Lecture 26 Error, Power And Sample Size Estimation

BIO210 Biostatistics

Xi Chen

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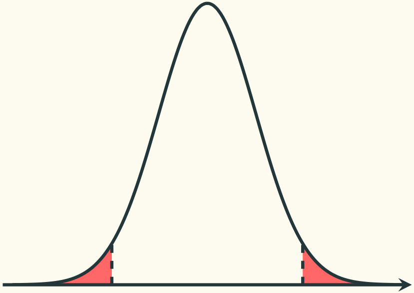
School of Life Sciences
Southern University of Science and Technology

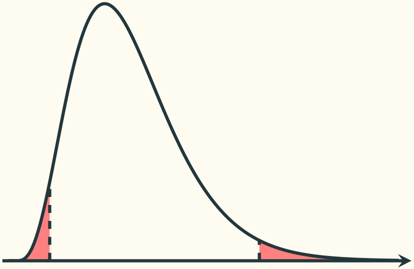
南方科技大学生命科学学院
SUSTech · SCHOOL OF
LIFE SCIENCES

One-sample Hypothesis Testing For Variance

Sampling distribution of the sample mean/proportion



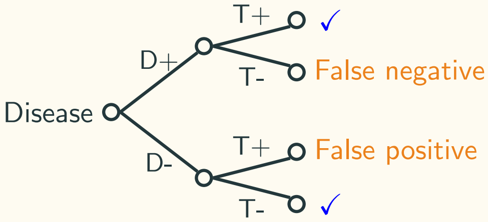
Sampling distribution of the sample variance



Significance level α: never 0 !

Types of Errors

Diagnostic testing



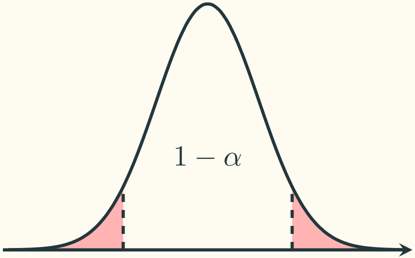
Hypothesis testing

Truth / Decision	Population	
	H_0 is true	H_0 is false
Reject H_0	Type I Error	✓
Do not reject H_0	✓	Type II Error

Probability of Making Different Errors

- $\mathbb{P}\,(\text{Type I Error}) = ?$
- $\mathbb{P}\,(\text{reject } H_0 \mid H_0 \text{ is true}) = ?$

When H_0 is true:



Type I error is also called the rejection error or α error.

- $\mathbb{P}\,(\text{Type II Error}) = ?$
- $\mathbb{P}\,(\text{Do not reject } H_0 \mid H_0 \text{ is false}) = \beta$
- $1-\beta$ is more useful in reality

Definition

The Power of the test is defined as

$\mathbb{P}\,(\text{reject } H_0 \mid H_0 \text{ is false}) = 1-\beta$

Calculating Power - Roughly

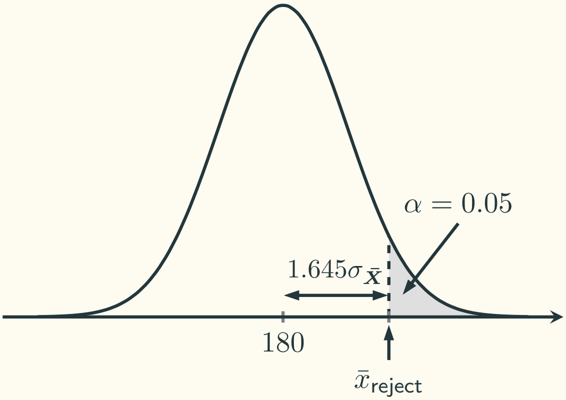
- Serum cholesterol level for 20- to 74-year-old males
- Truth about the whole population: normally distributed, mean 200 mg/100ml ($\mu = 200$), standard deviation 46 mg/100ml ($\sigma = 46$): we don't know this!
- A subpopulation (20- to 24-year-old males): normally distributed, mean 180 mg/100ml ($\mu = 180$), standard deviation 46 mg/100ml ($\sigma = 46$): we do know this from a previous study!
- Now we are interested in serum cholesterol level for 20- to 74-year-old males, and a random sample of size ($n = 25$) is drawn from the 20- to 74-year-old male population. We conduct a hypothesis testing about the mean.

Calculating Power - Roughly

- We have reasons to believe that the mean serum cholesterol level of 20- to 74-year-old male should be higher than 180 mg/100ml.

