Introduction to Discrete Mathematics

Dana C. Ernst $<$ John W. Hagood

Northern Arizona University

Version Spring 2024

This book is intended to be used for a one-semester discrete mathematics course. There is always a debt to be paid in creating a text, and this one is no different. The initial source for this book was John W. Hagood's notes by the title *MAT 226 Discrete Mathematics*.

Contents

1 Sets and Counting Principles 3

2 Functions 11

3 Permutations 16

4 Combinations 19

5 The Binomial Theorem 23

6 Pigeonhole Principle 25

7 Principle of Inclusion and Exclusion 26

8 Mathematical Induction 28

9 Sequences and Recurrence Relations 33

10 Introduction to Graph Theory 40

11 Additional Graph Theory 46

Chapter 1

Sets and Counting Principles

A **set** is a collection of objects called **elements**. Typically, braces are used to denote sets as in $\{b, c, d, f\}$, which is the same set as $\{b, d, c, f\}$, and the same as $\{c, f, c, f, d, b\}$. If A is a set and x is an element of A, we write $\boxed{x \in A}$. Otherwise, we write $\boxed{x \notin A}$. For example, $b \in \{b, d, c, f\}$ while $e \notin \{b, d, c, f\}$. The set containing no elements is called the **empty set**, and is denoted by the symbol $\boxed{\emptyset}$, but can also be written as $\boxed{\{\}}$. Any set that contains at least one element is referred to as a **nonempty set**. If we think of a set as a box potentially containing some stuff, then the empty set is a box with nothing in it.

The empty set and the set $\{b, c, d, f\}$ are examples of **finite sets**. Any set that is not finite is an **infinite set**. For example, the **natural numbers** defined by $\boxed{\mathbb{N} := \{1, 2, 3, \ldots\}}$ and **integers** defined by $\boxed{\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \ldots\}}$ are both examples of infinite sets. Some books will include zero in the set of natural numbers, but we do not. Since the set of natural numbers consists of the positive integers, the natural numbers are sometimes denoted by $\boxed{\mathbb{Z}^+}$. The **real numbers**, denoted by $\boxed{\mathbb{R}}$, is another example of an infinite set. If you look closely at our definitions for the natural numbers and integers you will notice that we wrote $\boxed{:=}$ instead of $=$. We use $:=$ to mean that the symbol or expression on the left is defined to be equal to the expression on the right.

The language associated to sets is specific. We will often define sets using the following notation, called **set-builder notation**:

$$\boxed{S = \{x \in A \mid P(x)\}},$$

where $P(x)$ is some predicate statement involving x. The first part "$x \in A$" denotes what type of x is being considered. The predicate to the right of the vertical bar determines the condition(s) that each x must satisfy in order to be a member of the set. This notation is read as "The set of all x in A such that $P(x)$."

Example 1.1. The set $\{x \in \mathbb{N} \mid x \text{ is even and } x \geq 8\}$ describes the collection of even natural numbers that are greater than or equal to 8.

Example 1.2. We can define the **rational numbers** using set-builder notation via

$$\boxed{\mathbb{Q} := \{a/b \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}}.$$

Problem 1.3. Write the set of perfect squares $\{1, 4, 9, 16, \ldots\}$ using set-builder notation.

Problem 1.4. Translate the following set defined with set-builder notation in English and list a few of its elements:

$$P = \{x \in \mathbb{N} \mid x = 2^n \text{ for some } n \in \mathbb{N}\}.$$

The **cardinality** of a set A, denoted by $\boxed{|A|}$, is the number of elements in A. For example, if $A = \{b, c, d, f\}$, then $|A| = 4$. We will mostly concern ourselves with the cardinality of finite sets in this book. Cardinality gets wildly more complicated if we consider infinite sets.

If A and B are sets, then we say that A is a **subset** of B, written $\boxed{A \subseteq B}$, provided that every element of A is an element of B.

Example 1.5. We have $\{c, f, g\} \subseteq \{b, c, d, f, g\}$.

Problem 1.6. Let A be a set. Determine whether each of the following statements is true or false. If a statement is true, explain why. If a statement is false, provide a specific counterexample.

(a) $A \subseteq A$.

(b) $\emptyset \subseteq A$.

Problem 1.7. Consider the set $A = \{a, b, c\}$. How many subsets does A have?

Theorem 1.8 (Transitivity of Subsets)**.** Suppose that A, B, and C are sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Note that two sets A and B are **equal**, denoted $\boxed{A = B}$, if the sets contain the same elements. It's clear that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. If $A \subseteq B$, then A is called a **proper subset** provided that $A \neq B$. In this case, we may write $\boxed{A \subset B}$ or $\boxed{A \subsetneq B}$. Some authors use $\subset$ to mean $\subseteq$, so some confusion could arise if you are not reading carefully.

Sometimes it is useful to fix a set to focus our attention on. The term **universe of discourse** (or **domain of discourse**) generally refers to the collection of objects being discussed in a specific context.

Let A and B be sets in some universe of discourse U. We define the following.

(a) The **union** of A and B is $\boxed{A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\}}$.

(b) The **intersection** of A and B is $\boxed{A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\}}$.

(c) If A and B have the property that $A \cap B = \emptyset$, then we say that A and B are **disjoint**.

(d) The **set difference** (relative to U) of the sets A and B is $\boxed{A \setminus B := \{x \in U \mid x \in A \text{ and } x \notin B\}}$.

(e) The **complement** (relative to U) of A is the set $\boxed{A^c := U \setminus A = \{x \in U \mid x \notin A\}}$.

Problem 1.9. Suppose that the universe of discourse is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5\}$, and $C = \{2, 4, 6, 8\}$. Find each of the following and then determine the cardinality of the set.

(a) $A \cap C$

(b) $B \cap C$

(c) $A \cup B$

(d) $A \setminus B$

(e) $B \setminus A$

(f) $C \setminus B$

(g) B^c

(h) A^c

(i) $(A \cup B)^c$

(j) $A^c \cap B^c$

Is any pair of the original sets A, B, and C disjoint?

The next theorem should be clear.

Theorem 1.10. If A is a subset of some universe of discourse U, then

(a) $\boxed{A \cup A^c = U}$;

(b) $\boxed{A \cap A^c = \emptyset}$ (i.e., A and A^c are disjoint).

The following principle is a natural notion.

Theorem 1.11 (Sum Principle for Disjoint Sets)**.** If A and B are disjoint sets, then

$$\boxed{|A \cup B| = |A| + |B|.}$$

More generally, if $A_1, \dots, A_n$ are pairwise disjoint sets, then

$$\boxed{|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|.}$$

Problem 1.12. A **domino** is a rectangular tile, usually with a line dividing its face into two square ends, and each end is marked with a number of dots (also called pips) or is blank. Figure 1.1 shows a few examples of dominoes. The first and second domino in the figure are actually the same domino since orientation of the domino is irrelevant! When tackling the problems below, consider handling disjoint cases. If you visualize the various cases, you may end up with a nice staircase-looking arrangement.

(a) How many dominoes are there in a set that includes double blank through double nine?

(b) How many dominoes are there in a set that includes double blank through double twelve?

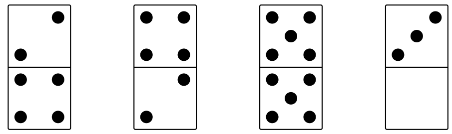


Figure 1.1: Examples of dominoes.

(c) How many dominoes are there in a set that includes double blank through double n?

Problem 1.13. How many students are enrolled in MAT 136 or MAT 226 (or both) if there are 409 in MAT 136 and 156 in MAT 226? Do you need to know that 40 students are in both courses?

The previous problems gives us an indication of how to generalize the Sum Principle (Theorem 1.11) to the case when the sets are not disjoint.

Theorem 1.14 (General Sum Principle)**.** If A and B are sets, then

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|,}$$

or equivalently

$$\boxed{|A \cup B| + |A \cap B| = |A| + |B|.}$$

Problem 1.15. I'm in the mood for a cup of coffee! Let's say that the nearest place to get a cup of coffee is four blocks East and three blocks North. Assuming I only walk East or North, how many different routes can I take to get there if:

(a) The last block I walk is heading North?

(b) The last block I walk is heading East?

(c) I don't care whether the last block I walk is North or East?

In Problem 1.15, we could describe paths to the coffee shop using expressions like $NNENEEE$, where we interpret the expression from left to right as instruction for which direction to walk at each step. Such expressions have a name. A **string** of objects of **length** k is a linearly ordered arrangements of k objects. The objects in a string may be repeated unless specified otherwise. For example, $NNENEEE$ is a string of length 7 consisting of N's and E's. If a string consists of letters, then it may be called a **word** (we do not require a word to be in the dictionary!). A **bit string** is a string consisting of 0's and 1's. Each occurrence of a 0 or 1 in a bit string is called a **bit**. The strings 010 and 100 are two different bit strings of length 3.

Example 1.16. How many two-letter words are there using only the letters b, c, d, f?

Solution. We can easily list these, but a different arrangement will illustrate a more powerful technique.

	b	c	d	f
b	bb	bc	bd	bf
c	cb	cc	cd	cf
d	db	dc	dd	df
f	fb	fc	fd	ff

The rectangular table makes it clear that there are 16 two-letter words using the letters b, c, d, and f. Another way to see this is to note that there are 4 choices for the first (left-hand) letter and for each of these there are 4 choices for the right-hand letter, so there are $4 \cdot 4 = 16$ words total. □

Example 1.17. How many "numbered words" are there consisting of two letters from b, c, d, and f, followed by one of the digits 1, 2, 3, 4, and 5?

Solution. A geometric display as in Example 1.16 would seem to require three dimensions, but we can take what we already know and just add a digit to the right:

	1	2	3	4	5
bb	$bb1$	$bb2$	$bb3$	$bb4$	$bb5$
bc	$bc1$	$bc2$	$bc3$	$bc4$	$bc5$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ff	$ff1$	$ff2$	$ff3$	$ff4$	$ff5$

There are 16 rows (each with a word from Example 1.16) and five columns, so there are $16 \cdot 5 = 80$ two-letter words using b, c, d, and f, followed by a digit from 1, 2, 3, 4, and 5. □

If A and B are sets, the **Cartesian product** of A and B, denoted $A \times B$ (read as "A times B" or "A cross B"), is the set of all **ordered pairs** where the first component is from A and the second component is from B. In set-builder notation, we have

$$\boxed{A \times B := \{(a, b) \mid a \in A, b \in B\}}.$$

Similarly, the Cartesian product of sets $A_1, \ldots, A_n$ sets is the collection of n**-tuples** given by

$$\boxed{A_1 \times \cdots \times A_n := \{(a_1, \ldots, a_n) \mid a_j \in A_j \text{ for all } 1 \le j \le n\}},$$

where A_i is referred to as the ith **factor** of the Cartesian product. As a special case, the set

$$\underbrace{A \times \cdots \times A}_{n \text{ factors}}$$

is often abbreviated as $\boxed{A^n}$.

Cartesian products are named after French philosopher and mathematician René Descartes (1596–1650).

Example 1.18. If $A = \{a, b, c\}$ and $B = \{☺, ☹\}$, then

$$A \times B = \{(a, ☺), (a, ☹), (b, ☺), (b, ☹), (c, ☺), (c, ☹)\}.$$

Example 1.19. The standard two-dimensional plane $\mathbb{R}^2$ and standard three space $\mathbb{R}^3$ are familiar examples of Cartesian products. In particular, we have

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$$

and

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

Problem 1.20. Consider the sets A and B from Example 1.18.

(a) Find $B \times A$.

(b) Find $B \times B$.

Problem 1.21. Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$, and $C = \{1, 3\}$. Find $A \times B \times C$.

Problem 1.22. If A is a set, then what is $A \times \emptyset$ equal to?

Problem 1.23. If A and B are both finite sets, find a nice formula for $|A \times B|$.

Problem 1.24. Are collections of words or strings that we encountered earlier in the chapter just Cartesian products in disguise? Explain.

Problem 1.25. Is a set of dominoes a Cartesian product in disguise? Explain.

We have been dancing around a fundamental counting principle. Let's state it officially. There are three progressively more general versions, the first of which we already encountered in Problem 1.23. The second version officially follows from the first version by induction, which is a topic we will encounter later. The third version follows from the second by making a careful use of notation to identify the set $\mathcal{O}$ with a Cartesian product.

Theorem 1.26 (Product Principle)**.** Each of the following are referred to as the **Product Principle**.

