Lecture 15 Sampling Distribution And The Central Limit Theorem

BIO210 Biostatistics

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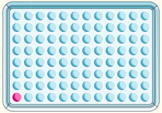
Use Sample Statistics To Estimate Population Parameters

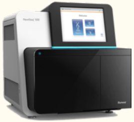
$$\text{Sample(s)} \xrightarrow[\text{Sampling distribution}]{?} \text{Population}$$

The scenario: we draw a sample of size n from the population. We observe the sample mean is $\bar{x}$ and the sample variance is s^2. **We want to answer the following type of questions:**

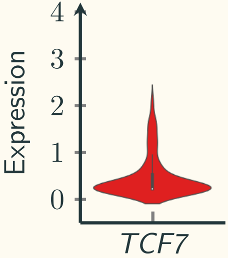
- **If the population mean were μ_0:**
  - what would be the probability of observing a sample of size n with a mean of $\bar{x}$?
  - what would be the probability of observing a sample of size n with a mean falling into $[a, b]$?
- **If the population variance were σ_0^2:**
  - what would be the probability of observing a sample of size n with a variance of s^2?
  - what would be the probability of observing a sample of size n with a variance of $[a, b]$?

Intuition of Sampling Distribution

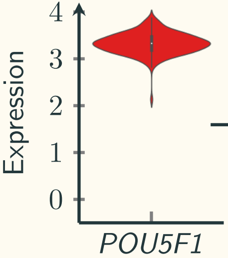




- Experiment_name
  - plate_1
    - cell_1
    - cell_2
    - cell_3
    - ⋮
  - plate_2
    - cell_1
    - cell_2
    - cell_3
    - ⋮

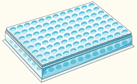


→ Not T cells

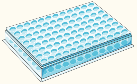


→ Probably ES cells

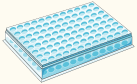
100 plates (samples)



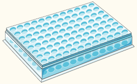
$n = 96$
$\bar{x} = 3.07$



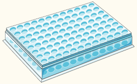
$n = 96$
$\bar{x} = 2.97$



$n = 96$
$\bar{x} = 2.80$



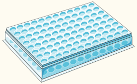
$n = 96$
$\bar{x} = 4.95$



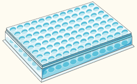
$n = 96$
$\bar{x} = 3.60$

$\vdots$

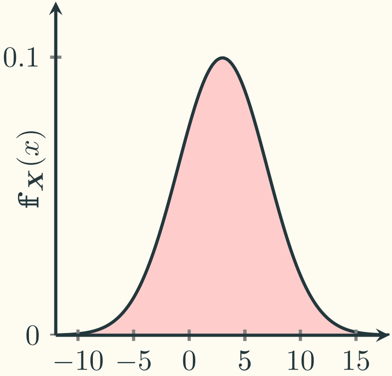
$\vdots$



$n = 96$
$\bar{x} = 1.05$

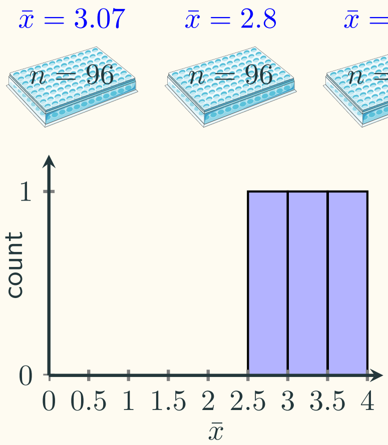


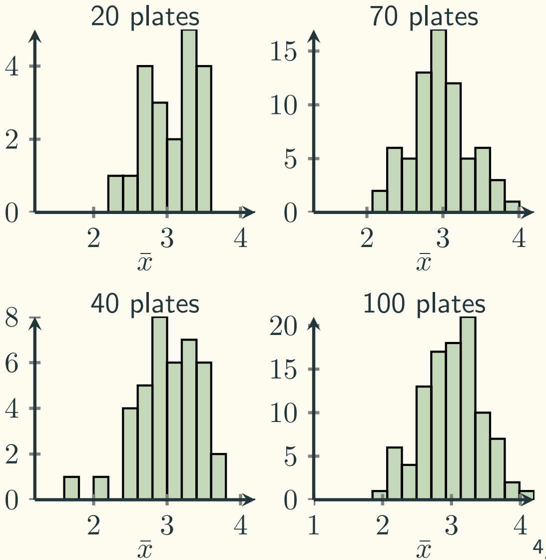
$n = 96$
$\bar{x} = 3.27$



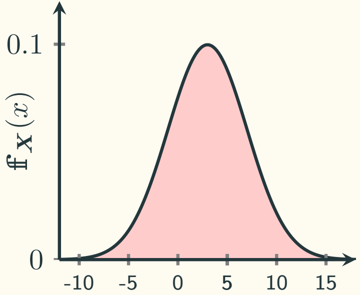
POU5F1 $\sim \mathcal{N}(\mu = 3, \sigma^2 = 4^2)$

