> Every time that a human being succeeds in making an effort of attention with the sole idea of increasing [their] grasp of truth, [they acquire] a greater aptitude for grasping it, even if [their] effort produces no visible fruit.
>
> Simone Weil, philosopher & political activist

Chapter 4

Induction

In this chapter, we introduce mathematical induction, which is a proof technique that is useful for proving statements of the form $(\forall n \in \mathbb{N})P(n)$, or more generally $(\forall n \in \mathbb{Z})(n \geq a \implies P(n))$, where $P(n)$ is some predicate and $a \in \mathbb{Z}$.

4.1 Introduction to Induction

Consider the claims:

(a) For all $n \in \mathbb{N}$, $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

(b) For all $n \in \mathbb{N}$, $n^2 + n + 41$ is prime.

Let's take a look at potential proofs.

"Proof" of (a). If $n = 1$, then $1 = \frac{1(1+1)}{2}$. If $n = 2$, then $1 + 2 = 3 = \frac{2(2+1)}{2}$. If $n = 3$, then $1 + 2 + 3 = 6 = \frac{3(3+1)}{2}$, and so on. □

"Proof" of (b). If $n = 1$, then $n^2 + n + 41 = 43$, which is prime. If $n = 2$, then $n^2 + n + 41 = 47$, which is prime. If $n = 3$, then $n^2 + n + 41 = 53$, which is prime, and so on. □

Are these actual proofs? No! In fact, the second claim isn't even true. If $n = 41$, then $n^2 + n + 41 = 41^2 + 41 + 41 = 41(41 + 1 + 1)$, which is not prime since it has 41 as a factor. It turns out that the first claim is true, but what we wrote cannot be a proof since the same type of reasoning when applied to the second claim seems to prove something that isn not actually true. We need a rigorous way of capturing "and so on" and a way to verify whether it really is "and so on."

Recall that an axiom is a basic mathematical assumption. The following axiom is one of the Peano Axioms, which is a collection of axioms for the natural numbers introduced in the 19th century by Italian mathematician Giuseppe Peano (1858–1932).

Axiom 4.1 (Axiom of Induction)**.** Let $S \subseteq \mathbb{N}$ such that both

(i) $1 \in S$, and

(ii) if $k \in S$, then $k+1 \in S$.

Then $S = \mathbb{N}$.

We can think of the set S as a ladder, where the first hypothesis as saying that we have a first rung of a ladder. The second hypothesis says that if we are on any arbitrary rung of the ladder, then we can always get to the next rung. Taken together, this says that we can get from the first rung to the second, from the second to the third, and in general, from any kth rung to the $(k+1)$st rung, so that our ladder is actually $\mathbb{N}$. Do you agree that the Axiom of Induction is a pretty reasonable assumption?

At the end of Section 3.2, we briefly discussed ZFC, which is the standard choice for axiomatic set theory. It turns out that one can prove the Axiom of Induction as a theorem in ZFC. However, that will not be the approach we take. Instead, we are assuming the Axiom of Induction is true. Using this axiom, we can prove the following theorem, known as the **Principle of Mathematical Induction**. One approach to proving this theorem is to let $S = \{k \in \mathbb{N} \mid P(k) \text{ is true}\}$ and use the Axiom of Induction. The set S is sometimes called the **truth set**. Your job is to show that the truth set is all of $\mathbb{N}$.

Theorem 4.2 (Principle of Mathematical Induction)**.** Let $P(1), P(2), P(3), \ldots$ be a sequence of statements, one for each natural number. Assume

(i) $P(1)$ is true, and

(ii) if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

The Principle of Mathematical Induction provides us with a process for proving statements of the form: "For all $n \in \mathbb{N}$, $P(n)$," where $P(n)$ is some predicate involving n. Hypothesis (i) above is called the **base step** (or **base case**) while (ii) is called the **inductive step**.

You should not confuse *mathematical induction* with *inductive reasoning* associated with the natural sciences. Inductive reasoning is a scientific method whereby one induces general principles from observations. On the other hand, mathematical induction is a deductive form of reasoning used to establish the validity of a proposition.

Skeleton Proof 4.3 (Proof of $(\forall n \in \mathbb{N})P(n)$ by Induction)**.** Here is the general structure for a proof by induction.

Proof. We proceed by induction.

(i) Base step: *[Verify that $P(1)$ is true. This often, but not always, amounts to plugging $n = 1$ into two sides of some claimed equation and verifying that both sides are actually equal.]*

(ii) Inductive step: *[Your goal is to prove "For all $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is true."]* Let $k \in \mathbb{N}$ and assume that $P(k)$ is true. *[Do something to derive that $P(k+1)$ is true.]* Therefore, $P(k+1)$ is true.

