Lecture 6 The Bayes' Theorem

BIO210 Biostatistics

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School of Life Sciences
Southern University of Science and Technology
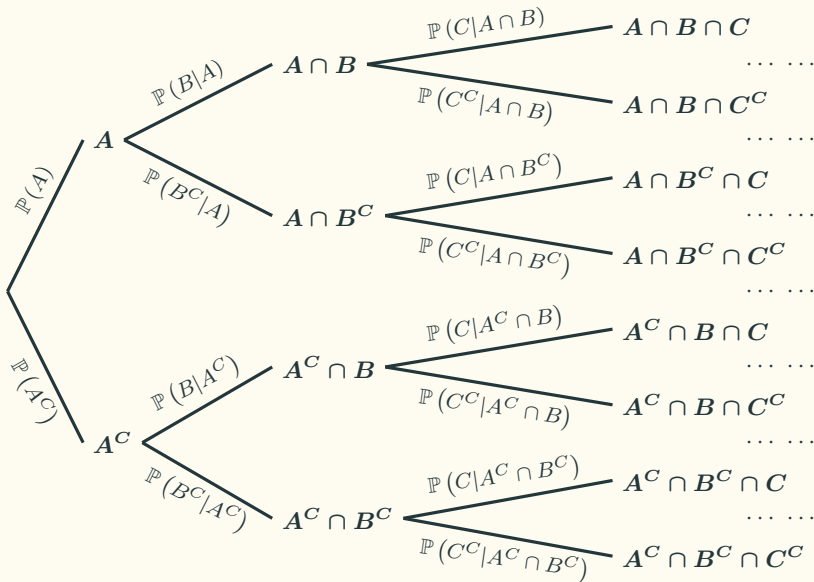
南方科技大学生命科学学院
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Basic components

Three basic components in conditional probability:

1. $\mathbb{P}(A \cap B)$
2. $\mathbb{P}(B)$
3. $\mathbb{P}(A|B)$

Generalisation of $\mathbb{P}\left(A \cap B \cap C \cap \cdots\right)$

The multiplication rule:

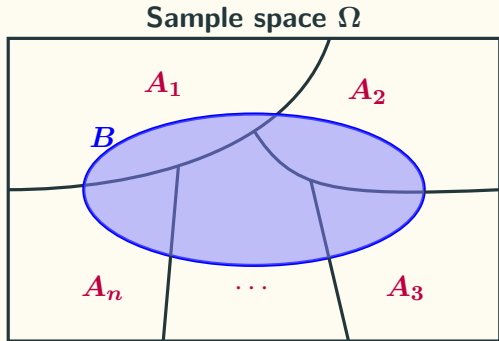$$P\left(\bigcap_{i=1}^{n} A_i\right) =$$

$$P(A_1)\cdot$$

$$P(A_2|A_1)\cdot$$

$$P(A_3|A_1 \cap A_2)\cdot$$

$$P(A_4|A_1 \cap A_2 \cap A_3)\cdot$$

$$\cdots$$

$$P\left(A_n| \bigcap_{i=1}^{n-1} A_i\right)$$
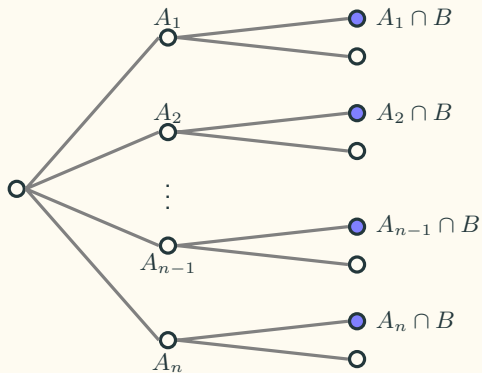
Sample space Ω

$$\begin{aligned}
\mathbb{P}(B) =& \mathbb{P}[(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)] \\
=& \mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B) + \cdots + \mathbb{P}(A_n \cap B) \\
=& \mathbb{P}(A_1)\,\mathbb{P}(B|A_1) + \\
& \mathbb{P}(A_2)\,\mathbb{P}(B|A_2) + \\
& \vdots \\
& \mathbb{P}(A_n)\,\mathbb{P}(B|A_n)
\end{aligned}$$

The total probability theorem:

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)\,\mathbb{P}(B|A_i)$$

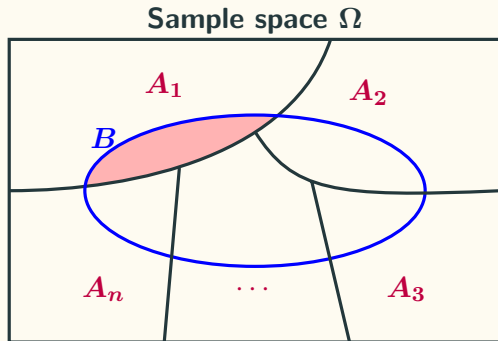# Generalisation of $\mathbb{P}(B)$

$$\begin{aligned}
\mathbb{P}(B) =& \mathbb{P}[(A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)] \\
=& \mathbb{P}(A_1 \cap B) + \mathbb{P}(A_2 \cap B) + \cdots + \mathbb{P}(A_n \cap B) \\
=& \mathbb{P}(A_1)\mathbb{P}(B|A_1) + \\
& \mathbb{P}(A_2)\mathbb{P}(B|A_2) + \\
& \quad \vdots \\
& \mathbb{P}(A_n)\mathbb{P}(B|A_n)
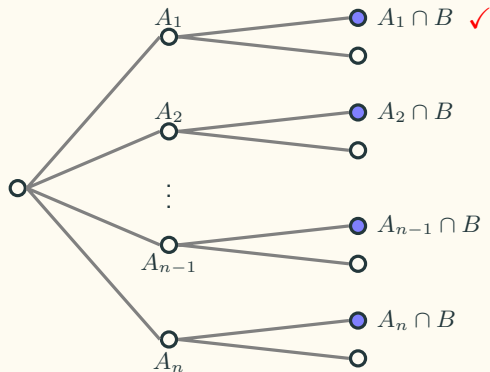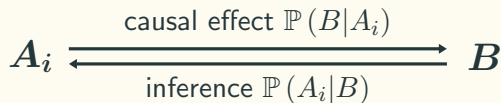\end{aligned}$$