Intuition of Sampling Distribution



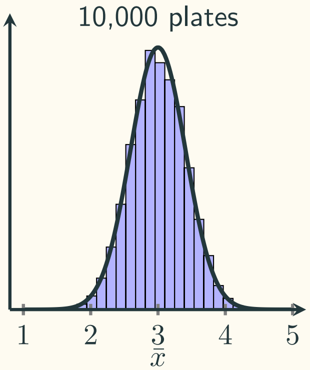


Sampling Distribution of The Sample Mean



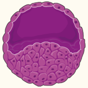
POU5F1 $\sim \mathcal{N}(\mu = 3, \sigma^2 = 4^2)$

Sample		Sample mean
$n = 96$	$\longmapsto$	$\bar{x} = 3.05$
$n = 96$	$\longmapsto$	$\bar{x} = 3.13$
$n = 96$	$\longmapsto$	$\bar{x} = 3.15$
$n = 96$	$\longmapsto$	$\bar{x} = 2.95$
$n = 96$	$\longmapsto$	$\bar{x} = 2.87$
$n = 96$	$\longmapsto$	$\bar{x} = 3.20$
$\vdots$	$\vdots$	$\vdots$
$n = 96$	$\longmapsto$	$\bar{x} = 2.50$
$\vdots$	$\vdots$	$\vdots$



$\bar{X} \sim$

Sampling distribution of the sample mean



Gene A expression $\sim \mathcal{D}(\mu, \sigma^2)$

$\boldsymbol{X}_1$	$\boldsymbol{X}_2$	$\boldsymbol{X}_3$	$\cdots$	$\cdots$	$\boldsymbol{X}_{n-1}$	$\boldsymbol{X}_n$	$\bar{\boldsymbol{X}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i$	
x_1	x_2	x_3			x_{n-1}	x_n	$\bar{x}$	
3.0	3.5	3.3	$\cdots$	$\cdots$	4.2	4.0	3.0	sample

$\boldsymbol{X}_1, \boldsymbol{X}_2, ..., \boldsymbol{X}_n$ are independent and identically distributed (**i.i.d.**) random variables.

$$\boldsymbol{X}_1 \sim \mathcal{D}(\mu, \sigma^2)$$
$$\boldsymbol{X}_2 \sim \mathcal{D}(\mu, \sigma^2)$$
$$\boldsymbol{X}_3 \sim \mathcal{D}(\mu, \sigma^2)$$
$$\vdots$$
$$\boldsymbol{X}_{n-1} \sim \mathcal{D}(\mu, \sigma^2)$$
$$\boldsymbol{X}_n \sim \mathcal{D}(\mu, \sigma^2)$$

$$\bar{\boldsymbol{X}} \sim ?(?, ?)$$

The Central Limit Theorem

By Pierre Simon de Laplace in 1810.

Theorem

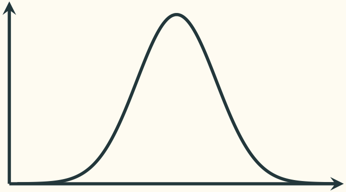
The sampling distribution of the sample mean of n independent and identically distributed (i.i.d.) random variables is approximately normal, even if original variables themselves are not normally distributed, provided that n is large enough.

$$\bar{\boldsymbol{X}} \dot{\sim} \mathcal{N}(\mu_{\bar{\boldsymbol{X}}}, \sigma^2_{\bar{\boldsymbol{X}}}), \text{ where } \mu_{\bar{\boldsymbol{X}}} = \mu, \sigma^2_{\bar{\boldsymbol{X}}} = \frac{\sigma^2}{n}$$

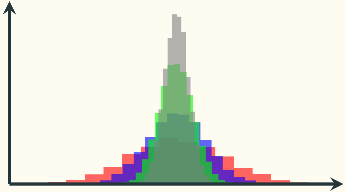
$\sigma_{\bar{\boldsymbol{X}}} = \dfrac{\sigma}{\sqrt{n}}$: **standard error**.