- $H_0 : \mu \leqslant \mu_0 = 180$ mg/100ml
- $H_1 : \mu > \mu_0 = 180$ mg/100ml

- Truth:
  $H_1 : \mu = \mu_1 = 200$ mg/100ml

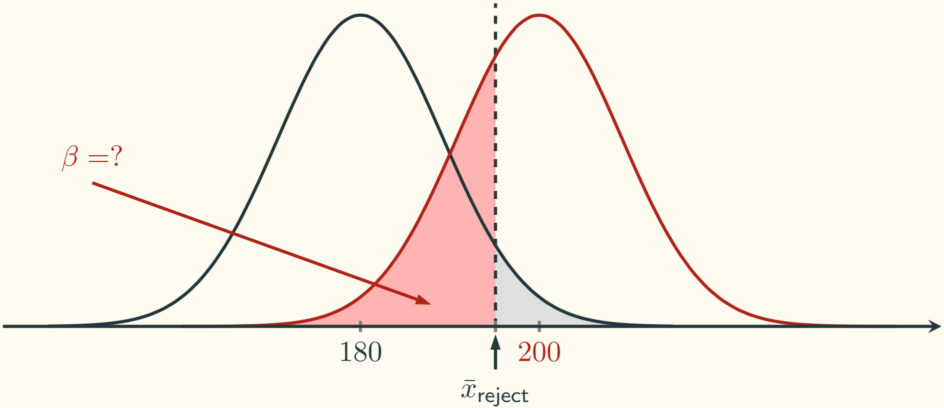


$$\bar{x}_{\text{reject}} = 180 + 1.645 \times \frac{46}{\sqrt{25}} = 195.1$$

Calculating Power - Roughly

What we want to calculate: **Power $(1-\beta)$:** $\mathbb{P}$ (reject $H_0 \mid H_0$ is false)

Truth: $H_1 : \mu = \mu_1 = 200$ mg/100ml



Calculating Power - Roughly

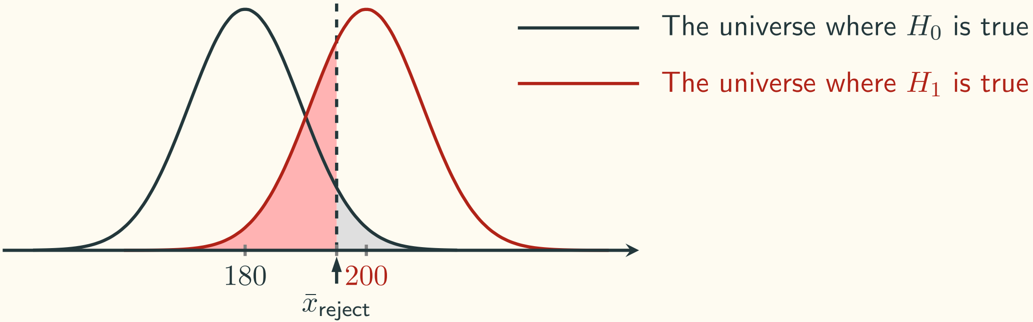
$$\beta = \mathbb{P}\left(\text{do not reject } H_0 \,|\, H_0 \text{ is false}\right)$$

$$= \mathbb{P}\left(\bar{x} \leqslant 195.1 \,|\, \mu_{\bar{\boldsymbol{X}}} = \mu_1 = 200,\ \sigma_{\bar{\boldsymbol{X}}} = \frac{\sigma}{\sqrt{25}} = \frac{46}{5} = 9.2\right)$$

$$= \mathbb{P}\left(z \leqslant \frac{195.1 - 200}{9.2}\right) = \mathbb{P}\left(z \leqslant -0.53\right) = 0.298$$

Power: $1 - \beta = 1 - 0.298 = 0.702$

How To Increase Power



Increasing the power of the test

1. Shift $\bar{x}_{\text{reject}}$ to the left
2. μ_0 shifts to the left, or μ_1 shifts to the right.
3. Make the sampling distribution narrower

Ways of achieving the goal

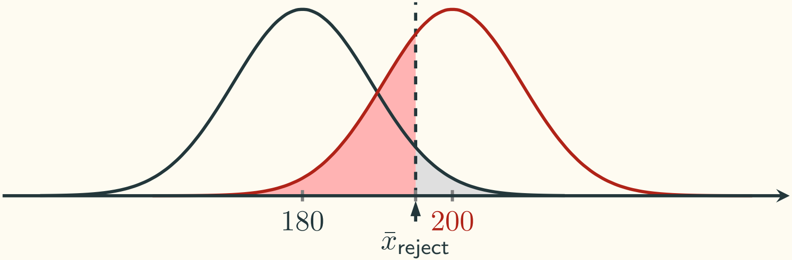
1. Increase α
2. Increase $|\mu_0 - \mu_1|$
3. Decrease $\sigma_{\bar{X}} \Leftrightarrow$ Increase n

Sample Size Estimation

In the previous example about the serum cholesterol level, suppose we want a significance level of 0.01 and a power of 0.95. What is the minimum sample size needed for the test?

