

# Lecture 4 Probability Axioms

BIO210 Biostatistics

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**Probability theory is nothing but common sense reduced to calculation.**

Laplace

## Set

A set is a well-defined collection of distinct objects.

$$S = \{ \text{list or description of the objects in the set} \}$$

## Sample space ( $\Omega$ )

Set of all possible outcomes

Outcomes: **mutually exclusive** and **collectively exhaustive**

# Sample space example 1

**Example 1:** flipping a coin four times

**Sample space**  $\Omega = \{ \text{HHHH, HHHT, HHTH, HTHH,}$   
 $\text{THHH, HHTT, HTHT, THHT,}$   
 $\text{THTH, TTHH, HTTH, TTTH,}$   
 $\text{TTHT, THTT, HTTT, TTTT} \}$

## Sample space example 2

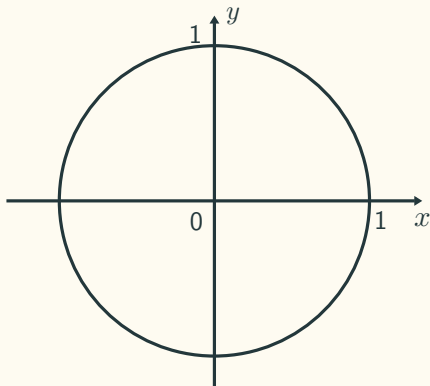
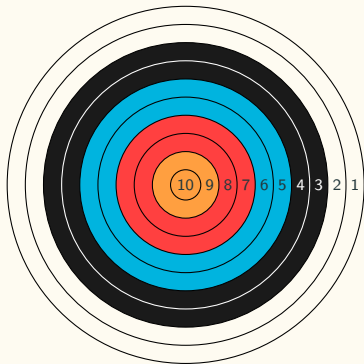
**Example 2:** an exam contained ten questions; each has 10 points; what is the total points you may get ?

**Sample space**  $\Omega = \{0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$

**Alternative sample space**  $\Omega = \{ 0, 10, 20, 30, 40, 50, 60, 70, 80, 90,$   
100 and you are using your lucky pen,  
100 and you are not using your lucky pen }

## Sample space example 3

**Example 3:** archery (positions on a target)



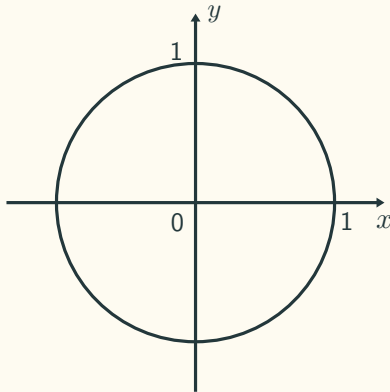
**Sample space**  $\Omega = \{ (x, y) \mid x^2 + y^2 \leq 1 \}$

## Assign probability to outcomes ... ?

Now, we can assign probability to individual outcomes ...

**Not exactly!**

What is the probability of hitting  $(0, 0)$ ?

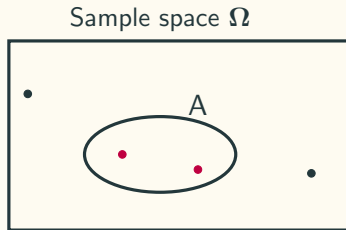




## Event

**An event ( $A, B, C, D, \text{etc.}$ ):** a subset of the sample space  $\Omega$

- Probabilities are assigned to events. The probability represents **our belief** on how likely we think **an event will occur**.
- Event  $A$  has occurred.  $\leftarrow$  what does this mean?



# Probability axioms

## FOUNDATIONS OF THE THEORY OF PROBABILITY

BY  
A. N. KOLMOGOROV

TRANSLATION EDITED BY  
NATHAN MORRISON

CHELSEA PUBLISHING COMPANY  
NEW YORK

1950

### § 1. Axioms<sup>2</sup>

Let  $\mathcal{E}$  be a collection of elements  $\xi, \eta, \zeta, \dots$ , which we shall call *elementary events*, and  $\mathfrak{F}$  a set of subsets of  $E$ ; the elements of the set  $\mathfrak{F}$  will be called *random events*.

- I.  $\mathfrak{F}$  is a field<sup>3</sup> of sets.
- II.  $\mathfrak{F}$  contains the set  $E$ .
- III. To each set  $A$  in  $\mathfrak{F}$  is assigned a non-negative real number  $P(A)$ . This number  $P(A)$  is called the *probability of the event  $A$* .
- IV.  $P(E)$  equals 1.
- V. If  $A$  and  $B$  have no element in common, then

$$P(A + B) = P(A) + P(B)$$

A system of sets,  $\mathfrak{F}$ , together with a definite assignment of numbers  $P(A)$ , satisfying Axioms I-V, is called a *field of probability*.

