Chapter 6

Sequences and Recurrence Relations

In this chapter we will study sequences of numbers that are built recursively. Technically, a **sequence** (of real numbers) is a function a from $\mathbb{N}$ to $\mathbb{R}$. If $n \in \mathbb{N}$, it is common to write $a_n := a(n)$. We refer to a_n as the nth **term** of the sequence. We will abuse notation and associate a sequence with its list of outputs, namely:

$$\boxed{(a_n)_{n=1}^{\infty} := (a_1, a_2, \ldots)},$$

which we may abbreviate as $\boxed{(a_n)}$. Sometimes we may start our sequences at $n = 0$ as opposed to $n = 1$. That is, we may allow the domain of a sequence to be $\mathbb{N} \cup \{0\}$.

Example 6.1. Define $a : \mathbb{N} \to \mathbb{R}$ via $a_n = \frac{1}{2^n}$. Then we have

$$a = \left(\frac{1}{2}, \frac{1}{4}, \ldots\right) = \left(\frac{1}{2^n}\right)_{n=1}^{\infty}.$$

It is important to point out that not every sequence has a description in terms of an algebraic formula. For example, we could form a sequence out of the digits to the right of the decimal in the decimal expansion of π, namely the nth term of the sequence is the nth digit to the right of the decimal. But then there is no nice algebraic formula for describing the nth term of this sequence.

Loosely speaking, a sequence of numbers is defined recursively if the nth term of the sequence is defined in terms of "earlier" terms of the sequence. We have already encountered one famous example of a sequence being defined recursively, namely the Fibonacci sequence (f_n), which we defined by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. The equation $f_n = f_{n-1} + f_{n-2}$ is the **recurrence relation** while $f_1 = 1$ and $f_2 = 1$ are the **initial conditions**. It is important to emphasize that we cannot define the Fibonacci number using only the recurrence relation since otherwise, we would not be able to "get started" with the recurrence.

We have also encountered a few recurrence relations of a different flavor that arise out of two-dimensional arrays of numbers. For example:

(a) Number of k-permutations of $[n]$: For $1 \leq k \leq n$,

$$P(n, k) = P(n-1, k) + kP(n-1, k-1).$$

(b) Number of k-subsets of $[n]$: For $1 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

(c) Number of set partitions of $[n]$ with k blocks: For $1 \leq k \leq n$,

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}.$$

Notice that each of the descriptions above are not sufficient without also providing a way to "get started". For the two-dimensional case, the initial conditions are often called **boundary conditions**. For the rest of this chapter, we will focus on one-dimensional sequences.

Here is an important general principle.

Theorem 6.2. If two sequences satisfy the same recurrence relation and initial conditions, then the two sequences must be equal.

Problem 6.3. Recall that a **composition** of n with k parts is an ordered list of k positive integers whose sum is n, denoted $\alpha = (\alpha_1, \dots, \alpha_k)$. We say that α_i is the ith part.

(a) How many compositions of n have only odd parts?

(b) How many compositions of n have parts of size 1 and 2 only?

Problem 6.4. Prove that $f_{n+1} = \displaystyle\sum_{k \geq 0} \binom{n-k}{k}$ by utilizing one of the parts from Problem 6.3. What does this identity tell us about Pascal's Triangle?

Problem 6.5. For each of the following recursively defined sequences, generate the first few terms. If possible, find an explicit formula for the terms of the sequence.

(a) $a_1 = 2$, $a_n = a_{n-1} + 7$ for $n \geq 2$.

(b) $a_0 = 1$, $a_n = 2a_{n-1}$ for $n \geq 1$.

(c) $a_0 = 0$, $a_n = na_{n-1}$ for $n \geq 1$.

(d) $a_0 = 0$, $a_n = a_{n-1} + n$ for $n \geq 1$.

(e) $a_0 = 0$, $a_n a_{n-1} + \displaystyle\sum_{i=0}^{n} (i + n)$ for $n \geq 1$.

By **solving** a recurrence relation together with its initial conditions we mean finding an explicit expression for the general term a_n as a function of n. The explicit expression for a_n is called the **solution** of the recurrence relation. For example, each time we found an explicit formula for the nth term of a sequence in the previous problem, we were solving the recurrence relation and the corresponding expression we found is the solution.

Sometimes solving a recurrence relation is straightforward and other times it can be quite complicated, or maybe even unknown or impossible! Notice that we have not yet solved the recurrence relation together with initial conditions for the Fibonacci sequence.

We now turn our attention to two special classes of recurrence relations. An **arithmetic progression** is a recurrence relation in which the first term a_1 (or a_0) and a **common difference** d are given and the corresponding recurrence relation is

$$\boxed{a_n = a_{n-1} + d.}$$

A **geometric progression** is a recurrence relation in which the first term a_1 (or a_0) and **common ratio** r are given and the corresponding recurrence relation is

$$\boxed{a_n = r \cdot a_{n-1}.}$$

Problem 6.6. Find examples of each of the following in previous problems.

(a) An arithmetic progression.

(b) A geometric progression.

(c) Two different recursively defined sequences that are neither arithmetic nor geometric progressions.

Problem 6.7. Compute the first few terms of each of the following and conjecture a solution.

(a) Arithmetic progression with $a_1 = 3$ and $d = 2$.

(b) Geometric progression with $a_1 = 3$ and $r = 2$.

Problem 6.8. Conjecture a solution to an arithmetic progression with first term a_1 and common difference d. Can you prove that your conjecture is correct?

Problem 6.9. Conjecture a solution to a geometric progression with first term a_1 and common ratio r. Can you prove that your conjecture is correct?

More coming soon...