

Lecture 26 Error, Power And Sample Size Estimation

BIO210 Biostatistics

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School of Life Sciences

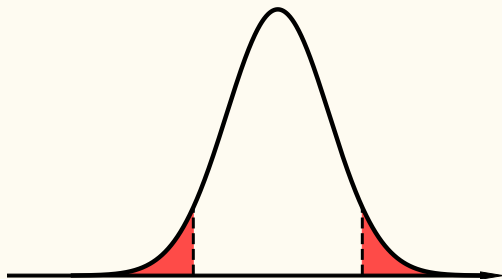
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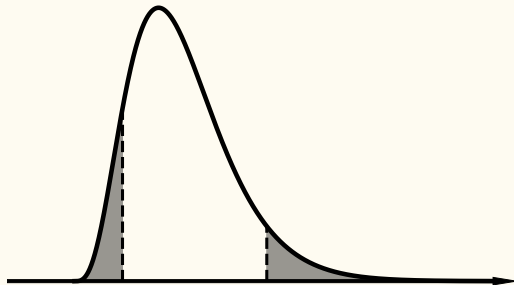
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One-sample Hypothesis Testing For Variance

Sampling distribution of the sample
mean/proportion



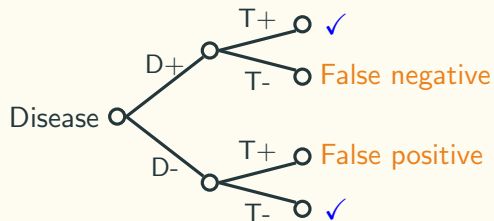
Sampling distribution of the sample
variance



Significance level α : never 0 !

Types of Errors

Diagnostic testing



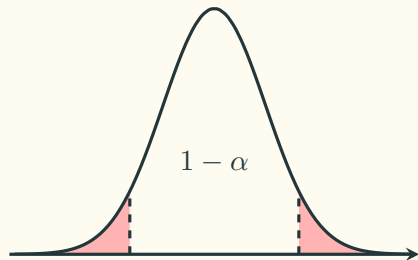
Hypothesis testing

Decision \ Truth	Population	
	H_0 is true	H_0 is false
Reject H_0	Type I Error	✓
Do not reject H_0	✓	Type II Error

Probability of Making Different Errors

- $\mathbb{P}(\text{Type I Error}) = ?$
- $\mathbb{P}(\text{reject } H_0 \mid H_0 \text{ is true}) = ?$

When H_0 is true:



Type I error is also called the **rejection error** or **α error**.

- $\mathbb{P}(\text{Type II Error}) = ?$
- $\mathbb{P}(\text{Do not reject } H_0 \mid H_0 \text{ is false}) = \beta$
- $1 - \beta$ is more useful in reality

Definition

The **Power** of the test is defined as
 $\mathbb{P}(\text{reject } H_0 \mid H_0 \text{ is false}) = 1 - \beta$

Calculating Power - Roughly

- Serum cholesterol level for 20- to 74-year-old males
- Truth about the whole population: normally distributed, mean 200 mg/100ml ($\mu = 200$), standard deviation 46 mg/100ml ($\sigma = 46$): we don't know this!
- A subpopulation (20- to 24-year-old males): normally distributed, mean 180 mg/100ml ($\mu = 180$), standard deviation 46 mg/100ml ($\sigma = 46$): we do know this from a previous study!
- Now we are interested in serum cholesterol level for 20- to 74-year-old males, and a random sample of size ($n = 25$) is drawn from the 20- to 74-year-old male population. We conduct a hypothesis testing about the mean.

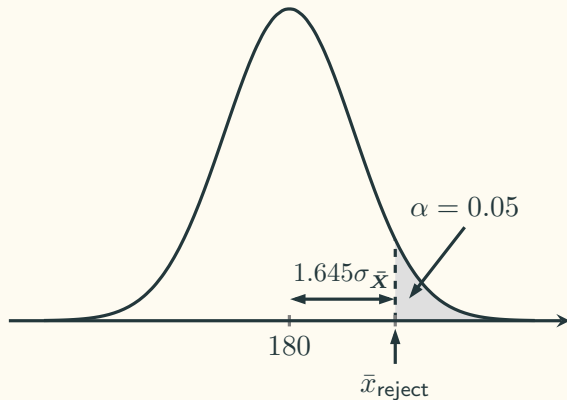
Calculating Power - Roughly

- We have reasons to believe that the mean serum cholesterol level of 20- to 74-year-old male should be higher than 180 mg/100ml.

