Chapter 8

Additional Graph Theory

A **digraph** (or **directed graph**) D consists of a set V of vertices and a set E of **directed edges** (or **arrows**), each of which is represented as an ordered pair (u, v), where $u, v \in V$. We say that u is the **initial vertex** and v is the **terminal vertex** of the directed edge (u, v). We write $D = (V, E)$ as we did with undirected graphs. The **indegree** of a vertex v in a digraph, denoted $\boxed{\deg^-(v)}$, is the number of directed edges have v as a terminal vertex while the **outdegree** of v, denoted $\boxed{\deg^+(v)}$, is the number of edges having v as an initial vertex.

As expected, we have the following result that is analogous to the Handshake Lemma (Theorem 7.12).

Theorem 8.1. If $D = (V, E)$ is a digraph, then

$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v).$$

Each graph/digraph is determined by its vertices and the manner in which they are connected by edges, not the way a graph/digraph might be sketched. We can represent a graph in a couple of ways.

The **adjacency list** of a simple graph lists all vertices in one column and all adjacent vertices in second column. For a digraph, the columns contain the initial vertices and the associated terminal vertices.

Problem 8.2. Make up a couple examples to explore adjacency lists for simple graphs and digraphs.

An $m \times n$ **matrix** A is a rectangular array of numbers with m rows and n columns. The entry in the ith row and jth column is indicated by $A_{i,j}$.

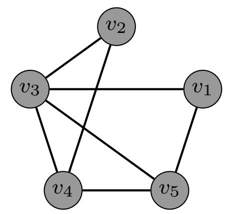
Example 8.3. The example below is a 2×3 matrix:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 10 & 6 & 7 \end{bmatrix}$$

In this example, $A_{1,2} = 3$.

The **adjacency matrix** A of a graph (respectively, digraph) G with vertices listed as $v_1, v_2, \ldots, v_n$ is the $n \times n$ matrix A whose entry $A_{i,j}$ in row i and column j is the number of edges connecting v_i and v_j (respectively, the number of edges from v_i to v_j).

Problem 8.4. Find the adjacency matrix for the following graph.



Problem 8.5. What properties will the adjacency matrix for a simple graph have?

Problem 8.6. Sketch a graph that has the following adjacency matrix.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 2 & 1 & 0 \end{bmatrix}$$

Problem 8.7. Sketch a digraph that has the following adjacency matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 \end{bmatrix}$$

Problem 8.8. What will the adjacency matrix for P_n look like, assuming the vertices are taken in the natural order (start at one end of the path and end at the other)? What about C_n? K_n?

Recall that a graph is not determined by a sketch since many sketches give the same graph. It may be hard to recognize from sketches whether two graphs are "essentially" the same even though the vertices may be different points. The notion of isomorphism (same form) gives us a way to deal with this. Two *simple* graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are the **isomorphic**, written $G_1 \cong G_2$, if there is a a bijection $f : V_1 \to V_2$ such that $\{u, v\}$ is an edge in G_1 if and only if $\{f(u), f(v)\}$ is an edge in G_2. The function f is called an **isomorphism**. For digraphs, we require that (u, v) is a directed edge in G_1 if and only if $(f(u), f(v))$ is a directed edge in G_2.

For $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, to show that $G_1 \cong G_2$:

1. State a vertex matching explicitly, and

2. Either

(a) Check adjacency for each pair of vertices in G_1 and the corresponding pair in G_2 (a total of $\binom{|V_1|}{2}$ checks). This could also be as simple as providing sketches for each graph that clearly exhibit the correspondence of vertices and edges.

(b) Demonstrate that the adjacency matrices of G_1 and G_2 are the same using an ordering that is compatible with the vertex matching.

Warning! The second method above usually involves much less writing, but be aware that the adjacency matrices may differ in one ordering but agree with a different ordering.

The simplest way to show that $G_1 \ncong G_2$ is to show that a feature preserved under isomorphism (called an **invariant**) holds for one graph but not the other. Here are a few isomorphic invariants:

(a) Order of the graph

(b) Number of edges in the graph

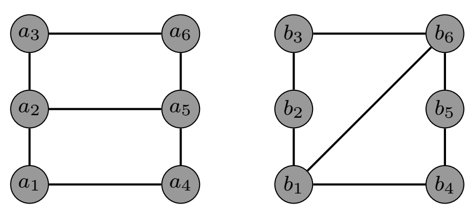
(c) Number of vertices of a given degree

(d) Degree sequence

(e) Vertices of degree k and ℓ are adjacent

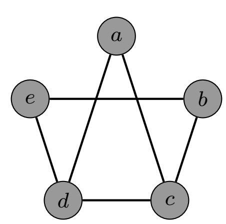
(f) Subgraph that is isomorphic to C_n or P_n.

Problem 8.9. Determine whether the following graphs are isomorphic.



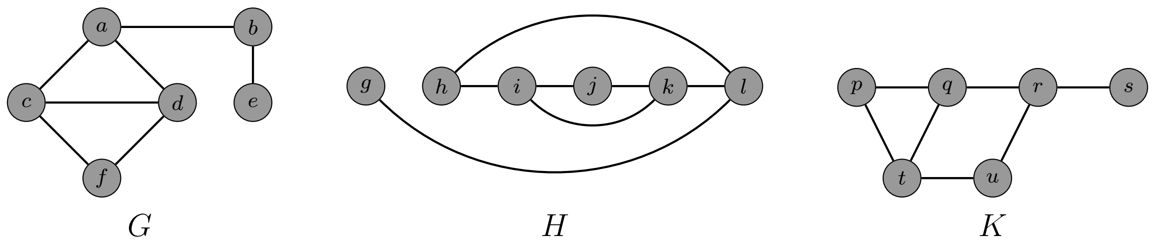
G $\qquad$ H

Problem 8.10. Let G be the graph with vertex set $V = \{a, b, c, d, e\}$ and edge set $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, d\}\}$ and let H be the following graph.

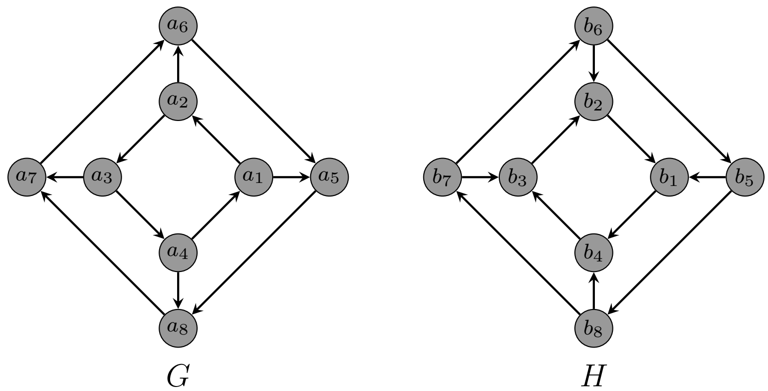


Determine whether G and H are isomorphic.

Problem 8.11. Determine which pairs of the following graphs are isomorphic.



Problem 8.12. Determine whether the following digraphs are isomorphic.



More coming soon...