

Lecture 20 Confidence Interval For The Variance

BIO210 Biostatistics

Xi Chen

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School of Life Sciences

Southern University of Science and Technology



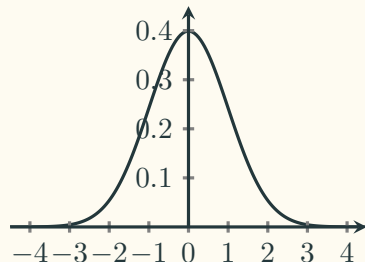
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Interval Estimation For The Variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \quad \text{Not normally distributed!}$$

Chi-squared Distribution

$$Z_1, Z_2, Z_3, \dots, Z_n \sim \mathcal{N}(0, 1)$$



$$U_1 = Z_1^2$$

$$U_1 \sim \chi^2(1)$$

$$U_2 = Z_1^2 + Z_2^2$$

$$U_2 \sim \chi^2(2)$$

$$U_3 = Z_1^2 + Z_2^2 + Z_3^2$$

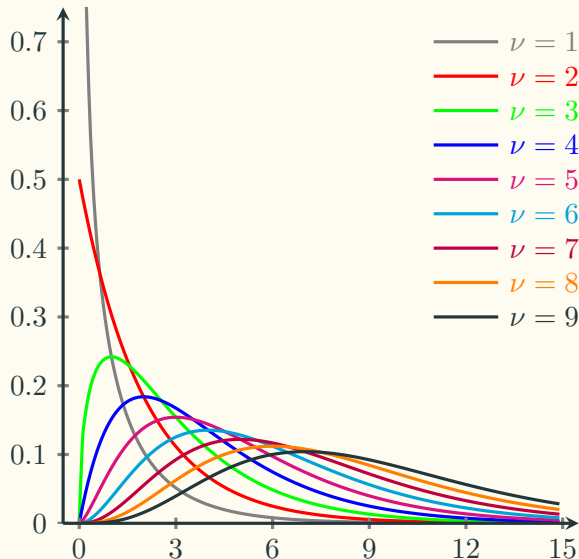
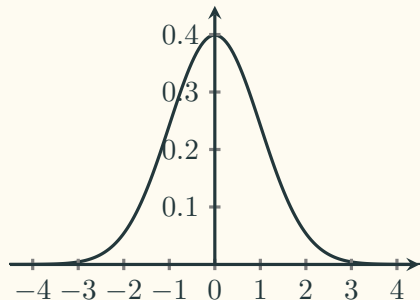
$$U_3 \sim \chi^2(3)$$

$$\vdots$$
$$\vdots$$

$$U_n = Z_1^2 + Z_2^2 + Z_3^2 + \dots + Z_n^2$$

$$U_n \sim \chi^2(n)$$

Chi-squared Distributions of Different ν



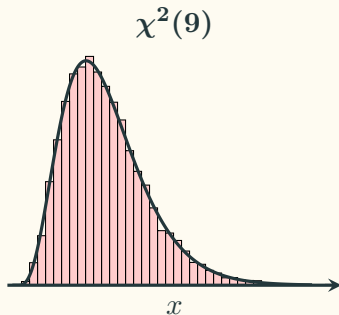
Sampling Distribution of The Sample Variance

Let X_1, X_2, \dots, X_n be a sample independently drawn from a normal distribution $\mathcal{N}(\mu, \sigma^2)$, and we define the sample mean and variance as:

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i, \quad S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

Then we have:

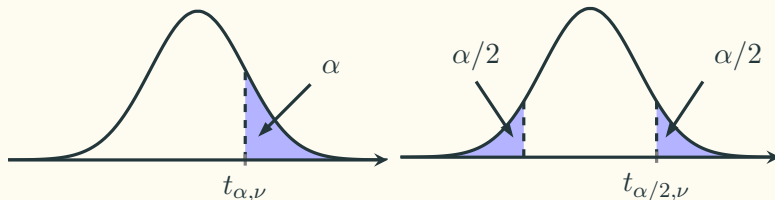
$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$



Confidence Interval For The Mean

Recall: construct $(1 - \alpha) \times 100\%$ confidence interval for the mean.

