Exercises 2.10

(15) $\int \cos(2x)\,dx = \frac{1}{2}\int 2\cos(2x)\,dx = \frac{1}{2}\sin(2x) + c$

$\frac{d}{dx} 2x = 2$

(16) $\int \operatorname{sen}\left(\frac{x}{2}\right) dx = 2\int \frac{1}{2} \operatorname{sen}\left(\frac{x}{2}\right) dx = -2\cos\frac{x}{2} + c$

$\frac{d}{dx}\frac{x}{2} = \frac{1}{2}$

(17) $\int \frac{1}{(1+x)^2}\,dx = \int (1+x)^{-2}\,dx = \frac{(1+x)^{-1}}{-1} = -\frac{1}{1+x} + c$

$\frac{d}{dx}(1+x) = 1$

(19) $\int \sqrt{2x+3}\,dx = \int (2x+3)^{1/2}\,dx = \frac{1}{2}\int 2(2x+3)^{1/2}\,dx$

$\frac{1}{2} + 1 = \frac{1+2}{2} = \frac{3}{2}$

$= \frac{1}{2}\left(\frac{(2x+3)^{3/2}}{3/2}\right) + c$

$= \frac{1}{2}\,\frac{2\cdot\sqrt[2]{(2x+3)^3}}{3} + c$

$= \frac{1}{3}\sqrt{(2x+3)^3} + c$

(22) $\int \frac{2x}{\sqrt{x^2+1}}\,dx = \int 2x(x^2+1)^{-1/2}\,dx = 2\sqrt{x^2+1} + c$

$\frac{d}{dx}(x^2+1) = 2x \qquad -\frac{1}{2} + 1 = \frac{1}{2}$

(24) $\int \operatorname{sen} x \cos x\,dx = \frac{1}{2}\int 2\operatorname{sen} x\cos x\,dx = \frac{1}{2}\int \operatorname{sen}(2x)\,dx$

$\operatorname{sen}(2x) = 2\operatorname{sen} x\cos x$

$= \frac{1}{4}\int 2\operatorname{sen}(2x)\,dx$

$= -\frac{1}{4}\cos(2x) + c$

Exercises 5.6

(5) $\int \frac{x}{(4x^2+1)^5}\,dx = \frac{1}{8}\int 8(4x^2+1)^{-5}\,dx = -\frac{1}{32}(4x^2+1)^{-4} + c$

$\frac{d}{dx}(4x^2) = 8x$

(6) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx = 2\int \frac{\sin(\sqrt{x})}{2\sqrt{x}}\,dx$

$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} \qquad = -2\cos(\sqrt{x}) + c$

(12) $\int \frac{\ln t}{t}\,dt = \int u\,du = \frac{1}{2}u^2 + c$

$u = \ln t \qquad = \frac{1}{2}\ln^2 t + c$

$du = \frac{1}{t}\,dt$

(39) $\int_0^4 x^3(x^2+1)^{-1/2}\,dx = \frac{1}{2}\int_0^4 x^2(x^2+1)^{-1/2}\,2x\,dx$

$u = x^2+1 \qquad = \frac{1}{2}\int_1^{17}(u-1)u^{-1/2}\,du$

$du = 2x\,dx$

$x^2 = u-1 \qquad = \frac{1}{2}\int_1^{17}(u^{1/2} - u^{-1/2})\,du$

$x^3 = x(x^2) =$

$= \frac{1}{2}\left(u^{3/2}\cdot\frac{2}{3} - 2u^{1/2} \right|_1^{17}$

x	u
0	1
4	17

(42) $\int_{\pi/4}^{\pi} \sin^5 x\,dx \qquad u = \cos x$

$du = -\sin x\,dx$

x	u
π	1
$\pi/4$	$1/\sqrt{2}$

$= \int_{\pi/4}^{\pi} (1-\cos^2 x)^2 \sin x\,dx$

$= -\int (1-\cos^2 x)^2\cdot(-\sin x)\,dx = -\int_1^{1} (1-u^2)^2\,du$

Exercises 6.1

(1) $\int x\cos x \, dx$

$$\int \frac{d}{dx}\left[f(x)g(x)\right]dx = \int\left[f'(x)g(x) + g'(x)f(x)\right]dx$$

$$= \int f'(x)g(x)\,dx + \int g'(x)f(x)\,dx$$

$$\downarrow$$

$$\int g'(x)f(x)\,dx = \int \frac{d}{dx}\left[f(x)g(x)\right] - \int f'(x)g(x)$$

$$\Rightarrow \int g'(x)f(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$u = f(x) \qquad v = g(x)$

$du = f'(x)dx \qquad dv = g'(x)dx$

$$\int u\,dv = uv - \int v\,du$$

$$\int x\cos x\,dx =$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$g(x) = \operatorname{sen} x \qquad g'(x) = \cos x$

