Maximum Likelihood Estimation (MLE) For Variance

BIO210 Biostatistics

Extra reading material for Lecture 18

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The random variable $\boldsymbol{X}$ denotes certain metric (*e.g.* height, weight) we are interested in from a population, and $\boldsymbol{X} \sim \mathcal{N}(\mu, \sigma^2)$. We draw a random sample of size n from the population. Like we discussed during the lecture, a random sample of size n can be thought as n **i.i.d.** random variables. That is:

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3, \cdots, \boldsymbol{X}_n \sim \mathcal{N}(\mu, \sigma^2)$$

We have seen that the maximum likelihood estimator for σ^2 is:

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{X}_i - \bar{\boldsymbol{X}})^2$$

Then, what is $E[\hat{\sigma^2}]$? If $E[\hat{\sigma^2}] = \sigma^2$, it is an unbiased estimator. Otherwise, it is a biased one.

Now let's have a look.

$$E[\hat{\sigma^2}] = E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \frac{1}{n}E\left[\sum_{i=1}^{n}(X_i^2 - 2\bar{X}X_i + \bar{X}^2)\right]$$
$$= \frac{1}{n}E\left[\sum_{i=1}^{n}X_i^2 - 2\bar{X}\sum_{i=1}^{n}X_i + \sum_{i=1}^{n}\bar{X}^2\right]$$

Note that: $\sum_{i=1}^{n} X_i = n\bar{X}$. Since $\bar{X}$ remains the same for each i, we have $\sum_{i=1}^{n} \bar{X}^2 = n\bar{X}^2$. Replacing the blue terms above, we have:

$$E[\hat{\sigma^2}] = \frac{1}{n}E\left[\sum_{i=1}^{n}X_i^2 - 2\bar{X}\cdot n\bar{X} + n\bar{X}^2\right] = \frac{1}{n}E\left[\sum_{i=1}^{n}X_i^2 - n\bar{X}^2\right]$$
$$= \frac{1}{n}\left(E\left[\sum_{i=1}^{n}X_i^2\right] - E[n\bar{X}^2]\right) \tag{1}$$

Since $var(X) = E[X^2] - (E[X])^2$, so we have $E[X^2] = var(X) + (E[X])^2$, then,

$$\begin{aligned}
E\left[\sum_{i=1}^{n} X_i^2\right] &= E[X_1^2] + E[X_2^2] + E[X_3^2] + \cdots + E[X_n^2] \\
&= var(X_1) + (E[X_1])^2 + var(X_2) + (E[X_2])^2 + \cdots \\
&\quad + var(X_n) + (E[X_n])^2 \\
&= \sigma^2 + \mu^2 + \sigma^2 + \mu^2 + \cdots + \sigma^2 + \mu^2 \\
&= n\sigma^2 + n\mu^2 \qquad (2)
\end{aligned}$$

Putting equation (2) into equation (1), we have:

$$\begin{aligned}
E[\hat{\sigma^2}] &= \sigma^2 + \mu^2 - \frac{1}{n} \cdot E[n\bar{X}^2] = \sigma^2 + \mu^2 - E[\bar{X}^2] \\
&= \sigma^2 + \mu^2 - (\sigma_{\bar{X}}^2 + \mu_{\bar{X}}^2) \qquad (3)
\end{aligned}$$

According to the central limit theorem, we have $\mu_{\bar{X}} = \mu$ and $\sigma_{\bar{X}}^2 = \frac{\sigma^2}{n}$. Therefore, equation (3) becomes:

$$E[\hat{\sigma^2}] = \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2 = \frac{n-1}{n}\sigma^2 \neq \sigma^2$$

Hence, it is not an unbiased estimator.