Introduction to Real Analysis

Dana C. Ernst < Karl-Dieter Crisman < W. Ted Mahavier < W. S. Mahavier

Northern Arizona University

Version Fall 2023

This book is intended to be a problem sequence for a one-semester undergraduate real analysis course that utilizes an inquiry-based learning (IBL) approach. There is always a debt to be paid in creating a text, and this one is no different. The primary source for these notes is Karl-Dieter Crisman's *One-Semester Real Analysis* notes[1], which were based on W. Ted Mahavier's *Analysis* notes[2], which in turn were based on notes his father (W. S. Mahavier) created.

Below is a partial list of people (alphabetical by last name) that I need to thank for supplying additional content, advice, and feedback.

- Anders Hendrickson (St. Norbert College). Anders is the original author of the content in Appendix A: Elements of Style for Proofs. The current version in Appendix A is a result of modifications made by myself with some suggestions from Dave Richeson.

- Crystal Kalinec-Craig (University of Texas at San Antonio). Section 1.5: Rights of the Learner is an adaptation of a similar list written by Crystal.

- Dave Richeson (Dickinson College). Dave is responsible for much of the content in Appendix B: Fancy Mathematical Terms and Appendix C: Definitions in Mathematics.

You may not modify the content of this book without expressed consent of all parties mentioned above.

[1]Journal of Inquiry-Based Learning in Mathematics, No. 46 (2013)
[2]Journal of Inquiry-Based Learning in Mathematics, No. 12 (2009)

Contents

Preface

Mathematics is not about calculations, but ideas. My goal as a teacher is to provide students with the opportunity to grapple with these ideas and to be immersed in the process of mathematical discovery. Repeatedly engaging in this process hones the mind and develops mental maturity marked by clear and rigorous thinking. Like music and art, mathematics provides an opportunity for enrichment, experiencing beauty, elegance, and aesthetic value. The medium of a painter is color and shape, whereas the medium of a mathematician is abstract thought. The creative aspect of mathematics is what captivates me and fuels my motivation to keep learning and exploring.

While the content we teach our students is important, it is not enough. An education must prepare individuals to ask and explore questions in contexts that do not yet exist and to be able to tackle problems they have never encountered. It is important that we put these issues front and center and place an explicit focus on students producing, rather than consuming, knowledge. If we truly want our students to be independent, inquisitive, and persistent, then we need to provide them with the means to acquire these skills. Their viability as a professional in the modern workforce depends on their ability to embrace this mindset.

When I started teaching, I mimicked the experiences I had as a student. Because it was all I knew, I lectured. By standard metrics, this seemed to work out just fine. Glowing student and peer evaluations, as well as reoccurring teaching awards, indicated that I was effectively doing my job. People consistently told me that I was an excellent teacher. However, two observations made me reconsider how well I was really doing. Namely, many of my students seemed to depend on me to be successful, and second, they retained only some of what I had taught them. In the words of Dylan Retsek:

> "Things my students claim that I taught them masterfully, they don't know."

Inspired by a desire to address these concerns, I began transitioning away from direct instruction towards a more student-centered approach. The goals and philosophy behind inquiry-based learning (IBL) resonate deeply with my ideals, which is why I have embraced this paradigm. According to the Academy of Inquiry-Based Learning, IBL is a method of teaching that engages students in sense-making activities. Students are given tasks requiring them to solve problems, conjecture, experiment, explore, create, and communicate—all those wonderful skills and habits of mind that mathematicians engage in regularly. This book has IBL baked into its core.

The primary objectives of this book are to:

- Expand the mathematical content knowledge of the reader,

- Provide an opportunity for the reader to experience the profound beauty of mathematics,
- Allow the reader to exercise creativity in producing and discovering mathematics,
- Enhance the ability of the reader to be a robust and persistent problem solver.

Ultimately, this is really a book about productive struggle and learning how to learn. Mathematics is simply the vehicle.

> Much more important than specific mathematical results are the habits of mind used by the people who create those results. ... Although it is necessary to infuse courses and curricula with modern content, what is even more important is to give students the tools they will need in order to use, understand, and even make mathematics that does not yet exist.
>
> Cuoco, Goldenberg, & Mark in *Habit of Mind: An Organizing Principle for Mathematics Curriculum*

> The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful.
>
> Henri Poincaré, mathematician & physicist

Chapter 1

Introduction

1.1 What is This Course All About?

This course introduces basic concepts and methods of analysis. The course focuses on the theory of the real number system and calculus of functions of a real variable. The content will include:

1. Axioms of the real numbers, supremum and infimum.
2. Topology of the real numbers including completeness and compactness.
3. Sequences and convergence, including the algebra of limits.
4. Limits of functions, including the algebra of limits.
5. Continuity, including the algebra of continuous functions, continuity of compositions, and uniform continuity.
6. Differentiation, including the algebra of derivatives and applications to behavior of functions.
7. Riemann integration, including linearity and order properties, integrability of continuous functions, Riemann sums, and the Fundamental Theorem of Calculus.

We will take an axiomatic approach (definition, theorem, and proof) to the subject, but along the way, you will develop intuition about the objects of real analysis and pick up more proof-writing skills. The emphasis of this course is on your ability to read, understand, and communicate mathematics in the context of real analysis.

Your progress will be fueled by your ability to wrestle with mathematical ideas and to prove theorems. As you work through the book, you will find that you have ideas for proofs, but you are unsure of them. Do not be afraid to tinker and make mistakes. You can always revisit your work as you become more proficient. Do not expect to do most things perfectly on your first—or even second or third—attempt. The material is too rich for a human being to completely understand immediately. Learning a new skill requires dedication and patience during periods of frustration. Moreover, solving genuine problems is difficult and takes time. But it is also rewarding!

> The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful.
>
> *Henri Poincaré*

1.2 An Inquiry-Based Approach

In many mathematics classrooms, "doing mathematics" means following the rules dictated by the teacher, and "knowing mathematics" means remembering and applying them. However, this is not a typical mathematics textbook and is likely a significant departure from your prior experience, where mimicking prefabricated examples led you to success. In order to promote a more active participation in your learning, this book adheres to an educational philosophy called inquiry-based learning (IBL). IBL is a student-centered method of teaching that engages students in sense-making activities and challenges them to create or discover mathematics. In this book, you will be expected to actively engage with the topics at hand and to construct your own understanding. You will be given tasks requiring you to solve problems, conjecture, experiment, explore, create, and communicate. Rather than showing facts or a clear, smooth path to a solution, this book will guide and mentor you through an adventure in mathematical discovery.

This book makes no assumptions about the specifics of how your instructor chooses to implement an IBL approach. Generally speaking, students are told which problems and theorems to grapple with for the next class sessions, and then the majority of class time is devoted to students working in groups on unresolved solutions/proofs or having students present their proposed solutions/proofs to the rest of the class. Students should—as much as possible—be responsible for guiding the acquisition of knowledge and validating the ideas presented. That is, you should not be looking to the instructor as the sole authority. In an IBL course, instructor and students have joint responsibility for the depth and progress of the course. While effective IBL courses come in a variety of forms, they all possess a few essential ingredients. According to Laursen and Rasmussen (2019), the Four Pillars of IBL are:

- Students engage deeply with coherent and meaningful mathematical tasks.
- Students collaboratively process mathematical ideas.
- Instructors inquire into student thinking.
- Instructors foster equity in their design and facilitation choices.

This book can only address the first pillar while it is the responsibility of your instructor and class to develop a culture that provides an adequate environment for the remaining pillars to take root. If you are studying this material independent of a classroom setting, I encourage you to find a community where you can collaborate and discuss your ideas.

Just like learning to play an instrument or sport, you will have to learn new skills and ideas. Along this journey, you should expect a cycle of victory and defeat, experiencing a full range of emotions. Sometimes you will feel exhilarated, other times you might be seemingly paralyzed by extreme confusion. You will experience struggle and failure before you experience understanding. This is part of the normal learning process. If you are doing things well, you should be confused on a regular basis. Productive struggle and mistakes provide opportunities for growth. As the author of this text, I am here to guide and challenge you, but I cannot do the learning for you, just as a music teacher cannot move your fingers and your heart for you. This is a very exciting time in your mathematical career. You will experience mathematics in a new and profound way. Be patient with yourself and others as you adjust to a new paradigm.

You could view this book as mountaineering guidebook. I have provided a list of mountains to summit, sometimes indicating which trailhead to start at or which trail to follow. There will always be multiple routes to top, some more challenging than others. Some summits you will attain quickly and easily, others might require a multi-day expedition. Oftentimes, your journey will be laced with false summits. Some summits will be obscured by clouds. Sometimes you will have to wait out a storm, perhaps turning around and attempting another route, or even attempting to summit on a different day after the weather has cleared. The strength, fitness, and endurance you gain along the way will allow you to take on more and more challenging, and often beautiful, terrain. Do not forget to take in the view from the top! The joy you feel from overcoming obstacles and reaching each summit under your own will and power has the potential to be life changing. But make no mistake, the journey is vastly more important than the destinations.

> Don’t fear failure. Not failure, but low aim, is the crime. In great attempts it is glorious even to fail.
>
> Bruce Lee, martial artist & actor

1.3 Structure of the Textbook

As you read this book, you will be required to digest the material in a meaningful way. It is your responsibility to read and understand new definitions and their related concepts. In addition, you will be asked to complete problems aimed at solidifying your understanding of the material. Most importantly, you will be asked to make conjectures, produce counterexamples, and prove theorems. All of these tasks will almost always be challenging.

The items labeled as **Definition**, **Axiom**, **Example**, and **Theorem** are meant to be read and digested. The items labeled as **Problem** require action on your part. Some Problems are computational in nature and aimed at improving your understanding of a particular concept while others ask you to provide a counterexample for a statement if it is false or to provide a proof if the statement is true. Other problems are exercises aimed at developing understanding of the content. Some problems will be easy, a few extremely hard. Most will be somewhere in between, to enhance your sense of confidence and accomplishment.

It is important to point out that there are very few examples in the notes. This is intentional. The goal of some of the problems is for you to produce the examples. There are a handful of items labeled as **Theorem**, which are results that we will take for granted. Lastly, there are many situations where you will want to refer to an earlier axiom, definition, theorem or problem. In this case, you should reference the statement by number. For example, you might write something like, "By Problem 2.20, we see that...."

One thing to always keep in mind is that every task in this book can be done by you, the student. But it may not be on your first try, or even your second. The overarching goal of this book is to help you develop a deep and meaningful understanding of the processes of producing mathematics by putting you in direct contact with mathematical phenomena.

> Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?
>
> Paul Halmos, mathematician

1.4 Rules of the Game

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing assignments for this course, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

> If you want to sharpen a sword, you have to remove a little metal.
>
> *Unknown*

1.5 Rights of the Learner

As a reader of this textbook, you have the right to:

1. be confused,
2. make a mistake and to revise your thinking,
3. speak, listen, and be heard, and
4. enjoy doing mathematics.

> You may encounter many defeats, but you must not be defeated.
>
> Maya Angelou, poet & activist

1.6 Some Minimal Guidance

Especially in the opening sections, it will not be clear what facts from your prior experience in mathematics you are "allowed" to use. Unfortunately, addressing this issue is difficult and is something we will sort out along the way. In addition, you are likely unfamiliar with how to structure a valid mathematical proof. So that you do not feel completely abandoned, here are some guidelines to keep in mind as you get started with writing proofs.

- The statement you are proving should be on the same page as the beginning of your proof.
- You should indicate where the proof begins by writing "*Proof.*" at the beginning.
- Make it clear to yourself and the reader what your assumptions are at the very beginning of your proof. Typically, these statements will start off "Assume...", "Suppose...", or "Let...". Sometimes there will be some implicit assumptions that we can omit, but at least in the beginning, you should get in the habit of clearly stating your assumptions up front.
- Carefully consider the order in which you write your proof. Each sentence should follow from an earlier sentence in your proof or possibly a result you have already proved.
- Unlike the experience many of you had writing proofs in your high school geometry class, our proofs should be written in complete sentences. You should break sections of a proof into paragraphs and use proper grammar. There are some pedantic conventions for doing this that will be pointed out along the way. Initially, this will be an issue that you may struggle with, but you will get the hang of it.
- There will be many situations where you will want to refer to an earlier definition, problem, theorem, or corollary. In this case, you should reference the statement by number, but it is also helpful to the reader to summarize the statement you are citing.

- There will be times when we will need to do some basic algebraic manipulations. You should feel free to do this whenever the need arises. But you should show sufficient work along the way. In addition, you should organize your calculations so that each step follows from the previous. The order in which we write things matters. You do not need to write down justifications for basic algebraic manipulations (e.g., adding 1 to both sides of an equation, adding and subtracting the same amount on the same side of an equation, adding like terms, factoring, basic simplification, etc.).

- On the other hand, you do need to make explicit justification of the logical steps in a proof. As stated above, you should cite a previous definition, theorem, etc. when necessary.

- Similar to making it clear where your proof begins, you should indicate where it ends. It is common to conclude a proof with the standard "proof box" (□ or ■). This little square at end of a proof is sometimes called a **tombstone** or **Halmos symbol** after Hungarian-born American mathematician Paul Halmos (1916–2006).

It is of utmost importance that you work to understand every proof. Questions—asked to your instructor, your peers, and yourself—are often your best tool for determining whether you understand a proof. Another way to help you process and understand a proof is to try and make observations and connections between different ideas, proof statements and methods, and to compare various approaches.

If you would like additional guidance before you dig in, look over the guidelines in Appendix A: Elements of Style for Proofs. It is suggested that you review this appendix occasionally as you progress through the book as some guidelines may not initially make sense or seem relevant. Be prepared to put in a lot of time and do all the work. Your effort will pay off in intellectual development. Now, go have fun and start exploring mathematics!

> Our greatest glory is not in never falling, but in rising every time we fall.
>
> Confucius, philosopher

Pure mathematics is the poetry of logical ideas.

Albert Einstein, theoretical physicist

Chapter 2

Preliminaries

In this chapter, we summarize some background material we need to be familiar with.

2.1 Sets

At its essence, all of mathematics is built on set theory.

A **set** is a collection of objects called **elements**. If A is a set and x is an element of A, we write $\boxed{x \in A}$. Otherwise, we write $\boxed{x \notin A}$. The set containing no elements is called the **empty set**, and is denoted by the symbol $\boxed{\emptyset}$. Any set that contains at least one element is referred to as a **nonempty set**.

If we think of a set as a box potentially containing some stuff, then the empty set is a box with nothing in it. One assumption we will make is that for any set A, $A \notin A$. The language associated to sets is specific. We will often define sets using the following notation, called **set-builder notation**:

$$\boxed{S = \{x \in A \mid P(x)\}},$$

where $P(x)$ is some predicate statement involving x. The first part "$x \in A$" denotes what type of x is being considered. The predicate to the right of the vertical bar (not to be confused with "divides") determines the condition(s) that each x must satisfy in order to be a member of the set. This notation is read as "The set of all x in A such that $P(x)$." As an example, the set $\{x \in \mathbb{N} \mid x \text{ is even and } x \geq 8\}$ describes the collection of even natural numbers that are greater than or equal to 8.

There are a few sets that are commonly discussed in mathematics and have predefined symbols to denote them. We've already encountered the integers, natural numbers, and real numbers. Notice that our definition of the rational numbers uses set-builder notation.

- **Natural numbers:** $\boxed{\mathbb{N} := \{1, 2, 3, \ldots\}}$. Some books will include zero in the set of natural numbers, but we do not.
- **Integers:** $\boxed{\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \ldots\}}$.
- **Rational Numbers:** $\boxed{\mathbb{Q} := \{a/b \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}}$.

- **Real Numbers:** $\boxed{\mathbb{R}}$ denotes the set of real numbers.

Since the set of natural numbers consists of the positive integers, the natural numbers are sometimes denoted by $\boxed{\mathbb{Z}^+}$.

If A and B are sets, then we say that A is a **subset** of B, written $\boxed{A \subseteq B}$, provided that every element of A is an element of B. Observe that $A \subseteq B$ is equivalent to "For all x in the universe of discourse, if $x \in A$, then $x \in B$."

Every nonempty set always has two rather boring subsets.

Problem 2.1. Let A be a set. Write a short proof for each of the following.

