Lecture 6 The Bayes' Theorem

BIO210 Biostatistics

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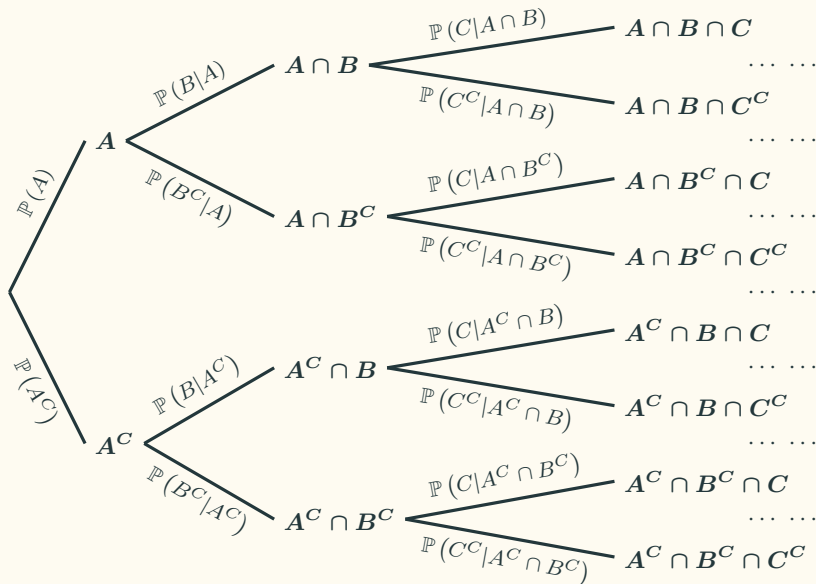
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Basic components

Three basic components in conditional probability:

1. $\mathbb{P}(A \cap B)$
2. $\mathbb{P}(B)$
3. $\mathbb{P}(A|B)$

Generalisation of $\mathbb{P}(A \cap B \cap C \cap \cdots)$

The multiplication rule:

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) =$$
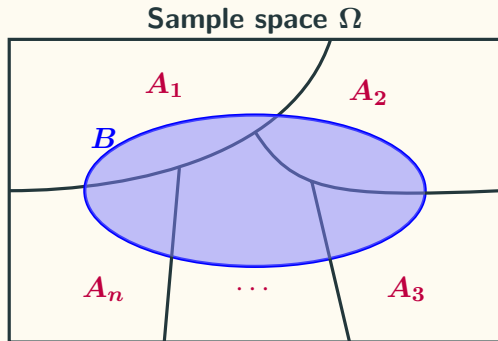
$\mathbb{P}(A_1) \cdot$

$\mathbb{P}(A_2|A_1) \cdot$

$\mathbb{P}(A_3|A_1 \cap A_2) \cdot$

$\mathbb{P}(A_4|A_1 \cap A_2 \cap A_3) \cdot$

$\cdots$

$$\mathbb{P}\left(A_n|\bigcap_{i=1}^{n-1} A_i\right)$$

Generalisation of $\mathbb{P}(B)$
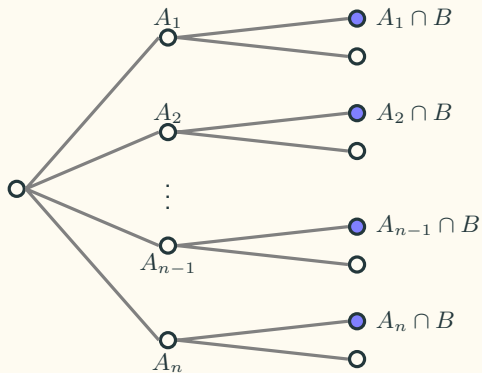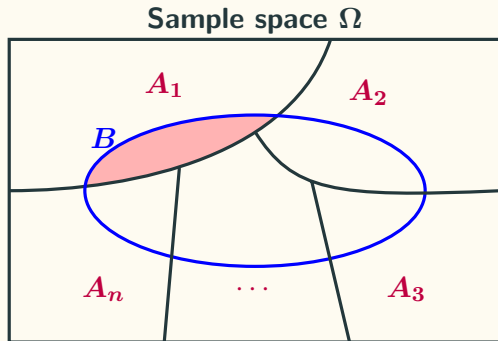
Sample space Ω

$$\begin{aligned} \mathbb{P}(B) =& \mathbb{P}[(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)] \\ =& \mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B) + \cdots + \mathbb{P}(A_n \cap B) \\ =& \mathbb{P}(A_1)\,\mathbb{P}(B|A_1) + \\ & \mathbb{P}(A_2)\,\mathbb{P}(B|A_2) + \\ & \vdots \\ & \mathbb{P}(A_n)\,\mathbb{P}(B|A_n) \end{aligned}$$

The total probability theorem:

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)\,\mathbb{P}(B|A_i)$$

Generalisation of $\mathbb{P}(B)$

$$\mathbb{P}(B) = \mathbb{P}[(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)]$$
$$= \mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B) + \cdots + \mathbb{P}(A_n \cap B)$$
$$= \mathbb{P}(A_1)\mathbb{P}(B|A_1) +$$
$$\mathbb{P}(A_2)\mathbb{P}(B|A_2) +$$
$$\vdots$$
$$\mathbb{P}(A_n)\mathbb{P}(B|A_n)$$