$f(x) = x \qquad f'(x) = 1$

$$= x\operatorname{sen} x - \int \operatorname{sen} x\,dx$$

$$= x\operatorname{sen} x + \cos x + C$$

(2) $\int (x+3)e^{2x}\,dx$

$\int e^{2x}$

$= \frac{1}{2}e^{2x} + c$

$f(x) = x+3 \qquad f'(x) = 1$

$g(x) = \frac{1}{2}e^{2x} \qquad g'(x) = e^{2x}$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$= \frac{1}{2}(x+3)e^{2x} - \int \frac{1}{2}e^{2x}\,dx$$

$$= \quad // \quad - \frac{1}{4}\int 2e^{2x}\,dx = \frac{1}{2}(x+3)e^{2x} - \frac{1}{4}e^{2x} + c$$

$$\frac{d}{dx}e^{2x} = 2e^{2x}$$

(5) $\int x^3 \ln x\,dx$

$f(x) = \ln x \qquad f'(x) = \frac{1}{x}$

$g'(x) = x^3 \qquad g(x) = \frac{1}{4}x^4$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$= \frac{1}{4}x^4\ln x - \int \frac{1}{4}x^3\,dx$$

$$= \frac{1}{4}x^4\ln x - \frac{1}{16}x^4\,dx + c$$

(3) $\int x^2\cos(\pi x)\,dx$

$\frac{1}{\pi}\int \pi\cos(\pi x)\,dx$

$f(x) = x^2 \qquad f'(x) = 2x$

$g'(x) = \cos(\pi x) \qquad g(x) = \frac{1}{\pi}\sin(\pi x) = \frac{1}{\pi}\sin(\pi x)\,dx$

$$= \frac{x^2}{\pi}\sin(\pi x) - \int \frac{2x}{\pi}\sin(\pi x)\,dx$$

$\frac{d}{dx}\frac{1}{\pi}\sin(\pi x)$

$= \cancel{\frac{1}{\pi}}\cos(\pi x)$

$$\frac{d}{dx}\ -\frac{1}{\pi}\cos(\pi x)$$
$$= -\frac{1}{\pi}(-\sin(\pi x))\cdot\pi$$

$$\frac{2}{\pi}\int x\cdot\sin(\pi x)\,dx$$

$f(x) = x \qquad f'(x) = 1$

$g'(x) = \sin(\pi x) \qquad g(x) = -\frac{1}{\pi}\cos(\pi x)$

$$= \frac{2}{\pi}\left[-\frac{x}{\pi}\cos(\pi x) - \int -\frac{1}{\pi}\cos(\pi x)\,dx\right]$$
$$= \frac{2}{\pi}\left[\ \text{"}\ + \int \frac{1}{\pi}\cos(\pi x)\,dx\right]$$
$$= \frac{2}{\pi}\left[\ \text{"}\ + \frac{1}{\pi^2}\sin(\pi x)\right]$$

$$= \frac{x^2}{\pi}\sin(\pi x) + \frac{2x}{\pi^2}\cos(\pi x) + \frac{2}{\pi^3}\sin(\pi x)$$

$$\frac{1}{\pi}\int\cos(\pi x)\,dx = \frac{1}{\pi}\sin(\pi x)$$
$$\frac{d}{dx}\sin(\pi x) = \pi\cos(\pi x)$$

(4) $\int (x^2 - 2x)e^{kx}\,dx$

$f(x) = x^2 - 2x \qquad f'(x)\ 2x - 2$

$g'(x) = e^{kx} \qquad g(x) = \frac{1}{k}e^{kx}$

$$\int\frac{1}{k}e^{kx}\,dx = \frac{1}{k^2}e^{kx}$$

$$= \frac{1}{k}(x^2-2x)e^{kx} - \frac{1}{k}\int(2x-2)e^{kx}\,dx$$

(b)

$$= \text{//} - \frac{2}{k}\int xe^{kx}\,dx + \frac{2}{k}\int e^{kx}\,dx$$

(a) (c)

(b) $f(x) = x \qquad f'(x) = 1$

$g'(x) = e^{kx} \qquad g(x) = \frac{1}{k}e^{kx}$

$$-\frac{2}{k}\int xe^{kx}\,dx = -\frac{2}{k}\left[\frac{x}{k}e^{kx} - \int\frac{1}{k}e^{kx}\,dx\right]$$

$$= -\frac{2}{k}\left[\frac{x}{k}e^{kx} - \frac{1}{k^2}e^{kx}\right] + C$$

(c) $\frac{2}{k}\int e^{kx}\,dx = \frac{2}{k}\left(\frac{1}{k}e^{kx}\right) = \frac{2}{k^2}e^{kx}$

$= (a) + (b) + (c)$

$$= \frac{1}{k}(x^2 - 2x)e^{kx} - \frac{2x}{k^2}e^{kx} + \frac{2}{k^3}e^{kx} + \frac{2}{k^2}e^{kx}$$

$$= \frac{1}{k}(x^2 - 2x)e^{kx} + \frac{2}{k^2}(1-x)e^{kx} + \frac{2}{k^3}e^{kx}$$ ✓

