Introduction to Real Analysis

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This book is intended to be a problem sequence for a one-semester undergraduate real analysis course that utilizes an inquiry-based learning (IBL) approach. There is always a debt to be paid in creating a text, and this one is no different. The primary source for these notes is Karl-Dieter Crisman's *One-Semester Real Analysis* notes[1], which were based on W. Ted Mahavier's *Analysis* notes[2], which in turn were based on notes his father (W. S. Mahavier) created.

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- Dave Richeson (Dickinson College). Dave is responsible for much of the content in Appendix B: Fancy Mathematical Terms and Appendix C: Definitions in Mathematics.

[1]Journal of Inquiry-Based Learning in Mathematics, No. 46 (2013)
[2]Journal of Inquiry-Based Learning in Mathematics, No. 12 (2009)

Contents

Preface

You are the creators. This book is a guide.

This book will not show you how to solve all the problems that are presented, but it should *enable* you to find solutions, on your own and working together. The material you are about to study did not come together fully formed at a single moment in history. It was composed gradually over the course of centuries, with various mathematicians building on the work of others, improving the subject while increasing its breadth and depth.

Mathematics is essentially a human endeavor. Whatever you may believe about the true nature of mathematics—does it exist eternally in a transcendent Platonic realm, or is it contingent upon our shared human consciousness?—our *experience* of mathematics is temporal, personal, and communal. Like music, mathematics that is encountered only as symbols on a page remains inert. Like music, mathematics must be created in the moment, and it takes time and practice to master each piece. The creation of mathematics takes place in writing, in conversations, in explanations, and most profoundly in the mental construction of its edifices on the basis of reason and observation.

To continue the musical analogy, you might think of these notes like a performer's score. Much is included to direct you towards particular ideas, but much is missing that can only be supplied by you: participation in the creative process that will make those ideas come alive. Moreover, your success will depend on the pursuit of both *individual* excellence and *collective* achievement. Like a musician in an orchestra, you should bring your best work and be prepared to blend it with others' contributions.

In any act of creation, there must be room for experimentation, and thus allowance for mistakes, even failure. A key goal of our community is that we support each other—sharpening each other's thinking but also bolstering each other's confidence—so that we can make failure a *productive* experience. Mistakes are inevitable, and they should not be an obstacle to further progress. It's normal to struggle and be confused as you work through new material. Accepting that means you can keep working even while feeling stuck, until you overcome and reach even greater accomplishments.

This book is a guide. You are the creators.

Chapter 1

Introduction

1.1 What is This Course All About?

This course introduces basic concepts and methods of analysis. The course focuses on the theory of the real number system and calculus of functions of a real variable. The content will include:

1. The Real Number System: axioms; supremum and infimum.
2. Topology of the real number system including completeness, compactness.
3. Sequences and Convergence, including the algebra of limits.
4. Limits of Functions, including the algebra of limits.
5. Continuity, including the algebra of continuous functions, continuity of compositions, and uniform continuity.
6. Differentiation, including the algebra of derivatives, chain rule, Mean Value Theorem, Inverse Function Theorem, applications to behavior of functions, Taylor's Theorem and L'Hospital's Rule.
7. Riemann integration, including linearity and order properties, integrability of continuous functions, Riemann sums, and the Fundamental Theorem of Calculus.

We will take an axiomatic approach (definition, theorem, and proof) to the subject, but along the way, you will develop intuition about the objects of real analysis and pick up more proof-writing skills. The emphasis of this course is on your ability to read, understand, and communicate mathematics in the context of real analysis.

> The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful.
>
> *Henri Poincaré*

1.2 An Inquiry-Based Approach

This is not a lecture-oriented class or one in which mimicking prefabricated examples will lead you to success. You will be expected to work actively to construct your own understanding of the topics at hand with the readily available help of me and your classmates. Many of the concepts you learn and problems you work on will be new to you and ask you to stretch your thinking. You will experience *frustration* and *failure* before you experience *understanding*. This is part of the normal learning process. If you are doing things well, you should be confused at different points in the semester. The material is too rich for a human being to completely understand it immediately. Your viability as a professional in the modern workforce depends on your ability to embrace this learning process and make it work for you.

> Don't fear failure. Not failure, but low aim, is the crime. In great attempts it is glorious even to fail.
>
> *Bruce Lee*

In order to promote a more active participation in your learning, we will incorporate ideas from an educational philosophy called inquiry-based learning (IBL). Loosely speaking, IBL is a student-centered method of teaching mathematics that engages students in sense-making activities. Students are given tasks requiring them to solve problems, conjecture, experiment, explore, create, and communicate. Rather than showing facts or a clear, smooth path to a solution, the instructor guides and mentors students via well-crafted problems through an adventure in mathematical discovery. According to Laursen and Rasmussen (2019), the Four Pillars of IBL are:

- Students engage deeply with coherent and meaningful mathematical tasks.
- Students collaboratively process mathematical ideas.
- Instructors inquire into student thinking.
- Instructors foster equity in their design and facilitation choices.

Much of the course will be devoted to students presenting their proposed solutions or proofs on the board and a significant portion of your grade will be determined by how much mathematics you produce. I use the word *produce* because I believe that the best way to learn mathematics is by doing mathematics. Someone cannot master a musical instrument or a martial art by simply watching, and in a similar fashion, you cannot master mathematics by simply watching; you must do mathematics!

In any act of creation, there must be room for experimentation, and thus allowance for mistakes, even failure. A key goal of our community is that we support each other—sharpening each other's thinking but also bolstering each other's confidence—so that we can make failure a productive experience. Mistakes are inevitable, and they should not be an obstacle to further progress. It's normal to struggle and be confused as you work through new material. Accepting that means you can keep working even while feeling stuck, until you overcome and reach even greater accomplishments.

> You will become clever through your mistakes.
>
> *German Proverb*

Furthermore, it is important to understand that solving genuine problems is difficult and takes time. You shouldn't expect to complete each problem in 10 minutes or less. Sometimes, you might have to stare at the problem for an hour before even understanding how to get started.
In this course, everyone will be required to

- read and interact with course notes and textbook on your own;
- write up quality solutions/proofs to assigned problems;
- present solutions/proofs on the board to the rest of the class;
- participate in discussions centered around a student's presented solution/proof;
- call upon your own prodigious mental faculties to respond in flexible, thoughtful, and creative ways to problems that may seem unfamiliar on first glance.

As the semester progresses, it should become clear to you what the expectations are.

> Tell me and I forget, teach me and I may remember, involve me and I learn.
>
> *Benjamin Franklin*

1.3 Rights of the Learner

As a student in this class, you have the right:

1. to be confused,
2. to make a mistake and to revise your thinking,
3. to speak, listen, and be heard, and
4. to enjoy doing mathematics.

> You may encounter many defeats, but you must not be defeated.
>
> *Maya Angelou*

1.4 Your Toolbox, Questions, and Observations

Throughout the semester, we will develop a list of *tools* that will help you understand and do mathematics. Your job is to keep a list of these tools, and it is suggested that you keep a running list someplace.

Next, it is of utmost importance that you work to understand every proof. (Every!) Questions are often your best tool for determining whether you understand a proof. Therefore, here are some sample questions that apply to any proof that you should be prepared to ask of yourself or the presenter:

- What method(s) of proof are you using?
- What form will the conclusion take?
- How did you know to set up that [equation, set, whatever]?
- How did you figure out what the problem was asking?
- Was this the first thing you tried?
- Can you explain how you went from this line to the next one?
- What were you thinking when you introduced this?
- Could we have ... instead?
- Would it be possible to ...?
- What if ...?

Another way to help you process and understand proofs is to try and make observations and connections between different ideas, proof statements and methods, and to compare approaches used by different people. Observations might sound like some of the following:

- When I tried this proof, I thought I needed to ... But I didn't need that, because ...
- I think that ... is important to this proof, because ...
- When I read the statement of this theorem, it seemed similar to this earlier theorem. Now I see that it [is/isn't] because ...

1.5 Rules of the Game

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing assignments for this course, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

1.6 Structure of the Notes

As you read the notes, you will be required to digest the material in a meaningful way. It is your responsibility to read and understand new definitions and their related concepts. However, you will be supported in this sometimes difficult endeavor. In addition, you will be asked to complete exercises aimed at solidifying your understanding of the material. Most importantly, you will be asked to make conjectures, produce counterexamples, and prove theorems.

Most items in the notes are labelled with a number. A **Definition** is just that. However, in this type of course, it is extremely important to use definitions very accurately, and we try to use as few as possible to avoid confusion and over-saturation. A **Problem** is a mixed bag of tasks. Some problems ask you to prove something. Some of these will be easy, a few extremely hard. Most will be somewhere in between, to enhance your sense of confidence and accomplishment. Other problems are exercises aimed at developing understanding of the content. It is important to point out that there are very few examples in the notes. This is intentional. The goal of some of the problems is for you to produce the examples. There are a handful of items labeled as **Theorem**, which are results that we will take for granted. Lastly, there are many situations where you will want to refer to an earlier definition or problem. In this case, you should reference the statement by number. For example, you might write something like, "By Problem 2.20, we see that...."

1.7 Some Minimal Guidance

Especially in the opening sections, it won't be clear what facts from your prior experience in mathematics we are "allowed" to use. Unfortunately, addressing this issue is difficult and is something we will sort out along the way. However, in general, here are some minimal and vague guidelines to keep in mind.

First, there are times when we will need to do some basic algebraic manipulations. You should feel free to do this whenever the need arises. But you should show sufficient work along the way. You do not need to write down justifications for basic algebraic manipulations (e.g., adding 1 to both sides of an equation, adding and subtracting the same amount on the same side of an equation, adding like terms, factoring, basic simplification, etc.).

On the other hand, you do need to make explicit justification of the logical steps in a proof. When necessary, you should cite a previous definition, theorem, etc. by number.

Unlike the experience many of you had writing proofs in geometry, our proofs will be written in complete sentences. You should break sections of a proof into paragraphs and use proper grammar. There are some pedantic conventions for doing this that I will point out along the way. Initially, this will be an issue that most students will struggle with, but after a few weeks everyone will get the hang of it.

Ideally, you should rewrite the statements of theorems before you start the proof. Moreover, for your sake and mine, you should label the statement with the appropriate number. I will expect you to indicate where the proof begins by writing "*Proof.*" at the beginning. Also, we will conclude our proofs with the standard "proof box" (i.e., $\square$ or $\blacksquare$), which is typically right-justified.

Lastly, every time you write a proof, you need to make sure that you are making your assumptions crystal clear. Sometimes there will be some implicit assumptions that we can omit, but at least in the beginning, you should get in the habit of stating your assumptions up front. Typically, these statements will start off "Assume..." or "Let...".

This should get you started. We will discuss more as the semester progresses. Now, go have fun and start exploring mathematics!

> If you want to sharpen a sword, you have to remove a little metal.
>
> *Unknown*

Chapter 2

Preliminaries

In this chapter, we summarize some background material we need to be familiar with. Sections 2.1 and 2.2 should mostly be review.

2.1 Sets

A **set** is a collection of objects called **elements**. If A is a set and x is an element of A, we write $\boxed{x \in A}$. Otherwise, we write $\boxed{x \notin A}$. The set containing no elements is called the **empty set**, and is denoted by the symbol $\boxed{\emptyset}$. Any set that contains at least one element is referred to as a **nonempty set**.

If we think of a set as a box potentially containing some stuff, then the empty set is a box with nothing in it. One assumption we will make is that for any set A, $A \notin A$. The language associated to sets is specific. We will often define sets using the following notation, called **set-builder notation**:

$$\boxed{S = \{x \in A \mid x \text{ satisfies some condition}\}}$$

The first part "$x \in A$" denotes what type of x is being considered. The statements to the right of the vertical bar (not to be confused with "divides") are the conditions that x must satisfy in order to be members of the set. This notation is read as "The set of all x in A such that x satisfies some condition," where "some condition" is something specific about the restrictions on x relative to A.

There are a few sets that are commonly discussed in mathematics and have predefined symbols to denote them. We've already encountered the integers, natural numbers, and real numbers. Notice that our definition of the rational numbers uses set-builder notation.

- **Natural numbers:** $\boxed{\mathbb{N} := \{1, 2, 3, \ldots\}}$. Some books will include zero in the set of natural numbers, but we do not.
- **Integers:** $\boxed{\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \ldots\}}$.
- **Rational Numbers:** $\boxed{\mathbb{Q} := \{a/b \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}}$.

- **Real Numbers:** $\boxed{\mathbb{R}}$ denotes the set of real numbers.

Since the set of natural numbers consists of the positive integers, the natural numbers are sometimes denoted by $\mathbb{Z}^+$.

If A and B are sets, then we say that A is a **subset** of B, written $\boxed{A \subseteq B}$, provided that every element of A is an element of B. Observe that $A \subseteq B$ is equivalent to "For all x in the universe of discourse, if $x \in A$, then $x \in B$."

Every nonempty set always has two rather boring subsets.

Problem 2.1. Let A be a set. Write a short proof for each of the following.

(a) $A \subseteq A$

(b) $\emptyset \subseteq A$

The next problem shows that "$\subseteq$" is a transitive relation.

Problem 2.2 (Transitivity of subsets)**.** Prove that if A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Let A and B be sets in some universe of discourse U. We define the following.

- The sets A and B are **equal**, denoted $\boxed{A = B}$, if and only if $A \subseteq B$ and $B \subseteq A$. Note that if we want to prove $A = B$, then we have to do two separate mini-proofs: one for $A \subseteq B$ and one for $B \subseteq A$. It is common to label each mini-proof with "($\subseteq$)" and "($\supseteq$)", respectively.

- If $A \subseteq B$, then A is called a **proper subset** provided that $A \neq B$. In this case, we may write $\boxed{A \subset B}$ or $\boxed{A \subsetneq B}$. *Warning:* Some books use $\subset$ to mean $\subseteq$.

- The **union** of the sets A and B is $\boxed{A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\}}$.

- The **intersection** of the sets A and B is $\boxed{A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\}}$.

- The **set difference** of the sets A and B is $\boxed{A \setminus B := \{x \in U \mid x \in A \text{ and } x \notin B\}}$.

- The **complement of** A (relative to U) is the set $\boxed{A^c := U \setminus A = \{x \in U \mid x \notin A\}}$.

- If $A \cap B = \emptyset$, then we say that A and B are **disjoint** sets.

Example 2.3. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of **irrational numbers**.

Problem 2.4. Prove that if A and B are sets such that $A \subseteq B$, then $B^c \subseteq A^c$.

Problem 2.5. Prove that if A and B are sets, then $A \setminus B = A \cap B^c$.