(a) If A and B are finite sets, then

$$\boxed{|A \times B| = |A| \cdot |B|.}$$

(b) If $A_1, \ldots, A_k$ are finite sets, then

$$\boxed{|A_1 \times \cdots \times A_k| = |A_1| \cdots |A_k|.}$$

(c) If $\mathcal{O}$ is the set of outcomes for a k-step process, where for $1 \leq i \leq k$, there are n_i choices for step i, no matter what earlier choices were made, then

$$\boxed{|\mathcal{O}| = n_1 n_2 \cdots n_k.}$$

The key difference between versions (b) and (c) of the Product Principle is that version (c) does not assume that the set of choices for step i is independent of the previous choices.

Problem 1.27. A fashion-challenged freshman has three pair of pants, five shirts (all t-shirts, but different patterns) and a pair of sandals. How many different ensembles are available for this freshman?

Problem 1.28. Suppose you flip a coin fives times in a row, recording the sequence of heads and tails that you see. How many different sequences of flips are possible?

Problem 1.29. Suppose we roll a six-sided die and then flip a coin. How many distinct outcomes are possible?

Problem 1.30. How many bit strings of length n are there? How many of these start and end with the bit 1?

Problem 1.31. How many subsets does a set with n elements have? Make sure you are taking the case $n = 0$ into account. You should be able to carefully justify your answer by cleverly utilizing the Product Principle.

Problem 1.32. How many four-letter words are there using lower-case English letter consonants (including y) that include exactly one occurrence of b?

Problem 1.33. A pass code consists of a string of two or three case-sensitive English letters followed by three digits (i.e., $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$). How many different possible pass codes are there?

The next result is simply a special case of Theorem 1.11. Sometimes it might be easier to count the opposite of what you are looking to count and then to subtract from the size of the universe of discourse. This idea is sometimes referred to as the **Subtraction Principle**. Keep this in mind as a method of attack for future problems.

Theorem 1.34 (Subtraction Principle)**.** If A is a subset of a universe of discourse U, then $\boxed{|U| = |A| + |A^c|}$, or equivalently, $\boxed{|A| = |U| - |A^c|}$.

Problem 1.35. How many words of length 4 are there consisting of lower-case English letters that utilize at least two x's? Find the answer in two different ways.

Problem 1.36. How many words of length 6 are there consisting of lower-case English letters subject to the following constraints?

(a) No constraints.

(b) Repetition not allowed.

(c) No instances of letter a.

(d) At least one occurrence of letter a.

Problem 1.37. Six friends sit on one side of long rectangular table.

(a) How many seating arrangements are there?

(b) How many seating arrangements are there if Sally and Maria always sit next to each other?

The next theorem provides a method for ignoring "unimportant" differences when counting things.

Theorem 1.38 (Division Principle)**.** If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $|A| = n/d$.

Problem 1.39. Six friends sit around a circle to play a game. Rotations of the group do not constitute different seating orders.

(a) How many circular seating arrangements are there?

(b) How many circular seating arrangements are there if Sally and Maria always sit next to each other?

Problem 1.40. How many ways can the letters of the word SLOPPY be arranged?

Problem 1.41. It's Halloween and five students arrive at my office begging for candy. I happen to have five pieces of candy. Depending on my mood, I may give away none of the candy, all of the candy, or any amount in between. Assuming I don't give any student more than one piece of candy, how many different ways can I distribute the candy? Does it matter if the pieces of candy are identical or not? If so, count both situations.

Chapter 2

Functions

Let X and Y be two nonempty sets. A **function** f **from** X **to** Y is a subset of $X \times Y$ such that for every $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in f$. The set X is called the **domain** of f and is denoted by $\boxed{\mathrm{Dom}(f)}$. The set Y is called the **codomain** of f and is denoted by $\boxed{\mathrm{Codom}(f)}$ while the subset of the codomain defined via

$$\boxed{\mathrm{Rng}(f) \coloneqq \{y \in Y \mid \text{there exists } x \text{ such that } (x, y) \in f\}}$$

is called the **range** of f or the **image of** X under f.

There is a variety of notation and terminology associated to functions. We will write $\boxed{f : X \to Y}$ to indicate that f is a function from X to Y. We will make use of statements such as "Let $f : X \to Y$ be the function defined via..." or "Define $f : X \to Y$ via...", where f is understood to be a function in the second statement. Sometimes the word **mapping** (or **map**) is used in place of the word function. If $(x, y) \in f$ for a function f, we often write $\boxed{f(x) = y}$ and say that "f maps x to y" or "f of x equals y". In this case, x may be called an **input** of f and is the **preimage** of y under f while y is called an **output** of f and is the **image** of x under f. Note that the domain of a function is the set of inputs while the range is the set of outputs for the function.

Sometimes we can represent functions visual representations called **function** (or **mapping**) **diagrams**, where the elements of the domain and codomain are indicated by labeled nodes and ordered pairs for the function are indicated by an arrow pointing from the node for input to the node for the output. When drawing function diagrams, it is standard practice to put the elements for the domain on the left and the elements for the codomain on the right, so that all directed edges point from left to right. We may also draw an additional arrow labeled by the name of the function from the domain to the codomain.

Example 2.1. Figure 2.1 depicts a function $f : X \to Y$ for the sets $X = \{a, b, c, d\}$ to $Y = \{1, 2, 3, 4\}$. In this case, we see that $\mathrm{Rng}(f) = \{1, 2, 4\}$. Moreover, we can write things like $f(a) = 2$ and $c \mapsto 4$, and say things like "f maps b to 4" and "the image of d is 1." Note that it is perfectly okay to have both b and c mapped to 4.

Sometimes we can define a function using a formula. For example, we can write $f(x) = x^2 - 1$ to mean that each x in the domain of f maps to $x^2 - 1$ in the codomain. However,

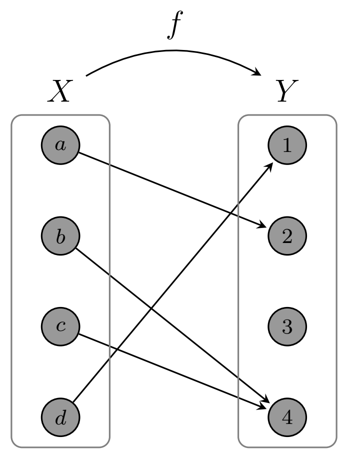


Figure 2.1: Function diagram for a function from $X = \{a, b, c, d, \}$ to $Y = \{1, 2, 3, 4\}$.

notice that providing only a formula is ambiguous! A function is determined by its domain, codomain, and the correspondence between these two sets. If we only provide a description for the correspondence, it is not clear what the domain and codomain are. Two functions that are defined by the same formula, but have different domains or codomains are *not* equal. It is important to point out that not every function can be described using a formula! Despite your prior experience, functions that can be represented succinctly using a formula are rare.

Example 2.2. The function $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x^2 - 1$ is not equal to the function $g : \mathbb{N} \to \mathbb{R}$ defined by $g(x) = x^2 - 1$ since the two functions do not have the same domain.

Problem 2.3. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs that represents the assignment, or write a formula as long as the domain and codomain are clear.

(a) A function f from a set with 4 elements to a set with 3 elements such that $\text{Rng}(f) = \text{Codom}(f)$.

(b) A function g from a set with 4 elements to a set with 3 elements such that $\text{Rng}(g)$ is strictly smaller than $\text{Codom}(g)$.

Problem 2.4. In high school you may have been told that a graph represents a function if it passes the **vertical line test**. Carefully state what the vertical line test says and then explain why it works.

A **piecewise-defined function** (or **piecewise function**) is a function defined by specifying its output on a partition (i.e., "disjoint chunks") of the domain. Note that "piecewise" is a way of expressing the function, rather than a property of the function itself.

Example 2.5. The function $f : \mathbb{R} \to \mathbb{R}$ defined via

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x \geq 0, \\ 17, & \text{if } -2 \leq x < 0, \\ -x, & \text{if } x < -2 \end{cases}$$

is an example of a piecewise-defined function.

Problem 2.6. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ via $f(x) = \frac{|x|}{x}$. Express f as a piecewise function.

Problem 2.7. Let $n \in \mathbb{N}$. Count all functions $f : \{1, 2, \ldots, n\} \to \{0, 1\}$. How is this problem related to Problem 1.30? How is this problem related to Problem 1.31?

Let $f : X \to Y$ be a function. We define the following.

(a) The function f is said to be **injective** (or **one-to-one**) if for all $y \in \mathrm{Rng}(f)$, there is a unique $x \in X$ such that $y = f(x)$. Said another way, f is one-to-one provided $f(x) = f(y)$ implies that $x = y$, or equivalently $x \neq y$ (in X) implies $f(x) \neq f(y)$. That is, different inputs produce different outputs.

(b) The function f is said to be **surjective** (or **onto**) if for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$.

(c) If f is both injective and surjective, we say that f is **bijective**.

Problem 2.8. Assume that X and Y are finite sets. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs, or write a formula as long as the domain and codomain are clear.

(a) A function $f : X \to Y$ that is injective but not surjective.

(b) A function $f : X \to Y$ that is surjective but not injective.

(c) A function $f : X \to Y$ that is a bijection.

(d) A function $f : X \to Y$ that is neither injective nor surjective.

Problem 2.9. Determine whether each of the following functions is injective, surjective, both, or neither.

(a) Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$

(b) Define $g : \mathbb{R} \to [0, \infty)$ via $g(x) = x^2$

(c) Define $h : \mathbb{R} \to \mathbb{R}$ via $h(x) = x^3$

(d) Define $k : \mathbb{R} \to \mathbb{R}$ via $k(x) = x^3 - x$

(e) Define $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via $c(x, y) = x^2 + y^2$

(f) Define $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ via $f(n) = (n, n)$

(g) Define $g : \mathbb{Z} \to \mathbb{Z}$ via

$$g(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

(h) Define $\ell : \mathbb{Z} \to \mathbb{N}$ via

$$\ell(n) = \begin{cases} 2n+1, & \text{if } n \geq 0 \\ -2n, & \text{if } n < 0 \end{cases}$$

Problem 2.10. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is ____________ if and only if every horizontal line hits the graph of f *at most once*.

This statement is often called the **horizontal line test**. Explain why the horizontal line test is true. What kind of theorems to we get if we replace "at most once" with either "at least once" or "exactly once"?

Problem 2.11. Suppose $f : A \to B$ is a function for finite sets A and B. Explain why each of the following statements is true.

(a) If f is an injection, then $|A| \leq |B|$.

(b) If f is a surjection, then $|A| \geq |B|$.

(c) If f is a bijection, then $|A| = |B|$.

Problem 2.12. Let A and B be nonempty finite sets with $|A| = m$ and $|B| = n$.

(a) How many different functions are there from A to B?

(b) If $m \leq n$, how many injective functions are there from A to B?

(c) If $m = n$, how many bijective functions are there from A to B?

(d) If $m \geq n$, do you think it would be challenging to count the number of surjective functions from A to B?

The next theorem states an important counting technique, which we refer to as the **Bijection Principle**.

Theorem 2.13 (Bijection Principle)**.** If $|B| = n$ and there exists a bijection between A and B, then $|A| = n$, as well.

Problem 2.14. Let A denote the set of ways to distribute candy in Problem 1.41 (in the situation where the candy is all identical) and let B denote the set of sequence of coin flips in Problem 1.28. Find a bijection $f : A \to B$.

Problem 2.15. Utilize a bijection to connect Problems 1.30 and 1.31.

The next problem illustrates that some care must be taken when attempting to define a function.

Problem 2.16. For each of the following, explain why the given description does not define a function.

(a) Define $f : \{1, 2, 3\} \to \{1, 2, 3\}$ via $f(a) = a - 1$.

(b) Define $g : \mathbb{N} \to \mathbb{Q}$ via $g(n) = \frac{n}{n-1}$.

(c) Let $A_1 = \{1, 2, 3\}$ and $A_2 = \{3, 4, 5\}$. Define $h : A_1 \cup A_2 \to \{1, 2\}$ via

$$h(x) = \begin{cases} 1, & \text{if } x \in A_1 \\ 2, & \text{if } x \in A_2. \end{cases}$$

(d) Define $s : \mathbb{Q} \to \mathbb{Z}$ via $s(a/b) = a + b$.

In mathematics, we say that an expression is **well defined** (or **unambiguous**) if its definition yields a unique interpretation. Otherwise, we say that the expression is not well defined (or is **ambiguous**). For example, if $a, b, c \in \mathbb{R}$, then the expression abc is well defined since it does not matter if we interpret this as $(ab)c$ or $a(bc)$ since the real numbers are associative under multiplication.

When we attempt to define a function, it may not be clear without doing some work that our definition really does yield a function. If there is some potential ambiguity in the definition of a function that ends up not causing any issues, we say that the function is well defined. However, this phrase is a bit of misnomer since all functions are well defined. The issue of whether a description for a proposed function is well defined often arises when defining things in terms of how an element of the domain is written. For example, the descriptions given in Parts (c) and (d) of Problem 2.16 are not well defined.