$\frac{d}{dx}(\ln x)^3 = 3(\ln x)^2 \cdot \frac{1}{x}$

(6) $\int x(\ln x)^3\,dx$

$f(x) = \ln^3 x \qquad f'(x) = \frac{3\ln^2 x}{x}$

$g'(x) = x \qquad g(x) = \frac{x^2}{2}$

$\int \frac{3}{x}\ln^2 x \cdot \frac{x^2}{2}$

$= \frac{3}{2}\int x \ln^2 x$

$= \underbrace{\frac{x^2}{2}\ln^3 x}\ \underbrace{-\frac{3}{2}}\int x^{2}\ln^2 x\,dx$

(a) $\int x^{3}\ln^2 x\,dx$

$f(x) = \ln^2 x \qquad f'(x) = \frac{2\ln x}{x}$

$g'(x) = x^{2} \qquad g(x) = \frac{x^2}{2}$

$= \underbrace{\frac{x^2}{2}\ln^2 x}\ \underbrace{-1}\int x^{2}\ln x\,dx$

(b) $\int x^{3}\ln x\,dx =$

$f(x) = \ln x \qquad f'(x) = \frac{1}{x}$

$g'(x) = x^{3} \qquad g(x) = \frac{x^2}{2}$

$= \frac{x^2}{2}\ln x - \int \frac{x^1}{2}\,dx$

$$= \underbrace{\frac{x^2}{2}\left(\ln x - \frac{x^2}{4}\right.} + c$$

$$= \frac{x^2}{2}\ln^3 x - \frac{3}{2}\left[\frac{x^2}{2}\ln^2 x - \frac{2}{2}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\right] + c$$

$$= \frac{x^2}{2}\ln^3 x - \frac{3}{2}\left[\frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \frac{x^2}{4}\right] + c$$

$$= \frac{x^2}{2}\ln^3 x - \frac{3x^2}{4}\ln^2 x + \frac{3x^2}{4}\ln x - \frac{3x^2}{8} + c$$

$$= \frac{x^2}{2}\left(\ln^3 x - \frac{3}{2}\ln^2 x + \frac{3}{2}\ln x - \frac{3}{4}\right) + c$$

$$= \frac{x^2}{2}\left(\ln^3 x - \frac{3}{2}x\ln^2 x + \frac{2}{3}x\ln x - \frac{2}{9}x\right) + c$$

$$= \frac{x^2}{2}\left(\ln^3 x - \frac{3}{2}\ln^2 x + \frac{3}{2}\ln x - \frac{3}{4}\right) + c$$

(10) $\int x^5 e^{-x^2}\,dx = \int x^4 \cdot x e^{-x^2}$

$f(x) = x^4 \qquad f'(x) = 4x^3$

$g'(x) = xe^{-x^2} \qquad g(x) = -\frac{1}{2}e^{-x^2}$

$-\frac{1}{2}\int -2xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}$

$$= -\frac{1}{2}x^4 e^{-x^2} - \int 4x^3 \cdot -\frac{1}{2}e^{-x^2}\,dx$$

$$(2) = \underbrace{-\frac{1}{2}x^4 e^{-x^2}} + 2\int x^3 e^{-x^2}\,dx$$

$f(x) = x^2 \qquad f'(x) = 2x$

$g'(x) = xe^{-x^2} \qquad g(x) = -\frac{1}{2}e^{-x^2}$

$$\int x^2 \cdot xe^{-x^2}\,dx = -\frac{x^2}{2}e^{-x^2} - \int -\frac{2}{2}xe^{-x^2}\,dx$$

(b) $= -\underbrace{\frac{x^2}{2}e^{-x^2}} + \int x e^{-x^2}\,dx$

(c) $-\frac{1}{2}\int -2xe^{-x^2}\,dx = \underbrace{-\frac{1}{2}e^{-x^2}} + c$

$\frac{d}{dx}\left(-\frac{1}{2}e^{-x^2}\right) = -\frac{1}{2}(-2x)e^{-x^2}$

$= xe^{-x^2}$

$= -\frac{1}{2}x^4 e^{-x^2} + 2\left(-\frac{x^2}{2}e^{-x^2} - \frac{1}{2}e^{-x^2}\right) + c$

$= -\frac{1}{2}x^4 e^{-x^2} - x^2 e^{-x^2} - e^{-x^2} + c$

$= \left(-\frac{1}{2}x^4 - x^2 - 1\right)e^{-x^2} + c$ ✓

(13) $\int e^{2x}\sin(3x)\,dx$

$f(x) = e^{2x}$ $\qquad f'(x) = 2e^{2x}$

$g'(x) = \sin(3x)$ $\qquad g(x) = -\frac{1}{3}\cos(3x)$

$\int \sin(3x)\,dx = \frac{1}{3}\int 3\sin(3x)\,dx = -\frac{1}{3}\cos(3x)\,dx$

$= -\frac{1}{3}e^{2x}\cos(3x) + \int 2e^{2x}\frac{1}{3}\cos(3x)$

$= \underbrace{-\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}}\int e^{2x}\cos(3x)\,dx$

