Lecture 16 Sampling Distribution of The Sample Variance

BIO210 Biostatistics

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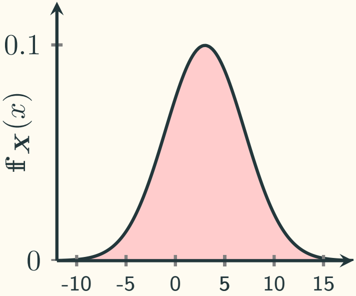
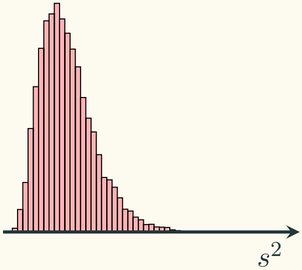
Sampling Distribution of The Sample Variance



POU5F1 $\sim \mathcal{N}(\mu = 3, \sigma^2 = 4^2)$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Sample		Sample mean
$n = 96$	$\rightarrow$	$s^2 = 15.11$
$n = 96$	$\rightarrow$	$s^2 = 15.75$
$n = 96$	$\rightarrow$	$s^2 = 17.44$
$n = 96$	$\rightarrow$	$s^2 = 16.85$
$n = 96$	$\rightarrow$	$s^2 = 13.31$
$n = 96$	$\rightarrow$	$s^2 = 16.36$
$\vdots$	$\vdots$	$\vdots$
$n = 96$	$\rightarrow$	$s^2 = 17.29$
$\vdots$	$\vdots$	$\vdots$



$S^2 \sim$?

Sampling distribution of the sample variance

Start With The Special Case

Task: We draw a sample of size n $(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n)$ from a population $(\boldsymbol{X} \sim \boldsymbol{\mathcal{D}})$, where $\mathbb{V}\mathrm{ar}\,(\boldsymbol{X}) = \sigma^2$, we want to figure out:

$$\boldsymbol{S}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\boldsymbol{X}_i - \bar{\boldsymbol{X}})^2 = \frac{1}{n-1}\Big[(\boldsymbol{X}_1 - \bar{\boldsymbol{X}})^2 + (\boldsymbol{X}_2 - \bar{\boldsymbol{X}})^2 + \cdots + (\boldsymbol{X}_n - \bar{\boldsymbol{X}})^2\Big]$$

Simplify: Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ be **i.i.d.** random variables from a normal population $\mathcal{N}(\mu, \sigma^2)$

$$\boldsymbol{S}^2 = \frac{1}{n-1}\Big[(\boldsymbol{X}_1 - \bar{\boldsymbol{X}})^2 + (\boldsymbol{X}_2 - \bar{\boldsymbol{X}})^2 + \cdots + (\boldsymbol{X}_n - \bar{\boldsymbol{X}})^2\Big]$$

The question becomes: what is the sum of a bunch of squared normal random variables?

The Standard Normal Squared

Let $\boldsymbol{Z}_1, \boldsymbol{Z}_2, \boldsymbol{Z}_3, \cdots, \boldsymbol{Z}_n$ be **i.i.d.** standard normal random variables: $\boldsymbol{Z}_i \sim \mathcal{N}(0, 1)$, then

- $\boldsymbol{Z}_1^2 \sim ?$
- $\boldsymbol{Z}_1^2 + \boldsymbol{Z}_2^2 \sim ?$
- $\vdots$
- $\sum_{i=1}^{n} \boldsymbol{Z}_i^2 \sim ?$

The Chi-squared (χ^2) Distribution

Friedrich Robert Helmert in 1876:

Number of Z_i^2	The PDF of the sum
1	$\frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} e^{-\frac{x}{2}} \, : \, \chi^2(1)$
2	$\frac{1}{2} e^{-\frac{x}{2}} \, : \, \chi^2(2)$
3	$\frac{1}{\sqrt{2\pi}} x^{\frac{1}{2}} e^{-\frac{x}{2}} \, : \, \chi^2(3)$
4	$\frac{1}{4} x e^{-\frac{x}{2}} \, : \, \chi^2(4)$
5	$\frac{1}{3\sqrt{2\pi}} x^{\frac{3}{2}} e^{-\frac{x}{2}} \, : \, \chi^2(5)$
$\vdots$	$\vdots$

by induction:

$$\chi^2(n) : \, \mathbb{f}_{\boldsymbol{X}}(x) = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, x \geqslant 0$$

where:

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} \mathrm{d}t, \;\; \alpha > 0$$

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

$\Gamma(k) = (k-1)!$, when k is an integer

One parameter - the degree of freedom:

the number of independent $\boldsymbol{Z}^2$ in the sum

The Distribution of S^2

By definition:

$$\sum_{i=1}^{n} \left(\frac{\boldsymbol{X}_i - \mu}{\sigma} \right)^2 \sim \chi^2(n)$$

Replacing μ with $\bar{X}$:

$$\sum_{i=1}^{n} \left(\frac{\boldsymbol{X}_i - \bar{\boldsymbol{X}}}{\sigma} \right)^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} (\boldsymbol{X}_i - \bar{\boldsymbol{X}})^2 \sim \chi^2(n-1)$$

Manipulate to get the sample variance:

$$\frac{(n-1)\boldsymbol{S}^2}{\sigma^2} \sim \chi^2(n-1)$$

Why $n-1$? part 1

$$\frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{X}_i - \bar{\boldsymbol{X}})^2 \leqslant \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{X}_i - \mu)^2$$

Why? Because:

$$\sum_{i=1}^{n}(x_i - m)^2 = n \cdot m^2 - \left(2\sum_{i=1}^{n} x_i\right) \cdot m + \sum_{i=1}^{n} x_i^2$$

But why exactly $n-1$? Wait until **part 2** in **Lecture 18**

The Degree of Freedom (DF, DOF, ν)

Typical definition: the number of values in the final calculation of a statistic that are free to vary; the number of independent pieces of information used to calculate the statistic.

There are two types of degrees of freedom:

$$\begin{cases} df \text{ of the data} & \text{- } df \text{ left (statistical cash)} \\ df \text{ of the statistical model} & \text{- } df \text{ spent (buy with cash)} \end{cases}$$

A statistical model: a mathematical process that attempts to describe the sample data that come from a population, allowing us to make predictions.

Different Types of df

Intuitive thinking: the number of cells that can vary in a `Spreadsheet`.

	Data	Model
	x_1	
	x_2	
	x_3	$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$
	$\vdots$	
	x_n	
df	n	1