{\pi}{6}\left(x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right).$$

Since $(\pi/5) - (\pi/6) = \pi/30$, our approximation is

$$\cos 36° = \cos\frac{\pi}{5} \approx L\left(\frac{\pi}{5}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{\pi}{30}\right) \approx 0.813\,67.$$

If $(\pi/6) < t < (\pi/5)$, then $f''(t) < 0$ and $|f''(t)| < \cos(\pi/6) = \sqrt{3}/2$. Therefore, $\cos 36° < 0.813\,67$ and

$$|E(36°)| < \frac{\sqrt{3}}{4}\left(\frac{\pi}{30}\right)^2 < 0.004\,75.$$

Thus, $0.813\,67 - 0.004\,75 < \cos 36° < 0.813\,67$, so $\cos 36°$ lies in the interval $(0.808\,92, 0.813\,67)$.

Remark The error in the linearization of $f(x)$ about $x = a$ can be interpreted in terms of differentials (see Section 2.7 and the beginning of this section) as follows: if $\Delta x = dx = x - a$, then the change in $f(x)$ as we pass from $x = a$ to $x = a + \Delta x$ is $f(a + \Delta x) - f(a) = \Delta y$, and the corresponding change in the linearization $L(x)$ is $f'(a)(x - a) = f'(a)\,dx$, which is just the value at $x = a$ of the differential $dy = f'(x)\,dx$. Thus,

$$E(x) = \Delta y - dy.$$

The error $E(x)$ is small compared with Δx as Δx approaches 0, as seen in Figure 4.64. In fact,

$$\lim_{\Delta x \to 0} \frac{\Delta y - dy}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x} - \frac{dy}{dx}\right) = \frac{dy}{dx} - \frac{dy}{dx} = 0.$$

If $|f''(t)| \le K$ (constant) near $t = a$, a stronger assertion can be made:

$$\left|\frac{\Delta y - dy}{(\Delta x)^2}\right| = \left|\frac{E(x)}{(\Delta x)^2}\right| \le \frac{K}{2}, \qquad \text{so} \qquad |\Delta y - dy| \le \frac{K}{2}(\Delta x)^2.$$

EXERCISES 4.9

In Exercises 1–10, find the linearization of the given function about the given point.

1. x^2 about $x = 3$

2. x^{-3} about $x = 2$

3. $\sqrt{4 - x}$ about $x = 0$

4. $\sqrt{3 + x^2}$ about $x = 1$

5. $1/(1 + x)^2$ about $x = 2$

6. $1/\sqrt{x}$ about $x = 4$

7. $\sin x$ about $x = \pi$

8. $\cos(2x)$ about $x = \pi/3$

9. $\sin^2 x$ about $x = \pi/6$

10. $\tan x$ about $x = \pi/4$

11. By approximately how much does the area of a square increase if its side length increases from 10 cm to 10.4 cm?

12. By about how much must the edge length of a cube decrease from 20 cm to reduce the volume of the cube by 12 cm^3?

13. A spacecraft orbits the earth at a distance of 4,100 miles from the centre of the earth. By about how much will the circumference of its orbit decrease if the radius decreases by 10 miles?

14. **(Acceleration of gravity)** The acceleration a of gravity at an altitude of h miles above the surface of the earth is given by

$$a = g\left(\frac{R}{R + h}\right)^2,$$

where $g \approx 32$ ft/s^2 is the acceleration at the surface of the earth, and $R \approx 3,960$ miles is the radius of the earth. By about what percentage will a decrease if h increases from 0 to 10 miles?

In Exercises 15–22, use a suitable linearization to approximate the indicated value. Determine the sign of the error and estimate its size. Use this information to specify an interval you can be sure contains the value.

15. $\sqrt{50}$

16. $\sqrt{47}$

17. $\sqrt[4]{85}$

18. $\dfrac{1}{2.003}$

19. $\cos 46°$

20. $\sin \dfrac{\pi}{5}$

21. $\sin(3.14)$

22. $\sin 33°$

Use Corollary C of Theorem 11 in the manner suggested in the remark following Example 4 to find better intervals and better approximations to the values in Exercises 23–26.