$f(x) = e^{2x}$ $\qquad f'(x) = 2e^{2x}$

$g'(x) = \cos 3x$ $\qquad g(x) = \frac{1}{3}\sin(3x)$

$\int \cos(3x)\,dx = \frac{1}{3}\sin(3x) + c$

$= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\int e^{2x}\sin(3x)\,dx\right)$

$$= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) - \frac{4}{9}\int e^{2x}\sin(3x)\,dx$$

$$= \int e^{2x}\sin(3x)\,dx$$

$$\int e^{2x}\sin(3x)\,dx + \frac{4}{9}\int e^{2x}\sin(3x)\,dx = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x)$$

$$\int e^{2x}\sin(3x)\,dx = \frac{\overset{3}{\cancel{9}}}{13}\cdot\frac{1}{\cancel{3}}\left(-e^{2x}\cos(3x) + \frac{2}{3}e^{2x}\sin(3x)\right) + c$$

$$= \frac{3\,\cancel{8}}{13\,\cancel{39}}\left(-e^{2x}\cos(3x) + \frac{2}{3}e^{2x}\sin(3x)\right) + c$$

$$\frac{1}{13}e^{2x}\left(2\sin(3x) - 3\cos(3x)\right)$$

Exercises 6.2

(i) $\int \frac{2}{2x-3}\,dx = \int \frac{2}{2x-3}\,dx = \ln|2x-3| + c$

$$\frac{d}{dx}(2x-3) = 2$$

(ii) $\int \frac{x^2}{x-4}\,dx = \int x\,dx + \int 4\,dx + \int \frac{16}{x-4}\,dx$

$$\frac{x^2}{x-4} = x+4+\frac{16}{x-4} \qquad = \frac{x^2}{2} + 4x + 16\ln|x-4| + c$$

(iii) $\int \frac{x-2}{x^2+x}\,dx = -2\int\frac{1}{x}\,dx + 3\int\frac{1}{x+1}\,dx =$

$$\frac{x-2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = -\frac{2}{x} + \frac{3}{x+1}$$

$$x-2 = A(x+1) + Bx$$

$$x-2 = Ax + A + Bx$$

$$= x(A+B) + A$$

$$= 3\ln|x+1| - 2\ln|x| + C$$

$$\begin{cases} A+B = 1 \\ A = -2 \end{cases} \qquad \begin{cases} B = 1-A \\ A = -2 \end{cases}$$

(13) $\int \frac{1}{9x^2-6x+1}\,dx = \int \frac{1}{(3x-1)^2}\,dx = \frac{1}{3}\int 3(3x-1)^{-2}\,dx$

$9x^2-6x+1 = (9x-3)(x-2)$

$\frac{d}{dx} = 3$

$\Delta/4 = 9-9=0$

$(3x-1)^2$

$= -\frac{1}{3}\frac{1}{(3x-1)}$

(7) $\int \frac{1}{a^2-x^2}\,dx = \int \frac{1}{(a-x)(a+x)}\,dx$

$\frac{1}{(a-x)(a+x)} = \frac{A}{(a-x)} + \frac{B}{(a+x)} = \frac{1}{2a(a-x)} + \frac{1}{2a(a+x)}$

$1 = A(a+x) + B(a-x)$

$1 = Aa + Ax + Ba - Bx$

$1 = Aa + Ba + Ax - Bx$

$\begin{cases} A-B=0 \\ Aa+Ba=1 \end{cases} \quad \begin{cases} A=B \\ 2Aa=1 \end{cases} \quad \begin{cases} A=B \\ A=\frac{1}{2a} \end{cases}$

$= \frac{1}{2a}\left[\int \frac{1}{a-x}\,dx + \int \frac{1}{a+x}\,dx\right]$

$= \frac{1}{2a}\left[-\ln|a-x| + \ln|a+x|\right] + c$

$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

(10) $\int \frac{x}{3x^2+8x-3}\,dx$

$\frac{d}{dx}(3x^2+8x-3) = 6x+8$

$3x^2+8x-3 =$

$\Delta/4 = 16+9 = 5^2$

$x_{1,2} = \frac{-4\pm5}{3} < \begin{matrix} -3 \\ 1/3 \end{matrix}$

$3(x-\frac{1}{3})(x+3)$

$(3x-1)(x+3)$

$$\frac{x}{3x^2+8x-3} = \frac{A}{3x-1} + \frac{B}{x+3} = \frac{1}{10(3x-1)} + \frac{3}{10(x+3)}$$

$$x = A(x+3) + B(3x-1)$$

$$x = Ax + 3A + 3Bx - B$$

$$x = Ax + 3Bx + 3A - B$$

$$\begin{cases} A + 3B = 1 \\ 3A - B = 0 \end{cases} \quad \begin{cases} A = 1 - 3B \\ 3(1-3B) - B = 0 \end{cases} \qquad \frac{10-9}{10} = \frac{1}{10}$$

