

The Maximum Likelihood Estimation For The Variance Is Biased

BIO210 Biostatistics

Extra Reading Material for Lecture 17

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The random variable \mathbf{X} denotes certain metric (*e.g.* height, weight) we are interested in from a population, and $\mathbf{X} \sim \mathcal{N}(\mu, \sigma^2)$. We draw a random sample of size n from the population. Like we discussed during the lecture, a random sample of size n can be thought as n **i.i.d.** random variables. That is:

$$\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \dots, \mathbf{X}_n \sim \mathcal{N}(\mu, \sigma^2)$$

We have seen that the maximum likelihood estimator for σ^2 is:

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (\mathbf{X}_i - \bar{\mathbf{X}})^2$$

Then, what is $\mathbb{E}[\hat{\sigma}^2]$? If $\mathbb{E}[\hat{\sigma}^2] = \sigma^2$, it is an unbiased estimator. Otherwise, it is a biased one.

Now let's have a look.

$$\begin{aligned} \mathbb{E}[\hat{\sigma}^2] &= \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n (\mathbf{X}_i - \bar{\mathbf{X}})^2\right] = \frac{1}{n} \mathbb{E}\left[\sum_{i=1}^n (\mathbf{X}_i^2 - 2\bar{\mathbf{X}}\mathbf{X}_i + \bar{\mathbf{X}}^2)\right] \\ &= \frac{1}{n} \mathbb{E}\left[\sum_{i=1}^n \mathbf{X}_i^2 - 2\bar{\mathbf{X}} \sum_{i=1}^n \mathbf{X}_i + \sum_{i=1}^n \bar{\mathbf{X}}^2\right] \end{aligned}$$

Note that: $\sum_{i=1}^n \mathbf{X}_i = n\bar{\mathbf{X}}$. Since $\bar{\mathbf{X}}$ remains the same for each i , we have $\sum_{i=1}^n \bar{\mathbf{X}}^2 = n\bar{\mathbf{X}}^2$. Replacing the blue terms above, we have:

$$\begin{aligned} \mathbb{E}[\hat{\sigma}^2] &= \frac{1}{n} \mathbb{E}\left[\sum_{i=1}^n \mathbf{X}_i^2 - 2\bar{\mathbf{X}} \cdot n\bar{\mathbf{X}} + n\bar{\mathbf{X}}^2\right] = \frac{1}{n} \mathbb{E}\left[\sum_{i=1}^n \mathbf{X}_i^2 - n\bar{\mathbf{X}}^2\right] \\ &= \frac{1}{n} \left(\mathbb{E}\left[\sum_{i=1}^n \mathbf{X}_i^2\right] - \mathbb{E}[n\bar{\mathbf{X}}^2] \right) \end{aligned} \tag{1}$$

Since $\text{Var}(\mathbf{X}) = \mathbb{E}[\mathbf{X}^2] - (\mathbb{E}[\mathbf{X}])^2$, so we have $\mathbb{E}[\mathbf{X}^2] = \text{Var}(\mathbf{X}) + (\mathbb{E}[\mathbf{X}])^2$,

then,

$$\begin{aligned}
 \mathbb{E} \left[\sum_{i=1}^n \mathbf{X}_i^2 \right] &= \mathbb{E} [\mathbf{X}_1^2] + \mathbb{E} [\mathbf{X}_2^2] + \mathbb{E} [\mathbf{X}_3^2] + \cdots + \mathbb{E} [\mathbf{X}_n^2] \\
 &= \text{Var}(\mathbf{X}_1) + (\mathbb{E}[\mathbf{X}_1])^2 + \text{Var}(\mathbf{X}_2) + (\mathbb{E}[\mathbf{X}_2])^2 + \cdots \\
 &\quad + \text{Var}(\mathbf{X}_n) + (\mathbb{E}[\mathbf{X}_n])^2 \\
 &= \sigma^2 + \mu^2 + \sigma^2 + \mu^2 + \cdots + \sigma^2 + \mu^2 \\
 &= n\sigma^2 + n\mu^2
 \end{aligned} \tag{2}$$

Putting equation (2) into equation (1), we have:

$$\begin{aligned}
 \mathbb{E} [\hat{\sigma}^2] &= \sigma^2 + \mu^2 - \frac{1}{n} \cdot \mathbb{E} [n\bar{\mathbf{X}}^2] = \sigma^2 + \mu^2 - \mathbb{E} [\bar{\mathbf{X}}^2] \\
 &= \sigma^2 + \mu^2 - (\sigma_{\bar{\mathbf{X}}}^2 + \mu_{\bar{\mathbf{X}}}^2)
 \end{aligned} \tag{3}$$

According to the central limit theorem, we have $\mu_{\bar{\mathbf{X}}} = \mu$ and $\sigma_{\bar{\mathbf{X}}}^2 = \frac{\sigma^2}{n}$. Therefore, equation (3) becomes:

$$\mathbb{E} [\hat{\sigma}^2] = \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2 = \frac{n-1}{n} \sigma^2 \neq \sigma^2$$

Hence, it is not an unbiased estimator.