Lecture 16 Sampling Distribution of The Sample Variance

BIO210 Biostatistics

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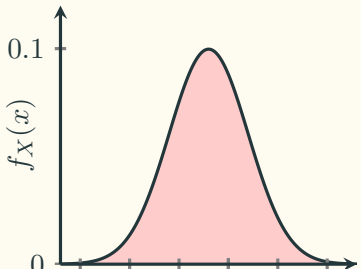
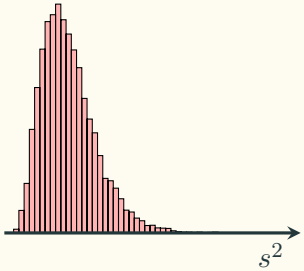
Sampling Distribution of The Sample Variance



$POU5F1 \sim \mathcal{N}(\mu = 3, \sigma^2 = 4^2)$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Sample		Sample mean
$n = 96$	$\rightarrow$	$s^2 = 15.11$
$n = 96$	$\rightarrow$	$s^2 = 15.75$
$n = 96$	$\rightarrow$	$s^2 = 17.44$
$n = 96$	$\rightarrow$	$s^2 = 16.85$
$n = 96$	$\rightarrow$	$s^2 = 13.31$
$n = 96$	$\rightarrow$	$s^2 = 16.36$
$\vdots$	$\vdots$	$\vdots$
$n = 96$	$\rightarrow$	$s^2 = 17.29$
$\vdots$	$\vdots$	$\vdots$



$S^2 \sim$?

Sampling distribution of the sample variance

Start With The Special Case

Task: We draw a sample of size n $(X_1, X_2, \cdots, X_n)$ from a population $(X \sim \mathcal{D})$, where $\mathbb{V}\mathrm{ar}\,(X) = \sigma^2$, we want to figure out:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n-1}\Big[(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + \cdots + (X_n - \bar{X})^2\Big]$$

Simplify: Let $X_1, X_2, \cdots, X_n$ be **i.i.d.** random variables from a normal population $\mathcal{N}(\mu, \sigma^2)$

$$S^2 = \frac{1}{n-1}\Big[(X_1 - \bar{X})^2 + (X_2 - \bar{X})^2 + \cdots + (X_n - \bar{X})^2\Big]$$

The question becomes: what is the sum of a bunch of squared normal random variables?

The Standard Normal Squared

Let $Z_1, Z_2, Z_3, \cdots, Z_n$ be **i.i.d.** standard normal random variables: $Z_i \sim \mathcal{N}(0, 1)$, then

- $Z_1^2 \sim ?$
- $Z_1^2 + Z_2^2 \sim ?$
- $\vdots$
- $\sum_{i=1}^{n} Z_i^2 \sim ?$

The Chi-squared (χ^2) Distribution

Friedrich Robert Helmert in 1876:

Number of Z_i^2	The PDF of the sum
1	$\frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} e^{-\frac{x}{2}} : \chi^2(1)$
2	$\frac{1}{2} e^{-\frac{x}{2}} : \chi^2(2)$
3	$\frac{1}{\sqrt{2\pi}} x^{\frac{1}{2}} e^{-\frac{x}{2}} : \chi^2(3)$
4	$\frac{1}{4} x e^{-\frac{x}{2}} : \chi^2(4)$
5	$\frac{1}{3\sqrt{2\pi}} x^{\frac{3}{2}} e^{-\frac{x}{2}} : \chi^2(5)$
$\vdots$	$\vdots$

by induction:

$$\chi^2(n) : \ f_X(x) = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}, x \geqslant 0$$

where:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \mathrm{d}t, \ \alpha > 0$$

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$

$$\Gamma(k) = (k-1)! \text{ , when } k \text{ is an integer}$$

One parameter - the degree of freedom:
the number of independent Z^2 in the sum

The Distribution of S^2

By definition:

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(n)$$

Replacing μ with $\bar{X}$:

$$\sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma} \right)^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \sim \chi^2(n-1)$$

Manipulate to get the sample variance:

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

Why $n-1$? part 1

$$\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2 \leqslant \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$$

Why? Because:

$$\sum_{i=1}^{n}(x_i-m)^2 = n\cdot m^2 - \left(2\sum_{i=1}^{n}x_i\right)\cdot m + \sum_{i=1}^{n}x_i^2$$

But why exactly $n-1$? Wait until **part 2** in **Lecture 18**

The Degree of Freedom (DF, DOF, ν)

Typical definition: the number of values in the final calculation of a statistic that are free to vary; the number of independent pieces of information used to calculate the statistic.

There are two types of degrees of freedom:

$$\begin{cases} df \text{ of the data} & \text{- } df \text{ left (statistical cash)} \\ df \text{ of the statistical model} & \text{- } df \text{ spent (buy with cash)} \end{cases}$$

Statistical models: a mathematical process that attempts to describe the population where the sample comes from, allowing us to make predictions.

Different Types of df

Intuitive thinking: the number of cells that can vary in a `Spreadsheet`.

	Data	Model
	x_1	
	x_2	$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$
	x_3	
	$\vdots$	
	x_n	
df	n	1