

Lecture 10 Random Variables, PMF, Expectation & Variance

BIO210 Biostatistics

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What is a random variable (r.v.) ?

- An assignment of a value (a real number) to every possible outcome in the sample space.
- **Mathematically:** A real-valued **function** defined on a sample space Ω . In a particular experiment, a random variable (r.v.) would be some function that assigns a real number to each possible outcome.

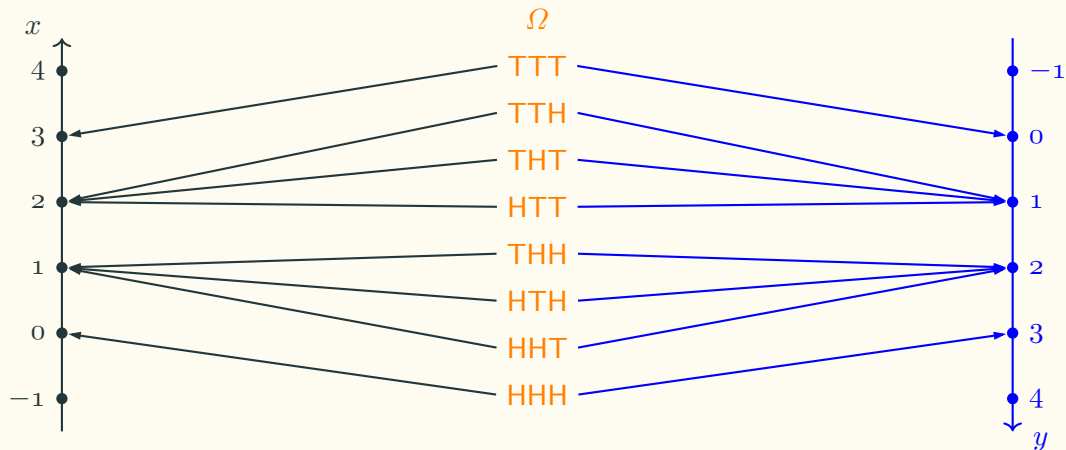
More about random variables

- Discrete or continuous.
- Can have several random variables defined on the same sample space.
- Notation
 - random variable X : function $\Omega \mapsto \mathbb{R}$
 - numerical value: x : value $\in \mathbb{R}$

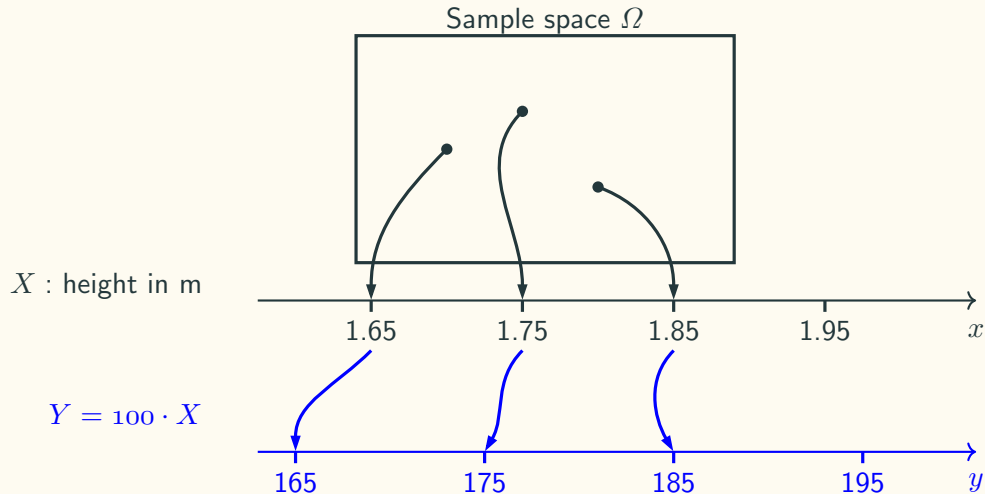
Different random variables on the same sample space