23. $\sqrt{50}$ as first approximated in Exercise 15.

24. $\sqrt{47}$ as first approximated in Exercise 16.

25. $\cos 36°$ as first approximated in Example 5.

26. $\sin 33°$ as first approximated in Exercise 22.

27. If $f(2) = 4$, $f'(2) = -1$, and $0 \le f''(x) \le 1/x$ for $x > 0$, find the smallest interval you can be sure contains $f(3)$.

28. If $f(2) = 4$, $f'(2) = -1$, and $\dfrac{1}{2x} \le f''(x) \le \dfrac{1}{x}$ for $2 \le x \le 3$, find the best approximation you can for $f(3)$.

29. If $g(2) = 1$, $g'(2) = 2$, and $|g''(x)| < 1 + (x-2)^2$ for all $x > 0$, find the best approximation you can for $g(1.8)$. How large can the error be?

30. Show that the linearization of $\sin\theta$ at $\theta = 0$ is $L(\theta) = \theta$. How large can the percentage error in the approximation $\sin\theta \approx \theta$ be if $|\theta|$ is less than $17°$?

31. A spherical balloon is inflated so that its radius increases from 20.00 cm to 20.20 cm in 1 min. By approximately how much has its volume increased in that minute?

4.10 Taylor Polynomials

The linearization of a function $f(x)$ about $x = a$, namely, the linear function

$$P_1(x) = L(x) = f(a) + f'(a)(x-a),$$

describes the behaviour of f near a better than any other polynomial of degree 1 because both P_1 and f have the same value and the same derivative at a:

$$P_1(a) = f(a) \qquad \text{and} \qquad P_1'(a) = f'(a).$$

(We are now using the symbol P_1 instead of L to stress the fact that the linearization is a polynomial of degree at most 1.)

We can obtain even better approximations to $f(x)$ by using quadratic or higher-degree polynomials and matching more derivatives at $x = a$. For example, if f is twice differentiable near a, then the polynomial

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

satisfies $P_2(a) = f(a)$, $P_2'(a) = f'(a)$, and $P_2''(a) = f''(a)$ and describes the behaviour of f near a better than any other polynomial of degree at most 2.

In general, if $f^{(n)}(x)$ exists in an open interval containing $x = a$, then the polynomial

$$\begin{aligned} P_n(x) = f(a) &+ \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &+ \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

matches f and its first n derivatives at $x = a$,

$$P_n(a) = f(a), \quad P_n'(a) = f'(a), \quad \ldots, \quad P_n^{(n)}(a) = f^{(n)}(a),$$

EXAMPLE 10 Evaluate $\lim_{x\to 1} \frac{\ln x}{x^2 - 1}$.

Solution This is also of type [0/0]. We begin by substituting $x = 1 + t$. Note that $x \to 1$ corresponds to $t \to 0$. We can use a known Maclaurin polynomial for $\ln(1+t)$. For this limit even the degree 1 polynomial $P_1(t) = t$ with error $O(t^2)$ will do.

$$\begin{aligned}\lim_{x\to 1} \frac{\ln x}{x^2 - 1} &= \lim_{t\to 0} \frac{\ln(1+t)}{(1+t)^2 - 1} = \lim_{t\to 0} \frac{\ln(1+t)}{2t + t^2} \\ &= \lim_{t\to 0} \frac{t + O(t^2)}{2t + t^2} = \lim_{t\to 0} \frac{1 + O(t)}{2 + t} = \frac{1}{2}.\end{aligned}$$

EXERCISES 4.10

Find the indicated Taylor polynomials for the functions in Exercises 1–8 by using the definition of Taylor polynomial.

1. for e^{-x} about $x = 0$, order 4.

2. for $\cos x$ about $x = \pi/4$, order 3.