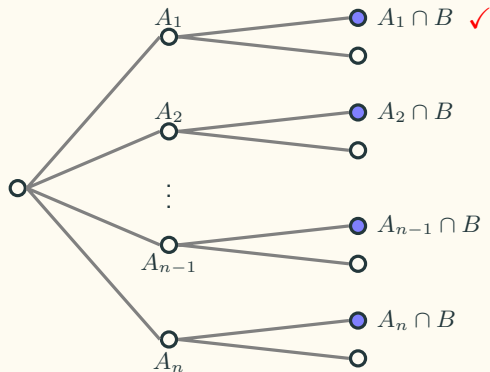**The total probability theorem:**

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$
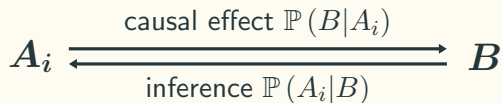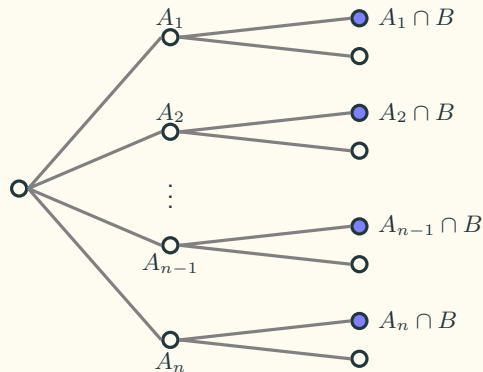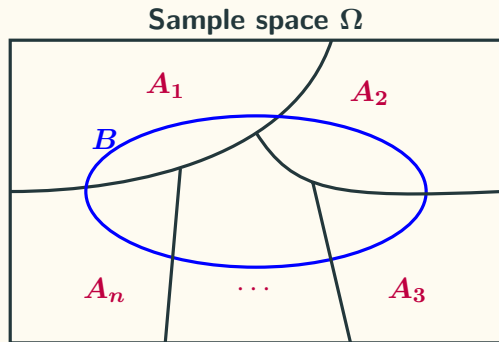
Generalisation of $\mathbb{P}(A|B)$

Sample space Ω

A_1 A_2

B

A_n $\cdots$ A_3

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)}$$

$$= \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\mathbb{P}(B)}$$

$$= \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\sum_{i=1}^{n}\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}$$

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)}$$

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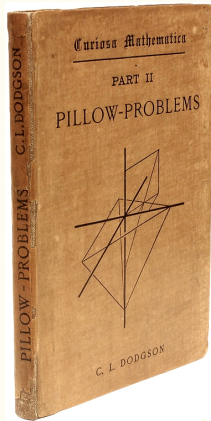
The Bayes Theorem

Sample space Ω

A_1 A_2

B

A_n $\cdots$ A_3

A_1 — $A_1 \cap B$

A_2 — $A_2 \cap B$

A_{n-1} — $A_{n-1} \cap B$

A_n — $A_n \cap B$

$$A_i \underset{\text{inference } \mathbb{P}\left(A_i|B\right)}{\overset{\text{causal effect } \mathbb{P}\left(B|A_i\right)}{\rightleftarrows}} B$$

The Bayes' Theorem

$$\mathbb{P}\left(A_i|B\right) = \frac{\mathbb{P}\left(A_i \cap B\right)}{\mathbb{P}\left(B\right)} = \frac{\mathbb{P}\left(A_i\right)\mathbb{P}\left(B|A_i\right)}{\sum_{i=1}^{n}\mathbb{P}\left(A_i\right)\mathbb{P}\left(B|A_i\right)}$$
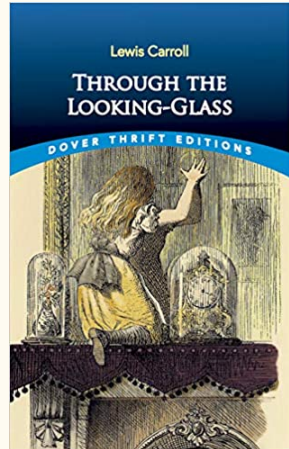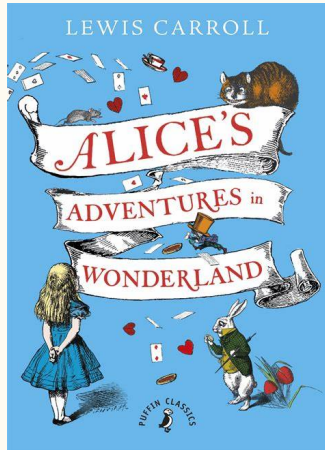
$\mathbb{P}\left(A_i\right)$: **prior probability**

$\mathbb{P}\left(A_i|B\right)$: **posterior probability**

Lewis Carroll's Pillow Problems

by Charles Dodgson

Question #5 (8th Sep 1887):

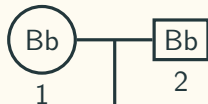**5.** (19, 31)

A bag contains one counter, known to be either white or black. A white counter is put in, the bag shaken, and a counter drawn out, which proves to be white. What is now the chance of drawing a white counter? [8/9/87
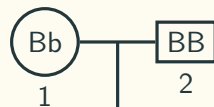
Pedigree Analysis