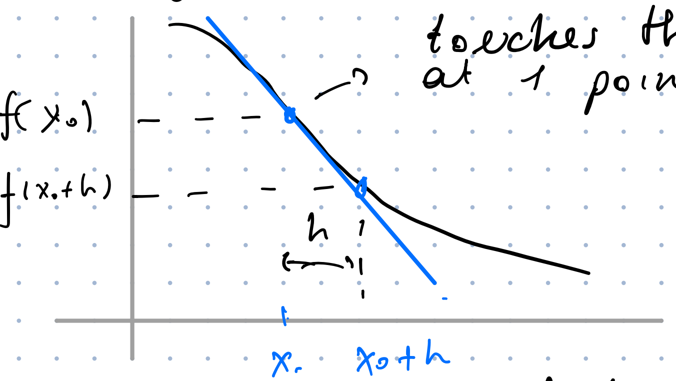
Differentiation

Tangent line



touches the curve at 1 point

the steeper the slope of the tangent line, the quicker the function changes

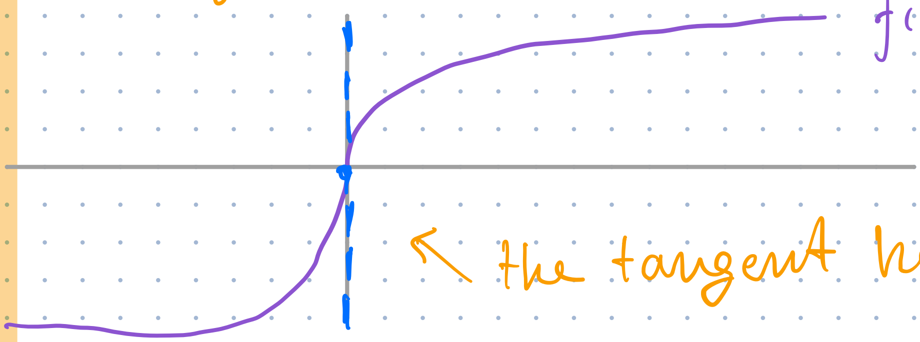
slope : $\frac{\Delta y}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{h}$

slope of tangent line

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

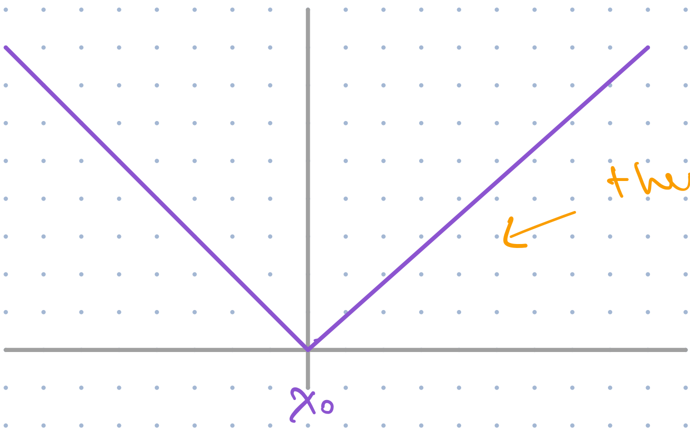
the tangent line CAN be vertical

$f(x) = \sqrt[3]{x}$



the tangent has slope of ∞

the tangent line DOES NOT always exist



there is no single tangent

Derivative

The derivative of f at point x_0 of domain is defined as the slope of the tangent of f at x_0

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

↳ the derivative only exists if the limit exists and is finite

Notations:

g', $f'(x)$, $\frac{d}{dx} f(x)$, $D_x f$.

