

Lecture 12 Continuous Probability Distribution

BIO210 Biostatistics

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Fall, 2023

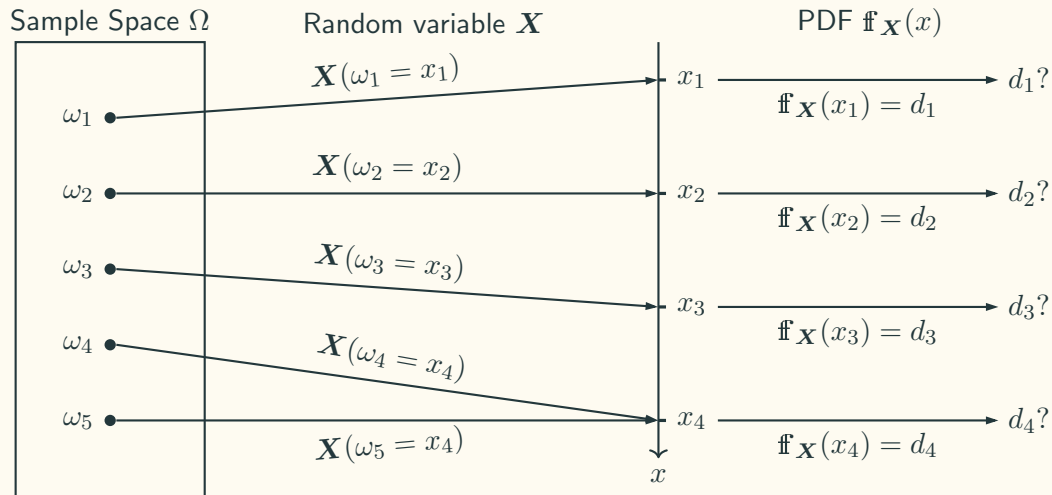
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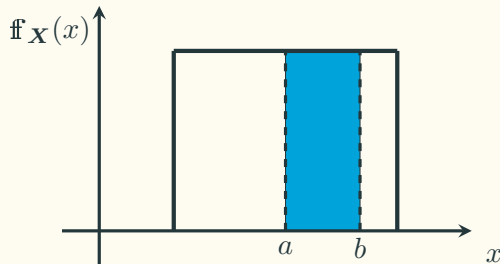
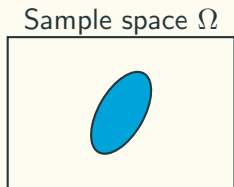


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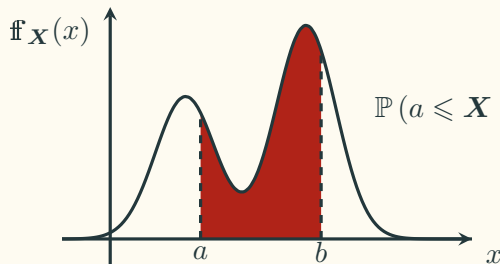
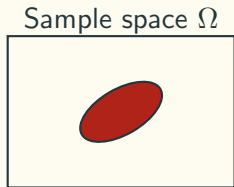
Probability Density Function (PDF)



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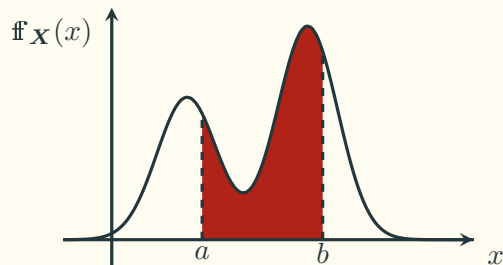


$$\mathbb{P}(a \leq X \leq b) \\ = f_X(x) \cdot (b - a)$$



$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x) dx$$

Probability Density Function (PDF)



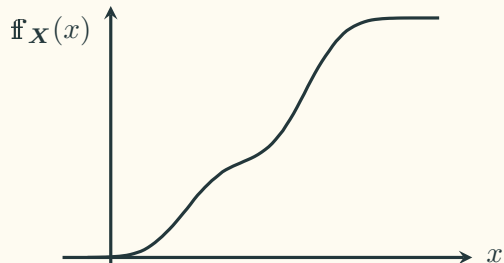
$$f_X(x) \geq 0, \quad \int_{-\infty}^{+\infty} f_X(x) dx = 1$$

