Problem Sequence for MAT 511

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1 Introduction to Groups

1.1 Binary Operations

Definition 1.1. A **binary operation** $*$ on a set A is a function from $A \times A$ into A. For each $(a,b) \in A \times A$, we denote the element $*(a,b)$ via $a * b$. If the context is clear, we may abbreviate $a * b$ as ab.

Do not misunderstand the use of $*$ in this context. We are not implying that $*$ is the ordinary multiplication of real numbers. We are using $*$ to represent a generic binary operation. Notice that since the codomain of a binary operation on a set A is A, binary operations require that we yield an element of A when combining two elements of A. In this case, we say that A is **closed** under $*$. Binary operations have this closure property by definition. Also, since binary operations are functions, any attempt to combine two elements from A should result in a *unique* element of A. Moreover, since the domain of $*$ is $A \times A$, it must be the case that $*$ is defined for *all* pairs of elements from A.

Problem 1.2. Let A be a set. Feel free to consult outside resources for parts (a) and (d).

(a) If $*$ is a binary operation on A, what does it mean for $*$ to be **associative**?

(b) Provide an example of a set together with a binary operation that is associative.

(c) Provide an example of a set together with a binary operation that is *not* associative.

(d) If $*$ is a binary operation on A, what does it mean for $*$ to be **commutative**?

(e) Provide an example of a set together with a binary operation that is commutative.

(f) Provide an example of a set together with a binary operation that is *not* commutative.

Problem 1.3. Provide an example of a set A and a binary operation $*$ on A such that $(a*b)^2 \neq a^2 * b^2$ for some $a, b \in A$. Under what conditions will $(a*b)^2 = a^2 * b^2$ for all $a, b \in A$? *Note:* The notation x^2 is shorthand for $x * x$.

Problem 1.4. Determine whether each of the following binary operations is (i) associative and (ii) commutative.

(a) The operation $\star$ on $\mathbb{R}$ defined via $a \star b = 1 + ab$. In this case, ab denotes the ordinary multiplication of the real numbers a and b.

(b) The operation $\circ$ on $\mathbb{Q}$ defined via $a \circ b = \dfrac{a+b}{5}$.

(c) The operation $\circledcirc$ on $\mathbb{Z} \times \mathbb{Z}$ defined via $(a,b) \circledcirc (c,d) = (ad + bc, bd)$.

(d) The operation $\circledast$ on $\mathbb{Q} \setminus \{0\}$ defined via $a \circledast b = \dfrac{a}{b}$.

(e) The operation $\ominus$ on $\mathbb{R}/I := \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ defined via $a \ominus b = a + b - \lfloor a + b \rfloor$ (i.e., $a \ominus b$ is the fractional part of $a + b$).

Problem 1.5. Prove that if A is a nonempty set and F is the set of functions from A to A, then function composition is an associative binary operation on F.

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When the set A is finite, we can represent a binary operation on A using a table in which the elements of the set are listed across the top and down the left side (in the same order). The entry in the ith row and jth column of the table represents the output of combining the element that labels the ith row with the element that labels the jth column (order matters).

Example 1.6. Consider the following table.

$*$	a	b	c
a	b	c	b
b	a	c	b
c	c	b	a

This table represents a binary operation on the set $A = \{a, b, c\}$. In this case, $a * b = c$ while $b * a = a$. This shows that $*$ is not commutative.

Problem 1.7. What property must the table for a binary operation have in order for the operation to be commutative?

Problem 1.8. Consider the following table that displays the binary operation $*$ on the set $\{x, y, z\}$.

$*$	x	y	z
x	x	y	z
y	y	x	x
z	y	x	x

(a) Determine whether $*$ is commutative.

(b) Determine whether $*$ is associative.

Problem 1.9. Let n be a fixed positive integer. Define $\equiv_n$ on $\mathbb{Z}$ via

$$a \equiv_n b \text{ if and only if } n \mid (b - a).$$

It turns out that $\equiv_n$ is an equivalence relation (you may take this for granted). If $a \equiv_n b$, then we say, "a is congruent to b mod n." The equivalence classes determined by $\equiv_n$ are defined via

$$\overline{a} = \{a + kn \mid k \in \mathbb{Z}\}.$$

There are precisely n equivalence classes mod n, namely $\overline{0}, \overline{1}, \ldots, \overline{n-1}$ determined by the possible remainders after division by n. We denote the collection of equivalence classes mod n by $\mathbb{Z}/n\mathbb{Z}$. For $\overline{a}, \overline{b} \in \mathbb{Z}/n\mathbb{Z}$, we define **addition mod** n via

$$\overline{a} + \overline{b} = \overline{a + b}.$$

Similarly, we define **multiplication mod** n via

$$\overline{a} \cdot \overline{b} = \overline{a \cdot b}.$$

Prove each of the following.

(a) Addition mod n is a well-defined binary operation on $\mathbb{Z}/n\mathbb{Z}$.

(b) Multiplication mod n is a well-defined binary operation on $\mathbb{Z}/n\mathbb{Z}$.

Problem 1.10. Write down the table that represents addition mod 4 on $\mathbb{Z}/4\mathbb{Z}$.

Definition 1.11. Suppose $*$ is a binary operation on a set A and let $T \subseteq A$. If the restriction of $*$ to T is a binary operation on T, then we say that T is **closed under** $*$.

Problem 1.12. Provide an example of a set A and a proper subset T of A together with a binary operation $*$ on A such that T is closed under $*$.

Problem 1.13. Provide an example of a set A and a proper subset T of A together with a binary operation $*$ on A such that T is *not* closed under $*$.

Problem 1.14. Suppose $*$ is an associative binary operation on A and let $T \subseteq A$ such that T is closed under $*$. Is $*$ an associative binary operation on T? Justify your assertion.

Problem 1.15. Suppose $*$ is a commutative binary operation on A and let $T \subseteq A$ such that T is closed under $*$. Is $*$ a commutative binary operation on T? Justify your assertion.

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1.2 Groups

Definition 1.16. A **group** $(G,*)$ is a set G together with a binary operation $*$ such that the following axioms hold.

(0) The set G is closed under $*$.

(1) The operation $*$ is associative.

(2) There is an element $e \in G$ such that for all $g \in G$, $e*g = g*e = g$. We call e the **identity**.[1]

(3) Corresponding to each $g \in G$, there is an element $g' \in G$ such that $g*g' = g'*g = e$. In this case, g' is said to be an **inverse** of g.

The **order** of G, denoted $|G|$, is the cardinality of the set G. If $|G|$ is finite, then we say that G has finite order. Otherwise, we say that G has infinite order.

In the definition of a group, the binary operation $*$ is not required to be commutative. If $*$ is commutative, then we say that G is **abelian**[2]. A few additional comments are in order.

- Axiom 2 forces G to be nonempty.
- If $(G,*)$ is a group, then we say that G **is a group under** $*$.
- We refer to $a*b$ as the **product** of a and b even if $*$ is not actually multiplication.
- For simplicity, if $(G,*)$ is a group, we will often refer to G as being the group and suppress any mention of $*$ whatsoever. In particular, we will often abbreviate $a*b$ as ab.
- We shall see that each $g \in G$ has a unique inverse. From that point on, we will denote *the* inverse of g by g^{-1}.

Problem 1.17. Explain why Axiom 0 is unnecessary.

Problem 1.18. Explain why every group is nonempty.

Problem 1.19. Consider a square puzzle piece that fits perfectly into a square hole. Let R_4 be the set of net actions consisting of the rotations of the square by an appropriate amount so that it fits back into the hole. For example, rotating by 90° clockwise and 270° counterclockwise are considered the same net action. Assume we can tell the corners of the square apart from each other so that if the square has been rotated and put back in the hole we can notice the difference. Each net action is called a **symmetry** of the square.

(a) Describe all of the distinct symmetries in R_4. How many distinct symmetries are in R_4?

(b) Explain why R_4 is a group under composition of symmetries.

(c) Describe the identity of this group.

(d) Describe the inverse of each element in this group.

(e) Is R_4 an abelian group?

Let's pause for a moment to make sure we understand our use of the word symmetry in this context. A fundamental question in mathematics is "When are two things the same?", where "things" can be whatever mathematical notion we happen to be thinking about at a particular moment. Right now we need to answer, "When do we want to consider two symmetries to be the same?" To be clear, this is a choice, and different choices can lead to different, interesting, and equally valid mathematics. For symmetries, one natural thought is that symmetries are equal when they produce the same net action on the square, meaning that when applied to a square in a particular starting position, they both yield the same ending position. In general, two symmetries are equal if they produce the same net action on the object in question. Notice that we are really defining an equivalence relation here.

The set R_4 is called the rotation group for the square. For $n \geq 3$, R_n is the **rotation group** for the regular n-gon and consists of the rotational symmetries for a regular n-gon. Every R_n really is a group under composition of symmetries.

[1] The origin of using the letter e for the identity of a group appears to be due to German mathematician Heinrich Weber, who uses "einheit" (German for "unit" or "unity") and e in his *Lehrbuch der Algebra* (1896).

[2] Commutative groups are called abelian in honor of the Norwegian mathematician Niels Abel (1802–1829).

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Problem 1.20. Consider a puzzle piece like the one in the previous problem, except this time, let's assume that the piece and the hole are an equilateral triangle. Let D_3 be the full set of symmetries that allow the triangle to fit back in the hole. In addition to rotations, we will also allow the triangle to be flipped over—called a reflection.

(a) Describe all of the distinct symmetries in D_3. How many distinct symmetries are in D_3?

(b) Explain why D_3 is a group under composition of symmetries.

(c) Describe the identity of this group.

(d) Describe the inverse of each element in this group.

(e) Is D_3 an abelian group?

Problem 1.21. Repeat the above problem, but do it for a square instead of a triangle. The corresponding group is called D_4.

The sets D_3 and D_4 are examples of dihedral groups. In general, for $n \geq 3$, D_n consists of the symmetries (rotations and reflections) of a regular n-gon and is called the **dihedral group of order** $2n$. Do you see why D_n consists of $2n$ net actions? As expected, every D_n really is a group.

Problem 1.22. Consider the set S_3 consisting of the net actions that permute the positions of three coins (without flipping them over) that are sitting side by side in a line. Assume that you can tell the coins apart.

(a) Write down all distinct net actions in S_3 using verbal descriptions. Some of these will be tricky to describe. How many distinct net actions are in S_3?

(b) Explain why S_3 is a group under composition of symmetries.

(c) Describe the identity of this group.

(d) Describe the inverse of each element in this group.

(e) Is S_3 an abelian group?

The set S_3 is an example of a symmetric group. In general, S_n is the **symmetric group on** n **objects** and consists of the net actions that rearrange the n objects. Such rearrangements are called **permutations**. Later we will prove that each S_n is a group under composition of permutations.

Problem 1.23. Determine whether each of the following is a group. If the pair is a group, determine the order, identify the identity, describe the inverses, and determine whether the group is abelian. If the pair is not a group, explain why.

(a) $(\mathbb{Z}, +)$

(b) $(\mathbb{N}, +)$

(c) $(\mathbb{Z}, \cdot)$

(d) $(\mathbb{Z}, \div)$

(e) $(\mathbb{R}, +)$

(f) $(\mathbb{C}, +)$

(g) $(\mathbb{R}, \cdot)$

(h) $(\mathbb{Q} \setminus \{0\}, \cdot)$

(i) $(\mathbb{Z} \setminus \{0\}, \cdot)$

(j) $(M_{2\times 2}(\mathbb{R}), +)$

(k) $(M_{2\times 2}(\mathbb{R}), *)$, where $*$ is matrix multiplication.

(l) $([0,1], *)$, where $a * b := \min(a, b)$

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(m) $(\{a,b,c\},*)$, where $*$ is the operation determined by the table in Example 1.6.

(n) $(\{x,y,z\},*)$, where $*$ is the operation determined by the table in Problem 1.8.

(o) $\mathbb{Z}/n\mathbb{Z}$ under addition mod n.

(p) $\mathbb{Z}/n\mathbb{Z}$ under multiplication mod n.

(q) Set of rational numbers in lowest terms whose denominators are odd under addition. *Note:* Since we can write $0 = 0/1$, 0 is included in this set.

