Lecture 10 Random Variables, PMF, Expectation & Variance

BIO210 Biostatistics

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Spring, 2023

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Random Variable

What is a random variable (r.v.) ?

- An assignment of a value (a real number) to every possible outcome in the sample space.
- **Mathematically**: A real-valued **function** defined on a sample space Ω. In a particular experiment, a random variable (**r.v.**) would be some function that assigns a real number to each possible outcome.

Random Variable

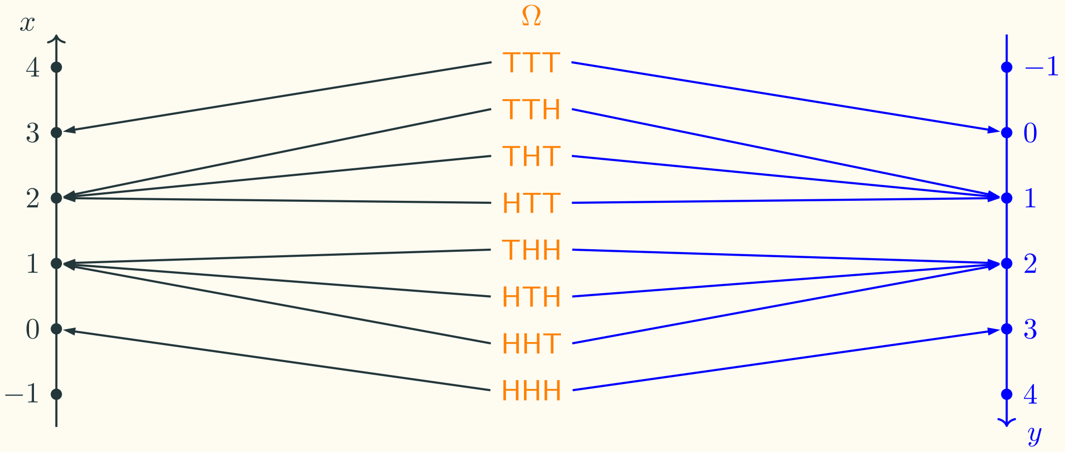
More about random variables

- Discrete or continuous.
- Can have several random variables defined on the same sample space.
- Notation
  - random variable X : function $\Omega \mapsto \mathbb{R}$
  - numerical value: x : value $\in \mathbb{R}$

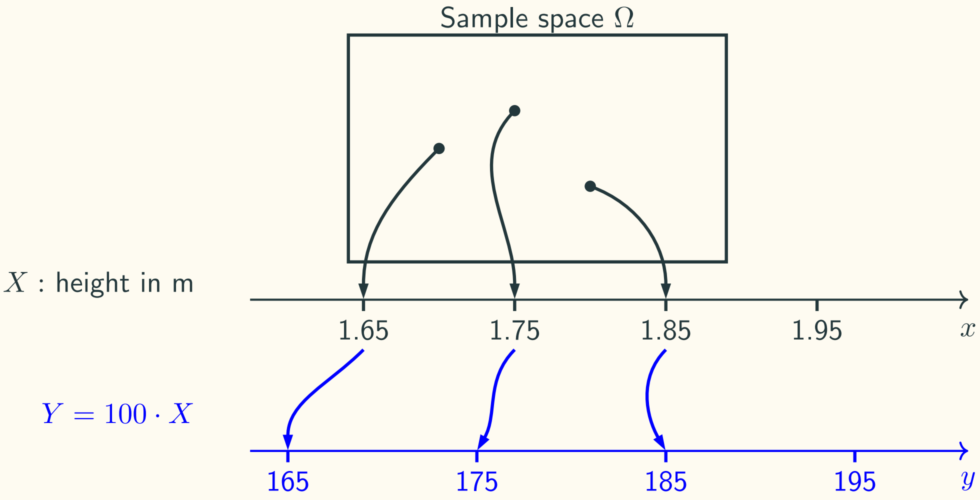
Different random variables on the same sample space