$$f'(x_0) = \frac{d}{dx} f(x) \Big|_{x=x_0}$$

Differentiability

f is differentiable at $x_0 \longrightarrow f'(x_0)$ exists

Singular points

x_0 is a singular point $\longrightarrow f'(x_0)$ does NOT exist

Left & Right Derivatives

$$f'_{\pm}(x_0) = \lim_{h \to 0^{\pm}} \frac{f(x_0+h) - f(x_0)}{h}$$

Power rule

Differentiation rules

- Differentiable implies continuous
- for f and g differentiable at x

$$(f+g)'(x) = f'(x) + g'(x)$$
$$(kf)'(x) = k f'(x)$$

- Product rule

$$(f(x)\,g(x))' = f'(x)\,g(x) + g'(x)\,f(x)$$

$$\frac{d}{dx} f(x)g(x) = \lim_{h \to 0} \frac{f(x+h)\cdot g(x+h) - f(x)g(x)}{h}$$

Power rule

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1} \quad \text{for } n \in \mathbb{R}$$

Proof for $n \in \mathbb{N}$

Base case : $\frac{d}{dx}(x^1) = 1$

Induction step: $P(n) \longrightarrow P(n+1)$

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

$$\left[\frac{d}{dx}(x^{n+1}) \stackrel{?}{=} (n+1)\, x^{(n+1)-1} \right]$$ we need to get here

$$\frac{d}{dx}(x^{n+1}) = \frac{d}{dx}(x^n \cdot x)$$

$$= \frac{d}{dx}(x^n) \cdot x + x^n \frac{d}{dx}(x)$$

we use hypotheses $\rightarrow$ $= \underbrace{n \cdot x^{n-1}} \cdot x + x^n = (n+1)\, x^n$

$f(x) = \sqrt{g(x)}$

$f'(x) = \dfrac{g'(x)}{2\sqrt{g(x)}}$

Chain rule

$$\frac{d}{dx}\left[f(g(x)) \right] = f'\left[g(x) \right] \cdot g'(x)$$

$$g(x) = u \longrightarrow \frac{d}{dx} f(u(x)) = \frac{df}{du} \cdot \frac{du}{dx}$$

Reciprocal rule:

$$\frac{d}{dx}\left[\frac{1}{f(x)} \right] = -\frac{1}{f(x)^2} \cdot f'(x)$$

Quotient rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)} \right] = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

Exercises

(1) $f(x) = \frac{1}{(x^2+1)}$

$$f'(x) = -\frac{2x}{(x^2+1)^2}$$

Derivatives of trigonometric functions

Introduction

- $\lim_{x\to 0} \frac{\sin(x)}{x}$

$x>0$: $\sin(x) < x < \tan(x)$

$\Leftrightarrow \frac{\sin(x)}{x} < 1 < \frac{\tan(x)}{x}$

$\frac{\tan(x)}{x} = \frac{\sin(x)}{\cos(x)\, x} \Rightarrow \cos(x) < \frac{\sin(x)}{x}$

$\underbrace{\cos(x)}_{=1,\ x\to 0} < \frac{\sin(x)}{x} < 1 \longrightarrow 1$ as $x\to 0$

$\frac{\sin(x)}{x}$ is included between $\cos(x)$ and 1, so the limit $\lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$

$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ since $\sin(x)$ is even

Derivatives

- $\frac{d}{dx}(\sin(x)) = \cos(x)$

$= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$

Simpson's rule

$\sin(a) - \sin(b)$

$= 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$= \lim_{h\to 0} \frac{2\cos\left(x + \cancel{\frac{h}{2}}^{\,0}\right)\sin\left(\frac{h}{2}\right)}{h} = \cos x$

$\to 1$

$\lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = 1$

- $\frac{d}{dx}(e^x) = e^x$

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h}$$

e is chosen such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

- $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

$y = \ln(x) \iff x = e^y$

Exercises

(1) $y = \cotan(x) = \frac{\cos(x)}{\sin(x)}$

$$f'(x) = \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{(\sin(x))^2}$$

$$= -\frac{\sin(x)^2 + \cos(x)^2}{\sin(x)^2}$$

$$= -\frac{1}{\sin(x)^2} =$$

(2) $f(x) = e^{-\sin(3x)}$ $\quad f'(x) = -3\cos(3x)\, e^{-\sin(3x)}$

$$f'(x) = e^{-\sin(3x)}(-3\cos(3x))$$

Higher order derivatives

- second derivative : $f''(x) = \frac{\partial^2 f}{\partial x^2}$