

Lecture 31 Analysis of Variance (ANOVA)

BIO210 Biostatistics

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School of Life Sciences

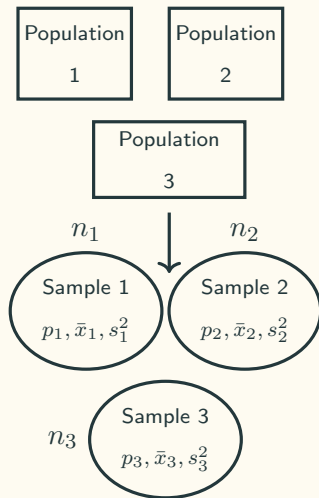
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Compare More Than Two Means

More than two-samples



Intuitive way: compare all possible pairs using two-sample independent t test:

Samples 1 vs 2: $H_0 : \mu_1 = \mu_2$; $H_1 : \mu_1 \neq \mu_2$

Samples 1 vs 3: $H_0 : \mu_1 = \mu_3$; $H_1 : \mu_1 \neq \mu_3$

Samples 2 vs 3: $H_0 : \mu_2 = \mu_3$; $H_1 : \mu_2 \neq \mu_3$

Good enough?

Compare More Than Two Means

What if we have 15 samples from 15 different populations ?

- **Intuitive way:** compare all possible pairs using two-sample independent t test:
 - Number of comparisons: $\binom{15}{2} = \frac{15 \times 14}{2} = 105$
 - Significance level: $\alpha = 0.05$
 - When we set $\alpha = 0.05$, we want to tolerate a 5% of chance of making a type I error. That is, the intended number of tests of making a type I error: ≈ 5
- Assume that the means are all the same, what is the probability of making a type I error in at least one test ?

$$\begin{aligned} & \mathbb{P}(\text{reject } H_0 \text{ in at least one test} \mid H_0 \text{ is true}) \\ &= 1 - \mathbb{P}(\text{not rejecting } H_0 \text{ in all tests} \mid H_0 \text{ is true}) \\ &= 1 - 0.95^{105} \\ &= 0.995 \end{aligned}$$

Source of Variation - Total

| Sample 1 | Sample 2 | Sample 3 |
|-----------------|-----------------|-----------------|
| 3 | 5 | 5 |
| 2 | 3 | 6 |
| 1 | 4 | 7 |
| $\bar{x}_1 = 2$ | $\bar{x}_2 = 4$ | $\bar{x}_3 = 6$ |

sum of squares (SS): add up the **squared distance** between an observation and the mean:

$$\sum (X - \bar{X})^2$$

SST: total sum of squares

The grand mean: $\bar{\bar{x}} = \frac{3 + 2 + 1 + 5 + 3 + 4 + 5 + 6 + 7}{9} = 4$

$$SST = (3 - 4)^2 + (2 - 4)^2 + (1 - 4)^2 + (5 - 4)^2 + (3 - 4)^2 + (4 - 4)^2 + (5 - 4)^2 + (6 - 4)^2 + (7 - 4)^2 = 30$$

What is the df ? $df_T = 9 - 1 = 8$

Source of Variation - Within Groups

| Sample 1 | Sample 2 | Sample 3 |
|-----------------|-----------------|-----------------|
| 3 | 5 | 5 |
| 2 | 3 | 6 |
| 1 | 4 | 7 |
| $\bar{x}_1 = 2$ | $\bar{x}_2 = 4$ | $\bar{x}_3 = 6$ |

sum of squares (SS): add up the **squared distance** between an observation and the mean:

$$\sum (X - \bar{X})^2$$

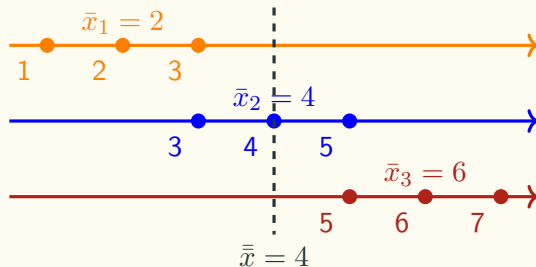
SSW: sum of squares within

$$\begin{aligned} \text{SSW} &= (3 - 2)^2 + (2 - 2)^2 + (1 - 2)^2 + (5 - 4)^2 + (3 - 4)^2 + (4 - 4)^2 + (5 - 6)^2 + (6 - 6)^2 + (7 - 6)^2 \\ &= df_1 \cdot s_1^2 + df_2 \cdot s_2^2 + df_3 \cdot s_3^2 \\ &= 6 \end{aligned}$$

What is the df ? $df_W = (3 - 1) + (3 - 1) + (3 - 1) = 6$

Source of Variation - Between Groups

| Sample 1 | Sample 2 | Sample 3 |
|-----------------|-----------------|-----------------|
| 3 | 5 | 5 |
| 2 | 3 | 6 |
| 1 | 4 | 7 |
| $\bar{x}_1 = 2$ | $\bar{x}_2 = 4$ | $\bar{x}_3 = 6$ |