Chapter 3

Permutations

For $n \in \mathbb{N}$, we define $\boxed{[n] := \{1, 2, \ldots, n\}}$. That is, $[n]$ is just clever shorthand for the set containing 1 through n. This notation is meant to resemble interval notation.

For $k \in \mathbb{N}$ and a nonempty set A, a **k-permutation** of A is an injective function $w : [k] \to A$. The set of all k-permutations of A is denoted by $\boxed{S_{A,k}}$. If A happens to be the set $[n]$, we use the notation $\boxed{S_{n,k}}$. And if $n = k$, we write $\boxed{S_n := S_{n,n}}$ and refer to each n-permutation in S_n as a **permutation**. Let $\boxed{P(n,k) := |S_{n,k}|}$. By convention, we set $\boxed{P(n,0) := 1}$, including the case when $n = 0$.

Problem 3.1. Complete the following.

(a) Write down all of the elements in S_3. What is $P(3,3)$?

(b) Write down all of the elements in $S_{4,3}$. What is $P(4,3)$?

Recall that for $n \in \mathbb{N}$, the **factorial** of n is defined $\boxed{n! := n \cdot (n-1) \cdots 2 \cdot 1}$, and we define $\boxed{0! := 1}$ for convenience.

Problem 3.2. Consider the collection of k-permutations in $S_{n,k}$ with $1 \leq k \leq n$. Explain why $P(n,k)$ is equal to the number of nonattacking rook arrangements on an $n \times k$ chess board. *Hint:* Establish a bijection between the collection of nonattacking rook arrangements on an $n \times k$ chess board and the collection of k-permutations.

Theorem 3.3. For $1 \leq k \leq n$, we have

$$\boxed{P(n,k) = n \cdot (n-1) \cdots (n+1-k) = \frac{n!}{(n-k)!}.}$$

Note that as a special case of the formula above, we have $|S_n| = P(n,n) = n!$ and we obtain

$$\boxed{P(0,0) = \frac{0!}{(0-0)!} = 1} \quad \text{and} \quad \boxed{P(n,0) = \frac{n!}{(n-0)!} = 1.}$$

We can think of a k-permutation as a linearly ordered arrangement (i.e., string) of k of n objects. That is, we can denote a k-permutation as a string $w = w(1)w(2)\cdots w(k)$, where each $w(i) \in [n]$ and $w(i) \neq w(j)$ for $i \neq j$. For example, if $n = 7$ and $k = 4$, then the string 7142 represents the 4-permutation $w : [4] \to [7]$ given by

$$w(1) = 7, w(2) = 1, w(3) = 4, w(4) = 2.$$

In the case when $n = k$, we can denote a permutation as a string $w = w(1)w(2)\cdots w(n)$, where each entry $w(i)$ appears once. For example, the string $w = 241365$ represents the bijection $w : [6] \to [6]$ given by

$$w(1) = 2, w(2) = 4, w(3) = 1, w(4) = 3, w(5) = 6, w(6) = 5.$$

Problem 3.4. How many strings of length three are there using letters from $\{a, b, c, d, e, f, g\}$ if the letters in the string are not repeated?

Problem 3.5. There are 8 finalists at the Olympic Games 100 meters sprint. Assume there are no ties.

(a) How many ways are there for the runners to finish?

(b) How many ways are there for the runners to get gold, silver, bronze?

(c) How many ways are there for the runners to get gold, silver, bronze given that Usain Bolt is sure to get the gold medal?

Problem 3.6. If $1 \leq k \leq n$, prove that $\boxed{P(n, k) = P(n-1, k) + kP(n-1, k-1)}$, both using the formula in Theorem 3.3, and separately using the definition of k-permutations together with Product and Sum Principles. The latter approach is an example of a **combinatorial proof**.

The formula in the previous problem is an example of a **recurrence relation**, which will be a topic of focus in a later chapter.

Interpreting a permutation as a linearly ordered arrangement of object (i.e., string), a **circular permutation** is similar to a permutation except the objects are arranged on a circle, so that there is no beginning or end. We can present a circular permutation w of length n as in Figure 3.1. Each $w(i)$ is a distinct value from $[n]$, and reading clockwise we encounter $w(1), w(2), \ldots, w(n)$, so that $w(n)$ is placed next to $w(1)$. Any circular rotation yields the same circular permutation.

We encountered circular permutations back in Problem 1.39 when we counted circular seating arrangements of six friends sitting around a circle to play a game. Recall that the trick in that problem was to make use of the Division Principle.

Problem 3.7. How many circular permutations are there of length n?

Moving away from circular permutations and back to k-permutations, recall that we can represent each k-permutation of $[n]$ as a string of length k, where each entry is from $[n]$ and no repeats are allowed. What if we allow repeats?

Problem 3.8. How many ways can the letters of the word PRESCOTT be arranged?

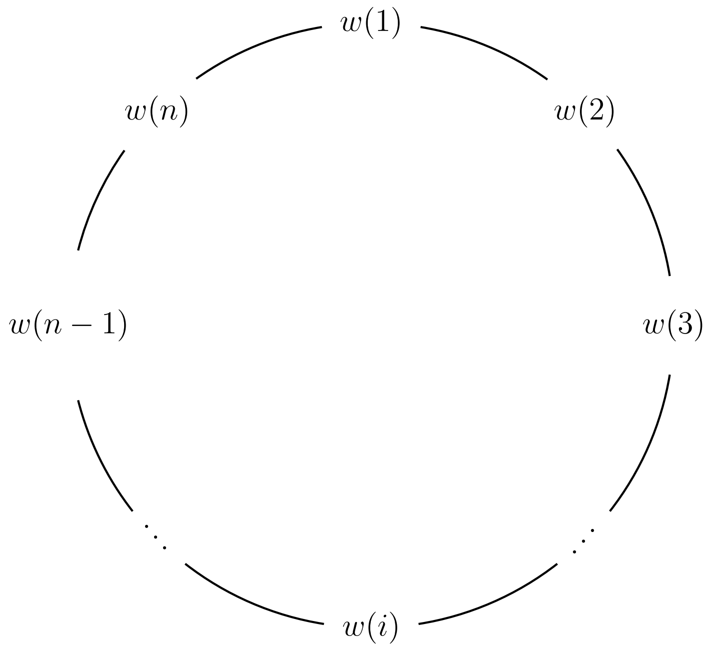


Figure 3.1: Representation of a circular permutation.

Problem 3.9. How many ways can the letters of the word POPPY be arranged? Try to solve this problem in two different ways.

Consider a set of n objects that are not necessarily distinct, with p different types objects and n_i objects of type i (for $i = 1, 2, \ldots, p$), so that $n = n_1 + \cdots + n_p$. An ordered arrangement of these n objects is called a **generalized permutation** and the number of such arrangements is denoted by $\boxed{P(n; n_1, \ldots, n_p)}$. For example, the number of words we can make out of the letters of POPPY is $P(5; 3, 1, 1)$. The following theorem follows immediately from the Division Principle.

Theorem 3.10. For $n, n_1, \ldots, n_p \in \mathbb{N}$ such that $n = n_1 + \cdots + n_p$, we have

$$\boxed{P(n; n_1, \ldots, n_p) = \frac{n!}{n_1! \cdots n_p!}.}$$

Problem 3.11. How many ways can the letters of the word MISSISSIPPI be arranged?

Problem 3.12. In Professor X's class of 9 graduate students she will give two A's, one B, and six C's. How many possible ways are there to do this?

Problem 3.13. Let's revisit Problem 1.15, which involved my walk to get coffee. When we attacked that problem, we did a lot of brute force. Do we now have an easier method?

Problem 3.14. In how many ways can a deck of 52 cards be dealt to four players, say N, E, S, and W?

Chapter 4

Combinations

The notion of k-permutations captures arrangements of distinct objects where order matters. But what should we do if we want to capture a situation where the order of the objects does not matter? Since the order of the objects in a set does not matter, this is the model we should use.

If A is a set and $B \subseteq A$ with $|B| = k$, we refer to B as a **k-subset** of A. The collection of all k-subsets of A is defined via

$$\boxed{\binom{A}{k} := \{B \subseteq A \mid |B| = k\}.}$$

The **binomial coefficient** is defined via

$$\boxed{\binom{n}{k} := \text{number of } k\text{-subsets of an } n\text{-element set.}}$$

In particular, if $|A| = n$, then $|\binom{A}{k}| = \binom{n}{k}$. We read "$\binom{n}{k}$" as "$n$ choose k". Alternate notations for binomial coefficients include $C(n,k)$ and ${}_nC_k$. We will see later why $\binom{n}{k}$ is referred to as a binomial coefficient.

Example 4.1. If $A = \{a, b, c, d\}$, then

$$\binom{A}{2} = \{\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}\},$$

which implies that $\binom{4}{2} = 6$.

Problem 4.2. For any A, including the empty set, what is $\binom{A}{0}$? For $n \geq 0$, what is $\binom{n}{0}$ equal to?

Problem 4.3. For $n \geq 0$, what is $\binom{n}{n}$ equal to?

If we let n and k vary, we can organize the binomial coefficients in a triangular array, often referred to as **Pascal's Triangle**. See Table 4.1.

$n \setminus k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Table 4.1: Pascal's Triangle of binomial coefficients.

Problem 4.4. Suppose you have a pool of 6 applicants for a job opening. Let's assume you believe the values in Table 4.1.

(a) How many ways can you choose 3 of the 6 applicants to interview?

(b) How many ways can you hire 3 of the 6 applicants for 3 distinct jobs?

Problem 4.5. What are the row sums in Pascal's Triangle? That is, what is the following sum equal to for any $n \geq 0$?

$$\sum_{k=0}^{n} \binom{n}{k} := \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

Problem 4.6. Using the meanings of k-subset and k-permutation, explain why

$$P(n,k) = \binom{n}{k} \cdot k!.$$

Using the previous problem, we immediately get the following handy formula for computing binomial coefficients.

Theorem 4.7. For $0 \leq k \leq n$, we have

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{P(n,k)}{k!}.}$$

In the last expression above, the numerator of $\frac{P(n,k)}{k!}$ is counting how many distinct arrangements (order matters) there are of k objects taken from n objects and the denominator is essentially unordering arrangements (by Division Principle) that consist of the same objects.

Problem 4.8. A state senate consists of 19 Republicans and 14 Democrats. In how many ways can a committee be chosen if:

(a) The committee contains 6 senators without regard to party?

(b) The committee contains 3 Republicans and 3 Democrats?

Problem 4.9. How many bit strings of length 10 have exactly three 1's?

Problem 4.10. How many bit strings of length 6 have an odd number of 0's?

Problem 4.11. As we noted earlier, we did quite a bit of brute force to determine how many paths I could take to get coffee in Problem 1.15. Find a solution that utilizes binomial coefficients.

Problem 4.12. How many strings of 10 lower-case English letters have exactly two g's and exactly three v's?

Problem 4.13. Assume $1 \leq k \leq n$.

(a) Using the definition of $\binom{n}{k}$ in terms of k-subsets (as opposed to the formula in Theorem 4.7), explain why

$$\boxed{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.}$$

This identity is often called **Pascal's Identity** (or **Pascal's Recurrence**).

(b) Connect the formula above with Problem 1.15 involving my walk to get coffee.

Problem 4.14. Assume $1 \leq k \leq n$. It turns out that

$$\boxed{\binom{n}{k} = \binom{n}{n-k}.}$$

(a) Prove the identity above using the formula for $\binom{n}{k}$ given in Theorem 4.7.

(b) Explain why the identity is true by using the definition of $\binom{n}{k}$ in terms of k-subsets.

The upshot is that each row of Pascal's Triangle is a palindrome.

Problem 4.15. Explain why

$$\boxed{1 + 2 + \cdots + n = \binom{n+1}{2}}$$

by counting the number of handshakes that could occur among a group of $n+1$ people in two different ways.

By the way, the number defined by $\boxed{t_n := 1 + 2 + \cdots + n}$ is called the nth **Triangular number** (due to the shape we get by representing each number in the sum by a stack of balls).

Problem 4.16. Consider the linear equation $x_1 + x_2 + x_3 = 11$. How many *integer* solutions are there if:

(a) $x_1, x_2, x_3 \geq 0$?

(b) $x_1, x_2, x_3 > 0$?

(c) $x_1 \geq 1, x_2 \geq 0, x_3 \geq 2$?

Problem 4.17. How many ways can you distribute 5 identical lollipops to 6 kids?

These last two problems illustrate a technique known as **stars and bars**. In general, n stars tally the number of objects and $k-1$ bars separate them into k distinct categories.