3. for $\ln x$ about $x = 2$, order 4.

4. for $\sec x$ about $x = 0$, order 3.

5. for $\sqrt{x}$ about $x = 4$, order 3.

6. for $1/(1-x)$ about $x = 0$, order n.

7. for $1/(2+x)$ about $x = 1$, order n.

8. for $\sin(2x)$ about $x = \pi/2$, order $2n - 1$.

In Exercises 9–14, use second order Taylor polynomials $P_2(x)$ for the given function about the point specified to approximate the indicated value. Estimate the error, and write the smallest interval you can be sure contains the value.

9. $f(x) = x^{1/3}$ about 8; approximate $9^{1/3}$.

10. $f(x) = \sqrt{x}$ about 64; approximate $\sqrt{61}$.

11. $f(x) = \frac{1}{x}$ about 1; approximate $\frac{1}{1.02}$.

12. $f(x) = \tan^{-1} x$ about 1; approximate $\tan^{-1}(0.97)$.

13. $f(x) = e^x$ about 0; approximate $e^{-0.5}$.

14. $f(x) = \sin x$ about $\pi/4$; approximate $\sin(47°)$.

In Exercises 15–20, write the indicated case of Taylor's formula for the given function. What is the Lagrange remainder in each case?

15. $f(x) = \sin x,\ a = 0,\ n = 7$

16. $f(x) = \cos x,\ a = 0,\ n = 6$

17. $f(x) = \sin x,\ a = \pi/4,\ n = 4$

18. $f(x) = \frac{1}{1-x},\ a = 0,\ n = 6$

19. $f(x) = \ln x,\ a = 1, n = 6$

20. $f(x) = \tan x,\ a = 0,\ n = 3$

Find the requested Taylor polynomials in Exercises 21–26 by using known Taylor or Maclaurin polynomials and changing variables as in Examples 6–8.

21. $P_3(x)$ for e^{3x} about $x = -1$.

22. $P_8(x)$ for e^{-x^2} about $x = 0$.

23. $P_4(x)$ for $\sin^2 x$ about $x = 0$. *Hint:* $\sin^2 x = \frac{1 - \cos(2x)}{2}$.

24. $P_5(x)$ for $\sin x$ about $x = \pi$.

25. $P_6(x)$ for $1/(1 + 2x^2)$ about $x = 0$

26. $P_8(x)$ for $\cos(3x - \pi)$ about $x = 0$.

27. Find all Maclaurin polynomials $P_n(x)$ for $f(x) = x^3$.

28. Find all Taylor polynomials $P_n(x)$ for $f(x) = x^3$ at $x = 1$.

29. Find the Maclaurin polynomial $P_{2n+1}(x)$ for $\sinh x$ by suitably combining polynomials for e^x and e^{-x}.

30. By suitably combining Maclaurin polynomials for $\ln(1 + x)$ and $\ln(1 - x)$, find the Maclaurin polynomial of order $2n + 1$ for $\tanh^{-1}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$.

31. Write Taylor's formula for $f(x) = e^{-x}$ with $a = 0$, and use it to calculate $1/e$ to 5 decimal places. (You may use a calculator but not the e^x function on it.)

! **32.** Write the general form of Taylor's formula for $f(x) = \sin x$ at $x = 0$ with Lagrange remainder. How large need n be taken to ensure that the corresponding Taylor polynomial approximation will give the sine of 1 radian correct to 5 decimal places?

33. What is the best order 2 approximation to the function $f(x) = (x - 1)^2$ at $x = 0$? What is the error in this approximation? Now answer the same questions for $g(x) = x^3 + 2x^2 + 3x + 4$. Can the constant $1/6 = 1/3!$, in the error formula for the degree 2 approximation, be improved (i.e., made smaller)?