Problem 2.6. Give an example where $A \neq B$ but $A \setminus B = \emptyset$.

Consider the following collection of sets:

$$\{a\}, \{a,b\}, \{a,b,c\}, \ldots, \{a,b,c,\ldots,z\}$$

This collection has a natural way for us to "index" the sets:

$$A_1 = \{a\}, A_2 = \{a,b\}, A_3 = \{a,b,c\}, \ldots, A_{26} = \{a,b,c,\ldots,z\}$$

In this case the sets are **indexed** by the set $\{1,2,\ldots,26\}$, where the subscripts are taken from the **index set**. If we wanted to talk about an arbitrary set from this indexed collection, we could use the notation A_n.

Using indexing sets in mathematics is an extremely useful notational tool, but it is important to keep straight the difference between the sets that are being indexed, the elements in each set being indexed, the indexing set, and the elements of the indexing set.

Any set (finite or infinite) can be used as an indexing set. Often capital Greek letters are used to denote arbitrary indexing sets and small Greek letters to represent elements of these sets. If the indexing set is a subset of $\mathbb{R}$, then it is common to use Roman letters as individual indices. Of course, these are merely conventions, not rules.

- If Δ is a set and we have a collection of sets indexed by Δ, then we may write $\{S_\alpha\}_{\alpha\in\Delta}$ to refer to this collection. We read this as "the set of S-alphas over alpha in Delta."
- If a collection of sets is indexed by $\mathbb{N}$, then we may write $\{U_n\}_{n\in\mathbb{N}}$ or $\{U_n\}_{n=1}^{\infty}$.
- Borrowing from this idea, a collection $\{A_1,\ldots,A_{26}\}$ may be written as $\{A_n\}_{n=1}^{26}$.

Suppose we have a collection $\{A_\alpha\}_{\alpha\in\Delta}$.

- The **union of the entire collection** is defined via

$$\boxed{\bigcup_{\alpha\in\Delta} A_\alpha = \{x \mid x \in A_\alpha \text{ for some } \alpha \in \Delta\}.}$$

- The **intersection of the entire collection** is defined via

$$\boxed{\bigcap_{\alpha\in\Delta} A_\alpha = \{x \mid x \in A_\alpha \text{ for all } \alpha \in \Delta\}.}$$

In the special case that $\Delta = \mathbb{N}$, we write

$$\bigcup_{n=1}^{\infty} A_n = \{x \mid x \in A_n \text{ for some } n \in \mathbb{N}\} = A_1 \cup A_2 \cup A_3 \cup \cdots$$

and

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid x \in A_n \text{ for all } n \in \mathbb{N}\} = A_1 \cap A_2 \cap A_3 \cap \cdots$$

Similarly, if $\Delta = \{1,2,3,4\}$, then

$$\bigcup_{n=1}^{4} A_n = A_1 \cup A_2 \cup A_3 \cup A_4 \quad \text{and} \quad \bigcap_{n=1}^{4} A_n = A_1 \cap A_2 \cap A_3 \cap A_4.$$

Notice the difference between "$\bigcup$" and "$\cup$" (respectively, "$\bigcap$" and "$\cap$").

Problem 2.7. Let $\{A_n\}_{n=1}^{26}$ be the collection from the discussion below Problem 2.6. Find each of the following.

(a) $\displaystyle\bigcup_{n=1}^{26} A_n$

(b) $\displaystyle\bigcap_{n=1}^{26} A_n$

Problem 2.8. For each $r \in \mathbb{Q}$ (the rational numbers), let N_r be the set containing all real numbers *except* r. Find each of the following.

(a) $\displaystyle\bigcup_{r\in\mathbb{Q}} N_r$

(b) $\displaystyle\bigcap_{r\in\mathbb{Q}} N_r$

A collection of sets $\{A_\alpha\}_{\alpha\in\Delta}$ is **pairwise disjoint** if $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$.

Problem 2.9. Draw a Venn diagram of a collection of three sets that are pairwise disjoint.

Problem 2.10. Provide an example of a collection of three sets, say $\{A_1, A_2, A_3\}$, such that the collection is *not* pairwise disjoint, but $\bigcap_{n=1}^{3} A_n = \emptyset$.

Problem 2.11. Find a collection of nonempty sets $S_i \subseteq \mathbb{N}$ indexed by $i \in \mathbb{N}$ such that $S_{i+1} \subsetneq S_i$ and $\bigcap_{i=1}^{\infty} S_i = \emptyset$.

Problem 2.12. Find a collection of nonempty sets $S_i \subseteq \mathbb{N}$ indexed by $i \in \mathbb{N}$ such that $S_i \subsetneq S_{i+1}$ but $\bigcup_{i=1}^{\infty} S_i \neq \mathbb{N}$.

Problem 2.13 (DeMorgan's Law)**.** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of sets. Prove one of the following.

(a) $\displaystyle\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^C = \bigcap_{\alpha\in\Delta} A_\alpha^C$

(b) $\displaystyle\left(\bigcap_{\alpha\in\Delta} A_\alpha\right)^C = \bigcup_{\alpha\in\Delta} A_\alpha^C$

Problem 2.14 (Distribution of Union and Intersection)**.** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of sets and let B be any set. Prove one of the following.

(a) $\displaystyle B \cup \left(\bigcap_{\alpha\in\Delta} A_\alpha\right) = \bigcap_{\alpha\in\Delta} (B \cup A_\alpha)$

(b) $\displaystyle B \cap \left(\bigcup_{\alpha\in\Delta} A_\alpha\right) = \bigcup_{\alpha\in\Delta} (B \cap A_\alpha)$

For each $n \in \mathbb{N}$, we define an n-**tuple** to be an ordered list of n elements of the form $(a_1, a_2, \ldots, a_n)$. We refer to a_i as the ith **component** (or **coordinate**) of $(a_1, a_2, \ldots, a_n)$. Two n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equal if and only if $a_i = b_i$ for all $1 \leq i \leq n$. A 2-tuple (a, b) is more commonly referred to as an **ordered pair** while a 3-tuple (a, b, c) is often called an **ordered triple**.

We can use the notion of n-tuples to construct new sets from existing sets. If A and B are sets, the **Cartesian product** (or **direct product**) of A and B, denoted $A \times B$ (read as "A times B" or "A cross B"), is the set of all ordered pairs where the first component is from A and the second component is from B. In set-builder notation, we have

$$\boxed{A \times B := \{(a,b) \mid a \in A, b \in B\}}.$$

We similarly define the Cartesian product of n sets, say $A_1, \ldots, A_n$, by

$$\boxed{\prod_{i=1}^{n} A_i := A_1 \times \cdots \times A_n := \{(a_1, \ldots, a_n) \mid a_j \in A_j \text{ for all } 1 \leq j \leq n\}},$$

where A_i is referred to as the ith **factor** of the Cartesian product. As a special case, the set

$$\underbrace{A \times \cdots \times A}_{n \text{ factors}}$$

is often abbreviated as A^n.

Example 2.15. The standard two-dimensional plane $\mathbb{R}^2$ and standard three space $\mathbb{R}^3$ are familiar examples of Cartesian products. In particular, we have

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x, y \in \mathbb{R}\}$$

and

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) \mid x, y, z \in \mathbb{R}\}.$$

Problem 2.16. If A is a set, then what is $A \times \emptyset$ equal to?

Problem 2.17. Given sets A and B, when will $A \times B$ be equal to $B \times A$?

We now turn our attention to subsets of Cartesian products.

Problem 2.18. Prove that if A, B, C, and D are sets such that $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

Problem 2.19. Is it true that if $A \times B \subseteq C \times D$, then $A \subseteq C$ and $B \subseteq D$? Don't forget to think about cases involving the empty set.

Problem 2.20. Is every subset of $C \times D$ of the form $A \times B$, where $A \subseteq C$ and $B \subseteq D$? If so, prove it. If not, find a counterexample.

Problem 2.21. If A, B, and C are nonempty sets, is $A \times B$ a subset of $A \times B \times C$?

Problem 2.22. Let A, B, C, and D be sets. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

(b) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$

(c) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(d) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(e) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

2.2 Functions

Let A and B be sets. A **relation** R **from** A **to** B is a subset of $A \times B$. If R is a relation from A to B and $(a,b) \in R$, then we say that a **is related to** b and we may write aRb in place of $(a,b) \in R$.

A function is a special type of relation, where the basic building blocks are a first set and a second set, say X and Y, and a "correspondence" that assigns *every* element of X to *exactly one* element of Y. More formally, if X and Y are nonempty sets, a **function** f **from** X **to** Y is a relation from X to Y such that for every $x \in X$, there exists a unique $y \in Y$ such that $(x,y) \in f$. The set X is called the **domain** of f and is denoted by $\boxed{\mathrm{Dom}(f)}$. The set Y is called the **codomain** of f and is denoted by $\boxed{\mathrm{Codom}(f)}$ while the subset of the codomain defined via

$$\boxed{\mathrm{Rng}(f) := \{y \in Y \mid \text{there exists } x \text{ such that } (x,y) \in f\}}$$

is called the **range** of f or the **image of** X under f.

There is a variety of notation and terminology associated to functions. We will write $\boxed{f : X \to Y}$ to indicate that f is a function from X to Y. We will make use of statements such as "Let $f : X \to Y$ be the function defined via..." or "Define $f : X \to Y$ via...", where f is understood to be a function in the second statement. Sometimes the word **mapping** (or **map**) is used in place of the word function. If $(a,b) \in f$ for a function f, we often write $\boxed{f(a) = b}$ and say that "f maps a to b" or "f of a equals b". In this case, a may be called an **input** of f and is the **preimage** of b under f while b is called an **output** of f and is the **image** of a under f. Note that the domain of a function is the set of inputs while the range is the set of outputs for the function.

Notice that we can interpret our definition of function in terms of existence and uniqueness. That is, $f : X \to Y$ is a function provided:

1. (Existence) For each $x \in X$, there exists $y \in Y$ such that $y = f(x)$, and
2. (Uniqueness) If $f(x) = y_1$ and $f(x) = y_2$, then $y_1 = y_2$.

In other words, every element of the domain is utilized and is utilized exactly once. However, there are no restrictions on whether an element of the codomain ever appears in the second coordinate of an ordered pair in the relation. Yet if an element of Y is in the range of f, it may appear in more than one ordered pair in the relation.

It follows immediately from the definition of function that two functions are equal if and only if they have the same domain, same codomain, and the same set of ordered pairs in the relation. That is, functions f and g are equal if and only if $\mathrm{Dom}(f) = \mathrm{Dom}(g)$, $\mathrm{Codom}(f) = \mathrm{Codom}(g)$, and $f(x) = g(x)$ for all $x \in X$.

Since functions are special types of relations, we can represent them using digraphs and graphs when practical. Digraphs for functions are often called **function** (or **mapping**) **diagrams**. When drawing function diagrams, it is standard practice to put the vertices for the domain on the left and the vertices for the codomain on the right, so that all directed edges point from left to right. We may also draw an additional arrow labeled by the name of the function from the domain to the codomain.

Example 2.23. Let $X = \{a, b, c, d\}$ to $Y = \{1, 2, 3, 4\}$ and define the relation f from X to Y via

$$f = \{(a, 2), (b, 4), (c, 4), (d, 1)\}.$$

Since each element X appears exactly once as a first coordinate, f is a function with domain X and codomain Y (i.e., $f : X \to Y$). In this case, we see that $\mathrm{Rng}(f) = \{1, 2, 4\}$. Moreover, we can write things like $f(a) = 2$ and $c \mapsto 4$, and say things like "f maps b to 4" and "the image of d is 1." The function diagram for f is depicted in Figure 2.1.

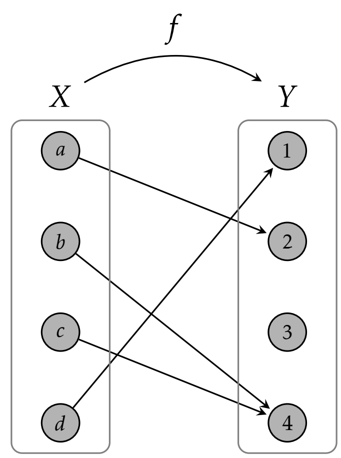

Figure 2.1: Function diagram for a function from $X = \{a, b, c, d, \}$ to $Y = \{1, 2, 3, 4\}$.

Problem 2.24. What properties does the digraph for a relation from X to Y need to have in order for it to represent a function?

Problem 2.25. In high school I am sure that you were told that a graph represents a function if it passes the **vertical line test**. Carefully state what the vertical line test says and then explain why it works.

Sometimes we can define a function using a formula. For example, we can write $f(x) = x^2 - 1$ to mean that each x in the domain of f maps to $x^2 - 1$ in the codomain. However, notice that providing only a formula is ambiguous! A function is determined by its domain, codomain, and the correspondence between these two sets. If we only provide a description for the correspondence, it is not clear what the domain and codomain are. Two functions that are defined by the same formula, but have different domains or codomains are *not* equal.

Example 2.26. The function $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x^2 - 1$ is not equal to the function $g : \mathbb{N} \to \mathbb{R}$ defined by $g(x) = x^2 - 1$ since the two functions do not have the same domain.

Sometimes we rely on context to interpret the domain and codomain. For example, in a calculus class, when we describe a function in terms of a formula, we are implicitly assuming that the domain is the largest allowable subset of $\mathbb{R}$—sometimes called the **default domain**—that makes sense for the given formula while the codomain is $\mathbb{R}$.

Example 2.27. If we write $f(x) = x^2 - 1$, $g(x) = \sqrt{x}$, and $h(x) = \frac{1}{x}$ without mentioning the domains, we would typically interpret these as the functions $f : \mathbb{R} \to \mathbb{R}$, $g : [0, \infty) \to \mathbb{R}$, and $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ that are determined by their respective formulas.

Problem 2.28. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs, or a write a formula as long as the domain and codomain are clear.

(a) A function f from a set with 4 elements to a set with 3 elements such that $\mathrm{Rng}(f) = \mathrm{Codom}(f)$.

(b) A function g from a set with 4 elements to a set with 3 elements such that $\mathrm{Rng}(g)$ is strictly smaller than $\mathrm{Codom}(g)$.

There are a few special functions that we should know the names of. Let X and Y be nonempty sets.

- If $X \subseteq Y$, then the function $\iota : X \to Y$ defined via $\iota(x) = x$ is called the **inclusion map from X into Y**. Note that "ι" is the Greek letter "iota".

