Lecture 20 Confidence Interval For The Variance

BIO210 Biostatistics

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Spring, 2025

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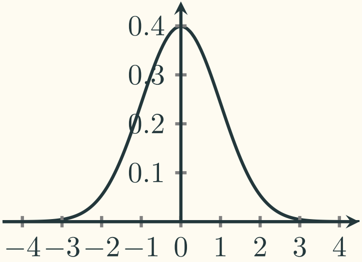
Interval Estimation For The Variance

$$\boldsymbol{S}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\boldsymbol{X}_i - \bar{\boldsymbol{X}})^2$$ Not normally distributed!

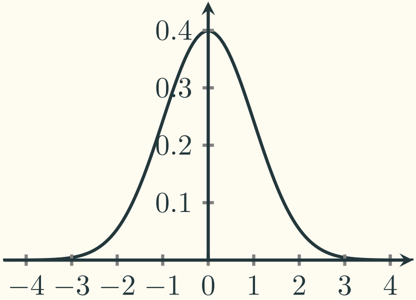
Chi-squared Distribution

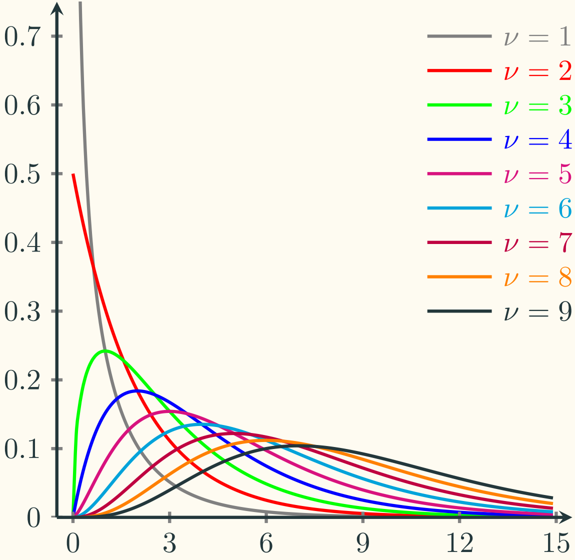
$$Z_1, Z_2, Z_3, ..., Z_n \sim \mathcal{N}(0, 1)$$



$$\begin{aligned}
U_1 &= Z_1^2 & U_1 &\sim \chi^2(1) \\
U_2 &= Z_1^2 + Z_2^2 & U_2 &\sim \chi^2(2) \\
U_3 &= Z_1^2 + Z_2^2 + Z_3^2 & U_3 &\sim \chi^2(3) \\
&\vdots & &\vdots \\
U_n &= Z_1^2 + Z_2^2 + Z_3^2 + \cdots Z_n^2 & U_n &\sim \chi^2(n)
\end{aligned}$$

Chi-squared Distributions of Different ν





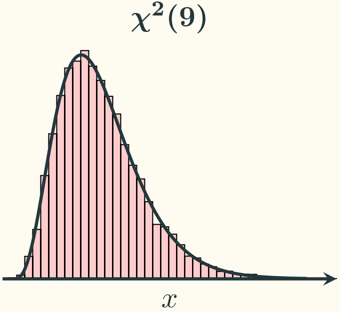
Sampling Distribution of The Sample Variance

Let $\boldsymbol{X}_1, \boldsymbol{X}_2, ..., \boldsymbol{X}_n$ be a sample independently drawn from a normal distribution $\mathcal{N}(\mu, \sigma^2)$, and we define the sample mean and variance as:

$$\bar{\boldsymbol{X}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i\,, \quad \boldsymbol{S}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\boldsymbol{X}_i - \bar{\boldsymbol{X}})^2$$

Then we have:

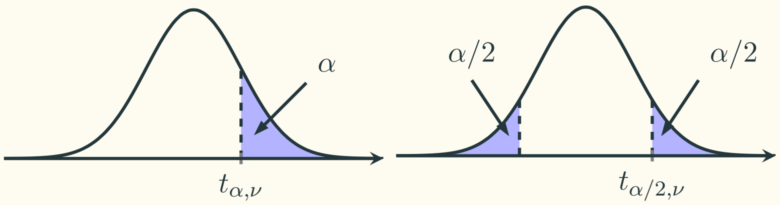
$$\frac{(n-1)\boldsymbol{S}^2}{\sigma^2} \sim \boldsymbol{\chi}^2(n-1)$$



Confidence Interval For The Mean

Recall: construct $(1-\alpha)\times 100\%$ confidence interval for the mean.

