

Lecture 37 Simple Linear Regression - The Idea

BIO210 Biostatistics

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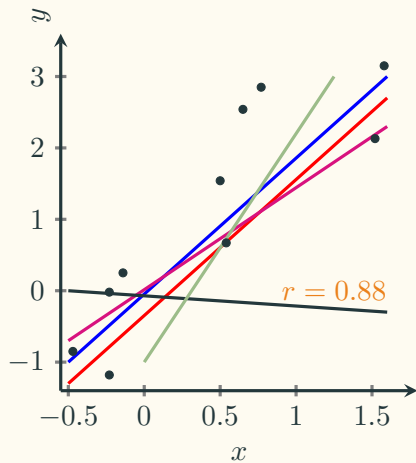
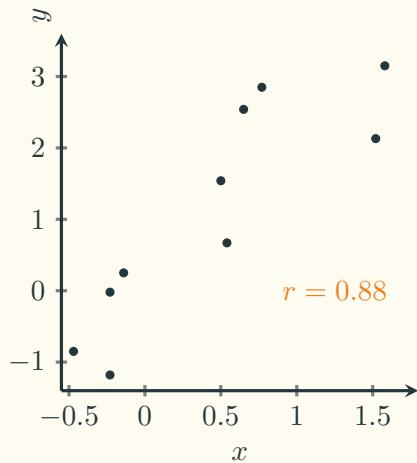
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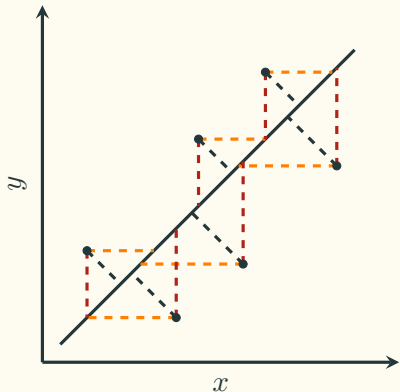


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Linear Regression



Best Fit Line



Our goal: minimise the “difference” between the data points and the line.

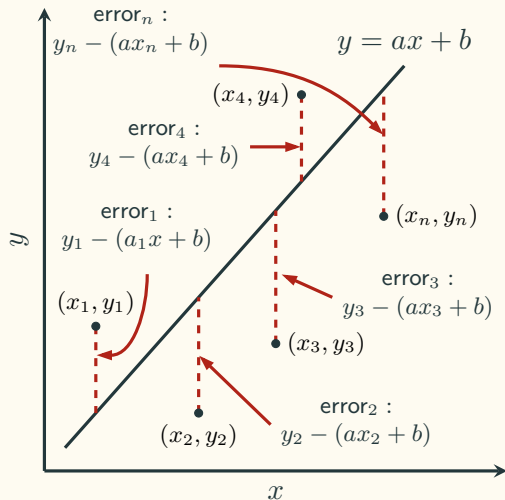
{ Deming regression, PCA etc.

{ Errors-in-variables models

{ Ordinary least squares (OLS) regression

In practice: minimise squared distance.

Best Fit Line

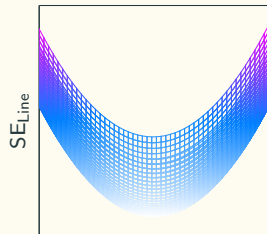
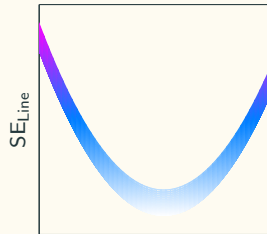
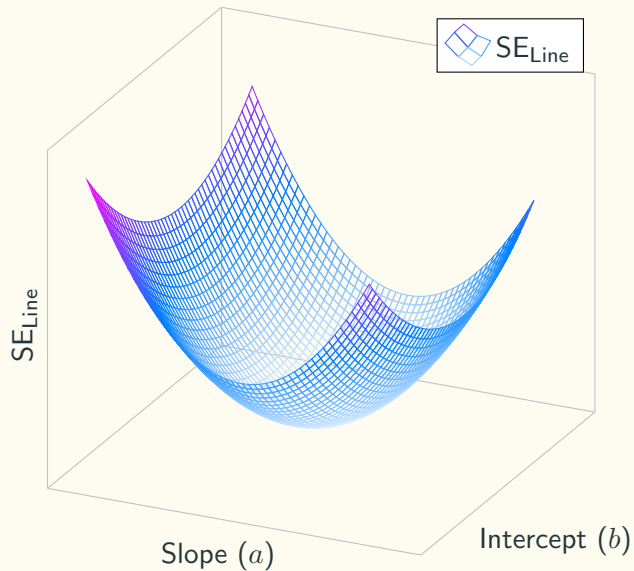


