Portfolio Optimization of Relativistic Value at Risk

Dany Cajas - September 2023

7 Color Mountain - Cusco - Peru

How to measure risk?

- **Deviation Risk Measures:** Variance, Mean Absolute Deviation (MAD), etc.
- **Downside Risk Measures:** Semivariance, Lower Partial Moments (LPMs), Value at Risk (VaR), Conditional Value at Risk (CVaR), etc.
- **Drawdown Risk Measures:** Drawdown at Risk (DaR), Conditional Drawdown at Risk (CDaR), Maximum Drawdown, etc.
- **Higher Moments:** Skewness, Kurtosis or Higher L-Moments.

Value at Risk

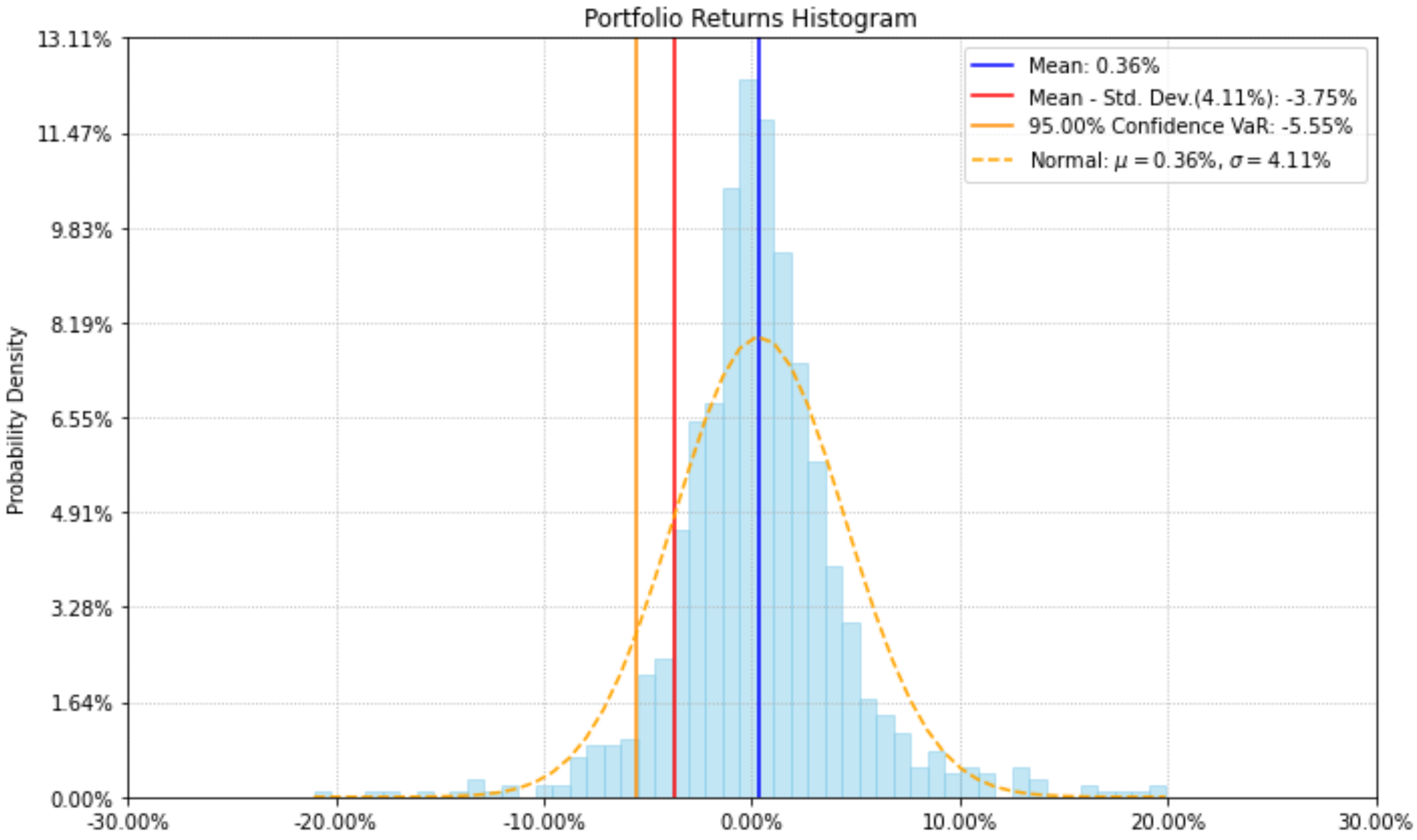
The Value at Risk (VaR) is defined as the (1-α) quantile of the distribution of losses -X. Formally, it is expressed as:

$$\text{VaR}_{\alpha}(X) = -\inf \left\{ x \in \mathbb{R} : F_X(x) > \alpha \right\}$$

$$\text{VaR}_{\alpha}(X) = F_{-X}^{-1}(1 - \alpha)$$

Where F_X is the cumulative probability distribution of X, α is the significance level and F_X^{-1} is the percent point function. It is widely used by practitioners and academics. The main critique of VaR is that not a coherent risk measures (lack the sub-additivity property).

Value at Risk



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Conditional Value at Risk

The Conditional Value at Risk (CVaR) was proposed by Rockafellar and Uryasev (2000). It is the average of losses higher than $\text{VaR}_{\alpha}(X)$. Formally, it is expressed as:

$$\text{CVaR}_{\alpha}(X) = \frac{1}{\alpha}\int_{0}^{\alpha} \text{VaR}_{\gamma}(X) d\gamma$$

$$\text{CVaR}_{\alpha}(X) = \inf_{t>0} \left\{ t + \frac{1}{\alpha T} \sum_{i=1}^{T} \max(-X_i - t, 0) \right\}$$

Where t is an auxiliary variable, α is the significance level and $\max(\,.\,)$ is the max operator.

