Student's t-Distributions

Note that like many other distributions, we are always using the plural term "**distributions**", because they are a family of distributions. There are many $\boldsymbol{t}$-distributions. The parameter of a $\boldsymbol{t}$-distribution is its ***degree of freedom***, or ***df***, or **DOF**, or ν.

1 Derivation of The PDF

We start where we left from the lecture. We say that the random variable T has the Student's $\boldsymbol{t}$-distribution with ν degree of freedom if:

$$\boldsymbol{T} = \frac{\boldsymbol{Z}}{\sqrt{\boldsymbol{U}/\nu}} \tag{1}$$

where $Z \sim \mathcal{N}(0,1)$, $U \sim \chi^2(\nu)$, and Z, U are independent. Therefore, what we know is

$$f_{\boldsymbol{Z}}(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \text{ and } f_{\boldsymbol{U}}(u) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}}} u^{\frac{\nu}{2}-1} e^{-\frac{u}{2}}$$

what we want now is the PDF of $\boldsymbol{T}$: $f_{\boldsymbol{T}}(t)$.

There are a number of ways of deriving the PDF of Student's $\boldsymbol{t}$-distributions. The first thing that comes to our mind should be the common trick to avoid integration that we have talked about in the **Extra Reading Material** from **Lecture 13**. That is, to work with the CDF and then take the derivative. Intuitively, we should start:

$$F_{\boldsymbol{T}}(t) = \mathbb{P}(\boldsymbol{T} \leqslant t) = \mathbb{P}\left(\frac{\sqrt{\nu}\boldsymbol{Z}}{\sqrt{\boldsymbol{U}}} \leqslant t\right)$$

but you see that there is a ratio of two random variables there, which is slightly more difficult to work with. We will come back to this method in a future lecture later.

Alternatively, we can do another trick that is similar to the procedures which we used to derive the ***convolution formula***. That is, we start with a joint PDF of $\boldsymbol{T}$ and another random variable, either $\boldsymbol{Z}$ or $\boldsymbol{U}$. Then we integrate over the other random variable to get the marginal PDF of $\boldsymbol{T}$, which is essentially $f_{\boldsymbol{T}}(t)$. If you forget, go back and check the **Extra Reading Material** from **Lecture 13**.

If you do some initial trials, you will realise that the joint PDF $f_{\boldsymbol{T},\boldsymbol{U}}(t,u)$ is slightly easier to compute compared to $f_{\boldsymbol{T},\boldsymbol{Z}}(t,z)$. Therefore, we start with the easier one.

By definition, we have:

$$f_{\boldsymbol{T},\boldsymbol{U}}(t,u) = f_{\boldsymbol{U}}(u) \cdot f_{\boldsymbol{T}\,|\,\boldsymbol{U}}(t|u) \tag{2}$$

Now we need to rewrite $f_{\boldsymbol{T}\,|\,\boldsymbol{U}}(t|u)$ in terms of $\boldsymbol{Z}$. Whenever the random variable $\boldsymbol{U}$ takes a number, $\boldsymbol{T}$ is a simple function of $\boldsymbol{Z}$, so we could easily work out $f_{\boldsymbol{T}\,|\,\boldsymbol{U}}(t|u)$ using the common trick of working out CDF and taking the derivative:

$$\begin{aligned} F_{\boldsymbol{T}\,|\,\boldsymbol{U}}(t|u) &= \mathbb{P}(\boldsymbol{T} \leqslant t \mid u) = \mathbb{P}\left(\frac{\sqrt{\nu}\boldsymbol{Z}}{\sqrt{u}} \leqslant t \,\middle|\, u\right) \\ &= \mathbb{P}\left(\boldsymbol{Z} \leqslant \frac{\sqrt{u}}{\sqrt{\nu}}t \,\middle|\, u\right) \\ &= F_{\boldsymbol{Z}\,|\,\boldsymbol{U}}\left(\frac{\sqrt{u}}{\sqrt{\nu}}t \,\middle|\, u\right) \end{aligned}$$

Take the derivate at both sides, we have (note the extra coefficient before t):

$$f_{\boldsymbol{T}\,|\,\boldsymbol{U}}(t|u) = \frac{\sqrt{u}}{\sqrt{\nu}} f_{\boldsymbol{Z}\,|\,\boldsymbol{U}}\left(\frac{\sqrt{u}}{\sqrt{\nu}}t \,\middle|\, u\right)$$

Since $\boldsymbol{Z}$ and $\boldsymbol{U}$ are independent, we could remove the conditioning between

