The Quotient of Two Independent Random Variables

1 The General Case

Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be two independent ***random variables*** with PDF $\mathbb{f}_{\boldsymbol{V}}(v)$ and $\mathbb{f}_{\boldsymbol{W}}(w)$, respectively.

Now we define a new random variable $\boldsymbol{Q} = \dfrac{\boldsymbol{V}}{\boldsymbol{W}}$. We want to compute the PDF of $\boldsymbol{Q}$:

$$\mathbb{f}_{\boldsymbol{Q}}(q)$$

Using similar techniques when we derived the ***convolution formula*** in the **Extra Reading Material** from **Lecture 13**, we start with the CDF of $\boldsymbol{Q}$. In addition, when $\boldsymbol{W}$ takes a value, $\boldsymbol{Q}$ is basically a linear function of $\boldsymbol{V}$. This makes the derivation a lot easier.

We start with the definition:

$$\mathbb{F}_{\boldsymbol{Q}|\boldsymbol{W}}(q \,|\, w) = \mathbb{P}\left(\boldsymbol{Q} \leqslant q \,|\, w\right) = \mathbb{P}\left(\frac{\boldsymbol{V}}{w} \leqslant q \,\middle|\, w\right) \tag{1}$$

1.1 When $\boldsymbol{w} > 0$

When $w > 0$, equation (1) becomes:

$$\mathbb{F}_{\boldsymbol{Q}|\boldsymbol{W}}(q \,|\, w) = \mathbb{P}\left(\boldsymbol{V} \leqslant wq \,|\, w\right) = F_{\boldsymbol{V}|\boldsymbol{W}}(wq \,|\, w) \tag{2}$$

Take the derivate at both sides with respect to $\boldsymbol{Q}$, equation (2) becomes:

$$\mathbb{f}_{\boldsymbol{Q}|\boldsymbol{W}}(q \,|\, w) = w \cdot \mathbb{f}_{\boldsymbol{V}|\boldsymbol{W}}(wq \,|\, w) \tag{3}$$

By definition, we have the joint PDF of $\boldsymbol{Q}$ and $\boldsymbol{W}$:

$$\mathbb{f}_{\boldsymbol{Q},\boldsymbol{W}}(q,w)=\mathbb{f}_{\boldsymbol{W}}(w)\cdot\mathbb{f}_{\boldsymbol{Q}|\boldsymbol{W}}(q\,|\,w) \tag{4}$$

Putting equation (3) into equation (4), we have:

$$\mathbb{f}_{\boldsymbol{Q},\boldsymbol{W}}(q,w)=\mathbb{f}_{\boldsymbol{W}}(w)\cdot w\cdot\mathbb{f}_{\boldsymbol{V}|\boldsymbol{W}}(wq\,|\,w)$$

Since $\boldsymbol{V}$ and $\boldsymbol{W}$ are independent, we can remove the conditioning:

$$\mathbb{f}_{\boldsymbol{Q},\boldsymbol{W}}(q,w)=w\cdot\mathbb{f}_{\boldsymbol{W}}(w)\mathbb{f}_{\boldsymbol{V}}(wq)$$

Integrate over $\boldsymbol{W}$ under the condition $w>0$, we get the marginal PDF of $\boldsymbol{X}$:

$$\mathbb{f}_{\boldsymbol{Q}}(q)=\int_0^\infty\mathbb{f}_{\boldsymbol{Q},\boldsymbol{W}}(q,w)\mathrm{d}w=\int_0^\infty w\cdot\mathbb{f}_{\boldsymbol{W}}(w)\mathbb{f}_{\boldsymbol{V}}(wq)\mathrm{d}w \tag{5}$$

1.2 When $\boldsymbol{W}<0$

When $w<0$, equation (1) becomes:

$$\mathbb{F}_{\boldsymbol{Q}|\boldsymbol{W}}(q\,|\,w)=\mathbb{P}\left(\boldsymbol{V}\geqslant wq\,|\,w\right)=1-\mathbb{F}_{\boldsymbol{V}|\boldsymbol{W}}(wq\,|\,w) \tag{6}$$