$$\begin{cases} A = 1 - 3B \\ 3 - 9B - B = 0 \end{cases} \quad \begin{cases} \text{---} \\ 3 - 10B = 0 \end{cases}$$

$$\begin{cases} \text{---} \\ B = \frac{3}{10} \end{cases} \quad \begin{cases} A = 1 - \frac{9}{10} = \frac{1}{10} \\ B = \frac{3}{10} \end{cases}$$

$$= \frac{1}{10}\left[\frac{1}{3}\int \frac{3}{3x-1}\,dx + 3\int \frac{1}{x+3}\,dx\right]$$

$$= \frac{1}{10}\left[\frac{1}{3}\ln|3x-1| + 3\ln|x+3|\right]$$

Exercises 6.5

(1) $$\int_2^{\infty} \frac{1}{(x-1)^3}\,dx = \lim_{t\to+\infty} \int_2^t \frac{1}{(x-1)^3}\,dx$$

$$\int \frac{1}{(x-1)^3}\,dx = \int (x-1)^{-3}\,dx = -\frac{1}{2}(x-1)^{-2}$$

$$\lim_{t\to+\infty}\left[-\frac{1}{2}(x-1)^{-2}\right]_2^t$$

$$-\frac{1}{2}\lim_{t\to+\infty}\left[(t-1)^{-2} - (2-1)^{-2}\right]$$

$$= -\frac{1}{2}\lim_{t\to+\infty}\left[\frac{1}{(t-1)^2} - 1\right]$$

$$= -\frac{1}{2}(-1) = \frac{1}{2}$$

(10) $\int_0^{\infty} xe^{-x}\,dx$

$= \lim_{t\to\infty} \int_0^t xe^{-x}\,dx$

$= \lim_{t\to\infty} \left[-e^{-x}(x+1)\right]_0^t$

$= -\lim_{t\to\infty} \left[e^{-t}(t+1) - e^{0}(1)\right]$

$= -\lim_{t\to\infty} \left[\cancel{e^{-t}(t+1)}^{\to 0} - 1\right]$

$= 1$

$\int xe^{-x}\,dx =$

$f(x) = x \qquad f'(x) = 1$

$g'(x) = e^{-x} \qquad g(x) = -e^{-x}$

$\underbrace{-\int -e^{-x}\,dx}$ $= \underbrace{-e^{-x}}$

$-xe^{-x} - \int -e^{-x}$

$= -xe^{-x} - e^{-x}$

$= e^{-x}(-1-x)$

$= -e^{-x}(x+1)$

(15) $\int_0^{\pi/2} \tan\,dx$

$= -\lim_{t\to\pi/2} \left[\ln|\cos t| - \cancel{\ln|\cos 0|}^{0}\right]$

$= -\lim_{t\to\pi/2} \left[\ln|\underbrace{\cos t}_{0}|\right]$

$-\infty$

$= +\infty$ (diverges)

$\int \tan dx = \int \frac{\sin x}{\cos x}$

$\frac{d}{dx}\cos x = -\sin x$

$= -\int \frac{-\sin x}{\cos x}$

$= -\ln|\cos x| + c$

$\ln 1 = 0$

(19) $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$ ← split the integral!

$= \lim_{t\to\infty} \int_{-t}^{t} \frac{x}{1+x^2}\,dx$

$= \frac{1}{2} \lim_{t\to\infty} \left[\ln|x^2+1|\right]_{-t}^{t}$

$= \frac{1}{2} \lim_{t\to\infty} \left[\ln|\cancel{t^2}^{\infty}+1| - \ln|\cancel{(-t)^2}^{\infty}+1|\right]$ $\quad \infty - \infty$

$= \frac{1}{2} \lim_{t\to\infty} \left[\ln\left|\frac{t^2+1}{t^2+1}\right|\right]$

$= \frac{1}{2} \lim_{t\to\infty} \left[\ln(1)\right] = \frac{e}{2}$

$\frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1|$

$$= \int_{-\infty}^{0} \frac{x}{1+x^2}\,dx + \int_{0}^{\infty} \frac{x}{1+x^2}\,dx$$

Sequences and Series

Notes

- Increasing / Dec. : $a_{n+1} \geq a_n$

  if $a_n = f(n)$, show f is increasing on $[1, \infty) \to f'(x) \geq 0$

- Convergence : $\lim_{n \to \infty} a_n = L$

- If $\{a_n\}$ converges, it is bounded

  Bounded + Non-Monotone = DIVERGENT

- If $\{a_n\}$ is bounded above/below and is increa./decr., then it is convergent