X : number tails

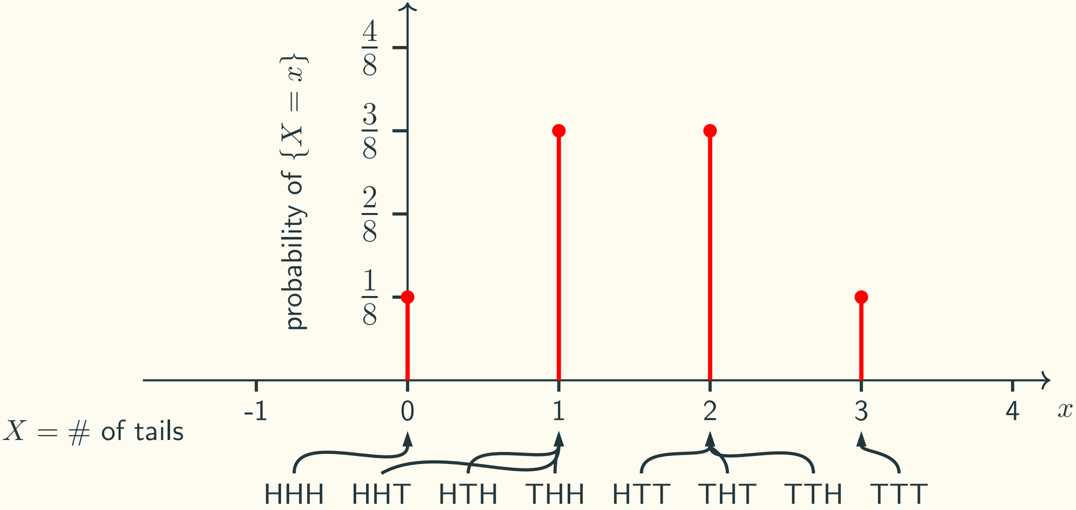
Y : number heads



Function of a random variable is an r.v.



Probability Mass Function (PMF)



Probability Mass Function (PMF)

The PMF of $X =$ number of tails after three flips

x	$\mathbb{P}(\{X = x\})$
0	$1/8$
1	$3/8$
2	$3/8$
3	$1/8$
otherwise	0

$$\mathbb{P}(\{X = x\}) = \begin{cases} \frac{1}{8}, & x = 0, 3 \\ \frac{3}{8}, & x = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

PMF Notation

Probability Mass Function

- Notation

$$\begin{aligned} p_X(x) &= \mathbb{P}(\{X = x\}) \\ &= \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\}) \end{aligned}$$

- Properties

$$p_X(x) \geqslant 0$$
$$\sum_x p_X(x) = 1$$

ω	$X(\omega) = x$	$p_X(x) = \mathbb{P}(\{X = x\})$
HHH	0	$\frac{1}{8}$
THH, HTH, HHT	1	$\frac{3}{8}$
TTH, THT, THH	2	$\frac{3}{8}$
TTT	3	$\frac{1}{8}$

Geometric PMF

Experiment: keep flipping a coin ($\mathbb{P}(H) = p$) until a head comes up for the first time. Let the random variable X be the number of flips.

ω	$X(\omega)$	$p_X(x)$
H	1	p
TH	2	$(1-p)p$
TTH	3	$(1-p)^2 p$
$\vdots$	$\vdots$	$\vdots$
$\underbrace{\text{TTT...TTT}}_{n-1}\text{H}$	n	$(1-p)^{n-1}p$

Geometric PMF. X: geometric random variable.

How to compute a PMF $p_X(x)$

To compute a PMF $p_X(x)$:

1. Collect all possible outcomes for which $X = x$;
2. add their probabilities;
3. repeat for all x.

Compute PMF

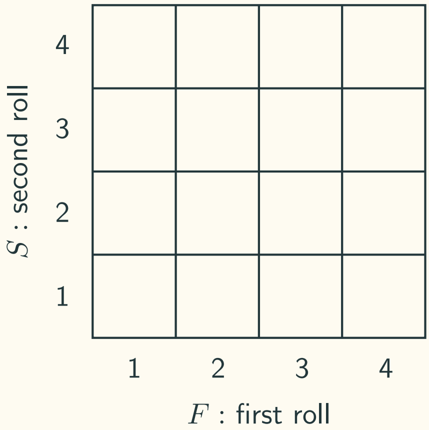
Experiment: two independent rolls of a fair tetrahedral die.

F: outcome of the first roll

S: outcome of the second roll

$X = min(F, S)$

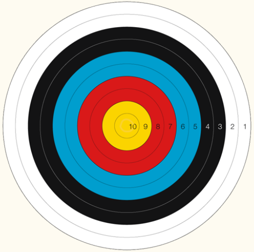
$p_X(x) = ?$



Expected value of a random variable (Expectation)

Experiment: archery

Let X be the score you get for each shot. What is the expected value of X ?



x	$p_X(x)$
1	0.19
2	0.17
3	0.15
4	0.13
5	0.11
6	0.09
7	0.07
8	0.05
9	0.03
10	0.01

Think: What is the average score you will get after a large number of trials?

