Homework 12

Discrete Mathematics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition.You are highly encouraged to seek out assistance by asking questions on our Discord server. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers. In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. Solve the recurrence $a_n = 7a_{n-1} - 10a_{n-2}$ with $a_0 = 2$ and $a_1 = 3$.

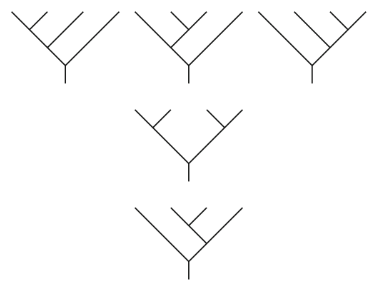
2. Solve the recurrence $a_n = 3a_{n-1} + 4a_{n-2}$ with $a_0 = 2$ and $a_1 = 3$.

3. Given a second-order linear constant-coefficient homogeneous recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$, the solution described in Theorem 9.18 only works if there are two distinct roots of the corresponding equation. It turns out if there is a repeated characteristic root r (i.e., the characteristic equation can be factored as $0 = (x - r)^2$, so that r is the only characteristic root), then the solution to the recurrence relation is
$$a_n = ar^n + bnr^n,$$
where a and b are constants determined by the initial conditions. Assuming this result, solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 4$.

4. The **Catalan numbers** are defined via $c_0 = 1$ and
$$c_n = \sum_{i=0}^{n-1} c_i c_{n-1-i}$$
for $n \geq 1$. The equation above is called the **Catalan recurrence**. Using the initial condition and the Catalan recurrence, we can generate the first several terms of the Catalan sequence:
$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786$$

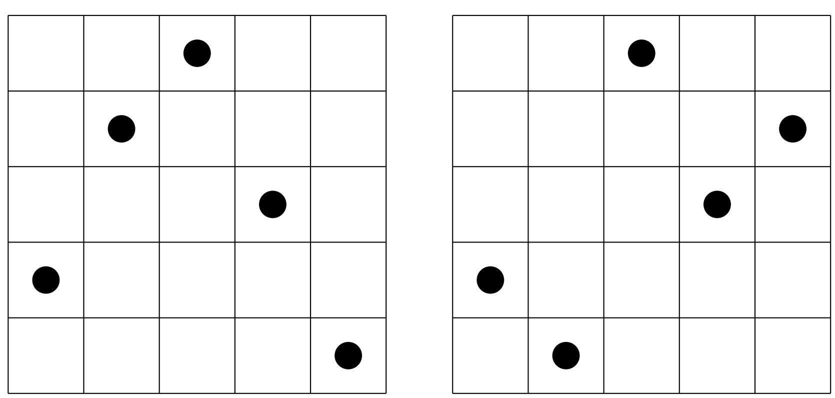
There are hundreds of interesting combinatorial objects counted by the Catalan numbers! For example, the number of Dyck paths from $(0,0)$ to (n,n) is counted by the Catalan numbers since d_n satisfied the Catalan recurrence above (namely, $d_n = \sum_{i=0}^{n-1} d_i d_{n-1-i}$ from a previous homework) and we have the same initial conditions (namely, $d_0 = 1 = c_0$). That is, $d_n = c_n$. Here are a few more examples of collection of objects counted by the Catalan numbers. Complete **one** of the following.

(a) A **planar binary tree** is a rooted tree such that every internal node has precisely two successors. An internal node is any node that has degree greater than 1. If there are n internal nodes, this means there are $n+1$ leaves. For example, below are all of the planar binary trees with 3 internal nodes.



Notice that two planar binary trees that are symmetric across the vertical midline are considered different. I'm happy to explain this further if the figures and description are not sufficient. Let p_n denote the number of planar binary trees with n internal nodes and let's define $p_0 := 1$. **Prove** that $p_n = c_n$. *Hint:* Verify initial conditions and recurrence for Catalan numbers.

(b) We say that a non-attacking rook arrangement on an $n \times n$ chessboard is **medium-high-low-avoiding** if when scanning the heights of the rooks from left to right, for each rook a we never see both a rook b to the right (not necessarily consecutively) that is higher than a followed (not necessarily consecutively) by a rook c that is lower than both a and b. For example, consider the non-attacking rook arrangements given below. The arrangement on the left has a few occurrences of a medium-high-low pattern, namely involving columns 1, 2, 5; columns 1, 3, 5; columns 1, 4, 5; columns 2, 3, 4; columns 2, 3, 5. On the other hand, the arrangement on the right is medium-high-low-avoiding.



Prove that the number of medium-high-low-avoiding non-attacking rook arrangements on an $n \times n$ chessboard is c_n. *Hint:* Verify initial conditions and recurrence for Catalan numbers.