Homework 8
Combinatorial Game Theory

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on Discord. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

You may freely use the following group theory fact:

- If G and H are finite groups, then $\Phi(G \times H) = \Phi(G) \times \Phi(H)$.

- If n has prime factorization $n = p_1^{n_1} \cdots p_k^{n_k}$, then the Frattini subgroup of $\mathbb{Z}_n$ is generated by $p_1 \cdots p_k$, and so it is isomorphic to the cyclic group of order $p_1^{n_1-1} \cdots p_k^{n_k-1}$.

- A subgroup of a finite abelian group is maximal if and only if it has prime index.

- The subgroups of the dihedral group $D_{2n} = \langle r, s \mid r^n = s^2 = e, rs = sr^{n-1} \rangle$ are either dihedral or cyclic. The maximal subgroups of D_{2n} are the cyclic group $\langle r \rangle$ and the dihedral groups of the form $\langle r^p, r^i s \rangle \cong D_{2(n/p)}$ with prime divisors p of n.

- If n has prime factorization $n = p_1^{n_1} \cdots p_k^{n_k}$, then the Frattini subgroup of D_{2n} is a cyclic group of order $p_1^{n_1-1} \cdots p_k^{n_k-1}$.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

1. Find the simplified structure diagram for **four** of the following (where at least one is chosen from (g), (h), or (i)) and identify the nim-value of the game. Justify your answer.

    (a) DNG($\mathbb{Z}_{4k}$) for $k \geq 1$.
    (b) DNG($\mathbb{Z}_{2k+1}$) for $k \geq 1$.
    (c) DNG($\mathbb{Z}_{4k+2}$) for $k \geq 1$.
    (d) DNG($D_{2(4k)}$) for $k \geq 1$.
    (e) DNG($D_{2(2k+1)}$) for $k \geq 1$.

    (f) $\mathsf{DNG}(D_{2(4k+2)})$ for $k \geq 1$.

    (g) $\mathsf{GEN}(D_{2(4k)})$ for $k \geq 1$.

    (h) $\mathsf{GEN}(D_{2(2k+1)})$ for $k \geq 1$.

    (i) $\mathsf{GEN}(D_{2(4k+2)})$ for $k \geq 1$.

2. What is the spectrum of nim-values for $\mathsf{DNG}(\mathbb{Z}_n)$? You might need to share information with each other depending on which structure diagrams you chose to tackle in the previous problem.

3. For $n \geq 2$, how are the structure diagrams for $\mathsf{DNG}(\mathbb{Z}_n)$ and $\mathsf{GEN}(\mathbb{Z}_n)$ related? Justify your answer.

4. For $n \geq 2$, prove that $\mathrm{nim}(\mathsf{GEN}(\mathbb{Z}_n)) = \mathrm{nim}(\mathsf{DNG}(\mathbb{Z}_n)) + 1$.

5. What is the spectrum of nim-values for $\mathsf{GEN}(\mathbb{Z}_n)$?

6. What is the spectrum of nim-values for $\mathsf{DNG}(D_{2n})$?

7. What is the spectrum of nim-values for $\mathsf{GEN}(D_{2n})$?