- If the domain and codomain are equal, the inclusion map has a special name. If X is a nonempty set, then the function $i_X : X \to X$ defined via $i_X(x) = x$ is called the **identity map** (or **identity function**) **on** X.

- Any function $f : X \to Y$ defined via $f(x) = c$ for a fixed $c \in Y$ is called a **constant function**.

- A **piecewise-defined function** (or **piecewise function**) is a function defined by specifying its output on a partition of the domain. Note that "piecewise" is a way of expressing the function, rather than a property of the function itself.

Example 2.29. The function $f : \mathbb{R} \to \mathbb{R}$ defined via

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x \text{ is negative,} \\ 17, & \text{if } x = 0, \\ -x, & \text{if } x \text{ is positive} \end{cases}$$

is an example of a piecewise-defined function.

It is important to point out that not every function can be described using a formula! Despite your prior experience, functions that can be represented succinctly using a formula are rare.

The next problem illustrates that some care must be taken when attempting to define a function.

Problem 2.30. For each of the following, explain why the given description does not define a function.

(a) Define $f : \{1,2,3\} \to \{1,2,3\}$ via $f(a) = a - 1$.

(b) Define $g : \mathbb{N} \to \mathbb{Q}$ via $g(n) = \frac{n}{n-1}$.

(c) Let $A_1 = \{1,2,3\}$ and $A_2 = \{3,4,5\}$. Define $h : A_1 \cup A_2 \to \{1,2\}$ via

$$h(x) = \begin{cases} 1, & \text{if } x \in A_1 \\ 2, & \text{if } x \in A_2. \end{cases}$$

(d) Define $s : \mathbb{Q} \to \mathbb{Z}$ via $s(a/b) = a + b$.

In mathematics, we say that an expression is **well defined** (or **unambiguous**) if its definition yields a unique interpretation. Otherwise, we say that the expression is not well defined (or is **ambiguous**). For example, if $a, b, c \in \mathbb{R}$, then the expression abc is well defined since it does not matter if we interpret this as $(ab)c$ or $a(bc)$ since the real numbers are associative under multiplication.

When we attempt to define a function, it may not be clear without doing some work that our definition really does yield a function. If there is some potential ambiguity in the definition of a function that ends up not causing any issues, we say that the function is well defined. However, this phrase is a bit of misnomer since all functions are well defined. The issue of whether a description for a proposed function is well defined often arises when defining things in terms of representatives of equivalence classes, or more generally in terms of how an element of the domain is written. For example, the descriptions given in parts (c) and (d) of Problem 2.30 are not well defined. To show that a potentially ambiguous description for a function $f : X \to Y$ is well defined prove that if a and b are two representations for the same element in X, then $f(a) = f(b)$.

Let $f : X \to Y$ be a function.

- The function f is said to be **injective** (or **one-to-one**) if for all $y \in \text{Rng}(f)$, there is a unique $x \in X$ such that $y = f(x)$.
- The function f is said to be **surjective** (or **onto**) if for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$.
- If f is both injective and surjective, we say that f is **bijective**.

An injective function is also called an **injection**, a surjective function is called a **surjection**, and a bijective function is called a **bijection** (or a **one-to-one correspondence**). A one-to-one correspondence should not be confused with a one-to-one function which may not be surjective. To prove that a function $f : X \to Y$ is an injection, we must prove that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. To show that f is surjective, you should start with an arbitrary $y \in Y$ and then work to show that there exists $x \in X$ such that $y = f(x)$.

Problem 2.31. Assume that X and Y are finite sets. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs, or a write a formula as long as the domain and codomain are clear.

(a) A function $f : X \to Y$ that is injective but not surjective.

(b) A function $f : X \to Y$ that is surjective but not injective.

(c) A function $f : X \to Y$ that is a bijection.

(d) A function $f : X \to Y$ that is neither injective nor surjective.

Problem 2.32. Provide an example of each of the following. You may either draw a graph or write down a formula. Make sure you have the correct domain.

(a) A function $f : \mathbb{R} \to \mathbb{R}$ that is injective but not surjective.

(b) A function $f : \mathbb{R} \to \mathbb{R}$ that is surjective but not injective.

(c) A function $f : \mathbb{R} \to \mathbb{R}$ that is a bijection.

(d) A function $f : \mathbb{R} \to \mathbb{R}$ that is neither injective nor surjective.

(e) A function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that is injective.

Problem 2.33. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is _________ if and only if every horizontal line hits the graph of f *at most once*.

This statement is often called the **horizontal line test**. Explain why the horizontal line test is true.

Problem 2.34. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is _________ if and only if every horizontal line hits the graph of f *at least once*.

Explain why this statement is true.

Problem 2.35. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is _________ if and only if every horizontal line hits the graph of f *exactly once*.

Explain why this statement is true.

Problem 2.36. Determine whether each of the following functions is injective, surjective, both, or neither. In each case, you should provide a proof or a counterexample as appropriate.

(a) Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$

(b) Define $g : \mathbb{R} \to [0, \infty)$ via $g(x) = x^2$

(c) Define $h : \mathbb{R} \to \mathbb{R}$ via $h(x) = x^3$

(d) Define $k : \mathbb{R} \to \mathbb{R}$ via $k(x) = x^3 - x$

(e) Define $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via $c(x, y) = x^2 + y^2$

(f) Define $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ via $f(n) = (n, n)$

(g) Define $g : \mathbb{Z} \to \mathbb{Z}$ via

$$g(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

(h) Define $\ell : \mathbb{Z} \to \mathbb{N}$ via

$$\ell(n) = \begin{cases} 2n + 1, & \text{if } n \geq 0 \\ -2n, & \text{if } n < 0 \end{cases}$$

The next two results should not come as as surprise.

Problem 2.37. Prove that the inclusion map $\iota : X \to Y$ for $X \subseteq Y$ is an injection.

Problem 2.38. Prove that the identity function $i_X : X \to X$ is a bijection.

If $f : X \to Y$ and $g : Y \to Z$ are functions, we define $g \circ f : X \to Z$ via $\boxed{(g \circ f)(x) = g(f(x))}$. The function $g \circ f$ is called the **composition of** f **and** g. It is important to notice that the function on the right is the one that "goes first." Moreover, we cannot compose any two random functions since the codomain of the first function must agree with the domain of the second function. In particular, $f \circ g$ may not be a sensible function even when $g \circ f$ exists. Figure 2.2 provides a visual representation of function composition in terms of function diagrams.

Example 2.39. Consider the inclusion map $\iota : X \to Y$ such that X is a proper subset of Y and suppose $f : Y \to Z$ is a function. Then the composite function $f \circ \iota : X \to Z$ is given by

$$f \circ \iota(x) = f(\iota(x)) = f(x)$$

for all $x \in X$. Notice that $f \circ \iota$ is simply the function f but with a smaller domain. In this case, we say that $f \circ \iota$ is the **restriction of** f **to** X, which is often denoted by $\boxed{f|_X}$.

The next problem illustrates that $f \circ g$ and $g \circ f$ need not be equal even when both composite functions exist.

Problem 2.40. Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$ and $g(x) = 3x-5$, respectively. Determine formulas for the composite functions $f \circ g$ and $g \circ f$.

The next problem tells us that function composition is associative.

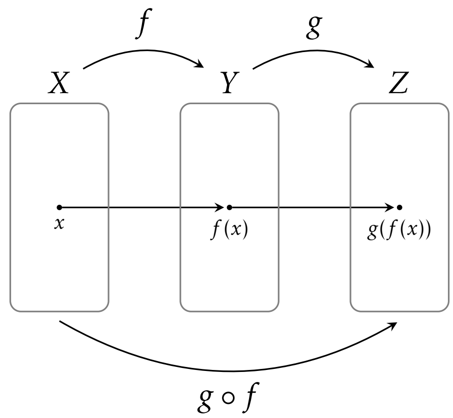

Figure 2.2: Visual representation of function composition.

Problem 2.41. Prove that if $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ are functions, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Problem 2.42. In each case, give examples of finite sets X, Y, and Z, and functions $f : X \to Y$ and $g : Y \to Z$ that satisfy the given conditions. Drawing a function diagram is sufficient.

(a) f is surjective, but $g \circ f$ is not surjective.

(b) g is surjective, but $g \circ f$ is not surjective.

(c) f is injective, but $g \circ f$ is not injective.

(d) g is injective, but $g \circ f$ is not injective.

Problem 2.43. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both surjective functions, then $g \circ f$ is also surjective.

Problem 2.44. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both injective functions, then $g \circ f$ is also injective.

Problem 2.45. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both bijections, then $g \circ f$ is also a bijection.

Problem 2.46. Assume that $f : X \to Y$ and $g : Y \to Z$ are both functions. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If $g \circ f$ is injective, then f is injective.

(b) If $g \circ f$ is injective, then g is injective.

(c) If $g \circ f$ is surjective, then f is surjective.

(d) If $g \circ f$ is surjective, then g is surjective.

There are two important types of sets related to functions. Let $f : X \to Y$ be a function.

- If $S \subseteq X$, the **image** of S under f is defined via

$$\boxed{f(S) := \{f(x) \mid x \in S\}}.$$

- If $T \subseteq Y$, the **preimage** (or **inverse image**) of T under f is defined via

$$\boxed{f^{-1}(T) := \{x \in X \mid f(x) \in T\}}.$$

The image of a subset S of the domain is simply the subset of the codomain we obtain by mapping the elements of S. It is important to emphasize that the function f maps *elements* of X to *elements* of Y, but we can apply f to a *subset* of X to yield a *subset* of Y. That is, if $S \subseteq X$, then $f(S) \subseteq Y$. Note that the image of the domain is the same as the range of the function. That is, $f(X) = \text{Rng}(f)$.

When it comes to preimages, the notation $f^{-1}(T)$ should not be confused with an inverse function (which may or may not exist for an arbitrary function f). For $T \subseteq Y$, $f^{-1}(T)$ is the set of elements in the domain that map to elements in T. As a special case, $f^{-1}(\{y\})$ is the set of elements in the domain that map to $y \in Y$. If $y \notin \text{Rng}(f)$, then $f^{-1}(\{y\}) = \emptyset$. Notice that if $y \in Y$, $f^{-1}(\{y\})$ is always a sensible thing to write while $f^{-1}(y)$ only makes sense if f^{-1} is a function. Also, note that the preimage of the codomain is the domain. That is, $f^{-1}(Y) = X$.

Problem 2.47. Define $f : \mathbb{Z} \to \mathbb{Z}$ via $f(x) = x^2$. List elements in each of the following sets.

(a) $f(\{0,1,2\})$

(b) $f^{-1}(\{0,1,4\})$

Problem 2.48. Find functions f and g and sets S and T such that $f(f^{-1}(T)) \neq T$ and $g^{-1}(g(S)) \neq S$.

Problem 2.49. Suppose $f : X \to Y$ is an injection and A and B are disjoint subsets of X. Are $f(A)$ and $f(B)$ necessarily disjoint subsets of Y? If so, prove it. Otherwise, provide a counterexample.

Problem 2.50. Let $f : X \to Y$ be a function and suppose $A, B \subseteq X$ and $C, D \subseteq Y$. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If $A \subseteq B$, then $f(A) \subseteq f(B)$.

(b) If $C \subseteq D$, then $f^{-1}(C) \subseteq f^{-1}(D)$.

(c) $f(A \cup B) \subseteq f(A) \cup f(B)$.

(d) $f(A \cup B) \supseteq f(A) \cup f(B)$.

(e) $f(A \cap B) \subseteq f(A) \cap f(B)$.

(f) $f(A \cap B) \supseteq f(A) \cap f(B)$.

(g) $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$.

(h) $f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$.

(i) $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$.

(j) $f^{-1}(C \cap D) \supseteq f^{-1}(C) \cap f^{-1}(D)$.

(k) $A \subseteq f^{-1}(f(A))$.

(l) $A \supseteq f^{-1}(f(A))$.

(m) $f(f^{-1}(C)) \subseteq C$.

(n) $f(f^{-1}(C)) \supseteq C$.

2.3 The Real Numbers

The real numbers form the foundation of mathematical analysis. It is worth pointing out that one can carefully construct the real numbers from the natural numbers. However, that will not be the approach we take. Instead, we will simply list the axioms that the real numbers satisfy. Recall that an axiom is a statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proved. Our axioms for the real numbers fall into three categories:

1. **Field Axioms:** These axioms provide the essential properties of arithmetic involving addition and subtraction.

2. **Order Axioms:** These axioms provide the necessary properties of inequalities.

3. **Completeness Axiom:** This axiom guarantees that the familiar number line representing the real numbers does not have any "gaps". We will not introduce this axiom until Chapter 3.

Field Axioms 2.51. There exist functions $(a, b) \mapsto a + b$ and $(a, b) \mapsto ab$ from $\mathbb{R}^2$ to $\mathbb{R}$ satisfying:

(F1) (Associativity for Addition) For all $a, b, c \in \mathbb{R}$ we have $(a + b) + c = a + (b + c)$;

(F2) (Commutativity for Addition) For all $a, b \in \mathbb{R}$, we have $a + b = b + a$;

(F3) (Additive Identity) There exists $0 \in \mathbb{R}$ such that for all $a \in \mathbb{R}$, $0 + a = a$;

(F4) (Additive Inverses) For all $a \in \mathbb{R}$ there exists $-a \in \mathbb{R}$ such that $a + (-a) = 0$;

(F5) (Associativity for Multiplication) For all $a, b, c \in \mathbb{R}$ we have $(ab)c = a(bc)$;

(F6) (Commutativity for Multiplication) For all $a, b \in \mathbb{R}$, we have $ab = ba$;

(F7) (Multiplicative Identity) There exists $1 \in \mathbb{R}$ such that $1 \neq 0$ and for all $a \in \mathbb{R}$, $1a = a$;

(F8) (Multiplicative Inverses) For all $a \in \mathbb{R} \setminus \{0\}$ there exists $a^{-1} \in \mathbb{R}$ such that $aa^{-1} = 1$.

(F9) (Distributive Property) For all $a, b, c \in \mathbb{R}$, $a(b + c) = ab + ac$;

In the language of abstract algebra, Axioms (F1)–(F4) and (F5)–(F8) make each of $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$ an abelian group under addition and multiplication, respectively. Axiom (F9) provides a way for the operations of addition and multiplication to interact. Collectively, Axioms (F1)–(F9) make the real numbers a **field**. It follows from the axioms that the elements 0 and 1 of $\mathbb{R}$ are the unique additive and multiplicative identities. For every $a \in \mathbb{R}$, the elements $-a$ and a^{-1} (as long as $a \neq 0$) are also the unique additive and multiplicative inverses. We will take these facts for granted. For every $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}$, we define the following:

- $a - b := a + (-b)$
- $\dfrac{a}{b} := ab^{-1}$ (for $b \neq 0$)
- $a^n := \begin{cases} \overbrace{aa\cdots a}^{n}, & \text{if } n \in \mathbb{N} \\ 1, & \text{if } n = 0 \text{ and } a \neq 0 \\ \dfrac{1}{a^{-n}}, & \text{if } -n \in \mathbb{N} \text{ and } a \neq 0 \end{cases}$

Using the Field Axioms, we could prove each of the statements in the following theorem. However, we will take each for granted.