(a) $A \subseteq A$

(b) $\emptyset \subseteq A$

The next problem shows that "$\subseteq$" is a transitive relation.

Problem 2.2 (Transitivity of subsets)**.** Prove that if A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Let A and B be sets in some universe of discourse U. We define the following.

- Two sets A and B are **equal**, denoted $\boxed{A = B}$, if the sets contain the same elements. That is, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. Note that if we want to prove $A = B$, then we have to do two separate subproofs: one for $A \subseteq B$ and one for $B \subseteq A$. It is common to label each mini-proof with "($\subseteq$)" and "($\supseteq$)", respectively.

- If $A \subseteq B$, then A is called a **proper subset** provided that $A \neq B$. In this case, we may write $\boxed{A \subset B}$ or $\boxed{A \subsetneq B}$. *Warning:* Some books use $\subset$ to mean $\subseteq$.

- The **union** of the sets A and B is $\boxed{A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\}}$.

- The **intersection** of the sets A and B is $\boxed{A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\}}$.

- The **set difference** of the sets A and B is $\boxed{A \setminus B := \{x \in U \mid x \in A \text{ and } x \notin B\}}$.

- The **complement of** A (relative to U) is the set $\boxed{A^c := U \setminus A = \{x \in U \mid x \notin A\}}$.

- If $A \cap B = \emptyset$, then we say that A and B are **disjoint** sets.

Example 2.3. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of **irrational numbers**.

Problem 2.4. Prove that if A and B are sets such that $A \subseteq B$, then $B^c \subseteq A^c$.

Problem 2.5. Prove that if A and B are sets, then $A \setminus B = A \cap B^c$.

Problem 2.6. Give an example where $A \neq B$ but $A \setminus B = \emptyset$.

Consider the following collection of sets:

$$\{a\},\{a,b\},\{a,b,c\},\ldots,\{a,b,c,\ldots,z\}$$

This collection has a natural way for us to "index" the sets:

$$A_1=\{a\},A_2=\{a,b\},A_3=\{a,b,c\},\ldots,A_{26}=\{a,b,c,\ldots,z\}$$

In this case the sets are **indexed** by the set $\{1,2,\ldots,26\}$, where the subscripts are taken from the **index set**. If we wanted to talk about an arbitrary set from this indexed collection, we could use the notation A_n.

Using indexing sets in mathematics is an extremely useful notational tool, but it is important to keep straight the difference between the sets that are being indexed, the elements in each set being indexed, the indexing set, and the elements of the indexing set.

Any set (finite or infinite) can be used as an indexing set. Often capital Greek letters are used to denote arbitrary indexing sets and small Greek letters to represent elements of these sets. If the indexing set is a subset of $\mathbb{R}$, then it is common to use Roman letters as individual indices. Of course, these are merely conventions, not rules.

- If Δ is a set and we have a collection of sets indexed by Δ, then we may write $\{S_\alpha\}_{\alpha\in\Delta}$ to refer to this collection. We read this as "the set of S-sub-alphas over alpha in Delta."
- If a collection of sets is indexed by $\mathbb{N}$, then we may write $\{U_n\}_{n\in\mathbb{N}}$ or $\{U_n\}_{n=1}^{\infty}$.
- Borrowing from this idea, a collection $\{A_1,\ldots,A_{26}\}$ may be written as $\{A_n\}_{n=1}^{26}$.

Suppose we have a collection $\{A_\alpha\}_{\alpha\in\Delta}$.

- The **union of the entire collection** is defined via

$$\boxed{\bigcup_{\alpha\in\Delta} A_\alpha=\{x \mid x\in A_\alpha \text{ for some } \alpha\in\Delta\}.}$$

- The **intersection of the entire collection** is defined via

$$\boxed{\bigcap_{\alpha\in\Delta} A_\alpha=\{x \mid x\in A_\alpha \text{ for all } \alpha\in\Delta\}.}$$

In the special case that $\Delta=\mathbb{N}$, we write

$$\bigcup_{n=1}^{\infty} A_n=\{x \mid x\in A_n \text{ for some } n\in\mathbb{N}\}=A_1\cup A_2\cup A_3\cup\cdots$$

and

$$\bigcap_{n=1}^{\infty} A_n=\{x \mid x\in A_n \text{ for all } n\in\mathbb{N}\}=A_1\cap A_2\cap A_3\cap\cdots$$

Similarly, if $\Delta = \{1, 2, 3, 4\}$, then

$$\bigcup_{n=1}^{4} A_n = A_1 \cup A_2 \cup A_3 \cup A_4 \quad \text{and} \quad \bigcap_{n=1}^{4} A_n = A_1 \cap A_2 \cap A_3 \cap A_4.$$

Notice the difference between "$\bigcup$" and "$\cup$" (respectively, "$\bigcap$" and "$\cap$").

Problem 2.7. Let $\{A_n\}_{n=1}^{26}$ be the collection from the discussion below Problem 2.6. Find each of the following.

(a) $\displaystyle\bigcup_{n=1}^{26} A_n$

(b) $\displaystyle\bigcap_{n=1}^{26} A_n$

Problem 2.8. For each $r \in \mathbb{Q}$ (the rational numbers), let N_r be the set containing all real numbers *except* r. Find each of the following.

(a) $\displaystyle\bigcup_{r\in\mathbb{Q}} N_r$

(b) $\displaystyle\bigcap_{r\in\mathbb{Q}} N_r$

A collection of sets $\{A_\alpha\}_{\alpha\in\Delta}$ is **pairwise disjoint** if $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$.

Problem 2.9. Draw a Venn diagram of a collection of three sets that are pairwise disjoint.

Problem 2.10. Provide an example of a collection of three sets, say $\{A_1, A_2, A_3\}$, such that the collection is *not* pairwise disjoint, but $\bigcap_{n=1}^{3} A_n = \emptyset$.

Problem 2.11. Find a collection of nonempty sets $S_i \subseteq \mathbb{N}$ indexed by $i \in \mathbb{N}$ such that $S_{i+1} \subsetneq S_i$ and $\bigcap_{i=1}^{\infty} S_i = \emptyset$.

Problem 2.12. Find a collection of nonempty sets $S_i \subseteq \mathbb{N}$ indexed by $i \in \mathbb{N}$ such that $S_i \subsetneq S_{i+1}$ but $\bigcup_{i=1}^{\infty} S_i \neq \mathbb{N}$.

Problem 2.13 (DeMorgan's Law)**.** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of sets. Prove one of the following.

(a) $\displaystyle\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^C = \bigcap_{\alpha\in\Delta} A_\alpha^C$

(b) $\displaystyle\left(\bigcap_{\alpha\in\Delta} A_\alpha\right)^C = \bigcup_{\alpha\in\Delta} A_\alpha^C$

Problem 2.14 (Distribution of Union and Intersection)**.** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of sets and let B be any set. Prove one of the following.

(a) $\displaystyle B \cup \left(\bigcap_{\alpha\in\Delta} A_\alpha\right) = \bigcap_{\alpha\in\Delta} (B \cup A_\alpha)$

(b) $\displaystyle B \cap \left(\bigcup_{\alpha\in\Delta} A_\alpha\right) = \bigcup_{\alpha\in\Delta} (B \cap A_\alpha)$

An **ordered pair** is an ordered list of two elements of the form (a, b). In this case, a is called the **first component** (or **first coordinate**) while b is called the **second component** (or **second coordinate**). We can use the notion of ordered pairs to construct new sets from existing sets. If A and B are sets, the **Cartesian product** (or **direct product**) of A and B, denoted $A \times B$ (read as "A times B" or "A cross B"), is the set of all ordered pairs where the first component is from A and the second component is from B. In set-builder notation, we have

$$\boxed{A \times B := \{(a, b) \mid a \in A, b \in B\}}.$$

Example 2.15. The standard two-dimensional plane $\mathbb{R}^2$ is a familiar example of Cartesian product. In particular, we have

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

Problem 2.16. If A is a set, then what is $A \times \emptyset$ equal to?

Problem 2.17. Given sets A and B, when will $A \times B$ be equal to $B \times A$?

> It does not matter how slowly you go as long as you do not stop.
>
> Confucius, philosopher

2.2 Induction and The Well-Ordering Principle

The following axiom is one of the Peano Axioms, which is a collection of axioms for the natural numbers introduced in the 19th century by Italian mathematician Giuseppe Peano (1858–1932).

Axiom 2.18 (Axiom of Induction)**.** Let $S \subseteq \mathbb{N}$ such that both

(i) $1 \in S$, and

(ii) if $k \in S$, then $k + 1 \in S$.

Then $S = \mathbb{N}$.

We can think of the set S as a ladder, where the first hypothesis as saying that we have a first rung of a ladder. The second hypothesis says that if we are on any arbitrary rung of the ladder, then we can always get to the next rung. Taken together, this says that we can get from the first rung to the second, from the second to the third, and in general, from any kth rung to the $(k + 1)$st rung, so that our ladder is actually $\mathbb{N}$. Do you agree that the Axiom of Induction is a pretty reasonable assumption?

Using the Axiom of Induction, we can prove the following theorem, known as the **Principle of Mathematical Induction**. One approach to proving this theorem is to let $S = \{k \in \mathbb{N} \mid P(k) \text{ is true}\}$ and use the Axiom of Induction. The set S is sometimes called the **truth set**. Your job is to show that the truth set is all of $\mathbb{N}$.

Problem 2.19 (Principle of Mathematical Induction)**.** Let $P(1), P(2), P(3), \ldots$ be a sequence of statements, one for each natural number. Assume

(i) $P(1)$ is true, and

(ii) if $P(k)$ is true, then $P(k+1)$ is true.

Prove that $P(n)$ is true for all $n \in \mathbb{N}$.

The Principle of Mathematical Induction provides us with a process for proving statements of the form: "For all $n \in \mathbb{N}$, $P(n)$," where $P(n)$ is some predicate involving n. Hypothesis (i) above is called the **base step** (or **base case**) while (ii) is called the **inductive step**.

You should not confuse *mathematical induction* with *inductive reasoning* associated with the natural sciences. Inductive reasoning is a scientific method whereby one induces general principles from observations. On the other hand, mathematical induction is a deductive form of reasoning used to establish the validity of a proposition.

There is another formulation of induction, where the inductive step begins with a set of assumptions rather than one single assumption. This method is sometimes called **complete induction** (or **strong induction**).

Problem 2.20 (Principle of Complete Mathematical Induction)**.** Let $P(1), P(2), P(3), \dots$ be a sequence of statements, one for each natural number. Assume that

(i) $P(1)$ is true, and

(ii) For all $k \in \mathbb{N}$, if $P(j)$ is true for all $j \in \mathbb{N}$ such that $j \leq k$, then $P(k+1)$ is true.

Prove that $P(n)$ is true for all $n \in \mathbb{N}$.

Note the difference between ordinary induction and complete induction. For the induction step of complete induction, we are not only assuming that $P(k)$ is true, but rather that $P(j)$ is true for all j from 1 to k. Despite the name, complete induction is not any stronger or more powerful than ordinary induction. It is worth pointing out that anytime ordinary induction is an appropriate proof technique, so is complete induction. So, when should we use complete induction?

In the inductive step, you need to reach $P(k+1)$, and you should ask yourself which of the previous cases you need to get there. If all you need, is the statement $P(k)$, then ordinary induction is the way to go. If two preceding cases, $P(k-1)$ and $P(k)$, are necessary to reach $P(k+1)$, then complete induction is appropriate. In the extreme, if one needs the full range of preceding cases (i.e., all statements $P(1), P(2), \dots, P(k)$), then again complete induction should be utilized.

Note that in situations where complete induction is appropriate, it might be the case that you need to verify more than one case in the base step. The number of base cases to be checked depends on how one needs to "look back" in the induction step.

The penultimate result of this section is known as the **Well-Ordering Principle**. This seemingly obvious theorem requires a bit of work to prove. It is worth noting that in some axiomatic systems, the Well-Ordering Principle is sometimes taken as an axiom. However, in our case, we will assume the Axiom of Induction and then prove the result using complete induction. Before stating the Well-Ordering Principle, we need an additional definition.

Definition 2.21. Let $A \subseteq \mathbb{R}$ and $m \in A$. Then m is called a **maximum** (or **greatest element**) of A if for all $a \in A$, we have $a \leq m$. Similarly, m is called **minimum** (or **least element**) of A if for all $a \in A$, we have $m \leq a$.

Not surprisingly, maximums and minimums are unique when they exist.

Problem 2.22. Prove that if $A \subseteq \mathbb{R}$ such that the maximum (respectively, minimum) of A exists, then the maximum (respectively, minimum) of A is unique.

If the maximum of a set A exists, then it is denoted by $\boxed{\max(A)}$. Similarly, if the minimum of a set A exists, then it is denoted by $\boxed{\min(A)}$.

Problem 2.23. Find the maximum and the minimum for each of the following sets when they exist.

(a) $\{5, 11, 17, 42, 103\}$

(b) $\mathbb{N}$

(c) $\mathbb{Z}$

(d) $(0, 1]$

(e) $(0, 1] \cap \mathbb{Q}$

(f) $(0, \infty)$

(g) $\{42\}$

(h) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(i) $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

(j) $\emptyset$

To prove the Well-Ordering Principle, consider a proof by contradiction. Suppose S is a nonempty subset of $\mathbb{N}$ that does not have a least element. Define the proposition $P(n) :=$"n is not an element of S" and then use complete induction to prove the result.

Problem 2.24 (Well-Ordering Principle)**.** Prove that every nonempty subset of the natural numbers has a least element.

It turns out that the Well-Ordering Principle and the Axiom of Induction are equivalent. In other words, one can prove the Well-Ordering Principle from the Axiom of Induction, as we have done, but one can also prove the Axiom of Induction if the Well-Ordering Principle is assumed.

The final result of this section can be thought of as a generalized version of the Well-Ordering Principle.

Problem 2.25. Prove that if A is a nonempty subset of the integers and there exists $b \in A^c$ such that $b \geq a$ for all $a \in A$, then A contains a greatest element.

In the previous problem, b is referred to as an upper bound for A. We will study upper bounds in Section 3.4.

> Nothing that's worth anything is ever easy.
>
> Mike Hall, ultra-distance cyclist

2.3 Functions

Let A and B be sets. A **relation** R **from** A **to** B is a subset of $A \times B$. If R is a relation from A to B and $(a,b) \in R$, then we say that a **is related to** b and we may write aRb in place of $(a,b) \in R$.

A function is a special type of relation, where the basic building blocks are a first set and a second set, say X and Y, and a "correspondence" that assigns *every* element of X to *exactly one* element of Y. More formally, if X and Y are nonempty sets, a **function** f **from** X **to** Y is a relation from X to Y such that for every $x \in X$, there exists a unique $y \in Y$ such that $(x,y) \in f$. The set X is called the **domain** of f and is denoted by $\boxed{\operatorname{Dom}(f)}$. The set Y is called the **codomain** of f and is denoted by $\boxed{\operatorname{Codom}(f)}$ while the subset of the codomain defined via

$$\boxed{\operatorname{Rng}(f) := \{y \in Y \mid \text{there exists } x \text{ such that } (x,y) \in f\}}$$

is called the **range** of f or the **image of** X under f.

There is a variety of notation and terminology associated to functions. We will write $\boxed{f : X \to Y}$ to indicate that f is a function from X to Y. We will make use of statements such as "Let $f : X \to Y$ be the function defined via…" or "Define $f : X \to Y$ via…", where f is understood to be a function in the second statement. Sometimes the word **mapping** (or **map**) is used in place of the word function. If $(a,b) \in f$ for a function f, we often write $\boxed{f(a) = b}$ and say that "f maps a to b" or "f of a equals b". In this case, a may be called an **input** of f and is the **preimage** of b under f while b is called an **output** of f and is the **image** of a under f. Note that the domain of a function is the set of inputs while the range is the set of outputs for the function.

According to our definition, if $f : X \to Y$ is a function, then every element of the domain is utilized exactly once. However, there are no restrictions on whether an element of the codomain ever appears in the second coordinate of an ordered pair in the relation. Yet if an element of Y is in the range of f, it may appear in more than one ordered pair in the relation.