(r) Set of rational numbers in lowest terms whose denominators are even together with 0 under addition.

(s) Set of rational numbers of absolute value less than 1 under addition.

(t) $\mathbb{R}/I$ under $\ominus$ as defined in Problem 1.4(e).

Problem 1.24. Let $G = \{a + b\sqrt{2} \mid a,b \in \mathbb{Q}\}$. Prove each of the following.

(a) The set G is a group under addition.

(b) If $H = G \setminus \{0\}$, then H is a group under multiplication.

Notice that in Axiom 2 of Definition 1.16, we said *the* identity and not *an* identity. Implicitly, this implies that the identity is unique. You'll notice that I even said "the identity" in Problems 1.19–1.23.

Problem 1.25. Prove that if G is a group, then there is a unique identity element in G. That is, there is only one element $e \in G$ such that $ge = eg = g$ for all $g \in G$.

Problem 1.26. Provide an example of a group of order 1. Can you find more than one such group?

Any group of order 1 is called a **trivial group**. It follows immediately from the definition of a group that the element of a trivial group must be the identity.

It is important to note that if we have an equation involving the product of group elements, we can still "do the same thing to both sides" and maintain equality. However, because general groups are not necessarily abelian, we have to be careful that we truly operate in the same way on each side. For example, if we have the equation $g = h$ in some group, then we also have $ag = ah$, where we "multiplied" both sides on the left by the group element a. We could not necessarily conclude that $ag = ha$, unless one pair of the elements happen to commute with each other.

The following theorem is crucial for proving many theorems about groups.

Problem 1.27 (Cancellation Law)**.** Let G be a group and let $g,x,y \in G$. Prove that $gx = gy$ if and only if $x = y$. Similarly, we have $xg = yg$ if and only if $x = y$.

Problem 1.28. Show that $(\mathbb{R},\cdot)$ fails the Cancellation Law confirming the fact that it is not a group.

Recall that Axiom (3) of Definition 1.16 states that each element of a group has at least one inverse. The next theorem tells us that each element has exactly one inverse. Again, you'll notice that I already cheated at wrote "the inverse" in Problems 1.19–1.23.

Problem 1.29. Prove that if G is a group, then each $g \in G$ has a unique inverse.

In light of the previous problem, the unique inverse of $g \in G$ will be denoted as g^{-1}. In groups, it turns out that inverses are always "two-sided". That is, if G is a group and $g,h \in G$ such that $gh = e$, then it must be the case that $hg = e$, as well. In this case, $g^{-1} = h$ and $h^{-1} = g$. However, there are mathematical structures where a "left inverse" exists but the "right inverse" does not.

Problem 1.30. Prove that if G is a group, then for all $g,h \in G$, the equation $gx = h$ has a unique solution for x in G. Similarly, the equation $xg = h$ has a unique solution.

The next result should not be surprising.

Problem 1.31. Prove that if G is a group, then $(g^{-1})^{-1} = g$ for all $g \in G$.

The next result is analogous to the "socks and shoes theorem" for composition of functions.

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Problem 1.32. Prove that if G is a group, then $(gh)^{-1} = h^{-1}g^{-1}$ for all $g, h \in G$.

Problem 1.33 (Generalized Associative Law)**.** Prove that if G is a group, then for any $g_1, g_2, \ldots, g_n \in G$, the value of $g_1 g_2 \cdots g_n$ is independent of how the product is bracketed. Consider using induction on n.

Definition 1.34. If G is a group and $g \in G$, then for all $n \in \mathbb{N}$, we define:

(a) $g^n = \underbrace{gg\cdots g}_{n \text{ factors}}$

(b) $g^{-n} = \underbrace{g^{-1}g^{-1}\cdots g^{-1}}_{n \text{ factors}}$

(c) $g^0 = e$

Remark 1.35. If G is a group under $+$, then we can reinterpret Definition 1.34 as:

(a) $ng = \underbrace{g + g + \cdots + g}_{n \text{ summands}}$

(b) $-ng = \underbrace{-g + -g + \cdots + -g}_{n \text{ summands}}$

(c) $0g = 0$

Notice all that we have done is taken the statements of Definition 1.34, which use multiplicative notation for the group operation, and translated what they say in the case that the group operation uses additive notation.

The good news is that the many of the rules of exponents you are familiar with still hold for groups.

Problem 1.36. Prove that if G is a group and $g \in G$, then for all $n, m \in \mathbb{Z}$, we have the following:

(a) $g^n g^m = g^{n+m}$,

(b) $(g^n)^{-1} = g^{-n}$,

(c) $(g^n)^m = g^{nm}$.

Problem 1.37. Reinterpret problem 1.36 if G is a group under addition.

Unfortunately, there are some rules of exponents that do not apply for general groups.

Problem 1.38. Assume G is a group and let $a, b \in G$. Is it true that $(ab)^n = a^n b^n$? If not, under what minimal conditions would it be true? Prove the statement that you think is true.

Problem 1.39. Assume G is a group. Prove that if $g^2 = e$ for all $g \in G$, then G is abelian. Is the converse true?

Problem 1.40. Assume $G = \{e, a, b, c\}$ is a group under $\star$ with the property that $x^2 = x^4$ for all $x \in G$ (where e is the identity). Complete the following **group table**, where $x \star y$ is defined to be the entry in the row labeled by x and the column labeled by y.

$\star$	e	a	b	c
e	e	a	b	c
a	a			
b	b			
c	c			

Is your table unique? That is, did you have to fill it out the way you did? Deduce that G is abelian.

Problem 1.41. Assume G is a finite group. Prove that every element of G must appear exactly once in every row and column of the group table for G. (Of course, we are not counting the row and column headings.)

Problem 1.42. Prove that if G is a group and $g \in G$, then the two functions $l_g(x) := gx$ and $r_g(x) := xg$ are both permutations of G (i.e., l_g and r_g are bijections from G to G).

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1.3 Generating Sets

In this section, we explore the concept of a generating set for a group.

Definition 1.43. Let G be a group and let S be a subset of G. A finite product (under the operation of G) consisting of elements from S or their inverses is called a **word** in S. That is, a word in S is of the form

$$s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n},$$

where each $s_i \in S$ and $\varepsilon_i \in \{\pm 1\}$. Each s_i is called a **letter** and the set S is called the **alphabet**. By convention, the identity of G can be represented by the **empty word**, which is the word having no letters. The set of elements of G that can be written as words in S is denoted by $\langle S \rangle$ and is called the **group generated by** S.

It is worth mentioning that we are slightly abusing notation here. For nonempty $S \subset G$, we can form infinitely many words in $\langle S \rangle$, but often there are many words that represent the same group element. We can partition the collection of words in the alphabet S into equivalence classes based on which group element a word represents. Strictly speaking, each group element is an equivalence class of words. When we say two words are equal in the group, what we really mean is that both words are in the same equivalence class.

Moreover, while S and $\langle S \rangle$ are both sets, the latter set is the set of elements we can build using letters and their inverses from S. It turns out that if S is itself a group, then $S = \langle S \rangle$. Otherwise, S is a proper subset of $\langle S \rangle$.

If we know what the elements of S actually are, then we will list them inside the angle brackets without the set braces. For example, if $S = \{a, b, c\}$, then we will write $\langle a, b, c \rangle$ instead of $\langle \{a, b, c\} \rangle$. In the special case when the generating set S consists of a single element, say g, we have

$$G = \langle g \rangle = \{g^k \mid k \in \mathbb{Z}\}$$

and say that G is a **cyclic group**. As we shall see, $\langle g \rangle$ may be finite or infinite.

Example 1.44. Suppose G is a group such that $a, b, c \in G$ and let $S = \{a, b, c\}$. Then ab, $c^{-1}acc$, and $ab^{-1}caa^{-1}bc^{-1}$ are words in $\langle S \rangle$. If any one of these words is not equal to a, b, or c, then S is a proper subset of $\langle S \rangle$.

Problem 1.45. Prove that if G is a group under $*$ and S is a subset of G, then $\langle S \rangle$ is also a group under $*$.

Definition 1.46. If G is a group and S is a subset of G such that $G = \langle S \rangle$, then S is called a **generating set** of G. In other words, S is a generating set of G if every element of G can be expressed as a word in S. In this case, we say S **generates** G. A generating set S for G is a **minimal generating set** if $S \setminus \{x\}$ is no longer a generating set for G for all $x \in S$.

A generating set for a group is analogous to a spanning set for a vector space and a minimal generating set for a group is analogous to a basis for a vector space.

Problem 1.47. Consider the rotation group R_4 that we introduced in Problem 1.19. Let r be the element of R_4 that rotates the square by 90° clockwise.

(a) Describe the action of r^{-1} on the square and express r^{-1} as a word using r only.

(b) Prove that $R_4 = \langle r \rangle$ by writing every element of R_4 as a word using r only.

(c) Is $\{r\}$ a minimal generating set for R_4?

(d) Is R_4 a cyclic group?

Problem 1.48. Consider the dihedral group D_3 introduced in Problem 1.20. To give us a common starting point, let's assume the triangle and hole are positioned so that one of the tips of the triangle is pointed up. Let r be rotation by 120° in the clockwise direction and let s be the reflection in D_3 that fixes the top of the triangle.

(a) Describe the action of r^{-1} on the triangle and express r^{-1} as a word using r only.

(b) Describe the action of s^{-1} on the triangle and express s^{-1} as a word using s only.

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(c) Prove that $D_3 = \langle r, s \rangle$ by writing every element of D_3 as a word in r or s.

(d) Is $\{r, s\}$ a minimal generating set for D_3?

(e) Explain why there is no single generating set for D_3 consisting of a single element. This proves that D_3 is not cyclic.

It is important to point out that the fact that $\{r, s\}$ is a minimal generating set for D_3 does not immediately imply that D_3 is not a cyclic group. There are examples of cyclic groups that have minimal generating sets consisting of more than one element as the next problem illustrates.

Problem 1.49. Let R_6 denote the group of rotational symmetries of a regular hexagon and let r be rotation by 60° clockwise.

(a) Is R_6 cyclic?

(b) Is R_6 abelian?

(c) Write r^{-1} as a word in r.

(d) Can you find a shorter word to describe r^8?

(e) Does r^2 generate the group?

(f) Does r^3 generate the group?

(g) Does r^5 generate the group?

(h) Is $\{r^2, r^3\}$ a minimal generating set for R_6?

Problem 1.50. Let's consider the group D_3 again. Let s be the same reflection as in Problem 1.48 and let s' be the reflection in D_3 that fixes the bottom right corner of the triangle.

(a) Express r as a word in s and s'.

(b) Use part (a) together with Problem 1.48 to prove that $\langle s, s' \rangle = D_3$.

Problem 1.51. Consider the dihedral group D_4 introduced in Problem 1.21. Let r be clockwise rotation by 90° and let s be the reflection over the vertical midline of the square.

(a) Describe the action of r^{-1} on the square and express r^{-1} as a word using r only.

(b) Describe the action of s^{-1} on the square and express s^{-1} as a word using s only.

(c) Prove that $\{r, s\}$ is generating set for D_4.

(d) Is $\{r, s\}$ a minimal generating set for D_4?

(e) Find a different generating set for D_4.

(f) Is D_4 a cyclic group?

Problem 1.52. Consider the symmetric group S_3 that was introduced in Problem 1.22. Let s_1 be the action that swaps the positions of the first and second coins and let s_2 be the action that swaps the positions of the second and third coins.

(a) Prove that $S_3 = \langle s_1, s_2 \rangle$.

(b) Is $\{s_1, s_2\}$ a minimal generating set for S_3?

Problem 1.53. Consider a rectangle (which may or may not be a square) oriented so that one side is parallel to the ground. Let h be the symmetry that reflects the rectangle over the horizontal midline and let v be the symmetry that reflects the rectangle over the vertical midline. Define $V_4 := \langle v, h \rangle$. This group is called the **Klein group** (or **Vierergruppe**, which is German for "four-group") after the German mathematician Felix Klein (1849–1925).

(a) Verify that $|V_4| = 4$ by describing the symmetries in the group.

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(b) Is V_4 abelian?

(c) Is V_4 cyclic?