Theorem 2.52. For all $a, b, c \in \mathbb{R}$, we have the following:

(a) $a = b$ if and only if $a + c = b + c$;

(b) $0a = 0$;

(c) $-a = (-1)a$;

(d) $(-1)^2 = 1$;

(e) $-(-a) = a$;

(f) If $a \neq 0$, then $(a^{-1})^{-1} = a$;

(g) If $a \neq 0$ and $ab = ac$, then $b = c$.

(h) If $ab = 0$, then either $a = 0$ or $b = 0$.

Problem 2.53. Carefully prove that for all $a, b \in \mathbb{R}$, we have $(a + b)(a - b) = a^2 - b^2$.

Order Axioms 2.54. For $a,b,c \in \mathbb{R}$, there is a relation $<$ on $\mathbb{R}$ satisfying:

(O1) (Trichotomy Law) If $a \neq b$, then either $a < b$ or $b < a$ but not both;

(O2) (Transitivity) If $a < b$ and $b < c$, then $a < c$;

(O3) If $a < b$, then $a + c < b + c$;

(O4) If $a < b$ and $0 < c$, then $ac < bc$;

Given Axioms (O1)–(O4) above, we say that the real numbers are **linearly ordered** (or **totally ordered**). We call numbers greater than zero **positive** and those greater than or equal to zero **nonnegative**. There are similar definitions for **negative** and **nonpositive**. For $a,b \in \mathbb{R}$, we define:

- $a > b$ if $b < a$;
- $a \leq b$ if $a < b$ or $a = b$;
- $a \geq b$ if $b \leq a$.

Using the Order Axioms, we can prove many familiar facts.

Problem 2.55. Prove that for all $a,b \in \mathbb{R}$, if $a,b > 0$, then $a + b > 0$, and if $a,b < 0$, then $a + b < 0$.

The next problem extends Axiom (O3).

Problem 2.56. Prove that for all $a,b,c,d \in \mathbb{R}$, if $a < b$ and $c < d$, then $a + c < b + d$.

Problem 2.57. For all $a \in \mathbb{R}$, $a > 0$ if and only if $-a < 0$.

Problem 2.58. Prove that if a, b, c, and d are positive real numbers such that $a < b$ and $c < d$, then $ac < bd$.

We will take the following theorem for granted. Both statements can be proved using the axioms above.

Theorem 2.59. For all $a,b \in \mathbb{R}$, we have the following:

(a) $ab > 0$ if and only if either $a,b > 0$ or $a,b < 0$;

(b) $ab < 0$ if and only if $a < 0 < b$ or $b < 0 < a$.

Problem 2.60. Prove that for all positive real numbers a and b, $a < b$ if an only if $a^2 < b^2$.

Consider using three cases when approaching the following problem.

Problem 2.61. Prove that for all $a \in \mathbb{R}$, we have $a^2 \geq 0$.

It might come as a surprise that the following result requires proof.

Problem 2.62. Prove that $0 < 1$.

The previous problem together with Problem 2.57 implies that $-1 < 0$ as you expect. It also follows from Axiom (O3) that for all $n \in \mathbb{Z}$, we have $n < n+1$. We assume that there are no integers between n and $n+1$.

Problem 2.63. Prove that for all $a \in \mathbb{R}$, if $a > 0$, then $a^{-1} > 0$, and if $a < 0$, then $a^{-1} < 0$.

Problem 2.64. Prove that for all $a, b \in \mathbb{R}$, if $a < b$, then $-b < -a$.

The last few results allow us to take for granted our usual understanding of which real numbers are positive and which are negative. The next problem yields a result that extends the previous problem.

Problem 2.65. Prove that for all $a, b, c \in \mathbb{R}$, if $a < b$ and $c < 0$, then $bc < ac$.

We could spend weeks building up from the axioms all of the machinery necessary for the rest of the course. Instead we will toss in a few additional axioms to save ourselves a little time.

Additional Order Axioms 2.66. The real numbers satisfy each of the following:

(O5) For every $x \in \mathbb{R}$, there exists $a, b \in \mathbb{R}$ such that $a < x < b$;

(O6) For every $a, b \in \mathbb{R}$, if $a < b$, there exists $x \in \mathbb{R}$ such that $a < x < b$ (in particular, $\frac{a+b}{2}$ is between a and b);

(O7) For every $a \in \mathbb{R}$, there exists $n \in \mathbb{Z}$ such that $n \leq a < n+1$.

Axiom (O7) is sometimes referred to as the **Archimedean Principle**. It turns out that we could derive this axiom from the **Completeness Axiom**, which we will introduce in the next chapter.

Problem 2.67. Prove that for any positive real number a, there exists $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < a$.

For $a, b \in \mathbb{R}$ with $a < b$, we define the following **intervals**:

- $\boxed{(a,b) := \{x \in \mathbb{R} \mid a < x < b\}}$
- $\boxed{(a,\infty) := \{x \in \mathbb{R} \mid a < x\}}$
- $\boxed{(-\infty,b) := \{x \in \mathbb{R} \mid x < b\}}$
- $\boxed{[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}}$

We analogously define $[a,b)$, $(a,b]$, $[a,\infty)$, and $(-\infty,b]$. Intervals of the form (a,b), $(-\infty,b)$, and (a,∞) are called **open intervals** while $[a,b]$, $(-\infty,b]$, and $[a,\infty)$ are referred to as **closed intervals**. A **finite length interval** is any interval of the form (a,b), $[a,b)$, $(a,b]$, and $[a,b]$. For finite length intervals, a and b are called the **endpoints** of the interval.

Notice that Axiom (O5) says that every real number is contained in a finite open interval. In particular, Axiom (O7) says that every non-integer is contained in an open interval with consecutive integer endpoints. Axiom (O6) tells us that every open interval is nonempty. In fact, repeated applications of Axiom (O6) implies that every open interval contains infinitely many points.

Problem 2.68. Assume that there is a positive element of the preimage of $\{2\}$ under the function $f(x) = x^2$ from the reals to the reals. That is, assume $\sqrt{2}$ exists. Prove $\sqrt{2} \in (1,2)$.

Recall that $\sqrt{2}$ is an irrational number. The previous problem provides an example of an irrational number occurring between a pair of distinct rational numbers. The following problems are a good challenge to generalize this.

Problem 2.69. Prove that between any two distinct real numbers there is a rational number.

Problem 2.70. Prove that between any two distinct real numbers there is an irrational number.

Repeated applications of the previous two problems implies that every open interval contains infinitely many rational numbers and infinitely many irrational numbers. In light of these two problems, we say that both the rationals and irrationals are **dense** in every open interval. In particular, they are dense in the real numbers.

There is a special function that we can now introduce. Given $a \in \mathbb{R}$, we define the **absolute value of** a, denoted $|a|$, via

$$|a| := \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

Problem 2.71. Prove that for all $a \in \mathbb{R}$, $|a| \geq 0$ with equality only if $a = 0$.

Problem 2.72. Prove that for all $a, b \in \mathbb{R}$, if $\pm a \leq b$, then $|a| \leq b$. *Note:* Writing $\pm a \leq b$ is an abbreviation for $a \leq b$ and $-a \leq b$.

Problem 2.73. Prove that for all $a \in \mathbb{R}$, $|a|^2 = a^2$.

Problem 2.74. Prove that for all $a \in \mathbb{R}$, $\pm a \leq |a|$.

Problem 2.75. Prove that for all $a, r \in \mathbb{R}$, $|a| \leq r$ if and only if $-r \leq a \leq r$.

In the previous problem, it must be the case that r is nonnegative. The letter r was used because it is the first letter of the word "radius". If r is positive, we can think of the interval $(-r, r)$ as the interior of a 1-dimensional circle with radius r centered at 0.

Problem 2.76. Prove that for all $a, b \in \mathbb{R}$, $|ab| = |a||b|$.

Consider using Problems 2.74 and 2.75 when attacking the next problem. This result is extremely useful.

Problem 2.77 (Triangle Inequality)**.** Prove that for all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

The next problem is related to the Triangle Inequality

Problem 2.78 (Reverse Triangle Inequality)**.** Prove that for all $a, b \in \mathbb{R}$, $|a - b| \geq ||a| - |b||$.

Chapter 3

Sequences and Completeness

Throughout this chapter, our universe of discourse will be the real numbers. Any time we refer to a generic set, we mean a subset of real numbers. We will often refer to an element in a subset of real numbers as a **point**. We begin with a definition.

Definition 3.1. If M is a set, we say that p is an **accumulation point of** M if *every* finite length open interval containing p also contains a point of M different from p.

That is, p is an accumulation point of M if and only if for each open interval O containing p, $(O \setminus \{p\}) \cap M \neq \emptyset$. Notice that if p is an accumulation point of M, then p may or may not be in M.

Problem 3.2. Show that if M is an open interval and $p \in M$, then p is an accumulation point of M.

Problem 3.3. Show that if M is a closed interval and $p \notin M$, then p is not an accumulation point of M.

Problem 3.4. Determine whether the endpoints of an open interval (a, b) are accumulation points of the interval.

It is worth exploring exactly how many points it is possible or impossible for M to have. The next two problems are just a start in investigating that.

Problem 3.5. Show that if M is a set having an accumulation point, then M contains at least two points. Determine whether M must contain at least three points.

Problem 3.6. Show that $\mathbb{Z}$ has no accumulation points.

Problem 3.7. Given sets H and K, determine whether each of the following is true or false. If the statement is true, prove it. Otherwise, provide a counterexample.

(a) If p is an accumulation point of $H \cap K$, then p is an accumulation point of both H and K.

(b) If p is an accumulation point of $H \cup K$, then p is an accumulation point of H or p is an accumulation point of K.

Problem 3.8. Prove that if M is the set of all reciprocals of elements of $\mathbb{N}$, then zero is an accumulation point of M.

We will now begin connecting the concepts of sets to more familiar ones from calculus, beginning with sequences.

Definition 3.9. A **sequence** (of real numbers) is a function p from $\mathbb{N}$ to $\mathbb{R}$.

If $n \in \mathbb{N}$, it is common to write $p_i := p(i)$. We refer to p_i as the ith **term** of the sequence. We will abuse notation and associate a sequence with its list of outputs, namely:

$$\boxed{(p_i)_{i=1}^{\infty} := (p_1, p_2, \ldots)},$$

which we may abbreviate as $\boxed{(p_i)}$.

Example 3.10. Define $p : \mathbb{N} \to \mathbb{R}$ via $p(i) = \frac{1}{2^i}$. Then we have

$$p = \left(\frac{1}{2}, \frac{1}{4}, \ldots\right) = \left(\frac{1}{2^i}\right)_{i=1}^{\infty}.$$

It is important to point out that not every sequence has a description in terms of an algebraic formula. For example, we could form a sequence out of the digits to the right of the decimal in the decimal expansion of π, namely the ith term of the sequence is the ith digit to the right of the decimal. But then there is no nice algebraic formula for describing the ith term of this sequence.

Problem 3.11. Write down several sequences p you are familiar with. If possible, give an algebraic formula for each p_i in terms of i.

Problem 3.12. Give an example of a sequence where the image set of a sequence $\{p_i\}_{i=1}^{\infty}$ is finite. In general, what's the difference between $\{p_i\}_{i=1}^{\infty}$ and $(p_i)_{i=1}^{\infty}$?

There is a deep connection between sequences and accumulation points, which the next few problems will elucidate. First, a definition—one you may have seen in calculus in a different form. When digesting the following definition, try to think about how this definition is capturing the notion that the sequence is getting "closer and closer" to the point that the sequence converges to.

Definition 3.13. We say that the sequence $p = (p_i)_{i=1}^{\infty}$ **converges to the point** x if for every open interval S containing x, there exists an $N \in \mathbb{N}$ such that for all natural numbers $n \geq N$, $p_n \in S$.

In the definition above, we sometimes refer to all p_n with $n \geq N$ as the **tail of the sequence**. Notice that the tail of the sequence depends on N, and hence on the interval S. Informally, we write $\boxed{p \to x}$ or $\boxed{(p_i) \to x}$ to mean that the sequence p converges to the point x. We simply say that p converges if there exists a point x to which the sequence converges. If a sequence does not converge to *any* point x, then we say it **diverges**.

The first problem about this should be used as a place to test ideas for how to prove convergence. Take a moment to recall all of our axioms and results from Chapter 2—you may need them! As you tackle the next few problems, it might be useful to begin by writing down the first several terms of the sequences.

Problem 3.14. Consider the sequence given by $p_n = \frac{1}{n}$ (remember, $n \in \mathbb{N}$ is part of the definition of a sequence). Show that $p = (p_i)_{i=1}^{\infty}$ converges to 0.

Problem 3.15. Consider the sequence given by $p_n = 1 - \frac{1}{n}$. Show that p converges to 1.

Problem 3.16. Consider the sequence with even terms $p_{2n} = \frac{1}{2n-1}$ and odd terms $p_{2n-1} = \frac{1}{2n}$. Show that p converges to 0.

Problem 3.17. Consider the sequence with odd terms $p_{2n-1} = \frac{1}{2n-1}$ and even terms $p_{2n} = 1 + \frac{1}{2n}$. Determine whether p converges to 0.

The following problem connects accumulation points and sequences. The most profound property of the real numbers is part of this connection, as we shall soon see.

Problem 3.18. Show that if p converges to the point x *and* for each $i \in \mathbb{N}$, $p_i \neq p_{i+1}$, then x is an accumulation point of the image set of $(p_i)_{i=1}^{\infty}$. Why do we need the restriction that $p_i \neq p_{i+1}$? Is this an absolutely necessary restriction for x to be an accumulation point of the image set?

Problem 3.19. Show that the sequence from Problem 3.14 does not converge to a point other than zero.

Problem 3.20. Show that if p converges to the point x and y is a point different from x, then p does *not* converge to y.

We now explore some basic facts concerning the convergence of sequences. In these proofs, you may have to think a little more explicitly about what the intervals around x look like in order to combine sequences. Try doing some examples with explicit numbers in order to get a sense of how to approach the proofs.