34. By factoring $1 - x^{n+1}$ (or by long division), show that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \frac{x^{n+1}}{1-x}. \quad (*)$$

Next, show that if $|x| \le K < 1$, then

$$\left|\frac{x^{n+1}}{1-x}\right| \le \frac{1}{1-K}|x^{n+1}|.$$

This implies that $x^{n+1}/(1-x) = O(x^{n+1})$ as $x \to 0$ and confirms formula (d) of Table 5. What does Theorem 13 then say about the nth-order Maclaurin polynomial for $1/(1-x)$?

first place. But this also means that careful calculations on computers constitute a full field of modern research, requiring considerable mathematical knowledge.

EXERCISES 4.11

1. Use Maple to repeat the plots of Figure 4.68, except using the mathematically equivalent function $(x-1)^2-(x^2-2x+1)$. Does the result look the same? Is the result surprising?

2. Use Maple to graph $f-P_4(x)$ where $f(x)=\cos x$ and $P_4(x)$ is the 4th degree Taylor polynomial of f about $x=0$. Use the interval $[-10^{-3}/2, 10^{-3}/2]$ for the plot and plot 1000 points. On the same plot, graph $\pm\epsilon f/2$ and $\pm\epsilon f/4$, where ϵ is machine epsilon. How does the result differ from what is expected mathematically?

! 3. If a real number x is represented on a computer, it is replaced by a floating-point number $F(x)$; x is said to be "floated" by the function F. Show that the relative error in floating for a base-two machine satisfies

$$|\text{error}| = |x - F(x)| \le \epsilon|x|,$$

where $\epsilon = 2^{-t}$ and t is the number of base-two digits (bits) in the floating-point number.

! 4. Consider two different but mathematically equivalent expressions, having the value C after evaluation. On a computer, with each step in the evaluation of each of the expressions, roundoff error is introduced as digits are discarded and rounded according to various rules. In subsequent steps, resulting error is added or subtracted according to the details of the expression producing a final error that depends in detail on the expression, the particular software package, the operating system, and the machine hardware. Computer errors are not equivalent for the two expressions, even when the expressions are mathematically equivalent.

   (a) If we suppose that the computer satisfactorily evaluates the expressions for many input values within an interval, all to within machine precision, why might we expect the difference of these expressions on a computer to have an error contained within an interval $[-\epsilon C, \epsilon C]$?

   (b) Is it possible for exceptional values of the error to lie outside that interval in some cases? Why?

   (c) Is it possible for the error to be much smaller than the interval indicates? Why?

CHAPTER REVIEW

Key Ideas

- **What do the following words, phrases, and statements mean?**
  - ◇ critical point of f
  - ◇ singular point of f
  - ◇ inflection point of f
  - ◇ f has absolute maximum value M
  - ◇ f has a local minimum value at $x=c$
  - ◇ vertical asymptote
  - ◇ horizontal asymptote
  - ◇ oblique asymptote
  - ◇ machine epsilon
  - ◇ the linearization of $f(x)$ about $x=a$
  - ◇ the Taylor polynomial of degree n of $f(x)$ about $x=a$
  - ◇ Taylor's formula with Lagrange remainder
  - ◇ $f(x)=O\big((x-a)^n\big)$ as $x\to a$
  - ◇ a root of $f(x)=0$
  - ◇ a fixed point of $f(x)$
  - ◇ an indeterminate form
  - ◇ l'Hôpital's Rules
- **Describe how to estimate the error in a linear (tangent line) approximation to the value of a function.**
- **Describe how to find a root of an equation $f(x)=0$ by using Newton's Method. When will this method work well?**

Review Exercises

1. If the radius r of a ball is increasing at a rate of 2 percent per minute, how fast is the volume V of the ball increasing?

2. **(Gravitational attraction)** The gravitational attraction of the earth on a mass m at distance r from the centre of the earth is a continuous function of r for $r\ge 0$, given by

$$F = \begin{cases} \dfrac{mgR^2}{r^2} & \text{if } r \ge R \\ mkr & \text{if } 0 \le r < R, \end{cases}$$

where R is the radius of the earth, and g is the acceleration due to gravity at the surface of the earth.

   (a) Find the constant k in terms of g and R.