- $H_0 : \mu \leq \mu_0 = 180 \text{ mg/100ml}$
- $H_1 : \mu > \mu_0 = 180 \text{ mg/100ml}$

- Truth:

$$H_1 : \mu = \mu_1 = 200 \text{ mg/100ml}$$

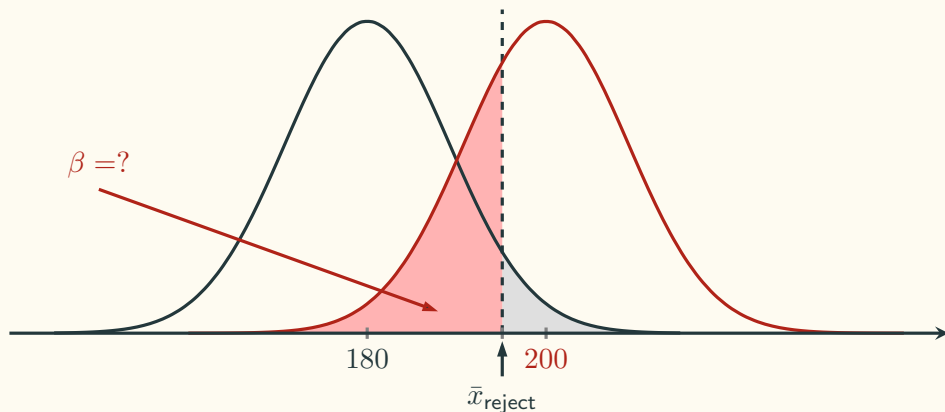


$$\bar{x}_{\text{reject}} = 180 + 1.645 \times \frac{46}{\sqrt{25}} = 195.1$$

Calculating Power - Roughly

What we want to calculate: **Power** ($1 - \beta$): $\mathbb{P}(\text{reject } H_0 \mid H_0 \text{ is false})$

Truth: $H_1 : \mu = \mu_1 = 200 \text{ mg/100ml}$

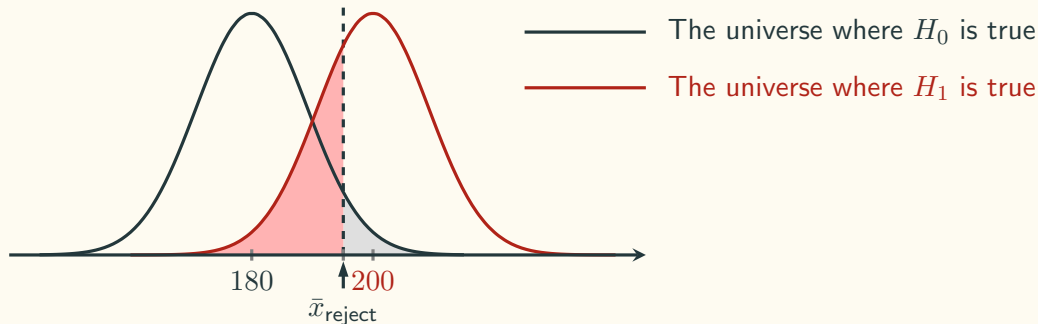


Calculating Power - Roughly

$$\begin{aligned}\beta &= \mathbb{P}(\text{do not reject } H_0 \mid H_0 \text{ is false}) \\ &= \mathbb{P}\left(\bar{x} \leq 195.1 \mid \mu_{\bar{X}} = \mu_1 = 200, \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{25}} = \frac{46}{5} = 9.2\right) \\ &= \mathbb{P}\left(z \leq \frac{195.1 - 200}{9.2}\right) = \mathbb{P}(z \leq -0.53) = 0.298\end{aligned}$$

Power: $1 - \beta = 1 - 0.298 = 0.702$

How To Increase Power



Increasing the power of the test

1. Shift \bar{x}_{reject} to the left
2. μ_0 shifts to the left, or μ_1 shifts to the right.
3. Make the sampling distribution narrower

Ways of achieving the goal

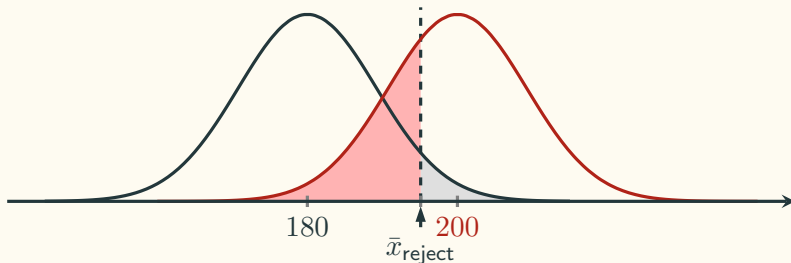
1. Increase α
2. Increase $|\mu_0 - \mu_1|$
3. Decrease $\sigma_{\bar{X}} \Leftrightarrow$ Increase n

Sample Size Estimation

In the previous example about the serum cholesterol level, suppose we want a significance level of 0.01 and a power of 0.95. What is the minimum sample size needed for the test?