Problem 3.21. Show that if c is a real number and $p = (p_i)_{i=1}^{\infty}$ converges to x, then the sequence $cp = (cp_i)_{i=1}^{\infty}$ converges to cx.

Problem 3.22. Show that if $p = (p_i)_{i=1}^{\infty}$ converges to x and $q = (q_i)_{i=1}^{\infty}$ converges to y, then $(p_i + q_i)_{i=1}^{\infty}$ converges to $x + y$.

Products and quotients of sequences behave like you think they will, as well. We will include one special case soon.

Now we introduce a few more definitions that will lead us to one of the key axioms for the real numbers (Completeness Axiom 3.37). We'll continue to see interplay between sequences and sets.

Definition 3.23. We say that a set M is **bounded** if M is a subset of some closed interval.

Definition 3.24. We say that a set M is **bounded above** if there is a point z such that if $x \in M$ then $x \leq z$; such a point is an **upper bound**.

Problem 3.25. The property of a set M being **bounded below** and the notion of a **lower bound** are defined similarly; try defining them.

Problem 3.26. Show that a set (in $\mathbb{R}$) being bounded is the same as it being bounded above and below.

Problem 3.27. Find all upper bounds for $(0,1)$, $[0,1]$, and $(0,1) \cap \mathbb{Q}^C$ (irrationals between 0 and 1).

Problem 3.28. If the sequence $(p_i)_{i=1}^{\infty}$ converges to the point x, then the image set $\{p_i\}_{i=1}^{\infty}$ is bounded.

You can use this concept to prove some of the more difficult sequence convergence properties.

Problem 3.29. Show that if q converges to 0 and p converges to x, then $(q_i \cdot p_i)_{i=1}^{\infty}$ converges to 0.

Now we start to approach the heart of why calculus works.

Definition 3.30. We say p is a **supremum** (or **least upper bound**) of a set M if p is an upper bound of M and $p \leq q$ for any other upper bound q of M. If the supremum of M exists, it is denoted $\boxed{\sup(M)}$.

Problem 3.31. Define the **infimum** (or **greatest lower bound**) by analogy. If the infimum of M exists, it is denoted $\boxed{\inf(M)}$.

Problem 3.32. Find the suprema of $(0,1)$, and $(0,1) \cap \mathbb{Q}^C$. If we could apply the definition of supremum to $\emptyset$, what would its supremum be?

Problem 3.33. Prove that the supremum of a set is unique, if it exists.

Problem 3.34. If M and N are sets with suprema, characterize the supremum of $M \cup N$.

If M and N are sets, define $cM := \{cx \mid x \in M\}$ and $M + N := \{x + y \mid x \in M, y \in N\}$.

Problem 3.35. Assuming M and N have suprema, prove one of the following.

(a) If $c > 0$, then $\sup(cM) = c\sup(M)$.

(b) $\sup(M + N) = \sup(M) + \sup(N)$.

Problem 3.36. Show that $c\inf(M) = \sup(cM)$ if $c < 0$. What other properties are there relating inf, sup, and c?

The reason the supremum is so important is because of the following fundamental axiom.

Completeness Axiom 3.37. If M is a nonempty set that is bounded above, then M has a supremum.

Given the Completeness Axiom, we say that the real numbers satisfy the **least upper bound property**.

Problem 3.38. This problem was a duplicate of Problem 3.40. Leaving this placeholder here so as not to mess up numbering.

Problem 3.39. Find an example of a sequence $(p_i)_{i=1}^{\infty}$ such that its image set $\{p_i\}_{i=1}^{\infty}$ is unbounded and hence does not have a supremum.

Problem 3.40. Show that the Completeness Axiom is *not* true if one requires that the supremum be a rational number. This show that the rationals do not satisfy the Completeness Axiom.

It will be useful in the future to have an equivalent way to formulate completeness in terms of sequences.

Definition 3.41. We say that a sequence p is **nondecreasing** if $p_i \leq p_{i+1}$ for all $i \in \mathbb{N}$. The concept of **nonincreasing** is defined similarly. A function that is either nondecreasing or nonincreasing is said to be **monotone**.

Problem 3.42. Replace $\leq$ above with $<$ to define the notion of (strictly) **increasing**. Find examples of nondecreasing sequences that are not increasing. Similarly, define (strictly) **decreasing**.

Problem 3.43 (Monotone Convergence Theorem)**.** Prove that if $(p_i)_{i=1}^{\infty}$ is a nondecreasing sequence such that the image set $\{p_i\}_{i=1}^{\infty}$ is bounded above, then $(p_i)_{i=1}^{\infty}$ converges to some point x.

The previous result is equivalent to the Completeness Axiom. The next problem asks you to verify this, but this is not a result that we need going forward, but rather is an interesting side story.

Problem 3.44. Assuming the result of Problem 3.43, prove the Completeness Axiom.

Why is all this so important? One reason is that we can use the completeness of the reals to *prove* Axiom (07) (sometimes called the Archimedean Principle). It may be thought of as the "real" reason why the following is true, since open intervals can be as small as we need them to be.

Problem 3.45. Using Problem 3.43, show that for any point x, there is an $n \in \mathbb{Z}$ such that $n > x$.[1]

At this point, it is not necessary that we complete the following problem, but you might find doing so to be an interesting challenge.

[1] *Hint:* Use a proof by contradiction.

Problem 3.46. Prove that Axiom (O7) follows from the Completeness Axiom.

Problem 3.47. Let M be a nonempty set that is bounded above, with supremum x. Prove that there exists a nondecreasing sequence (p_i) that converges to x, where the image set $\{p_i\} \subseteq M$.

Definition 3.48. A sequence $(b_k)_{k=1}^{\infty}$ is a **subsequence** of $(a_n)_{n=1}^{\infty}$ if there is a sequence of natural numbers $(n_i)_{i=1}^{\infty}$ with $n_i < n_{i+1}$ such that $b_k = a_{n_k}$.

Problem 3.49. Give some examples of subsequences of the sequence from Problem 3.14.

Problem 3.50. Prove that if a sequence converges to x, so does any subsequence of that sequence.

Problem 3.51. Suppose $(p_{i_k})_{k=1}^{\infty}$ is a subsequence of $(p_i)_{i=1}^{\infty}$. If p_{i_k} converges to x, does this imply that p_i converges to x? Justify your answer.

Problem 3.52. Provide an example of a sequence (p_i) with image set $\{p_i\} \subseteq \mathbb{N}$ such that *every* sequence of natural numbers is a subsequence of (p_i).

Problem 3.53. Prove that every sequence of real numbers has a nonincreasing or nondecreasing subsequence.

Problem 3.54 (Bolzano–Weierstrass Theorem)**.** Prove that every sequence with bounded image set has a convergent subsequence.

Chapter 4

Standard Topology of the Real Line

In this chapter, we will take a brief tour of the fascinating world of open and closed subsets of the real line.

Definition 4.1. A set M is defined to be an **open** set if for every point $x \in M$ there is an open interval that contains x and is a subset of M.

It is immediate from the definition that open intervals are in fact open sets.

Problem 4.2. Provide several examples of open sets that are not simply open intervals.

Problem 4.3. Show that the intersection of two (and hence, by induction, finitely many) open sets is open, but that the intersection of infinitely many open sets may not be open.

Definition 4.4. A set M is defined to be a **closed** set if every accumulation point of M is contained in M.

If M is a set, the set of accumulation points of M is sometimes denoted by M'. Using this notation, we can say that a set M is closed if and only if $M' \subseteq M$. Note that if a set M has no accumulation points, then it is vacuously closed.

Problem 4.5. Is every closed interval a closed set? Justify your answer.

Problem 4.6. Provide several examples of closed sets that are not closed intervals.

Problem 4.7. Provide an example of a set that is both open and closed.

One annoying feature of the terminology is that if a set is not open, it may or may not be closed. Similarly, if a set is not closed, it may or may not be open. That is, open and closed are not opposites of each other.

Problem 4.8. Provide an example of a set that is neither open nor closed.

Despite the fact that open and closed are not opposites of each other, there is a nice connection involving complements.

Problem 4.9. Prove that if M is a closed set such that $M \neq \mathbb{R}$, then M^c is an open set.

Problem 4.10. Prove that if M is an open set, then M^c is a closed set.

Definition 4.11. A set K is called **compact** if K is both closed and bounded.

It is important to point out that there is a more general definition of compact in an arbitrary topological space. However, using our notions of open and closed, it is a theorem that a subset of the real line is compact if and only if it is closed and bounded.

Problem 4.12. Provide several examples of sets that are compact and some that are not compact. Are finite sets compact?

Problem 4.13. Prove that if K is a nonempty compact set, then $\sup(K), \inf(K) \in K$.

The next problem is related to the Bolzano–Weierstrass Theorem.

Problem 4.14. Show that if K is a nonempty compact set, then any sequence with image set in K has a subsequence that converges to a point in K.

Problem 4.15. Come up with examples showing that if M is not closed or not bounded, then there exists a sequence with image set in M that does not have a subsequence converging to a point in M (or possibly not at all).

On the real line, compactness and satisfying the Bolzano–Weierstrass Theorem are equivalent. However, one can concoct examples of other mathematical spaces where they are not the same.

Chapter 5

Continuity

Definition 5.1. We say that a function f is **continuous at a point** x in its domain (or at the point $(x, f(x))$) if, for any open interval S containing $f(x)$, there is an open interval T containing x such that if $t \in T$ is in the domain of f, then $f(t) \in S$.

Definition 5.2. A function f is **continuous** if it is continuous at every point in its domain.

Let's show that this definition of continuity behaves the way we expect from calculus.

Problem 5.3. Show that each of the following functions is continuous.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x$.

(b) $g : \mathbb{R} \to \mathbb{R}$ defined via $g(x) = 2x$.

(c) $h : \mathbb{R} \to \mathbb{R}$ defined via $h(x) = x + 3$.

Problem 5.4. Show that any linear function given by $f(x) = mx + b$ is continuous for all $x \in \mathbb{R}$.

The next problem tells us that we can reframe continuity in terms of distance.

Problem 5.5. Let f be a function. Prove that f is continuous at x if and only if for every $\epsilon > 0$, then there exists $\delta > 0$ so that if t is in the domain of f and $|t - x| < \delta$, then $|f(t) - f(x)| < \epsilon$.

The previous characterization is typically referred to as the "ϵ–δ definition of continuity", although for us it is a theorem instead of a definition. This characterization is used as the definition of continuity in metric spaces.

Problem 5.6. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Find all points x where f is continuous and justify your answer.

Problem 5.7. Define $g : \{0\} \to \mathbb{R}$ via $g(0) = 0$. Show that g is continuous at $x = 0$.

Problem 5.8. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Find all points x where f is continuous and justify your answer.

Problem 5.9. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$. Prove that f is continuous.

Problem 5.10. Find a continuous function f and an open interval U such that the preimage $f^{-1}(U)$ is not an open interval.

Problem 5.11. Let f be a function. Prove that f is continuous if and only if the preimage $f^{-1}(U)$ of every open set U is an open set intersected with the domain of f.

The previous characterization of continuity is often referred to as the "open set definition of continuity" and is the definition used in topology.

It turns out that there is a deep connection between continuity and sequences!

Definition 5.12. We say that a function f is **sequentially continuous at a point** x if, for every sequence $(x_i)_{i=1}^{\infty}$ (in the domain of f) converging to x, it is also true that $(f(x_i))_{i=1}^{\infty}$ converges to $f(x)$.

Problem 5.13. Let f be a function. Prove that f is continuous at x if and only if f is sequentially continuous at x.

The upshot of the previous problem is that the notions of being *continuous at a point* and *sequentially continuous at a point* are equivalent on the real numbers. However, there are contexts in mathematics where the two are not equivalent. This is a topic in a branch of mathematics called **topology**. If you want to know more, check out the following YouTube video:

https://www.youtube.com/watch?v=sZ5fBHGYurg

The sequential way of thinking of continuity often makes proving some basic facts concerning continuity easier.

At this point, we have four different ways of thinking about continuity.

- Definition 5.1 using open intervals.
- Problem 5.5 using ϵ and δ.
- Problem 5.11 using inverse images of open sets.
- Problem 5.13 using sequential continuity.

You should take the time to review each one. Moreover, it is worth pointing out that three of the four characterizations involve continuity at a point. Which one does not? For the remainder of the book, feel free to use which ever characterization you'd like.

Problem 5.14. Suppose f and g are functions that are continuous at x and let $c \in \mathbb{R}$. Prove that each of the following functions are also continuous at x.

(a) cf

(b) $f+g$

(c) $f-g$

(d) fg

Problem 5.15. Prove that every polynomial is continuous on all of $\mathbb{R}$.

Problem 5.16. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and consider the closed interval $[a,b]$. Is the image $f([a,b])$ always a closed interval? If so, prove it. Otherwise, provide a counterexample.

Problem 5.17. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and K is a compact set, then the image $f(K)$ is compact.

The next result tells us that continuous functions always attain a maximum value on closed intervals. Of course, we have an analogous result involving minimums.

Problem 5.18 (Extreme Value Theorem)**.** Let $I = [a,b]$ be a closed interval. Prove that if f is continuous on I, then there exists $x_M \in I$ such that $f(x_M) \geq f(x)$ for all $x \in I$.

Problem 5.19. Is the hypothesis that I is closed needed in the Extreme Value Theorem? Justify your answer.

Problem 5.20. Is the converse of the Extreme Value Theorem true? That is, if a function attains a maximum value over a closed interval, does that imply that the function is continuous. If so, prove it. Otherwise, provide a counterexample.

Problem 5.21. Let $f : [0,1] \to \mathbb{R}$ and assume that the image $f([0,1])$ has a supremum. Show there is a sequence of points $(x_i)_{i=1}^{\infty}$ in $[0,1]$ such that $(f(x_i))_{i=1}^{\infty}$ converges to that supremum. Does this show that f is continuous on $[0,1]$?

Definition 5.22. We say that a set M is **disconnected** if there exists two disjoint open sets U_1 and U_2 such that $M \cap U_1$ and $M \cap U_2$ are nonempty but $M \subseteq U_1 \cup U_2$ (equivalently, $M = (M \cap U_1) \cup (M \cap U_2)$). If a set is non disconnected, then we say that it is **connected**.

Problem 5.23. Provide examples of sets that are disconnected. Also, provide some examples of sets that are connected. In each case, try to find examples with various other properties such as open, closed, neither open nor closed, bounded, unbounded, and compact. You do not need to worry about justifying your examples in this exercise.

Problem 5.24. Determine whether each of the following sets is connected or disconnected. Prove your answers.

