Course overview

- Logic (week 1)
- Proof techniques (weeks 1-2)
- Set theory (weeks 2-3)
- Relations (weeks 3-4)
- Functions yesterday and today
- Combinatorics (week 5)

Overview of today

- Functions: definition
- Injective functions
- Surjective functions
- Inverse of a function

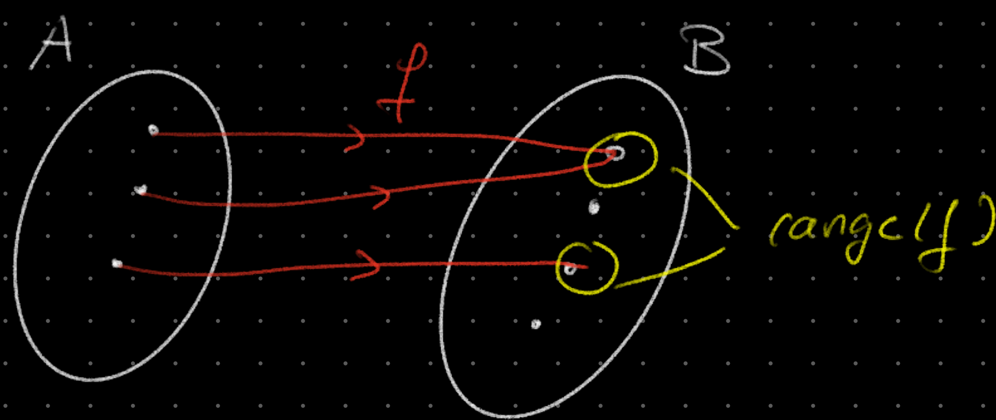
Book chapter 3, sections 3.6, 3.8

Recap

A function f: A -> B is a mapping from set A to set B. For all x in A,

- f(x) exists and is an element of B
- f(x) is unique
- Exactly one arrow leaves at every x in A.

- A: domain
- B: co-domain
- Range: {f(x), x in A}, range(f) $\subseteq$ B



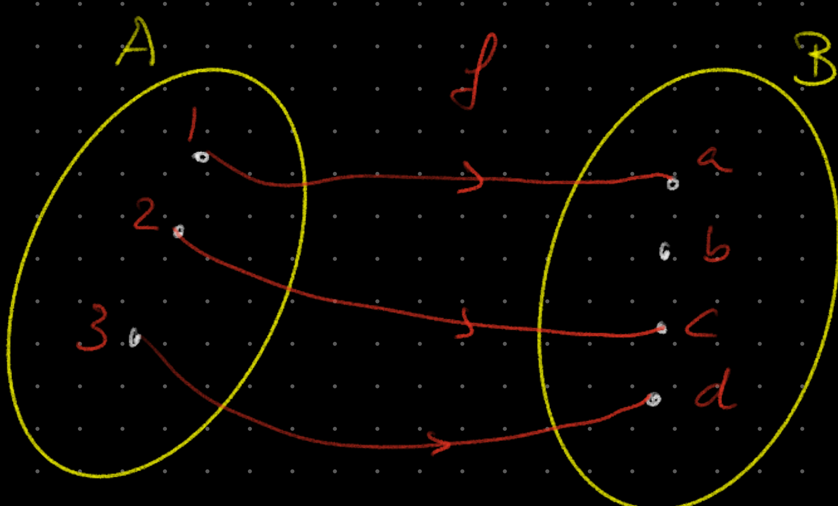
Injective functions (book: one-one)

A function f: A -> B is injective if two different elements have different function values.

(Maximally one arrow arrives at each y in B)

Definition : a function $f : A \to B$ is injective if $(\forall x, y \in A)(x \neq y \Rightarrow f(x) \neq f(y))$

Alternative definition (contra positive) $(\forall x, y \in A)(f(x) = f(y) \Rightarrow x = y)$



$f : \mathbb{N} \to \mathbb{Z}$, $f(x) = x - 1$

Let $x, y \in \mathbb{N}$, $x \neq y$

$\Rightarrow x - 1 \neq y - 1$

$\Rightarrow f(x) \neq f(y)$

injective !