- One-sided (lower bound): $\mathbb{P} \left(\mu \geq \bar{X} - t_{\alpha, \nu} \cdot \frac{s}{\sqrt{n}} \right) = 1 - \alpha$
- One-sided (upper bound): $\mathbb{P} \left(\mu \leq \bar{X} + t_{\alpha, \nu} \cdot \frac{s}{\sqrt{n}} \right) = 1 - \alpha$
- Two-sided: $\mathbb{P} \left(\bar{X} - t_{\alpha/2, \nu} \cdot \frac{s}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{\alpha/2, \nu} \cdot \frac{s}{\sqrt{n}} \right) = 1 - \alpha$



Confidence Interval For Variance

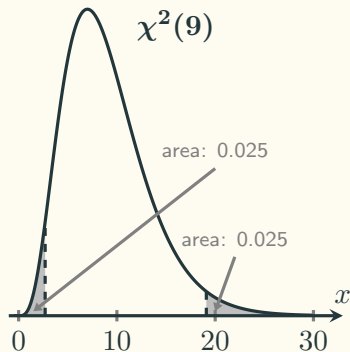
Goal: for a population with unknown variance σ^2 , find a and b , such that $\mathbb{P}(a \leq \sigma^2 \leq b) = 0.95$

$$\mathbb{P}(\chi_{0.975, \nu}^2 \leq \chi^2 \leq \chi_{0.025, \nu}^2) = 0.95$$

$$\mathbb{P}\left[\chi_{0.975, \nu}^2 \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi_{0.025, \nu}^2\right] = 0.95$$

$$\mathbb{P}\left[\frac{1}{\chi_{0.025, \nu}^2} \leq \frac{\sigma^2}{(n-1)S^2} \leq \frac{1}{\chi_{0.975, \nu}^2}\right] = 0.95$$

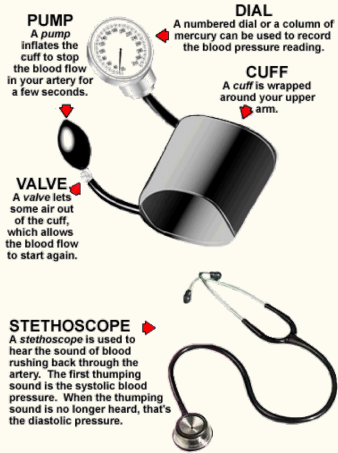
$$\mathbb{P}\left(\frac{n-1}{\chi_{0.025, \nu}^2} \cdot S^2 \leq \sigma^2 \leq \frac{n-1}{\chi_{0.975, \nu}^2} \cdot S^2\right) = 0.95$$



95% CI for σ^2 :

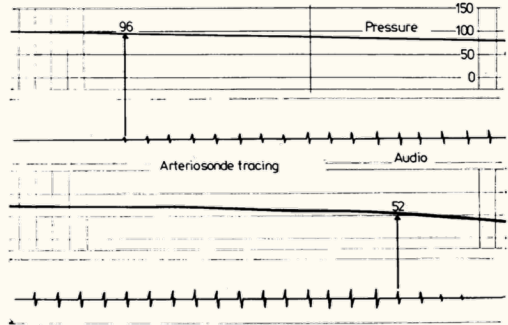
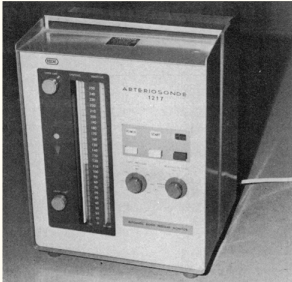
$$\left[\frac{n-1}{\chi_{0.025, \nu}^2} \cdot S^2, \frac{n-1}{\chi_{0.975, \nu}^2} \cdot S^2 \right]$$

Sphygmomanometer



Confidence interval for variance

Hypertension: An Arteriosonde machine “prints” blood-pressure readings on a tape so that the measurement can be read rather than heard. A major argument for using such a machine is that the variability of measurements obtained by different observers on the same person will be lower than with a standard blood-pressure cuff.