   (b) F decreases as m moves away from the surface of the earth, either upward or downward. Show that F decreases as r increases from R at twice the rate at which F decreases as r decreases from R.

3. **(Resistors in parallel)** Two variable resistors R_1 and R_2 are connected in parallel so that their combined resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

At an instant when $R_1 = 250$ ohms and $R_2 = 1{,}000$ ohms, R_1 is increasing at a rate of 100 ohms/min. How fast must R_2 be changing at that moment (a) to keep R constant? and (b) to enable R to increase at a rate of 10 ohms/min?

4. **(Gas law)** The volume V (in m^3), pressure P (in kilopascals, kPa), and temperature T (in kelvin, K) for a sample of a certain gas satisfy the equation $pV = 5.0T$.

   (a) How rapidly does the pressure increase if the temperature is 400 K and increasing at 4 K/min while the gas is kept confined in a volume of 2.0 m^3?

   (b) How rapidly does the pressure decrease if the volume is 2 m^3 and increases at 0.05 m^3/min while the temperature is kept constant at 400 K?

5. **(The size of a print run)** It costs a publisher \$10,000 to set up the presses for a print run of a book and \$8 to cover the material costs for each book printed. In addition, machinery servicing, labour, and warehousing add another $\$6.25\times10^{-7}x^2$ to the cost of each book if x copies are manufactured during the printing. How many copies should the publisher print in order to minimize the average cost per book?

6. **(Maximizing profit)** A bicycle wholesaler must pay the manufacturer \$75 for each bicycle. Market research tells the wholesaler that if she charges her customers $\$x$ per bicycle, she can expect to sell $N(x) = 4.5 \times 10^6/x^2$ of them. What price should she charge to maximize her profit, and how many bicycles should she order from the manufacturer?

7. Find the largest possible volume of a right-circular cone that can be inscribed in a sphere of radius R.

8. **(Minimizing production costs)** The cost $\$C(x)$ of production in a factory varies with the amount x of product manufactured. The cost may rise sharply with x when x is small, and more slowly for larger values of x because of economies of scale. However, if x becomes too large, the resources of the factory can be overtaxed, and the cost can begin to rise quickly again. Figure 4.70 shows the graph of a typical such cost function $C(x)$.

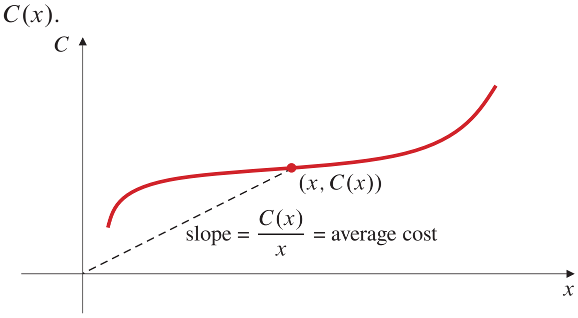

Figure 4.70

If x units are manufactured, the average cost per unit is $\$C(x)/x$, which is the slope of the line from the origin to the point $(x, C(x))$ on the graph.

   (a) If it is desired to choose x to minimize this average cost per unit (as would be the case if all units produced could be sold for the same price), show that x should be chosen to make the average cost equal to the marginal cost:

$$\frac{C(x)}{x} = C'(x).$$

   (b) Interpret the conclusion of (a) geometrically in the figure.

   (c) If the average cost equals the marginal cost for some x, does x necessarily minimize the average cost?

9. **(Box design)** Four squares are cut out of a rectangle of cardboard 50 cm by 80 cm, as shown in Figure 4.71, and the remaining piece is folded into a closed, rectangular box, with two extra flaps tucked in. What is the largest possible volume for such a box?

