Chapter 4

Additional Counting Methods

The **Pigeonhole Principle** is a very natural property. Here it is. If a collection of at least $n+1$ objects is put into n boxes, then there is a box with at least two things in it. The Pigeonhole Principle has surprisingly deep applications. We will start with a few examples.

Example 4.1. Back in Problem 1.47, we implicitly used the Pigeonhole Principle when we argued that if $f : A \to B$ is a function for finite sets A and B, then

(a) If f is an injection, then $|A| \leq |B|$.

(b) If f is a surjection, then $|A| \geq |B|$.

Problem 4.2. A box has blue, green, yellow, red, orange, and white balls. How many must be drawn without looking to be sure of getting at least two of the same color?

Problem 4.3. Prove that if seven distinct numbers are selected from $\{1, 2, \ldots, 11\}$, then some two of these numbers sum to 12.

We would like to generalize the Pigeonhole Principle, but first we need a useful function. The **ceiling function** of a real number x, written $\boxed{\lceil x \rceil}$, is the smallest integer greater than or equal to x. That is, $\lceil x \rceil$ is an integer, $\lceil x \rceil \geq x$, and there is no other integer between $\lceil x \rceil$ and x. You can think of it as the "round-up to an integer" function.

Example 4.4. For example, $\lceil \pi \rceil = 4$, $\lceil -\pi \rceil = -3$, and $\lceil 7 \rceil = 7$.

We can now generalize the Pigeonhole Principle as follows.

Theorem 4.5 (Generalized Pigeonhole Principle)**.** If n objects are placed in m boxes, then there is a box with at least $\lceil \frac{n}{m} \rceil$ objects.

Problem 4.6. If 20 buses seating at most 50 carry 621 passengers to a ball game, then some bus must have at least __________ passengers.

Problem 4.7. How many balls must be drawn from the box in Problem 4.2 in order to be sure of getting at least 4 of the same color?

Problem 4.8. Explain why a list of ten positive integers, $x_1, x_2, \ldots, x_{10}$ must have a sublist in the same order of the original ten whose sum is divisible by 10.

We now introduce a concept known as the **Principal of Inclusion and Exclusion**. Recall Theorem 1.16, which states that if A and B are sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Problem 4.9. How many integers between 1 and 881 inclusively are divisible by 3 or 5?

But what do we do if we have more than two sets? Let's first examine the situation with three sets.

Problem 4.10. If A, B, and C are sets, then find a formula in the same vein as Theorem 1.16 for $|A \cup B \cup C|$.

The upshot is that we add "singles" subtract "doubles" and add "triples".

Problem 4.11. In the Natteranian township, 750 of the residents have a smart phone, 620 have a laptop computer, 480 have a desktop computer, 420 have both a laptop and a smart phone, 390 have both a smart phone and a desktop, 212 have both a laptop and a desktop computer and 164 have all three items.

(a) How many residents have at least one of the three items?

(b) How many residents do not have desktop computer?

(c) How many residents have a smart phone or a laptop?

(d) How many have a smart phone or a laptop but not a desktop?

We can generalize to any finite number of sets.

Theorem 4.12 (Principal of Inclusion and Exclusion)**.** The number of elements in the union of sets $A_1, A_2, \ldots, A_n$ is

$$|A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cdots \cap A_n|.$$

Problem 4.13. How many nonnegative integer solutions does the equation $x_1 + x_2 + x_3 + x_4 = 25$ have such that $x_1 < 7$, $x_2 < 5$, and $x_4 < 8$?

We now discuss one important application of the Principle of Inclusion and Exclusion. Formally, a **derangement** is a permutation $w : [n] \to [n]$ such that $w(i) \neq i$ for all $1 \leq i \leq n$ (i.e., w has no fixed points). That is, a derangement is a special rearrangement of objects such that none is in its original spot.

Problem 4.14. How many derangements of CAT are there?

Let d_n denote the number of derangements of $[n]$. We set $d_0 := 1$.

Problem 4.15. For $1 \leq i \leq n$, let F_i be the set of permutations that fix i.

(a) Explain why

$$d_n = |F_1^c \cap \cdots \cap F_n^c|.$$

(b) Explain why

$$d_n = n! - \sum_i |F_i| + \sum_{i<j} |F_i \cap F_j| - \sum_{i<j<k} |F_i \cap F_j \cap F_k| + \cdots + (-1)^n |F_1 \cap \cdots \cap F_n|.$$

(c) Explain why the number of derangements of $[n]$ is

$$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Problem 4.16. Using the previous problem, verify that we got the right answer to Problem 4.14.

Problem 4.17. If 7 hats are left at the hat-check window, in how many ways can they be returned so that no one gets the correct hat?

Now, just for funsies…from second semester calculus, we know

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

which implies that

$$e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$

Using Part (c) of Problem 4.15, we see that

$$\lim_{n\to\infty} \frac{d_n}{n!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \frac{1}{e} \approx 0.367879.$$

In other words, when n is large, the probably of selecting a derangement at random from the collection of permutations of n is approximately $1/e$. As n increases, the approximation improves. Boom.