$\boldsymbol{Z}$ and $\boldsymbol{U}$. Therefore, we have[1]:

$$f_{\boldsymbol{T}\,|\,\boldsymbol{U}}(t|u) = \frac{\sqrt{u}}{\sqrt{\nu}} f_{\boldsymbol{Z}\,|\,\boldsymbol{U}}\left(\frac{\sqrt{u}}{\sqrt{\nu}}t \,\middle|\, u\right) = \frac{\sqrt{u}}{\sqrt{\nu}} f_{\boldsymbol{Z}}\left(\frac{\sqrt{u}}{\sqrt{\nu}}t\right)$$
$$= \frac{\sqrt{u}}{\sqrt{\nu}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{u}{2\nu}t^2} = \frac{\sqrt{u}}{\sqrt{2\pi\nu}} e^{-\frac{u}{2\nu}t^2} \tag{3}$$

Putting equation (3) back to equation (2), we have:

$$f_{\boldsymbol{T},\boldsymbol{U}}(t,u) = f_{\boldsymbol{U}}(u) \cdot f_{\boldsymbol{T}\,|\,\boldsymbol{U}}(t|u) = f_{\boldsymbol{U}}(u) \cdot \frac{\sqrt{u}}{\sqrt{2\pi\nu}} e^{-\frac{u}{2\nu}t^2}$$
$$= \frac{1}{\Gamma(\frac{\nu}{2})2^{\frac{\nu}{2}}} u^{\frac{\nu}{2}-1} e^{-\frac{u}{2}} \cdot \frac{\sqrt{u}}{\sqrt{2\pi\nu}} e^{-\frac{u}{2\nu}t^2}$$
$$= \frac{u^{\frac{\nu-1}{2}}}{2^{\frac{\nu+1}{2}}\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}} e^{-\frac{u}{2}\left(1+\frac{t^2}{\nu}\right)} \tag{4}$$

Now we integrate with respect to u at both sides of equation (4). Note that $\boldsymbol{U}$ is non-negative. At the left-hand side, we get the marginal PDF of $\boldsymbol{T}$, we have:

$$f_{\boldsymbol{T}}(t) = \frac{1}{2^{\frac{\nu+1}{2}}\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}} \int_0^\infty u^{\frac{\nu-1}{2}} e^{-\frac{u}{2}\left(1+\frac{t^2}{\nu}\right)} \mathrm{d}u \tag{5}$$

The blue part is very difficult to integrate, but it does look like a ***Gamma distribution*** that we talked about in a previous homework. Recall that the PDF of a Gamma distribution is:

$$f_{\boldsymbol{X}}(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}} \tag{6}$$

If you compare equation (6) to the blue part of equation (5), you will see some commonality. Now we let the shape parameter $\alpha = \dfrac{\nu+1}{2}$ and the scale

[1] If you think about it, equation (3) actually makes sense. When $\boldsymbol{U}$ takes the value u, then $\boldsymbol{T}$ simply becomes a linear function of $\boldsymbol{Z}$: $\frac{\sqrt{\nu}}{\sqrt{u}}\boldsymbol{Z}$. Since $\boldsymbol{Z} \sim \mathcal{N}(0,1)$, then it is easy to see $\boldsymbol{T} \sim \mathcal{N}\left(0, \frac{\sqrt{\nu}}{\sqrt{u}}\right)$.

parameter $\theta = \dfrac{2}{1+\frac{t^2}{\nu}}$. Then the Gamma distribution we have is:

$$f_{\boldsymbol{X}}(x) = \frac{1}{\Gamma\left(\dfrac{\nu+1}{2}\right)\cdot\left(\dfrac{2}{1+\frac{t^2}{\nu}}\right)^{\frac{\nu+1}{2}}} x^{\frac{\nu-1}{2}} e^{-\frac{x}{2}\left(1+\frac{t^2}{\nu}\right)}$$