It follows immediately from the definition of function that two functions are equal if and only if they have the same domain, same codomain, and the same set of ordered pairs in the relation. That is, functions f and g are equal if and only if $\operatorname{Dom}(f) = \operatorname{Dom}(g)$, $\operatorname{Codom}(f) = \operatorname{Codom}(g)$, and $f(x) = g(x)$ for all $x \in X$.

Since functions are special types of relations, we can represent them using digraphs and graphs when practical. Digraphs for functions are often called **function** (or **mapping**) **diagrams**. When drawing function diagrams, it is standard practice to put the vertices for the domain on the left and the vertices for the codomain on the right, so that all directed edges point from left to right. We may also draw an additional arrow labeled by the name of the function from the domain to the codomain.

Example 2.26. Let $X = \{a, b, c, d\}$ to $Y = \{1, 2, 3, 4\}$ and define the relation f from X to Y via

$$f = \{(a, 2), (b, 4), (c, 4), (d, 1)\}.$$

Since each element X appears exactly once as a first coordinate, f is a function with domain X and codomain Y (i.e., $f : X \to Y$). In this case, we see that $\operatorname{Rng}(f) = \{1, 2, 4\}$. Moreover, we can write things like $f(a) = 2$ and $c \mapsto 4$, and say things like "f maps b to 4" and "the image of d is 1." The function diagram for f is depicted in Figure 2.1.

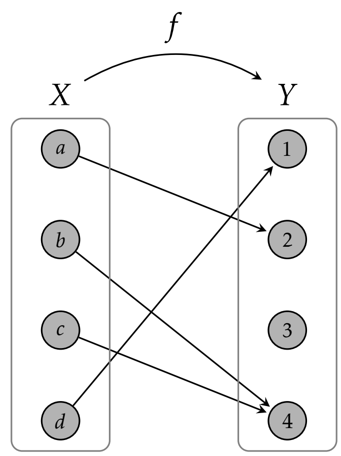

Figure 2.1: Function diagram for a function from $X = \{a, b, c, d, \}$ to $Y = \{1, 2, 3, 4\}$.

Problem 2.27. What properties does the digraph for a relation from X to Y need to have in order for it to represent a function?

Problem 2.28. In high school I am sure that you were told that a graph represents a function if it passes the **vertical line test**. Carefully state what the vertical line test says and then explain why it works.

Sometimes we can define a function using a formula. For example, we can write $f(x) = x^2 - 1$ to mean that each x in the domain of f maps to $x^2 - 1$ in the codomain. However, notice that providing only a formula is ambiguous! A function is determined by its domain, codomain, and the correspondence between these two sets. If we only provide a description for the correspondence, it is not clear what the domain and codomain are. Two functions that are defined by the same formula, but have different domains or codomains are *not* equal.

Example 2.29. The function $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x^2 - 1$ is not equal to the function $g : \mathbb{N} \to \mathbb{R}$ defined by $g(x) = x^2 - 1$ since the two functions do not have the same domain.

Sometimes we rely on context to interpret the domain and codomain. For example, in a calculus class, when we describe a function in terms of a formula, we are implicitly assuming that the domain is the largest allowable subset of $\mathbb{R}$—sometimes called the **default domain**—that makes sense for the given formula while the codomain is $\mathbb{R}$.

Example 2.30. If we write $f(x) = x^2 - 1$, $g(x) = \sqrt{x}$, and $h(x) = \frac{1}{x}$ without mentioning the domains, we would typically interpret these as the functions $f : \mathbb{R} \to \mathbb{R}$, $g : [0, \infty) \to \mathbb{R}$, and $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ that are determined by their respective formulas.

Problem 2.31. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs, or write a formula as long as the domain and codomain are clear.

(a) A function f from a set with 4 elements to a set with 3 elements such that $\mathrm{Rng}(f) = \mathrm{Codom}(f)$.

(b) A function g from a set with 4 elements to a set with 3 elements such that $\mathrm{Rng}(g)$ is strictly smaller than $\mathrm{Codom}(g)$.

There are a few special functions that we should know the names of. Let X and Y be nonempty sets.

- If $X \subseteq Y$, then the function $\iota : X \to Y$ defined via $\iota(x) = x$ is called the **inclusion map from X into Y**. Note that "ι" is the Greek letter "iota".
- If the domain and codomain are equal, the inclusion map has a special name. If X is a nonempty set, then the function $i_X : X \to X$ defined via $i_X(x) = x$ is called the **identity map** (or **identity function**) **on X**.
- Any function $f : X \to Y$ defined via $f(x) = c$ for a fixed $c \in Y$ is called a **constant function**.
- A **piecewise-defined function** (or **piecewise function**) is a function defined by specifying its output on a partition of the domain. Note that "piecewise" is a way of expressing the function, rather than a property of the function itself.

Example 2.32. The function $f : \mathbb{N} \to \mathbb{N}$ defined via

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x \text{ is even,} \\ 17, & \text{if } x \text{ is odd} \end{cases}$$

is an example of a piecewise-defined function.

It is important to point out that not every function can be described using a formula! Despite your prior experience, functions that can be represented succinctly using a formula are rare.

The next problem illustrates that some care must be taken when attempting to define a function.

Problem 2.33. For each of the following, explain why the given description does not define a function.

(a) Define $f : \{1,2,3\} \to \{1,2,3\}$ via $f(a) = a - 1$.

(b) Define $g : \mathbb{N} \to \mathbb{Q}$ via $g(n) = \frac{n}{n-1}$.

(c) Let $A_1 = \{1,2,3\}$ and $A_2 = \{3,4,5\}$. Define $h : A_1 \cup A_2 \to \{1,2\}$ via

$$h(x) = \begin{cases} 1, & \text{if } x \in A_1 \\ 2, & \text{if } x \in A_2. \end{cases}$$

(d) Define $s : \mathbb{Q} \to \mathbb{Z}$ via $s(a/b) = a + b$.

In mathematics, we say that an expression is **well defined** (or **unambiguous**) if its definition yields a unique interpretation. Otherwise, we say that the expression is not well defined (or is **ambiguous**). For example, if $a, b, c \in \mathbb{R}$, then the expression abc is well defined since it does not matter if we interpret this as $(ab)c$ or $a(bc)$ since the real numbers are associative under multiplication.

When we attempt to define a function, it may not be clear without doing some work that our definition really does yield a function. If there is some potential ambiguity in the definition of a function that ends up not causing any issues, we say that the function is well defined. However, this phrase is a bit of misnomer since all functions are well defined. The issue of whether a description for a proposed function is well defined often arises when defining things in terms of representatives of equivalence classes, or more generally in terms of how an element of the domain is written. For example, the descriptions given in parts (c) and (d) of Problem 2.33 are not well defined. To show that a potentially ambiguous description for a function $f : X \to Y$ is well defined prove that if a and b are two representations for the same element in X, then $f(a) = f(b)$.

Let $f : X \to Y$ be a function.

- The function f is said to be **injective** (or **one-to-one**) if for all $y \in \mathrm{Rng}(f)$, there is a unique $x \in X$ such that $y = f(x)$.
- The function f is said to be **surjective** (or **onto**) if for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$.
- If f is both injective and surjective, we say that f is **bijective**.

An injective function is also called an **injection**, a surjective function is called a **surjection**, and a bijective function is called a **bijection**. To prove that a function $f : X \to Y$ is an injection, we must prove that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. To show that f is surjective, you should start with an arbitrary $y \in Y$ and then work to show that there exists $x \in X$ such that $y = f(x)$.

Problem 2.34. Assume that X and Y are finite sets. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs, or write a formula as long as the domain and codomain are clear.

(a) A function $f : X \to Y$ that is injective but not surjective.

(b) A function $f : X \to Y$ that is surjective but not injective.

(c) A function $f : X \to Y$ that is a bijection.

(d) A function $f : X \to Y$ that is neither injective nor surjective.

Problem 2.35. Provide an example of each of the following. You may either draw a graph or write down a formula. Make sure you have the correct domain.

(a) A function $f : \mathbb{R} \to \mathbb{R}$ that is injective but not surjective.

(b) A function $f : \mathbb{R} \to \mathbb{R}$ that is surjective but not injective.

(c) A function $f : \mathbb{R} \to \mathbb{R}$ that is a bijection.

(d) A function $f : \mathbb{R} \to \mathbb{R}$ that is neither injective nor surjective.

(e) A function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that is injective.

Problem 2.36. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is _________ if and only if every horizontal line hits the graph of f *at most once*.

This statement is often called the **horizontal line test**. Explain why the horizontal line test is true.

Problem 2.37. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is _________ if and only if every horizontal line hits the graph of f *at least once*.

Explain why this statement is true.

Problem 2.38. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is _________ if and only if every horizontal line hits the graph of f *exactly once*.

Explain why this statement is true.

Problem 2.39. Determine whether each of the following functions is injective, surjective, both, or neither. In each case, you should provide a proof or a counterexample as appropriate. *Note:* You are probably not in a position to write a careful argument for surjectivity for Part (d).

(a) Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$

(b) Define $g : \mathbb{R} \to [0, \infty)$ via $g(x) = x^2$

(c) Define $h : \mathbb{R} \to \mathbb{R}$ via $h(x) = x^3$

(d) Define $k : \mathbb{R} \to \mathbb{R}$ via $k(x) = x^3 - x$

(e) Define $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via $c(x, y) = x^2 + y^2$

(f) Define $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ via $f(n) = (n, n)$

(g) Define $g : \mathbb{Z} \to \mathbb{Z}$ via

$$g(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

(h) Define $\ell : \mathbb{Z} \to \mathbb{N}$ via

$$\ell(n) = \begin{cases} 2n + 1, & \text{if } n \geq 0 \\ -2n, & \text{if } n < 0 \end{cases}$$

The next two results should not come as as surprise.

Problem 2.40. Prove that the inclusion map $\iota : X \to Y$ for $X \subseteq Y$ is an injection.

Problem 2.41. Prove that the identity function $i_X : X \to X$ is a bijection.

If $f : X \to Y$ and $g : Y \to Z$ are functions, we define $g \circ f : X \to Z$ via $\boxed{(g \circ f)(x) = g(f(x))}$. The function $g \circ f$ is called the **composition of** f **and** g. It is important to notice that the function on the right is the one that "goes first." Moreover, we cannot compose any two random functions since the codomain of the first function must agree with the domain of the second function. In particular, $f \circ g$ may not be a sensible function even when $g \circ f$ exists. Figure 2.2 provides a visual representation of function composition in terms of function diagrams.

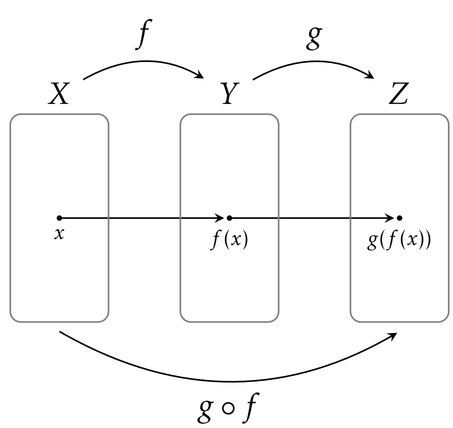

Figure 2.2: Visual representation of function composition.

Example 2.42. Consider the inclusion map $\iota : X \to Y$ such that X is a proper subset of Y and suppose $f : Y \to Z$ is a function. Then the composite function $f \circ \iota : X \to Z$ is given by

$$f \circ \iota(x) = f(\iota(x)) = f(x)$$

for all $x \in X$. Notice that $f \circ \iota$ is simply the function f but with a smaller domain. In this case, we say that $f \circ \iota$ is the **restriction of f to X**, which is often denoted by $\boxed{f|_X}$.

The next problem illustrates that $f \circ g$ and $g \circ f$ need not be equal even when both composite functions exist.

Problem 2.43. Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$ and $g(x) = 3x - 5$, respectively. Determine formulas for the composite functions $f \circ g$ and $g \circ f$.

The next problem tells us that function composition is associative.

Problem 2.44. Prove that if $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ are functions, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Problem 2.45. In each case, give examples of finite sets X, Y, and Z, and functions $f : X \to Y$ and $g : Y \to Z$ that satisfy the given conditions. Drawing a function diagram is sufficient.

(a) f is surjective, but $g \circ f$ is not surjective.

(b) g is surjective, but $g \circ f$ is not surjective.

(c) f is injective, but $g \circ f$ is not injective.

(d) g is injective, but $g \circ f$ is not injective.

Problem 2.46. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both surjective functions, then $g \circ f$ is also surjective.

Problem 2.47. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both injective functions, then $g \circ f$ is also injective.

Problem 2.48. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both bijections, then $g \circ f$ is also a bijection.

Problem 2.49. Assume that $f : X \to Y$ and $g : Y \to Z$ are both functions. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If $g \circ f$ is injective, then f is injective.

(b) If $g \circ f$ is injective, then g is injective.

(c) If $g \circ f$ is surjective, then f is surjective.

(d) If $g \circ f$ is surjective, then g is surjective.

There are two important types of sets related to functions. Let $f : X \to Y$ be a function.

- If $S \subseteq X$, the **image** of S under f is defined via

$$\boxed{f(S) := \{f(x) \mid x \in S\}}.$$

- If $T \subseteq Y$, the **preimage** (or **inverse image**) of T under f is defined via

$$\boxed{f^{-1}(T) := \{x \in X \mid f(x) \in T\}}.$$

The image of a subset S of the domain is simply the subset of the codomain we obtain by mapping the elements of S. It is important to emphasize that the function f maps *elements* of X to *elements* of Y, but we can apply f to a *subset* of X to yield a *subset* of Y. That is, if $S \subseteq X$, then $f(S) \subseteq Y$. Note that the image of the domain is the same as the range of the function. That is, $f(X) = \operatorname{Rng}(f)$.

When it comes to preimages, the notation $f^{-1}(T)$ should not be confused with an inverse function (which may or may not exist for an arbitrary function f). For $T \subseteq Y$, $f^{-1}(T)$ is the set of elements in the domain that map to elements in T. As a special case, $f^{-1}(\{y\})$ is the set of elements in the domain that map to $y \in Y$. If $y \notin \operatorname{Rng}(f)$, then $f^{-1}(\{y\}) = \emptyset$. Notice that if $y \in Y$, $f^{-1}(\{y\})$ is always a sensible thing to write while $f^{-1}(y)$ only makes sense if f^{-1} is a function. Also, note that the preimage of the codomain is the domain. That is, $f^{-1}(Y) = X$.

Problem 2.50. Define $f : \mathbb{Z} \to \mathbb{Z}$ via $f(x) = x^2$. List elements in each of the following sets.

(a) $f(\{0,1,2\})$

(b) $f^{-1}(\{0,1,4\})$

Problem 2.51. Find functions f and g and sets S and T such that $f(f^{-1}(T)) \neq T$ and $g^{-1}(g(S)) \neq S$.

Problem 2.52. Suppose $f : X \to Y$ is an injection and A and B are disjoint subsets of X. Are $f(A)$ and $f(B)$ necessarily disjoint subsets of Y? If so, prove it. Otherwise, provide a counterexample.

Problem 2.53. Let $f : X \to Y$ be a function and suppose $A, B \subseteq X$ and $C, D \subseteq Y$. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If $A \subseteq B$, then $f(A) \subseteq f(B)$.

(b) If $C \subseteq D$, then $f^{-1}(C) \subseteq f^{-1}(D)$.

(c) $f(A \cup B) \subseteq f(A) \cup f(B)$.

(d) $f(A \cup B) \supseteq f(A) \cup f(B)$.

(e) $f(A \cap B) \subseteq f(A) \cap f(B)$.

(f) $f(A \cap B) \supseteq f(A) \cap f(B)$.

(g) $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$.

(h) $f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$.

(i) $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$.

(j) $f^{-1}(C \cap D) \supseteq f^{-1}(C) \cap f^{-1}(D)$.

(k) $A \subseteq f^{-1}(f(A))$.

(l) $A \supseteq f^{-1}(f(A))$.

(m) $f(f^{-1}(C)) \subseteq C$.

(n) $f(f^{-1}(C)) \supseteq C$.

We can generalize the results above to handle arbitrary collections of sets.

Problem 2.54. Let $f : X \to Y$ be a function and suppose $\{A_\alpha\}_{\alpha \in \Delta}$ is a collection of subsets of X. Prove each of the following.