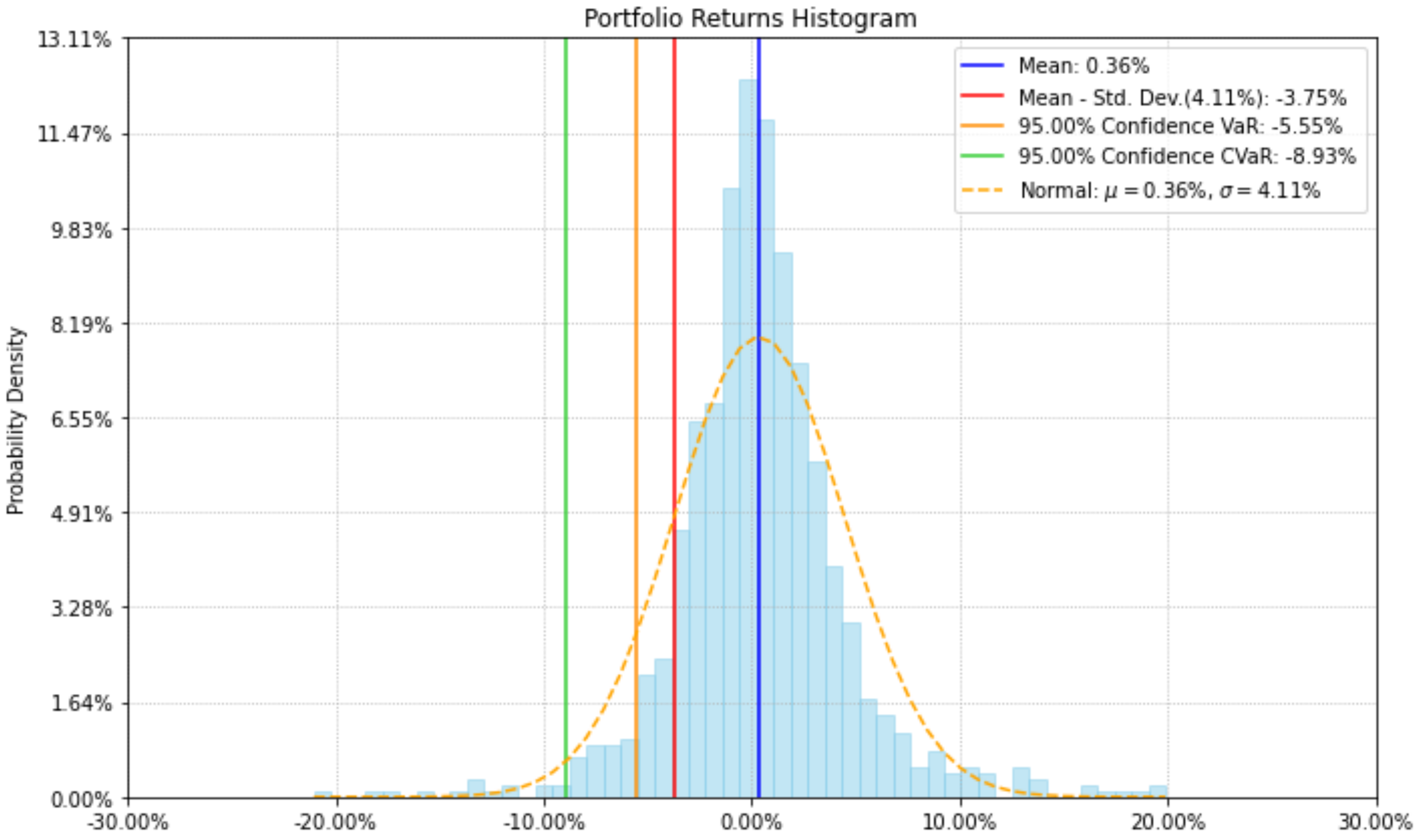
Conditional Value at Risk

The CVaR is a coherent risk measure, this means that CVaR satisfies the following four properties: translation invariance, monotonicity, sub-additivity and positive homogeneity of degree one.

The following inequality holds:

$$\text{VaR}_{\alpha}(X) \leq \text{CVaR}_{\alpha}(X) \leq \text{EssSup}(X)$$

Conditional Value at Risk



Entropic Value at Risk

The Entropic Value at Risk (EVaR) was proposed by Ahmadi-Javid (2012). It is defined as the upper bound of $\text{CVaR}_{\alpha}(X)$ based on Chernoff inequality. Formally, it is expressed as:

$$\text{EVaR}_{\alpha}(X) = \inf_{z>0} \left\{ z \ln \left(\frac{1}{\alpha} M_X \left(\frac{1}{z} \right) \right) \right\}$$

$$\text{EVaR}_{\alpha}(X) = \inf_{z>0} \left\{ z \ln \left(\sum_{i=1}^{T} \exp \left(\frac{-X_i}{z} \right) \right) + z \ln \left(\frac{1}{\alpha T} \right) \right\}$$

Where z is an auxiliary variable, α is the significance level and $M_X(z) = \mathbb{E}\left(e^{zX}\right)$ is the generating moment function.

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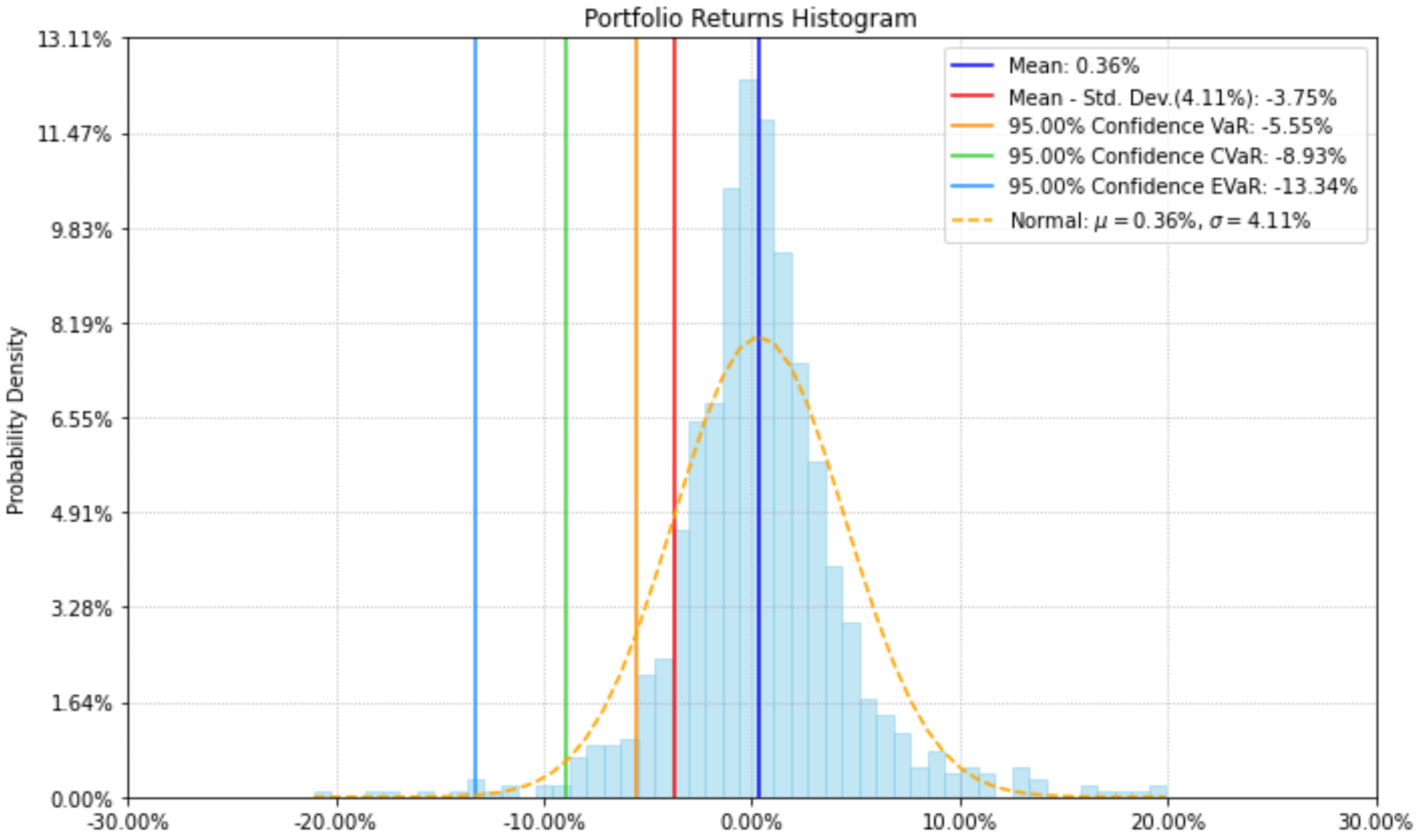
Entropic Value at Risk

The EVaR is a coherent risk measure.