## The Kolmogorov Axioms

1. Nonnegativity:  $\mathbb{P}(A) \geq 0$
2. Normalisation:  $\mathbb{P}(\Omega) = 1$
3. Additivity: if  $A$  and  $B$  are disjoint ( $A \cap B = \emptyset$ ), then  $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

## Nice properties

- The probability of any event is always between 0 and 1.
- If  $A_1, A_2, A_3, \dots, A_n$  are disjoint, then

$$\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \dots + \mathbb{P}(A_n)$$

- $s_1, s_2, s_3, \dots, s_k$  are individual outcomes from the sample space, then

$$\begin{aligned}\mathbb{P}(\{s_1, s_2, s_3, \dots, s_k\}) &= \mathbb{P}(\{s_1\}) + \mathbb{P}(\{s_2\}) + \dots + \mathbb{P}(\{s_k\}) \\ &= \mathbb{P}(s_1) + \mathbb{P}(s_2) + \dots + \mathbb{P}(s_k) \leftarrow \text{abuse notation}\end{aligned}$$

## Probabilities as long-term relative frequencies

If an experiment is repeated  $n$  times under essentially the identical conditions, and if the event  $A$  occurs  $m$  times, then as  $n$  grows large, the ratio  $\frac{m}{n}$  approaches a fixed limit that is the probability of  $A$ :

$$\mathbb{P}(A) = \frac{m}{n}, \text{ where } n \text{ is large.}$$

## Probabilities as a measure of belief

- The probability that my current manuscript about RGCs will get published without revision is 1%.
- The probability that you will get a full score in BIO210 is 5%.
- The probability that it rains tomorrow is 80%.

# Assigning probability

**Experiment 1:** flipping a fair coin four times

**Sample space**  $\Omega = \{ \text{HHHH, HHHT, HHTH, HTHH,}$   
 $\text{THHH, HHTT, HTHT, THHT,}$   
 $\text{HTHT, TTHH, HTTH, TTTH,}$   
 $\text{TTHT, THTT, HTTT, TTTT} \}$

All possible outcomes are **equally likely**, so we can let every possible outcome have a probability of  $1/16$ .

Calculate the probabilities of the following events:

$A = \{\text{all heads or tails}\}$

$B = \{\text{exactly two head}\}$

$C = \{\text{at least two tails}\}$

## Discrete Uniform Law

Let all outcomes be equally likely, then

$$\mathbb{P}(A) = \frac{\text{number of elements of } A}{\text{total number of sample points}} = \frac{|A|}{|\Omega|}$$

Computing probability is essentially just counting!



## Continuous uniform law

**Experiment 2:** archery

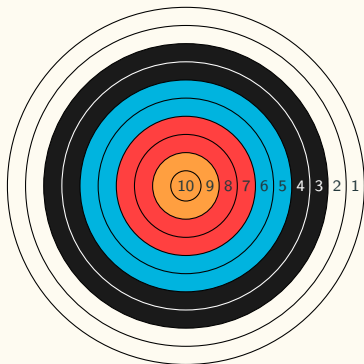
**Sample space**  $\Omega = \{ (x, y) \mid x^2 + y^2 \leq 1 \}$

**All possible outcomes are equally likely**, Then probability = the ratio of areas.

$A = \{\text{hitting the red area}\}$

$B = \{(x, y) \mid x + y \leq 1\}$

$C = \{(0, 1), (1, 0), (0, -1), (-1, 0)\}$



**Experiment 3:** keep flipping a fair coin until you obtain a head for the first time and stop.

**Sample space**  $\Omega = \{ H, TH, TTH, TTTH, TTTTH, \dots \}$

Let  $n$  be the number of flips,  $\mathbb{P}(n) = \frac{1}{2^n}$ ,  $n = 1, 2, 3, 4, \dots$

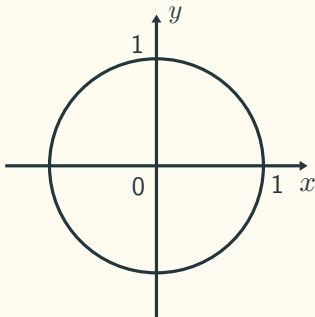
$A = \{ n \text{ is an even number} \}$ ,  $\mathbb{P}(A) = ?$

## Countable Additivity Axiom

If a sequence of events  $A_1, A_2, A_3, \dots$  are disjoint, then

$$\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \dots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) + \dots$$

# Countable additivity axiom



Sample space  $\Omega = \{ (x, y) \mid x^2 + y^2 \leq 1 \}$

## Paradox 1??

$$1 = \mathbb{P}(\Omega) = \mathbb{P}\left(\bigcup \{(x, y)\}\right) = \sum_{x, y} \mathbb{P}(\{(x, y)\}) = \sum_{x, y} 0 = 0$$

**Take-home message:**  $\{(x, y)\}$  is uncountable: it is not possible to list every single one of  $(x, y)$ .

## Paradox 2??

An experiment is performed, and the outcome is  $\left(\frac{1}{2}, \frac{1}{2}\right)$

**Take-home message:** probability of 0 does NOT mean impossible.