Theorem 4.18. The number of possible collections of n objects of k different types is

$$\boxed{\binom{n+k-1}{k-1} = \binom{n+k-1}{n}.}$$

Problem 4.19. Zittles come in 5 colors: green, yellow, red, orange, and purple. How many different collections of 32 Zittles are possible?

Chapter 5

The Binomial Theorem

Recall that $\binom{n}{k}$ counts the number of subsets of size k taken from a set of size n. Each $\binom{n}{k}$ is called a binomial coefficient, which likely seems like a strange name. Why are these numbers called binomial coefficients? In general a binomial is just a polynomial with two terms. Let's see what we can discover.

We see that

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= xx+xy+yx+yy\\ &= x^2+2xy+y^2.\end{aligned}$$

The coefficients in this expansion are $1, 2, 1$. Hey, we saw those in row $n = 2$ of Pascal's Triangle! Let's try a larger example. We see that

$$\begin{aligned}(x+y)^3 &= (x+y)^2(x+y)\\ &= (xx+xy+yx+yy)(x+y)\\ &= xxx+xyx+yxx+yyx+xxy+xyy+yxy+yyy\\ &= x^3+3x^2y+3xy^2+y^3.\end{aligned}$$

This time the coefficients are $1, 3, 3, 1$, which is the next row of Pascal's Triangle. What's going on here?

Consider the expansion of $(x+y)^n$. The key observation is that *before* commuting factors and collecting like terms, the terms in the expansion consist of *all possible* products where we choose either x or y from each factor. Each such term will consist of n letters. In particular, if there are k copies of x in a term, there will be $n-k$ copies of y (and vice versa). Moreover, every possible configuration of k copies x (i.e., location of the x's in the term *before* doing any commuting with x and y) will be represented. This means there will be precisely $\binom{n}{k}$ many terms with k copies of x (and $n-k$ copies of y). Thus, when we commute and then collect like terms, the coefficient on $x^k y^{n-k}$ will indeed be $\binom{n}{k}$. This leads us to the following remarkable fact, known as the **Binomial Theorem**.

Theorem 5.1 (The Binomial Theorem)**.** For $n \geq 0$, we have

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.}$$

In light of the Binomial Theorem, the binomial coefficients are the positive integers that occur as coefficients in the expansions of powers of binomials. Said another way, the coefficients in the expansion of $(x+y)^n$ correspond to the entries in the nth row of Pascal's Triangle.

Problem 5.2. Expand each of the following.

(a) $(a+b)^6$

(b) $(2x-3y)^4$

Problem 5.3. What is the coefficient of x^{12} in $(x+2)^{15}$?

Problem 5.4. What is the coefficient of x^6y^6 in $(x^3+2y^2)^5$?

Problem 5.5. In Problem 4.5, we discovered the following combinatorial identity:

$$2^n = \sum_{k=0}^{n} \binom{n}{k} := \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

Prove this fact using the Binomial Theorem.

Problem 5.6. Determine what the sum $\displaystyle\sum_{k=0}^{n} 2^n \binom{n}{k}$ is equal to.

Problem 5.7. Complete each of the following.

(a) Expand $(1+t)^n$.

(b) Assuming $n \geq 1$, take the derivative with respect to t of each side of the identity you discovered in Part (a).

(c) What happens if $t = 1$?

Problem 5.8. Suppose we have $n \geq 1$ people.

(a) Let $1 \leq k \leq n$. How many ways can we choose a team of k people to play dodgeball where one of the people on the team is designated team captain?

(b) How many ways can we choose of team of any size from 1 up to n where one of the people on the team is designated team captain?

Chapter 6

Pigeonhole Principle

The **Pigeonhole Principle** is a very natural idea. It says: If a collection of at least $n + 1$ objects is put into n boxes, then there is a box with at least two things in it. The Pigeonhole Principle has surprisingly deep applications. We will start with a few examples.

Example 6.1. Back in Problem 2.11, we implicitly used the Pigeonhole Principle when we argued that if $f : A \to B$ is a function for finite sets A and B, then

(a) If f is an injection, then $|A| \leq |B|$.

(b) If f is a surjection, then $|A| \geq |B|$.

Problem 6.2. A box has blue, green, yellow, red, orange, and white balls. How many must be drawn without looking to be sure of getting at least two of the same color?

Problem 6.3. Prove that if seven distinct numbers are selected from $\{1, 2, \ldots, 11\}$, then some two of these numbers sum to 12.

We would like to generalize the Pigeonhole Principle, but first we need a useful function. The **ceiling function** of a real number x, written $\boxed{\lceil x \rceil}$, is the smallest integer greater than or equal to x. That is, $\lceil x \rceil$ is an integer, $\lceil x \rceil \geq x$, and there is no other integer between $\lceil x \rceil$ and x. You can think of it as the "round-up to an integer" function.

Example 6.4. For example, $\lceil \pi \rceil = 4$, $\lceil -\pi \rceil = -3$, and $\lceil 7 \rceil = 7$.

We can now generalize the Pigeonhole Principle as follows.

Theorem 6.5 (Generalized Pigeonhole Principle)**.** If n objects are placed in m boxes, then there is a box with at least $\lceil \frac{n}{m} \rceil$ objects.

Problem 6.6. If 20 buses seating at most 50 carry 621 passengers to a ball game, then some bus must have at least __________ passengers.

Problem 6.7. How many balls must be drawn from the box in Problem 6.2 in order to be sure of getting at least 4 of the same color?

Problem 6.8. Explain why every collection of ten distinct integers $x_1, x_2, \ldots, x_{10}$ must have at least one subset whose sum of the elements in the subset is divisible by 10.

Chapter 7

Principle of Inclusion and Exclusion

We now introduce a concept known as the **Principal of Inclusion and Exclusion**. Recall Theorem 1.14, which states that if A and B are sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Problem 7.1. How many integers between 1 and 881 inclusively are divisible by 3 or 5?

But what do we do if we have more than two sets? Let's first examine the situation with three sets.

Problem 7.2. If A, B, and C are sets, then find a formula in the same vein as Theorem 1.14 for $|A \cup B \cup C|$.

The upshot is that we add "singles" subtract "doubles" and add "triples".

Problem 7.3. In the Natteranian township, 750 of the residents have a smart phone, 620 have a laptop computer, 480 have a desktop computer, 420 have both a laptop and a smart phone, 390 have both a smart phone and a desktop, 212 have both a laptop and a desktop computer and 164 have all three items.

(a) How many residents have at least one of the three items?

(b) How many residents do not have desktop computer?

(c) How many residents have a smart phone or a laptop?

(d) How many have a smart phone or a laptop but not a desktop?

We can generalize to any finite number of sets.

Theorem 7.4 (Principal of Inclusion and Exclusion)**.** The number of elements in the union of finite sets $A_1, A_2, \ldots, A_n$ is

$$|A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cdots \cap A_n|.$$

Problem 7.5. How many nonnegative integer solutions does the equation $x_1+x_2+x_3+x_4 = 25$ have such that $x_1 < 7$, $x_2 < 5$, and $x_4 < 8$?

We now discuss one important application of the Principle of Inclusion and Exclusion. Formally, a **derangement** is a permutation $w : [n] \to [n]$ such that $w(i) \neq i$ for all $1 \leq i \leq n$ (i.e., w has no fixed points). That is, a derangement is a special rearrangement of objects such that none is in its original spot.

Problem 7.6. How many derangements of CAT are there?

Let d_n denote the number of derangements of $[n]$. We set $d_0 := 1$.

Problem 7.7. For $1 \leq i \leq n$, let F_i be the set of permutations that fix i.

(a) Explain why
$$d_n = |F_1^c \cap \cdots \cap F_n^c|.$$

(b) Explain why
$$d_n = n! - \sum_i |F_i| + \sum_{i<j} |F_i \cap F_j| - \sum_{i<j<k} |F_i \cap F_j \cap F_k| + \cdots + (-1)^n |F_1 \cap \cdots \cap F_n|.$$

(c) Explain why the number of derangements of $[n]$ is
$$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Problem 7.8. Using the previous problem, verify that we got the right answer to Problem 7.6.

Problem 7.9. If 7 hats are left at the hat-check window, in how many ways can they be returned so that no one gets the correct hat?

Now, just for funsies... from second semester calculus, we know
$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$
which implies that
$$e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$
Using Part (c) of Problem 7.7, we see that
$$\lim_{n\to\infty} \frac{d_n}{n!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \frac{1}{e} \approx 0.367879.$$
In other words, when n is large, the probably of selecting a derangement at random from the collection of permutations of n is approximately $1/e$. As n increases, the approximation improves. Boom.

Chapter 8

Mathematical Induction

In this chapter, we introduce **mathematical induction**, which is a proof technique that is useful for proving statements of the form "For all natural numbers n, $P(n)$", or more generally "For all integers $n \geq a$, $P(n)$", where $P(n)$ is some predicate. Loosely speaking, a predicate $P(n)$ is some statement about n. For example, "n is prime" is a predicate.

Consider the claims:

(a) For all $n \in \mathbb{N}$, $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

(b) For all $n \in \mathbb{N}$, $n^2 + n + 41$ is prime.

Let's take a look at potential proofs.

"Proof" of (a). If $n = 1$, then $1 = \frac{1(1+1)}{2}$. If $n = 2$, then $1 + 2 = 3 = \frac{2(2+1)}{2}$. If $n = 3$, then $1 + 2 + 3 = 6 = \frac{3(3+1)}{2}$, and so on. $\square$

"Proof" of (b). If $n = 1$, then $n^2+n+41 = 43$, which is prime. If $n = 2$, then $n^2+n+41 = 47$, which is prime. If $n = 3$, then $n^2 + n + 41 = 53$, which is prime, and so on. $\square$

Are these actual proofs? No! In fact, the second claim is not even true. If $n = 41$, then $n^2 + n + 41 = 41^2 + 41 + 41 = 41(41 + 1 + 1)$, which is not prime since it has 41 as a factor. It turns out that the first claim is true, but what we wrote cannot be a proof since the same type of reasoning when applied to the second claim seems to prove something that is not actually true. We need a rigorous way of capturing "and so on" and a way to verify whether it really is "and so on."

We will not formally prove the following theorem, but instead rely on our intuition.

Theorem 8.1 (Principle of Mathematical Induction)**.** Let $P(1), P(2), P(3), \ldots$ be a sequence of statements, one for each natural number. Assume

(i) $P(1)$ is true, and

(ii) for all $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

The Principle of Mathematical Induction (or PMI for short) provides us with a process for proving statements of the form "For all natural numbers n, $P(n)$," where $P(n)$ is some predicate involving n. Hypothesis (i) above is called the **base step** (or **base case**) while (ii) is called the **inductive step**.

Intuitively, here is what the Principle of Mathematical Induction is saying. Think of the statements $P(1), P(2), P(3), \ldots$ as being rungs of a ladder. The base step indicates that we can step onto the first rung of the ladder while the inductive step tells us that if we are on a rung of the ladder we can always move up to the next rung. The Principle of Mathematical Induction asserts that if we can achieve these two things, then we can climb the entire infinite ladder. Do you agree that this seems reasonable?

You should not confuse *mathematical induction* with *inductive reasoning* associated with the natural sciences. Inductive reasoning is a scientific method whereby one induces general principles from observations. On the other hand, mathematical induction is a deductive form of reasoning used to establish the validity of a proposition.

Here is the general structure for a proof by induction.

Proof. We proceed by induction.

(i) **Base step:** *[Verify that $P(1)$ is true. This often, but not always, amounts to plugging $n = 1$ into two sides of some claimed equation and verifying that both sides are actually equal.]*

(ii) **Inductive step:** *[Your goal is to prove "For all $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true."]* Let $k \in \mathbb{N}$ and assume that $P(k)$ is true. *[Do something to derive that $P(k+1)$ is true.]* Therefore, $P(k+1)$ is true.

Thus, by induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

Problem 8.2. Conjecture a "nice" formula for the following summation and then prove your claim using induction:

$$\sum_{i=1}^{n}(2i-1) := 1 + 3 + 5 + \cdots + (2n-1).$$

Independent of induction, can you think of a nice visual proof of this result?

Problem 8.3. Prove the first claim that we introduced at the beginning of the chapter using induction. That is, prove that for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Problem 8.4. Prove that for all $n \in \mathbb{N}$, 3 divides $4^n - 1$.

Problem 8.5. Consider a grid of squares that is 2^n squares wide by 2^n squares long, where $n \in \mathbb{N}$. One of the squares has been cut out, but you do not know which one! You have a bunch of L-shapes made up of 3 squares. Prove that you can perfectly cover this chessboard with the L-shapes (with no overlap) for any $n \in \mathbb{N}$. Figure 8.1 depicts one possible covering for the case involving $n = 2$ and a fixed cut-out square.