- One-sided (lower bound): $\mathbb{P}\left(\mu \geqslant \bar{X} - t_{\alpha,\nu}\cdot\frac{s}{\sqrt{n}}\right) = 1-\alpha$
- One-sided (upper bound): $\mathbb{P}\left(\mu \leqslant \bar{X} + t_{\alpha,\nu}\cdot\frac{s}{\sqrt{n}}\right) = 1-\alpha$
- Two-sided: $\mathbb{P}\left(\bar{X} - t_{\alpha/2,\nu}\cdot\frac{s}{\sqrt{n}} \leqslant \mu \leqslant \bar{X} + t_{\alpha/2,\nu}\cdot\frac{s}{\sqrt{n}}\right) = 1-\alpha$



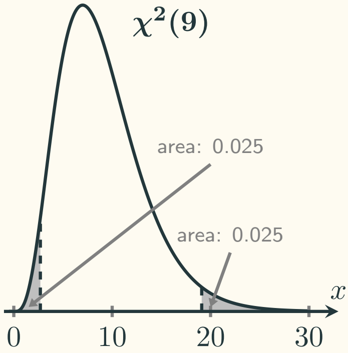
Goal: for a population with unknown variance σ^2, find a and b, such that $\mathbb{P}\left(a \leqslant \sigma^2 \leqslant b\right) = 0.95$

$$\mathbb{P}\left(\chi^2_{0.975,\nu} \leqslant \chi^2 \leqslant \chi^2_{0.025,\nu}\right) = 0.95$$

$$\mathbb{P}\left[\chi^2_{0.975,\nu} \leqslant \frac{(n-1)\boldsymbol{S}^2}{\sigma^2} \leqslant \chi^2_{0.025,\nu}\right] = 0.95$$

$$\mathbb{P}\left[\frac{1}{\chi^2_{0.025,\nu}} \leqslant \frac{\sigma^2}{(n-1)\boldsymbol{S}^2} \leqslant \frac{1}{\chi^2_{0.975,\nu}}\right] = 0.95$$

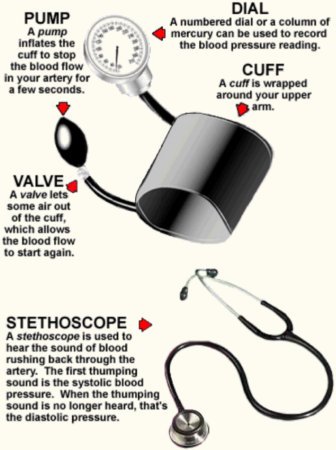
$$\mathbb{P}\left(\frac{n-1}{\chi^2_{0.025,\nu}} \cdot \boldsymbol{S}^2 \leqslant \sigma^2 \leqslant \frac{n-1}{\chi^2_{0.975,\nu}} \cdot \boldsymbol{S}^2\right) = 0.95$$



95% CI for σ^2 :

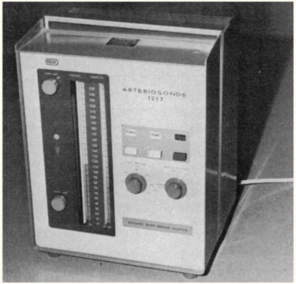
$$\left[\frac{n-1}{\chi^2_{0.025,\nu}} \cdot \boldsymbol{S}^2 , \frac{n-1}{\chi^2_{0.975,\nu}} \cdot \boldsymbol{S}^2\right]$$

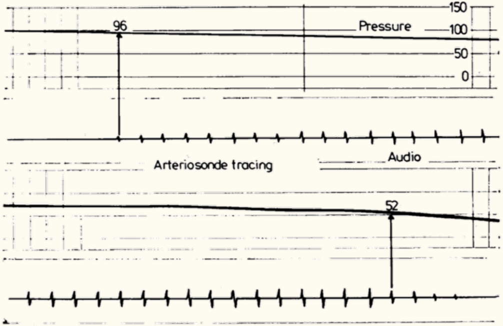
Sphygmomanometer



Confidence interval for variance

Hypertension: An Arteriosonde machine "prints" blood-pressure readings on a tape so that the measurement can be read rather than heard. A major argument for using such a machine is that the variability of measurements obtained by different observers on the same person will be lower than with a standard blood-pressure cuff.





