Exam 3

Your Name:

Names of Any Collaborators:

Instructions

This exam is worth a total of 37 points and is worth 15% of your overall grade. Your exam is due by midnight on **Saturday, April 11**. Submit your exam via BbLearn. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems/problems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 2.32, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

 Written by D.C. Ernst

The following problems are related to content from Chapters 4–7 of our textbook. Some of the problems are directly from the book. Note that you will need to independently digest some new material in Chapter 7, which involves functions. You may freely use any result that appears in any chapter as long as the result appears *before* the problem in question.

1. For any $n \in \mathbb{N}$, say that n straight lines are "safely drawn in the plane" if no two of them are parallel and no three of them meet in a single point. Let $S(n)$ be the number of regions formed when n straight lines are safely drawn in the plane.
   (a) (2 points) Compute $S(1)$, $S(2)$, $S(3)$, and $S(4)$.
   (b) (1 points) Conjecture a recursive formula for $S(n)$; that is, a formula for $S(n)$ which may involve some of the previous terms. If necessary, first compute a few more values of $S(n)$.
   (c) (3 points) Prove your conjecture from part (b).
2. (2 points each) Let $A = \{1, 2, 3, 4, 5, 6\}$ and $\Omega = \{\{1, 3, 4\}, \{2, 4\}, \{3, 4\}, \{6\}\}$.
   (a) Is Ω a partition of A? Briefly explain your answer.
   (b) Find $\sim_\Omega$ (i.e., the relation determined by Ω) by listing ordered pairs or drawing a digraph.
   (c) Is $\sim_\Omega$ an equivalence relation? Briefly explain your answer.
   (d) Is $\sim_\Omega$ a function from A to A?* Briefly explain your answer.
   (e) Find $\Omega_{\sim_\Omega}$ (i.e., the collection of subsets of A determined by $\sim_\Omega$).
3. (4 points each) Complete **two** of the following.
   (a) Let $n \in \mathbb{N}$ and define $\sim$ on $\mathbb{Z}$ via $a \sim b$ if and only n divides $a - b$. Prove that $\sim$ is an equivalence relation on $\mathbb{Z}$.
   (b) Define $\Psi := \{(a, b) \mid a, b \in \mathbb{Z}, b \neq 0\}$ and define $\sim$ on Ψ via $(a, b) \sim (c, d)$ if and only if $ad = bc$. Prove that $\sim$ is an equivalence relation on Ψ.†
   (c) Let $f : X \to Y$ be a function and define $\sim$ on X via $a \sim b$ if and only if $f(a) = f(b)$. Prove that $\sim$ is an equivalence relation on X.‡
   (d) Suppose R and S are both equivalence relations on a set A. Prove that $R \cap S$ is an equivalence relation on A.
4. (2 points) Let $A = \{a, b, c, d\}$. Provide examples of two different equivalence relations R and S on A such that $R \cup S$ is not an equivalence relation on A. Briefly justify your example.
5. (3 points each) Let A be a nonempty set and let Ω be a collection of subsets of A (not necessarily a partition). Complete **two** of the following.
   (a) Prove that $\sim_\Omega$ is symmetric.§
   (b) Prove that if $\displaystyle\bigcup_{R \in \Omega} R = A$, then $\sim_\Omega$ is reflexive.¶
   (c) Prove that if the elements of Ω are pairwise disjoint, then $\sim_\Omega$ is transitive.‖

*See Definition 7.1 for the definition of "function."

†Fun Fact! This is essentially an equivalence relation on the set of fractions of the form a/b, where $a, b \in \mathbb{Z}$ and $b \neq 0$. Formally, the set of rational numbers is a collection of equivalence classes determined by this relation.

‡This is part of Theorem 7.47.

§This is Theorem 6.55.

¶This is Theorem 6.57.

‖This is Theorem 6.58.

6. (3 points) Let A be a nonempty set and suppose $\sim$ is an equivalence relation on A. Prove that the function $\phi : A \to A/{\sim}$ defined via $\phi(x) = [x]$ is onto.[**]

7. (2 points) Complete **one** of the following.
   (a) Under what conditions would the function ϕ given in Problem 6 be one-to-one?[††]
   (b) Suppose A is a nonempty set and let $f : A \to A$ be a function. Under what conditions is f also an equivalence relation?

[**]This is Theorem 7.20. Recall that $A/{\sim}$ is the set of equivalence classes induced by the equivalence relation $\sim$. For the definition of "onto", see See Definition 7.11.

[††]For the definition of "one-to-one", see Definition 7.11.