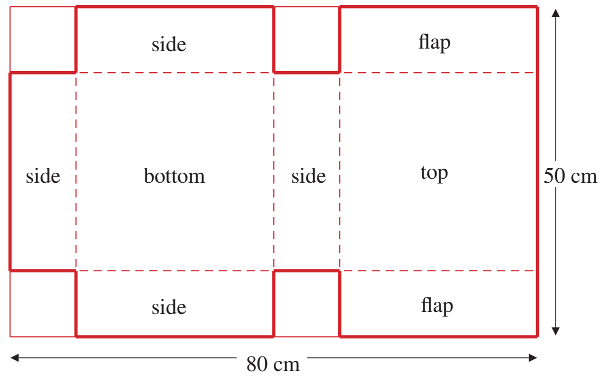

Figure 4.71

10. **(Yield from an orchard)** A certain orchard has 60 trees and produces an average of 800 apples per tree per year. If the density of trees is increased, the yield per tree drops; for each additional tree planted, the average yield per tree is reduced by 10 apples per year. How many more trees should be planted to maximize the total annual yield of apples from the orchard?

11. **(Rotation of a tracking antenna)** What is the maximum rate at which the antenna in Exercise 41 of Section 4.1 must be able to turn in order to track the rocket during its entire vertical ascent?

12. An oval table has its outer edge in the shape of the curve $x^2 + y^4 = 1/8$, where x and y are measured in metres. What is the width of the narrowest hallway in which the table can be turned horizontally through 180°?

13. A hollow iron ball whose shell is 2 cm thick weighs half as much as it would if it were solid iron throughout. What is the radius of the ball?

14. **(Range of a cannon fired from a hill)** A cannon ball is fired with a speed of 200 ft/s at an angle of 45° above the horizontal from the top of a hill whose height at a horizontal distance x ft from the top is $y = 1{,}000/(1 + (x/500)^2)$ ft above sea level. How far does the cannon ball travel horizontally before striking the ground?

15. **(Linear approximation for a pendulum)** Because $\sin\theta \approx \theta$ for small values of $|\theta|$, the nonlinear equation of motion of a simple pendulum

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta,$$

which determines the displacement angle $\theta(t)$ away from the vertical at time t for a simple pendulum, is frequently approximated by the simpler linear equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

when the maximum displacement of the pendulum is not large. What is the percentage error in the right side of the equation if $|\theta|$ does not exceed 20°?

16. Find the Taylor polynomial of degree 6 for $\sin^2 x$ about $x = 0$ and use it to help you evaluate

$$\lim_{x\to 0}\frac{3\sin^2 x - 3x^2 + x^4}{x^6}.$$

17. Use a second-order Taylor polynomial for $\tan^{-1} x$ about $x = 1$ to find an approximate value for $\tan^{-1}(1.1)$. Estimate the size of the error by using Taylor's formula.

18. The line $2y = 10x - 19$ is tangent to $y = f(x)$ at $x = 2$. If an initial approximation $x_0 = 2$ is made for a root of $f(x) = 0$ and Newton's Method is applied once, what will be the new approximation that results?

19. Find all solutions of the equation $\cos x = (x-1)^2$ to 10 decimal places.

20. Find the shortest distance from the point $(2, 0)$ to the curve $y = \ln x$.

21. A car is travelling at night along a level, curved road whose equation is $y = e^x$. At a certain instant its headlights illuminate a signpost located at the point $(1, 1)$. Where is the car at that instant?

Challenging Problems

1. (Growth of a crystal) A single cubical salt crystal is growing in a beaker of salt solution. The crystal's volume V increases at a rate proportional to its surface area and to the amount by which its volume is less than a limiting volume V_0:

$$\frac{dV}{dt} = kx^2(V_0 - V),$$

where x is the edge length of the crystal at time t.

(a) Using $V = x^3$, transform the equation above to one that gives the rate of change dx/dt of the edge length x in terms of x.

(b) Show that the growth rate of the edge of the crystal decreases with time but remains positive as long as $x < x_0 = V_0^{1/3}$.

(c) Find the volume of the crystal when its edge length is growing at half the rate it was initially.