Take the derivate at both sides with respect to $\boldsymbol{Q}$, equation (6) becomes:

$$\mathbb{f}_{\boldsymbol{Q}|\boldsymbol{W}}(q\,|\,w)=-w\cdot\mathbb{f}_{\boldsymbol{V}|\boldsymbol{W}}(wq\,|\,w) \tag{7}$$

Similarly, put equation (7) into equation (4), we have:

$$\mathbb{f}_{\boldsymbol{Q},\boldsymbol{W}}(q,w)=\mathbb{f}_{\boldsymbol{W}}(w)\cdot(-w)\cdot\mathbb{f}_{\boldsymbol{V}|\boldsymbol{W}}(wq\,|\,w)$$

Again, since $\boldsymbol{V}$ and $\boldsymbol{W}$ are independent, we can remove the conditioning:

$$\mathbb{f}_{\boldsymbol{Q},\boldsymbol{W}}(q,w)=-w\cdot\mathbb{f}_{\boldsymbol{W}}(w)\mathbb{f}_{\boldsymbol{V}}(wq)$$

Integrate over $\boldsymbol{W}$ under the condition $w < 0$, we get the marginal PDF of $\boldsymbol{Q}$:

$$\mathbb{f}_{\boldsymbol{Q}}(q) = \int_{-\infty}^{0} \mathbb{f}_{\boldsymbol{Q},\boldsymbol{W}}(q, w)\mathrm{d}w = \int_{-\infty}^{0} -w \cdot \mathbb{f}_{\boldsymbol{W}}(w)\mathbb{f}_{\boldsymbol{V}}(wq)\mathrm{d}w \tag{8}$$

Combining equations (5) and (8), we have the general case:

$$\mathbb{f}_{\boldsymbol{Q}}(q) = \int_{-\infty}^{\infty} |w| \cdot \mathbb{f}_{\boldsymbol{W}}(w)\mathbb{f}_{\boldsymbol{V}}(wq)\mathrm{d}w \tag{9}$$

2 The PDF of The $\boldsymbol{t}$-distribution

If we have a random variable $\boldsymbol{T}$ that satisfy:

$$\mathbb{f}_{\boldsymbol{T}}(t) = \frac{\boldsymbol{Z}}{\sqrt{\boldsymbol{U}/\nu}}$$

where $\boldsymbol{Z} \sim \mathcal{N}(0, 1)$, $\boldsymbol{U} \sim \chi^2(\nu)$ and $\nu > 0$ is the degree of freedom. Then we say $\boldsymbol{T}$ follows a $\boldsymbol{t}$-distribution of a degree of freedom ν:

$$\boldsymbol{T} \sim \mathcal{T}(\nu)$$

We want to figure out the PDF of $\boldsymbol{T}$: $\mathbb{f}_{\boldsymbol{T}}(t)$.

First, by definition, we have:

$$\mathbb{f}_{\boldsymbol{Z}}(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \text{ and } \mathbb{f}_{\boldsymbol{U}}(u) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}}} u^{\frac{\nu}{2}-1} e^{-\frac{u}{2}}$$

The denominator of the random variable $\boldsymbol{T}$ is $\sqrt{\boldsymbol{U}/\nu}$. We let $\boldsymbol{X} = \sqrt{\boldsymbol{U}/\nu}$. We need to figure out the PDF of $\boldsymbol{X}$ first. Using the exact same strategy, we start with:

$$\begin{aligned}\mathbb{F}_{\boldsymbol{X}}(x) &= \mathbb{P}(\boldsymbol{X} \leqslant x) = \mathbb{P}\left(\sqrt{\frac{\boldsymbol{U}}{\nu}} \leqslant x\right) = \mathbb{P}\left(\frac{\boldsymbol{U}}{\nu} \leqslant x^2\right) \\ &= \mathbb{P}\left(\boldsymbol{U} \leqslant \nu x^2\right) = \mathbb{F}_{\boldsymbol{U}}(\nu x^2)\end{aligned}$$