Thus, by induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

Prove the next few theorems using induction. The first result may look familiar from calculus. Recall that $\sum_{i=1}^{n} i = 1+2+3+\cdots+n$, by definition.

Theorem 4.4. For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Theorem 4.5. For all $n \in \mathbb{N}$, 3 divides $4^n - 1$.

Theorem 4.6. For all $n \in \mathbb{N}$, 6 divides $n^3 - n$.

Theorem 4.7. Let $p_1, p_2, \ldots, p_n$ be n distinct points arranged on a circle. Then the number of line segments joining all pairs of points is $\frac{n^2-n}{2}$.

Problem 4.8. Consider a grid of squares that is 2^n squares wide by 2^n squares long, where $n \in \mathbb{N}$. One of the squares has been cut out, but you do not know which one! You have a bunch of L-shapes made up of 3 squares. Prove that you can perfectly cover this chessboard with the L-shapes (with no overlap) for any $n \in \mathbb{N}$. Figure 4.1 depicts one possible covering for the case involving $n = 2$.

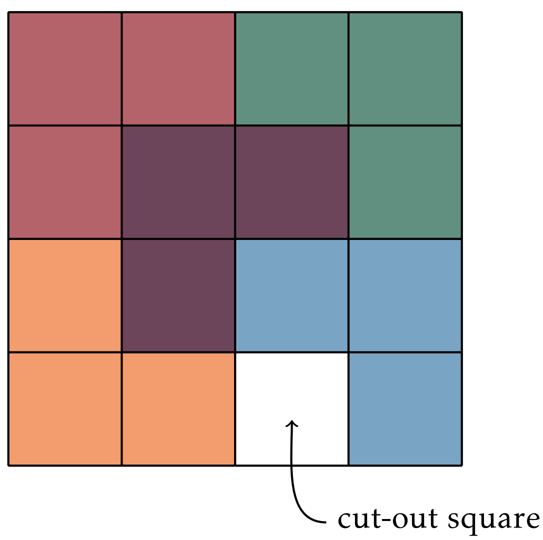

Figure 4.1: One possible covering for the case involving $n = 2$ for Problem 4.8.

> Do not stop thinking of life as an adventure. You have no security unless you can live bravely, excitingly, imaginatively; unless you can choose a challenge instead of competence.
>
> Eleanor Roosevelt, political figure & activist

4.2 More on Induction

In the previous section, we discussed proving statements of the form $(\forall n \in \mathbb{N})P(n)$. Mathematical induction can actually be used to prove a broader family of results; namely, those of the form

$$(\forall n \in \mathbb{Z})(n \geq a \Longrightarrow P(n))$$

for any value $a \in \mathbb{Z}$. Theorem 4.2 handles the special case when $a = 1$. The ladder analogy from the previous section holds for this more general situation, too. To prove the next theorem, mimic the proof of Theorem 4.2, but this time use the set $S = \{k \in \mathbb{N} \mid P(a+k-1) \text{ is true}\}$.

Theorem 4.9 (Principle of Mathematical Induction)**.** Let $P(a), P(a+1), P(a+2), \ldots$ be a sequence of statements, one for each integer greater than or equal to a. Assume that

(i) $P(a)$ is true, and

(ii) if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for all integers $n \geq a$.

Theorem 4.9 gives a process for proving statements of the form: "For all integers $n \geq a$, $P(n)$." As before, hypothesis (i) is called the **base step**, and (ii) is called the **inductive step**.

Skeleton Proof 4.10 (Proof of $(\forall n \in \mathbb{Z})(n \geq a \Longrightarrow P(n))$ by Induction)**.** Here is the general structure for a proof by induction when the base case does not necessarily involve $a = 1$.

Proof. We proceed by induction.

(i) Base step: *[Verify that $P(a)$ is true. This often, but not always, amounts to plugging $n = a$ into two sides of some claimed equation and <u>verifying</u> that both sides are actually equal.]*

(ii) Inductive step: *[Your goal is to prove "For all $k \in \mathbb{Z}$, if $P(k)$ is true, then $P(k+1)$ is true."]* Let $k \geq a$ be an integer and assume that $P(k)$ is true. *[Do something to derive that $P(k+1)$ is true.]* Therefore, $P(k+1)$ is true.

Thus, by induction, $P(n)$ is true for all integers $n \geq a$. □

We encountered the next theorem back in Section 3.3 (see Conjecture 3.29), but we did not prove it. When proving this theorem using induction, you will need to argue that if you add one more element to a finite set, then you end up with twice as many subsets. For your base case, consider the empty set.

Theorem 4.11. If A is a finite set with n elements, then $\mathcal{P}(A)$ is a set with 2^n elements.