2. (A review of calculus!) You are in a tank (the military variety) moving down the y-axis toward the origin. At time $t = 0$ you are 4 km from the origin, and 10 min later you are 2 km from the origin. Your speed is decreasing; it is proportional to your distance from the origin. You know that an enemy tank is waiting somewhere on the positive x-axis, but there is a high wall along the curve $xy = 1$ (all distances in kilometres) preventing you from seeing just where it is. How fast must your gun turret be capable of turning to maximize your chances of surviving the encounter?

3. (The economics of blood testing) Suppose that it is necessary to perform a blood test on a large number N of individuals to detect the presence of a virus. If each test costs \$$C$, then the total cost of the testing program is \$$NC$. If the proportion of people in the population who have the virus is not large, this cost can be greatly reduced by adopting the following strategy. Divide the N samples of blood into N/x groups of x samples each. Pool the blood in each group to make a single sample for that group and test it. If it tests negative, no further testing is necessary for individuals in that group. If the group sample tests positive, test all the individuals in that group.

Suppose that the fraction of individuals in the population infected with the virus is p, so the fraction uninfected is $q = 1 - p$. The probability that a given individual is unaffected is q, so the probability that all x individuals in a group are unaffected is q^x. Therefore, the probability that a pooled sample is infected is $1 - q^x$. Each group requires one test, and the infected groups require an extra x tests. Therefore, the expected total number of tests to be performed is

$$T = \frac{N}{x} + \frac{N}{x}(1 - q^x)x = N\left(\frac{1}{x} + 1 - q^x\right).$$

For example, if $p = 0.01$, so that $q = 0.99$ and $x = 20$, then the expected number of tests required is $T = 0.23N$, a reduction of over 75%. But maybe we can do better by making a different choice for x.

(a) For $q = 0.99$, find the number x of samples in a group that minimizes T (i.e., solve $dT/dx = 0$). Show that the minimizing value of x satisfies

$$x = \frac{(0.99)^{-x/2}}{\sqrt{-\ln(0.99)}}.$$

(b) Use the technique of fixed-point iteration (see Section 4.2) to solve the equation in (a) for x. Start with $x = 20$, say.

4. (Measuring variations in g) The period P of a pendulum of length L is given by

$$P = 2\pi\sqrt{L/g},$$

where g is the acceleration of gravity.

(a) Assuming that L remains fixed, show that a 1% increase in g results in approximately a 0.5% decrease in the period P. (Variations in the period of a pendulum can be used to detect small variations in g from place to place on the earth's surface.)

(b) For fixed g, what percentage change in L will produce a 1% increase in P?

5. (Torricelli's Law) The rate at which a tank drains is proportional to the square root of the depth of liquid in the tank above the level of the drain: if $V(t)$ is the volume of liquid in the tank at time t, and $y(t)$ is the height of the surface of the liquid above the drain, then $dV/dt = -k\sqrt{y}$, where k is a constant depending on the size of the drain. For a cylindrical tank with constant cross-sectional area A with drain at the bottom:

(a) Verify that the depth $y(t)$ of liquid in the tank at time t satisfies $dy/dt = -(k/A)\sqrt{y}$.

(b) Verify that if the depth of liquid in the tank at $t = 0$ is y_0, then the depth at subsequent times during the draining process is $y = \left(\sqrt{y_0} - \dfrac{kt}{2A}\right)^2$.

(c) If the tank drains completely in time T, express the depth $y(t)$ at time t in terms of y_0 and T.

(d) In terms of T, how long does it take for half the liquid in the tank to drain out?

6. If a conical tank with top radius R and depth H drains according to Torricelli's Law and empties in time T, show that the depth of liquid in the tank at time t $(0 < t < T)$ is

$$y = y_0\left(1 - \frac{t}{T}\right)^{2/5},$$

where y_0 is the depth at $t = 0$.

