Homework 6

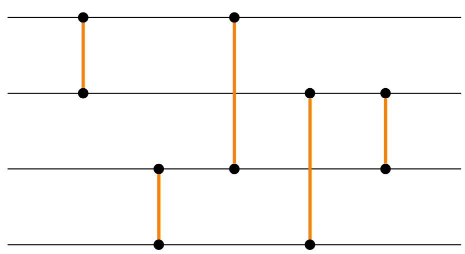
Combinatorics of Genome Rearrangements

You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in his or her own work. In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments with no questions asked**.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing assignments for this course, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.** Please read NAU's Academic Integrity Policy.

Complete the following problems.

1. Consider the following network of wires and comparators. Determine whether this network is a sorting network for S_4.



2. Define the **Mahonian polynomial** via

$$I_n(q) := \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)}.$$

(a) Explain why $I_n(q) = \sum_{k \geq 0} I_{n,k} q^k$, where $I_{n,k}$ denotes the Mahonian numbers (i.e., $I_{n,k}$ is the number of permutations in S_n with k inversions). Notice that the coefficient on q^k is the number of permutations in S_n with k inversions. This polynomial is an example of a **generating function** for a sequence. In this case, $I_n(q)$ is the generating function for the nth row of the array of Mahonian numbers. By Problem 1(c) on Homework 3, the coefficient on q^k also corresponds to $\mathrm{rk}_k(S_n, R_a)$ (i.e., the number of permutations of rank k in the sorting network for S_n using the collection of adjacent transpositions as generating set).

(b) Compute $I_4(q)$.

(c) What is the degree of $I_n(q)$?

(d) Prove that $\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = (1 + q + \cdots q^{n-1}) \sum_{\alpha \in S_{n-1}} q^{\mathrm{inv}(\alpha)}$. It follows that

$$I_n(q) = (1 + q + \cdots q^{n-1})(1 + q + \cdots + q^{n-2}) \cdots (1 + q).$$

The product on the right is sometimes written as $[n]_q!$ and is referred to as a q-**factorial**.

(e) What is the value of $I_n(1)$?

3. *This problem was broke and has been removed.*

4. The following is a greedy algorithm for sorting a permutation with transpositions. Find the largest value that is out of place, move it to its proper place, and repeat. More precisely, if $\pi_n = n$, do nothing and move on to sort $\pi_1\pi_2\cdots\pi_{n-1}$. Otherwise, if $\pi_i = n$ with $i < n$, apply the transposition (i,n) to get $\pi' = \pi \circ (i,n)$, so that $\pi'_n = n$. Now, we can sort $\pi'_1 \cdots \pi'_{n-1}$. This algorithm is sometimes called **straight selection sort**. Suppose $\rho_1, \ldots, \rho_k$ are the transpositions used in straight selection sort for some $\pi \in S_n$, so that $\pi \circ \rho_1 \circ \cdots \circ \rho_k$ is the identity, where $\rho_i = (\ell_i, m_i)$ with $\ell_i < m_i$. Define the **sorting index** for π via

$$\mathrm{sor}(\pi) := \sum_{i=0}^{k} (m_i - \ell_i).$$

Informally, the sorting index measures the "cost" of straight selection sort, with transpositions that are far away costing more than elements closer by. Define the **sorting index polynomial** via

$$\mathrm{Sor}_n(q) := \sum_{\pi \in S_n} q^{\mathrm{sor}(\pi)}.$$

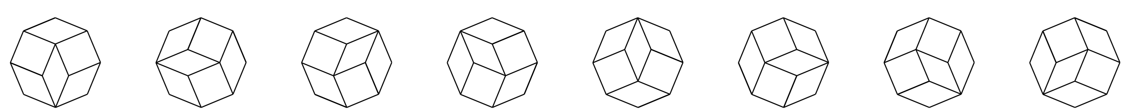
(a) Compute sor(3172546).

(b) Organize all of the permutations in S_4 according to their sorting index and then compute $\mathrm{Sor}_4(q)$.

(c) Is it true that $\mathrm{sor}(\pi) = \mathrm{inv}(\pi)$ for all $\pi \in S_4$? Justify your answer.

(d) Prove that $\mathrm{Sor}_n(q) = I_n(q)$. In light of this result, we say that the sorting index has a Mahonian distribution.

5. A **rhombic tiling** of a regular $2n$-gon is a tiling of the $2n$-gon using where all side lengths of the rhombi and the $2n$-gon are the same. For example, here all of the rhombic tilings of a regular octagon.



Construct a bijection between the collection of rhombic tilings of a regular $2n$-gon and the collection of primitive sorting networks of S_n.