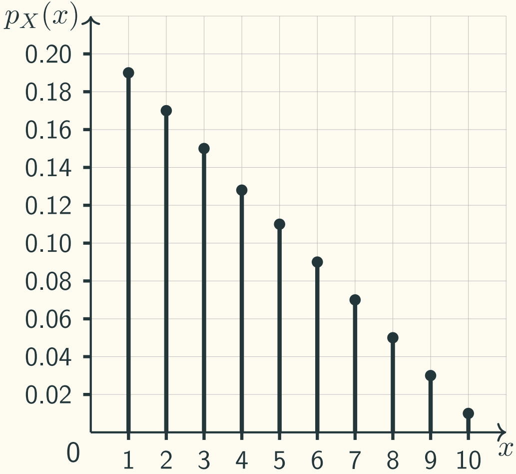
Expected value (Expectation)

Definition

$$\mathbb{E}[X] = \sum_x x p_X(x)$$

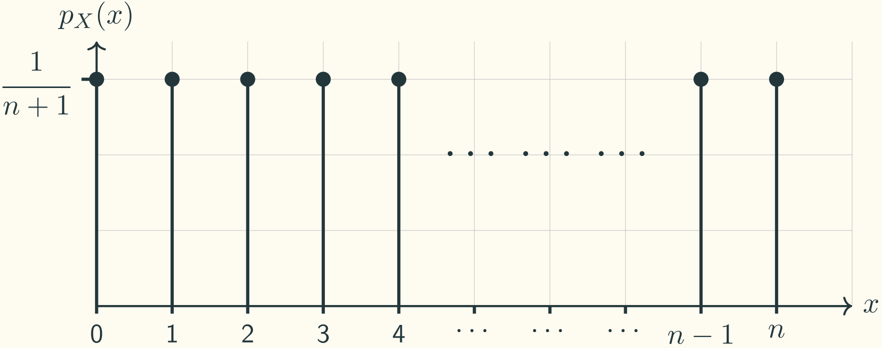
- Interpretation
    1. Centre of gravity of the PMF
    2. Average in large number of repetitions of the experiment

PMF of X from the archery experiment



Expectation of a Uniform Distribution

Example: a uniform discrete random variable X on $0, 1, 2, 3, ..., n$



What is $\mathbb{E}[X]$?

Properties of expectations

Let X be a random variable, and let $Y = g(X)$, what is $\mathbb{E}[Y]$?

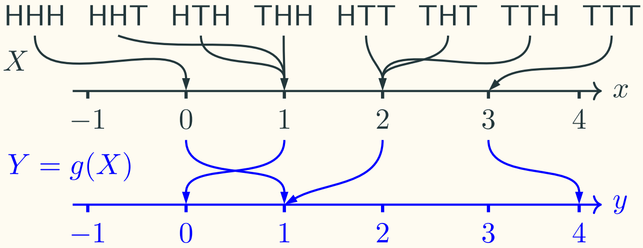
- The hard way:

$$\mathbb{E}[Y] = \sum_y y p_Y(y)$$

- The easy way:

$$\mathbb{E}[Y] = \sum_x g(x) p_X(x)$$

y	$p_Y(y)$
0	3/8
1	4/8
4	1/8



x	$g(x)$	$p_X(x)$
0	1	1/8
1	0	3/8
2	1	3/8
3	4	1/8

Expectation of a linear function of r.v.

- Caution: in general $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$
- Exception: if α, β are constants, then we have:

  - $\mathbb{E}[\alpha] = \alpha$
  - $\mathbb{E}[\alpha X] = \alpha \mathbb{E}[X]$
  - $\mathbb{E}[\alpha X + \beta] = \alpha \mathbb{E}[X] + \beta$

Variance and standard deviation of a random variable

Definition of Variance

$$\mathbb{V}\mathrm{ar}\left(X\right) = \mathbb{E}\left[\left(X - \mathbb{E}\left[X\right]\right)^2\right]$$

Properties of Variance

- $\mathbb{V}\mathrm{ar}\left(X\right) = \mathbb{E}\left[X^2\right] - \left(\mathbb{E}\left[X\right]\right)^2$
- If α, β are constants, then $\mathbb{V}\mathrm{ar}\left(\alpha X + \beta\right) = \alpha^2 \mathbb{V}\mathrm{ar}\left(X\right)$

Definition of Standard Deviation

$$\sigma_X = \sqrt{\mathbb{V}\mathrm{ar}\left(X\right)}$$

Random Variables (Summary slide)