(a) $f\left(\bigcup_{\alpha \in \Delta} A_\alpha\right) = \bigcup_{\alpha \in \Delta} f(A_\alpha)$.

(b) $f\left(\bigcap_{\alpha \in \Delta} A_\alpha\right) \subseteq \bigcap_{\alpha \in \Delta} f(A_\alpha)$.

Problem 2.55. Let $f : X \to Y$ be a function and suppose $\{C_\alpha\}_{\alpha \in \Delta}$ is a collection of subsets of Y. Prove each of the following.

(a) $f^{-1}\left(\bigcup_{\alpha \in \Delta} C_\alpha\right) = \bigcup_{\alpha \in \Delta} f^{-1}(C_\alpha)$.

(b) $f^{-1}\left(\bigcap_{\alpha \in \Delta} C_\alpha\right) = \bigcap_{\alpha \in \Delta} f^{-1}(C_\alpha)$.

> In mathematics the art of proposing a question must be held of higher value than solving it.
>
> Georg Cantor, mathematician

All truths are easy to understand once they are discovered; the point is to discover them.

Galileo Galilei, astronomer & physicist

Chapter 3

The Real Numbers

In this chapter we will take a deep dive into the structure of the real numbers by building up the multitude of properties you are familiar with by starting with a collection of fundamental axioms. Recall that an axiom is a statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proved. It is worth pointing out that one can carefully construct the real numbers from the natural numbers. However, that will not be the approach we take. Instead, we will simply list the axioms that the real numbers satisfy. Our axioms for the real numbers fall into three categories:

1. **Field Axioms:** These axioms provide the essential properties of arithmetic involving addition and multiplication.
2. **Order Axioms:** These axioms provide the necessary properties of inequalities.
3. **Completeness Axiom:** This axiom ensures that the familiar number line that we use to model the real numbers does not have any holes in it.

Throughout this book, our universe of discourse will be the real numbers. Any time we refer to a generic set, we mean a subset of real numbers. We will often refer to an element in a subset of real numbers as a **point**.

3.1 The Field Axioms

We begin with the Field Axioms.

Axioms 3.1 (Field Axioms)**.** There exist operations $+$ (addition) and $\cdot$ (multiplication) on $\mathbb{R}$ satisfying:

(F1) (Associativity for Addition) For all $a, b, c \in \mathbb{R}$ we have $(a+b)+c = a+(b+c)$;

(F2) (Commutativity for Addition) For all $a, b \in \mathbb{R}$, we have $a+b = b+a$;

(F3) (Additive Identity) There exists $0 \in \mathbb{R}$ such that for all $a \in \mathbb{R}$, $0 + a = a$;

(F4) (Additive Inverses) For all $a \in \mathbb{R}$ there exists $-a \in \mathbb{R}$ such that $a + (-a) = 0$;

(F5) (Associativity for Multiplication) For all $a, b, c \in \mathbb{R}$ we have $(ab)c = a(bc)$;

(F6) (Commutativity for Multiplication) For all $a, b \in \mathbb{R}$, we have $ab = ba$;

(F7) (Multiplicative Identity) There exists $1 \in \mathbb{R}$ such that $1 \neq 0$ and for all $a \in \mathbb{R}$, $1a = a$;

(F8) (Multiplicative Inverses) For all $a \in \mathbb{R} \setminus \{0\}$ there exists $a^{-1} \in \mathbb{R}$ such that $aa^{-1} = 1$.

(F9) (Distributive Property) For all $a, b, c \in \mathbb{R}$, $a(b + c) = ab + ac$;

In the language of abstract algebra, Axioms F1–F4 and F5–F8 make each of $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$ an abelian group under addition and multiplication, respectively. Axiom F9 provides a way for the operations of addition and multiplication to interact. Collectively, Axioms F1–F9 make the real numbers a **field**. Axioms F3 and F7 state the existence of additive and multiplicative identities, but these axioms do not assume that the elements are the unique elements with the specified properties. However, we can prove that this is the case. That is, 0 and 1 of $\mathbb{R}$ are the unique **additive** and **multiplicative identities** in $\mathbb{R}$. To prove the following theorem, suppose 0 and $0'$ are both additive identities in $\mathbb{R}$ and then show that $0 = 0'$. This shows that there can only be one additive identity. It is important to point out that we are not proving that the number 0 introduced in Axiom F3 is unique, but rather there is a unique number with the property specified in Axiom F3.

Problem 3.2. Prove that there exists a unique additive identity of $\mathbb{R}$.

For the next problem, mimic the approach you used to prove Problem 3.2.

Problem 3.3. Prove that there exists a unique multiplicative identity of $\mathbb{R}$.

Similar to Axioms F3 and F7, Axioms F4 and F8 state the existence of additive and multiplicative inverses, but these axioms do not assume that these elements are the unique elements with the specified properties. However, we can prove that for every $a \in \mathbb{R}$, the elements $-a$ and a^{-1} (as long as $a \neq 0$) are the unique **additive** and **multiplicative inverses**, respectively.

Problem 3.4. Prove that every real number has a unique additive inverse.

Note that since $0 + 0 = 0$ and additive inverses are unique, it must be the case that $-0 = 0$.

Problem 3.5. Prove that every nonzero real number has a unique multiplicative inverse.

In light of the last two problems, we now know that sticking a minus sign in front of $a \in \mathbb{R}$ or raising $a \in \mathbb{R} \setminus \{0\}$ to -1 each correspond to an operation that yields a unique element with the corresponding inverse property.

Since we are taking a formal axiomatic approach to the real numbers, we should make it clear how the natural numbers are embedded in $\mathbb{R}$.

Definition 3.6. We define the **natural numbers**, denoted by $\mathbb{N}$, to be the smallest subset of $\mathbb{R}$ satisfying:

(a) $1 \in \mathbb{N}$, and

(b) for all $n \in \mathbb{N}$, we have $n + 1 \in \mathbb{N}$.

Of course, we use the standard numeral system to represent the natural numbers, so that $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \ldots\}$.

Given the natural numbers, Axiom F3/Problem 3.2 and Axiom F4/Problem 3.4 together with the operation of addition allow us to define the **integers**, denoted by $\mathbb{Z}$, in the obvious way. That is, the integers consist of the natural numbers together with the additive identity and all of the additive inverses of the natural numbers.

We now introduce some common notation that you are likely familiar with. Take a moment to think about why the following is a definition as opposed to an axiom or theorem.

Definition 3.7. For every $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}$, we define the following:

(a) $\boxed{a - b := a + (-b)}$

(b) $\boxed{\dfrac{a}{b} := ab^{-1}}$ (for $b \neq 0$)

(c) $$\boxed{a^n := \begin{cases} \overbrace{aa\cdots a}^{n}, & \text{if } n \in \mathbb{N} \\ 1, & \text{if } n = 0 \text{ and } a \neq 0 \\ \dfrac{1}{a^{-n}}, & \text{if } -n \in \mathbb{N} \text{ and } a \neq 0 \end{cases}}$$

The set of **rational numbers**, denoted by $\mathbb{Q}$, is defined to be the collection of all real numbers having the form given in Part (b) of Definition 3.7. The **irrational numbers** are defined to be $\mathbb{R} \setminus \mathbb{Q}$.

Using the Field Axioms, we can prove each of the following statements.

Problem 3.8. Prove that for all $a, b, c \in \mathbb{R}$, we have the following:

(a) $a = b$ if and only if $a + c = b + c$;

(b) $0a = 0$;

(c) $-a = (-1)a$;

(d) $(-1)^2 = 1$;

(e) $-(-a) = a$;

(f) If $a \neq 0$, then $(a^{-1})^{-1} = a$;

(g) If $a \neq 0$ and $ab = ac$, then $b = c$.

(h) If $ab = 0$, then either $a = 0$ or $b = 0$.

Problem 3.9. Carefully prove that for all $a, b \in \mathbb{R}$, we have $(a+b)(a-b) = a^2 - b^2$. Explicitly cite where you are utilizing the Field Axioms and Problem 3.8.

> Like what you do, and then you will do your best.
>
> Katherine Johnson, mathematician

3.2 The Order Axioms

We now introduce the Order Axioms of the real numbers.

Axioms 3.10 (Order Axioms)**.** For $a, b, c \in \mathbb{R}$, there is a relation $\boxed{<}$ on $\mathbb{R}$ satisfying:

(O1) (Trichotomy Law) If $a \neq b$, then either $a < b$ or $b < a$ but not both;

(O2) (Transitivity) If $a < b$ and $b < c$, then $a < c$;

(O3) If $a < b$, then $a + c < b + c$;

(O4) If $a < b$ and $0 < c$, then $ac < bc$;

Given Axioms O1–O4, we say that the real numbers are a **linearly ordered field**. We call numbers greater than zero **positive** and those greater than or equal to zero **nonnegative**. There are similar definitions for **negative** and **nonpositive**.

Notice that the Order Axioms are phrased in terms of "<". We would also like to be able to utilize ">", "≤", and "≥".

Definition 3.11. For $a, b \in \mathbb{R}$, we define:

(a) $\boxed{a > b}$ if $b < a$;

(b) $\boxed{a \leq b}$ if $a < b$ or $a = b$;

(c) $\boxed{a \geq b}$ if $b \leq a$.

Using the inequalities on the real numbers, we can now define the following special sets.

Definition 3.12. For $a, b \in \mathbb{R}$ with $a < b$, we define the following sets, referred to as **intervals**.

(a) $\boxed{(a, b) := \{x \in \mathbb{R} \mid a < x < b\}}$

(b) $\boxed{[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}}$

(c) $\boxed{[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}}$

(d) $\boxed{(a,\infty) := \{x \in \mathbb{R} \mid a < x\}}$

(e) $\boxed{(-\infty,b) := \{x \in \mathbb{R} \mid x < b\}}$

(f) $\boxed{(-\infty,\infty) := \mathbb{R}}$

We analogously define $\boxed{(a,b]}$, $\boxed{[a,\infty)}$, and $\boxed{(-\infty,b]}$. Intervals of the form (a,b), $(-\infty,b)$, (a,∞), and $(-\infty,\infty)$ are called **open intervals** while $[a,b]$ is referred to as a **closed interval**. A **bounded interval** is any interval of the form (a,b), $[a,b)$, $(a,b]$, and $[a,b]$. For bounded intervals, a and b are called the **endpoints** of the interval.

We will always assume that any time we write $(a,b),[a,b],(a,b]$, or $[a,b)$ that $a<b$. We will see where the terminology of "open" and "closed" comes from in Chapter 4. Context will help us determine whether (a,b) represents a bounded open interval or an ordered pair.

Using the Order Axioms, we can prove many familiar facts.

Problem 3.13. Prove that for all $a,b \in \mathbb{R}$, if $a,b > 0$, then $a+b > 0$; and if $a,b < 0$, then $a+b < 0$.

The next result extends Axiom O3.

Problem 3.14. Prove that for all $a,b,c,d \in \mathbb{R}$, if $a<b$ and $c<d$, then $a+c<b+d$.

Problem 3.15. Prove that for all $a \in \mathbb{R}$, $a>0$ if and only if $-a<0$.

Problem 3.16. Prove that if a, b, c, and d are positive real numbers such that $a<b$ and $c<d$, then $ac<bd$.

Problem 3.17. Prove that for all $a,b \in \mathbb{R}$, we have the following:

(a) $ab>0$ if and only if either $a,b>0$ or $a,b<0$;

(b) $ab<0$ if and only if $a<0<b$ or $b<0<a$.

Problem 3.18. Prove that for all positive real numbers a and b, $a<b$ if and only if $a^2<b^2$.

Consider using three cases when approaching the following proof.

Problem 3.19. Prove that for all $a \in \mathbb{R}$, we have $a^2 \geq 0$.

It might come as a surprise that the following result requires proof.

Problem 3.20. Prove that $0<1$.

The previous problem together with Problem 3.15 implies that $-1<0$ as you expect. It also follows from Axiom O3 that for all $n \in \mathbb{Z}$, we have $n<n+1$. We assume that there are no integers between n and $n+1$.

Problem 3.21. Prove that for all $a \in \mathbb{R}$, if $a>0$, then $a^{-1}>0$, and if $a<0$, then $a^{-1}<0$.

Problem 3.22. Prove that for all $a, b \in \mathbb{R}$, if $a < b$, then $-b < -a$.

Problem. Prove that for all $a, b \in \mathbb{R} \setminus \{0\}$, if $a < b$, then $b^{-1} < a^{-1}$.

The last few results allow us to take for granted our usual understanding of which real numbers are positive and which are negative. The next problem yields a result that extends Problem 3.22.

Problem 3.23. Prove that for all $a, b, c \in \mathbb{R}$, if $a < b$ and $c < 0$, then $bc < ac$.

> Making learning easy does not necessarily ease learning.
>
> Manu Kapur, learning scientist

3.3 Absolute Value and the Triangle Inequality

There is a special function that we can now introduce.

Definition 3.24. Given $a \in \mathbb{R}$, we define the **absolute value of** a, denoted $|a|$, via

$$\boxed{|a| := \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}}$$

Problem 3.25. Prove that for all $a \in \mathbb{R}$, $|a| \geq 0$ with equality only if $a = 0$.

We can interpret $|a|$ as the distance between a and 0 as depicted in Figure 3.1.

(a) $a > 0$ (b) $a < 0$

Figure 3.1: Visual representation of $|a|$.

Problem 3.26. Prove that for all $a, b \in \mathbb{R}$, we have $|a - b| = |b - a|$.

Given two points a and b, $|a - b|$, and hence $|b - a|$ by the previous problem, is the distance between a and b as shown in Figure 3.2.

Problem 3.27. Prove that for all $a, b \in \mathbb{R}$, $|ab| = |a||b|$.

In the next problem, writing $\pm a \leq b$ is an abbreviation for $a \leq b$ and $-a \leq b$.

Problem 3.28. Prove that for all $a, b \in \mathbb{R}$, if $\pm a \leq b$, then $|a| \leq b$.

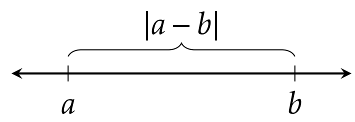

Figure 3.2: Visual representation of $|a-b|$.

Problem 3.29. Prove that for all $a \in \mathbb{R}$, $|a|^2 = a^2$.

Problem 3.30. Prove that for all $a \in \mathbb{R}$, $\pm a \leq |a|$.

Problem 3.31. Prove that for all $a, r \in \mathbb{R}$ with r nonnegative, $|a| \leq r$ if and only if $-r \leq a \leq r$.

The letter r was used in the previous problem because it is the first letter of the word "radius". If r is positive, we can think of the interval $(-r, r)$ as the interior of a one-dimensional circle with radius r centered at 0. Figure 3.3 provides a visual interpretation of Problem 3.31.

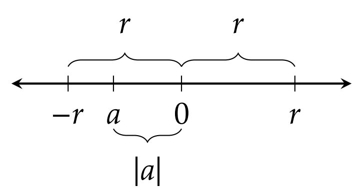

Figure 3.3: Visual representation of $|a| \leq r$.

Problem 3.32. Prove that for all $a, b, r \in \mathbb{R}$ with r nonnegative, $|a-b| \leq r$ if and only if $b - r \leq a \leq b + r$.

Since $|a-b|$ represents the distance between a and b, we can interpret $|a-b| \leq r$ as saying that the distance between a and b is less than or equal to r. In other words, a is within r units of b. See Figure 3.4.

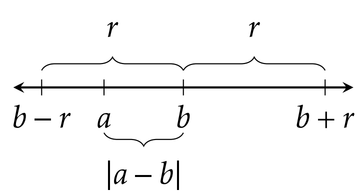

Figure 3.4: Visual representation of $|a-b| \leq r$.

Consider using Problems 3.30 and 3.31 when attacking the next result, which is known as the **Triangle Inequality**. This result can be extremely useful in some contexts.

Problem 3.33 (Triangle Inequality)**.** Prove that for all $a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$.

Figure 3.5 depicts two of the cases for the Triangle Inequality.

Problem 3.34. Under what conditions do we have equality for the Triangle Inequality?

(a) $a \geq 0, b \geq 0$ (b) $a < 0, b \geq 0$

Figure 3.5: Visual representation of two of the cases for the Triangle Inequality.

