

# Chapter 7

## Differentiation

It's time for derivatives!

**Definition 7.1.** Let  $f : A \rightarrow \mathbb{R}$  be a function and let  $a \in A$  such that  $f$  is defined on some open interval  $I$  containing  $a$  (i.e.,  $a \in I \subseteq A$ ). The **derivative** of  $f$  at  $a$  is defined via

$$f'(a) := \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists. If  $f'(a)$  exists, then we say that  $f$  is **differentiable** at  $a$ . More generally, we say that  $f$  is **differentiable** on  $B \subseteq A$  if  $f$  is differentiable at every point in  $B$ . As a special case,  $f$  is said to be **differentiable** if it is differentiable at every point in its domain. If  $f$  does indeed have a derivative at some points in its domain, then the **derivative** of  $f$  is the function denoted by  $f'$ , such that for each number  $x$  at which  $f$  is differentiable,  $f'(x)$  is the derivative of  $f$  at  $x$ . We may also write

$$\frac{d}{dx}[f(x)] := f'(x).$$

The lefthand side of the equation above is typically read as, “the derivative of  $f$  with respect to  $x$ .” The notation  $f'(x)$  is commonly referred to as “Newton’s notation” for the derivative while  $\frac{d}{dx}[f(x)]$  is often referred to as “Liebniz’s notation”.

Note that the definition of derivative automatically excludes the kind of behavior we saw with continuous functions, where a function defined only at a single point was continuous.

**Problem 7.2.** Find the derivative of  $f(x) = x^2 - x + 1$  at  $a = 2$ .

**Problem 7.3.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  via  $f(x) = c$  for some constant  $c \in \mathbb{R}$ . Prove that  $f$  is differentiable on  $\mathbb{R}$  and  $f'(x) = 0$  for all  $x \in \mathbb{R}$ .

**Problem 7.4.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  via  $f(x) = mx + b$  for some constants  $m, b \in \mathbb{R}$ . Prove that  $f$  is differentiable and  $f'(x) = m$  for all  $x \in \mathbb{R}$ .

**Problem 7.5.** Find and prove a formula for the derivative of  $f(x) = ax^2 + bx + c$  for any  $a, b, c \in \mathbb{R}$ .

**Problem 7.6.** Explain why any function defined only on  $\mathbb{Z}$  cannot have a derivative.

**Problem 7.7.** If  $f$  is differentiable at  $x$  and  $c \in \mathbb{R}$ , prove that the function  $cf$  also has a derivative at  $x$  and  $(cf)'(x) = cf'(x)$ .

**Problem 7.8.** If  $f$  and  $g$  are differentiable at  $x$ , show that the function  $f + g$  also has a derivative at  $x$  and  $(f + g)'(x) = f'(x) + g'(x)$ .

The next problem tells us that differentiability implies continuity.

**Problem 7.9.** Prove that if  $f$  has a derivative at  $x = a$ , then  $f$  is also continuous at  $x = a$ .

The converse of the previous theorem is not true. That is, continuity does not imply differentiability.

**Problem 7.10.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  via  $f(x) = |x|$ .

- (a) Prove that  $f$  is continuous at every point in its domain.
- (b) Prove that  $f$  is differentiable everywhere except at  $x = 0$ .

**Problem 7.11.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  via

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Show that  $f$  is continuous at  $x = 0$ , but not differentiable at  $x = 0$ .

The next problem states the well-known Product and Quotient Rules for Derivatives. You will need to use Problem 7.9 in their proofs.

**Problem 7.12.** Suppose  $f$  and  $g$  are differentiable at  $x$ . Prove each of the following:

- (a) (Product Rule) The function  $fg$  is differentiable at  $x$ . Moreover, its derivative function is given by

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

- (b) (Quotient Rule) The function  $f/g$  is differentiable at  $x$  provided  $g(x) \neq 0$ . Moreover, its derivative function is given by

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

The next problem is sure to make your head hurt.

**Problem 7.13.** Define  $g : \mathbb{R} \rightarrow \mathbb{R}$  via

$$g(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{otherwise.} \end{cases}$$

Now, define  $f : \mathbb{R} \rightarrow \mathbb{R}$  via  $f(x) = x^2g(x)$ . Determine where  $f$  is differentiable.

The next result tells us that if a differentiable function attains a maximum value at some point in an open interval contained in the domain of the function, then the derivative is zero at that point. In a calculus class, we would say that differentiable functions attain local maximums at critical numbers.

**Problem 7.14.** Let  $f : A \rightarrow \mathbb{R}$  be a function such that  $[a, b] \subseteq A$ ,  $f'(c)$  exists for some  $c \in (a, b)$ , and  $f(c) \geq f(x)$  for all  $x \in (a, b)$ . Prove that  $f'(c) = 0$ .

**Problem 7.15.** Let  $f : A \rightarrow \mathbb{R}$  be a function such that  $f'(c) = 0$  for some  $c \in A$ . Does this imply that there exists an open interval  $(a, b) \subseteq A$  containing  $c$  such that either  $f(x) \geq f(c)$  or  $f(x) \leq f(c)$  for all  $x \in (a, b)$ ? If so, prove it. Otherwise, provide a counterexample.

The next problem asks you to prove a result called Rolle's Theorem.

**Problem 7.16** (Rolle's Theorem). Let  $f : A \rightarrow \mathbb{R}$  be a function such that  $[a, b] \subseteq A$ . If  $f$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$ , and  $f(a) = f(b)$ , then prove that there exists a point  $c \in (a, b)$  such that  $f'(c) = 0$ .<sup>1</sup>

We can use Rolle's Theorem to prove the next result, which is the well-known Mean Value Theorem.

**Problem 7.17** (Mean Value Theorem). Let  $f : A \rightarrow \mathbb{R}$  be a function such that  $[a, b] \subseteq A$ . If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then prove that there exists a point  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.^2$$

**Problem 7.18.** Let  $f : A \rightarrow \mathbb{R}$  be a function such that  $[a, b] \subseteq A$ . If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  such that  $f'(x) = 0$  for all  $x \in (a, b)$ , then prove that  $f$  is constant over  $[a, b]$ .<sup>3</sup>

**Problem 7.19.** Let  $f : A \rightarrow \mathbb{R}$  and  $g : A \rightarrow \mathbb{R}$  such that  $[a, b] \subseteq A$ . Prove that if  $f'(x) = g'(x)$  for all  $x \in (a, b)$ , then there exists  $C \in \mathbb{R}$  such that  $f(x) = g(x) + C$ .

**Problem 7.20.** Is the converse of the previous problem true? If so, prove it. Otherwise, provide a counterexample.

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<sup>1</sup>*Hint:* First, apply the Extreme Value Theorem to  $f$  and  $-f$  to conclude that  $f$  attains both a maximum and minimum on  $[a, b]$ . If both the maximum and minimum are attained at the end points of  $[a, b]$ , then the maximum and minimum are the same and thus the function is constant. What does Problem 7.3 tell us in this case? But what if  $f$  is not constant over  $[a, b]$ ? Try using Problem 7.14.

<sup>2</sup>*Hint:* Cleverly define the function  $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$ . Is  $g$  continuous on  $[a, b]$ ? Is  $g$  differentiable on  $(a, b)$ ? Can we apply Rolle's Theorem to  $g$  using the interval  $[a, b]$ ? What can you conclude? Magic!

<sup>3</sup>*Hint:* Try applying the Mean Value Theorem to  $[a, t]$  for every  $t \in (a, b]$ .