$$\mathbb{P}(X = a) = ?$$

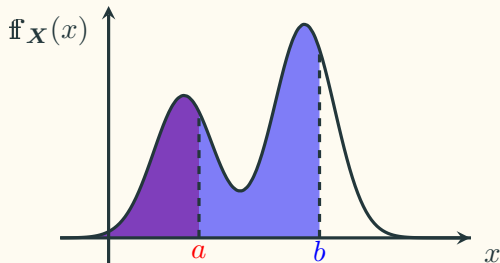
$$\begin{aligned} \mathbb{P}(x \leq X \leq x + \delta) \\ = \int_x^{x+\delta} f_X(x) dx = f_X(x) \cdot \delta \end{aligned}$$

$$f_X(x) = \frac{\mathbb{P}(x \leq X \leq X + \delta)}{\delta}$$

Cumulative Distribution Function



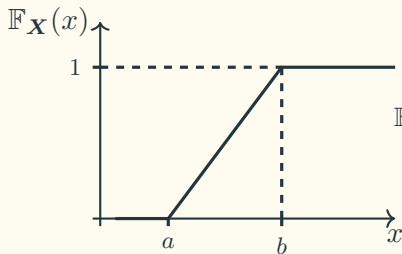
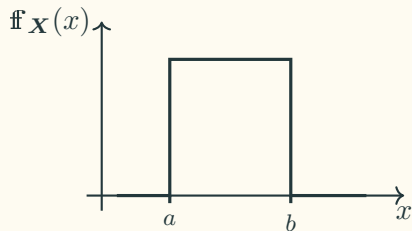
$$\mathbb{F}_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^x f_X(t) dt$$



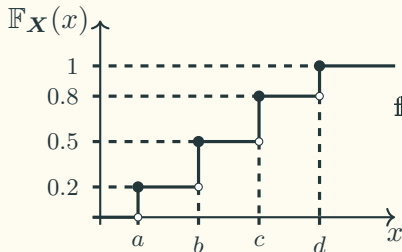
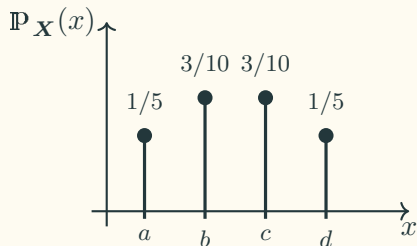
$$\mathbb{F}_X(a) = \mathbb{P}(X \leq a) = \int_{-\infty}^a f_X(x) dx$$

$$\mathbb{F}_X(b) = \mathbb{P}(X \leq b) = \int_{-\infty}^b f_X(x) dx$$

Cumulative Distribution Functions (CDFs)



$$F_X(x) = \mathbb{P}(X \leq x) \\ = \int_{-\infty}^x f_X(t) dt$$



$$f_X(x) = \mathbb{P}(X \leq x) \\ = \sum_{k \leq x} P_X(k)$$

Expectation and Variance

The continuous case

$$\mathbb{E}[\mathbf{X}] = \int_{-\infty}^{+\infty} x \mathbf{f}_{\mathbf{X}}(x) \, dx$$

$$\mathbb{E}[g(\mathbf{X})] = \int_{-\infty}^{+\infty} g(x) \mathbf{f}_{\mathbf{X}}(x) \, dx$$

$$\begin{aligned}\mathbb{V}\text{ar}(\mathbf{X}) &= \sigma_{\mathbf{X}}^2 = \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}])^2] \\ &= \int_{-\infty}^{+\infty} (\mathbf{X} - \mathbb{E}[\mathbf{X}])^2 \mathbf{f}_{\mathbf{X}}(x) \, dx \\ &= \mathbb{E}[\mathbf{X}^2] - (\mathbb{E}[\mathbf{X}])^2\end{aligned}$$

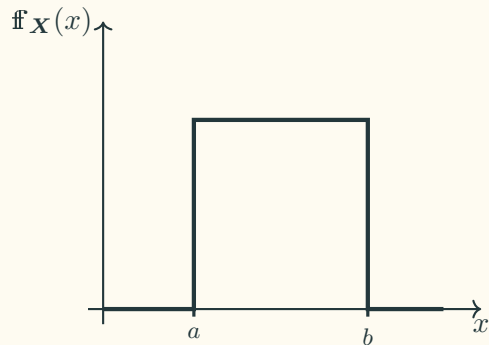
The discrete case

$$\mathbb{E}[\mathbf{X}] = \sum_x x \mathbf{P}_{\mathbf{X}}(x)$$

$$\mathbb{E}[g(\mathbf{X})] = \sum_x g(x) \mathbf{P}_{\mathbf{X}}(x)$$

$$\begin{aligned}\mathbb{V}\text{ar}(\mathbf{X}) &= \sigma_{\mathbf{X}}^2 = \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}])^2] \\ &= \sum_x (\mathbf{X} - \mathbb{E}[\mathbf{X}])^2 \mathbf{P}_{\mathbf{X}}(x) \\ &= \mathbb{E}[\mathbf{X}^2] - (\mathbb{E}[\mathbf{X}])^2\end{aligned}$$