Squared error against the line (SE_{line}):

$$\begin{aligned} SE_{line} = & [y_1 - (ax_1 + b)]^2 + \\ & [y_2 - (ax_2 + b)]^2 + \\ & [y_3 - (ax_3 + b)]^2 + \\ & [y_4 - (ax_4 + b)]^2 + \\ & \vdots \\ & [y_n - (ax_n + b)]^2 \end{aligned}$$

Find a, b to minimise this sum of squares

The Best Fit Line



Minimise SE_{line}

$$\begin{aligned} SE_{line} &= \sum_{i=1}^n [y_i - (ax_i + b)]^2 \\ &= \sum_{i=1}^n [y_i^2 - 2y_i(ax_i + b) + (ax_i + b)^2] \\ &= \sum_{i=1}^n (y_i^2 - 2ax_iy_i - 2by_i + a^2x_i^2 + 2abx_i + b^2) \\ &= \sum_{i=1}^n y_i^2 - 2a \sum_{i=1}^n x_iy_i - 2b \sum_{i=1}^n y_i + a^2 \sum_{i=1}^n x_i^2 + 2ab \sum_{i=1}^n x_i + nb^2 \end{aligned}$$

Minimise SE_{line}

$$SE_{line} = \left(\sum_{i=1}^n x_i^2 \right) \cdot a^2 + 2 \left(b \sum_{i=1}^n x_i - \sum_{i=1}^n x_i y_i \right) \cdot a + \left(\sum_{i=1}^n y_i^2 - 2b \sum_{i=1}^n y_i + nb^2 \right)$$

$$\Rightarrow \frac{\partial SE_{line}}{\partial a} = \left(2 \sum_{i=1}^n x_i^2 \right) \cdot a + 2 \left(b \sum_{i=1}^n x_i - \sum_{i=1}^n x_i y_i \right)$$

$$SE_{line} = n \cdot b^2 + \left(2a \sum_{i=1}^n x_i - 2 \sum_{i=1}^n y_i \right) \cdot b + \left(a^2 \sum_{i=1}^n x_i^2 - 2a \sum_{i=1}^n x_i y_i + \sum_{i=1}^n y_i^2 \right)$$

$$\Rightarrow \frac{\partial SE_{line}}{\partial b} = 2n \cdot b + \left(2a \sum_{i=1}^n x_i - 2 \sum_{i=1}^n y_i \right)$$

Minimise SE_{line}

$$\text{Let } \frac{\partial SE_{line}}{\partial a} = 0 \Rightarrow \left(2 \sum_{i=1}^n x_i^2 \right) \cdot a + 2 \left(b \sum_{i=1}^n x_i - \sum_{i=1}^n x_i y_i \right) = 0$$

$$\Rightarrow n \cdot \overline{x^2} \cdot a + b \cdot n \cdot \bar{x} - n \cdot \overline{xy} = 0$$

$$\Rightarrow \overline{x^2} \cdot a + b \cdot \bar{x} - \overline{xy} = 0$$

$$\text{Let } \frac{\partial SE_{line}}{\partial b} = 0 \Rightarrow 2n \cdot b + \left(2a \sum_{i=1}^n x_i - 2 \sum_{i=1}^n y_i \right) = 0$$

$$\Rightarrow n \cdot b + a \cdot n \cdot \bar{x} - n \cdot \bar{y} = 0$$

$$\Rightarrow b + a \cdot \bar{x} - \bar{y} = 0$$

Best Fit Line: $y = ax + b$

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To minimise SE_{line} , we need:

$$\begin{cases} a \cdot \overline{x^2} + b \cdot \bar{x} - \overline{xy} = 0 \\ b + a \cdot \bar{x} - \bar{y} = 0 \end{cases} \Rightarrow \begin{cases} a \cdot \frac{\overline{x^2}}{\bar{x}} + b = \frac{\overline{xy}}{\bar{x}} \\ a \cdot \bar{x} + b = \bar{y} \end{cases}$$

Points

(\bar{x}, \bar{y}) & $\left(\frac{\overline{x^2}}{\bar{x}}, \frac{\overline{xy}}{\bar{x}}\right)$
are on the best fit line !