7. Find the largest possible area of a right-angled triangle whose perimeter is P.

8. Find a tangent to the graph of $y = x^3 + ax^2 + bx + c$ that is not parallel to any other tangent.

9. (Branching angles for electric wires and pipes)

(a) The resistance offered by a wire to the flow of electric current through it is proportional to its length and inversely proportional to its cross-sectional area. Thus, the resistance R of a wire of length L and radius r is $R = kL/r^2$, where k is a positive constant. A long straight wire of length L and radius r_1 extends from A to B. A second straight wire of smaller radius r_2 is to be connected between a point P on AB and a point C at distance h from B such that CB is perpendicular to AB. (See Figure 4.72.) Find the value of the angle $\theta = \angle BPC$ that minimizes the total resistance of the path APC, that is, the resistance of AP plus the resistance of PC.

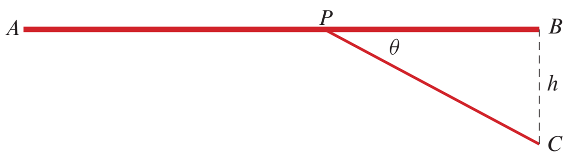

Figure 4.72

(b) The resistance of a pipe (e.g., a blood vessel) to the flow of liquid through it is, by Poiseuille's Law, proportional to its length and inversely proportional to the *fourth power* of its radius: $R = kL/r^4$. If the situation in part (a) represents pipes instead of wires, find the value of θ that minimizes the total resistance of the path APC. How does your answer relate to the answer for part (a)? Could you have predicted this relationship?

10. (The range of a spurt) A cylindrical water tank sitting on a horizontal table has a small hole located on its vertical wall at height h above the bottom of the tank. Water escapes from the tank horizontally through the hole and then curves down under the influence of gravity to strike the table at a distance R from the base of the tank, as shown in Figure 4.73. (We ignore air resistance.) Torricelli's Law implies that the speed v at which water escapes through the hole is proportional to the square root of the depth of the hole below the surface of the water: if the depth of water in the tank at time t is $y(t) > h$, then $v = k\sqrt{y - h}$, where the constant k depends on the size of the hole.

(a) Find the range R in terms of v and h.

(b) For a given depth y of water in the tank, how high should the hole be to maximize R?

(c) Suppose that the depth of water in the tank at time $t = 0$ is y_0, that the range R of the spurt is R_0 at that time, and that the water level drops to the height h of the hole in T minutes. Find, as a function of t, the range R of the water that escaped through the hole at time t.

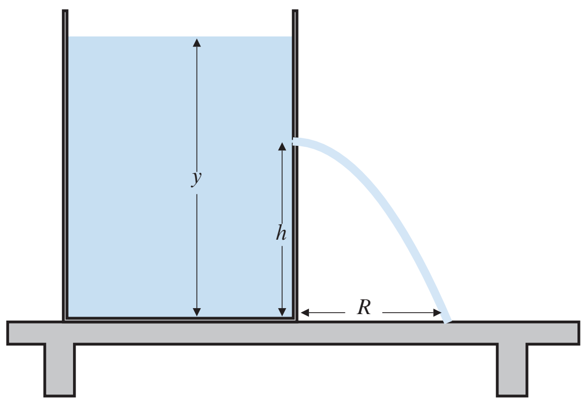

Figure 4.73

11. (Designing a dustpan) Equal squares are cut out of two adjacent corners of a square of sheet metal having sides of length 25 cm. The three resulting flaps are bent up, as shown in Figure 4.74, to form the sides of a dustpan. Find the maximum volume of a dustpan made in this way.

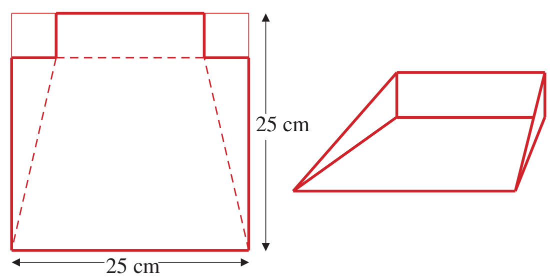

Figure 4.74