Taking the derivative with respect to $\boldsymbol{X}$, we have:

$$\mathbb{f}_{\boldsymbol{X}}(x) = 2\nu x \mathbb{f}_{\boldsymbol{U}}(\nu x^2)$$

Expanding the χ^2 PDF $\mathbb{f}_{\boldsymbol{U}}(\nu x^2)$, we have:

$$\mathbb{f}_{\boldsymbol{X}}(x) = 2\nu x \cdot \frac{1}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}}} (\nu x^2)^{\frac{\nu}{2}-1} e^{-\frac{\nu x^2}{2}} = 2\, \nu\, x \cdot \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} 2^{-\frac{\nu}{2}} \nu^{\frac{\nu}{2}-1} x^{\nu-2} e^{-\frac{\nu x^2}{2}}$$

Be patient and merge terms of the same colour, we have:

$$\mathbb{f}_{\boldsymbol{X}}(x) = \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \nu^{\frac{\nu}{2}} x^{\nu-1} e^{-\frac{\nu x^2}{2}} \tag{10}$$

Apparently, $\boldsymbol{T} = \dfrac{\boldsymbol{Z}}{\boldsymbol{X}}$. Using equation (9), we have:

$$\mathbb{f}_{\boldsymbol{T}}(t) = \int_{-\infty}^{\infty} |x| \mathbb{f}_{\boldsymbol{X}}(x) \mathbb{f}_{\boldsymbol{Z}}(tx) \mathrm{d}x$$

Put equation (10) into the above forulat, expand $f_{\boldsymbol{Z}}(tx)$ and notice that x is non-negative, we have:

$$\mathbb{f}_{\boldsymbol{T}}(t) = \int_{0}^{\infty} x \cdot \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \nu^{\frac{\nu}{2}} x^{\nu-1} e^{-\frac{\nu x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2 x^2}{2}} \mathrm{d}x$$

Bring the constant terms in front of the integration and merge similar terms, we have:

$$\begin{aligned}\mathbb{f}_{\boldsymbol{T}}(t) &= \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{2\pi}} \cdot \nu^{\frac{\nu}{2}} \int_{0}^{\infty} x \cdot x^{\nu-1} e^{-\frac{\nu x^2}{2}} \cdot e^{-\frac{t^2 x^2}{2}} \\ &= \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{2\pi}} \cdot \nu^{\frac{\nu}{2}} \int_{0}^{\infty} x^{\nu} e^{-\frac{\nu+t^2}{2} x^2} \mathrm{d}x \qquad (11)\end{aligned}$$

Now we need to figure out the blue part in equation (11). It resembles the **Gamma distribution**. The PDF of the Gamma distribution with a shape parameter α and a scale parameter θ is:

$$\frac{1}{\Gamma(\alpha)\theta^\alpha}x^{\alpha-1}e^{-\frac{x}{\theta}}$$

However, we have a x^2 in the exponent of e in the blue part of equation (11), so we need to modify a bit. We let $x^2 = y$. Then we have:

$$x^\nu = y^{\frac{\nu}{2}}$$
$$x^2 = y \quad \Rightarrow \quad 2x\mathrm{d}x = \mathrm{d}y \quad \Rightarrow \quad \mathrm{d}x = \frac{1}{2x}\mathrm{d}y = \frac{1}{2\sqrt{y}}\mathrm{d}y = \frac{1}{2}y^{-\frac{1}{2}}\mathrm{d}y$$

Put those into the blue part of equation (11), we have:

$$\int_0^\infty x^\nu e^{-\frac{\nu+t^2}{2}x^2}\mathrm{d}x = \int_0^\infty y^{\frac{\nu}{2}}e^{-\frac{\nu+t^2}{2}y}\cdot\frac{1}{2}y^{-\frac{1}{2}}\mathrm{d}y = \frac{1}{2}\int_0^\infty y^{\frac{\nu-1}{2}}e^{-\frac{\nu+t^2}{2}y}\,\mathrm{d}y$$
$$= \frac{1}{2}\int_0^\infty y^{\frac{\nu+1}{2}-1}e^{-\frac{y}{\frac{2}{\nu+t^2}}}\,\mathrm{d}y \tag{12}$$