The following inequality holds:

$$\mathrm{VaR}_{\alpha}(X) \leq \mathrm{CVaR}_{\alpha}(X) \leq \mathrm{EVaR}_{\alpha}(X) \leq \mathrm{EssSup}(X)$$

Entropic Value at Risk



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Dual Representation of CVaR and EVaR

Primal CVaR

$$\begin{aligned} \inf_{t,u} \quad & t + \frac{1}{\alpha T}\sum_{i=1}^{T} u_i \\ & u_i \geq -X_i - t \quad \forall\, i = 1,\ldots,T \\ & u \geq 0 \end{aligned}$$

Dual CVaR

$$\begin{aligned} \sup_{Z \in L} \quad & Z'(-X) \\ \text{s.t.} \quad & Z \geq 0 \\ & \mathbf{1}_T' Z = 1 \\ & Z_i \leq \frac{1}{\alpha T}, \quad \forall\, i = 1,\ldots,T \end{aligned}$$

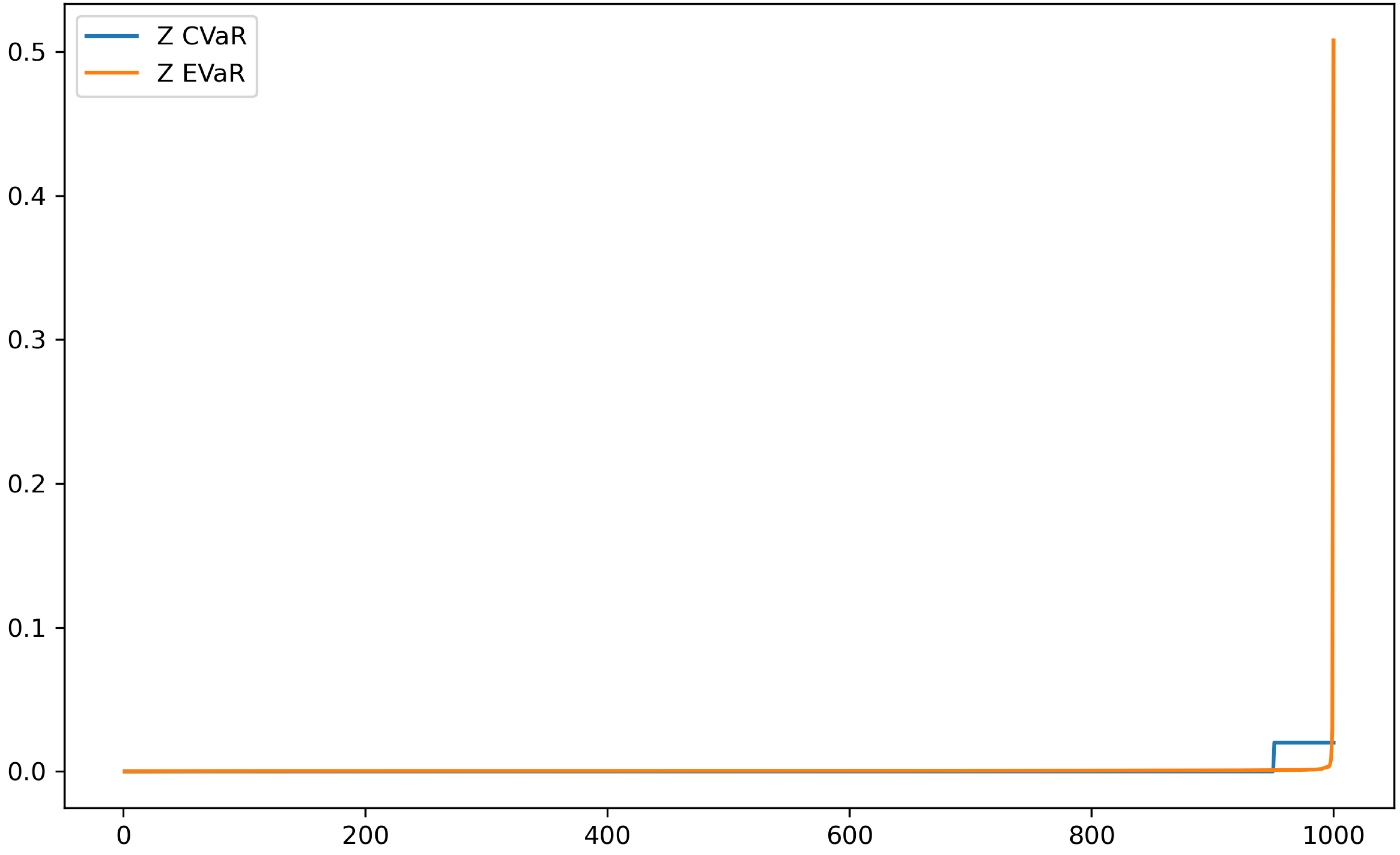
Primal EVaR

$$\begin{aligned} \inf_{x,z,t,u} \quad & t + z \ln\left(\frac{1}{\alpha T}\right) \\ \text{s.t.} \quad & z \geq \sum_{i=1}^{T} u_i \\ & u_i \geq z \exp(-X_i - t) \quad \forall\, i = 1,\ldots,T \end{aligned}$$