Exercises 9.1

every converging sequence is bounded

every bounded monotonous sequence converges

(a) bounded

(b) positive / negative

(c) increasing / dec / alternating

(d) convergent / divergent / divergent to $\pm\infty$

(1) $\left\{ \frac{2n^2}{n^2+1} \right\}$ ← Num degree grows faster than denominator

same degree → converges

(c) $\frac{2n^2}{n^2+1} \leq \frac{2(n+1)^2}{(n+1)^2+1}$ // $\leq \frac{2n^2+4n+2}{n^2+2n+2}$

// $\leq \frac{2(n^2+2n+1)}{(n^2+2n+1)+1}$

$2n^2(n^2+2n+2) \leq (n^2+1)(2n^2+4n+2)$

$\cancel{2n^4} + \cancel{4n^3} + \cancel{4n^2} \leq \cancel{2n^4} + \cancel{4n^3} + \cancel{2n^2} + \cancel{2n^2} + 4n + 2$

$0 \leq 4n + 2$

$-\frac{1}{2} \leq n$ ✓

↳ increasing

(d) $\lim_{x \to +\infty} \frac{2n^2}{n^2} = 2 \to$ convergent

(a) Bounded

(b) $\forall n \in \mathbb{N}$, $\frac{2n^2}{n^2+1} \geq 0$, positive

(a) $\{\sin\frac{1}{n}\}$

(d) $\lim_{x\to+\infty} \sin\frac{1}{x} = 0 \longrightarrow$ convergent

(a) bounded

(b) positive

(c) $\sin(\frac{1}{n}) \leq \sin(\frac{1}{n+1})$

$f(x) = \sin\frac{1}{x}$

$f'(x) = -\frac{1}{x^2}\sin\frac{1}{x}$

$-\frac{1}{x^2}\sin\frac{1}{x} \geq 0$

$-\frac{1}{x^2} \geq 0 \quad \nexists x \in R$

$\sin\frac{1}{x} \geq 0$

$\sin t \geq 0$

$0 \leq t \leq \pi$

$0 \leq \frac{1}{x} \leq \pi \quad \forall x \in N$

Decreasing

(8) $\left\{\frac{(-1)^n n}{e^n}\right\}$

(d) $\lim_{x\to\infty} \frac{(-1)^n n}{e^n} = \lim_{x\to\infty}(e^{-n}\cdot n) = 0$

linear function grows slower

$\frac{(-1)^n}{e^n}\cdot n = \left(-\frac{1}{e}\right)^n n = e^{-n} n$

converges to 0

(a) Bounded

(c) Alternating

(11) $\{n\cos(\frac{n\pi}{2})\}$

( ) $\lim_{x\to\infty} n\cos(\frac{n\pi}{2})$ → limit doesn't exist

↳ divergent sequence

(19) $a_n = \frac{e^n - e^{-n}}{e^n + e^{-n}} = \frac{e^n(1-e^{-2n})}{e^n(1+e^{-2n})} = \frac{1-e^{-2n}}{1+e^{-2n}}$

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{\cancel{e^n}(1-e^{-2n})}{\cancel{e^n}(1+e^{-2n})} = 1$$

(23) $a_n = \sqrt{n+1} - \sqrt{n}$

$$\lim_{x\to\infty} (\sqrt{n+1} - \sqrt{n})$$

$$= \lim_{x\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{x\to\infty} \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \lim_{x\to\infty} \frac{1}{\underbrace{\sqrt{n+1} + \sqrt{n}}_{\infty}} = 0$$

Exercises 9.2

Theory

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + \dots = \frac{a_{n+1}}{}$$

$$r = \frac{a_{n+1}}{a_n} = \frac{ar^n}{ar^{n-1}}$$

$$S_n = a + ar + ar^2 + \dots = \frac{a(1-r^n)}{1-r}$$

$$\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} a=0 \to \text{converges to } 0 \\ |r|<1 \to \text{converges to } \frac{a}{1-r} \\ r\geq 1 \wedge a>0 \to \text{diverges to } \infty \\ r\geq 1 \wedge a<0 \to \text{diverges to } -\infty \\ r\leq -1 \wedge a\neq 0 \to \text{diverges} \end{cases}$$

(1) $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$

$= \frac{1}{3}\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \dots\right)$

$r = \frac{1}{3} \rightarrow$ converges to $\frac{a}{1-r}$

(3) $\sum_{n=5}^{\infty} \frac{1}{(2+\pi)^{2n}} = \frac{1}{(2+\pi)^{10}} + \frac{1}{(2+\pi)^{12}} + \frac{1}{(2+\pi)^{14}} \dots$

$= \frac{1}{(2+\pi)^{10}}\left[1 + \frac{1}{(2+\pi)^2} + \left(\frac{1}{(2+\pi)^2}\right)^2\right]$

$\left.\begin{array}{ll} a = \frac{1}{(2+\pi)^{10}} & 0 < a < 1 \\ r = \frac{1}{(2+\pi)^2} & 0 < r < 1 \end{array}\right\}$ $|r| < 1$ converges to $\frac{a}{1-r}$