The Central Limit Theorem

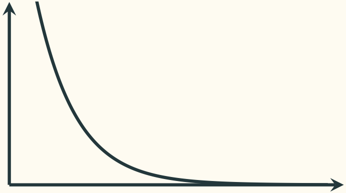
Population 1



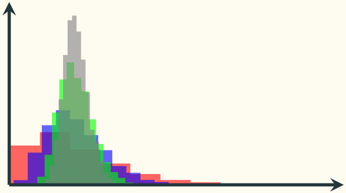
Samples from Pop. 1



Population 2



Samples from Pop. 2



$n = 2$

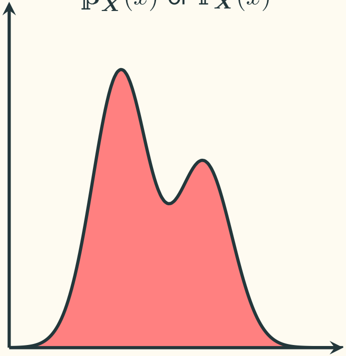
$n = 5$

$n = 15$

$n = 30$

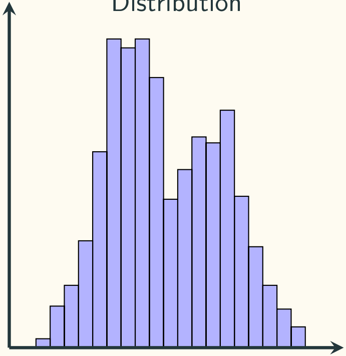
Three Distributions

Population Distribution
$\mathbb{p}_{\boldsymbol{X}}(x)$ or $\mathbb{f}_{\boldsymbol{X}}(x)$



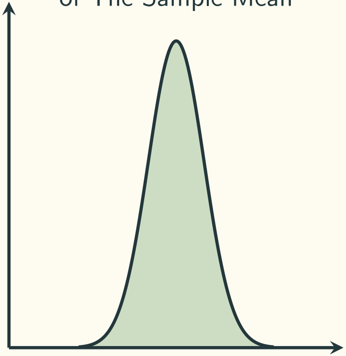
Population parameters
$\mathbb{E}\left[\boldsymbol{X}\right] = \mu,\ \mathbb{V}\mathrm{ar}\left(\boldsymbol{X}\right) = \sigma^2$

Sample
Distribution



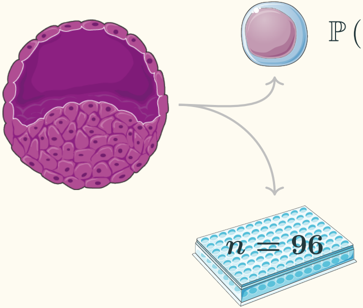
Sample statistics
$\bar{x},\ s^2$

Sampling Distribution
of The Sample Mean



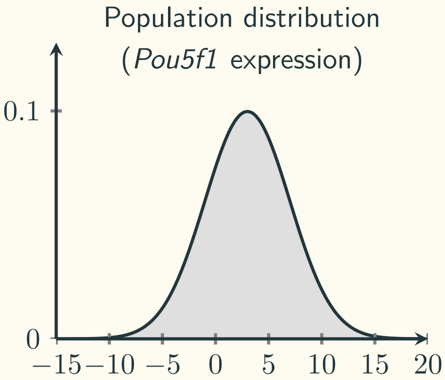
Parameters
$\mathbb{E}\left[\bar{\boldsymbol{X}}\right] = \mu_{\bar{\boldsymbol{X}}},\ \mathbb{V}\mathrm{ar}\left(\bar{\boldsymbol{X}}\right) = \sigma^2_{\bar{\boldsymbol{X}}}$

Based on the previous research, the expression of $Pou5f1$ in all ES cells follow a normal distribution with $\mu = 3$ and $\sigma^2 = 4^2$.

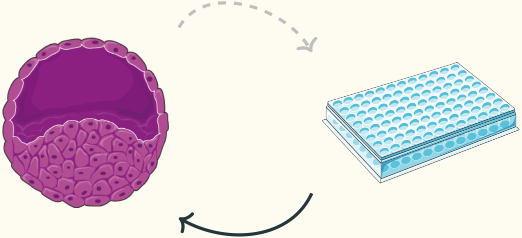


$\mathbb{P}\,(x \geqslant 8)$?

$\mathbb{P}\,(\bar{x} \geqslant 3.8)$?



Estimation



Use info. from the sample
to do a point estimation

Population parameter
μ, σ^2

Sample statistics
$\bar{x}, s^2$

- **Estimator**

$$\bar{\boldsymbol{X}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i$$

$$\boldsymbol{S}^2 = \frac{1}{n-1}\sum_{i=1}^{n} (\boldsymbol{X}_i - \bar{\boldsymbol{X}})^2$$

- **Estimate**

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

Unbiased Estimator

We say the following estimators are unbiased estimators:

$$\bar{\boldsymbol{X}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i$$

$$\boldsymbol{S}^2 = \frac{1}{n-1}\sum_{i=1}^{n} (\boldsymbol{X}_i - \bar{\boldsymbol{X}})^2$$

Because:

$$\mathbb{E}\left[\bar{\boldsymbol{X}}\right] = \mu$$

$$\mathbb{E}\left[\boldsymbol{S}^2\right] = \sigma^2$$