Lecture 6 The Bayes' Theorem

BIO210 Biostatistics

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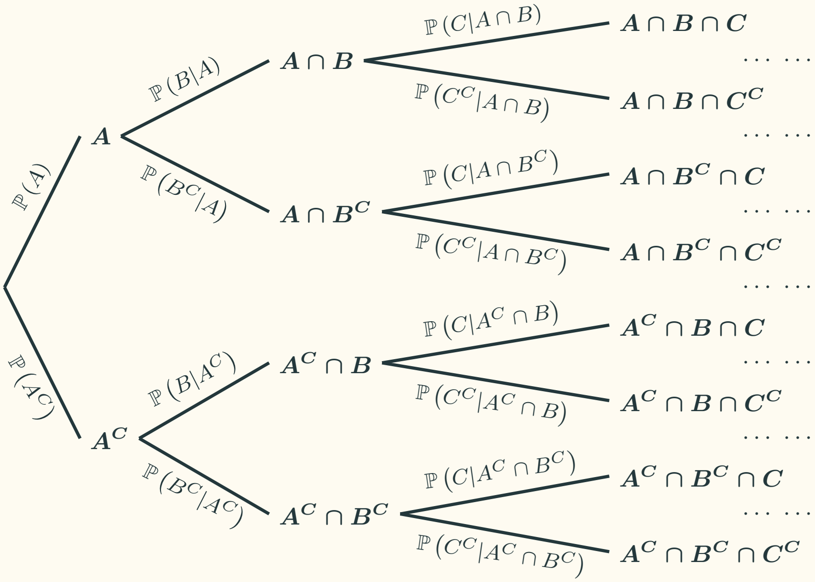
Basic components

Three basic components in conditional probability:

1. $\mathbb{P}\left(A \cap B\right)$
2. $\mathbb{P}\left(B\right)$
3. $\mathbb{P}\left(A|B\right)$

Generalisation of $\mathbb{P}(A \cap B \cap C \cap \cdots)$



The multiplication rule:

$$P\left(\bigcap_{i=1}^{n} A_i\right) =$$

$P(A_1)\cdot$

$P(A_2|A_1)\cdot$

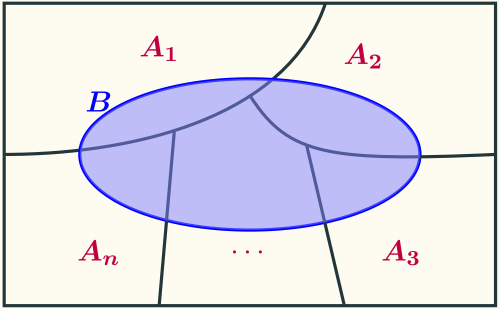
$P(A_3|A_1 \cap A_2)\cdot$

$P(A_4|A_1 \cap A_2 \cap A_3)\cdot$

$\cdots$

$$P\left(A_n|\bigcap_{i=1}^{n-1} A_i\right)$$

Generalisation of $\mathbb{P}(B)$

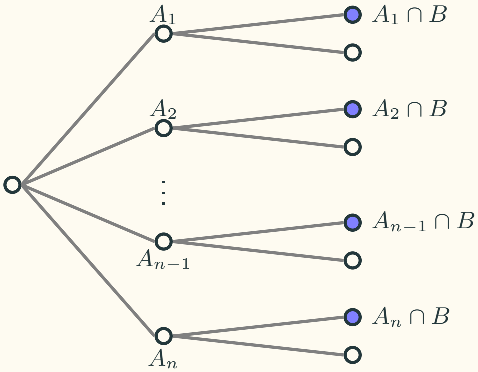


$$
\begin{aligned}
\mathbb{P}(B) =& \mathbb{P}[(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)] \\
=& \mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B) + \cdots + \mathbb{P}(A_n \cap B) \\
=& \mathbb{P}(A_1)\mathbb{P}(B|A_1) + \\
& \mathbb{P}(A_2)\mathbb{P}(B|A_2) + \\
& \vdots \\
& \mathbb{P}(A_n)\mathbb{P}(B|A_n)
\end{aligned}
$$

The total probability theorem:

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$

Generalisation of $\mathbb{P}(B)$



$$
\begin{aligned}
\mathbb{P}(B) =& \mathbb{P}[(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)] \\
=& \mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B) + \cdots + \mathbb{P}(A_n \cap B) \\
=& \mathbb{P}(A_1)\,\mathbb{P}(B|A_1) + \\
& \mathbb{P}(A_2)\,\mathbb{P}(B|A_2) + \\
& \vdots \\
& \mathbb{P}(A_n)\,\mathbb{P}(B|A_n)
\end{aligned}
$$

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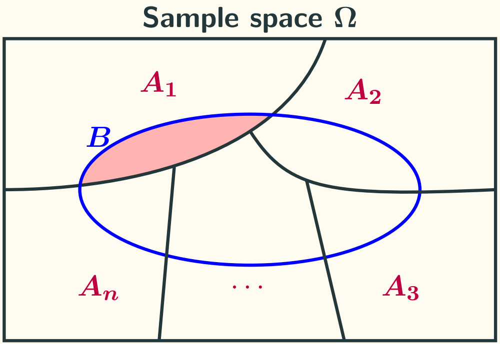
$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)\,\mathbb{P}(B|A_i)$$