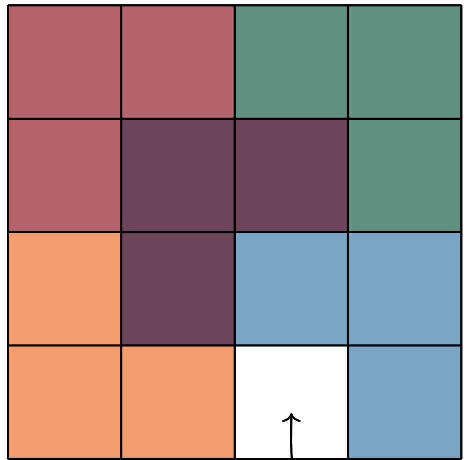


Figure 8.1: One possible covering for the case involving $n = 2$ for Problem 8.5.

Mathematical induction can actually be used to prove a broader family of results; namely, those of the form "For all integers $n \geq a$, $P(n)$", where $a \in \mathbb{Z}$. Theorem 8.1 handles the special case when $a = 1$. The ladder analogy from earlier holds for this more general situation, too.

Theorem 8.6 (Generalized PMI)**.** Let $P(a), P(a+1), P(a+2), \ldots$ be a sequence of statements, one for each integer greater than or equal to a. Assume that

(i) $P(a)$ is true, and

(ii) for all $k \geq a$, if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for all integers $n \geq a$.

Here is the general structure for a proof by induction when the base case does not necessarily involve $a = 1$.

Proof. We proceed by induction.

(i) **Base step:** *[Verify that $P(a)$ is true. This often, but not always, amounts to plugging $n = a$ into two sides of some claimed equation and verifying that both sides are actually equal.]*

(ii) **Inductive step:** *[Your goal is to prove "For all $k \geq a$, if $P(k)$ is true, then $P(k+1)$ is true."]* Let $k \geq a$ be an integer and assume that $P(k)$ is true. *[Do something to derive that $P(k+1)$ is true.]* Therefore, $P(k+1)$ is true.

Thus, by induction, $P(n)$ is true for all integers $n \geq a$. $\square$

We already encountered the next result back Problem 1.31, but let's see if we can use induction to prove it.

Problem 8.7. Use induction to prove that if A is a finite set with n elements, then A has 2^n subsets.

Problem 8.8. Determine when $n + 1 < n^2$ for integer values and prove the claim using mathematical induction.

Problem 8.9. Determine when $n^2 < 2^n$ for integer values and prove the claim using mathematical induction.

There is another formulation of induction, where the inductive step begins with a set of assumptions rather than one single assumption. This method is sometimes called **complete induction** or **strong induction**.

Theorem 8.10 (Principle of Complete Mathematical Induction)**.** Let $P(1), P(2), P(3), \ldots$ be a sequence of statements, one for each natural number. Assume that

(i) $P(1)$ is true, and

(ii) For all $k \in \mathbb{N}$, if $P(j)$ is true for all $j \in \mathbb{N}$ such that $j \leq k$, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Note the difference between ordinary induction (Theorems 8.1 and 8.6) and complete induction. For the induction step of complete induction, we are not only assuming that $P(k)$ is true, but rather that $P(j)$ is true for all j from 1 to k. Despite the name, complete induction is not any stronger or more powerful than ordinary induction. It is worth pointing out that anytime ordinary induction is an appropriate proof technique, so is complete induction. So, when should we use complete induction?

In the inductive step, you need to reach $P(k+1)$, and you should ask yourself which of the previous cases you need to get there. If all you need is the statement $P(k)$, then ordinary induction is the way to go. If two preceding cases, $P(k-1)$ and $P(k)$, are necessary to reach $P(k+1)$, then complete induction is appropriate. In the extreme, if one needs the full range of preceding cases (i.e., all statements $P(1), P(2), \ldots, P(k)$), then again complete induction should be utilized.

Note that in situations where complete induction is appropriate, it might be the case that you need to verify more than one case in the base step. The number of base cases to be checked depends on how one needs to "look back" in the induction step.

Here is the general structure for a proof by complete induction, where the base case includes $n = 1$ and possibly more.

Proof. We proceed by complete induction.

(i) **Base step:** *[Verify that $P(1)$ is true. Depending on the statement, you may also need to verify that $P(k)$ is true for other specific values of k.]*

For all $k \in \mathbb{N}$, if $P(j)$ is true for all $j \in \mathbb{N}$ such that $j \leq k$, then $P(k+1)$ is true

(ii) **Inductive step:** *[Your goal is to prove "For all $k \in \mathbb{N}$, if $P(j)$ is true for all $j \in \mathbb{N}$ such that $j \leq k$, then $P(k+1)$ is true."]* Let $k \in \mathbb{N}$ *[You may need to assume k is larger than the number of bases cases you verified]*. Suppose $P(j)$ is true for all $j \leq k$. *[Do something to derive that $P(k+1)$ is true.]* Therefore,

$P(k+1)$ is true.

Thus, by complete induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

When tackling the remaining problems in this chapter, think carefully about how many base steps you must verify.

Problem 8.11. The **Fibonacci sequence** is given by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all natural numbers $n \geq 3$. Prove that $f_n < 2^n$ for all $n \in \mathbb{N}$.

Recall that Theorem 8.6 generalized Theorem 8.1 and allowed us to handle situations where the base case was something other than $P(1)$. We can generalize complete induction in the same way, but we will not write this down as a formal theorem.

Problem 8.12. Prove that every amount of postage that is at least 12 cents can be made from 4-cent and 5-cent stamps.

Problem 8.13. Consider a grid of squares that is 2 squares wide and n squares long. Using n dominoes that are 1 square by 2 squares, there are many ways to perfectly cover this grid with no overlap. How many? Prove your answer.

One final thing worth mentioning is that we did not write down a rigorous proof of the Principle of Inclusion and Exclusion, which we encountered in Chapter 7. However, this omission could be remedied using induction. That is, using induction, we could prove that for all $n \in \mathbb{N}$ and finite sets $A_1, \ldots, A_n$, we have

$$|A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cdots \cap A_n|.$$

If $n = 1$, the expression above simply says $|A_1| = |A_1|$, which is certainly true. For $n = 2$, we have

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|,$$

which is the General Sum Principle that we encountered in Theorem 1.14. The inductive step is a bit "messy", so we will omit it, but if you are interested, you can find a complete proof here.

Chapter 9

Sequences and Recurrence Relations

In this chapter we will study sequences of numbers that are built recursively. Technically, a **sequence** (of real numbers) is a function a from $\mathbb{N}$ to $\mathbb{R}$. If $n \in \mathbb{N}$, it is common to write $a_n := a(n)$. We refer to a_n as the nth **term** of the sequence. We will abuse notation and associate a sequence with its list of outputs, namely:

$$\boxed{(a_n)_{n=1}^{\infty} := (a_1, a_2, \ldots)},$$

which we may abbreviate as $\boxed{(a_n)}$. Sometimes we may start our sequences at $n = 0$ as opposed to $n = 1$. That is, we may allow the domain of a sequence to be $\mathbb{N} \cup \{0\}$.

Example 9.1. Define $a : \mathbb{N} \to \mathbb{R}$ via $a_n = \frac{1}{2^n}$. Then we have

$$a = \left(\frac{1}{2}, \frac{1}{4}, \ldots\right) = \left(\frac{1}{2^n}\right)_{n=1}^{\infty}.$$

It is important to point out that not every sequence has a description in terms of an algebraic formula. For example, we could form a sequence out of the digits to the right of the decimal in the decimal expansion of π, namely the nth term of the sequence is the nth digit to the right of the decimal. But then there is no nice algebraic formula for describing the nth term of this sequence.

Loosely speaking, a sequence of numbers is defined recursively if the nth term of the sequence is defined in terms of "earlier" terms of the sequence. We have already encountered one famous example of a sequence being defined recursively, namely the Fibonacci sequence (f_n), which we defined by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. The equation $f_n = f_{n-1} + f_{n-2}$ is the **recurrence relation** while $f_1 = 1$ and $f_2 = 1$ are the **initial conditions**. It is important to emphasize that we cannot define the Fibonacci number using only the recurrence relation since otherwise, we would not be able to "get started" with the recurrence.

We have also encountered a few recurrence relations of a different flavor that arise out of two-dimensional arrays of numbers. For example:

(a) Number of k-permutations of $[n]$: For $1 \leq k \leq n$,

$$P(n, k) = P(n-1, k) + kP(n-1, k-1).$$

(b) Number of k-subsets of $[n]$: For $1 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

(c) Number of set partitions of $[n]$ with k blocks: For $1 \leq k \leq n$,

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\}.$$

Notice that each of the descriptions above are not sufficient without also providing a way to "get started". For the two-dimensional case, the initial conditions are often called **boundary conditions**. For the rest of this chapter, we will focus on one-dimensional sequences.

Here is an important general principle.

Theorem 9.2. If two sequences satisfy the same recurrence relation and initial conditions, then the two sequences must be equal.

Problem 9.3. Recall that a **composition** of n with k parts is an ordered list of k positive integers whose sum is n, denoted $\alpha = (\alpha_1, \ldots, \alpha_k)$. We say that α_i is the ith part.

(a) How many compositions of n have only odd parts?

(b) How many compositions of n have parts of size 1 and 2 only?

Problem 9.4. Prove that $f_{n+1} = \displaystyle\sum_{k \geq 0} \binom{n-k}{k}$ by utilizing one of the parts from Problem 9.3. What does this identity tell us about Pascal's Triangle?

Problem 9.5. For each of the following recursively defined sequences, generate the first few terms. If possible, find an explicit formula for the terms of the sequence.

(a) $a_1 = 2$, $a_n = a_{n-1} + 7$ for $n \geq 2$.

(b) $a_0 = 1$, $a_n = 2a_{n-1}$ for $n \geq 1$.

(c) $a_0 = 0$, $a_n = na_{n-1}$ for $n \geq 1$.

(d) $a_0 = 0$, $a_n = a_{n-1} + n$ for $n \geq 1$.

(e) $a_0 = 0$, $a_n = a_{n-1} + \displaystyle\sum_{i=0}^{n} (i + n)$ for $n \geq 1$.

By **solving** a recurrence relation together with its initial conditions we mean finding an explicit expression (sometimes called a **closed form**) for an arbitrary term a_n as a function of n (but no earlier terms of the sequence). The explicit expression for a_n is called the **solution** of the recurrence relation. For example, each time we found an explicit formula for the nth term of a sequence in the previous problem, we were solving the recurrence relation and the corresponding expression we found is the solution. By the **general solution** of a recurrence relation, we mean the set of its solutions given any initial conditions.

Problem 9.6. Find the general solution for $a_n = 2a_{n-1}$ if the first term of the sequence is a_0. What if the sequence starts at a_1?

It's important to point out that finding a solution to a recurrence relation can be quite complicated, maybe even impossible! However, verifying whether a proposed solution is correct or not is straightforward.

Problem 9.7. Consider the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$. Is $a_n = (-2)^n$ a solution? How about $a_n = 3^n$? How about $a_n = 5(-2)^n + 7 \cdot 3^n$?

We now turn our attention to two special classes of recurrence relations. An **arithmetic progression** is a recurrence relation in which the first term a_0 (or a_1) and a **common difference** d are given and the corresponding recurrence relation is

$$\boxed{a_n = a_{n-1} + d.}$$

A **geometric progression** is a recurrence relation in which the first term a_0 (or a_1) and **common ratio** r are given and the corresponding recurrence relation is

$$\boxed{a_n = r \cdot a_{n-1}.}$$

Problem 9.8. Compute the first few terms of each of the following and find the solution.

(a) Arithmetic progression with $a_0 = 3$ and $d = 2$.

(b) Geometric progression with $a_0 = 3$ and $r = 2$.

Problem 9.9. Conjecture a solution to an arithmetic progression with first term a_0 and common difference d. Can you prove that your conjecture is correct?

Problem 9.10. Conjecture a solution to a geometric progression with first term a_0 and common ratio r. Can you prove that your conjecture is correct?

Problem 9.11. Recall that the **triangular numbers** are defined via $t_n := 1 + 2 + \cdots + n$. The first few terms of this sequence are 1, 3, 6, 10, 15.

(a) Express the triangular numbers using a recurrence relation and initial condition.

(b) Is this sequence an arithmetic progression? Geometric progression?

(c) Notice that the sequence of triangular numbers is a sequence of partial sums of the arithmetic sequence $1, 2, 3, \ldots$. What happens if we add the partial sum expression for t_n to a second copy of t_n written in reverse? Can you recover the nice closed form for t_n we are already familiar with?