$\frac{1}{(2+\pi)^{10}} \Big/ 1 - \frac{1}{(2+\pi)^2} = \frac{1}{(2+\pi)^{10}} \Big/ \frac{(2+\pi)^2 - 1}{(2+\pi)^2}$

$= \frac{1}{(2+\pi)^{10}} \cdot \frac{(2+\pi)^2}{4 + 4\pi + \pi^2 - 1}$

$= \frac{1}{(2+\pi)^8 (4 + 4\pi + \pi^2 - 1)}$ ✓

(7) $\sum_{n=0}^{\infty} \frac{2^{n+3}}{e^{n-3}}$

$= \frac{2^3}{e^{-3}} + \frac{2^4}{e^{-2}} + \frac{2^5}{e^{-1}} + \frac{2^6}{e^0} + \frac{2^7}{e^1} \dots$

$= \frac{2^3}{e^{-3}}\left(1 + \frac{2}{e} + \frac{2^2}{e^2} \dots\right)$

$r = \frac{2}{e}$ $\quad a = \frac{2^3}{e^{-3}} = \frac{8}{\frac{1}{e^3}}$ $\quad r < 1, \; a > 1$

converges to $\frac{a}{1-r} = \frac{8e^3}{1-\frac{2}{e}}$

(15) $\sum_{n=1}^{\infty} \frac{1}{2n-1} = \frac{1}{2-1} + \frac{1}{4-1} + \frac{1}{6-1}$

$= \frac{1}{1} + \frac{1}{3} + \frac{1}{5}$

$\lambda = 0$

Converges to 0

Double Integrals

(1) $\int_0^1 dx \int_0^x (xy + y^2)\, dy$

$= \int_0^1 dx \cdot \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_0^x$

$= \int_0^1 dx \left[\left(\frac{x^3}{2} + \frac{x^3}{3} \right) - (0) \right]$

$= \int_0^1 \frac{5}{6} x^3\, dx$

$\frac{3x^3 + 2x^3}{6} = \frac{5x^3}{6}$

$= \frac{5}{6} \int_0^1 x^3 dx = \frac{5}{6} \left[\frac{x^4}{4} \right]_0^1$

$= \frac{5}{6} \left(\frac{1}{4} \right) = \frac{5}{24}$

(2) $\int_0^1 \underbrace{\int_0^y (xy + y^2)\, dx}\, dy$

$= \int_0^1 \left[\frac{yx^2}{2} + y^2 x \right]_0^y dy$

$= \int_0^1 \left(\frac{y^3}{2} + y^3 \right) dy$

$\frac{y^3 + 2y^3}{2} = \frac{3}{2} y^3$

$= \frac{3}{2} \int_0^1 y^3\, dy$

$= \frac{3}{2} \left(\frac{y^4}{4} \right)_0^1$

$= \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8}$

(3) $\int_0^\pi \underbrace{\int_{-x}^x \cos y\, dy}\, dx$

$= \int_0^\pi \left[\sin y \right]_{-x}^x dx$

$\frac{d}{dx} - \cos(-x)$

$= + \sin(-x)$

$= \int_0^\pi \left[\sin x - \sin(-x) \right] dx$

$= \int_0^\pi \left[2 \sin(x) \right] dx$

$= 2 \int_0^\pi \sin x\, dx$

$$= 2\left[-\cos x\right]_0^{\pi}$$
$$= 2\left[-\cos\pi + \cos 0\right]$$
$$= 2(+1+1) = 4$$

Fubini's Theorems

(1) $f(x,y)$ continuous on rectangular area R $\quad a \le x \le b,\ c \le y \le d$

$$\Rightarrow \iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$
$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

(2) $f(x,y)$ continuous on region R

$\rightarrow$ R is $a \le x \le b$, $g(x) \le y \le g_2(x)$

$$\Rightarrow \iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

$\rightarrow$ R is $a \le y \le b$, $h_1(y) \le x \le h_2(y)$

$$\Rightarrow \iint_R f(x,y)\,dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

(5) $\iint_R (x^2+y^2)\,dA \quad R: \begin{rcases} 0 \le x \le a \\ 0 \le y \le b \end{rcases}$ Rectangle

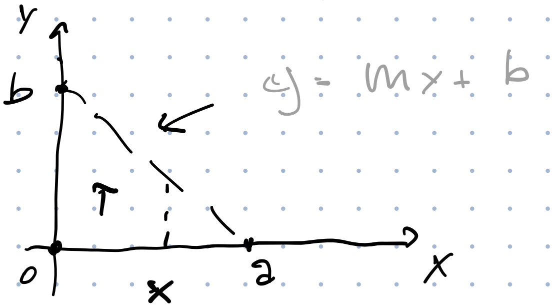
$$= \int_0^a \int_0^b (x^2+y^2)\,dy\,dx$$
$$= \int_0^a \left[x^2 y + \frac{y^3}{3}\right]_0^b dx =$$
$$= \int_0^a \left(bx^2 + \frac{b^3}{3}\right) dx$$
$$= \left[b\frac{x^3}{3} + \frac{b^3}{3}x\right]_0^a$$
$$= \left(\frac{1}{3}a^3 b + \frac{1}{3}ab^3\right)$$

