Exam 1 (Part 2)

Your Name:

Names of Any Collaborators:

Instructions

Answer each of the following questions. This part of Exam 1 is worth a total of 24 points and is worth 50% of your overall score on Exam 1. Your overall score on Exam 1 is worth 20% of your overall grade. This portion of Exam 1 is due at the beginning of class on **Monday, October 21**.

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts. Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing the following problems, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. Further information:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using.
2. Unless you prove them, you cannot use any results from the course notes/book that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.** To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

Before we proceed, we need a few definitions. For $n \geq 1$, a **Dyck path** on length $2n$ is a lattice path in the plane connecting $(0,0)$ and (n,n) that consists of n North steps and n East steps such that the lattice path never drops below the diagonal line $y = x$. Let $\mathrm{Dyck}(n)$ denote the collection of Dyck paths of length $2n$, and define $d_n := |\mathrm{Dyck}(n)|$ for $n \geq 1$ and $d_0 := 1$. A **peak** of a Dyck path is the location along the path (from left to right) where an occurrence of a North step is immediately followed by an East step.

For $1 \leq k \leq n$, define the **Narayana numbers** via

$$N_{n,k} := |\{w \in S_n(231) \mid \mathrm{des}(w) = k\}|.$$

That is, $N_{n,k}$ is the number of 231-avoiding permutations in S_n with k descents. For $n \geq 1$, we define the nth **Narayana polynomial** via

$$C_n(t) := \sum_{w \in S_n(231)} t^{\mathrm{des}(w)} = \sum_{k=0}^{n-1} N_{n,k} t^k,$$

and we define $C_0(t) := 1$.

1. (4 points each) Complete all of the following. For this problem, let $d_n := |\mathrm{Dyck}(n)|$.

   (a) Prove that $d_n = C_n$ by describing a bijection between $S_n(231)$ and $\mathrm{Dyck}(n)$ where maximal decreasing runs of a permutation correspond to peaks of a Dyck path. There are a few different ways to attack this problem, but one way is to think of 231-avoiding permutations as non-attacking rook arrangements (ask me if you don't know what this means). See if you can find a natural correspondence to a Dyck path for a given non-attacking rook arrangement for a 231-avoiding permutation. For your "reverse" map, be sure to argue that your resulting permutation truly is 231-avoiding.

   Observation: Notice that if $w \mapsto p$, then

   $$\mathrm{pk}(p) = \mathrm{decruns}(w) = \mathrm{asc}(w) + 1 = n - 1 - \mathrm{des}(w) + 1 = n - \mathrm{des}(w),$$

   and so $\mathrm{des}(w) = n - \mathrm{pk}(p)$. This verifies that

   $$N_{n,n-k-1} = |\{p \in \mathrm{Dyck}(n) \mid \mathrm{pk}(p) = k + 1\}|.$$

   (b) Prove that there is a bijection from the set of lattice paths from $(0,0)$ to (n,n) that pass below $y = x$ at least once to the set of lattice paths from $(0,0)$ to $(n+1, n-1)$. *Hint:* Consider the first point on lattice path from $(0,0)$ to (n,n) that passes below $y = x$. Reflect the remaining portion of the path over the appropriate line to get a path from $(0,0)$ to $(n+1, n-1)$.

   (c) Prove that $d_n = \binom{2n}{n} - \binom{2n}{n-1}$.

   Observation: You just proved that $C_n = \binom{2n}{n} - \binom{2n}{n-1}$, and it is easy to verify that $\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$, so that

   $$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

2. (4 points each) Complete **two** of the following.

   (a) Independent of the work done above, prove directly that d_n satisfies the Catalan recurrence for $n \geq 1$:

   $$d_n = \sum_{i=0}^{n-1} d_i d_{n-1-i}.$$

Hint: Consider the collection of Dyck paths that hit the line $y = x$ at $(i+1, i+1)$ for the first time after leaving $(0,0)$. Think about how many ways you can draw the Dyck path to get to $(i+1, i+1)$ versus how many ways you can draw the Dyck path from $(i+1, i+1)$ to (n, n).

(b) A **triangulation** of a convex $(n+2)$-gon is a dissection into n triangles using only lines from vertices to vertices. Think of the polygon as being fixed in space. Prove that the number of triangulations of a convex $(n+2)$-gon is C_n. Incidentally, this is the problem that Euler was interested in when he studied the Catalan numbers!

(c) Prove that C_n counts the number of ways we can stack coins in the plane such that the bottom row consists of n consecutive coins. You should think of this as the two-dimensional version of stacking apples. For example, here are all the ways to stack coins so that the bottom row has 3 coins.

(d) Let $L(k, n-k)$ denote the set of lattice paths p from $(0,0)$ to $(k, n-k)$ consisting of only North steps and East steps. Note that each path in $L(k, n-k)$ consists of k East steps and $n-k$ North steps for a total of n steps. Construct a bijection between $L(k, n-k)$ and the set $\{w \in S_n \mid \mathrm{Des}(w) \subseteq \{k\}\}$. *Note:* Notice that says "$\subseteq$" not "$=$".

3. (4 points) Complete **one** of the following.

(a) A **set composition** of a set S is a set partition with an ordering on its blocks. When writing a set composition of $[n]$, it is standard practice to write the elements in each block in increasing order. This allows us to abbreviate a set composition of $[n]$ as a sequence of increasing runs separated by vertical bars. For example, the set composition $(\{3\}, \{4, 6\}, \{1, 5, 2\})$ would be written as $3|46|125$. Using this model, each set composition of $[n]$ is associated with a permutation on n. For example, the "underlying" permutation in the example above is 346125. Notice that each permutation $w \in S_n$ is associated with potentially many set compositions and we must always place a bar in a descent position. Moreover, we can place additional bars in the gaps between numbers in the permutation, but we are only allowed to place a single bar (since blocks must be nonempty). Let $\mathcal{C}(w)$ denote the collection of set compositions with underlying permutation w. Prove that

$$\sum_{C \in \mathcal{C}(w)} t^{|C|} = t^{\mathrm{runs}(w)}(1+t)^{n-\mathrm{runs}(w)},$$

where $|C|$ denotes the number of blocks in the set composition C and $\mathrm{runs}(w)$ is the number of maximal increasing runs in w.

(b) Define the ordinary generating function for the Catalan numbers via

$$C(z) := \sum_{n \geq 0} C_n z^n.$$

Prove that $C(z) = \dfrac{1 - \sqrt{1-4z}}{2z}$. *Hint:* Peel off first term of sum, utilize Catalan recurrence to obtain a double sum, do some clever rearranging to obtain z times a product of two sums, each of which ends up equaling $C(z)$. At this point, you should end up with an equation that is quadratic in $C(z)$. Use the quadratic formula and ditch the solution that cannot occur.

(c) Prove that

$$C_n(t) = C_{n-1}(t) + t \sum_{i=0}^{n-2} C_i(t) C_{n-1-i}(t).$$

Hint: Refine the argument we used to prove the Catalan recurrence.