Theorem 4.12. For all integers $n \geq 0$, $n < 2^n$.

One consequence of the previous two theorems is that the power set of a finite set always consists of more elements than the original set.

Theorem 4.13. For all integers $n \geq 0$, 4 divides $9^n - 5$.

Theorem 4.14. For all integers $n \geq 0$, 4 divides $6 \cdot 7^n - 2 \cdot 3^n$.

Theorem 4.15. For all integers $n \geq 2$, $2^n > n + 1$.

Theorem 4.16. For all integers $n \geq 0$, $1 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.

Theorem 4.17. Fix a real number $r \neq 1$. For all integers $n \geq 0$,

$$1 + r^1 + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

Theorem 4.18. For all integers $n \geq 3$, $2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = \dfrac{(n-2)(n^2 + 2n + 3)}{3}$.

Theorem 4.19. For all integers $n \geq 1$, $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

Theorem 4.20. For all integers $n \geq 1$, $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$.

Theorem 4.21. For all integers $n \geq 0$, $3^{2n} - 1$ is divisible by 8.

Theorem 4.22. For all integers $n \geq 2$, $2^n < (n+1)!$.

Theorem 4.23. For all integers $n \geq 2$, $2 \cdot 9^n - 10 \cdot 3^n$ is divisible by 4.

We now consider an induction problem of a different sort, where you have to begin with some experimentation. For Part (c), consider using the results from Parts (a) and (b).

Problem 4.24. Suppose n lines are drawn in the plane so that no two lines are parallel and no three lines intersect at any one point. Such a collection of lines is said to be in **general position**. Every collection of lines in general position divides the plane into disjoint regions, some of which are polygons with finite area (bounded regions) and some of which are not (unbounded regions).

(a) Let $R(n)$ be the number of regions the plane is divided into by n lines in general position. Conjecture a formula for $R(n)$ and prove that your conjecture is correct.

(b) Let $U(n)$ be the number of unbounded regions the plane is divided into by n lines in general position. Conjecture a formula for $U(n)$ and prove that your conjecture is correct.

(c) Let $B(n)$ be the number of unbounded regions the plane is divided into by n lines in general position. Conjecture a formula for $B(n)$ and prove that your conjecture is correct.

(d) Suppose we color each of the regions (bounded and unbounded) so that no two adjacent regions (i.e., share a common edge) have the same color. What is the fewest colors we could use to accomplish this? Prove your assertion.

> If you don't learn to fail, you will fail to learn.
>
> Manu Kapur, learning scientist

4.3 Complete Induction and the Well-Ordering Principle

There is another formulation of induction, where the inductive step begins with a set of assumptions rather than one single assumption. This method is sometimes called **complete induction** or **strong induction**.

Theorem 4.25 (Principle of Complete Mathematical Induction)**.** Let $P(1), P(2), P(3), \ldots$ be a sequence of statements, one for each natural number. Assume that

(i) $P(1)$ is true, and

(ii) For all $k \in \mathbb{N}$, if $P(j)$ is true for all $j \in \mathbb{N}$ such that $j \leq k$, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Note the difference between ordinary induction (Theorems 4.2 and 4.9) and complete induction. For the induction step of complete induction, we are not only assuming that $P(k)$ is true, but rather that $P(j)$ is true for all j from 1 to k. Despite the name, complete induction is not any stronger or more powerful than ordinary induction. It is worth pointing out that anytime ordinary induction is an appropriate proof technique, so is complete induction. So, when should we use complete induction?

In the inductive step, you need to reach $P(k+1)$, and you should ask yourself which of the previous cases you need to get there. If all you need, is the statement $P(k)$, then ordinary induction is the way to go. If two preceding cases, $P(k-1)$ and $P(k)$, are necessary to reach $P(k+1)$, then complete induction is appropriate. In the extreme, if one needs the full range of preceding cases (i.e., all statements $P(1), P(2), \ldots, P(k)$), then again complete induction should be utilized.

Note that in situations where complete induction is appropriate, it might be the case that you need to verify more than one case in the base step. The number of base cases to be checked depends on how one needs to "look back" in the induction step.

Skeleton Proof 4.26 (Proof of $(\forall n \in \mathbb{N})P(n)$ by Complete Induction)**.** Here is the general structure for a proof by complete induction.

Proof. We proceed by induction.

(i) Base step: *[Verify that $P(1)$ is true. Depending on the statement, you may also need to verify that $P(k)$ is true for other specific values of k.]*

(ii) Inductive step: *[Your goal is to prove "For all $k \in \mathbb{N}$, if for each $k \in \mathbb{N}$, $P(j)$ is true for all $j \in \mathbb{N}$ such that $j \leq k$, then $P(k+1)$ is true."]* Let $k \in \mathbb{N}$. Suppose $P(j)$ is true for all $j \leq k$. *[Do something to derive that $P(k+1)$ is true.]* Therefore, $P(k+1)$ is true.