Problem 1.54. Prove that the group $(\mathbb{Z}/n\mathbb{Z}, + \bmod n)$ is cyclic.

Problem 1.55. Consider the group $(\mathbb{Z}, +)$.

(a) Find a generating set that consists of a single element. Is $\mathbb{Z}$ a cyclic group under addition?

(b) If possible, find a minimal generating set that consists of two elements. If this is not possible, explain way.

Problem 1.56. Consider the group $(\mathbb{Q}, +)$.

(a) Find a generating set that is a proper subset of $\mathbb{Q}$.

(b) Is your generating set a minimal generating set?

Problem 1.57. Prove that if G is a cyclic group, then G is abelian.

Problem 1.58. Is the converse of the previous problem true? If so, prove it. Otherwise, find a counterexample.

1.4 Group Presentations

In this section, we introduce the notion of a **presentation** of a group. We'll only touch the surface here. There's a lot more going on behind the scenes!

Definition 1.59. Let G be a group and suppose $S \subseteq G$ such that $G = \langle S \rangle$. Any equation that the generators satisfy is called a **relation**.

Example 1.60. Here are a few examples of relations.

(a) Recall that $D_3 = \langle r, s \rangle$, where r and s are the actions described in Problem 1.48. In D_3, it's easy to verify that $r^3 = e$, $s^2 = e$, and $sr = r^2 s$. Each of these equations is an example of a relation in D_3.

(b) We also have $D_3 = \langle s, s' \rangle$, where s and s' are the actions described in Problem 1.50. Using this set of generators, D_3 satisfies the relations $s^2 = e$ (same as part (a)), $(s')^2 = e$, and $ss's = s'ss'$.

(c) Similar to part (a), $D_4 = \langle r, s \rangle$. In this case, D_4 satisfies the relations $r^4 = e$, $s^2 = e$, and $sr = r^3 s$.

(d) According to Problem 1.52, $S_3 = \langle s_1, s_2 \rangle$. It is easy to verify that S_3 satisfies the relations $s_1^2 = e$, $s_2^2 = e$, and $s_1 s_2 s_1 = s_2 s_1 s_2$.

(e) Using the generating set $\{1\}$ for $\mathbb{Z}$, it turns out that there are no relations.

Problem 1.61. Complete each of the following.

(a) Prove that $r^5 = r^2$, $(sr)^2 = e$, and $(ss')^3 = e$ are relations in D_3 using the relations provided in parts (a) and (b) of Example 1.60.

(b) Prove that $sr^2 = r^2 s$ is a relation in D_4 using the relations provided in part (c) of Example 1.60.

(c) Prove that $(s_2 s_1)^3 = e$ is a relation in S_3 using the relations provided in part (d) of Example 1.60.

Definition 1.62. Let G be a group and suppose $S \subseteq G$ such that $G = \langle S \rangle$. If there is a collection of relations, say $w_1 = e, w_2 = e, \ldots, w_m = e$, where each w_i is a word consisting of elements from S or inverses of elements from S, such that any relation among the elements of S can be derived from $w_1 = e, w_2 = e, \ldots, w_m = e$, we say that $(S, w_1 = e, w_2 = e, \ldots, w_m = e)$ is a **presentation** of G and write

$$G = \langle S \mid w_1 = e, w_2 = e, \ldots, w_m = e \rangle.$$

Officially, this is a **finite presentation** for G since there are finitely many relations. If instead we utilize infinitely many relations, the corresponding presentation is an **infinite presentation**.

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Example 1.63. It is not immediately obvious, but it turns out that the relations described in Example 1.60 determine presentations for D_3, D_4, S_3, and $\mathbb{Z}$. In Problem 1.61, we verified that we can derive some additional relations from the ones given in Example 1.60. The not obvious part is that *every* relation in each of these groups can be deduced from the relations that were listed. That is, we have:

(a) $D_3 = \langle r, s \mid r^3 = e, s^2 = e, sr = r^2 s \rangle = \langle s, s' \mid s^2 = e, (s')^2 = e, ss's = s'ss' \rangle$.

(b) $D_4 = \langle r, s \mid r^4 = e, s^2 = e, sr = r^3 s \rangle$

(c) $S_3 = \langle s_1, s_2 \mid s_1^2 = e, s_2^2 = e, s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$

(d) $\mathbb{Z} = \langle 1 \rangle$

It turns out that that the relations in each of the presentations are also minimal in the sense that we cannot eliminate one of the relations and use the remaining ones to produce the eliminated one.

For $n \geq 3$, define the **dihedral group** D_n to be the group of symmetries of a regular n-gon. It's not too hard to prove that D_n consists of n distinct rotations and n distinct reflections, so that $|D_n| = 2n$. In particular, one can prove using geometric arguments that

$$D_n = \langle r, s \rangle = \{\underbrace{e, r, r^2, \ldots, r^{n-1}}_{\text{rotations}}, \underbrace{s, sr, sr^2, \ldots, sr^{n-1}}_{\text{reflections}}\},$$

where r is equal to rotation by $360°/n$ clockwise, s is any fixed reflection, and each of the elements listed above is distinct. It follows from geometric arguments that $r^{-1} = r^{n-1}$ and $s^{-1} = s$. Later, we will prove that

$$D_n = \langle r, s \mid r^n = s^2 = e, sr = r^{n-1} s \rangle$$

is a group presentation for D_n. It is easy to very that r and s satisfy the relations described in the presentations, but it is not obvious that these relations are enough to determine the group. In this section, we will take for granted that this is a presentation for D_n.

We can also define groups using presentations. If we define a group G via a presentation, say $G = \langle S \mid w_1 = e, w_2 = e, \ldots, w_m = e \rangle$, we mean that G is the group generated by S that satisfies all of the relations we can derive from $w_1 = e, w_2 = e, \ldots, w_m = e$. For example, we can define

$$D_2 := \langle r, s \mid r^2 = s^2 = e, sr = rs \rangle,$$

which fills in the case when $n = 2$ for the dihedral groups.

Problem 1.64. There's no such thing as a 2-gon, but can you describe an object that D_2 is the symmetry group for? There is more to proving your claim than you might expect. Don't worry about proving this carefully, but do consider what needs to be verified.

Problem 1.65. Consider D_n for $n \geq 3$.

(a) Prove that if $x \in D_n$ such that x is not a power of r, then $rx = xr^{-1}$.

(b) Assume $x \in D_n$ such that x is not a power of r. Geometrically, this implies that $x^2 = e$. Verify this fact using the relations provided in the presentation for D_n.

(c) Assume $n = 2k$ is even such that $n \geq 4$. By the description above for D_n in terms of rotations and reflection, we know $r^k \neq e$. Prove that $(r^k)^2 = e$. Moreover, prove that r^k is the only nonidentity element that commutes with every element of D_n.

(d) Assume n is odd such that $n \geq 3$. Prove that the identity is the only element of D_n that commutes with every element of D_n.

(e) Prove that $\langle a, b \mid a^2 = b^2 = (ab)^n = e \rangle$ is a presentation for D_n in terms of the generators $a = s$ and $b = sr$.

It follows from parts (c) and (d) of the previous problem that D_n is not abelian for all $n \geq 3$. This comes as no surprise since it is easy to see geometrically that $sr \neq rs$.

Utilizing presentations is tricky business. First, if you have a particular group in mind, it can often be difficult to find a presentation. Second, if you define a group using a presentation, it may be difficult

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(or even impossible!) to determine when two elements of the group (expressed as words in the generators) are equal. As a result, we may have some difficulty determining the order of a group given by a presentation. In particular, it may not be easy to determine whether the corresponding group is even finite or infinite!

Similar to the last part of Problem 1.65, one can show that $\langle a, b \mid a^2 = b^2 = (ab)^2 = e\rangle$ is a presentation for D_2 (where $a = r$ and $b = s$). It turns out that $|D_2| = 4$. In particular, any group with a similar presentation is finite (specifically order 4). However, if you consider the similar-looking presentation $\langle a, b \mid a^3 = b^3 = (ab)^3 = e\rangle$, it turns out that the corresponding group is infinite! This is not obvious at all. Loosely speaking, it must be the case that in the presentation $\langle a, b \mid a^2 = b^2 = (ab)^2 = e\rangle$ there are sufficiently many relations that can be deduced from the given relations that a massive amount of "collapsing" occurs to make the group finite. In contrast, not enough collapsing occurs in $\langle a, b \mid a^3 = b^3 = (ab)^3 = e\rangle$ to make the group finite.

This collapsing makes it difficult to even determine lower bounds on the order for the group being presented. Sometimes even innocent-looking presentations can collapse considerably.

Problem 1.66. Define $G = \langle x, y \mid x^4 = y^3 = e, xy = y^2x^2\rangle$.

(a) Show that $y^2 = y^{-1}$.

(b) Show that y commutes with x^3. *Hint:* Show that $y^2x^3y = x^3$ by writing the left hand side as $(y^2x^2)(xy)$ and using the relations to reduce this to the right hand side. Then use part (a).

(c) Show that y commutes with x. *Hint:* Show that $x^9 = x$ and then use part (b).

(d) Show that $xy = e$ *Hint:* Use part (c) and the last relation.

(e) Show that $x = e$, and then deduce that $y = e$. *Hint:* Use part (d) and the relation $x^4y^3 = e$.

(f) Conclude that $|G| = 1$.

1.5 Subgroups

According to Problem 1.45, if S is any subset of a group G under $*$, then $\langle S\rangle$ is also a group under $*$. However, notice that $\langle S\rangle$ may not be equal to G. That is, $\langle S\rangle$ may be a proper subset of G that is a group in its own right (using the same binary operation as G). We can give a name to this phenomenon.

Definition 1.67. Let G be a group and let H be a subset of G. Then H is a **subgroup** of G, written $H \leq G$, provided that H is a group in its own right under the binary operation inherited from G.

The phrase "under the binary operation inherited from G" means that to combine two elements in H, we should treat the elements as if they were in G and perform the binary operation of G.

As an example, the group of rotations of a square is a subgroup of the full group of symmetries of a square. That is, $R_4 \leq D_4$.

Problem 1.68. Let G be a group and let $H \subseteq G$. If we wanted to determine whether H is a subgroup of G, can we skip checking any of the axioms? Which axioms must we verify?

Let's make the observations of the previous problem a bit more formal.

Problem 1.69 (Two Step Subgroup Test)**.** Suppose G is a group and H is a nonempty subset of G. Prove that $H \leq G$ if and only if (i) for all $h \in H$, $h^{-1} \in H$, as well, and (ii) H is closed under the binary operation of G.

Notice that one of the hypotheses of Problem 1.69 is that H be nonempty. This means that if we want to prove that a certain subset H is a subgroup of a group G, then one of the things we must do is verify that H is in fact nonempty. In light of this, the "Two Step Subgroup Test" should probably be called the "Three Step Subgroup Test".

Problem 1.70. Suppose G is a group and H is a nonempty subset of G. Conjecture and prove a "One Step Subgroup Test" that streamlines Problem 1.69.

As Problems 1.71 and 1.72 will illustrate, there are a couple of subgroups that every group contains.

Problem 1.71. Prove that if G is a group, then $\{e\} \leq G$.

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The subgroup $\{e\}$ is referred to as the **trivial subgroup**. All other subgroups are called **nontrivial**. Subgroups are not required to be proper subsets of the "parent" group.

Problem 1.72. Prove that if G is a group, then $G \leq G$.

We refer to subgroups that are not equal to the whole group as **proper subgroups**. If H is a proper subgroup, then we may write $H < G$.

Let's take Problem 1.45 a step further.

Problem 1.73. Prove that if G is a group and $S \subseteq G$, then $\langle S \rangle$ is the smallest subgroup of G containing S.

The subgroup $\langle S \rangle$ is called the **subgroup generated by** S. In the special case when S equals a single element, say $S = \{g\}$, then

$$\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\},$$

which is called the (**cyclic**) **subgroup generated by** g. Every subgroup can be written in the "generated by" form. That is, if H is a subgroup of a group G, then there always exists a subset S of G such that $\langle S \rangle = H$. In particular, $\langle H \rangle = H$ for $H \leq G$, and as a special case, we have $\langle G \rangle = G$.