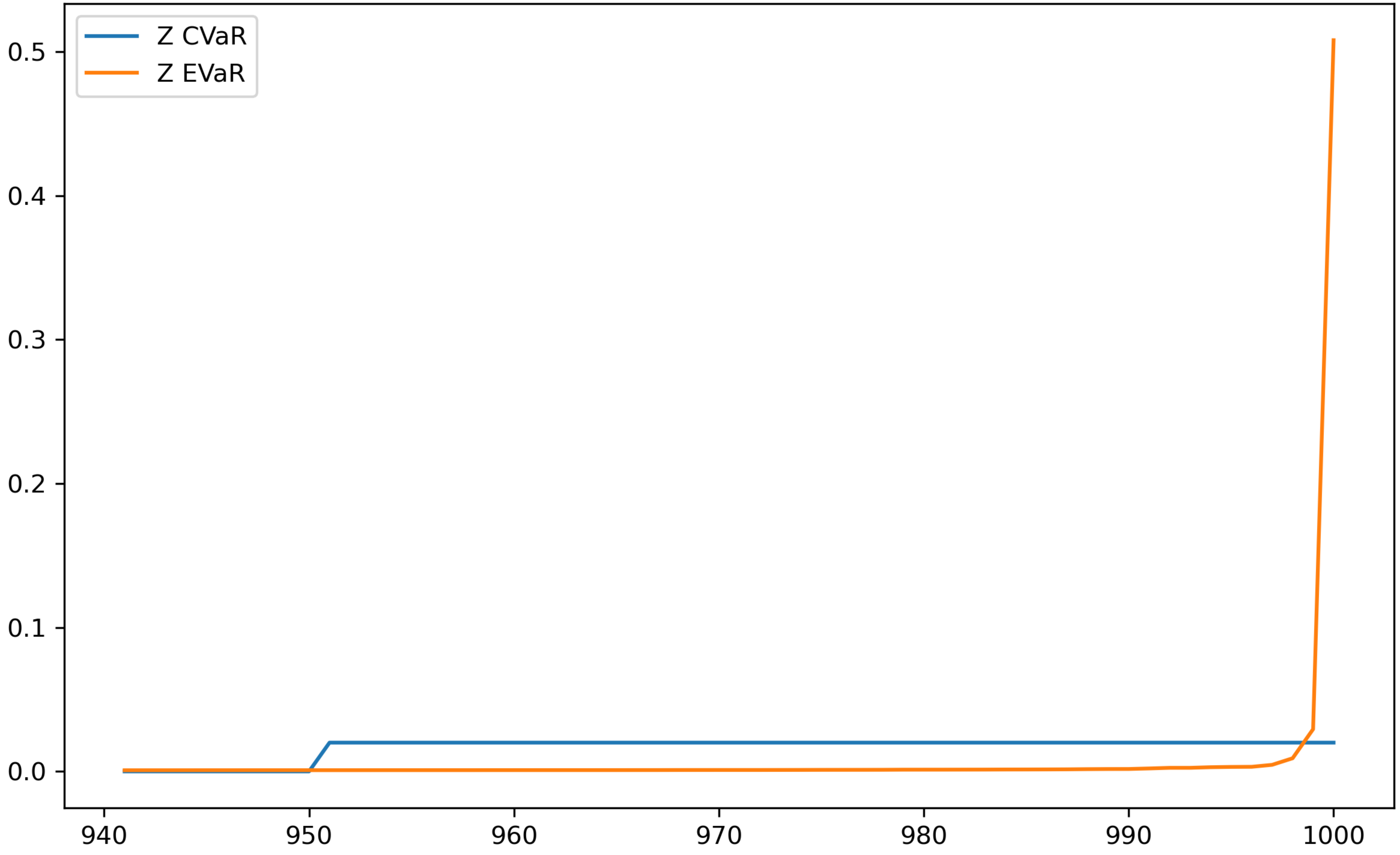
Dual EVaR

$$\begin{aligned} \sup_{Z \in L} \quad & Z'(-X) \\ \text{s.t.} \quad & Z \geq 0 \\ & \mathbf{1}' Z = 1 \\ & \sum_{i=1}^{T} Z_i \ln(Z_i) \leq \ln\left(\frac{1}{\alpha T}\right) \end{aligned}$$

Dual Weights of CVaR and EVaR



Dual Weights of CVaR and EVaR



φ-Divergence Risk Measures

Dommel and Pichler (2020) define a φ-divergence risk measure using the dual representation as:

$$\rho_{\varphi,\beta}(X) = \begin{cases} \sup\limits_{Z \in L} & Z'(-X) \\ \text{s.t.} & Z \geq 0 \\ & \mathbf{1}'Z = 1 \\ & \sum_{i=1}^{T} \varphi(Z_i) \leq \beta \end{cases}$$

Where φ is a divergence function, β is an appropriate constant and T the number of observations. φ-divergence risk measures are coherent risk measures.

Relativistic Value at Risk

The Relativistic Value at Risk (RLVaR) was proposed by Cajas (2023), it is a generalization of EVaR and a special case of φ-divergence risk measure based on Kaniadakis entropy. Formally it is expressed as:

$$\text{RLVaR}_{\alpha}^{\kappa}(X) = \left\{ \begin{array}{ll} \sup\limits_{Z \in L} & Z'X \\ \text{s.t.} & Z \geq 0 \\ & \mathbf{1}'Z = 1 \\ & \sum_{i=1}^{T} Z_i \ln_{\kappa}(Z_i) \leq \ln_{\kappa}\left(\frac{1}{\alpha T}\right) \end{array} \right.$$

Where α is the significance level, κ is the deformation parameter and

Relativistic Value at Risk

$\ln_\kappa$ is the Kaniadakis logarithm defined as:

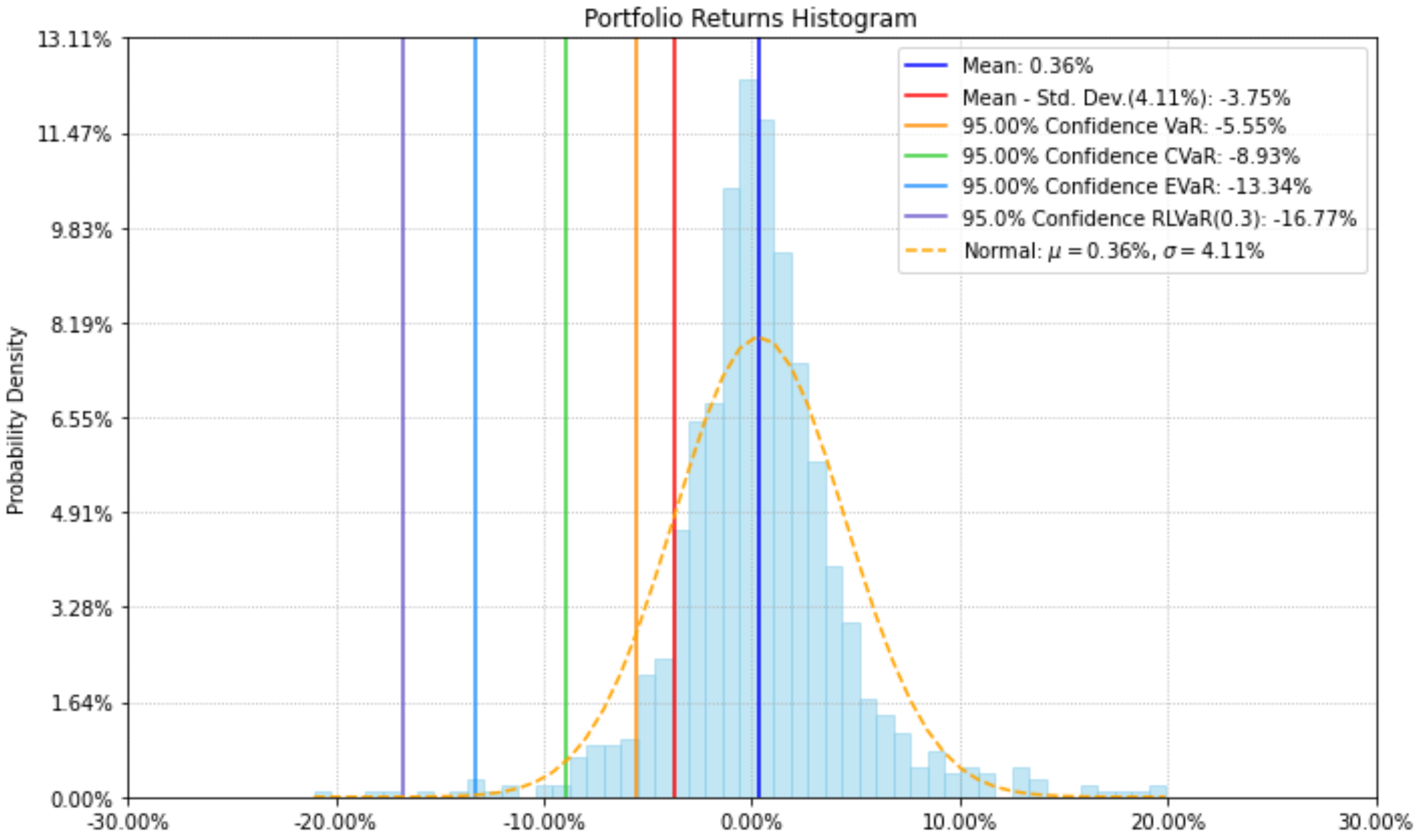
$$\ln_\kappa(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa}$$