Thus, by complete induction, $P(n)$ is true for all integers $n \geq a$. □

Recall that Theorem 4.9 generalized Theorem 4.2 and allowed us to handle situations where the base case was something other than $P(1)$. We can generalize complete induction in the same way, but we won't write this down as a formal theorem.

Theorem 4.27. Define a sequence of numbers by $a_1 = 1$, $a_2 = 3$, and $a_n = 3a_{n-1} - 2a_{n-2}$ for all natural numbers $n \geq 3$. Then $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

Theorem 4.28. Define a sequence of numbers by $a_1 = 3, a_2 = 5, a_3 = 9$ and $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for all natural numbers $n \geq 4$. Then $a_n = 2^n + 1$ for all $n \in \mathbb{N}$.

Theorem 4.29. Define a sequence of numbers by $a_1 = 1$, $a_2 = 3$, and $a_n = a_{n-1} + a_{n-2}$ for all natural numbers $n \geq 3$. Then $a_n < \left(\frac{7}{4}\right)^n$ for all $n \in \mathbb{N}$.

Theorem 4.30. Define a sequence of numbers by $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all natural numbers $n \geq 4$. Then $a_n < 2^n$ for all $n \in \mathbb{N}$.

Theorem 4.31. Define a sequence of numbers by $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for all natural numbers $n \geq 3$. Then $a_n < \left(\frac{5}{3}\right)^n$ for all $n \in \mathbb{N}$.

Problem 4.32. Prove that every amount of postage that is at least 12 cents can be made from 4-cent and 5-cent stamps.

Problem 4.33. Prove that for any $n \geq 4$, one can obtain n dollars using only \$2 bills and \$5 bills.

Problem 4.34. Consider a grid of squares that is 2 squares wide and n squares long. Using n dominoes that are 1 square by 2 squares, there are many ways to perfectly cover this chessboard with no overlap. How many? Prove your answer.

The penultimate theorem of this chapter is known as the **Well-Ordering Principle**. As you shall see, this seemingly obvious theorem requires a bit of work to prove. It is worth noting that in some axiomatic systems, the Well-Ordering Principle is sometimes taken as an axiom. However, in our case, the result follows from complete induction. Before stating the Well-Ordering Principle, we need an additional definition.

Definition 4.35. Let $A \subseteq \mathbb{R}$ and $m \in A$. Then m is called a **maximum** (or **greatest element**) of A if for all $a \in A$, we have $a \leq m$. Similarly, m is called **minimum** (or **least element**) of A if for all $a \in A$, we have $m \leq a$.

Not surprisingly, maximums and minimums are unique when they exist. It might be helpful to review Skeleton Proof 2.90 prior to attacking the next result.

Theorem 4.36. If $A \subseteq \mathbb{R}$ such that the maximum (respectively, minimum) of A exists, then the maximum (respectively, minimum) of A is unique.

If the maximum of a set A exists, then it is denoted by $\boxed{\max(A)}$. Similarly, if the minimum of a set A exists, then it is denoted by $\boxed{\min(A)}$.

Problem 4.37. Find the maximum and the minimum for each of the following sets when they exist.

(a) $\{5, 11, 17, 42, 103\}$

(b) $\mathbb{N}$

(c) $\mathbb{Z}$

(d) $(0, 1]$

(e) $(0, 1] \cap \mathbb{Q}$

(f) $(0, \infty)$

(g) $\{42\}$

(h) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(i) $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

(j) $\emptyset$

To prove the Well-Ordering Principle, consider a proof by contradiction. Suppose S is a nonempty subset of $\mathbb{N}$ that does not have a least element. Define the proposition $P(n)$:="n is not an element of S" and then use complete induction to prove the result.

Theorem 4.38 (Well-Ordering Principle)**.** Every nonempty subset of the natural numbers has a least element.

It turns out that the Well-Ordering Principle (Theorem 4.38) and the Axiom of Induction (Axiom 4.1) are equivalent. In other words, one can prove the Well-Ordering Principle from the Axiom of Induction, as we have done, but one can also prove the Axiom of Induction if the Well-Ordering Principle is assumed.

The final theorem of this section can be thought of as a generalized version of the Well-Ordering Principle.

Theorem 4.39. If A is a nonempty subset of the integers and there exists $b \in A^c$ such that $b \geq a$ for all $a \in A$, then A contains a greatest element.

In the previous theorem, b is referred to as an upper bound for A. We will study upper bounds in Section 5.1.

Nothing that's worth anything is ever easy.

Mike Hall, ultra-distance cyclist