SSB: sum of squares between

$$\begin{aligned} \text{SSB} &= (2 - 4)^2 + (2 - 4)^2 + (2 - 4)^2 + (4 - 4)^2 + (4 - 4)^2 + (4 - 4)^2 + (6 - 4)^2 + (6 - 4)^2 + (6 - 4)^2 \\ &= n_1 \cdot (\bar{x}_1 - \bar{\bar{x}})^2 + n_2 \cdot (\bar{x}_2 - \bar{\bar{x}})^2 + n_3 \cdot (\bar{x}_3 - \bar{\bar{x}})^2 \\ &= 24 \end{aligned}$$

What is the df ? $df_B = 3 - 1 = 2$

Summary of The Source of Variation

| Sample 1 | Sample 2 | Sample 3 |
|-----------------|-----------------|-----------------|
| 3 | 5 | 5 |
| 2 | 3 | 6 |
| 1 | 4 | 7 |
| $\bar{x}_1 = 2$ | $\bar{x}_2 = 4$ | $\bar{x}_3 = 6$ |

| Source of Variation | SS (sum of squares) | df | Variance-like |
|---------------------|---------------------|----|------------------|
| | | | MS (mean square) |
| Between | 24 | 2 | 12 |
| Within | 6 | 6 | 1 |
| Total | 30 | 8 | |

Multiple Samples From Multiple Populations

| | |
|----------------|--|
| Population 1 | Sample 1 (n_1, \bar{x}_1, s_1^2) |
| Population 2 | Sample 2 (n_2, \bar{x}_2, s_2^2) |
| Population 3 | Sample 3 (n_3, \bar{x}_3, s_3^2) |
| \vdots | \vdots |
| Population k | Sample k (n_k, \bar{x}_k, s_k^2) |

$$\bar{\bar{x}} = \frac{\sum_{i=1}^k \sum_{j=1}^{n_i} x_{ij}}{\sum_{i=1}^k n_i} = \frac{\sum_{i=1}^k n_i \bar{x}_i}{n}$$

The ANOVA Table

| Source of Variation | SS | df | MS |
|---------------------|--|---------|---------------------------|
| Between | $SSB = \sum_{i=1}^k n_i (\bar{x}_i - \bar{\bar{x}})^2$ | $k - 1$ | $MSB = \frac{SSB}{k - 1}$ |
| Within | $SSW = \sum_{i=1}^k \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 = \sum_{i=1}^k df_i s_i^2$ | $n - k$ | $MSW = \frac{SSW}{n - k}$ |
| Total | $SST = \sum_{i=1}^k \sum_{j=1}^{n_i} (x_{ij} - \bar{\bar{x}})^2 = SSB + SSW$ | $n - 1$ | |

The F -test

$$\begin{cases} H_0 : \mu_1 = \mu_2 = \cdots = \mu_k \\ H_1 : \text{not all equal} \end{cases} \Leftrightarrow \begin{cases} H_0 : \text{The main variation is from SSW} \\ H_1 : \text{The main variation is from SSB} \end{cases}$$

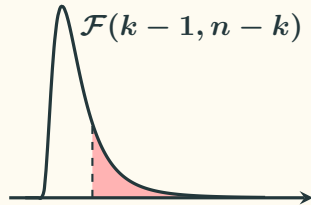
Under the null hypothesis:

$$\frac{\text{SSB}}{\sigma^2} \sim \chi^2(k-1) \text{ and } \frac{\text{SSW}}{\sigma^2} \sim \chi^2(n-k), \text{ where } \sigma^2 \text{ is the common variance}$$

The test statistic:

$$F = \frac{\frac{\text{SSB}}{(k-1)\sigma^2}}{\frac{\text{SSW}}{(n-k)\sigma^2}} = \frac{\text{MSB}}{\text{MSW}} \sim \mathcal{F}(k-1, n-k)$$

$$p\text{-value: } \mathbb{P}(\text{data} \mid H_0 \text{ is true}) = \mathbb{P}\left(F_{k-1, n-k} \geq \frac{\text{MSB}}{\text{MSW}}\right)$$



Summary of an ANOVA result

| Source of Variation | SS | df | MS | F | p-value |
|---------------------|--|---------|---------------------------|-------------------|--|
| Between | $SSB = \sum_{i=1}^k n_i (\bar{x}_i - \bar{\bar{x}})^2$ | $k - 1$ | $MSB = \frac{SSB}{k - 1}$ | $\frac{MSB}{MSW}$ | $\mathbb{P} \left(F \geq \frac{MSB}{MSW} \right)$ |
| Within | $SSW = \sum_{i=1}^k df_i s_i^2$ | $n - k$ | $MSW = \frac{SSW}{n - k}$ | | |
| Total | $SST = SSB + SSW$ | $n - 1$ | | | |