We can generalize the technique above for any sequence that is given by partial sums of an arithmetic sequence.

Problem 9.12. For $n \geq 2$, define $a_n = 6 + 10 + 14 + \cdots + (4n-2) = \sum_{i=2}^{n}(4i-2)$. Find a closed form for a_n.

What about sequences that are partial sums of geometric progressions? In this case, it turns out that we can multiply by the common ratio, shift, and subtract.

Problem 9.13. For $n \geq 0$, define $a_n = 3^0 + 3^1 + \cdots + 3^n = \displaystyle\sum_{i=0}^{n} 3^i$. Find a closed form for a_n.

A **linear constant-coefficient** recurrence relation of **order** r has the form

$$\boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r} + f(n),}$$

where $c_1, c_2, \ldots, c_r$ are real numbers with $c_r \neq 0$. Such a recurrence relation is said to be **homogeneous** if $f(n) = 0$, so that it can be written as

$$\boxed{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r},}$$

and is **non-homogeneous** otherwise.

Notice that every arithmetic progression and every geometric progression is a first order linear constant-coefficient recurrence relation. In particular, each geometric progression is homogeneous while each arithmetic progression is non-homogeneous.

Problem 9.14. Determine which of the following are linear constant-coefficient recurrence relations. For those that are, which are homogeneous and which are non-homogeneous?

(a) $a_n = n a_{n-1}$

(b) $a_n = a_{n-1} + d$

(c) $a_n = c a_{n-1}$

(d) $a_n = a_{n-1} + a_{n-2}$

(e) $a_n = \sum_{i=0}^{n-1} a_i a_{n-1-i}$

(f) $a_n = a_{n-1} - 4a_{n-2} + 7a_{n-3}$

(g) $a_n = a_{n-1}^2 + 7a_{n-2} + 2a_{n-6}$

Problem 9.15. Solve the first-order linear constant-coefficient non-homogeneous recurrence relation $a_n = 3a_{n-1} + 2$ with initial condition $a_0 = 1$.

Unfortunately, the technique of the previous example is difficult to generalize to higher orders.

The next theorem characterizes the phenomenon that we witnessed in Problem 9.7. This theorem can be proved by direct substitution and some algebraic manipulation.

Theorem 9.16 (Principle of Superposition)**.** If $s_1(n), \ldots, s_k(n)$ are solutions to the linear constant-coefficient homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r}$$

and $\alpha_1, \ldots, \alpha_k$ are real numbers, then the linear combination $\alpha_1 s_1(n) + \cdots + \alpha_k s_k(n)$ is also a solution.

We now focus on solving second-order linear constant-coefficient homogeneous recurrence relations. Given the second-order linear constant-coefficient homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2},$$

its corresponding **characteristic equation** is defined via

$$\boxed{x^2 - c_1 x - c_2 = 0.}$$

The solutions of the characteristic equation are called **characteristic roots**.

Example 9.17. The characteristic equation for the Fibonacci relation $f_n = f_{n-1} + f_{n-2}$ is $x^2 - x - 1 = 0$, which has characteristic roots $x = \frac{1\pm\sqrt{5}}{2}$. Note that the characteristic root $\frac{1+\sqrt{5}}{2} \approx 1.618$ is the well-known **golden ratio**.

We will utilize the following remarkable theorem without proving it.

Theorem 9.18. If r_1 and r_2 are two *distinct* characteristic roots (i.e., $r_1 \neq r_2$) of the characteristic equation for $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then the solution to the recurrence relation is

$$\boxed{a_n = a r_1^n + b r_2^n,}$$

where a and b are constants determined by the initial conditions.

Problem 9.19. Solve $a_n = a_{n-1} + a_{n-2}$ with initial conditions $a_0 = 0$ and $a_1 = 1$.

Problem 9.20. Use the previous problem to find a solution to the Fibonacci sequence given by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$. The closed form we just obtained for f_n is called **Binet's formula**.

Although we will not consider examples more complicated than these, this characteristic root technique can be applied to much more complicated recurrence relations.

We now turn our attention to one of my favorite sequences, which is defined by a recurrence relation of a different flavor. The **Catalan numbers** are defined via $c_0 = 1$ and

$$\boxed{c_n = \sum_{i=0}^{n-1} c_i c_{n-1-i}}$$

for $n \geq 1$. The equation above is called the **Catalan recurrence**. Using the initial condition and the Catalan recurrence, we can generate the first several terms of the Catalan sequence:

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786$$

There are hundreds of interesting combinatorial objects counted by the Catalan numbers! Let's explore a few.

Problem 9.21. A **Dyck path** of length $2n$ is a lattice path from $(0,0)$ to (n,n) that takes n steps East from (i,j) to $(i+1,j)$ and n steps North from (i,j) to $(i,j+1)$ such that all points on the path satisfy $i \leq j$. This sound more complicated that it really is. You can think of a Dyck path as one of our paths to get coffee that starts at $(0,0)$ and ends at (n,n) but never drops below the line $y = x$. Let $\text{Dyck}(n)$ denote set of all Dyck paths of length $2n$ and let $d_n := |\,\text{Dyck}(n)|$. We define $d_0 := 1$ for convenience. *Important:* Unfortunately, we also used d_n to denote the number of derangements of n. This problem is about Dyck path, not derangements.

(a) Compute d_1, d_2, d_3, and d_4 via brute force.

(b) Show that d_n satisfies the following recurrence for $n \geq 1$:

$$d_n = \sum_{i=0}^{n-1} d_i d_{n-1-i}.$$

Hint: Consider the collection of Dyck paths that hit the line $y = x$ at $(i+1, i+1)$ for the first time after leaving $(0,0)$. Think about how many ways you can draw the Dyck path to get to $(i+1, i+1)$ versus how many ways you can draw the Dyck path from $(i+1, i+1)$ to (n,n). The first case is trickier to think about. Notice that the portion of the Dyck path from $(0,0)$ to $(i+1, i+1)$ never hits the line $y = x$ along the way. Moreover, this portion necessarily starts with a North step and ends with an East step. What are the possible values for i?

Since d_n satisfies the same recurrence and initial conditions, it follows that $d_n = c_n$. That is, the number of Dyck paths is a Catalan number.

Problem 9.22. A sequence of parentheses is **balanced** if it can be parsed syntactically. In other words, there should be the same number of left parentheses "(" and right parentheses ")", and when reading from left to right there should never be more right parentheses than left. Here are the five balanced parenthesizations containing three pairs:

$$()()(), ()(()), (()()), (())(), ((())).$$

Prove that the number of balanced sequences of n pairs of parentheses is c_n. *Hint:* Use a bijection!

Problem 9.23. A **triangulation** of a convex $(n+2)$-gon is a dissection into n triangles using only lines from vertices to vertices. Think of the polygon as being fixed in space. Prove that the number of triangulations of a convex $(n+2)$-gon is c_n. Incidentally, this is the problem that Euler was interested in when he studied the Catalan numbers!

Let's see if we can find a closed form for the Catalan numbers!

Problem 9.24. Tackle each of the following.

(a) Argue that the number of lattice paths (not just Dyck paths) from $(0,0)$ to (n,n) is equal to $\binom{2n}{n}$.

(b) Argue that the number of lattice paths from $(0,0)$ to $(n+1, n-1)$ is equal to $\binom{2n}{n-1}$.

(c) Prove that there is a bijection from the set of lattice paths from $(0,0)$ to (n,n) that pass below $y = x$ at least once to the set of lattice paths from $(0,0)$ to $(n+1, n-1)$. *Hint:* Consider the first point on lattice path from $(0,0)$ to (n,n) that passes below $y = x$. Reflect the remaining portion of the path over the appropriate line to get a path from $(0,0)$ to $(n+1, n-1)$.

(d) Prove that $d_n = \binom{2n}{n} - \binom{2n}{n-1}$.

It is easy to verify that $\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$, and since $d_n = c_n$, we obtain

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Chapter 10

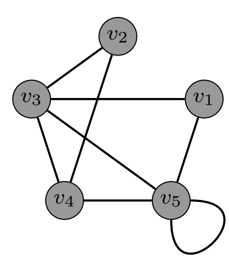
Introduction to Graph Theory

Loosely speaking, a graph is a collection of points called vertices and connecting segments called edges, each of which starts at a vertex, ends at a vertex and contains no other vertices beside these. More formally, we define the term as follows. A **graph** consists of two sets, a nonempty set V of points called **vertices** and a set E whose elements, called **edges**, are multisets of size two from V.

Each edge is associated with either one vertex which serves as both endpoints or two vertices as its endpoints. Technically, each edge is a multiset of the form $\boxed{\{\!\{u, v\}\!\}}$ where $u, v \in V$. We say that u and v are **endpoints** of the edge $\{\!\{u, v\}\!\}$. In an abuse of notation, it is customary to write $\boxed{\{u, v\}}$ even if $u = v$. In fact, we may abbreviate further and denote the edge by $\boxed{uv}$. Note that the order in which the vertices of an edge are listed is irrelevant. That is, $\{\!\{u, v\}\!\} = \{\!\{v, u\}\!\}$, $\{u, v\} = \{v, u\}$, and $uv = vu$. If G is the graph associated with the vertex set V and edge set E, we write $\boxed{G = (V, E)}$. It is worth pointing out that we assumed that V is nonempty, but E is allowed to be empty (i.e., the graph has no edges).

It is customary to represent a graph using visual representations, where each vertex is a dot and each edge is a connecting segment, not necessarily straight.

Example 10.1. Here is an example of a graph with vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$.



Problem 10.2. Find at least five different graphs with vertex sets $V = \{a, b, c\}$.

There is a lot of terminology associated to graphs! Here are some of the relevant concepts.

- Vertices u and v of a graph are **adjacent** if they are the endpoints of the same edge.
- If v is an endpoint of the edge e, we say that e is **incident** to v.

- If an edge e is incident to vertices u and v, we say that u and v are **connected** by edge e.
- An edge e that is incident to a single vertex (i.e., $e = uu$ for some $u \in V$) is called a **loop**.
- The **order** of a graph is the number of vertices in the graph. That is, if $G = (V, E)$, then the order of G is $|V|$.
- The **degree** of a vertex v, written $\deg(v)$, is the number of edges incident to v (i.e., the number of edges that have v as an endpoint). Note that a loop contributes 2 to a vertex's degree, one for each of the two ends of the edge. The degree of a vertex v is denoted $\boxed{\deg(v)}$.

Problem 10.3. Discuss each of the concepts introduced above in the context of Example 10.1.

Many graphs have similar properties that allow us to categorize them. Here are several families of graphs.

- Complete Graphs. The **complete graph** on $n \geq 1$ vertices, denoted $\boxed{K_n}$, is the graph of order n such that each pair of vertices is connected by exactly one edge, and there are no other edges (i.e., no loops).
- Cycle Graphs. The **cycle graph** on $n \geq 3$ vertices, denoted $\boxed{C_n}$, is the graph such that when the n vertices are suitably labeled $v_1, v_2, \ldots, v_n$, the edges of C_n are $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.
- Path Graphs. The **path** on $n \geq 1$ vertices, denoted $\boxed{P_n}$, has a description similar to C_n: for distinct vertices suitably labeled $v_1, v_2, \ldots, v_n$, the edges of P_n are $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$.
- Wheel Graphs. The **wheel graph** on $n \geq 4$ vertices, denoted $\boxed{W_n}$, is the graph C_{n-1} together with one additional vertex that is connected to each of the vertices of C_{n-1}.
- Hypercube Graphs. The **hypercube** of dimension $n \geq 1$, denoted $\boxed{Q_n}$, is the graph whose vertices are labeled with the 2^n bit strings of length n with an edge connecting two vertices if and only if their labels differ in exactly one bit.

Problem 10.4. Draw the first few graphs of each of the graph families above.

Problem 10.5. How many edges do each of the following have?

(a) K_n

(b) C_n

(c) P_n

(d) W_n

(e) Q_n

A **simple graph** is a graph in which each edge connects two distinct vertices and each pair of vertices is connected by at most one edge. Note that the graphs K_n, C_n, P_n, W_n, and Q_n are all simple graphs. A **pseudograph** (or **multigraph**) is like a graph but we allow **multiple edges** between a pair of vertices (i.e., E is a multiset instead of a set).

Problem 10.6. Draw examples of simple graphs, non-simple graphs, and psuedographs on 3 vertices.

A simple graph $G = (V, E)$ is **bipartite** if there is a partition of V into two nonempty sets V_1, V_2 (i.e., $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, $V_1 \cap V_2 = \emptyset$, and $V_1 \cup V_2 = V$) such that each edge of G connects a vertex in V_1 and a vertex in V_2. The pair (V_1, V_2) is called a **bipartition** of the graph.