Where did the Triangle Inequality get its name? Why "Triangle"? For any triangle (including degenerate triangles), the sum of the lengths of any two sides must be greater than or equal to the length of the remaining side. That is, if x, y, and z are the lengths of the sides of the triangle, then $z \leq x + y$, where we have equality only in the degenerate case of a triangle with no area. In linear algebra, the Triangle Inequality is a theorem about lengths of vectors. If **a** and **b** are vectors in $\mathbb{R}^n$, then the Triangle Inequality states that $\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$. Note that $\|\mathbf{a}\|$ denotes the length of vector **a**. See Figure 3.6. The version of the Triangle Inequality that we presented in Problem 3.33 is precisely the one-dimensional version of the Triangle Inequality in terms of vectors.

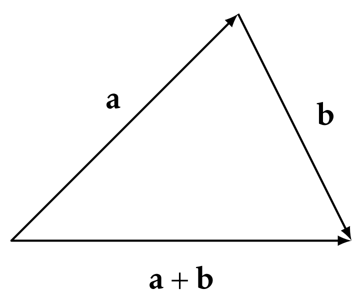

Figure 3.6: Triangle Inequality in terms of vectors.

The next result is sometimes called the **Reverse Triangle Inequality**.

Problem 3.35 (Reverse Triangle Inequality)**.** Prove that for all $a, b \in \mathbb{R}$, $|a-b| \geq ||a|-|b||$.

> I didn't want to just know names of things. I remember really wanting to know how it all worked.
>
> Elizabeth Blackburn, biologist

3.4 Suprema, Infima, and the Completeness Axiom

Before we introduce the Completeness Axiom, we need some additional terminology.

Definition 3.36. Let $A \subseteq \mathbb{R}$. A point b is called an **upper bound** of A if for all $a \in A$, $a \leq b$. The set A is said to be **bounded above** if it has an upper bound.

Problem 3.37. The notion of a **lower bound** and the property of a set being **bounded below** are defined similarly. Try defining them.

Problem 3.38. Find all upper bounds and all lower bounds for each of the following sets when they exist.

(a) $\{5, 11, 17, 42, 103\}$

(b) $\mathbb{N}$

(c) $\mathbb{Z}$

(d) $(0, 1]$

(e) $(0, 1] \cap \mathbb{Q}$

(f) $(0, \infty)$

(g) $\{42\}$

(h) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(i) $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

(j) $\emptyset$

Definition 3.39. A set $A \subseteq \mathbb{R}$ is **bounded** if A is bounded above and below.

Notice that a set $A \subseteq \mathbb{R}$ is bounded if and only if it is a subset of some bounded closed interval.

Definition 3.40. Let $A \subseteq \mathbb{R}$. A point p is a **supremum** (or **least upper bound**) of A if p is an upper bound of A and $p \leq b$ for every upper bound b of A. Analogously, a point p is an **infimum** (or **greatest lower bound**) of A if p is a lower bound of A and $p \geq b$ for every lower bound b of A.

Our next result tells us that a supremum of a set and an infimum of a set are unique when they exist.

Problem 3.41. Prove that if $A \subseteq \mathbb{R}$ such that a supremum (respectively, infimum) of A exists, then the supremum (respectively, infimum) of A is unique.

In light of the previous problem, if the supremum of A exists, it is denoted by $\boxed{\sup(A)}$. Similarly, if the infimum of A exists, it is denoted by $\boxed{\inf(A)}$.

Problem 3.42. Find the supremum and the infimum of each of the sets in Problem 3.38 when they exist.

It is important to recognize that the supremum or infimum of a set may or may not be contained in the set. In particular, we have the following theorem concerning suprema and maximums. The analogous result holds for infima and minimums.

Problem 3.43. Let $A \subseteq \mathbb{R}$. Prove that A has a maximum if and only if A has a supremum and $\sup(A) \in A$, in which case the $\max(A) = \sup(A)$.

Intuitively, a point is the supremum of a set A if and only if no point smaller than the supremum can be an upper bound of A. The next result makes this more precise.

Problem 3.44. Let $A \subseteq \mathbb{R}$ such that A is bounded above and let b be an upper bound of A. Prove that b is the supremum of A if and only if for every $\varepsilon > 0$, there exists $a \in A$ such that $b - \varepsilon < a$.

Problem 3.45. State and prove the analogous result to Problem 3.44 involving infimum.

The following axiom states that every nonempty subset of the real numbers that has an upper bound has a least upper bound.

Axiom 3.46 (Completeness Axiom)**.** If A is a nonempty subset of $\mathbb{R}$ that is bounded above, then $\sup(A)$ exists.

Given the Completeness Axiom, we say that the real numbers satisfy the **least upper bound property**. It is worth mentioning that we do not need the Completeness Axiom to conclude that every nonempty subset of the integers that is bounded above has a supremum, as this follows from a generalized version of the Well-Ordering Principle (see Problem 2.25).

Certainly, the real numbers also satisfy the analogous result involving infimum.

Problem 3.47. Prove that if A is a nonempty subset of $\mathbb{R}$ that is bounded below, then $\inf(A)$ exists.

Problem 3.48. Prove that the Completeness Axiom is *not* true if one requires that the supremum be a rational number. This shows that the rationals do not satisfy the Completeness Axiom.

Problem 3.49. If A and B are each bounded above, characterize the supremum of each of the following sets.

(a) $A \cup B$

(b) $A \cap B$

What are the analogous results involving infimum?

If A and B are sets, define $\boxed{A + B := \{a + b \mid a \in A, b \in B\}}$.

Problem 3.50. Prove each of the following.

(a) If A and B are each bounded above, then $\sup(A + B) = \sup(A) + \sup(B)$.

(b) If A and B are each bounded below, then $\inf(A+B) = \inf(A) + \inf(B)$.

For a set A and $c \in \mathbb{R}$, define $\boxed{cA := \{ca \mid a \in A\}}$.

Problem 3.51. Let A be a set and $c \in \mathbb{R}$. Prove each of the following.

(a) If $c > 0$ and A is bounded above, then $\sup(cA) = c\sup(A)$.

(b) If $c < 0$ and A is bounded below, then $c\inf(A) = \sup(cA)$.

What other properties are there relating inf, sup, and c?

> Time spent thinking about a problem is always time well spent. Even if you seem to make no progress at all.
>
> Paul Zeitz, mathematician

3.5 The Archimedean Property

Our next result, called the **Archimedean Property**, tells us that for every real number, we can always find a natural number that is larger. To prove this theorem, consider a proof by contradiction and then utilize the Completeness Axiom and Problem 3.44.

Problem 3.52 (Archimedean Property)**.** Prove that for every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$.

More generally, we can "squeeze" every real number between a pair of integers. The next result is sometimes referred to at the **Generalized Archimedean Property**.

Problem 3.53 (Generalized Archimedean Property)**.** Prove that for every $x \in \mathbb{R}$, there exists $k, n \in \mathbb{Z}$ such that $k < x < n$.

Problem 3.54. Prove that for any positive real number x, there exists $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < x$.

The next problem strengthens the Generalized Archimedean Property and says that every real number is either an integer or lies between a pair of consecutive integers. To tackle the next problem, let $x \in \mathbb{R}$ and define $L = \{k \in \mathbb{Z} \mid k \leq x\}$. Use the Generalized Archimedean Property to conclude that L is nonempty and then utilize Problem 2.25.

Problem 3.55. Prove that for every $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}$ such that $n \leq x < n+1$.

Recall that the set rational numbers is defined via

$$\mathbb{Q} := \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}.$$

For the next proof, let $a < b$, utilize Problem 3.54 on $b - a$ to obtain $N \in \mathbb{N}$ such that $\frac{1}{N} < b - a$, and then apply Problem 3.55 to Na to conclude that there exists $n \in \mathbb{N}$ such that $n \leq Na < n+1$. Lastly, argue that $\frac{n+1}{N}$ is the rational number you seek.

Problem 3.56. Prove that if (a, b) is an open interval, then there exists a rational number p such that $p \in (a, b)$.

The next problem tells us that the real numbers are **Hausdorff**.

Problem 3.57. Prove that if $a < b$, then there exists disjoint open intervals I and J such that $a \in I$ and $b \in J$.

Recall that the set of irrational numbers is given by $\mathbb{R} \setminus \mathbb{Q}$. It follows that the real numbers consist of rational and irrational numbers. However, it is not obvious that irrational numbers even exist. It is not too hard to prove that $\sqrt{2}$ is an irrational number. We will take this fact for granted. It turns out that $\sqrt{2} \approx 1.41421356237 \in (1, 2)$. This provides an example of an irrational number occurring between a pair of distinct rational numbers. The following problem is a good challenge to generalize this.

Problem 3.58. Prove that if (a, b) is an open interval, then there exists an irrational number p such that $p \in (a, b)$.

Repeated applications of Problems 3.56 and 3.58 implies that every open interval contains infinitely many rational numbers and infinitely many irrational numbers. In light of these two problems, we say that both the rationals and irrationals are **dense** in the real numbers. Of course, another consequence of these two problems is that between any pair of reals, there is a real number.

> If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.
>
> John von Neumann, mathematician

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

G.H. Hardy, mathematician

Chapter 4

Standard Topology of the Real Line

In this chapter, we will introduce the notions of open, closed, compact, and connected as they pertain to subsets of the real numbers. These properties form the underpinnings of a branch of mathematics called **topology** (derived from the Greek words *tópos*, meaning 'place, location', and *ology*, meaning 'study of'). Topology, sometimes called "rubber sheet geometry," is concerned with properties of spaces that are invariant under any continuous deformation (e.g., bending, twisting, and stretching like rubber while not allowing tearing apart or gluing together). The fundamental concepts in topology are continuity, compactness, and connectedness, which rely on ideas such as "arbitrary close" and "far apart". These ideas can be made precise using open sets.

Once considered an abstract branch of pure mathematics, topology now has applications in biology, computer science, physics, and robotics. The goal of this chapter is to introduce you to the basics of the set-theoretic definitions used in topology and to provide you with an opportunity to tinker with open and closed subsets of the real numbers. In Chapter 6, we will revisit these concepts when we explore continuous functions.

4.1 Open Sets

Definition 4.1. A set U is called an **open set** if for every $x \in U$, there exists a bounded open interval (a,b) containing x such that $(a,b) \subseteq U$.

It follows immediately from the definition that every open set is a union of bounded open intervals.

Problem 4.2. Determine whether each of the following sets is open. Justify your assertions.

(a) $(1,2)$

(b) $(1,\infty)$

(c) $(1,2) \cup (\pi,5)$

(d) $[1,2]$

(e) $(-\infty,\sqrt{2}]$

(f) $\{4,17,42\}$

(g) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(h) $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

(i) $\mathbb{R}$

(j) $\mathbb{Q}$

(k) $\mathbb{Z}$

(l) $\emptyset$

As expected, every open interval (i.e., intervals of the form (a,b), $(-\infty,b)$, (a,∞), or $(-\infty,\infty)$) is an open set.

Problem 4.3. Prove that every open interval is an open set.

However, it is important to point out that open sets can be more complicated than a single open interval.

Problem 4.4. Provide an example of an open set that is not a single open interval.

Problem 4.5. Prove that if U and V are open sets, then

(a) $U \cup V$ is an open set, and

(b) $U \cap V$ is an open set.

According to the next two problems, the union of arbitrarily many open sets is open while the intersection of a finite number of open sets is open.

Problem 4.6. Prove that if $\{U_\alpha\}_{\alpha\in\Delta}$ is a collection of open sets, then $\bigcup_{\alpha\in\Delta} U_\alpha$ is an open set.

Consider using induction on the next problem.

Problem 4.7. Prove that if $\{U_i\}_{i=1}^n$ is a finite collection of open sets for $n \in \mathbb{N}$, then $\bigcap_{i=1}^n U_i$ is an open set.

Problem 4.8. Explain why we cannot utilize induction to prove that the intersection of infinitely many open sets indexed by the natural numbers is open.

Problem 4.9. Give an example of each of the following.

(a) A collection of open sets $\{U_\alpha\}_{\alpha\in\Delta}$ such that $\bigcap_{\alpha\in\Delta} U_\alpha$ is an open set.

(b) A collection of open sets $\{U_\alpha\}_{\alpha\in\Delta}$ such that $\bigcap_{\alpha\in\Delta} U_\alpha$ is not an open set.

According to the previous problem, the intersection of infinitely many open sets may or may not be open. So, we know that there is no theorem that states that the intersection of arbitrarily many open sets is open. We only know for certain that the intersection of finitely many open sets is open by Problem 4.7.

> Any creative endeavor is built on the ash heap of failure.
>
> Michael Starbird, mathematician

4.2 Accumulation Points and Closed Sets

Definition 4.10. Suppose $A \subseteq \mathbb{R}$. A point $p \in \mathbb{R}$ is an **accumulation point** of A if for every bounded open interval (a,b) containing p, there exists a point $q \in (a,b) \cap A$ such that $q \neq p$.

Notice that if p is an accumulation point of A, then p may or may not be in A. Loosely speaking, p is an accumulation point of a set A if there are points in A arbitrarily close to p. That is, if we zoom in on p, we should always see points in A nearby. If A is a set, the set of accumulation points of A is sometimes denoted by $\boxed{A'}$.

Problem 4.11. Consider the open interval $I = (1,2)$. Prove each of the following.

(a) The points 1 and 2 are accumulation points of I.

(b) If $p \in I$, then p is an accumulation point of I.

(c) If $p < 1$ or $p > 2$, then p is not an accumulation point of I.

Problem 4.12. A point p is an accumulation point of the intervals (a,b), $(a,b]$, $[a,b)$, and $[a,b]$ if and only if $p \in [a,b]$.

Problem 4.13. Prove that the point $p = 0$ is an accumulation point of $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Are there any other accumulation points of A?

Problem 4.14. For each set A, find the set of accumulation points A'. In each case, sketch a proof to justify your assertion.

(a) $A = \{-17, 1, \pi, 42\}$

(b) $A = [1,3) \cup (3,5)$

(c) $A = [1,3) \cup \{17\}$

(d) $A = \mathbb{R}$

(e) $A = \mathbb{Q}$

(f) $A = \mathbb{Z}$

(g) $M = \displaystyle\bigcup_{n=1}^{\infty}(-1-\frac{1}{n}, 1+\frac{1}{n})$

(h) $M = \displaystyle\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$

Problem 4.15. Prove that if A is finite, then A has no accumulation points. That is, if A is finite, then $A' = \emptyset$.

Problem 4.16. State the contrapositive of the previous problem.

The upshot of the previous problem is that any set with an accumulation point must be infinite.

Problem 4.17. Determine whether the converse of Problem 4.16 is true or false. Justify your assertion.

Problem 4.18. Provide an example of a set A with exactly two accumulation points.

Problem 4.19. Given sets A and B, determine whether each of the following is true or false. If the statement is true, prove it. Otherwise, provide a counterexample.

(a) If p is an accumulation point of $A \cap B$, then p is an accumulation point of both A and B.

(b) If p is an accumulation point of $A \cup B$, then p is an accumulation point of A or p is an accumulation point of B.

Definition 4.20. A set $A \subseteq \mathbb{R}$ is called **closed** if A contains all of its accumulation points.

That is, a set A is closed if and only if $A' \subseteq A$. Note that if a set A has no accumulation points, then it is vacuously closed.

Problem 4.21. Determine whether each of the sets in Problem 4.2 is closed. Justify your assertions.

The upshot of Parts (i) and (l) of Problems 4.2 and 4.21 is that $\mathbb{R}$ and $\emptyset$ are both open and closed. It turns out that these are the only two subsets of the real numbers with this property. One issue with the terminology that could potentially create confusion is that the open interval $(-\infty, \infty)$ (i.e., the real numbers $\mathbb{R}$) is both an open and a closed set.

Problem 4.22. Provide an example of each of the following. You do not need to prove that your answers are correct.

(a) A set that is open but not closed.

(b) A set that is closed but not open.

(c) A set that is neither open nor closed.

Another potentially annoying feature of the terminology illustrated by Problem 4.22 is that if a set is not open, it may or may not be closed. Similarly, if a set is not closed, it may or may not be open. That is, open and closed are not opposites of each other.