Generalisation of $\mathbb{P}(A|B)$

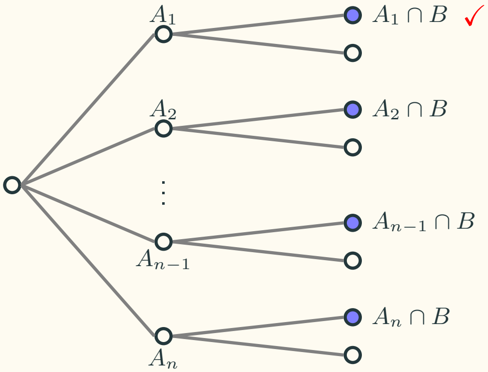


$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)}$$

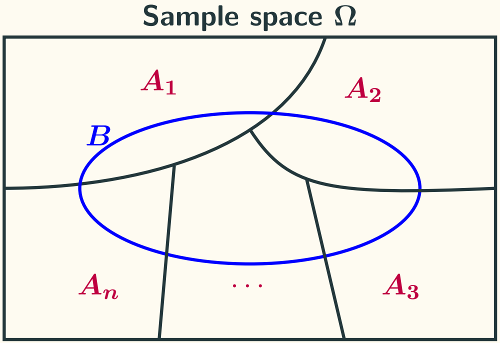
$$= \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\mathbb{P}(B)}$$

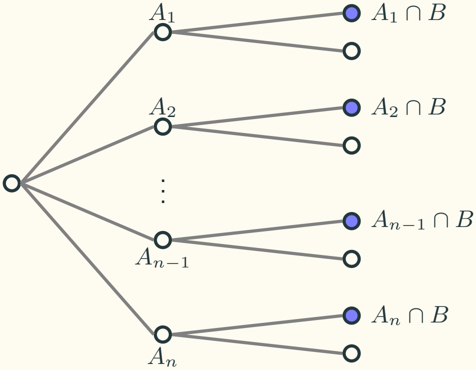
$$= \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\sum_{i=1}^{n} \mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}$$

Generalisation of $\mathbb{P}(A|B)$



$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\sum_{i=1}^{n} \mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}$$





$$A_i \underset{\text{inference } \mathbb{P}(A_i|B)}{\overset{\text{causal effect } \mathbb{P}(B|A_i)}{\rightleftarrows}} B$$

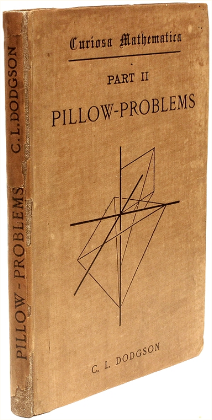
The Bayes' Theorem

$$\mathbb{P}\left(A_i|B\right) = \frac{\mathbb{P}\left(A_i \cap B\right)}{\mathbb{P}\left(B\right)} = \frac{\mathbb{P}\left(A_i\right)\mathbb{P}\left(B|A_i\right)}{\sum_{i=1}^{n}\mathbb{P}\left(A_i\right)\mathbb{P}\left(B|A_i\right)}$$

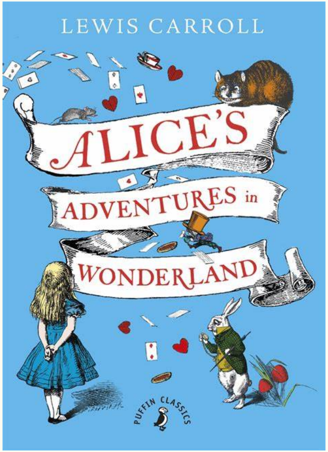
$\mathbb{P}\left(A_i\right)$: **prior probability**

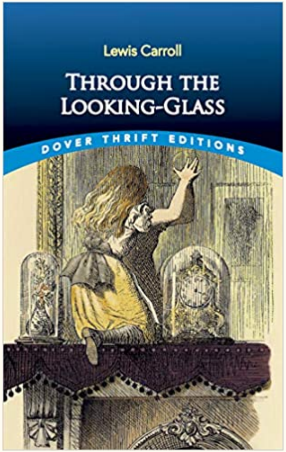
$\mathbb{P}\left(A_i|B\right)$: **posterior probability**

Lewis Carroll's Pillow Problems



by Charles Dodgson





Question #5 (8th Sep 1887):

A bag contains one counter, known to be either white or black. A white counter is put in, the bag shaken, and a counter drawn out, which proves to be white. What is now the chance of drawing a white counter?

Carroll's Pillow Problem #5

There is a ball inside a non-transparent bag. The colour of the ball is unknown, but it is equally likely to be either blue or red. Now you put a red ball into the bag, shake the bag, and take a ball without looking inside. The ball you have just taken out is red. What is the probability that the colour of the remaining ball that is still inside the bag is red?

Pedigree Analysis