Lecture 4 Probability Axioms

BIO210 Biostatistics

Xi Chen

Spring, 2022

School of Life Sciences
Southern University of Science and Technology

南方科技大学生命科学学院
SUSTech · SCHOOL OF
LIFE SCIENCES

Probability

Probability theory is nothing but common sense reduced to calculation.

Laplace

Notations

Set

A set is a well-defined collection of distinct objects.

$S = \{$ list or description of the objects in the set $\}$

Definitions

Sample space (Ω)

Set of all possible outcomes

Outcomes: **mutually exclusive** and **collectively exhaustive**

Sample space example 1

Example 1: flipping a coin four times

Sample space $\Omega = \{$ HHHH, HHHT, HHTH, HTHH,
THHH, HHTT, HTHT, THHT,
THTH, TTHH, HTTH, TTTH,
TTHT, THTT, HTTT, TTTT $\}$

Sample space example 2

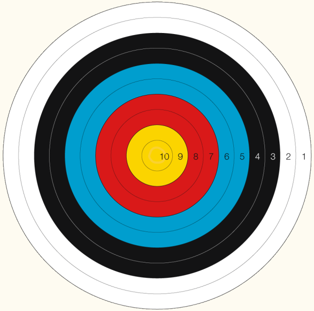
Example 2: an exam contained ten questions; each has 10 points; what is the total points you may get ?

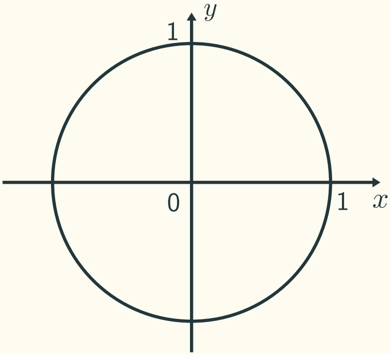
Sample space Ω $= \{0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100\}$

Alternative sample space Ω $= \{$ 0, 10, 20, 30, 40, 50, 60, 70, 80, 90,
100 and you are using your lucky pen,
100 and you are not using your lucky pen $\}$

Sample space example 3

Example 3: archery (positions on a target)





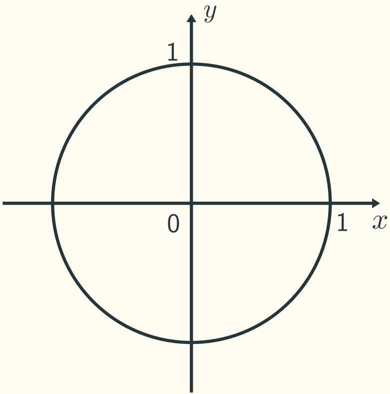
Sample space $\Omega = \{\ (x,\ y) \mid x^2 + y^2 \leqslant 1\ \}$

Assign probability to outcomes ... ?

Now, we can assign probability to individual outcomes ...

Not exactly!

What is the probability of hitting (0, 0)?

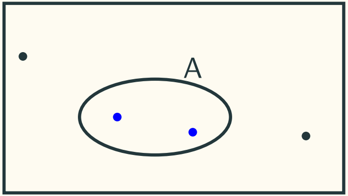


Event

An event (A, B, C, D, *etc.*): a subset of the sample space Ω

- Probabilities are assigned to events. The probability represents our belief on how likely we think an event will occur.
- Event A has occurred. $\leftarrow$ what does this mean?

Sample space Ω



Probability axioms

FOUNDATIONS

OF THE

THEORY OF PROBABILITY

BY

A. N. KOLMOGOROV

TRANSLATION EDITED BY

NATHAN MORRISON

CHELSEA PUBLISHING COMPANY
NEW YORK
1950

§ 1. Axioms[2]

Let E be a collection of elements $\xi, \eta, \zeta, \ldots$, which we shall call *elementary events*, and $\mathfrak{F}$ a set of subsets of E; the elements of the set $\mathfrak{F}$ will be called *random events*.

I. *$\mathfrak{F}$ is a field[3] of sets.*

II. *$\mathfrak{F}$ contains the set E.*

III. *To each set A in $\mathfrak{F}$ is assigned a non-negative real number $\mathsf{P}(A)$. This number $\mathsf{P}(A)$ is called the probability of the event A.*

IV. *$\mathsf{P}(E)$ equals 1.*

V. *If A and B have no element in common, then*

$$\mathsf{P}(A+B) = \mathsf{P}(A) + \mathsf{P}(B)$$

A system of sets, $\mathfrak{F}$, together with a definite assignment of numbers $\mathsf{P}(A)$, satisfying Axioms I-V, is called a *field of probability*.