X : number tails

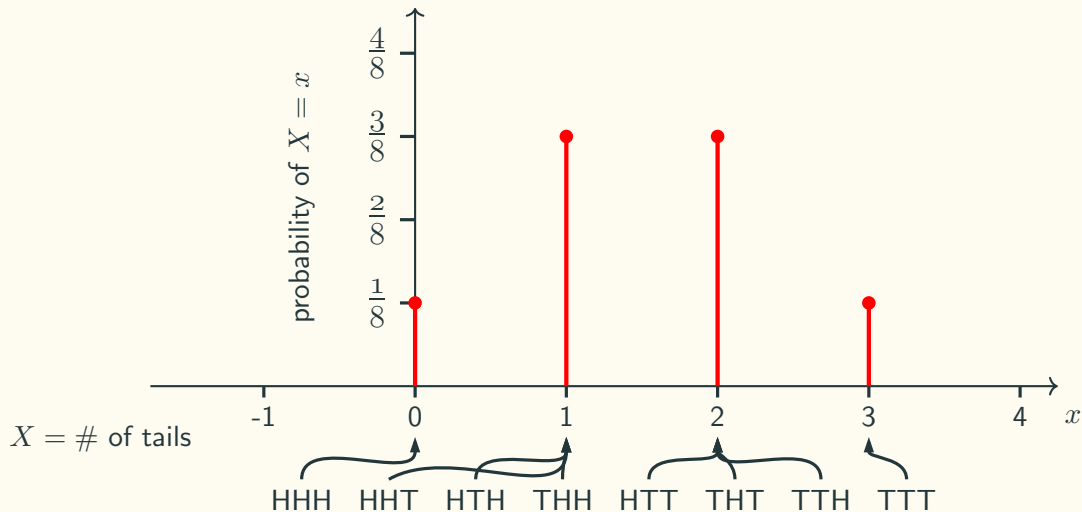
Y : number heads



Function of a random variable is an r.v.



Probability Mass Function (PMF)



Probability Mass Function (PMF)

The PMF of X = number of tails after three flips

x	$P(X = x)$
0	$1/8$
1	$3/8$
2	$3/8$
3	$1/8$
otherwise	0

$$P(X = x) = \begin{cases} \frac{1}{8}, & x = 0, 3 \\ \frac{3}{8}, & x = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

Probability Mass Function

- Notation

$$\begin{aligned}p_X(x) &= P(X = x) \\ &= P(\{\omega \in \Omega \mid X(\omega) = x\})\end{aligned}$$

- Properties

$$\begin{aligned}p_X(x) &\geq 0 \\ \sum_x p_X(x) &= 1\end{aligned}$$

ω	$X(\omega) = x$	$p_X(x) = p$
HHH	0	$\frac{1}{8}$
THH, HTH, HHT	1	$\frac{3}{8}$
TTH, THT, TTH	2	$\frac{3}{8}$
TTT	3	$\frac{1}{8}$

Geometric PMF

Experiment: keep flipping a coin ($P(H) = p$) until a head comes up for the first time.
Let the random variable X be the number of flips.

ω	$X(\omega)$	$p_X(x)$
H	1	p
TH	2	$(1 - p)p$
THH	3	$(1 - p)^2 p$
\vdots	\vdots	\vdots
$\underbrace{\text{TTT} \dots \text{TTT}}_{n-1} \text{H}$	n	$(1 - p)^{n-1} p$

Geometric PMF. X : geometric random variable.

How to compute a PMF $p_X(x)$

To compute a PMF $p_X(x)$:

1. Collect all possible outcomes for which $X = x$;
2. add their probabilities;
3. repeat for all x .

Compute PMF

Experiment: two independent rolls of a fair tetrahedral die.

F : outcome of the first roll

S : outcome of the second roll

$$X = \min(F, S)$$

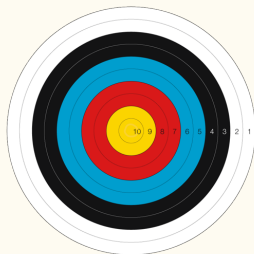
$$p_X(x) = ?$$

4				
3				
2				
1				
	1	2	3	4
F : first roll				

Expected value of a random variable (Expectation)

Experiment: archery

Let X be the score you get for each shot. What is the expected value of X ?



