Lecture 18 The Error Curve Derived By MLE

BIO210 Biostatistics

Xi Chen

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School of Life Sciences
Southern University of Science and Technology

南方科技大学生命科学学院
SUSTech · SCHOOL OF LIFE SCIENCES

Multiple Measurements & Measurement Errors

- **Ptolemy**: picked the value that fits the theory
- **Tycho Brahe**: introduced the technique of repeating and combining observations, but how ? - Not sure.
- **Galileo**: the measurement errors deserved a systematic and scientific treatment

Repeated Experiments Or Not?

Robert Boyle (the president of the Royal Society), circa 1660:

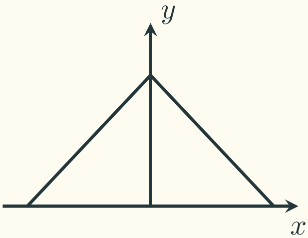
... experiments ought to be estimated by their value, not their number; ... a single experiment ... may as well deserve an entire treatise ... As one of those large and orient pearls ... may outvalue a very great number of those little ... pearls, that are to be bought by the ounce ...

Galileo's Summary

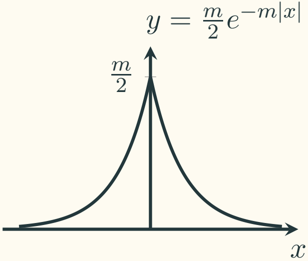
1. The observations are distributed **symmetrically about the true value**; that is, the errors are distributed symmetrically about zero.
2. Small errors occur more frequently than large errors.

Examples of The Error Curves

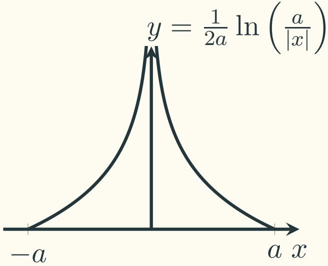
Simpson's error curve



Laplace's first error curve



Laplace's second error curve



The Error Curve

- The true location of a heavenly body is m
- We have n independent measurements $(X_1, X_2, X_3, \cdots, X_n)$ on the same heavenly body
- The measurements have errors $\mathrm{E}_i = X_i - m$
- The errors have a pattern $f_{\mathbf{E}}(\varepsilon)$
- We know that $f_{\mathbf{E}}(\varepsilon) = f_{\mathbf{E}}(-\varepsilon)$ and $f'_{\mathbf{E}}(\varepsilon) = -f'_{\mathbf{E}}(-\varepsilon)$

We want to figure out $f_{\mathbf{E}}(\varepsilon)$

Using The MLE

We write out the likelihood function:

$$\boldsymbol{\mathcal{L}}(m; \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = f(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n; m) = \prod_{i=1}^{n} f_{\mathbf{E}}(\varepsilon_i)$$

We convert it to the log-likelihood function:

$$\ell = \sum_{i=1}^{n} \ln f_{\mathbf{E}}(\varepsilon_i)$$

We take the derivate with respect to ε, and let it be 0:

$$\frac{\mathrm{d}\ell}{\mathrm{d}\varepsilon} = \sum_{i=1}^{n} \frac{f'_{\mathbf{E}}(\varepsilon_i)}{f_{\mathbf{E}}(\varepsilon_i)} = 0$$

The Arithmetic Mean

this cannot be defined *a priori*, we will, approaching the subject from another point of view, inquire upon what function, tacitly, as it were, assumed as a base, the common principle, the excellence of which is generally acknowledged, depends. It has been customary certainly to regard as an axiom the hypothesis that if any quantity has been determined by several direct observations, made under the same circumstances and with equal care, the arithmetical mean of the observed values affords the most probable value, if not rigorously, yet very nearly at least, so that it is always most safe to adhere to it. By putting, therefore,

$$V = V' = V'' \text{ etc.} = p,$$

The Gaussian Integral

Moreover, it is readily perceived that k must be negative, in order that Ω may really become a maximum, for which reason we shall put

$$\tfrac{1}{2}k = -hh;$$

and since, by the elegant theorem first discovered by LAPLACE, the integral

$$\int e^{-hh\Delta\Delta}\,d\Delta$$

from $\Delta = -\infty$ to $\Delta = +\infty$ is $\frac{\sqrt{\pi}}{h}$, (denoting by π the semicircumference of the circle the radius of which is unity), our function becomes

$$\varphi\Delta = \frac{h}{\sqrt{\pi}}e^{-hh\Delta\Delta}.$$

MLE For Parameters of Normal Distributions

Practice: Compute the MLE for μ and σ^2 of a normal distribution based on the observation $x_1, x_2, x_3, ..., x_n$.

1. $\theta : \mu, \sigma^2$
2. $\mathbf{\Omega} : \{(\mu, \sigma^2) \mid \mu \in (-\infty, +\infty), \sigma^2 \geqslant 0\}$
3. $f_X(x) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}$
4. $\boldsymbol{\mathcal{L}} = f(x_1, x_2, x_3, ..., x_n; \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \cdot e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}$

$$\hat{\boldsymbol{\mu}} = \bar{X}, \qquad \hat{\boldsymbol{\sigma^2}} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$\hat{\mu} = \bar{x}, \qquad \hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Why $n-1$, part 2

Unbiased Variance Estimator

$$\hat{\boldsymbol{\sigma}}^{\mathbf{2}} = \boxed{\frac{n}{n-1}} \cdot \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Bessel's Correction - Friedrich Bessel