

Lecture 10 Random Variables, PMF, Expectation & Variance

BIO210 Biostatistics

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What is a random variable (r.v.) ?

- An assignment of a value (a real number) to every possible outcome in the sample space.
- **Mathematically:** A real-valued **function** defined on a sample space Ω . In a particular experiment, a random variable (r.v.) would be some function that assigns a real number to each possible outcome.

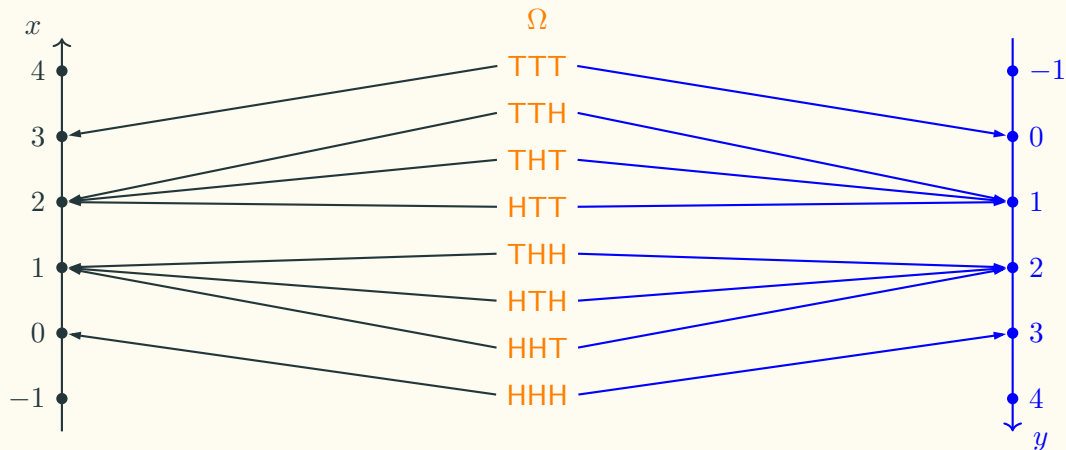
More about random variables

- Discrete or continuous.
- Can have several random variables defined on the same sample space.
- Notation
 - random variable X : function $\Omega \mapsto \mathbb{R}$
 - numerical value: x : value $\in \mathbb{R}$

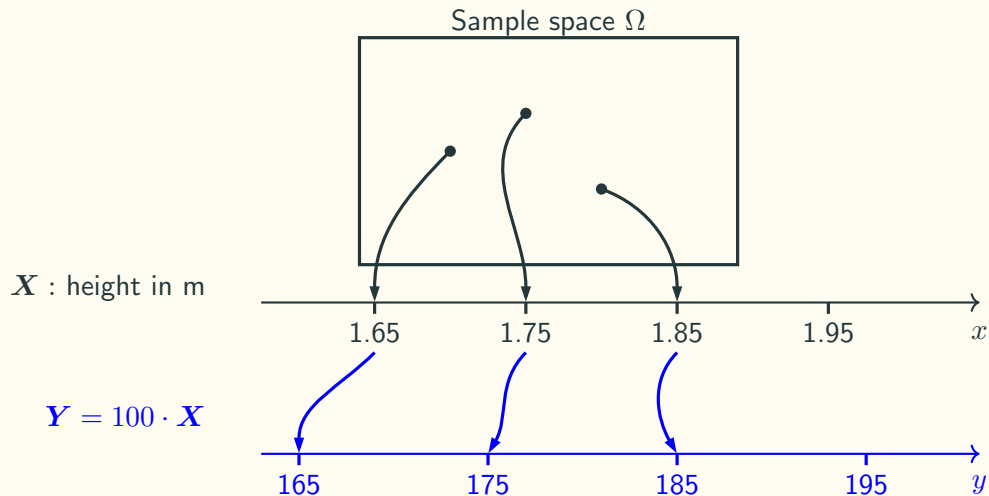
Different random variables on the same sample space