Think: What is the average score you will get after a large number of trials?

x	$p_X(x)$
1	0.19
2	0.17
3	0.15
4	0.13
5	0.11
6	0.09
7	0.07
8	0.05
9	0.03
10	0.01

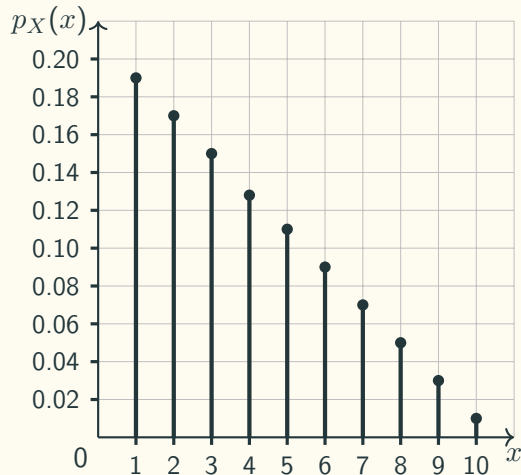
Expected value (Expectation)

Definition

$$E[X] = \sum_x xp_X(x)$$

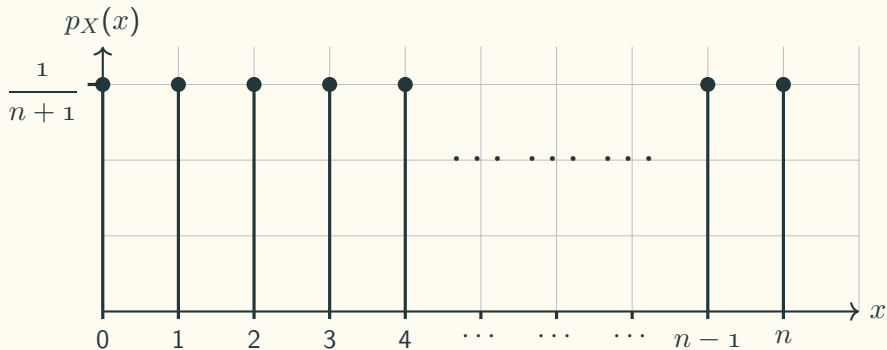
- Interpretation
 1. Centre of gravity of the PMF
 2. Average in large number of repetitions of the experiment

PMF of X from the archery experiment



Expectation of a Uniform Distribution

Example: a uniform discrete random variable X on $0, 1, 2, 3, \dots, n$



What is $E[X]$?

Properties of expectations

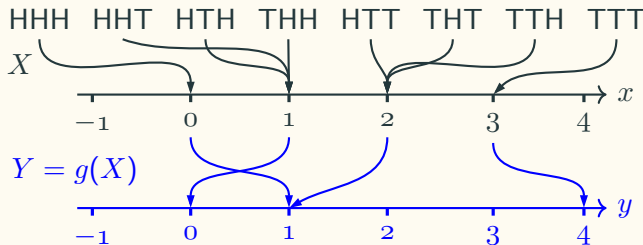
Let X be a random variable, and let $Y = g(X)$, what is $E[Y]$?

- The hard way:

$$E[Y] = \sum_y p_Y(y)$$

- The easy way:

$$E[Y] = \sum_x g(x)p_X(x)$$



y	$p_Y(y)$
0	3/8
1	4/8
4	1/8

x	$g(x)$	$p_X(x)$
0	1	1/8
1	0	3/8
2	1	3/8
3	4	1/8

Expectation of a linear function of r.v.

- Caution: in general $E[g(x)] \neq g(E[X])$
- Exception: if α, β are constants, then we have:
 - $E[\alpha] = \alpha$
 - $E[\alpha X] = \alpha E[X]$
 - $E[\alpha X + \beta] = \alpha E[X] + \beta$

Variance of a random variable

Definition of Variance

$$\text{var}(X) = E[(X - E[X])^2]$$

Properties of Variance

- $\text{var}(X) = E[X^2] - (E[X])^2$
- If α, β are constants, then $\text{var}(\alpha X + \beta) = \alpha^2 \text{var}(X)$

Random Variables (Summary slide)