The RLVaR is a coherent risk measure.

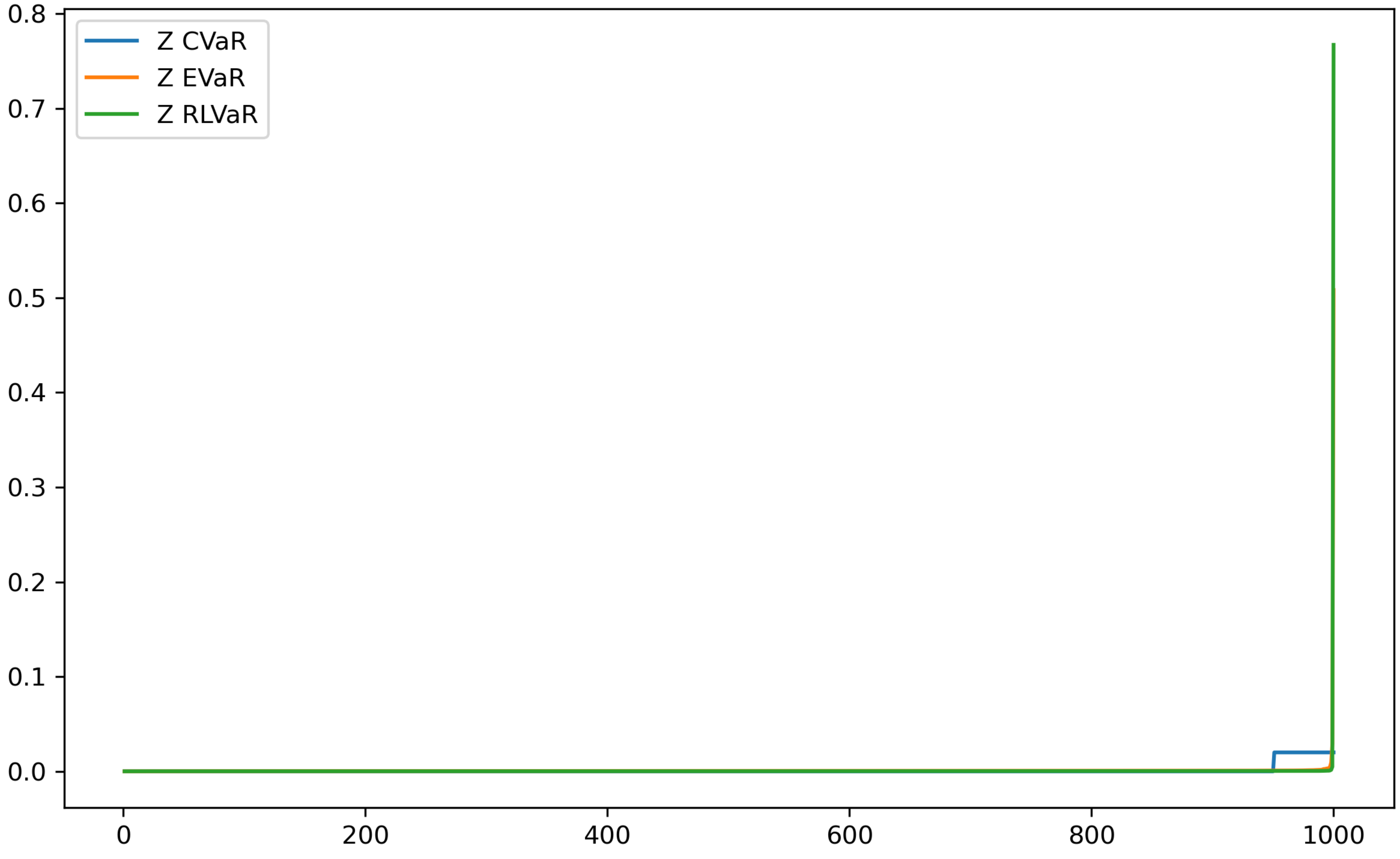
The following inequality holds:

$$\text{VaR}_\alpha(X) \leq \text{CVaR}_\alpha(X) \leq \text{EVaR}_\alpha(X) \leq \text{RLVaR}_\alpha^\kappa(X) \leq \text{EssSup}(X)$$