Problem 10.7. Provide an example of a bipartite graph with 5 vertices.

The following theorem provides a nice characterization of bipartite graphs.

Theorem 10.8. A graph is bipartite if each vertex can be colored with one of two colors so that each pair of adjacent vertices have different colors.

Problem 10.9. Which complete graphs are bipartite?

Problem 10.10. Which path graphs are bipartite?

Problem 10.11. Which cycle graphs are bipartite?

Problem 10.12. Is Q_3 bipartite?

A bipartite graph with bipartition (V_1, V_2) such that $|V_1| = m$ and $|V_2| = n$ is the **complete bipartite graph** $K_{m,n}$ if it contains each edge $\{u, v\}$ for every pair $u \in V_1$ and $v \in V_2$. Note that $K_{m,n} = K_{n,m}$.

Problem 10.13. Draw $K_{1,1}$, $K_{1,2}$, $K_{2,2}$, $K_{2,3}$, $K_{3,3}$.

The next result is sometimes referred to as the **Handshake Lemma**. Do you see why?

Theorem 10.14 (Degree Sum Formula)**.** In any graph, the sum of the degrees of vertices in the graph is always twice the number of edges. In other words, in a graph $G = (V, E)$,

$$2|E| = \sum_{v \in V} \deg(v).$$

Problem 10.15. At a recent party, 9 people greeted each other by shaking hands. Is it possible that each person shook hands with exactly 7 people at the party?

Sometimes it is convenient to use the term **even vertex** or **odd vertex** to refer to a vertex whose degree is even or odd, respectively.

Problem 10.16. Explain why every graph has an even number of odd vertices.

The **degree sequence** of a graph is the list of the degrees of the vertices of the graph in descending order. A finite list of nonnegative integers in descending order is **graphic** if it is the degree sequence of a simple graph.

Problem 10.17. Find the degree sequences for K_n $(n \geq 1)$, C_n $(n \geq 3)$, P_n $(n \geq 1)$, W_n $(n \geq 4)$, and Q_n $(n \geq 1)$.

Problem 10.18. Which of the following are graphic sequences?

(a) 3332

(b) 3331

(c) 44332

Problem 10.19. Find two different graphs that have 32222111 as their degree sequence.

Theorem 10.20. If $d_1 d_2 \cdots d_n$ is the degree sequence for a graph G of order n, then $\sum_{i=1}^{n} d_i$ must be even.

One consequence of the previous theorem is that any sequence with an odd sum (e.g., 331) is not graphic. It turns out that if a sequence has an even sum, it is the degree sequence of a multigraph. The construction of such a graph is straightforward: connect vertices with odd degrees in pairs, and fill out the remaining even degree counts by self-loops. The question of whether a given degree sequence can be realized by a simple graph is more challenging. This problem is also called **graph realization problem** and can be solved by either the Erdös–Gallai theorem or the Havel–Hakimi algorithm. Unfortunately, this is beyond the scope of this course.

We will now focus on making new graphs from old. Below are several definitions.

- A graph $H = (V_H, E_H)$ is a **subgraph** of a graph $G = (V_G, E_G)$ if $V_H \subseteq V_G$ and $E_H \subseteq E_G$ (i.e., the vertices of H are vertices of G and the edges of H are edges of G). If H is a subgraph of G, we may write $\boxed{H \subseteq G}$.

- A graph $H = (V_H, E_H)$ is an **induced subgraph** of a graph $G = (V_G, E_G)$ if $V_H \subseteq V_G$ and E_H consists of all of the edges in E_G that have both endpoints in E_H. That is, for any two vertices $u, v \in V_H$, u and v are adjacent in H if and only if u and v are adjacent in G.

- If $G = (V, E)$ is a graph and $S \subseteq V$, then the **subgraph of G induced by S**, denoted $\boxed{G[S]}$, is the induced subgraph of G with vertex set S.

- The **union** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

- The **complement** of a *simple* graph G of order n is the graph $\boxed{\overline{G}}$ on the same n vertices such for each pair of distinct vertices u and v, $\{u, v\}$ is an edge of $\overline{G}$ if and only if it is not an edge of G.

Problem 10.21. Consider the graph C_4. Label the vertices clockwise as a, b, c, and d.

1. Find all induced subgraphs of C_4.
2. Find a subgraph of C_4 that is not an induced subgraph of C_4.

Problem 10.22. Make up a few examples to explore the concepts of union and complement of graphs.

Problem 10.23. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) Any subgraph of a complete graph is also complete.

(b) Any induced subgraph of a complete graph is also complete.

(c) Any subgraph of a bipartite graph is bipartite.

If $G = (V, E)$ is a graph and $S \subseteq V$, the **neighborhood** of S, denoted $\boxed{N(S)}$ is the set of all vertices in V adjacent to at least one member of S. Of course, we can consider the neighborhood of a single vertex v, which is denoted $\boxed{N(v)}$. A neighborhood of a single vertex does not include itself, and is more specifically the **open neighborhood** of v. It is also possible to define a neighborhood in which v itself is also included. This is called the **closed neighborhood** of v, sometimes denoted by $\boxed{N[v]}$. Otherwise stated otherwise, a neighborhood is assumed to be open.

Problem 10.24. Make up a few examples of graphs and explore the concept of neighborhood.

Problem 10.25. Consider the graph $K_{3,5}$ with bipartition V_1 and V_2, where $|V_1| = 3$ and $|V_2| = 5$.

(a) For $v \in V_1$, what is $N(v)$?

(b) What is $N(V_1)$?

(c) What is $N(V_2)$?

A **matching** (or **independent edge set**) in a graph $G = (V, E)$ is a subset of edges $M \subseteq E$ without common vertices. That is, a subset of the edges is a matching if each vertex appears as an endpoint in at most one edge of that matching. If M is a matching, a vertex is said to be **matched** if it is an endpoint of one of the edges in M. Otherwise, the vertex is called **unmatched**. We say that M **covers** a subset $S \subseteq V$ if every vertex of S is matched by M.

We now explore a type of matching problem. Suppose we have a bipartite graph $G = (V, E)$ with a bipartition (V_1, V_2). We want to match up each element $v_1 \in V_1$ with exactly one element $v_2 \in V_2$ that is adjacent to it in G and that is not matched to any other element of V_1.

A **total matching** from V_1 to V_2 is matching that covers V_1. In other words, for each $v_1 \in V_1$, there is a unique edge $m \in M$ and a unique $v_2 \in V_2$ such that m is incident to v_1 and v_2. We can think of M as specifying an injective function from V_1 to V_2. If every vertex in V_2 is also matched (i.e., V_2 is covered), then the matching is called a **perfect matching**.

Finding a matching in a bipartite graph can be treated as a network flow problem.

Problem 10.26. Find an example of a bipartite graph G with bipartition (V_1, V_2) that has a total matching from V_1 to V_2.

Problem 10.27. Find an example of a bipartite graph G with bipartition (V_1, V_2) that does not have a total matching from V_1 to V_2.

Problem 10.28. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If a bipartite graph $G = (V, E)$ has a perfect matching, then $|V|$ is even.

(b) If $G = (V, E)$ is a bipartite graph such that $|V|$ is even, then G has a perfect matching.

In order for there to be a total matching from V_1 to V_2, we need $\deg(v_1) > 0$ for each $v_1 \in V_1$. However, that is not enough. Phillip Hall (1904–1982) discovered the following condition, known as **Hall's Marriage Theorem**, needed for a total matching.

Theorem 10.29 (Hall's Marriage Theorem)**.** A bipartite graph with bipartition (V_1, V_2) has a total matching from V_1 to V_2 if and only if $|N(S)| \geq |S|$ for all subsets $S \subseteq V_1$. In other words, every subset S of V_1 must have sufficiently many neighbors in V_2.

Hall actually stated and proved a more general theorem, but we have given its formulation in the context of graph theory.

A graph $G = (V, E)$ is called k**-regular** if $\deg(v) = k$ for every $v \in V$.

Example 10.30. The cycle graph C_n is 2-regular, the complete graph K_n is $(n-1)$-regular, the hypercube graph Q_n is n-regular, and the complete bipartite graph $K_{n,n}$ is n-regular.

Problem 10.31. Prove that if G is a k-regular bipartite graph with bipartition (V_1, V_2), then $|V_1| = |V_2|$.

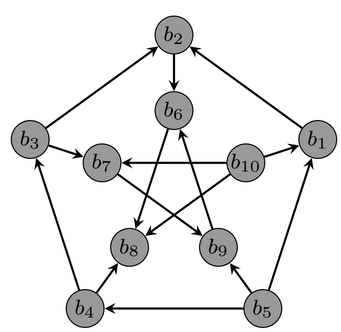
Problem 10.32. Prove that if G is a k-regular bipartite graph with $k > 1$, then G has a perfect matching.

Chapter 11

Additional Graph Theory

A **digraph** (or **directed graph**) D consists of a set V of vertices and a set E of **directed edges** (or **arrows**), each of which is represented as an ordered pair (u, v), where $u, v \in V$. We say that u is the **initial vertex** and v is the **terminal vertex** of the directed edge (u, v). We write $D = (V, E)$ as we did with undirected graphs. The **indegree** of a vertex v in a digraph, denoted $\boxed{\deg^-(v)}$, is the number of directed edges have v as a terminal vertex while the **outdegree** of v, denoted $\boxed{\deg^+(v)}$, is the number of edges having v as an initial vertex.

Problem 11.1. Find the indegree and outdegree of each vertex in the following graph.



As expected, we have the following result that is analogous to the Handshake Lemma (Theorem 10.14).

Theorem 11.2. If $D = (V, E)$ is a digraph, then

$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v).$$

Each graph/digraph is determined by its vertices and the manner in which they are connected by edges, not the way a graph/digraph might be sketched. We can represent a graph in a couple of ways.

The **adjacency list** of a simple graph lists all vertices in one column and all adjacent vertices in second column. For a digraph, the columns contain the initial vertices and the associated terminal vertices.

Problem 11.3. Make up a couple examples to explore adjacency lists for simple graphs and digraphs.

An $m \times n$ **matrix** A is a rectangular array of numbers with m rows and n columns. The entry in the ith row and jth column is indicated by $A_{i,j}$.

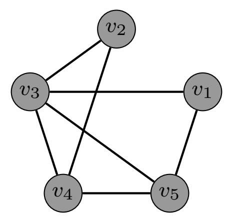
Example 11.4. The example below is a 2×3 matrix:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 10 & 6 & 7 \end{bmatrix}$$

In this example, $A_{1,2} = 3$.

The **adjacency matrix** A of a graph (respectively, digraph) G with vertices listed as $v_1, v_2, \ldots, v_n$ is the $n \times n$ matrix A whose entry $A_{i,j}$ in row i and column j is the number of edges connecting v_i and v_j (respectively, the number of edges from v_i to v_j).

Problem 11.5. Find the adjacency matrix for the following graph.



Problem 11.6. What properties will the adjacency matrix for a simple graph have?

Problem 11.7. Sketch a graph that has the following adjacency matrix.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 \end{bmatrix}$$

Problem 11.8. Sketch a digraph that has the following adjacency matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 \end{bmatrix}$$

Problem 11.9. What will the adjacency matrix for P_n look like, assuming the vertices are taken in the natural order (start at one end of the path and end at the other)? What about C_n? K_n?

Recall that a graph is not determined by a sketch since many sketches give the same graph. It may be hard to recognize from sketches whether two graphs are "essentially" the same even though the vertices may be different points. The notion of isomorphism (same form) gives us a way to deal with this. Two *simple* graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are the **isomorphic**, written $G_1 \cong G_2$, if there is a a bijection $f : V_1 \to V_2$ such that $\{u, v\}$ is an edge in G_1 if and only if $\{f(u), f(v)\}$ is an edge in G_2. The function f is called an **isomorphism**. For digraphs, we require that (u, v) is a directed edge in G_1 if and only if $(f(u), f(v))$ is a directed edge in G_2.

For $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, to show that $G_1 \cong G_2$:

1. State a vertex matching explicitly, and
2. Either
   (a) Check adjacency for each pair of vertices in G_1 and the corresponding pair in G_2 (a total of $\binom{|V_1|}{2}$ checks). This could also be as simple as providing sketches for each graph that clearly exhibit the correspondence of vertices and edges.
   (b) Demonstrate that the adjacency matrices of G_1 and G_2 are the same using an ordering that is compatible with the vertex matching.

Warning! The second method above usually involves much less writing, but be aware that the adjacency matrices may differ in one ordering but agree with a different ordering.