Confidence interval for variance

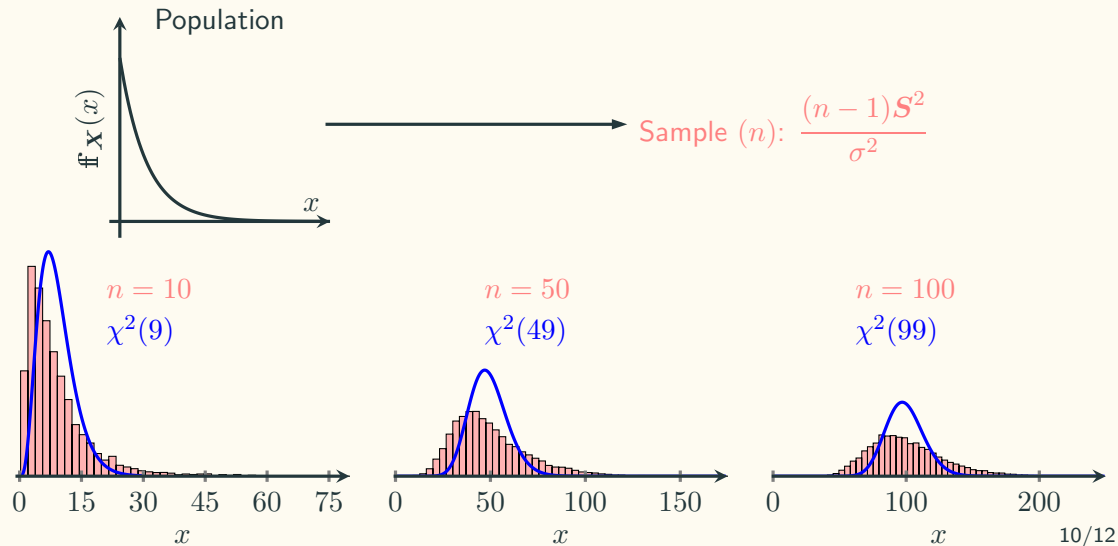
| Persion (i) | Observer #1 | Observer #2 | Difference (d_i) |
|----------------|----------------|----------------|-------------------------|
| 1 | 194 | 200 | -6 |
| 2 | 126 | 123 | 3 |
| 3 | 130 | 128 | 2 |
| 4 | 98 | 101 | -3 |
| 5 | 136 | 135 | 1 |
| 6 | 145 | 145 | 0 |
| 7 | 110 | 111 | -1 |
| 8 | 108 | 107 | 1 |
| 9 | 102 | 99 | 3 |
| 10 | 126 | 128 | -2 |

Does it make sense to construct a 95% CI for the variance?

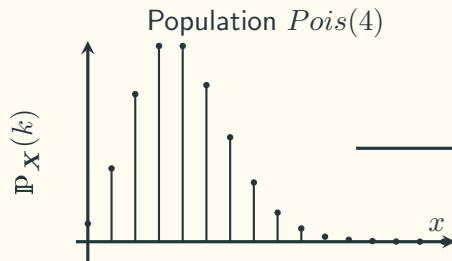
95% CI for σ^2 :

$$\left[\frac{n-1}{\chi_{0.025,\nu}^2} \cdot s^2, \frac{n-1}{\chi_{0.975,\nu}^2} \cdot s^2 \right]$$

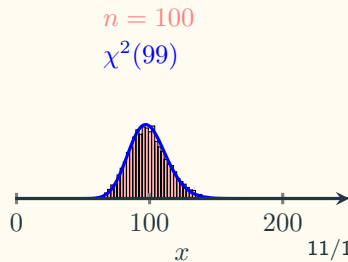
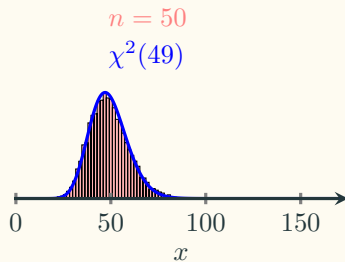
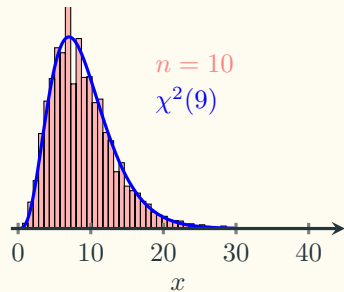
Sampling Distribution of The Sample Variance



Sampling Distribution of The Sample Variance



Sample (n) : $\frac{(n-1)S^2}{\sigma^2}$



Conditions For Valid Confidence Intervals For The Variance

1. Random Samples
2. Independence ($n < 10\%$ population size)
3. Original population distribution must be normal