$f : \mathbb{N} \to \mathbb{N}$, $f(x) = x^2$

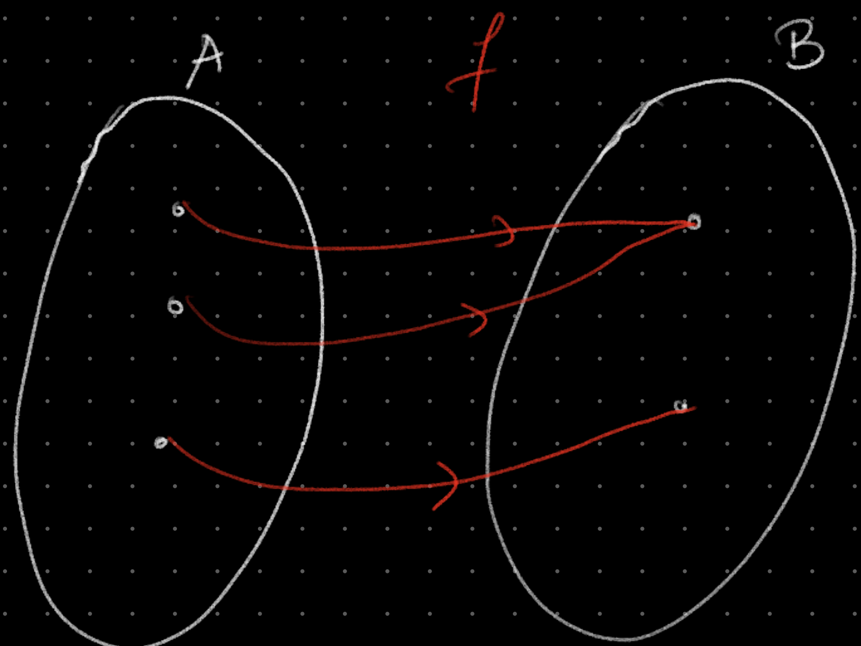
injective

Surjective functions (book: onto)

A function f: A -> B is surjective if every element of B is the image of an element of A, i.e. the range is equal to the co-domain.

(At every element of B, at least one arrow arrives.)

Definition : a function $f : A \to B$ is surjective if $(\forall y \in B)(\exists x \in A)(y = f(x))$



- $f : \mathbb{Z} \to \mathbb{Z} \quad f(x) = 2x+1$

$-1 \to -1$
$0 \to 1$
$1 \to 3$
$2 \to 5$
$3 \to 7$

→ we cannot reach even numbers ! NOT SURJECTIVE

Counter-example : $0 \in \mathbb{Z}$ (co-domain)

$f(x) = 0 \Leftrightarrow 2x+1 = 0 \Leftrightarrow x = \frac{-1}{2}$

↳ so $\neg \exists\, x \in \mathbb{Z}$ (domain) $: f(x) = 0$

* $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = -\frac{x}{2}$ for x even
  $f(x) = \frac{-x+1}{2}$ for x odd.

$$\begin{aligned} -2 &\to 1 \\ -1 &\to 1 \\ 0 &\to 0 \\ 1 &\to 0 \\ 2 &\to -1 \\ 3 &\to -1 \\ &\vdots \end{aligned}$$

$\left.\begin{matrix} -1 \\ -1 \end{matrix}\right\}$ not injective , $f(2) = f(3)$

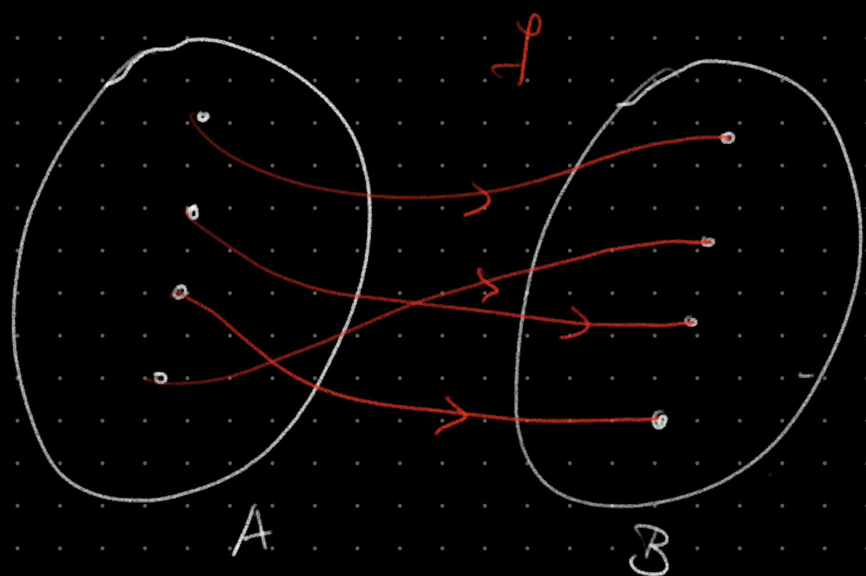
* Let $y \in \mathbb{Z}$ (co-domain)
  then $x = -2y \in \mathbb{Z}$ (domain).
  x is even. So $f(x) = -\frac{x}{2} = \frac{-(-2y)}{2} = y$.
  ↳ we found x in the domain, such that $y = f(x)$
  ⟶ f is surjective

Alternative: Let $y \in \mathbb{Z}$ (co-domain). Choose $x = -2y + 1 \in \mathbb{Z}$ (domain)
x is odd, so $(x) = \frac{-x+1}{2} = \frac{-(-2y+1)+1}{2} = y$

Bijections

A function f: A-> B is a bijection if it is injective and surjective

(Exactly one arrow arrives at every element B)



1) $f : \mathbb{N} \to \mathbb{N}$, $f(x) = 3x - 4$

not a properly defined function!

since $1 \in \mathbb{N}$ (domain) but $f(1) = -1 \notin \mathbb{N}$ (co-domain)

2) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 3x - 4$

- injective $\forall x_1, x_2 \in \mathbb{Z} : x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ (def.)