$\alpha = 0.01, \beta = 0.05, \text{power} = 0.95$: when H_0 is true, we want to risk a 1% chance of rejecting it; when H_0 is false, we want to risk a 5% chance of failing to reject it. The power of the test is 95%.

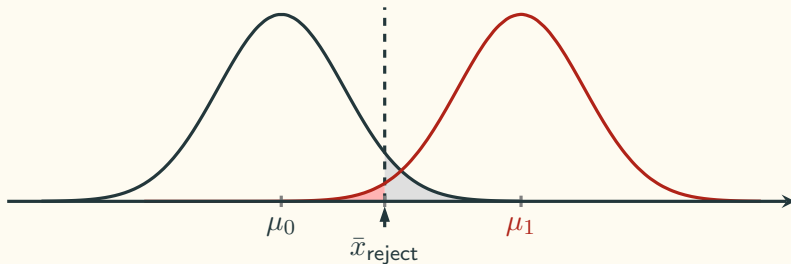
Sample Size Estimation



1. The minimum sample mean to reject H_0 : $\bar{x}_{\text{reject}} = 180 + 2.32 \times \frac{46}{\sqrt{n}}$
2. Calculate β :

$$\begin{aligned}\beta &= \mathbb{P}(\bar{x} \leq \bar{x}_{\text{reject}} \mid H_0 \text{ is false}) = \mathbb{P}\left(\bar{x} \leq \bar{x}_{\text{reject}} \mid \mu_{\bar{X}} = 200, \sigma_{\bar{X}} = \frac{46}{\sqrt{n}}\right) \\ &= \mathbb{P}\left(z \leq \frac{180 + 2.32 \times \frac{46}{\sqrt{n}} - 200}{\frac{46}{\sqrt{n}}}\right) = 0.05 \Rightarrow n = 83.165\end{aligned}$$

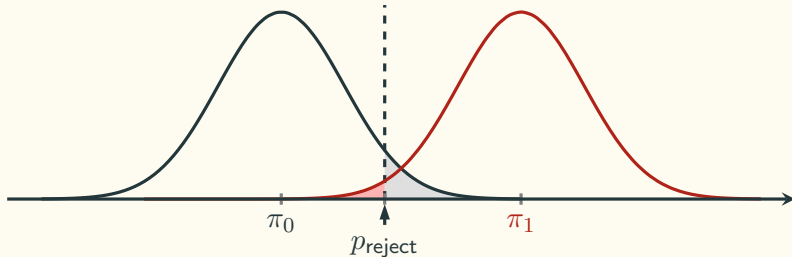
Sample Size Estimation For One Sided Test For The Mean



1. The minimum sample mean to reject H_0 : $\bar{x}_{\text{reject}} = \mu_0 + Z_\alpha \times \frac{\sigma}{\sqrt{n}}$
2. Calculate β :

$$\begin{aligned}\beta &= \mathbb{P}(\bar{x} \leq \bar{x}_{\text{reject}} \mid H_0 \text{ is false}) = \mathbb{P}\left(\bar{x} \leq \bar{x}_{\text{reject}} \mid \mu_{\bar{X}} = \mu_1, \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}\right) \\ &= \mathbb{P}\left(z \leq \frac{\mu_0 + Z_\alpha \times \frac{\sigma}{\sqrt{n}} - \mu_1}{\frac{\sigma}{\sqrt{n}}}\right) \Rightarrow n = \left[\frac{(Z_\alpha + Z_\beta)\sigma}{\mu_1 - \mu_0}\right]^2\end{aligned}$$

Sample Size Estimation For One Sided Test For The Proportion



1. The minimum sample mean to reject H_0 : $p_{\text{reject}} = \pi_0 + Z_\alpha \times \sqrt{\frac{\pi_0(1 - \pi_0)}{n}}$
2. Calculate β :

$$\beta = \mathbb{P}(p \leq p_{\text{reject}} \mid H_0 \text{ is false}) = \mathbb{P}\left(p \leq p_{\text{reject}} \mid \mu_p = \pi_1, \sigma_p = \sqrt{\frac{\pi_1(1 - \pi_1)}{n}}\right)$$

$$\Rightarrow n = \left[\frac{Z_\alpha \sqrt{\pi_0(1 - \pi_0)} + Z_\beta \sqrt{\pi_1(1 - \pi_1)}}{\pi_1 - \pi_0} \right]^2$$