Problem 1.74. Consider D_4. Let h be the reflection of the square over the horizontal midline and let v be the reflection over the vertical midline. Which of the following are subgroups of D_4? In each case, justify your answer. If a subset is a subgroup, try to find a minimal generating set.

(a) $\{e, r^2\}$

(b) $\{e, h\}$

(c) $\{e, h, v\}$

(d) $\{e, h, v, r^2\}$

Problem 1.75. Consider $(\mathbb{R}^3, +)$, where $\mathbb{R}^3$ is the set of all 3-entry row vectors with real number entries (e.g., (a, b, c) where $a, b, c \in \mathbb{R}$) and $+$ is ordinary vector addition. It turns out that $(\mathbb{R}^3, +)$ is an abelian group with identity $(0, 0, 0)$.

(a) Let H be the subset of $\mathbb{R}^3$ consisting of vectors with first coordinate 0. Is H a subgroup of $\mathbb{R}^3$? Prove your answer.

(b) Let K be the subset of $\mathbb{R}^3$ consisting of vectors whose entries sum to 0. Is K a subgroup of $\mathbb{R}^3$? Prove your answer.

(c) Construct a subset of $\mathbb{R}^3$ (different from H and K) that is *not* a subgroup of $\mathbb{R}^3$.

Problem 1.76. Consider the group $(\mathbb{Z}, +)$ (under ordinary addition).

(a) Show that the odd integers are not a subgroup of $\mathbb{Z}$.

(b) Show that all subsets of the form $n\mathbb{Z} := \{nk \mid k \in \mathbb{Z}\}$ for $n \in \mathbb{Z}$ are subgroups of $\mathbb{Z}$.

(c) For $n \in \mathbb{Z}$, write the subgroup $n\mathbb{Z}$ in the "generated by" notation. That is, find a set S such that $\langle S \rangle = n\mathbb{Z}$. Can you find more than one way to do it?

(d) Find n such that $\langle 6, 9 \rangle = n\mathbb{Z}$. Justify your assertion.

(e) Are there any other subgroups besides the ones listed in part (b)? State a conjecture and perhaps prove it.

Problem 1.77. Consider the groups $(\mathbb{R}, +)$ and $(\mathbb{R} \setminus \{0\}, \cdot)$. Explain why $\mathbb{R} \setminus \{0\}$ is not a subgroup of $\mathbb{R}$ despite the fact that $\mathbb{R} \setminus \{0\} \subseteq \mathbb{R}$ and both are groups (under the respective binary operations).

Problem 1.78. Prove that if G is an abelian group and $H \leq G$, then H is an abelian subgroup.

Problem 1.79. Is the converse of the previous theorem true? If so, prove it. Otherwise, provide a counterexample.

Recall that the order of a group G, denoted $|G|$, is the number of elements in G.

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Definition 1.80. We define the **order** of an element g, written $|g|$, to be the order of $\langle g \rangle$. That is, $|g| = |\langle g \rangle|$.

It is clear that a group G is cyclic with generator g if and only if $|G| = |g|$.

Problem 1.81. What is the order of the identity in any group?

Problem 1.82. Find the orders of each of the elements in each of the following groups.

(a) S_2

(b) R_3

(c) R_4

(d) V_4

(e) R_5

(f) D_3

(g) S_3

(h) D_4

Problem 1.83. Consider the group $(\mathbb{Z}, +)$. What is the order of 1? Are there any elements in $\mathbb{Z}$ with finite order?

Problem 1.84. Prove that if G is a group and $g \in G$, then $\langle g \rangle = \langle g^{-1} \rangle$.

The next result follows immediately from Problem 1.84.

Problem 1.85. Prove that if G is a group and $g \in G$, then $|g| = |g^{-1}|$.

1.6 Centers, Centralizers, and Normalizers

In this section, we introduce three special subgroups. As we've seen, some groups are abelian and some are not.

Definition 1.86. If G is a group, then we define the **center** of G to be

$$Z(G) := \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

Notice that if G is abelian, then $Z(G) = G$. However, if G is not abelian, then $Z(G)$ will be a proper subset of G. In some sense, the center of a group is a measure of how close G is to being abelian.

Problem 1.87. Prove that if G is a group, then $Z(G)$ is an abelian subgroup of G.

Problem 1.88. Find the center of each of the following groups.

(a) S_2

(b) V_4

(c) S_3

(d) $\mathbb{Z}/n\mathbb{Z}$

(e) R_n

(f) D_n

(g) $(\mathbb{Z}, +)$

(h) $(\mathbb{R} \setminus \{0\}, \cdot)$

Definition 1.89. Let G be a group and let A be a nonempty subset of G. Define the **centralizer** of A in G via

$$C_G(A) := \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}.$$

If $A = \{a\}$, we will write $C_G(a)$ instead of $C_G(\{a\})$.

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Since $gag^{-1} = a$ if and only if $ga = ag$, $C_G(A)$ is the set of elements of G that commute with every element of A. The product gag^{-1} is called the **conjugate** of a by g. Notice that $C_G(G) = Z(G)$.

Problem 1.90. Prove that if G is a group and A be a nonempty subset of G, then $C_G(A)$ is a subgroup of G.

Definition 1.91. Let G be a group and let A be a nonempty subset of G. Define the **normalizer** of A in G via

$$N_G(A) := \{g \in G \mid gAg^{-1} = A\},$$

where $gAg^{-1} := \{gag^{-1} \mid a \in A\}$.

Notice that $C_G(A)$ is the set of elements of G that fix the set A *pointwise* using conjugation while $N_G(A)$ is the set of elements of G that fix the set A *setwise* using conjugation. It's clear that $C_G(A) \subseteq N_G(A)$.

Problem 1.92. Prove that if G is a group and A be a nonempty subset of G, then $N_G(A)$ is a subgroup of G. Is it true that $C_G(A)$ is a subgroup of $N_G(A)$?

Problem 1.93. Let G be a group and let A be a nonempty subset of G. Determine whether the following statement is true or false: $Z(G) \leq C_G(A) \leq N_G(A)$. If it is true, prove it. Otherwise, find a counterexample.

Problem 1.94. Suppose G is an abelian group and A is a nonempty subset of G. What can you say about $C_G(A)$ and $N_G(A)$?

Problem 1.95. For each group G and subset A, determine $C_G(A)$ and $N_G(A)$.

(a) $G = D_4$, $A = \langle r \rangle$

(b) $G = D_4$, $A = \langle s \rangle$

(c) $G = D_4$, $A = \langle s, r^2 \rangle$

(d) $G = S_3$, $A = \langle s_1 \rangle$

Problem 1.96. Let G be a group and let A and B be subsets of G such that $A \subseteq B$. What is the relationship between $C_G(A)$ and $C_G(B)$? Justify your assertion.

Problem 1.97. Let H be a subgroup of a group G.

(a) Prove that $H \leq N_G(H)$.

(b) Prove that $H \leq C_G(H)$ if and only if H is abelian.

1.7 Subgroup Lattices

Suppose we wanted to find all of the subgroups of a finite group G. Problems 1.71 and 1.72 tell us that $\{e\}$ and G itself are subgroups of G, but there may be others. Problem 1.69 tells us that if we want to find other subgroups of G, we need to find nonempty subsets of G that are closed and contain all the necessary inverses. So, one method for finding subgroups would be to find all possible nonempty subsets of G and then go about determining which subsets are subgroups by verifying whether a given subset is closed under inverses and closed under the operation of G. This is very time consuming!

Another approach would be to utilize the fact that every subgroup H of G has a generating set. That is, if H is a subgroup of a group G, then there always exists a subset S of G such that $\langle S \rangle = H$. Given a subset S of G, $\langle S \rangle$ is guaranteed to be closed under inverses and the operation of the group G. So, we could determine all of the subgroups of G by generating groups with various subsets S of G. Of course, one drawback is that it might take a bit of effort to determine what $\langle S \rangle$ actually is. Another drawback is that two different subsets S and T may generate the same subgroup.

Let's make this a bit more concrete by exploring an example.

Example 1.98. Consider the group R_4. What are the subgroups of R_4? Since the order of R_4 is 4, we know that there are $2^4 - 1 = 15$ nonempty subsets of R_4. Some of these are subgroups, but most of them are not. We know that $\{e\}$ and R_4 itself are subgroups of R_4. That's 2 out of 15 so far. Are there any others? Let's do an exhaustive search by playing with generating sets. We can certainly be more efficient, but below we list all of the possible subgroups we can generate using subsets of R_4. As you scan the list, you should take a moment to convince yourself that the list is accurate.

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$\langle e \rangle = \{e\}$

$\langle r \rangle = \{e, r, r^2, r^3\}$

$\langle r^2 \rangle = \{e, r^2\}$

$\langle r^3 \rangle = \{e, r^3, r^2, r\}$

$\langle e, r \rangle = \{e, r, r^2, r^3\}$

$\langle e, r^2 \rangle = \{e, r^2\}$

$\langle e, r^3 \rangle = \{e, r^3, r^2, r\}$

$\langle r, r^2 \rangle = \{e, r, r^2, r^3\}$

$\langle r, r^3 \rangle = \{e, r, r^2, r^3\}$

$\langle r^2, r^3 \rangle = \{e, r, r^2, r^3\}$

$\langle e, r, r^2 \rangle = \{e, r, r^2, r^3\}$

$\langle e, r, r^3 \rangle = \{e, r, r^2, r^3\}$

$\langle e, r^2, r^3 \rangle = \{e, r, r^2, r^3\}$

$\langle r, r^2, r^3 \rangle = \{e, r, r^2, r^3\}$

$\langle e, r, r^2, r^3 \rangle = \{e, r, r^2, r^3\}$

Let's make a few observations. Scanning the list, we see only three distinct subgroups:

$$\{e\}, \{e, r^2\}, \{e, r, r^2, r^3\}.$$

Out of 15 nonempty subsets of R_4, only 3 subsets are subgroups. Our exhaustive search guarantees that these are the only subgroups of R_4. It is also worth pointing out that if a subset contains either r or r^3, then that subset generates all of R_4. The reason for this is that $\{r\}$ and $\{r^3\}$ are each minimal generating sets for R_4. More generally, observe that if we increase the size of the generating subset using an element that was already contained in the subgroup generated by the set, then we don't get anything new. For example, consider $\langle r^2 \rangle = \{e, r^2\}$. Since $e \in \langle r^2 \rangle$, we don't get anything new by including e in our generating set.

Problem 1.99. Let G be a group and let $S \subseteq G$. Prove that if $x \in \langle S \rangle$, then $\langle S \rangle = \langle S \cup \{x\} \rangle$.

In the previous theorem, we are not claiming that S is a generating set for G—although this may be the case. Instead, are simply making a statement about the subgroup $\langle S \rangle$, whatever it may be.

We can capture the overall relationship between the subgroups of a group G using a **subgroup lattice**. Given a group G, the **lattice of subgroups** of G is the partially ordered set whose elements are the subgroups of G with the partial order relation being set inclusion. It is common to depict the subgroup lattice for a group using a **Hasse diagram**. The Hasse diagram of subgroup lattice is drawn as follows:

(1) Each subgroup H of G is a vertex.

(2) Vertices corresponding to subgroups with smaller order are placed lower in the diagram than vertices corresponding to subgroups with larger order. In particular, the vertex for $\{e\}$ is placed at the bottom of the diagram and the vertex for G is placed at the top.

(3) There is an edge going up from H to K if $H \leq K$ and there is no subgroup L such that $H \leq L \leq K$ with $L \neq H, K$.

Notice that there is an upward path of edges in the Hasse diagram from H to K if and only if $H \leq K$. For convenience we will not make a distinction between the subgroup lattice for a group G and the corresponding Hasse diagram.

Example 1.100. The Hasse diagram for the subgroup lattice for R_4 is given in Figure 1.

Example 1.101. Let's see what we can do with $V_4 = \{e, v, h, vh\}$. Using an exhaustive search, we find that there are five subgroups:

$\langle e \rangle = \{e\}$

$\langle h \rangle = \{e, h\}$

$\langle v \rangle = \{e, v\}$

$\langle vh \rangle = \{e, vh\}$

$\langle v, h \rangle = \langle v, vh \rangle = \langle h, vh \rangle = \{e, v, h, vh\} = V_4$

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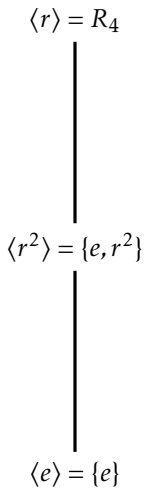

Figure 1: Subgroup lattice for R_4.