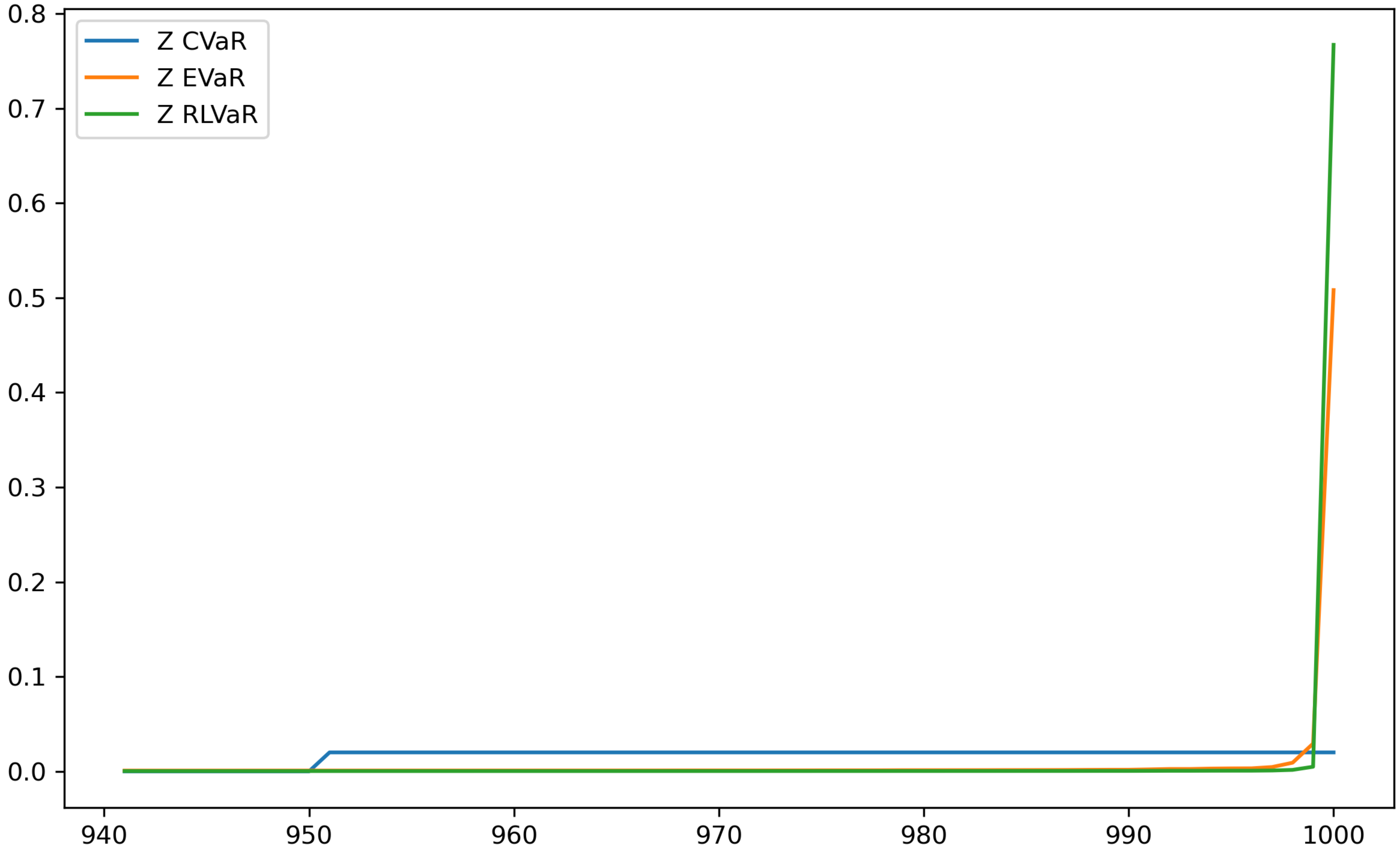
Relativistic Value at Risk



Dual Weights of CVaR, EVaR and RLVaR



Dual Weights of CVaR, EVaR and RLVaR



Relativistic Value at Risk Portfolio Model

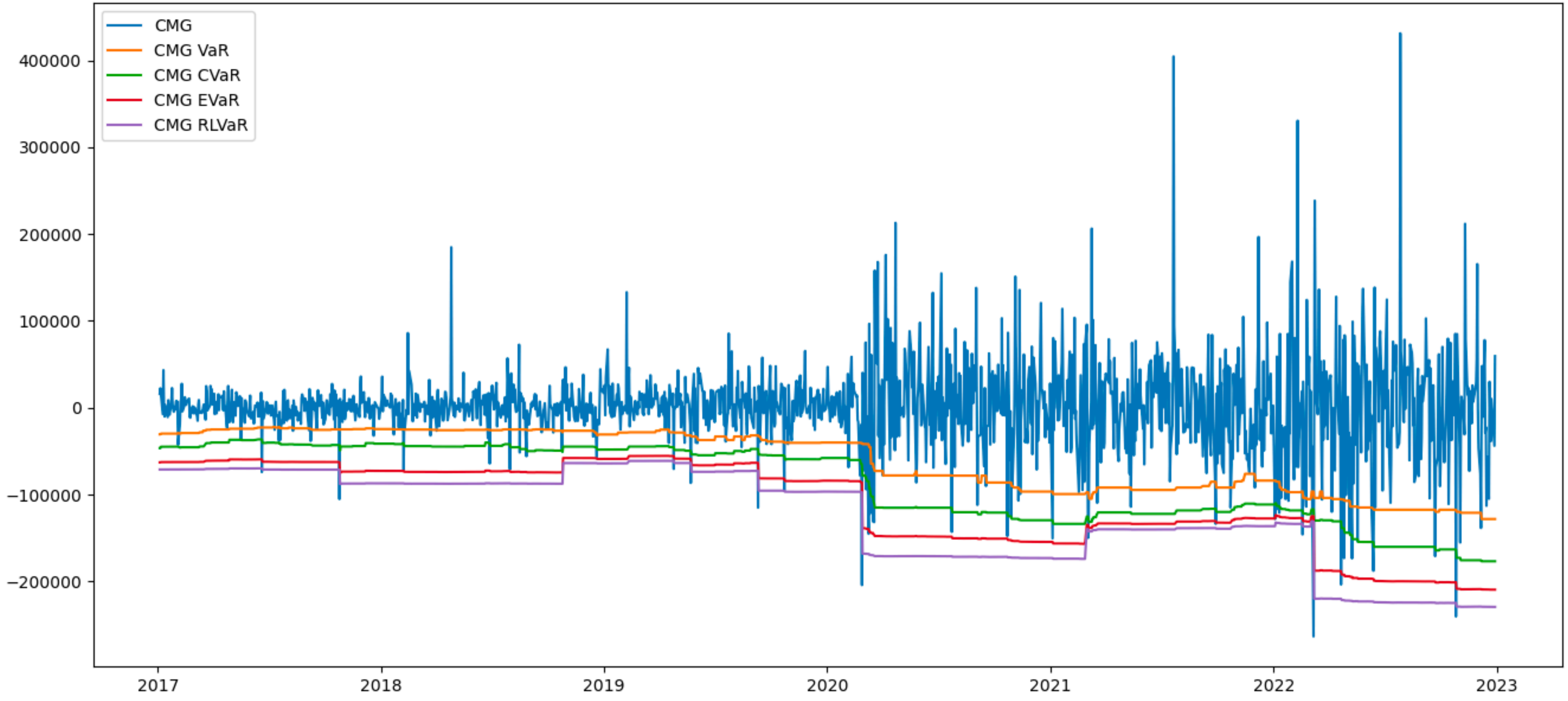
The convex portfolio optimization model for RLVaR was proposed by Cajas (2023):

$$
\begin{aligned}
\min_{x,z,t,\psi,\theta,\varepsilon,\omega} \quad & t + z \ln_{\kappa}\left(\frac{1}{\alpha T}\right) + \sum_{i=1}^{T} \left(\psi_i + \theta_i\right) \\
\text{s.t.} \quad & -r_i x - t + \varepsilon_i + \omega_i \leq 0, \quad \forall\, i = 1,\ldots,T \\
& \left(z\left(\frac{1+\kappa}{2\kappa}\right), \psi_i\left(\frac{1+\kappa}{\kappa}\right), \varepsilon_i\right) \in \mathcal{P}_3^{1/(1+\kappa),\, \kappa/(1+\kappa)}, \quad \forall\, i = 1,\ldots,T \\
& \left(\frac{\omega_i}{1-\kappa}, \frac{\theta_i}{\kappa}, \frac{-z}{2\kappa}\right) \in \mathcal{P}_3^{1-\kappa,\, \kappa}, \quad \forall\, i = 1,\ldots,T \\
& z \geq 0 \\
& x \in \mathcal{X}
\end{aligned}
$$