The next result justifies referring to $[a, b]$ as a closed interval.

Problem 4.23. Prove that every interval of the form $[a, b]$, $(-\infty, b]$, $[a, \infty)$, or $(-\infty, \infty)$ is a closed set.

Problem 4.24. Prove that if A is finite, then A is a closed set.

Problem 4.25. Prove that $\mathbb{Z}$ is a closed set.

Despite the fact that open and closed are not opposites of each other, there is a nice relationship between open and closed sets in terms of complements.

Problem 4.26. Let $U \subseteq \mathbb{R}$. Prove that U is open if and only if U^C is closed.

Problem 4.27. Prove that if A and B are closed sets, then

(a) $A \cup B$ is a closed set, and

(b) $A \cap B$ is a closed set.

The next two problems are analogous to Problems 4.6 and 4.7.

Problem 4.28. Prove that if $\{A_\alpha\}_{\alpha\in\Delta}$ is a collection of closed sets, then $\bigcap_{\alpha\in\Delta} A_\alpha$ is a closed set.

Problem 4.29. Prove that if $\{A_i\}_{i=1}^n$ is a finite collection of closed sets for $n \in \mathbb{N}$, then $\bigcup_{i=1}^n A_i$ is a closed set.

Problem 4.30. Provide an example of a collection of closed sets $\{A_\alpha\}_{\alpha\in\Delta}$ such that $\bigcup_{\alpha\in\Delta} A_\alpha$ is not a closed set.

Problem 4.31. Determine whether each of the following sets is open, closed, both, or neither.

(a) $V = \displaystyle\bigcup_{n=2}^{\infty}\left(n-\frac{1}{2}, n\right)$

(b) $W = \displaystyle\bigcap_{n=2}^{\infty}\left(n-\frac{1}{2}, n\right)$

(c) $X = \displaystyle\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$

(d) $Y = \displaystyle\bigcap_{n=1}^{\infty}(-n, n)$

(e) $Z = (0,1) \cap \mathbb{Q}$

Problem 4.32. Prove or provide a counterexample: Every non-closed set has at least one accumulation point.

> Getting better is not pretty. To get good we have to be down to struggle, seek out challenges, make some mistakes, to train ugly.
>
> Trevor Ragan, thelearnerlab.com

4.3 Compact and Connected Sets

We now introduce three special classes of subsets of $\mathbb{R}$: compact, connected, and disconnected.

Definition 4.33. A set $K \subseteq \mathbb{R}$ is called **compact** if K is both closed and bounded.

It is important to point out that there is a more general definition of compact in an arbitrary topological space. However, using our notions of open and closed, it is a theorem that a subset of the real line is compact if and only if it is closed and bounded.

Problem 4.34. Determine whether each of the following sets is compact. Briefly justify your assertions.

(a) $[0,1) \cup [2,3]$

(b) $[0,1) \cup (1,2]$

(c) $[0,1) \cup [1,2]$

(d) $\mathbb{R}$

(e) $\mathbb{Q}$

(f) $\mathbb{R} \setminus \mathbb{Q}$

(g) $\mathbb{Z}$

(h) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(i) $[0,1] \cup \{1 + \frac{1}{n} \mid n \in \mathbb{N}\}$

(j) $\{17, 42\}$

(k) $\{17\}$

(l) $\emptyset$

Problem 4.35. Is every finite set compact? Justify your assertion.

The next problem says that every nonempty compact set contains its greatest lower bound and its least upper bound. That is, every nonempty compact set attains a minimum and a maximum value.

Problem 4.36. Prove that if K is a nonempty compact subset of $\mathbb{R}$, then $\sup(K), \inf(K) \in K$.

Definition 4.37. Let A be a subset of real numbers and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Delta}$ be a collection of open subsets of $\mathbb{R}$. Then $\mathcal{U}$ is an **open cover** of A if $A \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. An **open subcover** $\mathcal{V}$ of an open cover $\mathcal{U}$ of A is a subcollection of $\mathcal{U}$ whose elements form an open cover of A. If $\mathcal{V}$ is an open subcover of $\mathcal{U}$ consisting of a finite number of open sets, then we say that $\mathcal{V}$ is a **finite open subcover**.

For example, the collections $\mathcal{U}_1 = \{(-n, n) \mid n \in \mathbb{N}\}$ and $\mathcal{U}_2 = \{(n, n+2) \mid n \in \mathbb{Z}\}$ are each open subcovers of the real numbers. The collection $\mathcal{V}_1 = \{(-n, n) \mid n \in \mathbb{N} \text{ and } n \geq 17\}$ is an example of an open subcover of $\mathcal{U}_1$. Does $\mathcal{U}_1$ have an example of a finite open subcover? Does $\mathcal{U}_2$ have an example of a proper open subcover (i.e., not the entire collection)?

Problem 4.38. Let $A = [0,1]$, $\mathcal{U}_1 = \{(-\frac{n}{2}, \frac{n}{2})\}$, and $\mathcal{U}_2 = \{(-\frac{1}{n}, \frac{1}{n})\} \cup \{(-\frac{1}{n} + 1, \frac{1}{n} + 1)\}$. It turns out that both $\mathcal{U}_1$ and $\mathcal{U}_2$ are open covers of A.

(a) If possible, find a finite open subcover of $\mathcal{U}_1$. If this is not possible, explain why.

(b) If possible, find a finite open subcover of $\mathcal{U}_2$. If this is not possible, explain why.

Problem 4.39. Let $A = (0,1)$, $\mathcal{V}_1 = \{(-\frac{n}{2}, \frac{n}{2})\}$, and $\mathcal{V}_2 = \{(\frac{1}{n}, 1)\}$. It turns out that both $\mathcal{V}_1$ and $\mathcal{V}_2$ are open covers of A.

(a) If possible, find a finite open subcover of $\mathcal{V}_1$. If this is not possible, explain why.

(b) If possible, find a finite open subcover of $\mathcal{V}_2$. If this is not possible, explain why.

Problem 4.40. Prove that a set A is compact if and only if every open cover $\mathcal{U}$ of A has a finite open subcover.

The characterization of compact in given in the previous problem is actually the standard definition of compact used in topology. If we had used this as our definition, we could have proved that a set is compact if and only if it is closed and bounded. This theorem is known as the **Heine–Borel Theorem**. Instead we took this as our definition to simplify things slightly.

Definition 4.41. A set $A \subseteq \mathbb{R}$ is **disconnected** if there exists two disjoint open sets U_1 and U_2 such that $A \cap U_1$ and $A \cap U_2$ are nonempty but $A \subseteq U_1 \cup U_2$ (equivalently, $A = (A \cap U_1) \cup (A \cap U_2)$). If a set is not disconnected, then we say that it is **connected**.

In other words, a set is disconnected if it can be partitioned into two nonempty subsets such that each subset does not contain points of the other and does not contain any accumulation points of the other. Showing that a set is disconnected is generally easier than showing a set is connected. To prove that a set is disconnected, you simply need to exhibit two open sets with the necessary properties. However, to prove that a set is connected, you need to prove that no such pair of open sets exists.

Problem 4.42. Determine whether each of the sets in Problem 4.34 is is connected or disconnected. Briefly justify your assertions.

Problem 4.43. Prove that if $a \in \mathbb{R}$, then $\{a\}$ is connected.

The next proof is harder than you might expect. Consider a proof by contradiction and try to make use of the Completeness Axiom.

Problem 4.44. Prove that every closed interval $[a,b]$ is connected.

It turns out that every connected set in $\mathbb{R}$ is either a singleton or an interval. We have not officially proved this claim, but we do have the tools to do so. Feel free to try your hand at proving this fact.

> If you learn how to learn, it's the ultimate meta skill and I believe you can learn how to be healthy, you can learn how to be fit, you can learn how to be happy, you can learn how to have good relationships, you can learn how to be successful. These are all things that can be learned. So if you can learn that is a trump card, it's an ace, it's a joker, it's a wild card. You can trade it for any other skill.
>
> Naval Ravikant, entrepreneur & investor

> Pass on what you have learned. Strength, mastery. But weakness, folly, failure also. Yes, failure most of all. The greatest teacher, failure is.
>
> Yoda, Jedi master

Chapter 5

Sequences

We will now begin connecting the concepts of sets to more familiar ones from calculus, beginning with sequences.

5.1 Introduction to Sequences

Definition 5.1. A **sequence** (of real numbers) is a function p from $\mathbb{N}$ to $\mathbb{R}$.

If $n \in \mathbb{N}$, it is common to write $p_n := p(n)$. We refer to p_n as the nth **term** of the sequence. We will abuse notation and associate a sequence with its list of outputs, namely:

$$\boxed{(p_n)_{n=1}^{\infty} := (p_1, p_2, \ldots)},$$

which we may abbreviate as $\boxed{(p_n)}$.

Example 5.2. Define $p : \mathbb{N} \to \mathbb{R}$ via $p(n) = \frac{1}{2^n}$. Then we have

$$p = \left(\frac{1}{2}, \frac{1}{4}, \ldots\right) = \left(\frac{1}{2^n}\right)_{n=1}^{\infty}.$$

It is important to point out that not every sequence has a description in terms of an algebraic formula. For example, we could form a sequence out of the digits to the right of the decimal in the decimal expansion of π, namely the nth term of the sequence is the nth digit to the right of the decimal. But then there is no nice algebraic formula for describing the nth term of this sequence.

Problem 5.3. Write down several sequences $(p_n)_{n=1}^{\infty}$ you are familiar with. If possible, give an algebraic formula for each p_n in terms of n.

Note that the **image** (or **range**) of a sequence $(p_n)_{n=1}^{\infty}$ is the set $\{p_n\}_{n=1}^{\infty}$.

Problem 5.4. Explain the difference between $\{p_n\}_{n=1}^{\infty}$ and $(p_n)_{n=1}^{\infty}$? Give an example of a sequence whose image set is finite.

There is a deep connection between sequences and accumulation points, which the next few problems will elucidate. First, a definition—one you may have seen in calculus in a different form. When digesting the following definition, try to think about how this definition is capturing the notion that the sequence is getting "closer and closer" to the point that the sequence converges to.

Definition 5.5. We say that the sequence $(p_n)_{n=1}^{\infty}$ **converges to the point** x if for every bounded open interval (a,b) containing x, there exists an $N \in \mathbb{N}$ such that for all natural numbers $n \geq N$, $p_n \in (a,b)$.

In the definition above, we sometimes refer to N as the **cutoff point** for the sequence relative to (a,b) and to all p_n with $n \geq N$ as the **tail of the sequence**. It is important to emphasize that the cutoff point and the tail of the sequence depend on the interval (a,b). Informally, we write $\boxed{(p_n) \to x}$ to mean that the sequence $(p_n)_{n=1}^{\infty}$ converges to the point x. We simply say that $(p_n)_{n=1}^{\infty}$ converges if there exists a point x to which the sequence converges. If a sequence does not converge to *any* point x, then we say it **diverges**.

The first problem about this should be used as a place to test ideas for how to prove convergence. As you tackle the next few problems, it might be useful to begin by writing down the first several terms of the sequences.

Problem 5.6. Consider the sequence given by $p_n = \frac{1}{n}$. Prove that $(p_n)_{n=1}^{\infty}$ converges to 0.

Problem 5.7. Consider the sequence given by $p_n = 1 - \frac{1}{n}$. Prove that $(p_n)_{n=1}^{\infty}$ converges to 1.

Problem 5.8. Consider the sequence with even terms $p_{2i} = \frac{1}{2i-1}$ and odd terms $p_{2i-1} = \frac{1}{2i}$. Prove that $(p_n)_{n=1}^{\infty}$ converges to 0.

Problem 5.9. Consider the sequence with odd terms $p_{2i-1} = \frac{1}{2i-1}$ and even terms $p_{2i} = 1 + \frac{1}{2i}$. Determine whether $(p_n)_{n=1}^{\infty}$ converges to 0.

The following problem connects accumulation points and sequences.

Problem 5.10. Prove that if $(p_n)_{n=1}^{\infty}$ converges to the point x *and* for each $n \in \mathbb{N}$, $p_n \neq p_{n+1}$, then x is an accumulation point of the image set $\{p_n\}_{n=1}^{\infty}$ of the sequence. Can we weaken the requirement that $p_n \neq p_{n+1}$ for all $n \in \mathbb{N}$ and still conclude that x is an accumulation point of the image set?

Problem 5.11. Prove that if $(p_n)_{n=1}^{\infty}$ converges to the point x and y is a point different from x, then $(p_n)_{n=1}^{\infty}$ does *not* converge to y.

Problem 5.12. Let $(p_n)_{n=1}^{\infty}$ be a sequence. Prove that if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|p_n - x| < \varepsilon$ whenever $n \geq N$, then $(p_n)_{n=1}^{\infty}$ converges to x.

5.2 Properties of Convergent Sequences

We now explore some basic facts concerning the convergence of sequences.

Problem 5.13. Prove that if c is a real number and $(p_n)_{n=1}^{\infty}$ converges to x, then the sequence $(cp_n)_{n=1}^{\infty}$ converges to cx.

Problem 5.14. Prove that if $(p_n)_{n=1}^{\infty}$ converges to x and $(q_n)_{n=1}^{\infty}$ converges to y, then the sequence $(p_n + q_n)_{n=1}^{\infty}$ converges to $x + y$.

Products and quotients of sequences behave like you think they will, as well. We will include one special case soon.

Problem 5.15. If the sequence $(p_n)_{n=1}^{\infty}$ converges to the point x, then the image set $\{p_n\}_{n=1}^{\infty}$ is bounded.

Problem 5.16. Find an example of a sequence $(p_n)_{n=1}^{\infty}$ such that its image set $\{p_n\}_{n=1}^{\infty}$ is unbounded and hence does not have a supremum.

Consider using Problem 5.15 when approaching the next problem.

Problem 5.17. Prove that if $(p_n)_{n=1}^{\infty}$ converges to x and $(q_n)_{n=1}^{\infty}$ converges to 0, then $(p_n \cdot q_n)_{n=1}^{\infty}$ converges to 0.

5.3 Monotone Convergence Theorem

Definition 5.18. We say that a sequence $(p_n)_{n=1}^{\infty}$ is **nondecreasing** if $p_n \leq p_{n+1}$ for all $n \in \mathbb{N}$. The concept of **nonincreasing** is defined similarly. A function that is either nondecreasing or nonincreasing is said to be **monotone**.

Problem 5.19. Replace $\leq$ above with $<$ to define the notion of (strictly) **increasing**. Find examples of nondecreasing sequences that are not increasing. Similarly, define (strictly) **decreasing**.

Problem 5.20 (Monotone Convergence Theorem)**.** Prove that if $(p_n)_{n=1}^{\infty}$ is a nondecreasing sequence such that the image set $\{p_n\}_{n=1}^{\infty}$ is bounded above, then $(p_n)_{n=1}^{\infty}$ converges to some point x.

It turns out that the previous result is equivalent to the Completeness Axiom. The next problem asks you to verify this, but this is not a result that we need going forward, but rather is an interesting side story.

Problem 5.21. Assuming the result of Problem 5.20, prove the Completeness Axiom.

Problem 5.22. Let A be a nonempty set that is bounded above. Prove that there exists a nondecreasing sequence $(p_n)_{n=1}^{\infty}$ that converges to $\sup(A)$, where the image set $\{p_n\} \subseteq A$.

5.4 Subsequences and the Bolzano–Weierstrass Theorem

Definition 5.23. A sequence $(b_k)_{k=1}^{\infty}$ is a **subsequence** of $(a_n)_{n=1}^{\infty}$ if there is a sequence of natural numbers $(n_k)_{k=1}^{\infty}$ with $n_k < n_{k+1}$ such that $b_k = a_{n_k}$.

Problem 5.24. Give some examples of subsequences of the sequence from Problem 5.6.

Problem 5.25. Prove that if a sequence converges to x, so does any subsequence of that sequence.

Problem 5.26. Suppose $(p_{n_k})_{k=1}^{\infty}$ is a subsequence of $(p_n)_{n=1}^{\infty}$. If p_{n_k} converges to x, does this imply that p_n converges to x? Justify your answer.