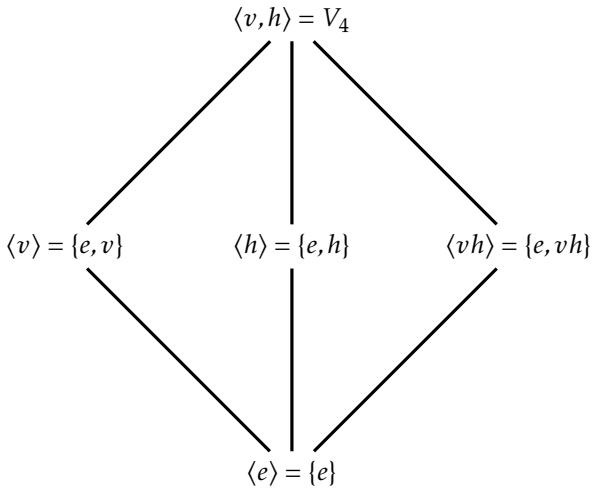

Figure 2: Subgroup lattice for V_4.

For each subgroup above, we've used minimal generating sets to determine the subgroup. The subgroup lattice for V_4 is given in Figure 2. Notice that there are no edges among $\langle v \rangle, \langle h \rangle$, and $\langle vh \rangle$. The reason for this is that none of these groups are subgroups of each other.

The next two problems provide some further insight into the overall structure of subgroups of a group.

Problem 1.102. Prove that if G is a group such that $H, K \leq G$, then $H \cap K \leq G$. Moreover, $H \cap K$ is the largest subgroup contained in both H and K.

It turns out that we cannot simply replace "intersection" with "union" in the previous problem.

Problem 1.103. Provide an example of a group G and subgroups H and K such that $H \cup K$ is not a subgroup of G.

Problem 1.104. Prove that if G is a group such that $H, K \leq G$, then $\langle H \cup K \rangle \leq G$. Moreover, $\langle H \cup K \rangle \leq G$ is the smallest subgroup containing both H and K.

Problems 1.102 and 1.104 justify the use of the word "lattice" in "subgroup lattice". In general, a lattice is a partially ordered set in which every two elements have a unique **meet** (also called a **greatest lower bound** or **infimum**) and a unique **join** (also called a **least upper bound** or **supremum**). In the case of a subgroup lattice for a group G, the meet of subgroups H and K is $H \cap K$ and the join is $\langle H \cup K \rangle$.

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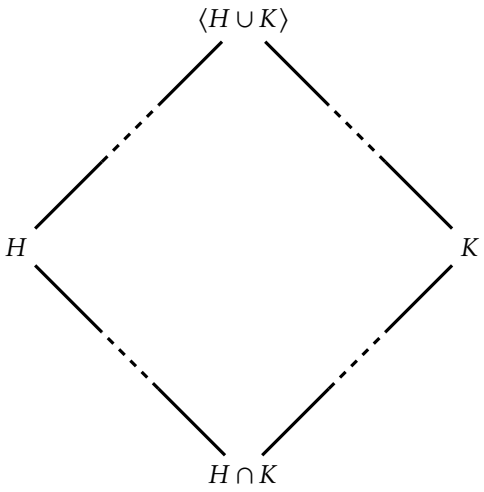

Figure 3: Meet and join for subgroups H and K.

Figure 3 illustrates the meet (Problem 1.102) and join (Problem 1.104) in the case when H and K are not comparable.

In the next problem, you are asked to create subgroup lattices. As you do this, try to minimize the amount of work it takes to come up with all the subgroups.

Problem 1.105. Find all the subgroups for each of the following groups and then draw the subgroup lattice.

(a) R_5

(b) R_6

(c) D_3

(d) S_3

(e) D_4

1.8 Cayley Diagrams

In this section, we will introduce visual way of encoding the abstract structure of the group in terms of a specified generating set.

Definition 1.106. Suppose G is a group and S is a generating set of G. The **Cayley diagram**[3] for G with generating set S is a colored directed graph constructed as follows:

(a) The vertices correspond to elements of G.

(b) Each generator $s \in S$ is assigned a color, say c_s.

(c) For $g \in G$ and $s \in S$, there is a directed edge from g to sg with color c_s.

Note that following the arrow from g to sg with color c_s corresponds to applying the action of s to g. Moreover, following the arrow backwards from sg to g corresponds to applying s^{-1} to sg. If a generator is its own inverse, then the arrows corresponding to that generator are two-way arrows.

Example 1.107. Let R_6 denote the group of rotational symmetries of a regular hexagon and let r be rotation by 60° clockwise. It's not too hard to see that $R_6 = \langle r \rangle$ and $|R_6| = 6$. The Cayley diagram for R_6 with generating set $\{r\}$ is given in Figure 4. In the diagram, following a red (solid) arrow backwards corresponds to the element r^{-1}.

[3] Cayley diagrams are named after their inventor Arthur Cayley, a nineteenth century British mathematician.

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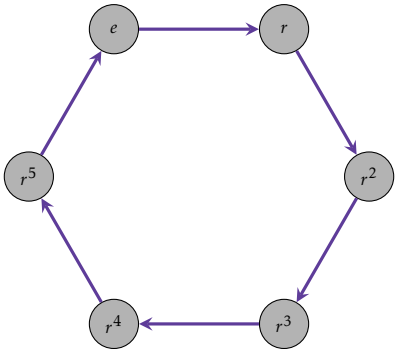

Figure 4: Cayley diagram for R_6 with generating set $\{r\}$.

Example 1.108. The Cayley diagram for the group D_3 with generating set $\{r, s\}$ is given in Figure 5. Notice that we labeled the lower right corner of the Cayley diagram with the word r^2s. This means that we first followed a blue (dashed) arrow out of e and then two red arrows. However, we could also get to this vertex by first doing a red (solid) arrow out of e followed by a blue arrow. So, we could also have labeled this vertex with the word sr. The upshot is that the diagram exhibits the relation $r^2s = sr$. Notice that this relation is true no matter where we start in the diagram!

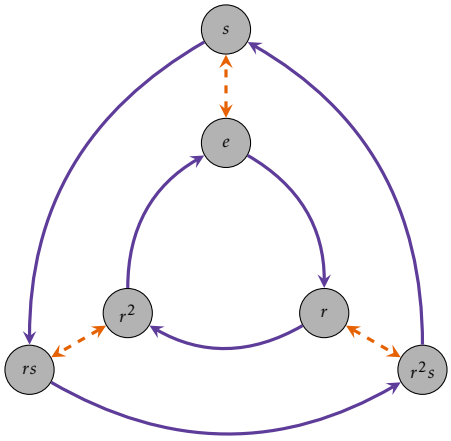

Figure 5: Cayley diagram for D_3 with generating set $\{r, s\}$.

Problem 1.109. Construct a Cayley diagram for each of the following groups using the specified generating set.

(a) S_2 with generating set $\{s\}$

(b) R_4 with generating set $\{r\}$

(c) V_4 with generating set $\{v, h\}$

(d) D_3 with generating set $\{s, s'\}$

(e) S_3 with generating set $\{s_1, s_2\}$

(f) D_4 with generating set $\{r, s\}$

(g) D_4 with generating set $\{s, sr\}$

Problem 1.110. Consider two light switches on a wall side by side. Let ℓ_1 be the action that toggles the position of left light switch and let ℓ_2 be the action that toggles the position of right light switch. Define $\text{Light}_2 := \langle \ell_1, \ell_2 \rangle$. Draw the Cayley diagram for Light_2 using the generating set $\{\ell_1, \ell_2\}$.

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Problem 1.111. Consider three light switches on a wall side by side. Let ℓ_1 be the action that toggles the position of left light switch, let ℓ_2 be the action that toggles the position of middle light switch, and let ℓ_3 be the action that toggles the position of middle light switch. Define $\text{Light}_3 := \langle \ell_1, \ell_2, \ell_3 \rangle$. Draw the Cayley diagram for Light_3 using the generating set $\{\ell_1, \ell_2, \ell_3\}$.

Problem 1.112. Consider two coins sitting side by side. Let s be the action that swaps the positions of the two coins and let t be the action that flips over the left coin. Define $\text{Coin}_2 := \langle s, t \rangle$. Draw the Cayley diagram for Coin_2 using the generating set $\{s, t\}$.

Not only are Cayley diagrams visually appealing, but they provide a map for the group in question. That is, they provide a method for navigating the group. Following sequences of arrows tells us how to achieve a net action. However, each Cayley diagram very much depends on the set of generators that are chosen to generate the group. If we change the generating set, we may end up with a very different looking Cayley diagram. For example, compare the Cayley diagram for D_3 that you constructed in Problem 1.109(d) with the one in Example 1.108. Similarly, compare the Cayley diagrams for D_4 that you constructed in Parts (f) and (g) of Problem 1.109.

Problem 1.113. Consider the group $(\mathbb{Z}, +)$.

(a) Construct a portion of the Cayley diagram for $(\mathbb{Z}, +)$ with generating set $\{1\}$.

(b) Construct a portion of the Cayley diagram for $(\mathbb{Z}, +)$ with generating set $\{-1\}$. How does this diagram compare to the one in part (a)?

(c) It turns out that $\mathbb{Z} = \langle 2, 3 \rangle$. Construct a portion of the Cayley diagram for $(\mathbb{Z}, +)$ with generating set $\{2, 3\}$.

Problem 1.114. Assume G is a group. Suppose that S and S' are two different sets that generate G. If you draw the Cayley diagram for G using S and then draw the Cayley diagram for G using S', what features of the two graphs are the same and which are potentially different?

Problem 1.115. Consider the diagram given in Figures 6. Explain why this diagram is not the Cayley diagram for a group.

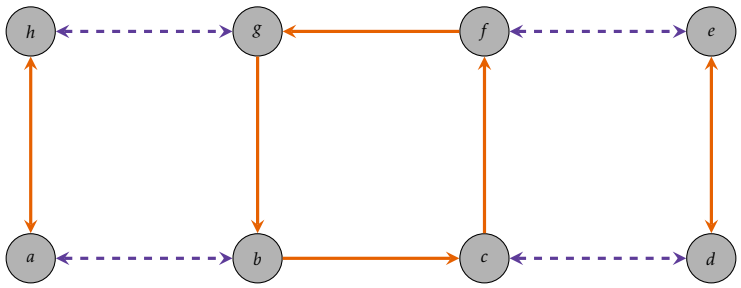

Figure 6: Example of a diagram that is not a Cayley diagram for a group.

Problem 1.116. Suppose G is a group with generating set S and consider the corresponding Cayley diagram. Prove each of the following.

(a) For every $g \in G$ and $s \in S$, there is exactly one arrow with color c_s pointing from $s^{-1}g$ to g and exactly one arrow with color c_s pointing from g to sg.

(b) The Cayley diagram for G with generating set S is connected. That is, for every pair of vertices g and h, there is a path of forward or backward arrows connecting g and h.

(c) Suppose G is a *finite* group and $s \in S$. If we follow a sequence of (forward) arrows of color c_s out of e, we eventually end up back at e after a finite number of steps.

(d) Every relation involving the generators is realized as a sequence of arrows from a vertex in the Cayley diagram back to the same vertex. Moreover, the sequence is independent of starting vertex. Loosely speaking, this says that every local pattern holds globally in the Cayley diagram.

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Problem 1.117. Suppose S is a generating set for a group G. Can you determine whether S is a minimal generating set using the Cayley diagram for G with generating set S?

Problem 1.118. Suppose $\{g_1,\ldots,g_n\}$ is a generating set for a group G.

(a) Explain why $\{g_1^{-1},\ldots,g_n^{-1}\}$ is also a generating set for G.

(b) How does the Cayley diagram for G with generating set $\{g_1,\ldots,g_n\}$ compare to the Cayley diagram with generating set $\{g_1^{-1},\ldots,g_n^{-1}\}$?

