The aim of argument, or of discussion, should not be victory, but progress.

Joseph Joubert, French moralist and essayist

Chapter 7

Limits

We are now prepared to dig into limits, which you are likely familiar with from calculus. However, chances are that you were never introduced to the formal definition.

7.1 Introduction to Limits

Definition 7.1. Let f be a real function. The **limit** of f as x approaches a is L if the following two conditions hold:

1. The point a is an accumulation point of $\mathrm{Dom}(f)$, and
2. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in \mathrm{Dom}(f)$ and $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

Notationally, we write this as

$$\boxed{\lim_{x \to a} f(x) = L.}$$

Problem 7.2. Why do we require $0 < |x-a|$ in Definition 7.1?

Problem 7.3. Why do you think we require a to be an accumulation point of the domain of f? What happens if $a \in \mathrm{Dom}(f)$ but a is not an accumulation point of $\mathrm{Dom}(f)$? Such points are called **isolated points** of the domain of f.

Notice that if $a \in \mathrm{Dom}(f)$ is an accumulation point of $\mathrm{Dom}(f)$, then the continuity of f at a is equivalent to the condition that

$$\lim_{x \to a} f(x) = f(a),$$

meaning that the limit of f as x approaches a exists and is equal to the value of f at a. However, it is important to notice that f may be continuous at a despite the fact that the limit of f as x approaches a is undefined. This happens when a is an isolated point of the domain.

Example 7.4. It should come as no surprise to you that $\lim_{x\to 5}(3x+2) = 17$. Let's prove this using Definition 7.1. First, notice that the default domain of $f(x) = 3x+2$ is the set of real numbers. So, any x-value we choose will be in the domain of the function. Now, let $\varepsilon > 0$. Choose $\delta = \varepsilon/3$. You'll see in a moment why this is a good choice for δ. Suppose $x \in \mathbb{R}$ such that $0 < |x-5| < \delta$. We see that

$$|(3x+2)-17| = |3x-15| = 3\cdot|x-5| < 3\cdot\delta = 3\cdot\varepsilon/3 = \varepsilon.$$

This proves the desired result.

Example 7.5. Let's try something a little more difficult. Let's prove that $\lim_{x\to 3} x^2 = 9$. As in the previous example, the default domain of our function is the set of real numbers. Our goal is to prove that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in \mathbb{R}$ such that $0 < |x-3| < \delta$, then $|x^2-9| < \varepsilon$. Let $\varepsilon > 0$. We need to figure out what δ needs to be. Notice that

$$|x^2-9| = |x+3|\cdot|x-3|.$$

The quantity $|x-3|$ is something we can control with δ, but the quantity $|x+3|$ seems to be problematic.

To get a handle on what's going on, let's temporarily assume that $\delta = 1$ and suppose that $0 < |x-3| < 1$. This means that x is within 1 unit of 3. In other words, $2 < x < 4$. But this implies that $5 < x+3 < 7$, which in turn implies that $|x+3|$ is bounded above by 7. That is, $|x+3| < 7$ when $0 < |x-3| < 1$. It's easy to see that we still have $|x+3| < 7$ even if we choose δ smaller than 1. That is, we have $|x+3| < 7$ when $0 < |x-3| < \delta \leq 1$. Putting this altogether, if we suppose that $0 < |x-3| < \delta \leq 1$, then we can conclude that

$$|x^2-9| = |x+3|\cdot|x-3| < 7\cdot|x-3|.$$

This work informs our choice of δ, but remember our scratch work above hinged on knowing that $\delta \leq 1$. If $\varepsilon/7 \leq 1$, we should choose $\delta = \varepsilon/7$. However, if $\varepsilon/7 > 1$, the easiest thing to do is to just let $\delta = 1$. Let's button it all up.

Let $\varepsilon > 0$. Choose $\delta = \min\{1, \varepsilon/7\}$ and suppose $0 < |x-3| < \delta$. We see that

$$|x^2-9| = |x+3|\cdot|x-3| < 7\cdot|x-3| < 7\cdot\delta \leq \varepsilon$$

since

$$7\cdot\delta = \begin{cases} 7, & \text{if } \varepsilon > 7 \\ 7\cdot\varepsilon/7, & \text{if } \varepsilon \leq 7. \end{cases}$$

Therefore, $\lim_{x\to 3} x^2 = 9$, as expected.

Problem 7.6. Prove that $\lim\limits_{x\to 1}(17x-42) = -25$ using Definition 7.1.

Problem 7.7. Prove that $\lim\limits_{x\to 2} x^3 = 8$ using Definition 7.1.

Problem 7.8. Define $f:\mathbb{R}\to\mathbb{R}$ via

$$f(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 17, & \text{if } x = 0. \end{cases}$$

Prove that $\lim\limits_{x\to 0} f(x) = 0$ using Definition 7.1.

Problem 7.9. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ -1, & \text{if } x > 0. \end{cases}$$

Using Definition 7.1, prove that $\lim_{x \to 0} f(x)$ does not exist.

Problem 7.10. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Using Definition 7.1, prove that $\lim_{x \to a} f(x)$ does not exist for all $a \in \mathbb{R}$.

Like the limits of sequences, limits of functions are unique when they exist.

Problem 7.11. Let f be a real function. Prove that if $\lim_{x \to a} f(x)$ exists, then the limit is unique.

> An ounce of practice is worth more than tons of preaching.
>
> Mahatma Gandhi, political activist

7.2 Limit Laws

Perhaps not surprisingly, there is a nice connection between limits and sequences.

Problem 7.12. Let f be a real function and let a be an accumulation point of $\mathrm{Dom}(f)$. Prove that $\lim_{x \to a} f(x)$ exists if and only if for every sequence (p_n) in $\mathrm{Dom}(f) \setminus \{a\}$ converging to a, the sequence $(f(p_n))$ converges, in which case, $\lim_{x \to a} f(x)$ equals the limit of the sequence $(f(x_n))$. This is often written as

$$\lim_{x \to a} f(x) = \lim_{n \to \infty} f(p_n).$$

In order for limits to be a useful tool, we need to prove a few important facts.

Problem 7.13 (Limit Laws)**.** Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be real functions. Prove each of the following using Definition 7.1 or Problem 7.12.

(a) If $c \in \mathbb{R}$, then $\lim_{x \to a} c = c$.

(b) If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, then

$$\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x).$$

(c) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then

$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x).$$

(d) If $c \in \mathbb{R}$ and $\lim_{x\to a} f(x)$ exists, then

$$\lim_{x\to a}(c\cdot f(x)) = c\cdot \lim_{x\to a} f(x).$$

(e) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and $\lim_{x\to a} g(x) \neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

(f) If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

The next problem is extremely useful. It allows us to simplify our calculations when computing limits.

Problem 7.14. Let f and g be real functions with $A = \mathrm{Dom}(f) = \mathrm{Dom}(g)$ and let a be an accumulation point of A. Prove that if there exists an open interval J such that $f(x) = g(x)$ for all $x \in (J\cap A)\setminus\{a\}$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

provided one of the limits exists.

> Vulnerability is not winning or losing; it's having the courage to show up and be seen when we have no control over the outcome.
>
> Brené Brown, storyteller & author