Homework 9

Discrete Mathematics

Please review the *Rules of the Game* from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition.You are highly encouraged to seek out assistance by asking questions in our Q&A Discussion Board in Canvas. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to three late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. **Unless explicitly stated otherwise, you are expected to justify your answers.** In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. How many compositions of n have all their parts greater than 1, except possibly the last entry. For example, for $n = 9$, $(3,4,2)$ and $(3,3,2,1)$ are acceptable, but not $(3,3,1,2)$.

2. Solve the following recurrence relations.

   (a) $a_n = a_{n-1} + 11$ with initial condition $a_1 = 7$.

   (b) $a_n = -5a_{n-1}$ with initial condition $a_1 = 40$.

3. Let s_n denote the number of subsets of $[n]$ that contain no two consecutive elements. Determine s_n in terms of a recurrence, a closed form, or by describing a bijection to a set we already know how to count.

4. A **Dyck path** of length $2n$ is a lattice path from $(0,0)$ to (n,n) that takes n steps East from (i,j) to $(i+1,j)$ and n steps North from (i,j) to $(i,j+1)$ such that all points on the path satisfy $i \leq j$. This sound more complicated that it really is. You can think of a Dyck path as one of our paths to get coffee that starts at $(0,0)$ and ends at (n,n) but never drops below the line $y = x$. Let $\text{Dyck}(n)$ denote set of all Dyck paths of length $2n$ and let $d_n := |\text{Dyck}(n)|$. We define $d_0 := 1$ for convenience. *Important:* Unfortunately, we also used d_n to denote the number of derangements of n. This problem is about Dyck path, not derangements.

   (a) Compute d_1, d_2, d_3, and d_4 via brute force.

(b) Show that d_n satisfies the following recurrence for $n \geq 1$:

$$d_n = \sum_{i=0}^{n-1} d_i d_{n-1-i}.$$

Hint: Consider the collection of Dyck paths that hit the line $y = x$ at $(i+1, i+1)$ for the first time after leaving $(0,0)$. Think about how many ways you can draw the Dyck path to get to $(i+1, i+1)$ versus how many ways you can draw the Dyck path from $(i+1, i+1)$ to (n, n). The first case is trickier to think about. Notice that the portion of the Dyck path from $(0,0)$ to $(i+1, i+1)$ never hits the line $y = x$ along the way. Moreover, this portion necessarily starts with a North step and ends with an East step. What are the possible values for i?