Lecture 22 Confidence Interval For The Proportion

BIO210 Biostatistics

Xi Chen

Spring, 2022

School of Life Sciences
Southern University of Science and Technology

南方科技大学生命科学学院
SUSTech · SCHOOL OF
LIFE SCIENCES

Population Parameters We Have Learnt

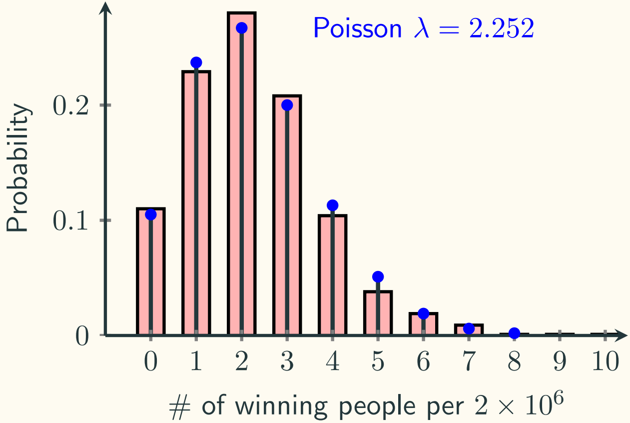
Population parameters	**Sample statistics**
μ	$\bar{x}$
σ^2	s^2
σ	s
π or p	p or $\hat{p}$

Sample Proportion Example

Lottery: We know the winning probability of *Daily Play* is $\pi = 0.000001126088083$. If we take a random sample of $n = 2,000,000$ people, what is the sampling distribution of proportion of the winning people?

$$\mathcal{N}(\mu = 1.12 \times 10^{-6},$$
$$\sigma^2 = 5.625 \times 10^{-13})?$$

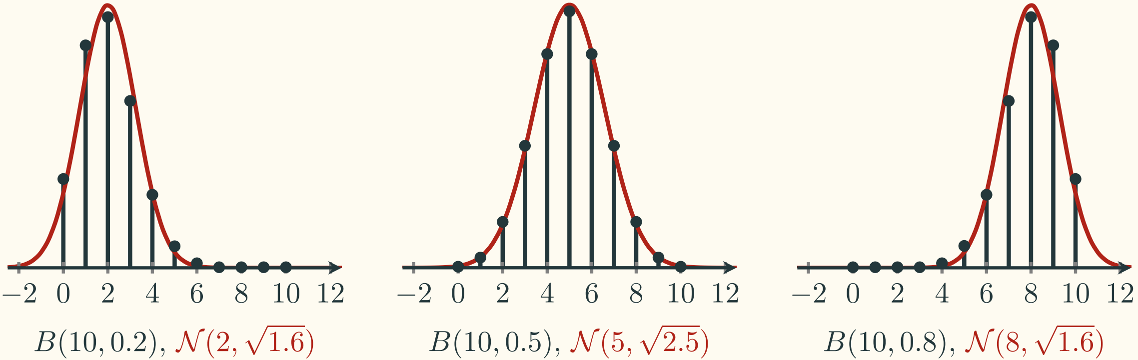
Results from 1,000 samples:
$(n = 2,000,000)$



Approximation of The Binomial Distribution

$$B(n,p)\begin{cases} \dot{\sim}\ \mathcal{N}(\mu = np, \sigma^2 = npq) & \text{, when } np \geqslant 10 \text{ and } nq \geqslant 10 \\ \\ \dot{\sim}\ Pois(\lambda = np) & \text{, when } n \text{ is large, and } p \text{ is small,} \\ & \text{ such that } np \text{ is between } 0 \text{ and } 10. \\ \\ \sim B(n,p) & \text{, otherwise} \end{cases}$$

The Limitations on np and nq



The Limitations on np and nq

- Binomial: all data are within $[0, n]$
- Normal: no bounds $(-\infty, \ +\infty)$ for data, but most are within $[\mu - 3\sigma, \ \mu + 3\sigma]$
- Intuitively: when $[\mu - 3\sigma, \ \mu + 3\sigma]$ is within $[0, n]$, the approximation works well!

$$\begin{aligned} \mu - 3\sigma &> 0 \\ np - 3\sqrt{npq} &> 0 \\ np &> 3\sqrt{npq} \\ n^2p^2 &> 9npq \\ np &> 9q \\ np &> 9(1-p) = 9 - 9p \end{aligned}$$

$$\begin{aligned} \mu + 3\sigma &< n \\ np + 3\sqrt{npq} &< n \\ n(1-p) &> 3\sqrt{npq} \\ n^2q^2 &> 9npq \\ nq &> 9p \\ nq &> 9(1-q) = 9 - 9q \end{aligned}$$

Interval Estimation For The Proportion

Goal: for a population containing an unknown proportion (π) of data of our interest, find a and b, such that $P(a \leqslant \pi \leqslant b) = 0.95$.