(8) $\iint_T (x-3y)\,dA$

T: $(0,0)$, $(a,0)$, $(0,b)$

$0 \le x \le a$

$0 \le y \le b - \frac{b}{a}x$



$y = mx + b$

$m = \frac{0-b}{a-0} = -\frac{b}{a}$

$y = -\frac{b}{a}x + b$

$y = b - \frac{b}{a}x$

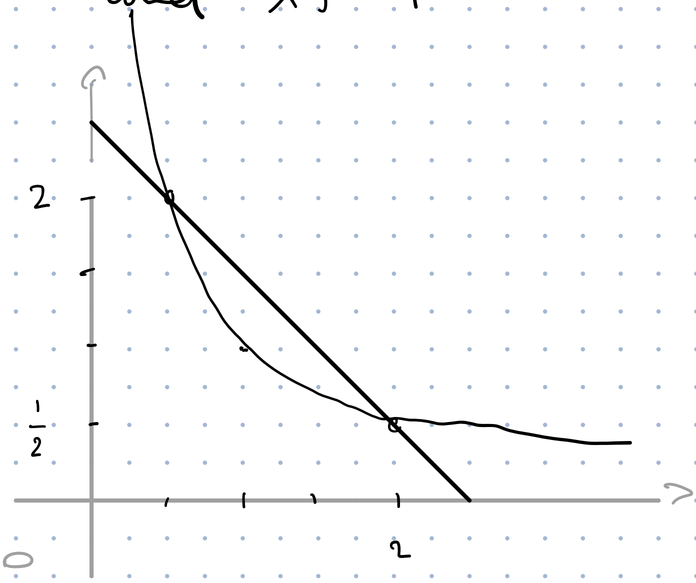
$$\iint_T (x-3y)\,dA = \int_0^a \int_0^{b-\frac{b}{a}x} (x-3y)\,dy\,dx$$

(11) $\iint_D \ln x\,dA$ D is bounded by $2x+2y=5$ and $xy=1$

(1) find intersections

$\begin{cases} 2x+2y=5 \\ xy=1 \end{cases}$

$\begin{cases} y = -x + \frac{5}{2} \\ y = \frac{1}{x} \end{cases}$



$-x + \frac{5}{2} = \frac{1}{x}$

$\frac{-2x+5}{2} = \frac{1}{x} \rightarrow -2x^2+5x-2=0$

$2x^2-5x+2=0$

$\Delta = 25 - 4(4) = 25-16 = 3^2$

$x_{1,2} = \frac{5 \pm 3}{4} \swarrow^{2}_{\frac{1}{2}}$

$(2, \frac{1}{2})$ v $(\frac{1}{2}, 2)$

$$\iint_D \ln x\,dA = \int_{1/2}^{2} \int_{1/x}^{5/2-x} \ln x\,dy\,dx$$

$$= \int_{1/2}^{2} \left[y \ln x \right]_{1/x}^{5/2-x} dx$$

$$\int \ln x = \int 1 \cdot \ln x$$

$f(x) = \ln x \qquad f'(x) = \frac{1}{x}$

$g'(x) = 1 \qquad g(x) = x$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + c$$

$$= x(\ln x - 1)$$

$$= \int_{1/2}^{2} \left[\left(\frac{5}{2} - x \right) \ln x - \frac{1}{x} \ln x \right] dx$$

$$= \int_{1/2}^{2} \frac{5}{2} \ln x \, dx - \int_{1/2}^{2} x \ln x \, dx - \int_{1/2}^{2} \frac{1}{x} \ln x \, dx$$

$$= \frac{5}{2} \Big[x(\ln x - 1) \Big]_{1/2}^{2} - \left[\frac{x^2}{2} (\ln x - 1) \right]_{1/2}^{2} - \left[\frac{\ln^2 x}{2} \right]_{1/2}^{2}$$

$$\int x \ln x = \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx$$

$f(x) = \ln x \qquad f'(x) \; \frac{1}{x} \qquad \frac{x^2}{2} \cdot \frac{1}{x} = \frac{x}{2}$

$g'(x) = x \qquad g(x) = \frac{x^2}{2}$

$$= \frac{x^2}{2} \ln x - \frac{1}{2} x^2$$

$$= \frac{x^2}{2} (\ln x - 1)$$

$$\int \frac{1}{x} \ln x \, dx$$

$u = \ln x$

$du = \frac{1}{x} \, dx$

$$\int u \, du = \frac{u^2}{2} = \frac{\ln^2 x}{2} + c$$

$$= \frac{5}{2} \left[(2 \ln 2 - 2) - \left(\frac{1}{2} \ln \frac{1}{2} - \frac{1}{2} \right) \right] - \Big[(2 \ln 2 - 2) - ($$