Confidence interval for variance

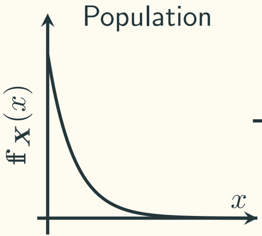
Persion (i)	Observer #1	Observer #2	Difference (d_i)
1	194	200	-6
2	126	123	3
3	130	128	2
4	98	101	-3
5	136	135	1
6	145	145	0
7	110	111	-1
8	108	107	1
9	102	99	3
10	126	128	-2

Does it make sense to construct a 95% CI for the variance?

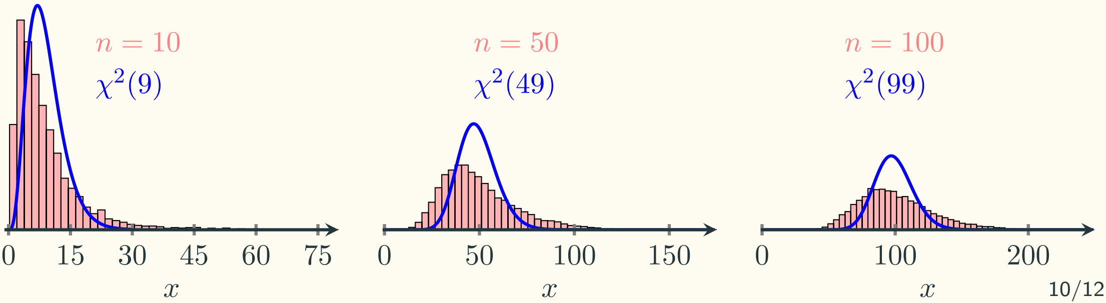
95% CI for σ^2 :

$$\left[\frac{n-1}{\chi^2_{0.025,\nu}} \cdot s^2 \, , \, \frac{n-1}{\chi^2_{0.975,\nu}} \cdot s^2\right]$$

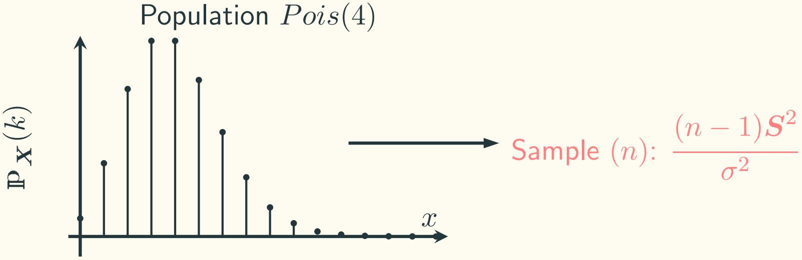
Sampling Distribution of The Sample Variance

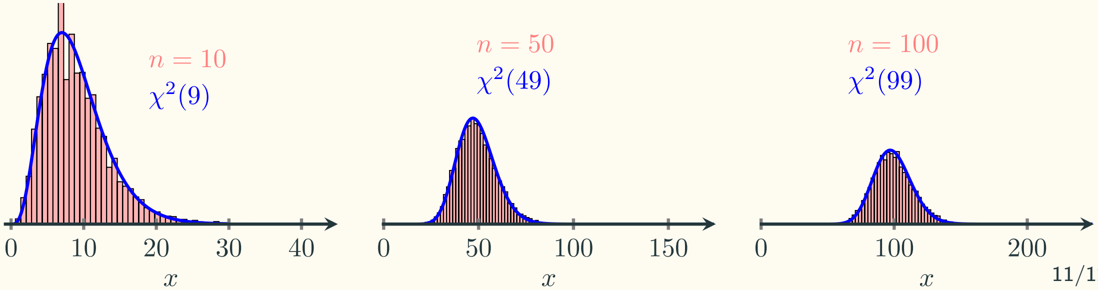


Sample (n): $\dfrac{(n-1)\boldsymbol{S}^2}{\sigma^2}$



Sampling Distribution of The Sample Variance





Conditions For Valid Confidence Intervals For The Variance

1. Random Samples
2. Independence ($n < 10\%$ population size)
3. Original population distribution must be normal