Again, recall that the Gamma distribution describes the behaviour of non-negative random variables. Therefore, if we integrate from 0 to ∞, we should get 1 (Check **Section 3** for the proof if you are not sure). That is:

$$\int_0^\infty \frac{1}{\Gamma\left(\frac{\nu+1}{2}\right)\cdot\left(\frac{2}{1+\frac{t^2}{\nu}}\right)^{\frac{\nu+1}{2}}} x^{\frac{\nu-1}{2}} e^{-\frac{x}{2}\left(1+\frac{t^2}{\nu}\right)} = 1$$

$$\frac{1}{\Gamma\left(\frac{\nu+1}{2}\right)\cdot\left(\frac{2}{1+\frac{t^2}{\nu}}\right)^{\frac{\nu+1}{2}}} \int_0^\infty x^{\frac{\nu-1}{2}} e^{-\frac{x}{2}\left(1+\frac{t^2}{\nu}\right)} = 1$$

$$\int_0^\infty x^{\frac{\nu-1}{2}} e^{-\frac{x}{2}\left(1+\frac{t^2}{\nu}\right)} = \Gamma\left(\frac{\nu+1}{2}\right)\cdot\left(\frac{2}{1+\frac{t^2}{\nu}}\right)^{\frac{\nu+1}{2}} \tag{7}$$

Note that the red part in equation (7) is the same as the blue part in equation (5). Therefore, replacing it and equation (5) becomes:

$$\begin{aligned} f_{\boldsymbol{T}}(t) &= \frac{1}{2^{\frac{\nu+1}{2}}\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}\Gamma\left(\frac{\nu+1}{2}\right)\cdot\left(\frac{2}{1+\frac{t^2}{\nu}}\right)^{\frac{\nu+1}{2}} \\ &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\cancel{2^{\frac{\nu+1}{2}}}}\cdot\frac{\cancel{2^{\frac{\nu+1}{2}}}}{\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}} \\ &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}\cdot\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} \end{aligned}$$

Finally, that's the scary PDF of Student's $\boldsymbol{t}$-distribution.

2 When $\nu \to \infty$

The PDF $f_{\boldsymbol{T}}(t)$ is a product consists of two basic terms. If the limit of each term exist, the limit of the product is basically the product of the limit. Now let's look at them one by one.

We start with $\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$ which is easier:

$$\begin{aligned}\lim_{\nu\to\infty}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} &= \lim_{\nu\to\infty}\frac{1}{\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}\\ &= \lim_{\nu\to\infty}\frac{1}{\left(1+\frac{t^2}{\nu}\right)^{\frac{1}{2}}\cdot\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu}{2}}}\\ &= \lim_{\nu\to\infty}\frac{1}{\left(1+\frac{t^2}{\nu}\right)^{\frac{1}{2}}\cdot\left(1+\frac{t^2/2}{\nu/2}\right)^{\frac{\nu}{2}}} \qquad (8)\end{aligned}$$

Note that $\lim_{\nu\to\infty}\left(1+\frac{t^2}{\nu}\right)^{\frac{1}{2}} = 1$. Then recall that when we derived the Poisson PMF, we did a quick review of

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$$

Now we let $a=\frac{t^2}{2}$ and $n=\frac{\nu}{2}$, and put back to equation (8), we have:

$$\lim_{\nu\to\infty}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} = \lim_{\nu\to\infty}\frac{1}{\left(1+\frac{t^2}{\nu}\right)^{\frac{1}{2}}\cdot\left(1+\frac{t^2/2}{\nu/2}\right)^{\frac{\nu}{2}}} = \frac{1}{1\cdot e^{\frac{t^2}{2}}} = e^{-\frac{t^2}{2}} \qquad (9)$$

Let's see $\dfrac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}}$ now. From the ***Stirling's formula***, we have the following approximation:

$$\Gamma(n) \approx \sqrt{\frac{2\pi}{n}} \left(\frac{n}{e}\right)^n$$

Using that, and let $k = \frac{\nu}{2}$ we have

$$\lim_{\nu\to\infty} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}} = \lim_{k\to\infty} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k\right)\sqrt{2\pi k}} = \lim_{k\to\infty} \frac{\sqrt{\frac{2\pi}{k+\frac{1}{2}}} \cdot \left(\frac{k+\frac{1}{2}}{e}\right)^{k+\frac{1}{2}}}{\sqrt{\frac{2\pi}{k}} \cdot \left(\frac{k}{e}\right)^k \cdot \sqrt{2\pi k}}$$

$$= \lim_{k\to\infty} \frac{\frac{\sqrt{2\pi}}{\sqrt{k+\frac{1}{2}}} \left(\frac{k+\frac{1}{2}}{e}\right)^k \cdot \left(\frac{k+\frac{1}{2}}{e}\right)^{\frac{1}{2}}}{\sqrt{\frac{2\pi}{k}} \cdot \sqrt{2\pi k} \cdot \left(\frac{k}{e}\right)^k}$$