The simplest way to show that $G_1 \not\cong G_2$ is to show that a feature preserved under isomorphism (called an **invariant**) holds for one graph but not the other. Here are a few isomorphic invariants:

(a) Order of the graph

(b) Number of edges in the graph

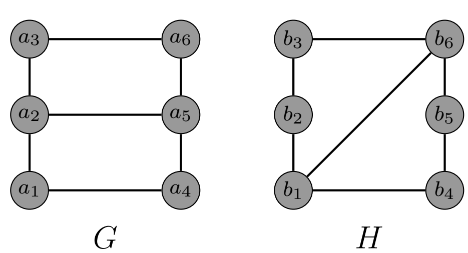
(c) Number of vertices of a given degree

(d) Degree sequence

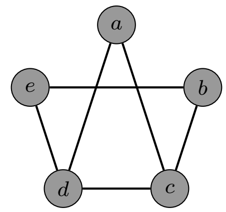
(e) Vertices of degree k and ℓ are adjacent

(f) Subgraph that is isomorphic to C_n or P_n.

Problem 11.10. Determine whether the following graphs are isomorphic.

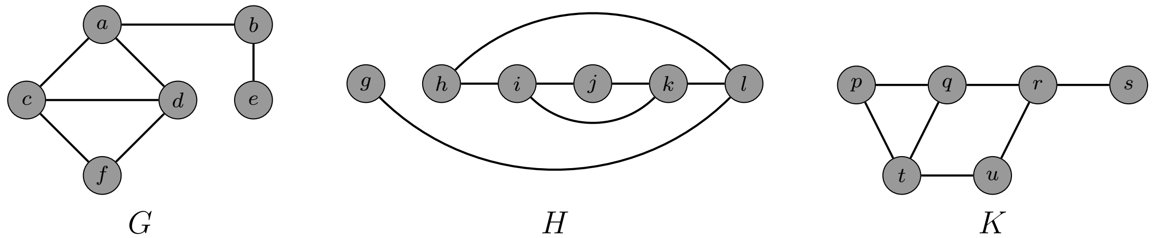


Problem 11.11. Let G be the graph with vertex set $V = \{a, b, c, d, e\}$ and edge set $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, d\}\}$ and let H be the following graph.

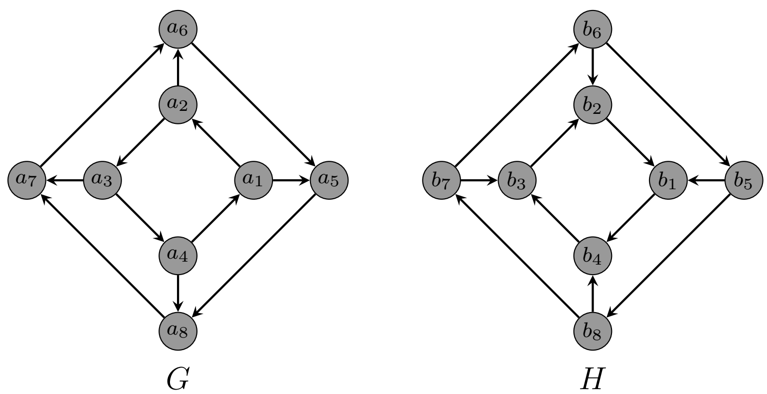


Determine whether G and H are isomorphic.

Problem 11.12. Determine which pairs of the following graphs are isomorphic.



Problem 11.13. Determine whether the following digraphs are isomorphic.



We now introduce several new terms.

- A **walk** in a graph is an alternating sequence of vertices and edges that starts with a vertex and ends with a vertex such that consecutive vertices in the walk are the endpoints of the edge that separates them. In a simple graph, a walk can be specified by a sequence of vertices.
- The **length** of a walk is the number of edges in the walk.
- If the initial and terminal vertices of a walk are the same, then the walk is a **closed walk**.
- A **trail** is a walk with distinct edges (no repeated edges).
- A **circuit** is a closed trail, that is, a closed walk with no repeated edges.
- A **path** is a walk with distinct vertices. This is a subgraph isomorphic to P_n for some n.

- A **cycle** is a closed walk with distinct vertices except the initial and terminal vertices. This corresponds to a subgraph isomorphic to C_n for some n.
- A graph G is **connected** if for each pair of distinct vertices u and v, there is a walk from u to v. A **component** of a graph is a connected subgraph that is not contained in a larger connected subgraph.
- A **cut vertex** of a connected graph G is a vertex which when removed along with all incident edges results in a disconnected graph.
- A **bridge** (or **cut edge**) is an edge of a connected graph which when removed results in a disconnected graph.

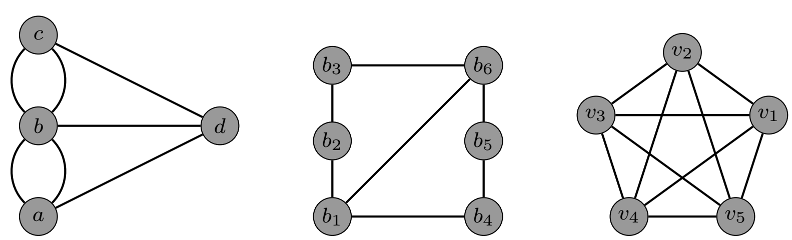
The following theorem likely does not come as a surprise.

Theorem 11.14. A graph G is connected if and only if for each pair of distinct vertices u and v, there is a path from u to v.

A digraph is **strongly connected** if for each pair of distinct vertices u and v there is a (directed) walk from u to v. A digraph is **weakly connected** if the underlying undirected graph in which the direction of edges is removed is connected. Note that a strongly connected digraph will always be weakly connected. A **strongly connected component** of a digraph is a maximal strongly connected subgraph.

We now introduce a couple of important circuits that a graph may or may not possess. An **Euler circuit** in a graph G is a circuit that contains every edge of the graph. An **Euler trail** in a graph is a trail that contains every edge of the graph. Note that an Euler circuit is also an Euler trail as well.

Problem 11.15. Determine whether each of the following graphs has an Euler trail. How about an Euler circuit?



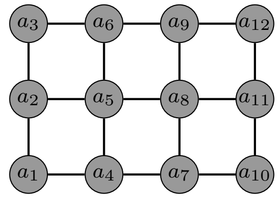
G $\qquad$ H $\qquad$ K

Theorem 11.16. If G is a connected graph of order $n \geq 2$, then G has an Euler circuit if and only if every vertex is even.

Corollary 11.17. If G is a connected graph of order $n \geq 2$, then G has an Euler trail that is not a circuit if and only if G has exactly two odd vertices.

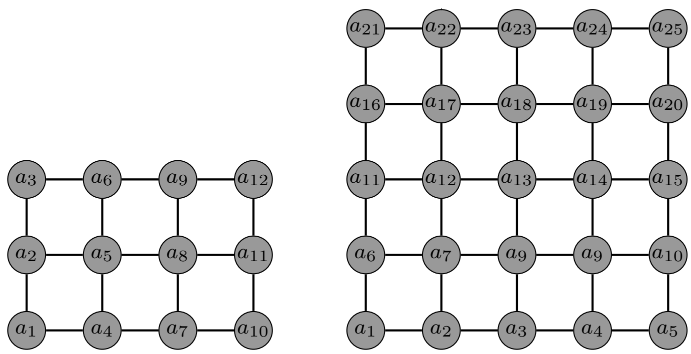
Edges can be added to a connected graph in order to cause it to have an Euler circuit (or trail). In particular, an **Eulerization** of a connected graph is the addition of suitable multiple edges (i.e., duplicate existing edges) to permit an Euler circuit, mimicking what must be done to complete a circuit such as a postal route or other delivery/pick-up route.

Problem 11.18. Eulerize the following graph.



A cycle in a graph that passes through every vertex is a **Hamilton cycle**. This is often called a **Hamilton circuit**. A **Hamilton path** is a path in a graph that includes every vertex.

Problem 11.19. Determine whether each of the following graphs has a Hamilton circuit or a Hamilton path that is not a circuit.



G H

Unfortunately, unlike the situation for Euler circuits, there is no known simple necessary and sufficient condition for a Hamilton cycle to exist in a graph. We can state some simple cases when one cannot exist, and there are some theorems for the existence of a Hamilton cycle, but these do not cover all possibilities.

Theorem 11.20 (Dirac's Theorem)**.** If G is a simple graph of order $n \geq 3$ in which $\deg(v) \geq n/2$ for each vertex, then G has a Hamilton cycle.

Theorem 11.21 (Ore's Theorem)**.** If G is a simple graph of order $n \geq 3$ in which $\deg(u) + \deg(v) \geq n$ for each pair of vertices u and v, then G has a Hamilton cycle.

Problem 11.22. How many distinct Hamilton cycles does K_n have that start/end at a fixed vertex?

We now turn our attention to trees. A **tree** is a connected graph that has no cycles. A **forest** is a graph in which every connected component is a tree. Trees provide a useful structure for organizing data, for displaying organization, and for decision processes.

Problem 11.23. Is every tree necessarily a simple graph?

Theorem 11.24. A graph G is a tree if and only if for each pair of distinct vertices u and v, there is a unique path from u to v.

Theorem 11.25. Some properties of trees.

(a) A tree of order 2 or more has at least two vertices of degree 1.

(b) Every edge of a tree is a bridge.

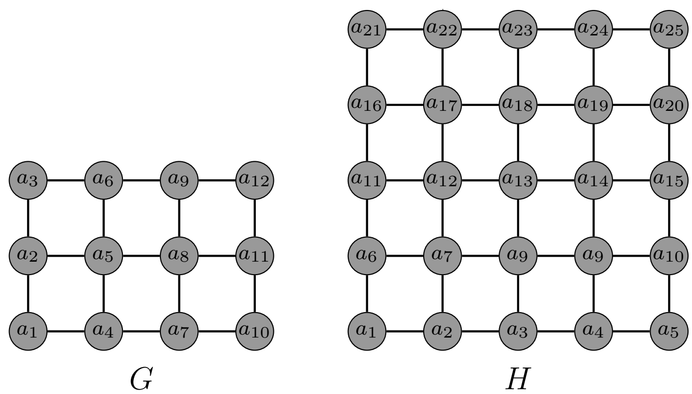
(c) A connected graph in which every edge is a bridge is a tree.

(d) A tree of order n has $n-1$ edges.

(e) A connected graph of order n with $n-1$ edges is a tree.

A **spanning tree** of a simple graph G is a subgraph T of G such that T is a tree contains every vertex of G.

Problem 11.26. Find a spanning tree for each of the following.



Problem 11.27. Do you think every connected simple graph contains a spanning tree?

In fact, we have the following theorem.

Theorem 11.28. A simple graph G is connected if and only if it contains a spanning tree.

Below, we provide informal descriptions of the depth-first and breadth-first algorithms for identifying a spanning tree in a connected simple graph G with vertices ordered as $v_1, v_2, \ldots, v_n$.

Depth-First Search. Initialize: $T = \{v_1\}$, $v = v_1$. As long as T does not contain all vertices, do the following:

- Choose the first vertex w in the ordered list that is adjacent to v and is not yet in T. If there are no vertices adjacent to v that are not yet in T, return to the most recently added previous vertex u and let $v = u$ and repeat this step with the revised v.
- Add w and edge $\{v, w\}$ to T.
- Set $v = w$.

Repeat the steps above (with revised v's) as often as needed.

Breadth-First Search. Initialize: $T = \{v_1\}$, $L = v_1$. As long as T does not contain all vertices, do the following.

- Put $v =$ the first vertex in L.
- Remove that vertex from L. (On paper, just mark out.)
- For all vertices adjacent to v not yet in T,
  - If w is the first vertex in order that is adjacent to v and not yet in T, add w and edge $\{v, w\}$ to T and put w at the end of list L.
  - Repeat until all vertices adjacent to v not yet in T have been examined.

Repeat the above steps as needed.

A **weighted graph** is a graph with a positive numerical values assigned to each edge of the graph. A **minimal spanning tree** of a connected weighted graph is a spanning tree that has the smallest possible sum of weights for its edges.

Minimal spanning trees arose in the practical matter of designing an efficient electrical network, and the concept applies to similar network notions in many areas such as transportation, utilities, and others. We will discuss one early, elementary, and effective algorithms for finding a minimal spanning tree in a connected undirected weighted graph.

Prim's Algorithm (Jarnik 1930, Prim 1957). Assume G is a connected, undirected, weighted loopless graph of order n.

1. Initialize a tree with a single vertex, chosen arbitrarily from the graph.
2. Grow the tree by one edge: Of the edges that connect the tree to vertices not yet in the tree, find a minimum-weight edge, and transfer it to the tree.
3. Repeat step 2 (until all vertices are in the tree).

Prim's Algorithm is an example of a greedy algorithm.

Problem 11.29. Make up an example of a connected, undirected, weighted loopless graph and find a minimal spanning tree.