Continuous Uniform Distribution

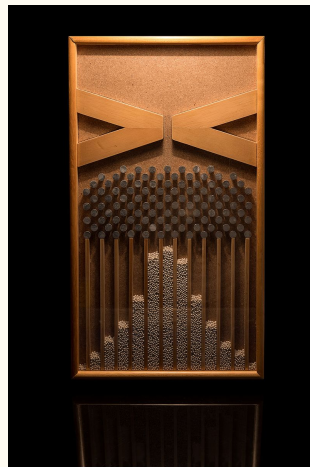
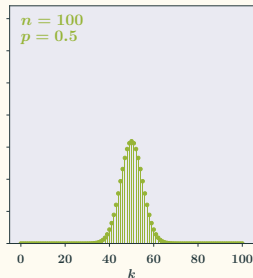
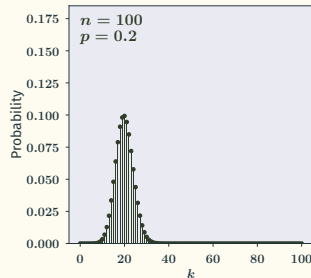
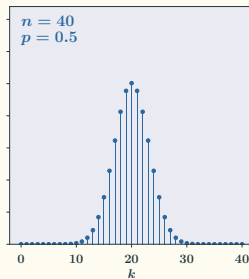
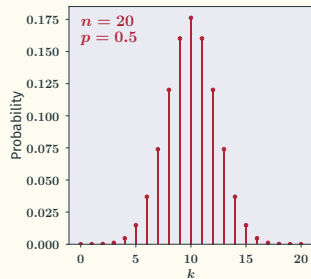


$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & x < a \text{ or } x > b \end{cases}$$

$$\mathbb{E}[X] = ?$$

$$\mathbb{V}\text{ar}(X) = ?$$

The Idea of The Normal Distributions

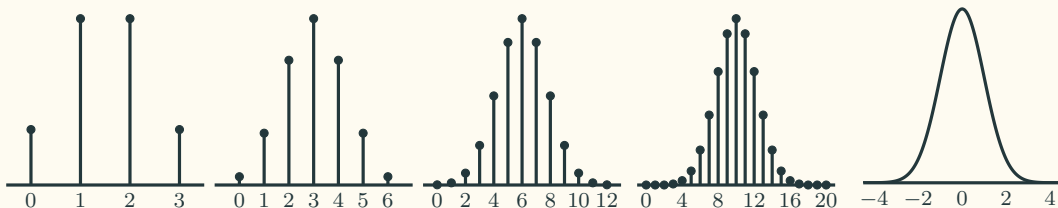


The Bean Machine
by Francis Galton

A Little History of The Normal Distribution - Binomial Approximation

Abraham de Moivre: The Doctrine of Chances, 1738

- The middle term of the binomial coefficient is approximately: $\frac{2}{\sqrt{2\pi n}}$
- Stirling's formula: $n! \simeq n^n e^{-n} \sqrt{2\pi n}$



The de Moivre–Laplace Theorem

When n becomes large, and np, nq are also large:

$$\mathbb{P}_{\mathbf{X}}(k) = \binom{n}{k} p^k q^{n-k} \approx \frac{1}{\sqrt{2\pi} \sqrt{npq}} \cdot e^{-\frac{(k - np)^2}{2npq}}, \text{ where } q = 1 - p$$

A Little History of the Normal Distribution - The Error Curve

Carl Friedrich Gauss: Theoria motus corporum coelestium in sectionibus conicis solem ambientium (Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections), 1809.

- least squares
- maximum likelihood
- normal distribution

Pierre Simon de Laplace

- In 1782: $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$
- In 1810: the central limit theorem

A Little History of the Normal Distribution - Beyond Errors

Adolphe Quetelet: the average man - Letters addressed to H.R.H. the grand duke of Saxe Coburg and Gotha, on the Theory of Probabilities as Applied to the Moral and Political Sciences, 1846.

TABLE 1: Chest measurement of Scottish soldiers

Girth	Frequency
33	3
34	18
35	81
36	185
37	420
38	749
39	1,073
40	1,079
41	934
42	658
43	370
44	92
45	50
46	21
47	4
48	1