Applications

Historical Backtesting of Losses

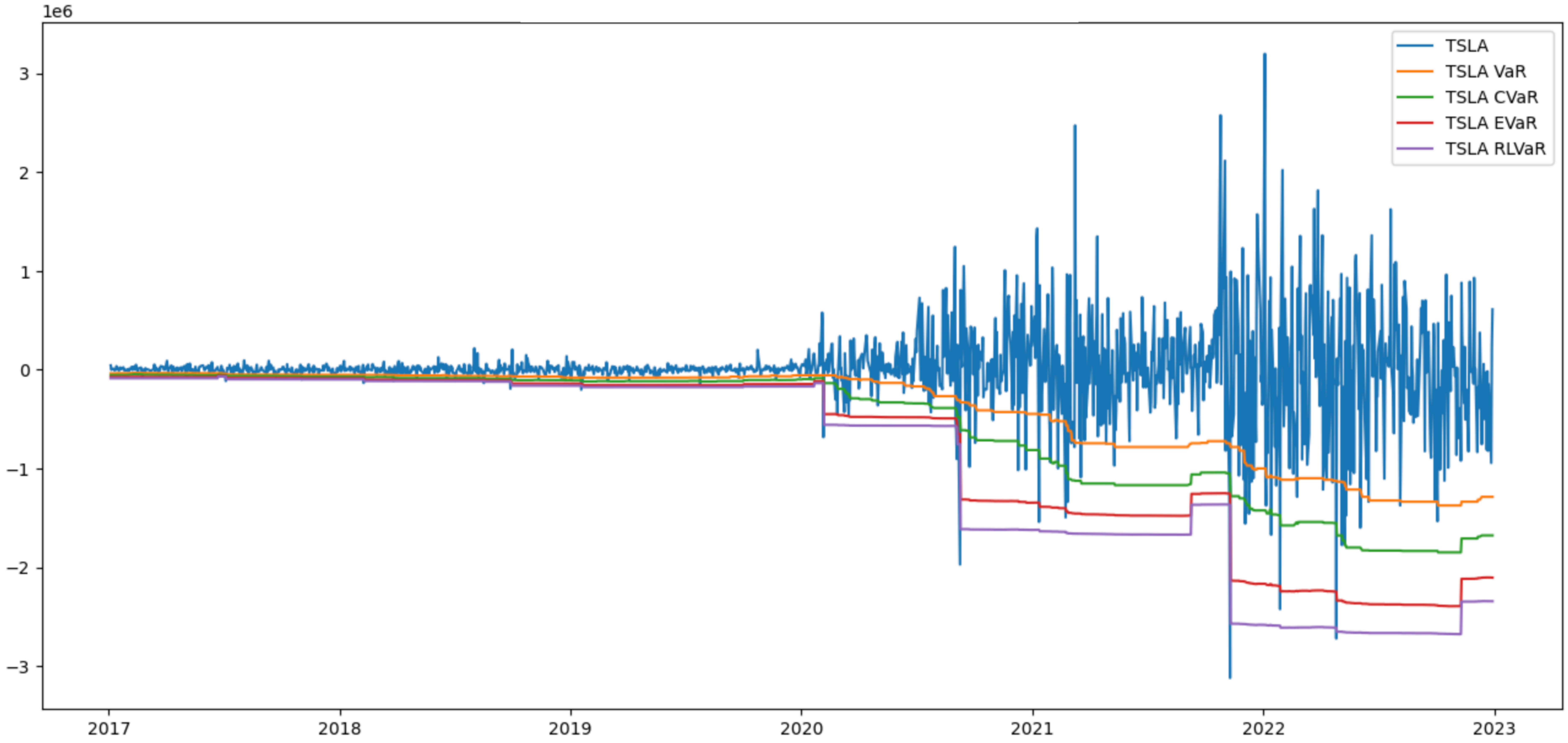
Profit and losses of an investment of 1MM of CMG



	CMG
Failure Ratio VaR	0.064987
Failure Ratio CVaR	0.026525
Failure Ratio EVaR	0.010610
Failure Ratio RLVaR	0.006631
Bernoulli Test VaR stat	6.541820
Bernoulli Test VaR p-value	0.010540
Bernoulli Test CVaR stat	20.953400
Bernoulli Test CVaR p-value	0.000000
Bernoulli Test EVaR stat	71.622560
Bernoulli Test EVaR p-value	0.000000
Bernoulli Test RLVaR stat	93.336730
Bernoulli Test RLVaR p-value	0.000000

Historical Backtesting of Losses

Profit and losses of an investment of 1MM of TSLA



	TSLA
Failure Ratio VaR	0.083554
Failure Ratio CVaR	0.033156
Failure Ratio EVaR	0.015252
Failure Ratio RLVaR	0.008621
Bernoulli Test VaR stat	30.004240
Bernoulli Test VaR p-value	0.000000
Bernoulli Test CVaR stat	10.169280
Bernoulli Test CVaR p-value	0.001430
Bernoulli Test EVaR stat	52.077280
Bernoulli Test EVaR p-value	0.000000
Bernoulli Test RLVaR stat	81.775040
Bernoulli Test RLVaR p-value	0.000000

