Homework 5

Combinatorics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions Discord. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

1. Define a **multiset** M to be an unordered collection of elements that may be repeated. To distinguish between sets and multisets, we will use the notation $\{\}$ versus $\{\{\}\}$. For example, $M = \{\{a,a,a,b,c,c\}\}$ is a multiset (not to be confused with the set containing the set $\{a,b,c\}$; mathematical notation is both awesome and frustrating at times). The primes occurring in the prime factorization of a natural number greater than 1 is an example of multiset. The cardinality of a multiset is its number of elements counted with multiplicity. So, in our example, $|M| = 3+1+2 = 6$. If S is a set, then M is a **multiset on** S if every element of M is an element of S. For $n \geq 1$ and $k \geq 0$, define

$$\left(\!\!\binom{n}{k}\!\!\right) := \text{number of multisets on } [n] \text{ of size } k.$$

Prove that

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}.$$

Hint: Here is one possible approach. Construct a bijection from the collection of multisets on $[n]$ of size k to the collection of subsets of $[n+k-1]$ of size k. To do this, assume your multisets on $[n]$ are canonically written in nondecreasing order, say $\{\{m_1, m_2, \ldots, m_k\}\}$, where $m_1 \leq m_2 \leq \cdots \leq m_k$. Try to map each such multiset to a k-subset of $[n+k-1]$. You will need to make sure the elements in your image set are distinct. You may also need to argue that your map is well defined.

2. For fixed $n \in \mathbb{N}$, let $M_n(t)$ denote the generating function for the number of multisets of $[n]$ of size k:
$$M_n(t) := \sum_{k \geq 0} \left(\!\!\binom{n}{k}\!\!\right) t^k.$$
Also, define $\mathcal{M}_n := \{A \mid A \text{ is a multiset on } [n]\}$.
   (a) Explain why $M_n(t) = \sum_{A \in \mathcal{M}_n} t^{|A|}$, where $|A|$ is the size of the multiset A.
   (b) Explain why $\sum_{A \in \mathcal{M}_n} t^{|A|} = (1 + t + t^2 + \cdots)^n$.
   (c) Explain why $M_n(t) = \dfrac{1}{(1-t)^n}$.

   Hint: For part (b), notice that if you were to expand $(1 + t + t^2 + \cdots)^n$, each resulting term (prior to collecting like terms) corresponds to choosing a term from each of the n factors and then multiplying them together. Each such product of choices corresponds to a unique multiset. In particular, think of the ith factor of $(1 + t + t^2 + \cdots)^n$ as corresponding to $i \in [n]$. Choosing t^j in the ith factor corresponds to having i occur with multiplicity j in a multiset.

3. Give a combinatorial or bijective proof that $\left\langle {n \atop 1} \right\rangle = 2^n - n - 1$.

4. Prove that the following permutation statistics are Eulerian. Complete any **two** of the following.
   (a) The number of **ascents** of a permutation w, $\operatorname{asc}(w) := |\{i \mid w(i) < w(i+1)\}|$.
   (b) The number of **maximal increasing runs** of a permutation, denoted $\operatorname{runs}(w)$, where a maximal increasing run is a substring $w(i) < w(i+1) < \cdots < w(i+r)$ such that $w(i-1)$ is not smaller than $w(i)$ and $w(i+r)$ is not smaller than $w(i+r+1)$.
   (c) The number of **excedances**, $\operatorname{exc}(w) := |\{i \mid w(i) > i\}|$.
   (d) The number of inversion sequences of length n counted according to ascents, where an **inversion sequence** of length n is a vector $s = (s_1, \ldots, s_n)$ such that $0 \leq s_i \leq i - 1$ and $\operatorname{asc}(s) := |\{i \mid s_i < s_{i+1}\}|$.

5. We define a **barred permutation** on n as follows. Given $w \in S_n$, we must place at least one vertical bar after each descent position and we can place finitely many additional vertical bars in gaps that do not correspond to descents. For example, 1|5|237||46 is a barred permutation with 4 vertical bars. A single vertical bar was required after 5 and 7, respectively, and the other two bars were optional.
   (a) Explain why there is a bijection between the collection of barred permutations on n with k bars and the collection of configurations of n labeled balls into $k+1$ labeled boxes.
   (b) Fix $n \in \mathbb{N}$. Prove that the generating function for the number of barred permutations that have a fixed $w \in S_n$ as their underlying permutation is given by
$$\frac{t^{\operatorname{des}(w)}}{(1-t)^{n+1}}.$$

Hint: As an example, the barred permutation 1|5|237||46 corresponds to placing 7 balls into 5 boxes such that the first box contains ball 1, the second box contains ball 5, the third box contains balls 2, 3, 7, the fourth box is empty, and the fifth box has balls 4 and 6. The underlying permutation for this example is $w = 1523746$. For part (b), mimic the approach in Problem 2(b). Utilize a factor of $1 + t + t^2 + \cdots$ when there is no descent and a factor of $t + t^2 + t^3 + \cdots$ if there is a descent.