(a) $\mathbb{Q}$

(b) $[0,1] \cup [2,3]$

(c) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(d) $\{17\}$

(e) $\emptyset$

The next problem is harder than it looks.

Problem 5.25. Prove that every closed interval $[a,b]$ is connected.

The next problem is analogous to Problem 5.17. It also likely captures your intuition about continuity from high school and calculus.

Problem 5.26. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and M is a connected set, then the image $f(M)$ is connected.

Problem 5.27 (Intermediate Value Theorem)**.** Let $I = [a,b]$ be a closed interval. Prove that if f is continuous on I such that $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, then there exists $r \in I$ such that $f(r) = 0$.

We can generalize the previous result, which is also often referred to as the Intermediate Value Theorem.

Problem 5.28. Let $I = [a,b]$ be a closed interval. Prove that if f is continuous on I such that $f(a) < c < f(b)$ or $f(a) > c > f(b)$, then there exists $r \in I$ such that $f(r) = c$.

Problem 5.29. Is the hypothesis that I is closed needed in the Intermediate Value Theorem? Justify your answer.

Problem 5.30. Is the converse of the Intermediate Value Theorem true? If so, prove it. Otherwise, provide a counterexample.

Problem 5.31. Let $f : [0,1] \to \mathbb{R}$ be a function such that $f(0) = -1$, $f(1) = 1$, and $f([0,1]) = \{-1,1\}$. Prove that there exists $x \in [0,1]$ such that f is not continuous at x.

Chapter 6

Limits

We are now prepared to dig into limits, which you are likely familiar with from calculus. However, chances are that you were never introduced to the formal definition.

Definition 6.1. Let $f : A \to \mathbb{R}$ be a function, where $A \subseteq \mathbb{R}$. The **limit** of f as x approaches a is L if the following two conditions hold:

1. The point a is an accumulation point of A, and
2. For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Notationally, we write this as

$$\boxed{\lim_{x \to a} f(x) = L.}$$

It turns out that limits are unique if they exist. You may assume this going forward.

Problem 6.2. Why do we require $0 < |x - a|$ in Definition 6.1?

Problem 6.3. Why do you think we require a to be an accumulation point of the domain of f? What happens if $a \in A$ but a is not an accumulation point of A (such points are called **isolated points** of A)?

Example 6.4. It should come as no surprise to you that $\lim_{x \to 5}(3x + 2) = 17$. Let's prove this using Definition 6.1. First, notice that the default domain of $f(x) = 3x + 2$ is the set of real numbers. So, any x-value we choose will be in the domain of the function. Now, let $\epsilon > 0$. Choose $\delta = \epsilon/3$. You'll see in a moment why this is a good choose for δ. Suppose $x \in \mathbb{R}$ such that $0 < |x - 5| < \delta$. We see that

$$|(3x + 2) - 17| = |3x - 15| = 3 \cdot |x - 5| < 3 \cdot \delta = 3 \cdot \epsilon/3 = \epsilon.$$

This proves the desired result.

Example 6.5. Let's try something a little more difficult. Let's prove that $\lim_{x \to 3} x^2 = 9$. As in the previous example, the default domain of our function is the set of real numbers. Our goal is to prove that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in \mathbb{R}$ such that

$0 < |x-3| < \delta$, then $|x^2-9| < \epsilon$. Let $\epsilon > 0$. We need to figure out what δ needs to be. Notice that

$$|x^2-9| = |x+3| \cdot |x-3|.$$

The quantity $|x-3|$ is something we can control with δ, but the quantity $|x+3|$ seems to be problematic.

To get a handle on what's going on, let's temporarily assume that $\delta = 1$ and suppose that $0 < |x-3| < 1$. This means that x is within 1 unit of 3. In other words, $2 < x < 4$. But this implies that $5 < x+3 < 7$, which in turn implies that $|x+3|$ is bounded above by 7. That is, $|x+3| < 7$ when $0 < |x-3| < 1$. It's easy to see that we still have $|x+3| < 7$ even if we choose δ smaller than 1. That is, we have $|x+3| < 7$ when $0 < |x-3| < \delta \leq 1$. Putting this altogether, if we suppose that $0 < |x-3| < \delta \leq 1$, then we can conclude that

$$|x^2-9| = |x+3| \cdot |x-3| < 7 \cdot |x-3|.$$

This work informs our choice of δ, but remember our scratch work above hinged on knowing that $\delta \leq 1$. If $\epsilon/7 \leq 1$, we should choose $\delta = \epsilon/7$. However, if $\epsilon/7 > 1$, the easiest thing to do is to just let $\delta = 1$. Let's button it all up.

Let $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon/7\}$ and suppose $0 < |x-3| < \delta$. We see that

$$|x^2-9| = |x+3| \cdot |x-3| < 7 \cdot |x-3| < 7 \cdot \delta \leq \epsilon$$

since

$$7 \cdot \delta = \begin{cases} 7, & \text{if } \epsilon > 7 \\ 7 \cdot \epsilon/7, & \text{if } \epsilon \leq 7. \end{cases}$$

Therefore, $\lim_{x \to 3} x^2 = 9$, as expected.

Problem 6.6. Prove that $\lim\limits_{x \to 1}(17x - 42) = -25$ using Definition 6.1.

Problem 6.7. Prove that $\lim\limits_{x \to 2} x^3 = 8$ using Definition 6.1.

Problem 6.8. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 17, & \text{if } x = 0. \end{cases}$$

Using Definition 6.1, prove that $\lim_{x \to 0} f(x) = 0$.

Problem 6.9. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ -1, & \text{if } x > 0. \end{cases}$$

Using Definition 6.1, prove that $\lim\limits_{x \to 0} f(x)$ does not exist.

Problem 6.10. Define $f:\mathbb{R}\to\mathbb{R}$ via

$$f(x)=\begin{cases}1, & \text{if } x\in\mathbb{Q}\\ 0, & \text{otherwise.}\end{cases}$$

Using Definition 6.1, prove that $\lim\limits_{x\to a} f(x)$ does not exist for all $a\in\mathbb{R}$.

Problem 6.11. Let $f:A\to\mathbb{R}$ be a function. Prove that if $\lim\limits_{x\to a} f(x)$ exists, then the limit is unique.

The $\epsilon-\delta$ approach to a function f being continuous at $x=a$ (see Problem 5.5) looks awfully similar to the definition of the limit of f as x approaches a. Let's explore this a bit.

Problem 6.12. Explain the similarities and differences between the definitions of continuity at $x=a$ versus the limit as x approaches a. State a theorem about continuity involving limits. You will have to make a special statement about isolated points of the domain.

Perhaps not surprisingly, there is a nice connection between limits and sequences.

Problem 6.13. Let $f:A\to\mathbb{R}$ be a function and let a be an accumulation point of A. Then $\lim_{x\to a} f(x)$ exists if and only if for every sequence (x_n) in $A\setminus\{a\}$ converging to a, the sequence $(f(x_n))$ converges, in which case, $\lim_{x\to a} f(x)$ equals the limit of the sequence $(f(x_n))$. This is often written as

$$\lim_{x\to a} f(x)=\lim_{n\to\infty} f(x_n).$$

In order for limits to be a useful tool, we need to prove a few important facts.

Problem 6.14 (Limit Laws)**.** Let $f:A\to\mathbb{R}$ and $g:B\to\mathbb{R}$ be functions. Prove each of the following using Definition 6.1.

(a) If $c\in\mathbb{R}$, then $\lim\limits_{x\to a} c=c$.

(b) If $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exist, then

$$\lim_{x\to a}(f(x)\pm g(x))=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x).$$

(c) If $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exist, then

$$\lim_{x\to a}(f(x)\cdot g(x))=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x).$$

(d) If $c\in\mathbb{R}$ and $\lim\limits_{x\to a} f(x)$ exists, then

$$\lim_{x\to a}(c\cdot f(x))=c\cdot\lim_{x\to a} f(x).$$

(e) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and $\lim_{x\to a} g(x) \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

(f) If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

The next problem is extremely useful. It allows us to simplify our calculations when computing limits.

Problem 6.15. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions and let a be an accumulation point of A. If there exists an open interval S such that $f(x) = g(x)$ for all $x \in (S \cap A) \setminus \{a\}$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

provided one of the limits exists.

Chapter 7

Differentiation

It's time for derivatives!

Definition 7.1. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$ such that f is defined on some open interval I containing a (i.e., $a \in I \subseteq A$). The **derivative** of f at a is defined via

$$\boxed{f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

provided this limit exists. If $f'(a)$ exists, then we say that f is **differentiable** at a. More generally, we say that f is **differentiable** on $B \subseteq A$ if f is differentiable at every point in B. As a special case, f is said to be **differentiable** if it is differentiable at every point in its domain. If f does indeed have a derivative at some points in its domain, then the **derivative** of f is the function denoted by f', such that for each number x at which f is differentiable, $f'(x)$ is the derivative of f at x. We may also write

$$\frac{d}{dx}[f(x)] := f'(x).$$

The lefthand side of the equation above is typically read as, "the derivative of f with respect to x." The notation $f'(x)$ is commonly referred to as "Newton's notation" for the derivative while $\frac{d}{dx}[f(x)]$ is often referred to as "Liebniz's notation".

Note that the definition of derivative automatically excludes the kind of behavior we saw with continuous functions, where a function defined only at a single point was continuous.

Problem 7.2. Find the derivative of $f(x) = x^2 - x + 1$ at $a = 2$.

Problem 7.3. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = c$ for some constant $c \in \mathbb{R}$. Prove that f is differentiable on $\mathbb{R}$ and $f'(x) = 0$ for all $x \in \mathbb{R}$.

Problem 7.4. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = mx + b$ for some constants $m, b \in \mathbb{R}$. Prove that f is differentiable and $f'(x) = m$ for all $x \in \mathbb{R}$.

Problem 7.5. Find and prove a formula for the derivative of $f(x) = ax^2 + bx + c$ for any $a, b, c \in \mathbb{R}$.

Problem 7.6. Explain why any function defined only on $\mathbb{Z}$ cannot have a derivative.

Problem 7.7. If f is differentiable at x and $c \in \mathbb{R}$, prove that the function cf also has a derivative at x and $(cf)'(x) = cf'(x)$.

Problem 7.8. If f and g are differentiable at x, show that the function $f+g$ also has a derivative at x and $(f+g)'(x) = f'(x) + g'(x)$.

The next problem tells us that differentiability implies continuity.

Problem 7.9. Prove that if f has a derivative at $x = a$, then f is also continuous at $x = a$.

The converse of the previous theorem is not true. That is, continuity does not imply differentiability.

Problem 7.10. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = |x|$.

(a) Prove that f is continuous at every point in its domain.

(b) Prove that f is differentiable everywhere except at $x = 0$.

Problem 7.11. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is continuous at $x = 0$, but not differentiable at $x = 0$.

The next problem states the well-known Product and Quotient Rules for Derivatives. You will need to use Problem 7.9 in their proofs.

Problem 7.12. Suppose f and g are differentiable at x. Prove each of the following:

(a) (Product Rule) The function fg is differentiable at x. Moreover, its derivative function is given by

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

(b) (Quotient Rule) The function f/g is differentiable at x provided $g(x) \neq 0$. Moreover, its derivative function is given by

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

The next problem is sure to make your head hurt.

Problem 7.13. Define $g : \mathbb{R} \to \mathbb{R}$ via

$$g(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{otherwise.} \end{cases}$$

Now, define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2 g(x)$. Determine where f is differentiable.

The next result tells us that if a differentiable function attains a maximum value at some point in an open interval contained in the domain of the function, then the derivative is zero at that point. In a calculus class, we would say that differentiable functions attain local maximums at critical numbers.

Problem 7.14. Let $f : A \to \mathbb{R}$ be a function such that $[a,b] \subseteq A$, $f'(c)$ exists for some $c \in (a,b)$, and $f(c) \geq f(x)$ for all $x \in (a,b)$. Prove that $f'(c) = 0$.

Problem 7.15. Let $f : A \to \mathbb{R}$ be a function such that $f'(c) = 0$ for some $c \in A$. Does this imply that there exists an open interval $(a,b) \subseteq A$ containing c such that either $f(x) \geq f(c)$ or $f(x) \leq f(c)$ for all $x \in (a,b)$? If so, prove it. Otherwise, provide a counterexample.

The next problem asks you to prove a result called Rolle's Theorem.

Problem 7.16 (Rolle's Theorem)**.** Let $f : A \to \mathbb{R}$ be a function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, then prove that there exists a point $c \in (a,b)$ such that $f'(c) = 0$.[1]

We can use Rolle's Theorem to prove the next result, which is the well-known Mean Value Theorem.

Problem 7.17 (Mean Value Theorem)**.** Let $f : A \to \mathbb{R}$ be a function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$ and differentiable on (a,b), then prove that there exists a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$[2]

Problem 7.18. Let $f : A \to \mathbb{R}$ be a function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$ and differentiable on (a,b) such that $f'(x) = 0$ for all $x \in (a,b)$, then prove that f is constant over $[a,b]$.[3]

Problem 7.19. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ such that $[a,b] \subseteq A$. Prove that if f and g are continuous on $[a,b]$ and $f'(x) = g'(x)$ for all $x \in (a,b)$, then there exists $C \in \mathbb{R}$ such that $f(x) = g(x) + C$ for all $x \in [a,b]$.

Problem 7.20. Is the converse of the previous problem true? If so, prove it. Otherwise, provide a counterexample.

[1] *Hint:* First, apply the Extreme Value Theorem to f and $-f$ to conclude that f attains both a maximum and minimum on $[a,b]$. If both the maximum and minimum are attained at the end points of $[a,b]$, then the maximum and minimum are the same and thus the function is constant. What does Problem 7.3 tell us in this case? But what if f is not constant over $[a,b]$? Try using Problem 7.14.

[2] *Hint:* Cleverly define the function $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. Is g continuous on $[a,b]$? Is g differentiable on (a,b)? Can we apply Rolle's Theorem to g using the interval $[a,b]$? What can you conclude? Magic!

[3] *Hint:* Try applying the Mean Value Theorem to $[a,t]$ for every $t \in (a,b)$.

Chapter 8

Integration

Unlike with differentiation, we will need a number of auxiliary definitions for beginning integration.

Definition 8.1. A set of points $P = \{t_0, t_1, \ldots, t_n\}$ is a **partition** of the closed interval $[a, b]$ if $a = t_0 < t_1 < \cdots t_{n-1} < t_n = b$. If $t_i - t_{i-1} = \frac{b-a}{n}$ for all i, we say that the partition is a **regular partition** of $[a, b]$. In this case, we may use the notation $\Delta t := t_i - t_{i-1}$.