Let $x_1, x_2 \in \mathbb{Z}$, $x_1 \neq x_2$.

then $3x_1 - 4 \neq 3 \cdot x_2 - 4$

so $f(x_1) \neq f(x_2)$ INJECTIVE!

- surjective $\forall y \in \mathbb{Z} \ \exists x \in \mathbb{Z} : y = f(x)$ (def.)

counter-example

$0 \in \mathbb{Z}$ (co-domain) . $0 = f(x)$

$\Leftrightarrow 0 = 3x - 4$

$\Leftrightarrow x = \frac{4}{3} \notin \mathbb{Z}$

↳ NOT in domain!!

3) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x - 4$

- Let $x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$.

  then $3x_1 - 4 \neq 3 \cdot x_2 - 4$

  so $f(x_1) \neq f(x_2)$ INJECTIVE!

- surjective

  Let $y \in \mathbb{R}$ (co-domain).

  choose $x = \frac{y+4}{3} \in \mathbb{R}$ (domain)

  and $f(x) = 3\left(\frac{y+4}{3}\right) - 4 = y$

$y = f(x)$
$= 3x - 4$
$\Leftrightarrow x = \frac{y+4}{3}$

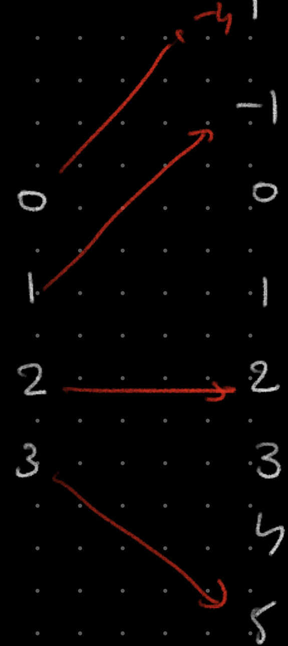
$\longrightarrow$ BIJECTIVE!

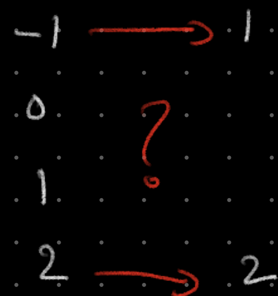
The inverse function $f^{-1}: \mathbb{R} \to \mathbb{R}$ is given by $f^{-1}(x) = \frac{x+4}{3}$

Inverse functions

The inverse of a function 'undoes' or 'reverses' the function.

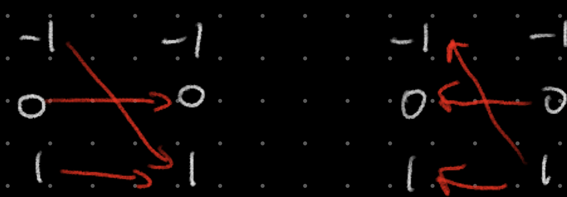
$f: \mathbb{Z} \to \mathbb{Z}$ $f(x) = 3x - 4$





not surjective
so the inverse function is not properly defined

$f: \{-1, 0, 1\} \to \{-1, 0, 1\}$
$f(x) = x^2$



not injective
so the inverse function is not properly defined

Let $f: A \to B$. The following are equivalent

1. f is a bijection
2. Reversing all arrows from A to B to be arrows from B to A gives a well-defined function from B to A.
3. There exists a function $g: B \to A$ such that for all x in A, $x = g(f(x))$ and for all y in B, $y=f(g(y))$.

If there exists such a function $g: B \to A$, then it is unique and we call it the inverse of f.

Example: $f: \mathbb{R}\setminus\{2\} \to \mathbb{R}$, $f(x) = \frac{1}{x-2}$

1) We start by calculating the inverse function

$$y = f(x) = \frac{1}{x-2} \iff x-2 = \frac{1}{y} \iff x = \frac{1}{y}+2$$

↳ so, $f^{-1}(y) = \frac{1}{y}+2$

2) Is this inverse function defined on domain $\mathbb{R}$ (co-domain of f)?

↳ NO, it is not defined for $y=0$

→ this means that $f: \mathbb{R}\setminus\{2\} \to \mathbb{R}$ is not surjective.

(the equation $f(x)=0$ has no real solution)

3) However $f: \mathbb{R}\setminus\{2\} \to \mathbb{R}\setminus\{0\}$, $f(x) = \frac{1}{x-2}$

$f^{-1}: \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{2\}$, $f^{-1}(y) = \frac{1}{y}+2$

are inverse functions, and therefore they are both bijective

we can verify this by calculating $f(f^{-1}(y))$, for $y \in \mathbb{R}\setminus\{0\}$

$$f(f^{-1}(y)) = f\left(\frac{1}{y}+2\right) = \frac{1}{\left(\frac{1}{y}+2\right)-2} = \frac{1}{\frac{1}{y}} = y$$

and $f^{-1}(f(x))$, for $x \in \mathbb{R}\setminus\{2\}$ (DIY)

Checklist

- Do you know what injective, surjective means?
- Can you prove that a function is injective?
- Can you prove that a function is surjective?
- Do you know what a bijection is?
- Do you understand that only bijective functions are invertible?
- Do you know how to check that a function is the inverse?