Problem 5.27. Provide an example of a sequence $(p_n)_{n=1}^{\infty}$ with image set $\{p_n\} \subseteq \mathbb{N}$ such that *every* sequence of natural numbers is a subsequence of (p_n).

Problem 5.28. Prove that every sequence of real numbers has a nonincreasing or nondecreasing subsequence.

Problem 5.29 (Bolzano–Weierstrass Theorem)**.** Prove that every sequence with bounded image set has a convergent subsequence.

The next problem is related to the Bolzano–Weierstrass Theorem.

Problem 5.30. Prove that if K is a nonempty compact subset of $\mathbb{R}$, then any sequence with image set in K has a subsequence that converges to a point in K.

Problem 5.31. Come up with examples showing that if A is not closed or not bounded, then there exists a sequence with image set in A that does not have a subsequence converging to a point in A (or possibly not at all).

On the real line, compactness and satisfying the Bolzano–Weierstrass Theorem are equivalent. However, one can concoct examples of other mathematical spaces where they are not the same.

> It does not matter how slowly you go as long as you do not stop.
>
> Confucius, philosopher

The impediment to action advances action.
What stands in the way becomes the way.

Marcus Aurelius, Roman emperor

Chapter 6

Continuity

In this chapter, we will explore the concept of continuity, which you likely encountered in high school.

6.1 Introduction to Continuity

We begin with the definition of a specific type of function, namely one whose domain and range is a subset of the real numbers.

Definition 6.1. A **real function** is any function $f : A \to \mathbb{R}$ such that A is a nonempty subset of $\mathbb{R}$.

There are many equivalent definitions of continuity for real functions. We will take the following definition as our starting point and then develop several equivalent characterizations of continuity.

Definition 6.2. Suppose f is a real function such that $a \in \mathrm{Dom}(f)$. We say that f is **continuous at** a if for every bounded open interval I containing $f(a)$, there is a bounded open interval J containing a such that if $x \in \mathrm{Dom}(f) \cap J$, then $f(x) \in I$. If f is continuous at every point in $B \subseteq \mathrm{Dom}(f)$, then we say that f is **continuous on** B. If f is continuous on the entire domain, we simply say that f is **continuous**.

Loosely speaking, a real function f is continuous at the point $a \in \mathrm{Dom}(f)$ if we can get $f(x)$ arbitrarily close to $f(a)$ by considering all $x \in \mathrm{Dom}(f)$ sufficiently close to a. The interval I is indicating how close to $f(a)$ we need to be while the interval J is providing the "window" around a needed to guarantee that all points in the window (and in the domain) yield outputs in I. Figure 6.1 illustrates our definition of continuity. Note that in the figure, the point a is fixed while we need to consider all $x \in \mathrm{Dom}(f) \cap J$. The dashed box in the figure has dimensions the length of J by the length of I. Intuitively, the function is continuous at a since given I, we can find J so that the graph of the function never exits the top or bottom of the dashed box.

Perhaps you have encountered the phrase "a function is continuous if you can draw its graph without lifting your pencil." While this description provides some intuition about what continuity of a function means, it is neither accurate nor precise enough to capture the meaning of continuity.

Let's show that our definition of continuity behaves the way we expect.

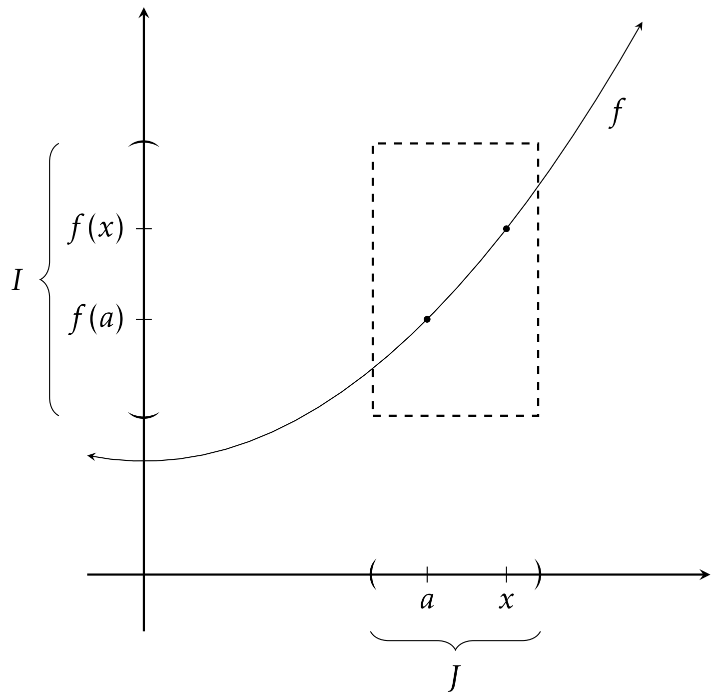

Figure 6.1: Visual representation of continuity of f at a.

Problem 6.3. Prove that each of the following real functions is continuous using Definition 6.2.

(a) $f:\mathbb{R}\to\mathbb{R}$ defined via $f(x)=x$.

(b) $g:\mathbb{R}\to\mathbb{R}$ defined via $g(x)=2x$.

(c) $h:\mathbb{R}\to\mathbb{R}$ defined via $h(x)=x+3$.

Problem 6.4. Prove that every linear function $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=mx+b$ is continuous.

After completing the next problem, reflect on the statement "a function is continuous if you can draw its graph without lifting your pencil."

Problem 6.5. Define $f:\mathbb{N}\to\mathbb{R}$ via $f(x)=1$. Notice the domain! Determine where f is continuous and justify your assertion.

The next problem illustrates that the order in which the sets I and J are considered in the definition of continuity is crucial.

Problem 6.6. Find an example of a real function f that satisfies each of the following:

(i) $1 \in \mathrm{Dom}(f)$;

(ii) For any open interval J containing 1, there is an open interval I containing $f(1)$ such that if $x \in \mathrm{Dom}(f) \cap J$, then $f(x) \in I$;

(iii) f is not continuous at 1.

The obstacle is the path.

Zen saying, Author Unknown

6.2 Additional Characterizations of Continuity

The next problem tells us that we can reframe continuity in terms of distance.

Problem 6.7. Suppose f is a real function such that $a \in \mathrm{Dom}(f)$. Prove that f is continuous at a if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in \mathrm{Dom}(f)$ and $|x-a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

The previous characterization is typically referred to as the "$\varepsilon - \delta$ definition of continuity", although for us it is a theorem instead of a definition. This characterization is used as the definition of continuity in metric spaces.

Problem 6.8. Draw a figure in the spirit of Figure 6.1 that captures the essence of Problem 6.7

When approaching the next three problems, either utilize Definition 6.2 or Problem 6.7.

Problem 6.9. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Determine where f is continuous and justify your assertion.

Problem 6.10. Define $g : \{0\} \to \mathbb{R}$ via $g(0) = 0$. Prove that g is continuous at 0.

Problem 6.11. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Determine where f is continuous and justify your assertion.

Problem 6.12. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$. Prove that f is continuous.

Problem 6.13. Find a continuous real function f and an open interval I such that the preimage $f^{-1}(I)$ is not an open interval.

Problem 6.14. Suppose f is a real function. Prove that f is continuous if and only if the preimage $f^{-1}(U)$ of every open set U is an open set intersected with the domain of f.

The previous characterization of continuity is often referred to as the "open set definition of continuity" and is the definition used in topology. Since every open set is the union of bounded open intervals (Definition 4.1), the union of open sets is open (Problem 4.6), and preimages respect unions (Problem 2.55), we can strengthen Problem 6.14 into a slightly more useful result.

Problem 6.15. Suppose f is a real function. Prove that f is continuous if and only if the preimage $f^{-1}(I)$ of every bounded open interval I is an open set intersected with the domain of f.

Problem 6.16. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ via $f(x) = \frac{1}{x}$. Determine where f is continuous and justify your assertion.

The previous problem once again calls into question the phrase "a function is continuous if you can draw its graph without lifting your pencil."

It turns out that there is a deep connection between continuity and sequences!

Definition 6.17. Suppose f is a real function such that $a \in \mathrm{Dom}(f)$. We say that f is **sequentially continuous** at a if, for every sequence $(p_n)_{n=1}^{\infty}$ in the domain of f converging to a, the sequence $(f(p_n))_{n=1}^{\infty}$ converges to $f(a)$.

Problem 6.18. Suppose f is a real function such that $a \in \mathrm{Dom}(f)$. Prove that f is continuous at a if and only if f is sequentially continuous at a.

The upshot of the previous problem is that the notions of being *continuous at a point* and *sequentially continuous at a point* are equivalent on the real numbers. However, there are contexts in mathematics where the two are not equivalent. This is a topic in a branch of mathematics called **topology**. If you want to know more, check out the following YouTube video:

https://www.youtube.com/watch?v=sZ5fBHGYurg

The sequential way of thinking of continuity often makes proving some basic facts concerning continuity easier.

At this point, we have four different ways of thinking about continuity.

- Definition 6.2 using open intervals.
- Problem 6.7 using ε and δ.
- Problem 6.14/Problem 6.15 using preimages of open sets.
- Problem 6.18 using sequential continuity.

You should take the time to review each one. For the remainder of the book, feel free to use whichever characterization suits your needs.

Problem 6.19. Suppose f and g are real functions that are continuous at a and let $c \in \mathbb{R}$. Prove that each of the following functions is also continuous at a.

(a) cf

(b) $f+g$

(c) $f-g$

(d) fg

Problem 6.20. Prove that every polynomial is continuous on all of $\mathbb{R}$.

> The most difficult thing is the decision to act.
> The rest is merely tenacity.
>
> Amelia Earhart, aviation pioneer

6.3 Extreme Value Theorem

Problem 6.21. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and consider the closed interval $[a,b]$. Is the image $f([a,b])$ always a closed interval? If so, prove it. Otherwise, provide a counterexample.

Problem 6.22. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and K is a compact set, then the image $f(K)$ is compact.

The next result tells us that continuous functions always attain a maximum value on closed intervals. Of course, we have an analogous result involving minimums.

Problem 6.23 (Extreme Value Theorem)**.** Suppose f is a real function and let $I = [a,b]$ be a closed interval. Prove that if f is continuous on I, then there exists $x_M \in I$ such that $f(x_M) \geq f(x)$ for all $x \in I$.

Problem 6.24. Is the hypothesis that I is closed needed in the Extreme Value Theorem? Justify your assertion.

Problem 6.25. Is the converse of the Extreme Value Theorem true? That is, if a function attains a maximum value over a closed interval, does that imply that the function is continuous. If so, prove it. Otherwise, provide a counterexample.

Problem 6.26. Let $f : [0,1] \to \mathbb{R}$ and assume that the image $f([0,1])$ has a supremum. Prove that there is a sequence of points $(p_n)_{n=1}^{\infty}$ in $[0,1]$ such that $(f(p_n))_{n=1}^{\infty}$ converges to that supremum. Does this show that f is continuous on $[0,1]$?

> God created infinity, and man, unable to understand infinity, had to invent finite sets.
>
> Gian-Carlo Rota, mathematician & philosopher

6.4 Intermediate Value Theorem

The next problem is analogous to Problem 6.22. It also likely captures your intuition about continuity from high school and calculus.

Problem 6.27. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and C is a connected set, then the image $f(C)$ is connected.

The next result is a special case of the well-known **Intermediate Value Theorem**, which states that if f is a continuous real function whose domain contains the bounded closed interval $[a,b]$, then f attains every value between $f(a)$ and $f(b)$ at some point within the interval $[a,b]$. To prove the special case, utilize Problems 4.44 and 6.27 together with a proof by contradiction.

Problem 6.28. Suppose f is a real function and let $I = [a,b]$ be a closed interval. Prove that if f is continuous on I such that $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, then there exists $r \in I$ such that $f(r) = 0$.

If we generalize the previous result, we obtain the Intermediate Value Theorem.

Problem 6.29 (Intermediate Value Theorem)**.** Suppose f is a real function and let $I = [a,b]$ be a closed interval. Prove that if f is continuous on I such that $f(a) < c < f(b)$ or $f(a) > c > f(b)$, then there exists $r \in I$ such that $f(r) = c$.

Problem 6.30. Is the converse of the Intermediate Value Theorem true? If so, prove it. Otherwise, provide a counterexample.

Problem 6.31. Let $f : [0,1] \to \mathbb{R}$ be a function such that $f(0) = -1$, $f(1) = 1$, and $f([0,1]) = \{-1, 1\}$. Prove that there exists $a \in [0,1]$ such that f is not continuous at a.

> You will become clever through your mistakes.
>
> German Proverb

6.5 Uniform Continuity

In $\varepsilon - \delta$ characterization of continuity at a given in Problem 6.7, each choice of δ depends on both ε and a. In the next definition, the choice of δ only depends on ε and is independent of $a \in \mathrm{Dom}(f)$.

Definition 6.32. Suppose f is a real function. We say that f is **uniformly continuous** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \mathrm{Dom}(f)$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Continuity itself is a *local property* of a real function. That is, a real function f is either continuous or not at a particular point. Whether f is continuous at a particular point can be determined by looking only at the values of the function "near" that specific point. In this case, the choice of δ may depend on the specific point in question. When we speak of a function being continuous on a set, we mean that it is continuous at each point of the set. In contrast, uniform continuity is a *global property* of f in the sense that the definition refers to pairs of points rather than individual points, and the choice of δ only depends on ϵ.

Loosely speaking, a function is uniformly continuous if and only if there is a one-size fits all δ for each ϵ. In other words, a function is uniformly continuous if for every $x \in \mathrm{Dom}(f)$, the graph of f never exits the top or bottom of the rectangle of dimensions 2δ by 2ε centered at $(x, f(x))$ on the graph of f.

Problem 6.33. Let f be a real function. Prove that if f is uniformly continuous, then f is continuous.

Problem 6.34. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = 5x - 3$. Prove that f is uniformly continuous.

Problem 6.35. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$. Prove that f is not uniformly continuous.

Problem 6.36. Define $f : [0, \infty) \to \mathbb{R}$ via $f(x) = \sqrt{x}$. Prove that f is uniformly continuous.

Problem 6.37. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ via $f(x) = \frac{1}{x}$. Determine whether f is uniformly continuous.

The next result tells us that every function that is continuous on a compact set is uniformly continuous.

Problem 6.38. Let $f : K \to \mathbb{R}$ be a continuous real function such that K is compact. Prove that f is uniformly continuous.

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.
>
> Eugene Paul Wigner, theoretical physicist

The aim of argument, or of discussion, should not be victory, but progress.

Joseph Joubert, French moralist and essayist

Chapter 7

Limits

We are now prepared to dig into limits, which you are likely familiar with from calculus. However, chances are that you were never introduced to the formal definition.

7.1 Introduction to Limits

Definition 7.1. Let f be a real function. The **limit** of f as x approaches a is L if the following two conditions hold:

1. The point a is an accumulation point of $\mathrm{Dom}(f)$, and
2. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in \mathrm{Dom}(f)$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Notationally, we write this as

$$\boxed{\lim_{x\to a} f(x) = L.}$$

Problem 7.2. Why do we require $0 < |x - a|$ in Definition 7.1?

Problem 7.3. Why do you think we require a to be an accumulation point of the domain of f? What happens if $a \in \mathrm{Dom}(f)$ but a is not an accumulation point of $\mathrm{Dom}(f)$? Such points are called **isolated points** of the domain of f.

Notice that if $a \in \mathrm{Dom}(f)$ is an accumulation point of $\mathrm{Dom}(f)$, then the continuity of f at a is equivalent to the condition that

$$\lim_{x\to a} f(x) = f(a),$$

meaning that the limit of f as x approaches a exists and is equal to the value of f at a. However, it is important to notice that f may be continuous at a despite the fact that the limit of f as x approaches a is undefined. This happens when a is an isolated point of the domain.

Example 7.4. It should come as no surprise to you that $\lim_{x\to 5}(3x+2) = 17$. Let's prove this using Definition 7.1. First, notice that the default domain of $f(x) = 3x+2$ is the set of real numbers. So, any x-value we choose will be in the domain of the function. Now, let $\varepsilon > 0$. Choose $\delta = \varepsilon/3$. You'll see in a moment why this is a good choice for δ. Suppose $x \in \mathbb{R}$ such that $0 < |x-5| < \delta$. We see that

$$|(3x+2)-17| = |3x-15| = 3\cdot|x-5| < 3\cdot\delta = 3\cdot\varepsilon/3 = \varepsilon.$$

This proves the desired result.