Problem 8.2. Give some partitions, regular and not regular, of $[0, 1]$, $[2, 4]$, and $[-1, 0]$.

Definition 8.3. We say that a function is **bounded** if it has bounded image set.

Important! For the next four definitions, we assume that f is a bounded function with domain equal to some closed interval $[a, b]$.

Definition 8.4. Let f be a bounded function with domain $[a, b]$ and let $\{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. We say that any sum S of the form

$$S = \sum_{i=1}^{n} f(x_i)(t_i - t_{i-1}),$$

where $x_i \in [t_{i-1}, t_i]$ is a **Riemann sum** for f on $[a, b]$.

Definition 8.5. Let f be a bounded function with domain $[a, b]$ and let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. For each $i \in \{1, 2, \ldots, n\}$, define $M_i := \sup\{f(x) \mid x \in [t_{i-1}, t_i]\}$. We say that the sum

$$U_P(f) := \sum_{i=1}^{n} M_i(t_i - t_{i-1}),$$

is the **upper Riemann sum** for f with partition P.

Definition 8.6. Let f be a bounded function with domain $[a, b]$ and let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. For each $i \in \{1, 2, \ldots, n\}$, define $m_i := \inf\{f(x) \mid x \in [t_{i-1}, t_i]\}$. We say that the sum

$$L_P(f) := \sum_{i=1}^{n} m_i(t_i - t_{i-1}),$$

is the **lower Riemann sum** for f with partition P.

Problem 8.7. Draw pictures that capture the concepts of upper and lower Riemann sums.

Contrary to the name, upper and lower Riemann sums are not always Riemann sums.

Problem 8.8. Give an example of an interval $[a,b]$, partition P, and bounded function f such that $U_p(f)$ is not a Riemann sum.

Problem 8.9. Define $f : [0,1] \to \mathbb{R}$ via

$$f(x) = \begin{cases} 0, & x \in (0,1] \\ 1, & x = 0. \end{cases}$$

(a) Show that $U_P(f) > 0$ for all partitions of $[0,1]$.

(b) Show that for any positive number ϵ there is a partition P_ϵ such that $U_{P_\epsilon}(f) < \epsilon$.

(c) Fully describe all lower sums of f on $[0,1]$.

Problem 8.10. Define $f : [0,1] \to \mathbb{R}$ via $f(x) = x$. For each $n \in \mathbb{N}$, let P_n be the regular partition of $[0,1]$ given by $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$.

(a) Compute $U_{P_5}(f)$.

(b) Give a formula for $U_{P_n}(f)$.[1]

(c) Compute $L_{P_5}(f)$.

(d) Give a formula for $L_{P_n}(f)$.

Problem 8.11. Suppose that f is a bounded function on $[a,b]$ with lower bound m and upper bound M. Show that for any partition P of $[a,b]$, $U_P(f) \leq M(b-a)$ and $L_P(f) \geq m(b-a)$.

Problem 8.12. Suppose that f is a bounded function on $[a,b]$ and P is a partition of $[a,b]$. Show that $L_P(f) \leq U_P(f)$.

One consequence of Problem 8.11 is that the set of all upper, respectively lower, sums of f over $[a,b]$ is a bounded set. This implies that if f is a bounded function on $[a,b]$, then the following supremum and infinum exist:

$$\inf\{U_P(f) \mid P \text{ is a partition of } [a,b]\}$$

$$\sup\{L_P(f) \mid P \text{ is a partition of } [a,b]\}$$

This leads to the following definition.

[1] Recall that the sum $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Definition 8.13. Let f be a bounded function with domain $[a,b]$. The **upper integral** of f from a to b is defined via

$$\overline{\int_a^b} f := \inf\{U_P(f) \mid P \text{ is a partition of } [a,b]\}.$$

Similarly, the **lower integral** of f from a to b is defined via

$$\underline{\int_a^b} f := \sup\{L_P(f) \mid P \text{ is a partition of } [a,b]\}.$$

Problem 8.14. Compute the upper and lower integrals for the function in Problem 8.9.

Problem 8.15. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that $\underline{\int_0^1} f < \overline{\int_0^1} f$.

Definition 8.16. If P and Q are partitions of $[a,b]$ such that $P \subseteq Q$, then we we say that Q is a **refinement** of P, or that Q **refines** P.

Problem 8.17. Let f be a bounded function with domain $[a,b]$. Prove that if P and Q are partitions of $[a,b]$ such that Q is a refinement of P, then $L_P(f) \leq L_Q(f)$ and $U_P(f) \geq U_Q(f)$.

Problem 8.18. Suppose f is a bounded function on $[a,b]$. Use the previous problem to prove that

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Problem 8.19. Suppose f is continuous on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and that for some $z \in [a,b]$, $f(z) > 0$. Explain why $\int_a^b f$ exists and then show that $\int_a^b f > 0$.

Definition 8.20. Let f be a bounded function with domain $[a,b]$. We say that f is (**Riemann**) **integrable** on $[a,b]$ if

$$\overline{\int_a^b} f = \underline{\int_a^b} f.$$

If f is integrable on $[a,b]$, then the common value of the upper and lower integrals is called the (**Riemann**) **integral** of f on $[a,b]$, which we denote via

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)\,dx.$$

Technically, we have defined the **Darboux integral**, with Riemann integrals coming from so-called Riemann sums. The two notions can be proved to be equivalent.

Problem 8.21. Give an example of a function f and an interval $[a,b]$ for which we know $\int_a^b f$ does not exist.

Problem 8.22. Is the function in Problem 8.9 integrable over $[0,1]$? If so, determine the value of the corresponding integral. If not, explain why.

There are so many facts about integrals, and unfortunately, we do not have time to prove them all! Nonetheless, we will hit some of the key results.

Problem 8.23. Prove that every constant function is integrable over every interval $[a,b]$.

The following theorem is a useful characterization of when a function is integrable over a closed interval.

Problem 8.24. Suppose f is a bounded function on $[a,b]$. Then f is (Riemann) integrable if and only if for every $\epsilon > 0$, there exists a partition P of $[a,b]$ such that $U_P(f) - L_P(f) < \epsilon$.

It is important to recognize that the previous problem provides us with a technique for determining whether a function is integrable over a closed interval, but does not necessarily help us with determining the value of a particular integral.

Problem 8.25. Define $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x$. Prove that f is integrable on $[0,1]$ and compute the value of the integral.[2]

The next sets of theorems will vastly expand our repertoire of functions known to be integrable. First, we need a few definitions, which resemble the corresponding concepts we defined for sequences in Chapter 3.

Definition 8.26. A function f is (strictly) **increasing** if for each pair of points x and y in the domain of f satisfying $x < y$, we have $f(x) < f(y)$. The function is **nondecreasing** if under the same assumptions we have $f(x) \leq f(y)$. The notions of (strictly) **decreasing** and **nonincreasing** are defined analogously. We say that f is a **monotonic** function if f is either nondecreasing or nonincreasing.

Problem 8.27. Prove that if f is a bounded monotonic function on $[a,b]$, then f is integrable on $[a,b]$.

Problem 8.28. Prove that each of the following exist. Do you know the value of any of these integrals knowing what we know now and perhaps some well-known area formulas?

(a) $\displaystyle\int_1^2 x^2\,dx$

(b) $\displaystyle\int_1^{17} e^{-x}\,dx$

[2]You need to use the tools we currently have at our disposal.

(c) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

(d) $\displaystyle\int_0^1 \sqrt{1+x^4}\,dx$

The next problem tells us that the integral respects scalar multiplication and sums and differences of integrable functions.

Problem 8.29. Suppose f and g be integrable on $[a,b]$ and let $c \in \mathbb{R}$. Prove each of the following:

(a) The function cf is integrable on $[a,b]$ and $\displaystyle\int_a^b cf = c\int_a^b f$.

(b) The function $f+g$ is integrable on $[a,b]$ and $\displaystyle\int_a^b (f+g) = \int_a^b f + \int_a^b g$.[3]

(c) The function $f-g$ is integrable on $[a,b]$ and $\displaystyle\int_a^b (f-g) = \int_a^b f - \int_a^b g$.[4]

Unfortunately, products of integrable functions are not well behaved.

Problem 8.30. Find two functions f and g which are integrable on $[0,1]$ such that fg is also integrable on $[0,1]$ but

$$\left(\int_0^1 f\right)\left(\int_0^1 g\right) \neq \int_0^1 fg.$$

Problem 8.31. Assume that $[a,b]$ is a closed interval and suppose f is integrable on $[a,c]$ and $[c,b]$ for $c \in (a,b)$. Show that f is integrable on $[a,b]$ and that

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Problem 8.32. Suppose f is integrable on $[a,b]$. Prove that for every $c \in \mathbb{R}$, the function g defined via $g(x) = f(x-c)$ is integrable on $[a+c, b+c]$ and

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx.$$

Let's turn our attention to continuous functions.

Problem 8.33. Suppose f is continuous on $[a,b]$. Prove that for every $\epsilon > 0$, there exists a partition $P = \{t_0 = a, t_1, \ldots, t_{n-1}, t_n = b\}$ of $[a,b]$ such that for each $1 \leq i \leq n$, if $u, v \in [t_{i-1}, t_i]$, then $|f(u) - f(v)| \leq \epsilon$.

[3]Proving this one is much harder than it looks!

[4]Use parts (a) and (b) to prove this one.

Problem 8.34. Prove that if f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Problem 8.35. Is the converse of the previous problem true? If so, prove it. Otherwise, provide a counterexample.

Definition 8.36. If f is integrable on $[a,b]$, then we define

$$\int_b^a f = -\int_a^b f \quad \text{and} \quad \int_a^a f = 0.$$

The next result is often referred to as the Mean Value Theorem for Integrals. Do you see why?

Problem 8.37 (Mean Value Theorem for Integrals)**.** Suppose f is continuous on $[a,b]$. Prove that there exists $c \in [a,b]$ such that

$$\int_a^b f = f(c)(b-a).$$

Can you draw a picture to capture the essence of this theorem?

The next two problems are the crowning achievement of calculus and of this course. Collectively, these two problems are known as the Fundamental Theorem of Calculus.

Problem 8.38 (Fundamental Theorem of Calculus, Part 1)**.** Suppose f is continuous on $[a,b]$ and define $F : [a,b] \to \mathbb{R}$ via

$$F(x) = \int_a^x f.$$

Prove that for each $c \in [a,b]$, F is differentiable at c and $F'(c) = f(c)$.

Problem 8.39 (Fundamental Theorem of Calculus, Part 2)**.** Suppose f is a function on $[a,b]$ such that f is differentiable at each point of $[a,b]$, and the function f' is continuous at each point in $[a,b]$. Then show that

$$\int_a^b f' = f(b) - f(a).$$

It is important to point out that the function we are integrating in Problem 8.39 needs to be continuous. Moreover, this function must be some other function's derivative. Given f' in Problem 8.39, there is an entire family of functions that have the same derivative as f, each differing by a constant, according to Problem 7.19. Each of the functions in this family is referred to as an **antiderivative** of f' and any one of them can be used to compute $\int_a^b f'$ using the Fundamental Theorem of Calculus.

The crux of using the Fundamental Theorem of Calculus boils down to finding an antiderivative of the function you are integrating. Some functions do not have nice antiderivatives! For example, in part (d) of Problem 8.28, we argued that the function given

by $f(x) = \sqrt{1+x^4}$ is integrable on $[0,1]$. However, this function does not have an antiderivative that you would recognize. Try asking WolframAlpha for the antiderivative of $f(x) = \sqrt{1+x^4}$ and see what you get.
Most functions you are familiar with are called elementary functions. Loosely speaking, a function is an **elementary function** if it is equal to a sum, product, and/or composition of finitely many polynomials, rational functions, trigonometric functions, exponential functions, and their inverse functions. These are the functions you typically encounter in high school, precalculus, and calculus. However, many functions are not elementary. For example, the function given in Problem 8.15 is not elementary. To complicate matters, many elementary functions do not have elementary antiderivatives. In fact, some rather innocent looking elementary functions do not have elementary antiderivatives. The function from part (d) of Problem 8.28 is such an example. Here are a few more elementary functions that do not have elementary antiderivatives:

- $\sqrt{1-x^4}$
- $\dfrac{1}{\ln(x)}$
- $\sin(x^2)$ and $\cos(x^2)$
- $\dfrac{\sin(x)}{x}$
- $\dfrac{e^x}{x}$
- e^{e^x}

Determining which elementary functions have elementary antiderivatives is not an easy task. The upshot is that utilizing the the Fundamental Theorem of Calculus to compute an integral may be difficult for seemingly innocent looking functions.

Problem 8.40. Using Problem 8.39 and your knowledge of antiderivatives from first semester calculus, compute the integrals in parts (a) and (b) of Problem 8.28.

Problem 8.41. According to WolframAlpha,

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = 2.$$

Explain why the techniques of this chapter cannot be used to verify this. How one might go about computing this integral? What definitions are needed?

Appendix A

Elements of Style for Proofs

Years of elementary school math taught us incorrectly that the answer to a math problem is just a single number, "the right answer." It is time to unlearn those lessons; those days are over. From here on out, mathematics is about discovering proofs and writing them clearly and compellingly.

The following rules apply whenever you write a proof. Keep these rules handy so that you may refer to them as you write your proofs.

1. **The burden of communication lies on you, not on your reader.** It is your job to explain your thoughts; it is not your reader's job to guess them from a few hints. You are trying to convince a skeptical reader who doesn't believe you, so you need to argue with airtight logic in crystal clear language; otherwise the reader will continue to doubt. If you didn't write something on the paper, then (a) you didn't communicate it, (b) the reader didn't learn it, and (c) the grader has to assume you didn't know it in the first place.

2. **Tell the reader what you're proving.** The reader doesn't necessarily know or remember what "Theorem 2.13" is. Even a professor grading a stack of papers might lose track from time to time. Therefore, the statement you are proving should be on the same page as the beginning of your proof. For an exam this won't be a problem, of course, but on your homework, recopy the claim you are proving. This has the additional advantage that when you study for exams by reviewing your homework, you won't have to flip back in the notes/textbook to know what you were proving.

3. **Use English words.** Although there will usually be equations or mathematical statements in your proofs, use English sentences to connect them and display their logical relationships. If you look in your notes/textbook, you'll see that each proof consists mostly of English words.

4. **Use complete sentences.** If you wrote a history essay in sentence fragments, the reader would not understand what you meant; likewise in mathematics you must use complete sentences, with verbs, to convey your logical train of thought.