X : number tails

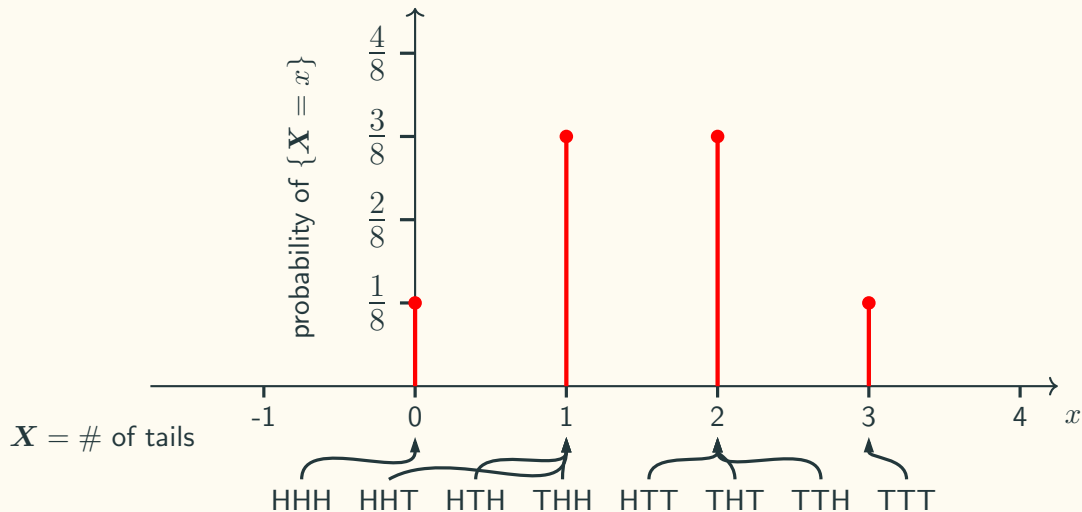
Y : number heads



Function of a random variable is an r.v.



Probability Mass Function (PMF)



Probability Mass Function (PMF)

The PMF of \mathbf{X} = number of tails after three flips

x	$\mathbb{P}(\{\mathbf{X} = x\})$
0	1/8
1	3/8
2	3/8
3	1/8
otherwise	0

$$\mathbb{P}(\{\mathbf{X} = x\}) = \begin{cases} \frac{1}{8}, & x = 0, 3 \\ \frac{3}{8}, & x = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

PMF Notation

Probability Mass Function

- Notation

$$\begin{aligned}\mathbb{P}_{\mathbf{X}}(x) &= \mathbb{P}(\{\mathbf{X} = x\}) \\ &= \mathbb{P}(\{\omega \in \Omega \mid \mathbf{X}(\omega) = x\})\end{aligned}$$

- Properties

$$\begin{aligned}\mathbb{P}_{\mathbf{X}}(x) &\geq 0 \\ \sum_x \mathbb{P}_{\mathbf{X}}(x) &= 1\end{aligned}$$

ω	$\mathbf{X}(\omega) = x$	$\mathbb{P}_{\mathbf{X}}(x) = \mathbb{P}(\{\mathbf{X} = x\})$
HHH	0	$\frac{1}{8}$
TTH, HTH, HHT	1	$\frac{3}{8}$
TTH, THT, TTH	2	$\frac{3}{8}$
TTT	3	$\frac{1}{8}$

Geometric PMF

Experiment: keep flipping a coin ($\mathbb{P}(H) = p$) until a head comes up for the first time.
Let the random variable X be the number of flips.

ω	$X(\omega)$	$\mathbb{P}_X(x)$
H	1	p
TH	2	$(1-p)p$
TTH	3	$(1-p)^2p$
\vdots	\vdots	\vdots
$\underbrace{\text{TTT}\dots\text{TTT}}_{n-1}\text{H}$	n	$(1-p)^{n-1}p$

Geometric PMF. X : geometric random variable.

How to compute a PMF $\mathbb{P}_X(x)$

To compute a PMF $\mathbb{P}_X(x)$:

1. Collect all possible outcomes for which $X = x$;
2. add their probabilities;
3. repeat for all x .

Compute PMF

Experiment: two independent rolls of a fair tetrahedral die.

F : outcome of the first roll

S : outcome of the second roll

$$X = \min(F, S)$$

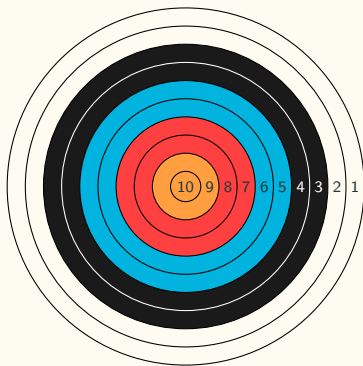
$$\mathbb{P}_X(x) = ?$$

S : second roll	4				
	3				
	2				
	1				
		1	2	3	4
		F : first roll			

Expected value of a random variable (Expectation)

Experiment: archery

Let X be the score you get for each shot. What is the expected value of X ?



x	$\mathbb{P}_X(x)$
1	0.19
2	0.17
3	0.15
4	0.13
5	0.11
6	0.09
7	0.07
8	0.05
9	0.03
10	0.01

Think: What is the average score you will get after a large number of trials?