$$= \lim_{k\to\infty} \frac{\sqrt{2\pi}\left(k+\frac{1}{2}\right)^{-\frac{1}{2}} \frac{\left(k+\frac{1}{2}\right)^k}{e^k} \cdot \frac{\left(k+\frac{1}{2}\right)^{\frac{1}{2}}}{e^{\frac{1}{2}}}}{2\pi \cdot \frac{k^k}{e^k}}$$

$$= \lim_{k\to\infty} \frac{\sqrt{2\pi}\cancel{\left(k+\frac{1}{2}\right)^{-\frac{1}{2}}} \frac{\left(k+\frac{1}{2}\right)^k}{\cancel{e^k}} \cdot \frac{\cancel{\left(k+\frac{1}{2}\right)^{\frac{1}{2}}}}{e^{\frac{1}{2}}}}{2\pi \cdot \frac{k^k}{\cancel{e^k}}}$$

$$= \lim_{k\to\infty} \frac{\sqrt{2\pi}\left(k+\frac{1}{2}\right)^k}{2\pi k^k e^{\frac{1}{2}}}$$

$$= \lim_{k\to\infty} \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{k+\frac{1}{2}}{k}\right)^k \cdot e^{-\frac{1}{2}} = \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}} \cdot \lim_{k\to\infty}\left(1+\frac{\frac{1}{2}}{k}\right)^k$$

Again, we see the same similar limit again. The cyan part is $e^{\frac{1}{2}}$. Therefore, we have:

$$\lim_{\nu\to\infty} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}} = \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}} \cdot e^{\frac{1}{2}} = \frac{1}{\sqrt{2\pi}} \tag{10}$$

Putting equations (8) and (10) back to the original PDF of Student's $\boldsymbol{t}$-distribution, we have:

$$f_{\boldsymbol{T}}(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

meaning that $\boldsymbol{T}$ becomes the standard normal when $\nu \to \infty$.

3 The Gamma PDF Satisfies The Normalisation Property

The **Gamma distribution** with a shape parameter α and a scale parameter θ is defined as:

$$f_{\boldsymbol{X}}(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}} \text{ , where } x, \alpha, \theta \text{ are all positive}$$

Alternatively, it can be parameterised with a shape parameter α and a rate parameter λ:

$$f_{\boldsymbol{X}}(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \text{ , where } x, \alpha, \lambda \text{ are all positive}$$

Proof. Using the parameterisation of α and θ:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}} \mathrm{d}x = \frac{1}{\Gamma(\alpha)\theta^\alpha} \int_0^\infty x^{\alpha-1} e^{-\frac{x}{\theta}} \mathrm{d}x &= \frac{1}{\Gamma(\alpha)\theta} \int_0^\infty \frac{x^{\alpha-1}}{\theta^{\alpha-1}} e^{-\frac{x}{\theta}} \mathrm{d}x \\ &= \frac{1}{\Gamma(\alpha)\theta} \int_0^\infty \left(\frac{x}{\theta}\right)^{\alpha-1} e^{-\frac{x}{\theta}} \mathrm{d}x\end{aligned}$$

Let $t = \dfrac{x}{\theta}$, then $\mathrm{d}x = \theta \mathrm{d}t$. We have:

$$\int_{-\infty}^{\infty} \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\theta}} \mathrm{d}x = \frac{1}{\Gamma(\alpha)\theta} \int_{0}^{\infty} t^{\alpha-1} e^{-t} \theta \mathrm{d}t = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1} e^{-t} \mathrm{d}t$$

By definition, the blue part is $\Gamma(\alpha)$. Therefore:

$$\int_{-\infty}^{\infty} \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\theta}} \mathrm{d}x = \frac{1}{\Gamma(\alpha)} \cdot \Gamma(\alpha) = 1$$

□

Proof. Similarly, using the parameterisation of α and λ:

$$\int_{-\infty}^{\infty} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \mathrm{d}x = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_{0}^{\infty} x^{\alpha-1} e^{-\lambda x} \mathrm{d}x = \frac{\lambda}{\Gamma(\alpha)} \int_{0}^{\infty} (\lambda x)^{\alpha-1} e^{-\lambda x} \mathrm{d}x$$

Again, let $t = \lambda x$, then $\mathrm{d}x = \dfrac{1}{\lambda} \mathrm{d}t$. We have:

$$\int_{-\infty}^{\infty} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \mathrm{d}x = \frac{\lambda}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1} e^{-t} \frac{1}{\lambda} \mathrm{d}t = 1$$

□