   Some complete sentences can be written purely in mathematical symbols, such as equations (e.g., $a^3 = b^{-1}$), inequalities (e.g., $x < 5$), and other relations (like $5 \mid 10$ or

$7 \in \mathbb{Z}$). These statements usually express a relationship between two mathematical *objects*, like numbers or sets. However, it is considered bad style to begin a sentence with symbols. A common phrase to use to avoid starting a sentence with mathematical symbols is "We see that..."

5. **Show the logical connections among your sentences.** Use phrases like "Therefore" or "because" or "if..., then..." or "if and only if" to connect your sentences.

6. **Know the difference between statements and objects.** A mathematical object is a *thing*, a noun, such as a group, an element, a vector space, a number, an ordered pair, etc. Objects either exist or don't exist. Statements, on the other hand, are mathematical *sentences*: they can be true or false.

   When you see or write a cluster of math symbols, be sure you know whether it's an object (e.g., "$x^2 + 3$") or a statement (e.g., "$x^2 + 3 < 7$"). One way to tell is that every mathematical statement includes a verb, such as $=$, $\leq$, "divides", etc.

7. **The symbol "=" means "equals".** Don't write $A = B$ unless you mean that A actually equals B. This rule seems obvious, but there is a great temptation to be sloppy. In calculus, for example, some people might write $f(x) = x^2 = 2x$ (which is false), when they really mean that "if $f(x) = x^2$, then $f'(x) = 2x$."

8. **Don't interchange = and $\implies$.** The equals sign connects two *objects*, as in "$x^2 = b$"; the symbol "$\implies$" is an abbreviation for "implies" and connects two *statements*, as in "$a + b = a \implies b = 0$." You should avoid using $\implies$ in your formal write-ups.

9. **Avoid logical symbols in your proofs.** Similar to $\implies$, you should avoid using the logical symbols $\forall, \exists, \vee, \wedge$, and $\iff$ in your formal write-ups. These symbols are useful for abbreviating in your scratch work.

10. **Say exactly what you mean.** Just as $=$ is sometimes abused, so too people sometimes write $A \in B$ when they mean $A \subseteq B$, or write $a_{ij} \in A$ when they mean that a_{ij} is an entry in matrix A. Mathematics is a very precise language, and there is a way to say exactly what you mean; find it and use it.

11. **Don't write anything unproven.** Every statement on your paper should be something you *know* to be true. The reader expects your proof to be a series of statements, each proven by the statements that came before it. If you ever need to write something you don't yet know is true, you *must* preface it with words like "assume," "suppose," or "if" (if you are temporarily assuming it), or with words like "we need to show that" or "we claim that" (if it is your goal). Otherwise the reader will think they have missed part of your proof.

12. **Write strings of equalities (or inequalities) in the proper order.** When your reader sees something like
$$A = B \leq C = D,$$
   he/she expects to understand easily why $A = B$, why $B \leq C$, and why $C = D$, and he/she expects the *point* of the entire line to be the more complicated fact that $A \leq$

D. For example, if you were computing the distance d of the point $(12,5)$ from the origin, you could write

$$d = \sqrt{12^2 + 5^2} = 13.$$

In this string of equalities, the first equals sign is true by the Pythagorean theorem, the second is just arithmetic, and the *point* is that the first item equals the last item: $d = 13$.

A common error is to write strings of equations in the wrong order. For example, if you were to write "$\sqrt{12^2 + 5^2} = 13 = d$", your reader would understand the first equals sign, would be baffled as to how we know $d = 13$, and would be utterly perplexed as to why you wanted or needed to go through 13 to prove that $\sqrt{12^2 + 5^2} = d$.

13. **Avoid circularity.** Be sure that no step in your proof makes use of the conclusion!

14. **Don't write the proof backwards.** Beginning students often attempt to write "proofs" like the following, which attempts to prove that $\tan^2(x) = \sec^2(x) - 1$:

$$\begin{aligned}
\tan^2(x) &= \sec^2(x) - 1 \\
\left(\frac{\sin(x)}{\cos(x)}\right)^2 &= \frac{1}{\cos^2(x)} - 1 \\
\frac{\sin^2(x)}{\cos^2(x)} &= \frac{1 - \cos^2(x)}{\cos^2(x)} \\
\sin^2(x) &= 1 - \cos^2(x) \\
\sin^2(x) + \cos^2(x) &= 1 \\
1 &= 1
\end{aligned}$$

Notice what has happened here: the student *started* with the conclusion, and deduced the true statement "$1 = 1$." In other words, he/she has proved "If $\tan^2(x) = \sec^2(x) - 1$, then $1 = 1$," which is true but highly uninteresting.

Now this isn't a bad way of *finding* a proof. Working backwards from your goal often is a good strategy *on your scratch paper*, but when it's time to *write* your proof, you have to start with the hypotheses and work to the conclusion.

Here is an example of a suitable proof for the desired result, where each expression follows from the one immediately proceeding it:

$$\begin{aligned}
\sec^2(x) - 1 &= \frac{1}{\cos^2(x)} - 1 \\
&= \frac{1 - \cos^2(x)}{\cos^2(x)} \\
&= \frac{\sin^2(x)}{\cos^2(x)} \\
&= \left(\frac{\sin(x)}{\cos(x)}\right)^2 \\
&= (\tan(x))^2 \\
&= \tan^2(x).
\end{aligned}$$

15. **Be concise.** Most students err by writing their proofs too short, so that the reader can't understand their logic. It is nevertheless quite possible to be too wordy, and if you find yourself writing a full-page essay, it's probably because you don't really have a proof, but just an intuition. When you find a way to turn that intuition into a formal proof, it will be much shorter.

16. **Introduce every symbol you use.** If you use the letter "k," the reader should know exactly what k is. Good phrases for introducing symbols include "Let $n \in \mathbb{N}$," "Let k be the least integer such that...," "For every real number a...," and "Suppose that X is a counterexample."

17. **Use appropriate quantifiers (once).** When you introduce a variable $x \in S$, it must be clear to your reader whether you mean "for all $x \in S$" or just "for some $x \in S$." If you just say something like "$y = x^2$ where $x \in S$," the word "where" doesn't indicate whether you mean "for all" or "some".

    Phrases indicating the quantifier "for all" include "Let $x \in S$"; "for all $x \in S$"; "for every $x \in S$"; "for each $x \in S$"; etc. Phrases indicating the quantifier "some" (or "there exists") include "for some $x \in S$"; "there exists an $x \in S$"; "for a suitable choice of $x \in S$"; etc.

    On the other hand, don't introduce a variable more than once! Once you have said "Let $x \in S$," the letter x has its meaning defined. You don't *need* to say "for all $x \in S$" again, and you definitely should *not* say "let $x \in S$" again.

18. **Use a symbol to mean only one thing.** Once you use the letter x once, its meaning is fixed for the duration of your proof. You cannot use x to mean anything else.

19. **Don't "prove by example."** Most problems ask you to prove that something is true "for all"—You *cannot* prove this by giving a single example, or even a hundred. Your answer will need to be a logical argument that holds for *every example there possibly could be*.

On the other hand, if the claim that you are trying to prove involves the existence of a mathematical object with a particular property, then providing a specific example is sufficient.

20. **Write "Let $x = \dots$," not "Let $\cdots = x$."** When you have an existing expression, say a^2, and you want to give it a new, simpler name like b, you should write "Let $b = a^2$," which means, "Let the new symbol b mean a^2." This convention makes it clear to the reader that b is the brand-new symbol and a^2 is the old expression he/she already understands.

    If you were to write it backwards, saying "Let $a^2 = b$," then your startled reader would ask, "What if $a^2 \neq b$?"

21. **Make your counterexamples concrete and specific.** Proofs need to be entirely general, but counterexamples should be absolutely concrete. When you provide an example or counterexample, make it as specific as possible. For a set, for example, you must name its elements, and for a function you must give its rule. Do not say things like "θ could be one-to-one but not onto"; instead, provide an actual function θ that *is* one-to-one but not onto.

22. **Don't include examples in proofs.** Including an example very rarely adds anything to your proof. If your logic is sound, then it doesn't need an example to back it up. If your logic is bad, a dozen examples won't help it (see rule 19). There are only two valid reasons to include an example in a proof: if it is a *counterexample* disproving something, or if you are performing complicated manipulations in a general setting and the example is just to help the reader understand what you are saying.

23. **Use scratch paper.** Finding your proof will be a long, potentially messy process, full of false starts and dead ends. Do all that on scratch paper until you find a real proof, and only then break out your clean paper to write your final proof carefully.

    Only sentences that actually contribute to your proof should be part of the proof. Do not just perform a "brain dump," throwing everything you know onto the paper before showing the logical steps that prove the conclusion. *That is what scratch paper is for.*

Appendix B

Fancy Mathematical Terms

Here are some important mathematical terms that you will encounter in this course and throughout your mathematical career.

1. **Definition**—a precise and unambiguous description of the meaning of a mathematical term. It characterizes the meaning of a word by giving all the properties and only those properties that must be true.

2. **Theorem**—a mathematical statement that is proved using rigorous mathematical reasoning. In a mathematical paper, the term theorem is often reserved for the most important results.

3. **Lemma**—a minor result whose sole purpose is to help in proving a theorem. It is a stepping stone on the path to proving a theorem. Very occasionally lemmas can take on a life of their own (Zorn's lemma, Urysohn's lemma, Burnside's lemma, Sperner's lemma).

4. **Corollary**—a result in which the (usually short) proof relies heavily on a given theorem (we often say that "this is a corollary of Theorem A").

5. **Proposition**—a proved and often interesting result, but generally less important than a theorem.

6. **Conjecture**—a statement that is unproved, but is believed to be true (Collatz conjecture, Goldbach conjecture, twin prime conjecture).

7. **Claim**—an assertion that is then proved. It is often used like an informal lemma.

8. **Axiom/Postulate**—a statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proved (Euclid's five postulates, Zermelo-Frankel axioms, Peano axioms).

9. **Identity**—a mathematical expression giving the equality of two (often variable) quantities (trigonometric identities, Euler's identity).

10. **Paradox**—a statement that can be shown, using a given set of axioms and definitions, to be both true and false. Paradoxes are often used to show the inconsistencies in a flawed theory (Russell's paradox). The term paradox is often used informally to describe a surprising or counterintuitive result that follows from a given set of rules (Banach-Tarski paradox, Alabama paradox, Gabriel's horn).

Appendix C

Definitions in Mathematics

It is difficult to overstate the importance of definitions in mathematics. Definitions play a different role in mathematics than they do in everyday life.

Suppose you give your friend a piece of paper containing the definition of the rarely-used word **rodomontade**. According to the Oxford English Dictionary[1] (OED) it is:

> A vainglorious brag or boast; an extravagantly boastful, arrogant, or bombastic speech or piece of writing; an arrogant act.

Give your friend some time to study the definition. Then take away the paper. Ten minutes later ask her to define rodomontade. Most likely she will be able to give a reasonably accurate definition. Maybe she'd say something like, "It is a speech or act or piece of writing created by a pompous or egotistical person who wants to show off how great they are." It is unlikely that she will have quoted the OED word-for-word. In everyday English that is fine—you would probably agree that your friend knows the meaning of the rodomontade. This is because most definitions are *descriptive*. They describe the common usage of a word.

Let us take a mathematical example. The OED[2] gives this definition of *continuous*.

> Characterized by continuity; extending in space without interruption of substance; having no interstices or breaks; having its parts in immediate connection; connected, unbroken.

Likewise, we often hear calculus students speak of a continuous function as one whose graph can be drawn "without picking up the pencil." This definition is descriptive. (As we learned in calculus the picking-up-the-pencil description is not a perfect description of continuous functions.) This is not a mathematical definition.

Mathematical definitions are *prescriptive*. The definition must prescribe the exact and correct meaning of a word. Contrast the OED's descriptive definition of continuous with the the definition of continuous found in a real analysis textbook.

> A function $f : A \to \mathbb{R}$ is **continuous at a point** $c \in A$ if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x-c| < \delta$ (and $x \in A$) it follows that $|f(x)-f(c)| < \varepsilon$. If f

[1] http://www.oed.com/view/Entry/166837
[2] http://www.oed.com/view/Entry/40280

is continuous at every point in the domain A, then we say that f is **continuous on** A.[3]

In mathematics there is very little freedom in definitions. Mathematics is a deductive theory; it is impossible to state and prove theorems without clear definitions of the mathematical terms. The definition of a term must completely, accurately, and unambiguously describe the term. Each word is chosen very carefully and the order of the words is critical. In the definition of continuity changing "there exists" to "for all," changing the orders of quantifiers, changing $<$ to $\leq$ or $>$, or changing $\mathbb{R}$ to $\mathbb{Z}$ would completely change the meaning of the definition.

What does this mean for you, the student? Our recommendation is that at this stage you memorize the definitions word-for-word. It is the safest way to guarantee that you have it correct. As you gain confidence and familiarity with the subject you may be ready to modify the wording. You may want to change "for all" to "given any" or you may want to change $|x-c| < \delta$ to $-\delta < x - c < \delta$ or to "the distance between x and c is less than δ."

Of course, memorization is not enough; you must have a conceptual understanding of the term, you must see how the formal definition matches up with your conceptual understanding, and you must know how to work with the definition. It is perhaps with the first of these that descriptive definitions are useful. They are useful for building intuition and for painting the "big picture." Only after days (weeks, months, years?) of experience does one get an intuitive feel for the ε, δ-definition of continuity; most mathematicians have the "picking-up-the-pencil" definitions in their head. This is fine as long as we know that it is imperfect, and that when we prove theorems about continuous functions in mathematics we use the mathematical definition.

We end this discussion with an amusing real-life example in which a descriptive definition was not sufficient. In 2003 the German version of the game show *Who wants to be a millionaire?* contained the following question: "Every rectangle is: (a) a rhombus, (b) a trapezoid, (c) a square, (d) a parallelogram."

The confused contestant decided to skip the question and left with €4000. Afterward the show received letters from irate viewers. Why were the contestant and the viewers upset with this problem? Clearly a rectangle is a parallelogram, so (d) is the answer. But what about (b)? Is a rectangle a trapezoid? We would describe a trapezoid as a quadrilateral with a pair of parallel sides. But this leaves open the question: can a trapezoid have *two* pairs of parallel sides or must there only be *one* pair? The viewers said two pairs is allowed, the producers of the television show said it is not. This is a case in which a clear, precise, mathematical definition is required.

[3]This definition is taken from page 109 of Stephen Abbott's *Understanding Analysis*, but the definition would be essentially the same in any modern real analysis textbook.