The total probability theorem:

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)\mathbb{P}(B|A_i)$$
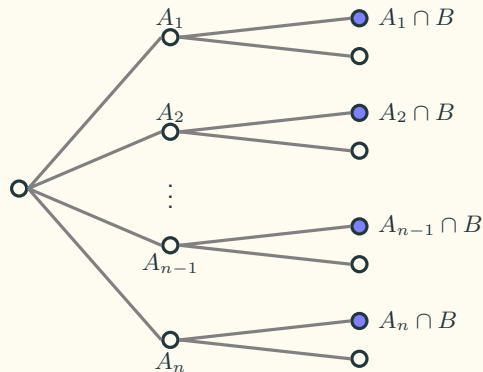
Sample space Ω

A_1 A_2 B A_n $\cdots$ A_3

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)}$$

$$= \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\mathbb{P}(B)}$$

$$= \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\sum_{i=1}^{n}\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}$$
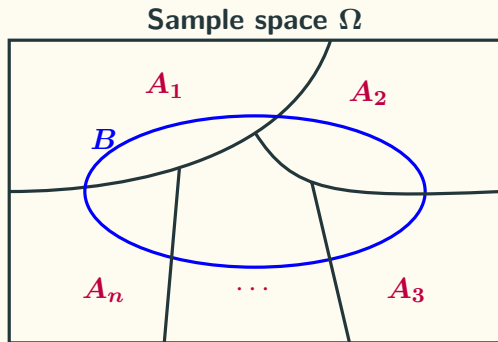
Generalisation of $\mathbb{P}(A|B)$

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)}$$

$$= \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\mathbb{P}(B)}$$

$$= \frac{\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}{\sum_{i=1}^{n}\mathbb{P}(A_i)\,\mathbb{P}(B|A_i)}$$
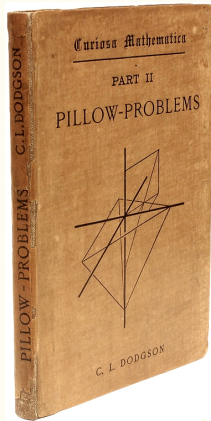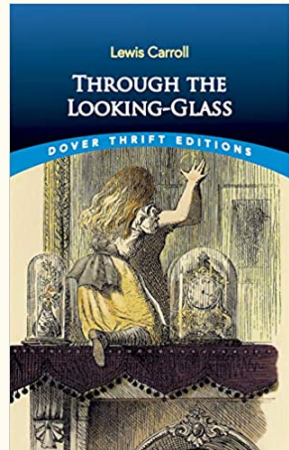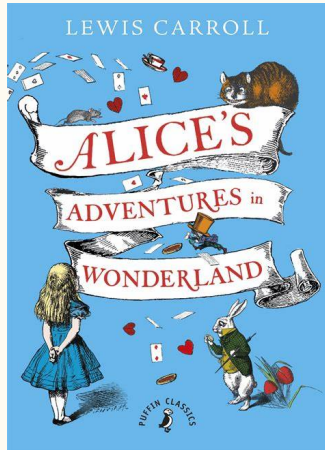
The Bayes Theorem

$$A_i \; \xmapsto[\text{inference } \mathbb{P}\left(A_i|B\right)]{\text{causal effect } \mathbb{P}\left(B|A_i\right)} \; B$$

The Bayes' Theorem

$$\mathbb{P}\left(A_i|B\right) = \frac{\mathbb{P}\left(A_i \cap B\right)}{\mathbb{P}\left(B\right)} = \frac{\mathbb{P}\left(A_i\right)\mathbb{P}\left(B|A_i\right)}{\sum_{i=1}^{n}\mathbb{P}\left(A_i\right)\mathbb{P}\left(B|A_i\right)}$$

$\mathbb{P}\left(A_i\right)$: **prior probability**

$\mathbb{P}\left(A_i|B\right)$: **posterior probability**

Lewis Carroll's Pillow Problems

by Charles Dodgson

Lewis Carroll's Pillow Problems
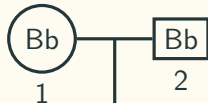
Question #5 (8th Sep 1887):

A bag contains one counter, known to be either white or black. A white counter is put in, the bag shaken, and a counter drawn out, which proves to be white. What is now the chance of drawing a white counter?

Carroll's Pillow Problem #5

There is a ball inside a non-transparent bag. The colour of the ball is unknown, but it is equally likely to be either blue or red. Now you put a red ball into the bag, shake the bag, and take a ball without looking inside. The ball you have just taken out is red. What is the probability that the colour of the remaining ball that is still inside the bag is red?
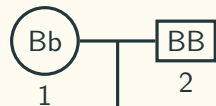
Pedigree Analysis