Backtesting of Min Risk Portfolios

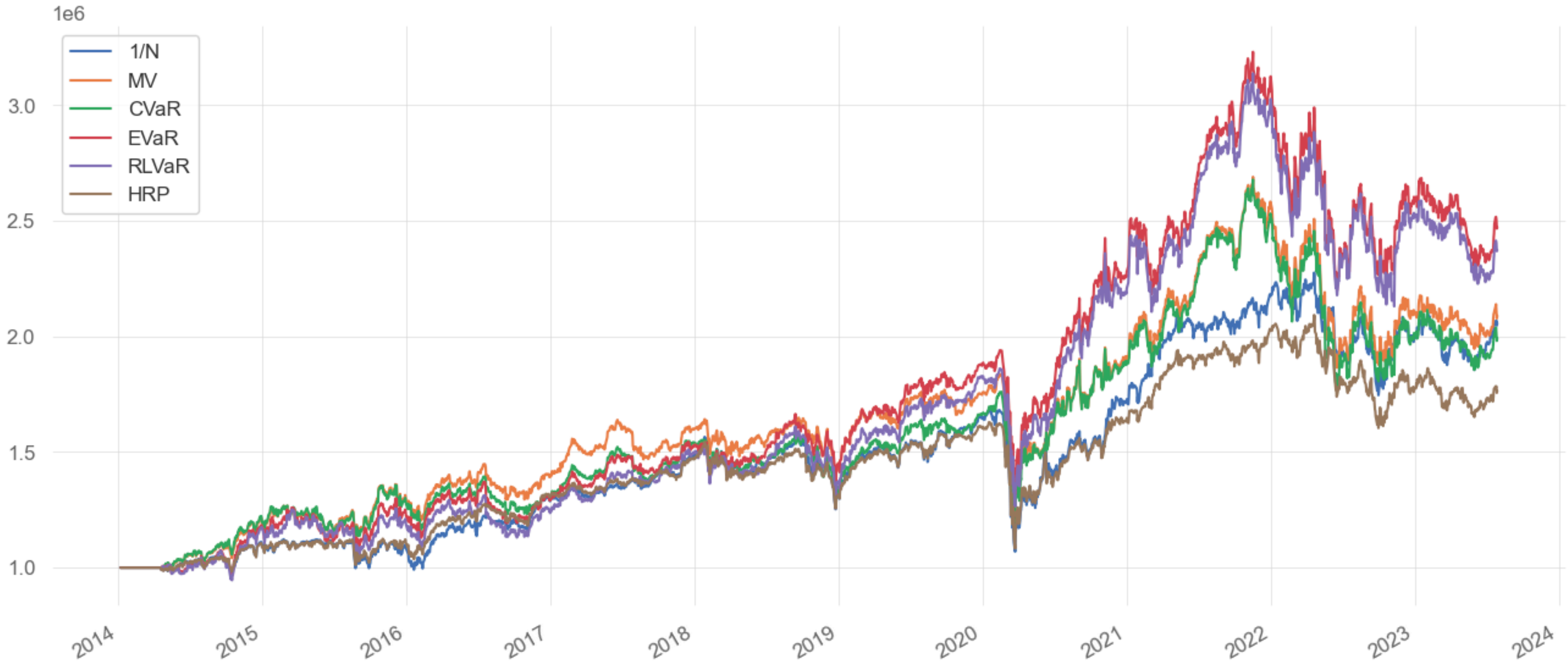
1MM investment portfolios, 90 days rebalancing period, fees=0.05%, slippage= 1%

Backtesting of Min Risk Portfolios

	MV	CVaR	EVaR	RLVaR
Start	2010-01-04 00:00:00	2010-01-04 00:00:00	2010-01-04 00:00:00	2010-01-04 00:00:00
End	2023-07-28 00:00:00	2023-07-28 00:00:00	2023-07-28 00:00:00	2023-07-28 00:00:00
Period	3415 days 00:00:00	3415 days 00:00:00	3415 days 00:00:00	3415 days 00:00:00
Start Value	1000000.0	1000000.0	1000000.0	1000000.0
End Value	1298387.849233	1327597.601921	1282859.855526	1701415.428956
Total Return [%]	29.838785	32.75976	28.285986	70.141543
Benchmark Return [%]	281.635025	281.635025	281.635025	281.635025
Annualized Return [%]	1.945568	2.11307	1.855098	3.999675
Annualized Volatility [%]	12.400804	12.09708	12.743255	13.403211
Total Fees Paid	3880.210752	5388.007454	5840.352571	7006.058452
Max Drawdown [%]	31.464018	31.891736	28.751384	29.342012
Max Drawdown Duration	378 days 00:00:00	393 days 00:00:00	654 days 00:00:00	510 days 00:00:00
Sharpe Ratio	0.217728	0.233583	0.208187	0.359897
Calmar Ratio	0.061835	0.066258	0.064522	0.136312
Omega Ratio	1.051592	1.053575	1.04728	1.082132
Sortino Ratio	0.298319	0.323192	0.288794	0.502488
Skew	-0.798781	-0.564549	-0.497812	-0.441573
Kurtosis	26.250776	17.213831	15.497679	14.089034
Tail Ratio	0.980417	0.958148	0.93547	0.933097
Common Sense Ratio	0.999492	0.978395	0.952824	0.970418
Value at Risk	-0.011237	-0.011477	-0.012163	-0.013165

Backtesting of Max Return/Risk Portfolios

1MM investment portfolios, 90 days rebalancing period, fees=0.05%, slippage= 1%



Backtesting of Max Return/Risk Portfolios

	1/N	MV	CVaR	EVaR	RLVaR	HRP
Start	2010-01-04 00:00:00	2010-01-04 00:00:00	2010-01-04 00:00:00	2010-01-04 00:00:00	2010-01-04 00:00:00	2010-01-04 00:00:00
End	2023-07-28 00:00:00	2023-07-28 00:00:00	2023-07-28 00:00:00	2023-07-28 00:00:00	2023-07-28 00:00:00	2023-07-28 00:00:00
Period	3415 days 00:00:00	3415 days 00:00:00	3415 days 00:00:00	3415 days 00:00:00	3415 days 00:00:00	3415 days 00:00:00
Start Value	1000000.0	1000000.0	1000000.0	1000000.0	1000000.0	1000000.0
End Value	2057162.616998	2089706.047414	1992608.032692	2471446.513472	2373802.08805	1764837.036747
Total Return [%]	105.716262	108.970605	99.260803	147.144651	137.380209	76.483704
Benchmark Return [%]	281.635025	281.635025	281.635025	281.635025	281.635025	281.635025
Annualized Return [%]	5.467038	5.589263	5.219193	6.904674	6.587147	4.280919
Annualized Volatility [%]	14.454055	15.998793	16.343791	16.715576	16.829377	13.078744
Total Fees Paid	2333.769532	10220.735531	11211.982246	10142.753295	12201.222686	4151.343399
Max Drawdown [%]	36.603404	32.610238	33.389733	32.032403	32.115386	33.60841
Max Drawdown Duration	406 days 00:00:00	425 days 00:00:00	425 days 00:00:00	425 days 00:00:00	425 days 00:00:00	359 days 00:00:00
Sharpe Ratio	0.441053	0.420448	0.393534	0.48347	0.463618	0.38633
Calmar Ratio	0.149359	0.171396	0.156311	0.215553	0.205109	0.127376
Omega Ratio	1.105888	1.099577	1.092794	1.113069	1.10804	1.093433
Sortino Ratio	0.609566	0.58466	0.546616	0.682729	0.654442	0.532143
Skew	-0.803838	-0.622916	-0.644708	-0.445326	-0.389071	-0.798744
Kurtosis	18.55712	18.209052	18.146391	14.546192	13.198804	21.760171
Tail Ratio	1.017844	0.967665	0.977912	1.003058	0.960435	0.998255
Common Sense Ratio	1.07349	1.021751	1.028951	1.072316	1.0237	1.04099
Value at Risk	-0.013482	-0.015403	-0.015677	-0.015705	-0.016345	-0.012098

Riskfolio-Lib Links

- Source code is available in https://github.com/dcajasn/Riskfolio-Lib
- Documentation is available in https://riskfolio-lib.readthedocs.io/
- Examples are available in https://riskfolio-lib.readthedocs.io/en/latest/examples.html
- Pypi page for installation is available in https://pypi.org/project/Riskfolio-Lib/
- Support this project (Donations):
    - https://github.com/sponsors/dcajasn
    - https://ko-fi.com/riskfolio

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Thanks

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Questions and Answers