$$P(-1.96 \leqslant Z \leqslant 1.96) = 0.95$$

$$P\left(-1.96 \leqslant \frac{p - \mu_p}{\sigma_p} \leqslant 1.96\right) = 0.95$$

$$P\left(-1.96 \leqslant \frac{p - \pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} \leqslant 1.96\right) = 0.95$$

$$P\left(p - 1.96\sqrt{\frac{\pi(1-\pi)}{n}} \leqslant \pi \leqslant p + 1.96\sqrt{\frac{\pi(1-\pi)}{n}}\right) = 0.95$$

Confidence Interval For The Proportion

95% CI For The Sample Proportion

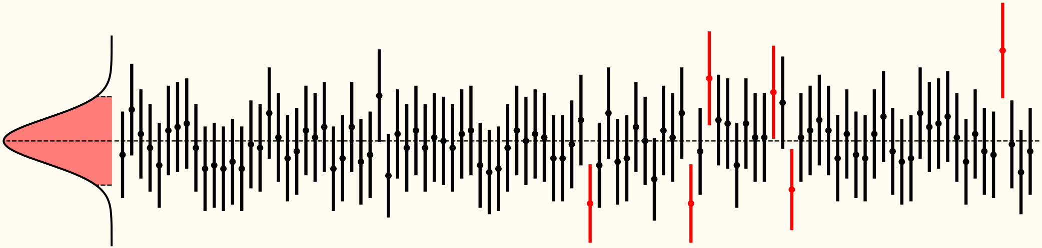
The Wald Interval:

$$\left[p - 1.96\sqrt{\frac{p(1-p)}{n}},\ p + 1.96\sqrt{\frac{p(1-p)}{n}}\ \right]$$

- **Not using t-distribution?** - You don't need to! Remember $\sigma_p = \sqrt{\frac{\pi(1-\pi)}{n}}$, and when p is calculated to estimate π, then σ_p is automatically determined, unlike in the situation of the mean, where you have to do extra calculation of s to estimate σ, which causes the extra error.

Simulation of 95% CI For The Proportion

100 95% CI for the proportion, constructed using the Wald interval



An Example in Lecture 1

Probability vs. Statistics

Probability: Previous studies showed that the drug was 80% effective. Then we can anticipate that for a study on 100 patients, on average 80 will be cured and at least 65 will be cured with 99.99% chance.

Statistics: We observe that 78/100 patients were cured by the drug. We will be able to conclude that we are 95% confident that for other studies the drug will be effective on between 69.88% and 86.11% of patients.

Sample Size Estimation Using Confidence Interval of The Proportion

Estimate Sample Size: We want to estimate the percentage of people cured by the drug. Suppose we could draw a truly random sample, and we want a 95% confidence interval estimation with a margin of error no more than $\pm 2\%$. What is the smallest sample size required to obtain the desired margin of error ?

$$95\% \text{ confidence interval: } p \pm 1.96\sqrt{\frac{p(1-p)}{n}}$$

Goal: find the smallest n such that it guarantees that $1.96\sqrt{\frac{p(1-p)}{n}} \leqslant 0.02$

Conditions For Interval Estimation For The Proportion

1. Random Samples

2. Normal Condition: the sampling distribution of p needs to be normal
    - $np \geqslant 10$
    - $nq \geqslant 10$

3. Independence ($n < 10\%$ population size)

Violation of The Conditions

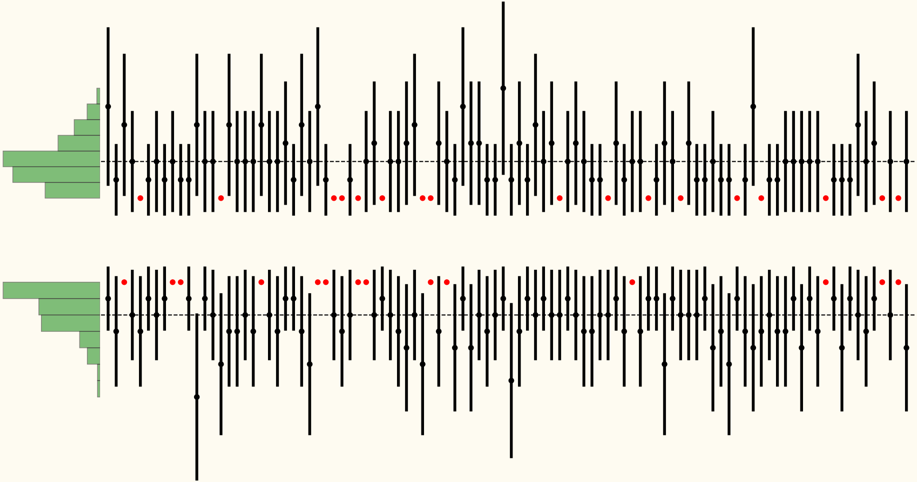
100 simulated 95% CI using the Wald interval

$n = 200$

$p = 0.01$

$n = 200$

$p = 0.99$



What to do when the normal condition is not met?

- Wilson score interval
- Jeffreys interval
- Agresti–Coull interval
- Arcsine transformation
- Clopper–Pearson interval (the exact method)