Apparently, the above integration looks like a Gamma distribution with a shape parameter $\alpha = \dfrac{\nu+1}{2}$ and a scale parameter $\theta = \dfrac{2}{\nu+t^2}$. Therefore, we could rewrite equation (12) as:

$$\int_0^\infty x^\nu e^{-\frac{\nu+t^2}{2}x^2}\mathrm{d}x = \frac{\Gamma\left(\frac{\nu+1}{2}\right)\cdot\left(\frac{2}{\nu+t^2}\right)^{\frac{\nu+1}{2}}}{2}\int_0^\infty \frac{1}{\Gamma\left(\frac{\nu+1}{2}\right)\cdot\left(\frac{2}{\nu+t^2}\right)^{\frac{\nu+1}{2}}}y^{\frac{\nu+1}{2}-1}e^{-\frac{y}{\frac{2}{\nu+t^2}}}\,\mathrm{d}y$$

The red part is a full model of a Gamma distribution, so its value is 1. Therefore, we have

$$\int_0^\infty x^\nu e^{-\frac{\nu+t^2}{2}x^2}\mathrm{d}x = \frac{\Gamma\left(\frac{\nu+1}{2}\right)\cdot\left(\frac{2}{\nu+t^2}\right)^{\frac{\nu+1}{2}}}{2}$$

Put it back to equation (11) and rearrange the terms:

$$\begin{aligned}
\mathbb{f}_{\boldsymbol{T}}(t) &= \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{2\pi}} \cdot \nu^{\frac{\nu}{2}} \cdot \frac{\Gamma\left(\frac{\nu+1}{2}\right) \cdot \left(\frac{2}{\nu+t^2}\right)^{\frac{\nu+1}{2}}}{2} \\
&= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{2^{1-\frac{\nu}{2}}}{2\sqrt{2\pi}} \cdot \nu^{\frac{\nu}{2}} \cdot \left(\frac{2}{\nu+t^2}\right)^{\frac{\nu+1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{2^{-\frac{\nu+1}{2}}}{\sqrt{\pi}} \cdot \nu^{-\frac{1}{2}} \cdot \nu^{\frac{\nu+1}{2}} \cdot \left(\frac{2}{\nu+t^2}\right)^{\frac{\nu+1}{2}} \\
&= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{2^{-\frac{\nu+1}{2}}}{\sqrt{\pi\nu}} \cdot \frac{2^{\frac{\nu+1}{2}} \cdot \nu^{\frac{\nu+1}{2}}}{(\nu+t^2)^{\frac{\nu+1}{2}}} \\
&= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}} \cdot \left(\frac{\nu}{\nu+t^2}\right)^{\frac{\nu+1}{2}} \qquad (13)
\end{aligned}$$

That's the PDF of $\boldsymbol{\mathcal{T}}(\nu)$. Typically, it is often written as:

$$\mathbb{f}_{\boldsymbol{T}}(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}} \cdot \left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

3 The PDF of The $\boldsymbol{\mathcal{F}}$-distribution

If we have a random variable $\boldsymbol{X}$ that satisfy:

$$\mathbb{f}_{\boldsymbol{X}}(x) = \frac{\boldsymbol{U_1}/n}{\boldsymbol{U_2}/m}$$

where $\boldsymbol{U_1} \sim \chi^2(n)$, $\boldsymbol{U_2} \sim \chi^2(m)$ and $n, m > 0$ are the degrees of freedom of $\boldsymbol{U_1}$ and $\boldsymbol{U_2}$, respectively. Then we say $\boldsymbol{X}$ follows an $\boldsymbol{F}$-distribution of a degree of freedom n in the numerator and m in the denominator:

$$\boldsymbol{X} \sim \boldsymbol{\mathcal{F}}(n, m)$$

We want to figure out the PDF of $\boldsymbol{X}$: $\mathbb{f}_{\boldsymbol{X}}(x)$.

Since n and m are just some numbers, let's first use equation (9) to derive the PDF of $\frac{\boldsymbol{U_1}}{\boldsymbol{U_2}}$. Let $\boldsymbol{Y} = \frac{\boldsymbol{U_1}}{\boldsymbol{U_2}}$. Using equation (9), we have:

$$\mathbb{f}_{\boldsymbol{Y}}(y) = \int_{-\infty}^{\infty} |u_2| \mathbb{f}_{\boldsymbol{U_2}}(u_2) \mathbb{f}_{\boldsymbol{U_1}}(yu_2) \mathrm{d}u_2$$