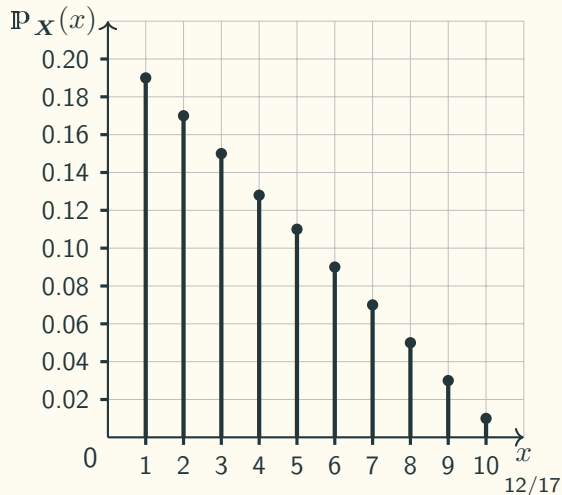
Expected value (Expectation)

Definition

$$\mathbb{E}[X] = \sum_x x \mathbb{P}_X(x)$$

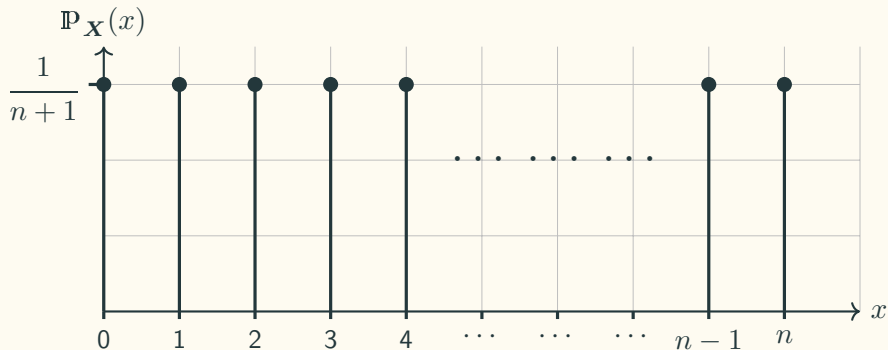
- Interpretation
 1. Centre of gravity of the PMF
 2. Average in large number of repetitions of the experiment

PMF of X from the archery experiment



Expectation of a Uniform Distribution

Example: a uniform discrete random variable X on $0, 1, 2, 3, \dots, n$



What is $\mathbb{E}[X]$?

Properties of expectations

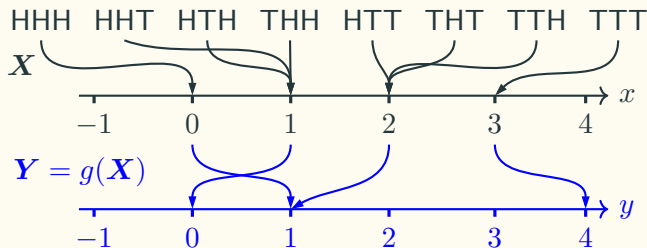
Let X be a random variable, and let $Y = g(X)$, what is $\mathbb{E}[Y]$?

- The hard way:

$$\mathbb{E}[Y] = \sum_y y \mathbb{P}_Y(y)$$

- The easy way:

$$\mathbb{E}[Y] = \sum_x g(x) \mathbb{P}_X(x)$$



y	$\mathbb{P}_Y(y)$
0	3/8
1	4/8
4	1/8

x	$g(x)$	$\mathbb{P}_X(x)$
0	1	1/8
1	0	3/8
2	1	3/8
3	4	1/8

Expectation of a linear function of r.v.

- Caution: in general $\mathbb{E}[g(\mathbf{X})] \neq g(\mathbb{E}[\mathbf{X}])$
- Exception: if α, β are constants, then we have:
 - $\mathbb{E}[\alpha] = \alpha$
 - $\mathbb{E}[\alpha \mathbf{X}] = \alpha \mathbb{E}[\mathbf{X}]$
 - $\mathbb{E}[\alpha \mathbf{X} + \beta] = \alpha \mathbb{E}[\mathbf{X}] + \beta$

Variance and standard deviation of a random variable

Definition of Variance

$$\text{Var}(\mathbf{X}) = \mathbb{E}[\mathbf{X} - \mathbb{E}[\mathbf{X}]]^2]$$

Properties of Variance

- $\text{Var}(\mathbf{X}) = \mathbb{E}[\mathbf{X}^2] - (\mathbb{E}[\mathbf{X}])^2$
- If α, β are constants, then $\text{Var}(\alpha\mathbf{X} + \beta) = \alpha^2 \text{Var}(\mathbf{X})$

Definition of Standard Deviation

$$\sigma_{\mathbf{X}} = \sqrt{\text{Var}(\mathbf{X})}$$

Discrete Random Variables (Summary slide)