Best Fit Line: $y = ax + b$

Best fit line: $y = ax + b$

$$\begin{cases} a \cdot \frac{\overline{x^2}}{\bar{x}} + b = \frac{\overline{xy}}{\bar{x}} \\ a \cdot \bar{x} + b = \bar{y} \end{cases} \Rightarrow a \left(\bar{x} - \frac{\overline{x^2}}{\bar{x}} \right) = \bar{y} - \frac{\overline{xy}}{\bar{x}} \Rightarrow a = \frac{\bar{y} - \overline{xy}/\bar{x}}{\bar{x} - \overline{x^2}/\bar{x}} = \frac{\bar{x} \cdot \bar{y} - \overline{xy}}{(\bar{x})^2 - \overline{x^2}}$$

- More widely used form:
$$a = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{Cov(x, y)}{Cov(x, x)}$$

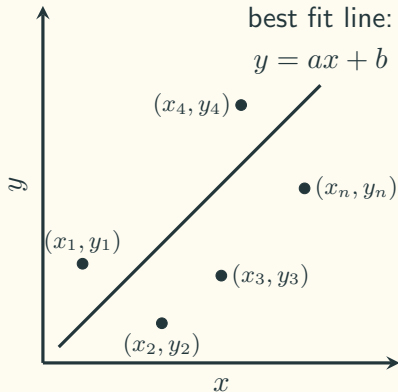
Two Forms of The Slope

$$\begin{aligned}\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) &= \sum_{i=1}^n (x_i y_i - \bar{x} \cdot y_i - \bar{y} \cdot x_i + \bar{x} \cdot \bar{y}) \\&= \sum_{i=1}^n x_i y_i - \bar{x} \sum_{i=1}^n y_i - \bar{y} \sum_{i=1}^n x_i + \sum_{i=1}^n \bar{x} \cdot \bar{y} \\&= n \cdot \overline{xy} - \bar{x} \cdot n \cdot \bar{y} - \bar{y} \cdot n \cdot \bar{x} + n \cdot \bar{x} \cdot \bar{y} \\&= n \cdot \overline{xy} - n \cdot \bar{x} \cdot \bar{y}\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^n (x_i - \bar{x})^2 &= \sum_{i=1}^n [x_i^2 - 2 \cdot \bar{x} \cdot x_i + (\bar{x})^2] = \sum_{i=1}^n x_i^2 - 2 \cdot \bar{x} \sum_{i=1}^n x_i + \sum_{i=1}^n (\bar{x})^2 \\&= n \cdot \bar{x} - 2n \cdot (\bar{x})^2 + n \cdot (\bar{x})^2 = n \cdot \bar{x} - n \cdot (\bar{x})^2\end{aligned}$$

Relationship Between The Slope & Pearson's r

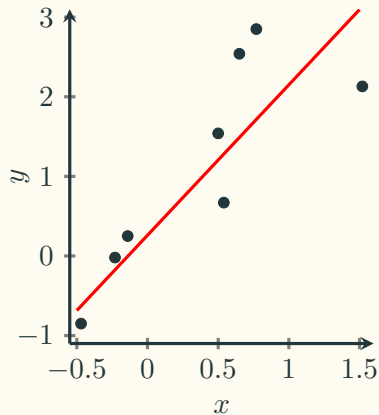
Using OLS regression:



$$\begin{cases} a = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} \\ b = \bar{y} - a \cdot \bar{x} \end{cases}$$

$$\begin{aligned} a &= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \cdot \sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}} \\ &= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \cdot \sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}} \cdot \frac{\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}{\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}} \\ &= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \cdot \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}} \cdot \frac{\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}} \\ &= r \cdot \frac{\sqrt{\sum_{i=1}^n (y_i - \bar{y})^2 / (n-1)}}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 / (n-1)}} = r \cdot \frac{s_y}{s_x} \end{aligned}$$

Best Fit Line



x	y	$x - \bar{x}$	$y - \bar{y}$	$(x - \bar{x}) \cdot (y - \bar{y})$	$(x - \bar{x})^2$
0.5	1.54	0.05	0.43	0.0215	0.0025
-0.14	0.25	-0.59	-0.86	0.5074	0.3481
0.65	2.54	0.2	1.43	0.286	0.04
1.52	2.13	1.07	1.02	1.0914	1.1449
-0.23	-1.18	-0.68	-2.29	1.5572	0.4624
-0.23	-0.02	-0.68	-1.13	0.7684	0.4624
1.58	3.15	1.13	2.04	2.3052	1.2769
0.77	2.85	0.32	1.74	0.5568	0.1024
-0.47	-0.85	-0.92	-1.96	1.8032	0.8464
0.54	0.67	0.09	-0.44	-0.0396	0.0081

$$\bar{x} = 0.45, \bar{y} = 1.11, s_x = 0.72, s_y = 1.55$$

$$a = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x - \bar{x})^2}, b = \bar{y} - a \cdot \bar{x}$$