Probability axioms

The Kolmogorov Axioms

1. Nonnegativity: $P(A) \geqslant 0$
2. Normalisation: $P(\mathbf{\Omega}) = 1$
3. Additivity: if A and B are distjoint $(A \cap B = \emptyset)$, then $P(A \cup B) = P(A) + P(B)$

Nice properties

- The probability of any event is always between 0 and 1.
- If A_1, A_2, A_3, $\cdots$, A_n are disjoint, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \cdots + P(A_n)$$

- s_1, s_2, s_3, $\cdots$, s_k are individual outcomes from the sample space, then

$$\begin{aligned} P(\{s_1,\ s_2,\ s_3,\ \cdots,\ s_k\}) &= P(\{s_1\}) + P(\{s_2\}) + \cdots + P(\{s_k\}) \\ &= P(s_1) + P(s_2) + \cdots + P(s_k) \leftarrow \text{abuse notation} \end{aligned}$$

Assigning probability

Experiment 1: flipping a fair coin four times

Sample space $\boldsymbol{\Omega}$ $= \{$ HHHH, HHHT, HHTH, HTHH,
THHH, HHTT, HTHT, THHT,
THTH, TTHH, HTTH, TTTH,
TTHT, THTT, HTTT, TTTT $\}$

All possible outcomes are **equally likely**, so we can let every possible outcome have a probability of $1/16$.

Calculate the probabilities of the following events:

$A = \{$all heads or tails$\}$
$B = \{$exactly two head$\}$
$C = \{$at least two tails$\}$

Discrete uniform law

Discrete Uniform Law

Let all outcomes be equally likely, then

$$P(A) = \frac{\text{number of elements of } A}{\text{total number of sample points}} = \frac{|A|}{|\Omega|}$$

Computing probability is essentially just counting!

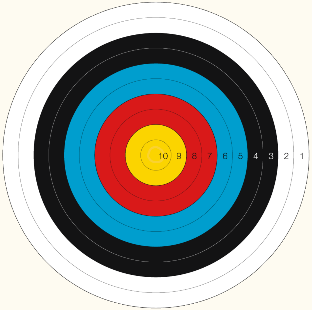
Continuous uniform law

Experiment 2: archery

Sample space $\boldsymbol{\Omega} = \{\ (x,\ y) \mid x^2 + y^2 \leqslant 1\ \}$

All possible outcomes are equally likely, Then probability = the ratio of areas.

$A = \{\text{hitting the red area}\},\ P(A) = ?$
$B = \{(x,\ y) \mid x + y \leqslant 1\},\ P(B) = ?$
$C = \{(0,\ 0)\},\ P(C) = ?$



Countable additive axiom

Experiment 3: keep flipping a fair coin until you obtain a head for the first time and stop.

Sample space $\boldsymbol{\Omega}$ $= \{$ H, TH, TTH, TTTH, TTTTH, $\cdots\}$

Let n be the number of flips, $P(n) = \dfrac{1}{2^n}$, $n = 1, 2, 3, 4, \cdots$

$A = \{\ n$ is an even number $\},\ P(A) = ?$

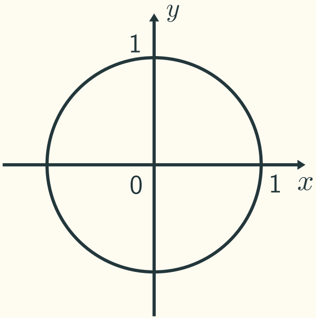
Countable additivity axiom

Countable Additivity Axiom

If a sequence of events $A_1, A_2, A_3, \cdots$ are disjoint, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

Countable additivivty axiom



Sample space $\boldsymbol{\Omega} = \{\ (x,\ y) \mid x^2 + y^2 \leqslant 1\ \}$

Paradox 1??

$$1 = P(\boldsymbol{\Omega}) = P\left(\bigcup \{(x,y)\}\right) = \sum_{x,y} P(\{(x,y)\}) = \sum_{x,y} 0 = 0$$

Take-home message: $\{(x,\ y)\}$ is uncountable: it is not possible to list every single one of $(x,\ y)$.

Paradox 2??

An experiment is performed, and the outcome is $\left(\frac{1}{2}, \frac{1}{2}\right)$

Take-home message: probability of 0 does NOT mean impossible.

Frequentist interpretation

The frequentist definition of probability

If an experiment is repeated n times under essentially the identical conditions, and if the event A occurs m times, then as n grows large, the ratio $\frac{m}{n}$ approaches a fixed limit that is the probability of A:

$$P(A) = \frac{m}{n}$$