$\alpha = 0.01, \beta = 0.05, \text{power} = 0.95$: when H_0 is true, we want to risk a 1% chance of rejecting it; when H_0 is false, we want to risk a 5% chance of failing to reject it. The power of the test is 95%.

Sample Size Estimation

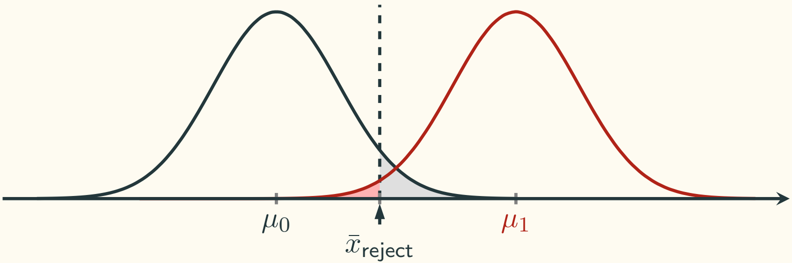


1. The minimum sample mean to reject H_0: $\bar{x}_{\text{reject}} = 180 + 2.32 \times \dfrac{46}{\sqrt{n}}$
2. Calculate β:

$$\beta = \mathbb{P}\left(\bar{x} \leqslant \bar{x}_{\text{reject}} \,\middle|\, H_0 \text{ is false}\right) = \mathbb{P}\left(\bar{x} \leqslant \bar{x}_{\text{reject}} \,\middle|\, \mu_{\bar{\boldsymbol{X}}} = 200, \sigma_{\bar{\boldsymbol{X}}} = \frac{46}{\sqrt{n}}\right)$$

$$= \mathbb{P}\left(z \leqslant \frac{180 + 2.32 \times \frac{46}{\sqrt{n}} - 200}{\frac{46}{\sqrt{n}}}\right) = 0.05 \;\Rightarrow\; n = 83.165$$

Sample Size Estimation For One Sided Test For The Mean

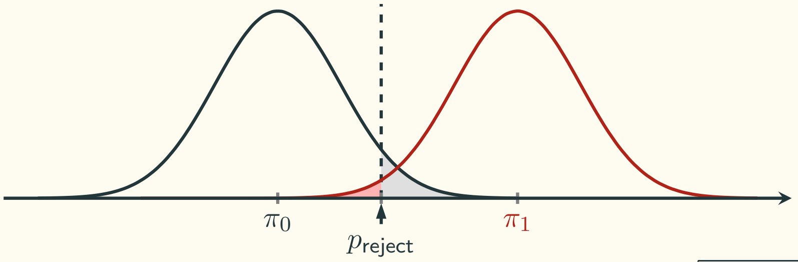


1. The minimum sample mean to reject H_0: $\bar{x}_{\text{reject}} = \mu_0 + Z_\alpha \times \dfrac{\sigma}{\sqrt{n}}$
2. Calculate β:

$$\beta = \mathbb{P}\left(\bar{x} \leqslant \bar{x}_{\text{reject}} \,\middle|\, H_0 \text{ is false}\right) = \mathbb{P}\left(\bar{x} \leqslant \bar{x}_{\text{reject}} \,\middle|\, \mu_{\bar{\boldsymbol{X}}} = \mu_1, \sigma_{\bar{\boldsymbol{X}}} = \frac{\sigma}{\sqrt{n}}\right)$$

$$= \mathbb{P}\left(z \leqslant \frac{\mu_0 + Z_\alpha \times \frac{\sigma}{\sqrt{n}} - \mu_1}{\frac{\sigma}{\sqrt{n}}}\right) \Rightarrow n = \left[\frac{(Z_\alpha + Z_\beta)\sigma}{\mu_1 - \mu_0}\right]^2$$

Sample Size Estimation For One Sided Test For The Proportion



1. The minimum sample mean to reject H_0: $p_{\text{reject}} = \pi_0 + Z_\alpha \times \sqrt{\dfrac{\pi_0(1-\pi_0)}{n}}$
2. Calculate β:

$$\beta = \mathbb{P}\left(p \leqslant p_{\text{reject}} \mid H_0 \text{ is false}\right) = \mathbb{P}\left(p \leqslant p_{\text{reject}} \,\middle|\, \mu_p = \pi_1, \sigma_p = \sqrt{\frac{\pi_1(1-\pi_1)}{n}}\right)$$

$$\Rightarrow \; n = \left[\frac{Z_\alpha\sqrt{\pi_0(1-\pi_0)} + Z_\beta\sqrt{\pi_1(1-\pi_1)}}{\pi_1 - \pi_0}\right]^2$$