Note both u_1 and u_2 are non-negative and put the values into the χ^2 PDFs:

$$\begin{aligned}
\mathbb{f}_{\boldsymbol{Y}}(y) &= \int_0^{\infty} u_2 \cdot \frac{1}{\Gamma\left(\frac{m}{2}\right) 2^{\frac{m}{2}}} u_2^{\frac{m}{2}-1} e^{-\frac{u_2}{2}} \cdot \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} (yu_2)^{\frac{n}{2}-1} e^{-\frac{yu_2}{2}} \mathrm{d}u_2 \\
&= \frac{y^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right) 2^{\frac{n+m}{2}}} \int_0^{\infty} u_2 \cdot u_2^{\frac{m}{2}-1} \cdot u_2^{\frac{n}{2}-1} \cdot e^{-\frac{u_2}{2}} \cdot e^{-\frac{yu_2}{2}} \mathrm{d}u_2 \\
&= \frac{y^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right) 2^{\frac{n+m}{2}}} \int_0^{\infty} u_2^{\frac{n+m}{2}-1} e^{-\frac{y+1}{2}u_2} \mathrm{d}u_2
\end{aligned} \tag{14}$$

Similar to the previous examples, the blue part of equation (14) looks like a Gamma distribution. We re-write it as:

$$\begin{aligned}
\mathbb{f}_{\boldsymbol{Y}}(y) &= \frac{y^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right) 2^{\frac{n+m}{2}}} \int_0^{\infty} u_2^{\frac{n+m}{2}-1} e^{-\frac{u_2}{\frac{2}{y+1}}} \mathrm{d}u_2 \\
&= \frac{y^{\frac{n}{2}-1} \cdot \Gamma\left(\frac{n+m}{2}\right) \cdot \left(\frac{2}{y+1}\right)^{\frac{n+m}{2}}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right) 2^{\frac{n+m}{2}}} \int_0^{\infty} \frac{1}{\Gamma\left(\frac{n+m}{2}\right) \cdot \left(\frac{2}{y+1}\right)^{\frac{n+m}{2}}} u_2^{\frac{n+m}{2}-1} e^{-\frac{u_2}{\frac{2}{y+1}}} \mathrm{d}u_2
\end{aligned} \tag{15}$$

The blue part of equation (15) is a full model of a Gamma distribution with the shape parameter $\alpha = \frac{n+m}{2}$ and a scale parameter of $\theta = \frac{2}{y+1}$. It is equal to 1. Therefore, we have:

$$\mathbb{f}_{\boldsymbol{Y}}(y) = \frac{y^{\frac{n}{2}-1} \cdot \Gamma\left(\frac{n+m}{2}\right) \cdot \left(\frac{2}{y+1}\right)^{\frac{n+m}{2}}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right) 2^{\frac{n+m}{2}}} = \frac{\Gamma\left(\frac{n+m}{2}\right) y^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)(y+1)^{\frac{n+m}{2}}} \tag{16}$$

Now we need to figure out the PDF of $\boldsymbol{X} = \frac{m}{n}\boldsymbol{Y}$, which is straightforward based on **Section 5.1** from the **Extra Reading Material** from **Lecture 13**. Put into equation (16), we have:

$$
\begin{aligned}
\mathbb{f}_{\boldsymbol{X}}(x) &= \frac{1}{\frac{m}{n}} \cdot f_{\boldsymbol{Y}}\left(\frac{x}{\frac{m}{n}}\right) = \frac{n}{m} \cdot f_{\boldsymbol{Y}}\left(\frac{n}{m}x\right) \\
&= \frac{n}{m} \cdot \frac{\Gamma\left(\frac{n+m}{2}\right)\left(\frac{n}{m}x\right)^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)\left(\frac{n}{m}x+1\right)^{\frac{n+m}{2}}} \\
&= \frac{n}{m} \cdot \frac{\Gamma\left(\frac{n+m}{2}\right)\left(\frac{n}{m}\right)^{\frac{n}{2}-1} x^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)\left(\frac{n}{m}x+1\right)^{\frac{n+m}{2}}} \\
&= \frac{\Gamma\left(\frac{n+m}{2}\right)\left(\frac{n}{m}\right)^{\frac{n}{2}} x^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)\left(\frac{n}{m}x+1\right)^{\frac{n+m}{2}}}
\end{aligned}
$$