Homework 13

Discrete Mathematics

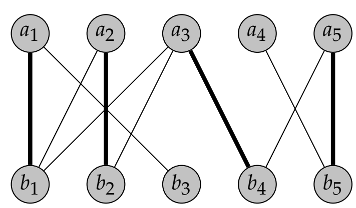
Please review the **Rules of the Game** from the syllabus.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers. In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. Suppose a shuffled deck of 52 regular playing cards are dealt into 13 piles of 4 cards each. **Explain** why it is possible to select one card from each pile to get one of each of the 13 card values Ace, 2, 3, ..., 10, Jack, Queen, and King (as a set, not necessarily in that order). *Hint:* We can model the situation with a bipartite graph G with 13 vertices in the set V_1, each representing one of the 13 card values, and 13 vertices in the set V_2, each representing one of the 13 piles. There is an edge between a vertex $v_1 \in V_1$ and a vertex $v_2 \in V_2$ if a card with value v_1 is in pile v_2. Use Hall's Marriage Theorem.

2. As we have seen, some bipartite graphs have a total matching and some do not. We say that a bipartite graph $G = (V, E)$ has a **partial matching** if there is subset $F \subseteq E$ of edges such that a vertex is the endpoint of at most one edge of F. Certainly, every total matching is a partial matching, but a partial matching may not be a total matching. Every bipartite graph with at least one edge has a partial matching, so we can look for the largest partial matching in a graph (i.e., a partial matching involving the largest number of edges).

   Bob claims that he has found the largest partial matching for the graph below (his matching is in bold). He explains that no other edge can be added, because all the edges not used in his partial matching are connected to matched vertices. Is he correct? **Explain**.



3. Consider the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ given by the following data.

   G_1: $V_1 = \{a, b, c, d, e, f, g\}$
   $E_1 = \{\{a,b\}, \{a,d\}, \{b,c\}, \{b,d\}, \{b,e\}, \{b,f\}, \{c,g\}, \{d,e\}, \{e,f\}, \{f,g\}\}$

   G_2: $V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$
   $E_2 = \{\{v_1,v_4\}, \{v_1,v_5\}, \{v_1,v_7\}, \{v_2,v_3\}, \{v_2,v_6\}, \{v_3,v_5\}, \{v_3,v_7\}, \{v_4,v_5\}, \{v_5,v_6\}, \{v_5,v_7\}\}$

   (a) Let $m : V_1 \to V_2$ be a function given by

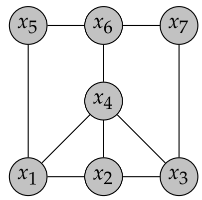
   $$m(a) = v_4, m(b) = v_5, m(c) = v_1, m(d) = v_6, m(e) = v_2, m(f) = v_3, m(g) = v_7.$$

   Is m an isomorphism between G_1 and G_2? **Briefly explain**.

(b) Using the ordering $v_1, v_2, v_3, v_4, v_5, v_6, v_7$, write down the adjacency matrix for G_2. **No justification required.**

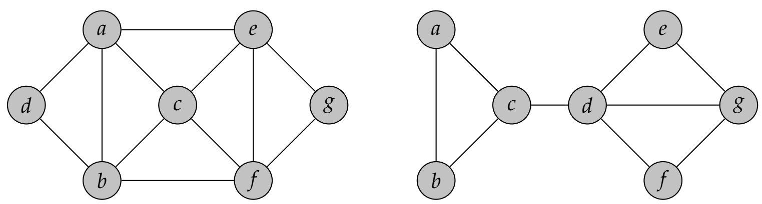
(c) Find an ordering on the vertices of V_1 and write down the corresponding adjacency matrix for G_1 that makes it clear that $G_1 \cong G_2$. **No justification required.**

(d) Is the graph G_3 given below isomorphic to G_1 and G_2? **Briefly explain**.



G_3

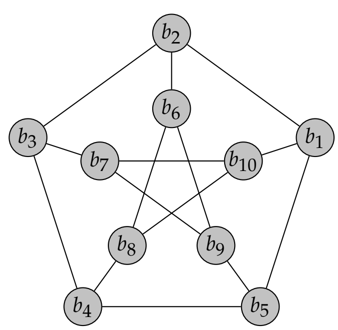
4. Determine whether each of the following graphs has an Euler circuit, an Euler trail that is not a circuit, or neither. If the graph has an Euler circuit or an Euler trail that is not a circuit, find one. Otherwise, briefly explain why neither exists.



G H

5. For which m and n does the graph $K_{m,n}$ contain an Euler circuit? An Euler trail that is not a circuit? **No justification required.**

6. Determine whether the following graph has a Hamilton path that is not a cycle, a Hamilton cycle, or neither. If the graph has Hamilton path that is not a cycle or a Hamilton cycle, find one. Otherwise, briefly explain why neither exists.



7. If possible, find an example of a graph that has a Hamilton cycle but not an Euler circuit. If no such example exists, briefly explain why.

8. If possible, find an example of a graph that has an Euler circuit but not a Hamilton cycle. If no such example exists, briefly explain why.