Practice Exam Questions Week 5, **Linear Algebra,**

1. Consider the following matrix A and vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 3 & -1 & -2 & -1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

   a. Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are both eigenvectors of A. What are the corresponding eigenvalues?
   b. Show that 0 is an eigenvalue of A.
   c. Compute a basis for the eigenspace of A for the eigenvalue 0.
   d. Compute a basis for Nul A.
   e. Is the matrix A diagonalizable? If it is, determine a matrix M such that $M^{-1}AM$ is a diagonal matrix. If it is not, explain why not.
   f. Compute the matrix A^9.

2. True or false? If the given statement is true, give a brief explanation. If it is false, give a counterexample.

   a. If $A = QR$ and Q is invertible, then A is similar to $B = RQ$;
   b. An elementary row operation on A does not change the determinant of A.
   c. If λ is an eigenvalue of A, then it is also an eigenvalue of A^{T}
   d. Each eigenvalue of A is also an eigenvalue of A^2.
   e. If M is a (2×2) matrix such that dim Nul A equals 1, then M has one eigenvalue equal to 0.
   f. Let B be an $(n \times n)$ matrix. Let $\mathbf{e}_1$ be the first column of the identity matrix I_n. If $\mathbf{e}_1$ is an eigenvector of B with eigenvalue 1, then the first column of B is $\mathbf{e}_1$.
   g. Any invertible matrix can also be diagonalized.
   h. If A and B are 2×2 matrices which can both be diagonalized, then their sum $C = A + B$ can also be diagonalized.
   i. If K and L are 3×3 matrices and $\mathbf{v}$ is an eigenvector of K and also of L, then $\mathbf{v}$ is an eigenvector of the matrix product KL.
   j. If $\mathbf{x}$ is an eigenvector of an invertible matrix P, then it is also an eigenvector of P^{-1}

3. Consider the following matrix Q:

$$Q = \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

   Find the characteristic polynomial and the eigenvalues of Q.