Problem 1.119. Suppose G is an abelian group with generating set S and consider the corresponding Cayley diagram.

(a) If $s,t \in S$, then what relationship must be true about the corresponding arrows?

(b) Is the converse of your claim in part (a) true? That is, if every pair of arrows in the Cayley diagram for G has the property you stated above, will the group be abelian?

Let's introduce a group we haven't seen yet. Define the **quaternion group** to be the group $Q_8 = \{1,-1,i,-i,j,-j,k,-k\}$ having the Cayley diagram with generating set $\{i,j,-1\}$ given in Figure 7. In this case, 1 is the identity of the group.

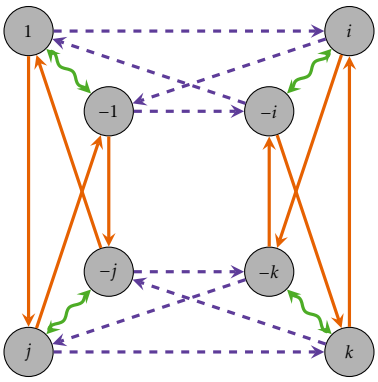

Figure 7: Cayley diagram for Q_8 with generating set $\{-1,i,j\}$.

Notice that I didn't mention what the actions actually do. For now, let's not worry about that. The relationship between the arrows and vertices tells us everything we need to know. Also, let's take it for granted that Q_8 actually is a group.

Problem 1.120. Consider the Cayley diagram for Q_8 given in Figure 7.

(a) Which arrows correspond to which generators in our Cayley diagram for Q_8?

(b) What is i^2 equal to? That is, what element of $\{1,-1,i,-i,j,-j,k,-k\}$ is i^2 equal to? How about i^3, i^4, and i^5?

(c) What are j^2, j^3, j^4, and j^5 equal to?

(d) What is $(-1)^2$ equal to?

(e) What is ij equal to? How about ji?

(f) Can you determine what k^2 and ik are equal to?

(g) Can you identify a generating set consisting of only two elements? Can you find more than one?

(h) What subgroups of Q_8 can you visually witness in the Cayley diagram in Figure 7?

(i) Find a subgroup of Q_8 that is not easily seen in the Cayley diagram in Figure 7.

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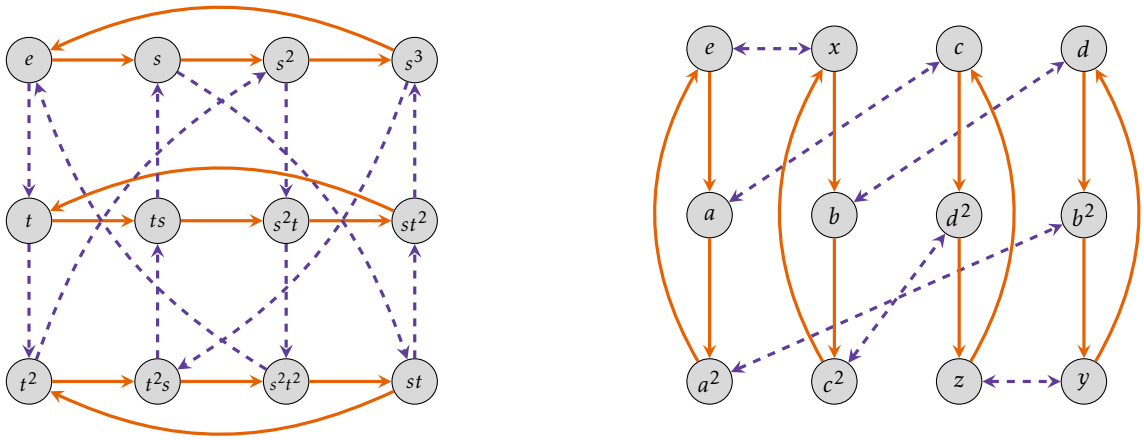
Problem 1.121. The Cayley diagrams of two groups of order 12 are shown below.



(a) Create a group table for each group. For consistency, please order the elements in the first group by $e, t, t^2, s, ts, t^2s, s^2, s^2t, s^2t^2, s^3, st^2, st$ and those in second by $e, x, y, z, a, b, c, d, a^2, b^2, c^2, d^2$.)

(b) Find the inverse of each element.

(c) Find the order of each element in each group.

(d) Find a presentation for each group.

(e) Squint your eyes. Do you see any patterns in these tables?

Problem 1.122. Consider the groups given by the following presentations:

(a) $G_1 = \langle g, h \mid h^2 = e, gh = hg \rangle$

(b) $G_2 = \langle g, h \mid h^2 = e, gh = hg^{-1} \rangle$

(c) $G_3 = \langle g, h \mid gh = hg \rangle$

(d) $G_4 = \langle g, h \mid g^2 = e, h^2 = e, gh = hg \rangle$

(e) $G_5 = \langle g, h \mid g^2 = e, h^2 = e \rangle$

(f) $G_6 = \langle g, h \mid \rangle$ (no relations!)

Draw the Cayley diagram for each group with the specified generating set. If the group is finite, draw the whole diagram. If the group is infinite, draw a portion of the diagram. At least one of these groups provides a presentation for the **infinite dihedral group**, which we have not officially defined. Can you take a guess which one(s)?

1.9 Lagrange's Theorem

The goal of this section is to prove Lagrange's Theorem, which states that for finite groups the order of an element divides the order of the group. We begin with defining a relation on a group that depends on a subgoup.

Let G be a group and let $H \leq G$. Define $\sim$ on G via

$$a \sim b \text{ if and only if } b = ah \text{ for some } h \in H.$$

Problem 1.123. Prove that $\sim$ is an equivalence relation on G.

Since $\sim$ is an equivalence relation, the corresponding equivalence classes form a partition of G. If $a \in G$, then the equivalence class containing a is given by

$$[a] = \{b \in G \mid a \sim b\}.$$

The next problem provides a nice description for $[a]$.

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Problem 1.124. Let G be a group and let $H \leq G$. Prove that $[a] = aH := \{ah \mid h \in H\}$.

In the previous problem, the set aH is called a **left coset**. Note that if $a \notin H$, then $aH \neq H$. In particular, if $a \notin H$, then aH is not a subgroup of G. We will explore cosets in more detail in a later section.

Problem 1.125. Let G be a group, $H \leq G$, and $a \in G$. Define $\phi : H \to aH$ via $\phi(h) = ah$. Prove that ϕ is a bijection.

It follows immediately from the previous problem that all of the left cosets of H are the same cardinality as H. In other words $\#(aH) = |H|$ for all $a \in G$, where $\#(aH)$ denotes the cardinality of aH. Note that everything works out just fine even if H has infinite order.

We're finally ready to state Lagrange's Theorem, which is named after the Italian born mathematician Joseph Louis Lagrange. It turns out that Lagrange did not actually prove the theorem that is named after him. The theorem was actually proved by Carl Friedrich Gauss in 1801.

Problem 1.126 (Lagrange's Theorem)**.** Prove that if G is a finite group and $H \leq G$, then $|H|$ divides $|G|$.

This simple sounding theorem is extremely powerful. One consequence is that groups and subgroups have a fairly rigid structure. Suppose G is a finite group and let $H \leq G$. Since G is finite, there must be a finite number of distinct left cosets, say $H, a_2H, \ldots, a_nH$. By earlier discussion, we know that each of these cosets has the same cardinality. In particular, Lagrange's Theorem implies that for each $i \in \{1, \ldots, n\}$, $|a_iH| = |G|/n$, or equivalently $n = |G|/|a_iH|$. This is depicted in Figure 8, where each rectangle represents a coset and we've labeled a single coset representative in each case.

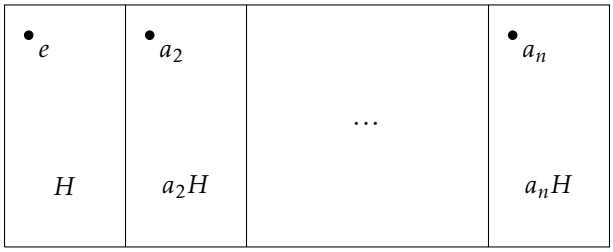

Figure 8: Left cosets as a partition of G into equal size blocks.

One important consequence of Lagrange's Theorem is that it narrows down the possible sizes for subgroups.

Problem 1.127. Suppose G is a group of order 48. What are the possible orders for subgroups of G?

Lagrange's Theorem tells us what the possible orders of a subgroup are, but if $k \in \mathbb{N}$ is a divisor of the order of a group, it does not guarantee that there is a subgroup of order k. It's not too hard to show that the converse of Lagrange's Theorem is true for cyclic groups. However, it's not true, in general. We'll discover a counterexample in a later section.

The next problem is a corollary of Lagrange's Theorem.

Problem 1.128. Prove that if G is a finite group and $a \in G$, then $|a|$ divides $|G|$.

The converse to the previous problem is not true either.

Problem 1.129. Provide an example of a finite group G and a divisor $k \in \mathbb{N}$ of $|G|$ such that G does not have an element of order k.

Problem 1.130. Suppose G is a group such that $|G| = p$, where p is prime. Prove that G is cyclic.

1.10 More on Cyclic Groups

Recall that if G is a group and $g \in G$, then the **cyclic subgroup generated by** g is given by

$$\langle g \rangle = \{g^k \mid k \in \mathbb{Z}\}.$$

It is important to point out that $\langle g \rangle$ may be finite or infinite. In the finite case, the Cayley diagram with generator g gives us a good indication of where the word "cyclic" comes from. If there exists $g \in G$ such that $G = \langle g \rangle$, then we say that G is a **cyclic group**.

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Problem 1.131. Prove that if G is a group such that G has no proper nontrivial subgroups, then G is cyclic.

The next result should look familiar and will come in handy a few times in this section. We'll take the result for granted and not worry about proving it.

Theorem 1.132 (Division Algorithm)**.** If n is a positive integer and m is any integer, then there exist unique integers q (called the **quotient**) and r (called the **remainder**) such that $m = nq+r$, where $0 \leq r < n$.

Problem 1.133. Suppose G is a group and let $g \in G$. Prove that the subgroup $\langle g \rangle$ is finite if and only if there exists $n \in \mathbb{N}$ such that $g^n = e$. It immediately follows that if G is a finite group, then for all $g \in G$, there exists $n \in \mathbb{N}$ such that $g^n = e$.

Problem 1.134. Suppose G is a group and let $g \in G$ such that $\langle g \rangle$ is a finite group. Prove that if n is the smallest positive integer such that $g^n = e$, then $\langle g \rangle = \{e, g, g^2, \ldots, g^{n-1}\}$ and this set contains n distinct elements. Note that the previous problem together with the Well-Ordering Principle guarantees the existence of a smallest positive integer n such that $g^n = e$.

The next result provides an extremely useful interpretation of the order of an element.

Problem 1.135. Prove that if G is a group and $g \in G$ such that $\langle g \rangle$ is a finite subgroup, then the order of g is the smallest positive integer n such that $g^n = e$.

Problem 1.136. Suppose G is a group and $x, y \in G$ such that $|x| = m$ and $|y| = n$. Is it true that $|xy| = mn$? If this is true, provide a proof. If this is not true, then provide a counterexample.

Problem 1.137. Suppose G is a group and let $g \in G$. Prove that the subgroup $\langle g \rangle$ is infinite if and only if each g^k is distinct for all $k \in \mathbb{Z}$.

Problem 1.138. Suppose G is a cyclic group with generator $g \in G$.

(a) If G is finite, what conclusions can you make about Cayley diagram for G with generating set $\{g\}$?

(b) If G is infinite, what conclusions can you make about Cayley diagram for G with generating set $\{g\}$?

The Division Algorithm should come in handy when proving the next theorem.

Problem 1.139. Suppose G is a group and let $g \in G$ such that $|g| = n$. Prove that $g^i = g^j$ if and only if n divides $i - j$. It immediately follows that $g^i = g^j$ if and only if $i \equiv j \pmod{n}$.

Problem 1.140. Suppose G is a group and let $g \in G$ such that $|g| = n$. Prove that if $g^k = e$, then n divides k.

Problem 1.141. Suppose G is a cyclic group. Prove that if $H \leq G$, then H is also cyclic.

It turns out that for proper subgroups, the converse of Problem 1.141 is not true.

Problem 1.142. Provide an example of a group G such that G is not cyclic, but all proper subgroups of G are cyclic.

One implication of the previous problem is that the subgroups of $\mathbb{Z}$ are precisely the groups $n\mathbb{Z} = \langle n \rangle$ for $n \in \mathbb{Z}$.

Problem 1.143. Prove that if G is a finite cyclic group with generator g such that $|G| = n$, then for all $m \in \mathbb{Z}$, $|g^m| = \dfrac{n}{\gcd(n, m)}$.

Problem 1.144. Prove that if G is a finite cyclic group with generator g such that $|G| = n$, then $\langle g^m \rangle = \langle g^k \rangle$ if and only if $\gcd(m, n) = \gcd(k, n)$. Use Problem 1.143 for the forward implication. For the reverse implication, first prove that for all $m \in \mathbb{Z}$, $\langle g^m \rangle = \langle g^{\gcd(m,n)} \rangle$ by proving two set containments. To show $\langle g^m \rangle \subseteq \langle g^{\gcd(m,n)} \rangle$, use the fact that there exists an integer q such that $m = q \cdot \gcd(m, n)$. For the reverse containment, you may freely use a fact known as Bezout's Lemma, which states that $\gcd(m, n) = nx + my$ for some integers x and y.

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Suppose G is a finite cyclic group with generator g such that $|G| = n$. It follows from Problem 1.144 that $\langle g \rangle = \langle g^k \rangle$ if and only if n and k are relatively prime. That is, g^k generates G if and only if n and k are relatively prime.

Problem 1.145. Suppose G is a cyclic group of order 12 with generator g.

(a) Find the orders of each of the following elements: g^2, g^7, g^8.

(b) Which elements of G individually generate G?

Problem 1.146. Consider $\mathbb{Z}/18\mathbb{Z}$.

(a) Find all of the elements of $\mathbb{Z}/18\mathbb{Z}$ that individually generate all of $\mathbb{Z}_{18}$.

(b) Draw the subgroup lattice for $\mathbb{Z}/18\mathbb{Z}$. For each subgroup, list the elements of the corresponding set. Moreover, circle the elements in each subgroup that individually generate that subgroup.

Problem 1.147. Suppose G is a finite cyclic group of order n and let k be a positive divisor of n. Prove that there is exactly one subgroup of order k for every positive divisor k of n.

Problem 1.148. Let G be a cyclic group of order n and let k be an integer relatively prime to n. Prove that the map $x \mapsto x^k$ is surjective. What if G is an abelian group, not necessarily cyclic, of order n?

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2 Group Homorphisms

2.1 Isomorphisms

As we have been exploring various groups, I'm sure you've noticed that some groups seem to look and behave the same.

Suppose G is a finite group and consider the group table for G. A **coloring** for the group table is an assignment of a unique color to each element of the group. For example, Figure 9 depicts a coloring for the group table of V_4.

$\circ$	e	v	h	vh
e	e	v	h	vh
v	v	e	vh	h
h	h	vh	e	v
vh	vh	h	v	e

Figure 9: A coloring for the group table of V_4.

We say that two finite groups have an **identical table coloring**, if we can arrange the rows and columns of each table and choose colorings for each table so that the pattern of colors is the same for both tables. Clearly, this is only possible if the two groups have the same order.

Problem 2.1. Determine whether V_4 and Light_2 (see Problem 1.110) have an identical table coloring.

Since a group table encodes all of the information about a group, if two groups have an identical table coloring, then the two groups have the same exact structure while the elements may have different names. In particular, if two finite groups G_1 and G_2 have an identical table coloring, then

the product of corresponding elements yields the corresponding result.

This is the essence of what it means for two groups to have the same structure.

Let's try to make this a little more precise. Suppose $(G_1,*)$ and $(G_2,\odot)$ are two finite groups that have an identical table coloring and let $x_1, y_1 \in G_1$. Then these two elements have corresponding elements in the group table for G_2, say x_2 and y_2, respectively. In other words, x_1 and x_2 have the same color while y_1 and y_2 have the same color. Since G_1 is closed under its binary operation $*$, there exists $z_1 \in G_1$ such that $z_1 = x_1 * y_1$. But then there must exist a $z_2 \in G_2$ such that z_2 has the same color as z_1. What must be true of $x_2 \odot y_2$? Since the two tables exhibit the same color pattern, it must be the case that $z_2 = x_2 \odot y_2$. This is what it means for the product of corresponding elements to yield the corresponding result. Figure 10 illustrates this phenomenon for group tables.

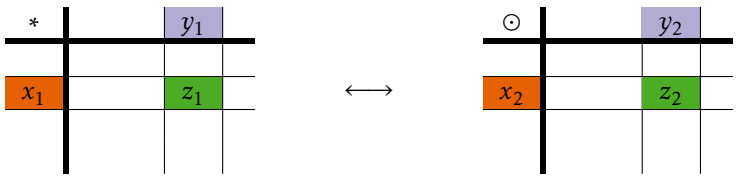

Figure 10:

We can describe the identical table matching between G_1 and G_2 using a function. Let $\phi : G_1 \to G_2$ be the one-to-one and onto function that maps elements of G_1 to their corresponding elements in G_2. Then $\phi(x_1) = x_2$, $\phi(y_1) = y_2$, and $\phi(z_1) = z_2$. Since $z_2 = x_2 \odot y_2$, we obtain

$$\phi(x_1 * y_1) = \phi(z_1) = z_2 = x_2 \odot y_2 = \phi(x_1) \odot \phi(y_1).$$

In summary, it must be the case that

$$\phi(x_1 * y_1) = \phi(x_1) \odot \phi(y_1).$$

We are now prepared to state a formal definition of what it means for two groups to be isomorphic.

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Definition 2.2. Let $(G_1, *)$ and $(G_2, \odot)$ be two groups. Then G_1 is **isomorphic** to G_2, written $G_1 \cong G_2$, if and only if there exists a one-to-one and onto function $\phi : G_1 \to G_2$ such that

$$\phi(x * y) = \phi(x) \odot \phi(y). \tag{1}$$

The function ϕ is referred to as an **isomorphism**. Equation 1 is often referred to as the **homomorphic property**.

Problem 2.3. Consider the groups $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$, where $\mathbb{R}^+$ is the set of positive real numbers. It turns out that these two groups are isomorphic, but this would be difficult to discover using our previous techniques because the groups are infinite. Define $\phi : \mathbb{R} \to \mathbb{R}^+$ via $\phi(r) = e^r$ (where e is the natural base, not the identity). Prove that ϕ is an isomorphism.

Problem 2.4. Show that the groups $(\mathbb{Z}, +)$ and $(2\mathbb{Z}, +)$ are isomorphic.

Perhaps one surprising consequence of the previous problem is that when dealing with infinite groups, a group may have a proper subgroup that it is isomorphic to. Of course, this never happens with finite groups.

Problem 2.5. Suppose $\phi : G_1 \to G_2$ is an isomorphism from the group $(G_1, *)$ to the group $(G_2, \odot)$. Prove each of the following:

(a) $|g| = |\phi(g)|$ for all $g \in G$.

(b) G_1 and G_2 have the same number of elements of order n for each $n \in \mathbb{N}$.

(c) G_1 is abelian if and only if G_2 is abelian.

Problem 2.6. For each pair of groups given below, explain why the groups are not isomorphic to each other.

(a) D_4 and Q_8

(b) D_4 and Light_3

(c) $\mathbb{R}$ and $\mathbb{Q}$ (both additive groups)

(d) $\mathbb{Z}$ and $\mathbb{Q}$ (both additive groups)

(e) $\mathbb{R} \setminus \{0\}$ and $\mathbb{C} \setminus \{0\}$ (both multiplicative groups)

Problem 2.7. Prove that any two cyclic groups of the same order are isomorphic. Be sure to handle both the finite and infinite cases. In the finite case, do you need to worry about your proposed function being well defined?

It turns out that "isomorphic" ($\cong$) determines an equivalence relation on the class of all possible groups. The next three problems justify that $\cong$ is reflexive, symmetric, and transitive.

Problem 2.8. Prove that if G is a group, then the identity map from G to G is an isomorphism.

Problem 2.9. Prove that if $\phi : G_1 \to G_2$ is an isomorphism from the group $(G_1, *)$ to the group $(G_2, \odot)$, then the function $\phi^{-1} : G_2 \to G_1$ is an isomorphism.

Problem 2.10. Prove that if $\phi : G_1 \to G_2$ and $\psi : G_2 \to G_3$ are isomorphisms from the groups $(G_1, *)$ to $(G_2, \odot)$ and $(G_2, \odot)$ to $(G_3, \star)$, respectively, then the composite function $\psi \circ \phi$ is an isomorphism of G_1 and G_3.

In light of the previous problem, we now know that if $\mathcal{G}$ is any nonempty collection of groups, then the relation $\cong$ is an equivalence relation on $\mathcal{G}$.

Mathematicians love to classify things. In particular, mathematicians want to classify groups. One can think of this pursuit as a taxonomy of groups. In order to simplify the task, one can classify isomorphism classes (i.e., the equivalence classes determined by $\cong$) instead of classifying groups individually. If two groups are isomorphic, then we say that the groups are **the same up to isomorphism**. If there are k isomorphism classes of order n, then we say that there are k **groups of order** n **up to isomorphism**. A natural question to ask is: how many groups are there of order n?

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Certainly, there is only one group of order 1 up to isomorphism. It follows from Problems 1.130 and 2.7 that up to isomorphism, there is only one group of order p if p is prime.

It turns out that up to isomorphism, there are two groups of order 4. In particular, every group of order 4 is isomorphic to R_4 if cyclic or isomorphic to V_4 if not cyclic. This isn't too hard to prove by examining all the possible ways one could fill out a group table for a group of order 4.

How about order 6? We've seen four groups of order 6, namely $\mathbb{Z}/6\mathbb{Z}$, R_6, D_3, and S_3. Since $\mathbb{Z}/6\mathbb{Z}$ and R_6 are both cyclic groups of order 6, these two groups are isomorphic. We know that neither D_3 nor S_3 are cyclic, so neither of these groups below to the same isomorphism class as R_6. Below, we will justify that $D_3 \cong S_3$. This implies that there are at least two groups up to isomorphism of order 6. But are there others? It turns out that the answer is no, but why?

We've encountered several groups of order 8, namely D_4, Coin_2, Q_8, R_8, and Light_3. Of these, only D_4 and Coin_2 are isomorphic (see Problem 2.11). Thus, there are at least four groups up to isomorphism of order 8. Are these the only isomorphism types? It turns out that there are five groups of order 8 up to isomorphism. We will encounter the fifth isomorphism type later.

You can also utilize Cayley diagrams to determine whether two finite groups are isomorphic. If two groups G_1 and G_2 have generating sets T_1 and T_2 such that we can color the edges of the corresponding Cayley diagrams so that the diagrams are identical up to relabeling of the vertices, then we say that there is a **matching** between G_1 and G_2 (or more formally, the Cayley diagrams are isomorphic as directed and colored graphs).

For example, Figure 11 makes it easy to describe a matching between D_3 and S_3. It's important to emphasize that the existence of a matching between two groups depends on our choice of generating set! If two Cayley diagrams do not look alike, it does not immediately imply that there is not a matching between the groups since it might be the case that choosing different generating sets for the two groups leads to a matching. It turns out that two groups G_1 and G_2 are isomorphic if and only if there exists choices of generating sets T_1 and T_2 for G_1 and G_2, respectively, such that there is a matching between the corresponding Cayley diagrams. Do you see why this is true?

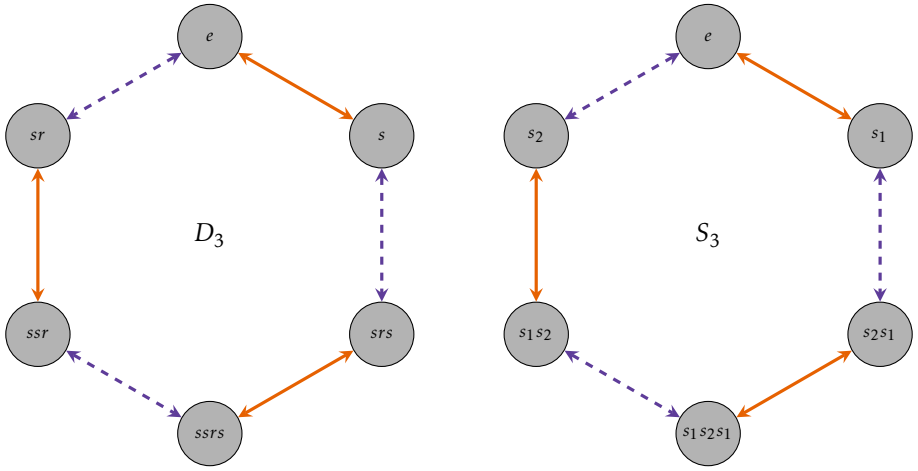

Figure 11: Cayley diagrams for D_3 and S_3 with generating sets $\{s, sr\}$ and $\{s_1, s_2\}$, respectively.

Problem 2.11. Explain why D_4 and Coin_2 are isomorphic.

Problem 2.12. Determine which of the following groups are isomorphic to each other: S_2, $\langle -1 \rangle$ in Q_8, R_3, R_4, V_4, Light_2, $\langle i \rangle$ in Q_8, $\langle sr, sr^3 \rangle$ in D_4, R_5, R_6, D_3, S_3, R_7, R_8, D_4, Coin_2, Q_8, Light_3.

Problem 2.13. What claims can be made about the subgroup lattices of two groups that are isomorphic? What claims can be made about the subgroup lattices of two groups that are not isomorphic? What claims can be made about two groups if their subgroup lattices look nothing alike?

2.2 Homomorphisms

Recall that a group isomorphism is a bijection between two groups that satisfies the homomorphic property. What if we drop the one-to-one and onto requirements?

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Definition 2.14. Let $(G_1, *)$ and $(G_2, \odot)$ be groups. A function $\phi : G_1 \to G_2$ is a **homomorphism** if and only if ϕ satisfies the homomorphic property:

$$\phi(x * y) = \phi(x) \odot \phi(y)$$

for all $x, y \in G_1$. At the risk of introducing ambiguity, we will usually omit making explicit reference to the binary operations and write the homomorphic property as

$$\phi(xy) = \phi(x)\phi(y).$$

Group homomorphisms are analogous to linear transformations on vector spaces that one encounters in linear algebra.

Problem 2.15. Define $\phi : \mathbb{Z}/3\mathbb{Z} \to D_3$ via $\phi(\overline{k}) = r^k$. Prove that ϕ is a well-defined homomorphism and then determine whether ϕ is one-to-one or onto. Also, try to draw a picture of the homomorphism in terms of Cayley diagrams.

There is always at least one homomorphism between two groups.

Problem 2.16. Let G_1 and G_2 be groups. Define $\phi : G_1 \to G_2$ via $\phi(g) = e_2$ (where e_2 is the identity of G_2). This function is often referred to as the **trivial homomorphism** or the 0-**map**. Prove that ϕ is a homomorphism.

Problem 2.17. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Prove each of the following.

(a) If e_1 and e_2 are the identity elements of G_1 and G_2, respectively, then $\phi(e_1) = e_2$.

(b) For all $g \in G_1$, we have $\phi(g^{-1}) = [\phi(g)]^{-1}$.

(c) For all $g \in G_1$ and $n \in \mathbb{Z}$, we have $\phi(g^n) = \phi(g)^n$.

(d) If $H \leq G_1$, then $\phi(H) \leq G_2$, where

$$\phi(H) := \{y \in G_2 \mid \text{there exists } h \in H \text{ such that } \phi(h) = y\}.$$

Note that $\phi(H)$ is the **image** of H. A special case is when $H = G_1$. Notice that ϕ is onto exactly when $\phi(G_1) = G_2$.

(e) If ϕ is injective, then $G_1 \cong \phi(G_1)$.

The following theorem is a consequence of Lagrange's Theorem.

Problem 2.18. Let G_1 and G_2 be groups such that G_2 is finite and let $H \leq G_1$. Prove that if $\phi : G_1 \to G_2$ is a homomorphism, then $|\phi(H)|$ divides $|G_2|$.

The next theorem tells us that under a homomorphism, the order of the image of an element must divide the order of the pre-image of that element.

Problem 2.19. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Prove that if $g \in G_1$ such that $|g|$ is finite, then $|\phi(g)|$ divides $|g|$.

Every homomorphism has an important subset of the domain associated with it.

Definition 2.20. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. The **kernel** of ϕ is defined via

$$\ker(\phi) := \{g \in G_1 \mid \phi(g) = e_2\} = \phi^{-1}(\{e_2\}).$$

The kernel of a homomorphism is analogous to the null space of a linear transformation of vector spaces.

Problem 2.21. Identify the kernel and image for the homomorphism given in Problem 2.15.

Problem 2.22. What is the kernel of a trivial homomorphism.

Problem 2.23. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Prove each of the following.

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(a) $\ker(\phi) \leq G_1$.

(b) $N_{G_1}(\ker(\phi)) = G_1$.

Part (b) of the previous problem says that the kernel of every homomorphism is fixed setwise under the action of conjugation by every element of G. The next theorem tells us that two elements in the domain of a group homomorphism map to the same element in the codomain if and only if they are in the same left (or right) coset of the kernel.

Problem 2.24. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Prove that $\phi(a) = \phi(b)$ if and only if $a \in b\ker(\phi)$.

One consequence of Problem 2.24 is that if the kernel of a homomorphism has order k, then the homomorphism is k-to-1 (since left cosets of $\ker(\phi)$ always have the same cardinality; see Problem 1.125). That is, every element in the range has exactly k elements from the domain that map to it. In particular, each of these collections of k elements corresponds to a left coset of the kernel.

Problem 2.25. Suppose $\phi : \mathbb{Z}/20\mathbb{Z} \to \mathbb{Z}/20\mathbb{Z}$ is a group homomorphism such that $\ker(\phi) = \{\overline{0}, \overline{5}, \overline{10}, \overline{15}\}$. If $\phi(\overline{13}) = \overline{8}$, determine all elements that ϕ maps to $\overline{8}$.

The next result is a special case of Problem 2.24.

Problem 2.26. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Prove that ϕ is one-to-one if and only if $\ker(\phi) = \{e_1\}$, where e_1 is the identity in G_1.

Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Given a generating set for G_1, the homomorphism ϕ is uniquely determined by its action on the generating set for G_1. In particular, if you have a word for a group element written in terms of the generators, just apply the homomorphic property to the word to find the image of the corresponding group element. However, it is important to point out that you can't just map the generators about willy nilly and expect to get a homomorphism.

Problem 2.27. Let $\phi : Q_8 \to V_4$ be the map satisfying $\phi(i) = h$ and $\phi(j) = v$. It turns out that ϕ is a group homomorphism.

(a) Find $\phi(1)$, $\phi(-1)$, $\phi(k)$, $\phi(-i)$, $\phi(-j)$, and $\phi(-k)$.

(b) Find $\ker(\phi)$.

(c) Draw a picture of this homomorphism in terms of Cayley diagrams?

(d) There is a natural bijection from the collection of left cosets of $\ker(\phi)$ to the image of ϕ. Describe this map.

The next four problems will be a good test of our current understanding.

Problem 2.28. Let G be a group. Prove that the map $\phi : G \to G$ defined via $\phi(g) = g^2$ is a group homomorphism if and only if G is abelian.

Problem 2.29. Find a non-trivial homomorphism from $\mathbb{Z}/10\mathbb{Z}$ to $\mathbb{Z}/6\mathbb{Z}$.

Problem 2.30. Find all non-trivial homomorphisms from $\mathbb{Z}/3\mathbb{Z}$ to $\mathbb{Z}/6\mathbb{Z}$.

Problem 2.31. Prove that the only homomorphism from D_3 to $\mathbb{Z}/3\mathbb{Z}$ is the trivial homomorphism.

3 Automorphisms

Recall that two groups G_1 and G_2 are isomorphic if there exists a bijection $\phi : G_1 \to G_2$ that satisfies the homomorphic property. In Problem 2.8, we learned that a group G is isomorphic to itself by using the identity map. However, it is possible that there are additional isomorphisms from a group to itself.

Definition 3.1. Let G be a group. If $\phi : G \to G$ is an isomorphism, then we say that ϕ is an **automorphism** of G. The collection of all automorphisms of G is denoted by $\mathrm{Aut}(G)$.

By Problem 2.8, we know that $\mathrm{Aut}(G)$ is always nonempty.

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Problem 3.2. Prove that for each $k \in \mathbb{Q}$, the map $\phi_k : \mathbb{Q} \to \mathbb{Q}$ defined via $\phi_k(q) = kq$ is an automorphism of $\mathbb{Q}$.

Problem 3.3. Let G be an abelian group and let $k \in \mathbb{Z}$.

(a) Prove the map $\phi_k : G \to G$ defined via $\phi_k(g) = g^k$ is a homomorphism.

(b) Prove that if $k = -1$, the homomorphism ϕ_k is actually an automorphism of G.

(c) Are there other situations under which ϕ_k will be an automorphism?

Problem 3.4. Let G be a group. Prove that $\mathrm{Aut}(G)$ is a group under function composition. We call $\mathrm{Aut}(G)$ the **automorphism group** of G.

Suppose G is a group and let $g \in G$. Define $\phi_g : G \to G$ via $\phi_g(x) = gxg^{-1}$. The map ϕ_g is called **conjugation** by g.

Problem 3.5. Prove that if G is a group and $g \in G$, then conjugation by g is an automorphism of G.

Problem 3.6. Prove that if H is any subgroup of G and $g \in G$, then $H \cong gHg^{-1}$.

Problem 3.5 tells us that conjugation by any element in G is always an element of $\mathrm{Aut}(G)$. Let $\mathrm{Inn}(G)$ denote the subset of $\mathrm{Aut}(G)$ consisting of all automorphisms of G that are equal to conjugation by g for some $g \in G$. Each conjugation is sometimes called an **inner automorphism** of G, which is where the notation $\mathrm{Inn}(G)$ comes from.

Problem 3.7. Prove that $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$.

Problem 3.8. Explain why G is abelian if and only if $\mathrm{Inn}(G)$ is trivial.

Problem 3.9. Prove one of the following.

(a) $\mathrm{Inn}(Q_8) \cong V_4$

(b) $\mathrm{Inn}(D_4) \cong V_4$

Problem 3.10. Let G be a finite group σ such that $\sigma(g) = g$ if and only if $g = e$ (i.e., the identity is the only element fixed by σ). Prove that if σ^2 is the identity map from G to G, then G must be abelian. *Hint:* Show that every element in G is of the form $x^{-1}\sigma(x)$ and apply σ.

In preparation for the next problem, define $U_n := \{\overline{k} \in \mathbb{Z}/n\mathbb{Z} \mid \gcd(n,k) = 1\}$. The set is also sometimes denoted by $(\mathbb{Z}/n\mathbb{Z})^\times$. For example, $U_{12} = \{\overline{1}, \overline{5}, \overline{7}, \overline{11}\}$. It turns out that the set U_n is an abelian group under multiplication mod n. We will take this fact for granted. Notice that the order of U_n is the number of elements from $\{1, 2 \ldots, n-1\}$ that are relatively prime to n. That is, $|U_n|$ the value of Euler's ϕ-function at n.

Problem 3.11. Let G be a cyclic group of order n. For each $k \in \mathbb{Z}$, define $\sigma_k : G \to G$ via $\sigma_k(x) = x^k$ for all $x \in G$.

(a) Prove that σ_k is an automorphism of G if and only if n and k are relatively prime.

(b) Prove that $\sigma_k = \sigma_m$ if and only if $k \equiv m \pmod{n}$.

(c) Prove that every automorphism of G is equal to σ_k for some $k \in \mathbb{Z}$.

(d) Prove that $\sigma_k \circ \sigma_m = \sigma_{km}$.

(e) Deduce that $\mathrm{Aut}(G)$ is isomorphic to U_n.

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