Example 7.5. Let's try something a little more difficult. Let's prove that $\lim_{x\to 3} x^2 = 9$. As in the previous example, the default domain of our function is the set of real numbers. Our goal is to prove that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in \mathbb{R}$ such that $0 < |x-3| < \delta$, then $|x^2-9| < \varepsilon$. Let $\varepsilon > 0$. We need to figure out what δ needs to be. Notice that

$$|x^2-9| = |x+3|\cdot|x-3|.$$

The quantity $|x-3|$ is something we can control with δ, but the quantity $|x+3|$ seems to be problematic.

To get a handle on what's going on, let's temporarily assume that $\delta = 1$ and suppose that $0 < |x-3| < 1$. This means that x is within 1 unit of 3. In other words, $2 < x < 4$. But this implies that $5 < x+3 < 7$, which in turn implies that $|x+3|$ is bounded above by 7. That is, $|x+3| < 7$ when $0 < |x-3| < 1$. It's easy to see that we still have $|x+3| < 7$ even if we choose δ smaller than 1. That is, we have $|x+3| < 7$ when $0 < |x-3| < \delta \leq 1$. Putting this altogether, if we suppose that $0 < |x-3| < \delta \leq 1$, then we can conclude that

$$|x^2-9| = |x+3|\cdot|x-3| < 7\cdot|x-3|.$$

This work informs our choice of δ, but remember our scratch work above hinged on knowing that $\delta \leq 1$. If $\varepsilon/7 \leq 1$, we should choose $\delta = \varepsilon/7$. However, if $\varepsilon/7 > 1$, the easiest thing to do is to just let $\delta = 1$. Let's button it all up.

Let $\varepsilon > 0$. Choose $\delta = \min\{1, \varepsilon/7\}$ and suppose $0 < |x-3| < \delta$. We see that

$$|x^2-9| = |x+3|\cdot|x-3| < 7\cdot|x-3| < 7\cdot\delta \leq \varepsilon$$

since

$$7\cdot\delta = \begin{cases} 7, & \text{if } \varepsilon > 7 \\ 7\cdot\varepsilon/7, & \text{if } \varepsilon \leq 7. \end{cases}$$

Therefore, $\lim_{x\to 3} x^2 = 9$, as expected.

Problem 7.6. Prove that $\lim\limits_{x\to 1}(17x-42) = -25$ using Definition 7.1.

Problem 7.7. Prove that $\lim\limits_{x\to 2} x^3 = 8$ using Definition 7.1.

Problem 7.8. Define $f: \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 17, & \text{if } x = 0. \end{cases}$$

Prove that $\lim\limits_{x\to 0} f(x) = 0$ using Definition 7.1.

Problem 7.9. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ -1, & \text{if } x > 0. \end{cases}$$

Using Definition 7.1, prove that $\lim_{x \to 0} f(x)$ does not exist.

Problem 7.10. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Using Definition 7.1, prove that $\lim_{x \to a} f(x)$ does not exist for all $a \in \mathbb{R}$.

Like the limits of sequences, limits of functions are unique when they exist.

Problem 7.11. Let f be a real function. Prove that if $\lim_{x \to a} f(x)$ exists, then the limit is unique.

> An ounce of practice is worth more than tons of preaching.
>
> Mahatma Gandhi, political activist

7.2 Limit Laws

Perhaps not surprisingly, there is a nice connection between limits and sequences.

Problem 7.12. Let f be a real function and let a be an accumulation point of $\text{Dom}(f)$. Prove that $\lim_{x \to a} f(x)$ exists if and only if for every sequence (p_n) in $\text{Dom}(f) \setminus \{a\}$ converging to a, the sequence $(f(p_n))$ converges, in which case, $\lim_{x \to a} f(x)$ equals the limit of the sequence $(f(p_n))$. This is often written as

$$\lim_{x \to a} f(x) = \lim_{n \to \infty} f(p_n).$$

In order for limits to be a useful tool, we need to prove a few important facts.

Problem 7.13 (Limit Laws)**.** Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be real functions. Prove each of the following using Definition 7.1 or Problem 7.12.

(a) If $c \in \mathbb{R}$, then $\lim_{x \to a} c = c$.

(b) If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, then

$$\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x).$$

(c) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then

$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x).$$

(d) If $c \in \mathbb{R}$ and $\lim_{x\to a} f(x)$ exists, then

$$\lim_{x\to a}(c\cdot f(x)) = c\cdot \lim_{x\to a} f(x).$$

(e) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and $\lim_{x\to a} g(x) \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

(f) If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

The next problem is extremely useful. It allows us to simplify our calculations when computing limits.

Problem 7.14. Let f and g be real functions with $A = \mathrm{Dom}(f) = \mathrm{Dom}(g)$ and let a be an accumulation point of A. Prove that if there exists an open interval J containing a such that $f(x) = g(x)$ for all $x \in (J \cap A) \setminus \{a\}$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

provided one of the limits exists.

> Vulnerability is not winning or losing; it's having the courage to show up and be seen when we have no control over the outcome.
>
> Brené Brown, storyteller & author

If you want to sharpen a sword, you have to remove a little metal.

Author Unknown

Chapter 8

Differentiation

It's time for derivatives!

8.1 Introduction to Differentiation

Definition 8.1. Let $f : A \to \mathbb{R}$ be a real function and let $a \in A$ such that f is defined on some open interval I containing a (i.e., $a \in I \subseteq A$). The **derivative** of f at a is defined via

$$\boxed{f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

provided this limit exists. If $f'(a)$ exists, then we say that f is **differentiable** at a. More generally, we say that f is **differentiable** on $B \subseteq A$ if f is differentiable at every point in B. As a special case, f is said to be **differentiable** if it is differentiable at every point in its domain. If f does indeed have a derivative at some points in its domain, then the **derivative** of f is the function denoted by f', such that for each number x at which f is differentiable, $f'(x)$ is the derivative of f at x. We may also write

$$\boxed{\frac{d}{dx}[f(x)] := f'(x).}$$

The lefthand side of the equation above is typically read as, "the derivative of f with respect to x." The notation $f'(x)$ is commonly referred to as "Newton's notation" for the derivative while $\frac{d}{dx}[f(x)]$ is often referred to as "Liebniz's notation".

Note that the definition of the derivative automatically excludes the kind of behavior we saw with continuous functions, where a function defined only at a single point was continuous.

Problem 8.2. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2 - x + 1$ at $a = 2$. Prove that f is differentiable on $\mathbb{R}$ and find a formula for the derivative of f.

Problem 8.3. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = c$ for some constant $c \in \mathbb{R}$. Prove that f is differentiable on $\mathbb{R}$ and $f'(x) = 0$ for all $x \in \mathbb{R}$.

Problem 8.4. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = mx + b$, where $m, b \in \mathbb{R}$. Prove that f is differentiable and $f'(x) = m$ for all $x \in \mathbb{R}$.

Problem 8.5. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$. Prove that f is differentiable on $\mathbb{R}$ and find a formula for the derivative of f.

In the previous four problems, note that if we restrict the domain of the functions to a closed interval $[a, b]$, then we can conclude that we get the expected derivatives for all $x \in (a, b)$.

Problem 8.6. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^3$. Prove that f is differentiable at 0 and $f'(0) = 0$.

Problem 8.7. Explain why any function defined only on $\mathbb{Z}$ cannot have a derivative.

The next problem tells us that differentiability implies continuity.

Problem 8.8. Prove that if f has a derivative at $x = a$, then f is also continuous at $x = a$.

The converse of the previous theorem is not true. That is, continuity does not imply differentiability.

Problem 8.9. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = |x|$.

(a) Prove that f is continuous at every point in its domain.

(b) Prove that f is differentiable everywhere except at $x = 0$.

Problem 8.10. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is continuous at $x = 0$, but not differentiable at $x = 0$.

> Don't let anyone rob you of your imagination, your creativity, or your curiosity. It's your place in the world; it's your life. Go on and do all you can with it, and make it the life you want to live.
>
> Mae Jemison, NASA astronaut

8.2 Derivative Rules

In this section, we prove a few of the well-known derivative rules from first-semester calculus.

Problem 8.11. If f is differentiable at x and $c \in \mathbb{R}$, prove that the function cf also has a derivative at x and $(cf)'(x) = cf'(x)$.

Problem 8.12. If f and g are differentiable at x, show that the function $f+g$ also has a derivative at x and $(f+g)'(x) = f'(x) + g'(x)$.

The next problem states the well-known Product and Quotient Rules for Derivatives. You will need to use Problem 8.8 in their proofs.

Problem 8.13. Suppose f and g are differentiable at x. Prove each of the following:

(a) (Product Rule) The function fg is differentiable at x. Moreover, its derivative function is given by
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

(b) (Quotient Rule) The function f/g is differentiable at x provided $g(x) \neq 0$. Moreover, its derivative function is given by
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

The next problem is sure to make your head hurt.

Problem 8.14. Define $g : \mathbb{R} \to \mathbb{R}$ via
$$g(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{otherwise.} \end{cases}$$
Now, define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2 g(x)$. Determine where f is differentiable.

> I write one page of masterpiece to ninety-one pages of shit.
>
> Ernest Hemingway, novelist & journalist

8.3 The Mean Value Theorem

The next result tells us that if a differentiable function attains a maximum value at some point in an open interval contained in the domain of the function, then the derivative is zero at that point. In a calculus class, we would say that differentiable functions attain local maximums at critical numbers.

Problem 8.15. Let $f : A \to \mathbb{R}$ be a real function such that $[a,b] \subseteq A$, $f'(c)$ exists for some $c \in (a,b)$, and $f(c) \geq f(x)$ for all $x \in (a,b)$. Prove that $f'(c) = 0$.

Problem 8.16. Let $f : A \to \mathbb{R}$ be a real function such that $f'(c) = 0$ for some $c \in A$. Does this imply that there exists an open interval $(a,b) \subseteq A$ containing c such that either $f(x) \geq f(c)$ or $f(x) \leq f(c)$ for all $x \in (a,b)$? If so, prove it. Otherwise, provide a counterexample.

The next problem asks you to prove a result called Rolle's Theorem. To prove Rolle's Theorem, first, apply the Extreme Value Theorem to f and $-f$ to conclude that f attains both a maximum and minimum on $[a,b]$. If both the maximum and minimum are attained at the end points of $[a,b]$, then the maximum and minimum are the same, and hence the function is constant. What does Problem 8.3 tell us in this case? But what if f is not constant over $[a,b]$? Try using Problem 8.15.

Problem 8.17 (Rolle's Theorem)**.** Let $f : A \to \mathbb{R}$ be a real function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, then prove that there exists a point $c \in (a,b)$ such that $f'(c) = 0$.

We can use Rolle's Theorem to prove the next result, which is the well-known Mean Value Theorem.

Problem 8.18 (Mean Value Theorem)**.** Let $f : A \to \mathbb{R}$ be a real function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$ and differentiable on (a,b), then prove that there exists a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Here's one approach. Cleverly define the function $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. Is g continuous on $[a,b]$? Is g differentiable on (a,b)? Can we apply Rolle's Theorem to g using the interval $[a,b]$? What can you conclude? Magic!

Problem 8.19. Let $f : A \to \mathbb{R}$ be a real function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$ and differentiable on (a,b) such that $f'(x) = 0$ for all $x \in (a,b)$, then prove that f is constant over $[a,b]$. Try applying the Mean Value Theorem to $[a,t]$ for every $t \in (a,b]$.

Problem 8.20. Let $f : A \to \mathbb{R}$ be a real function such that $(a,b) \subseteq A$ and f is differentiable at every point of (a,b). Prove that if $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on (a,b). What if $f'(x) < 0$ for all $x \in (a,b)$?

The converse of the previous result is not true in general!

Problem 8.21. Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that f is strictly increasing on its domain yet $f'(x)$ is not positive for all $x \in \mathbb{R}$.

Problem 8.22. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ such that $[a,b] \subseteq A$. Prove that if f and g are continuous on $[a,b]$ and $f'(x) = g'(x)$ for all $x \in (a,b)$, then there exists $C \in \mathbb{R}$ such that $f(x) = g(x) + C$ for all $x \in [a,b]$.

Problem 8.23. Is the converse of the previous problem true? If so, prove it. Otherwise, provide a counterexample.

Problem 8.24. Let $f : A \to \mathbb{R}$ be a real function such that $(a,b) \subseteq A$ and f is differentiable at every point of (a,b). Prove that if there exists a nonnegative real number M such that $|f'(x)| \leq M$ for all $x \in (a,b)$ (i.e., f has bounded derivative on (a,b)), then for $x, y \in (a,b)$, we have $|f(x) - f(y)| \leq M|x - y|$.

Problem 8.25. Let $f : A \to \mathbb{R}$ be a real function such that $(a,b) \subseteq A$ and f is differentiable at every point of (a,b). Prove that if there exists a nonnegative real number M such that $|f'(x)| \leq M$ for all $x \in (a,b)$ (i.e., f has bounded derivative on (a,b)), then f is uniformly continuous on (a,b).

Problem 8.26. If f is a differentiable real function that is also uniformly continuous, does this imply that the derivative is bounded? Justify your assertion.

> It is not the critic who counts; not the man who points out how the strong man stumbles, or where the doer of deeds could have done them better. The credit belongs to the man who is actually in the arena, whose face is marred by dust and sweat and blood; who strives valiantly; who errs, who comes short again and again, because there is no effort without error and shortcoming; but who does actually strive to do the deeds; who knows great enthusiasms, the great devotions; who spends himself in a worthy cause; who at the best knows in the end the triumph of high achievement, and who at the worst, if he fails, at least fails while daring greatly, so that his place shall never be with those cold and timid souls who neither know victory nor defeat.
>
> Theodore Roosevelt, statesman & conservationist

> Most of what we believe, we believe because it was told to us by someone we trusted. What I would like to suggest, however, is that if we rely too much on that kind of education, we could find in the end that we have never really learned anything.
>
> Paul Wallace, physicist & theologian

Chapter 9

Integration

9.1 Introduction to Integration

Unlike with differentiation, we will need a number of auxiliary definitions for beginning integration.

Definition 9.1. A set of points $P = \{t_0, t_1, \ldots, t_n\}$ is a **partition** of the closed interval $[a, b]$ if $a = t_0 < t_1 < \cdots t_{n-1} < t_n = b$. If $t_i - t_{i-1} = \frac{b-a}{n}$ for all i, we say that the partition is a **regular partition** of $[a, b]$. In this case, we may use the notation $\boxed{\Delta t := t_i - t_{i-1}}$.

Problem 9.2. Give some partitions, regular and not regular, of $[0, 1]$, $[2, 4]$, and $[-1, 0]$.

Definition 9.3. We say that a real function is **bounded** if it has bounded image set.

Important! For the next four definitions, we assume that f is a bounded real function with domain equal to some closed interval $[a, b]$.

Definition 9.4. Let f be a bounded real function with domain $[a, b]$ and let $\{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. We say that any sum S of the form

$$\boxed{S = \sum_{i=1}^{n} f(x_i)(t_i - t_{i-1}),}$$

where $x_i \in [t_{i-1}, t_i]$ is a **Riemann sum** for f on $[a, b]$.

Definition 9.5. Let f be a bounded real function with domain $[a, b]$ and let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. For each $i \in \{1, 2, \ldots, n\}$, define $\boxed{M_i := \sup\{f(x) \mid x \in [t_{i-1}, t_i]\}}$. We say that the sum

$$\boxed{U_P(f) := \sum_{i=1}^{n} M_i(t_i - t_{i-1}),}$$

is the **upper Riemann sum** for f with partition P.

Definition 9.6. Let f be a bounded real function with domain $[a,b]$ and let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a,b]$. For each $i \in \{1,2,\ldots,n\}$, define $\boxed{m_i := \inf\{f(x) \mid x \in [t_{i-1}, t_i]\}}$. We say that the sum

$$\boxed{L_P(f) := \sum_{i=1}^{n} m_i(t_i - t_{i-1}),}$$

is the **lower Riemann sum** for f with partition P.

Problem 9.7. Draw pictures that capture the concepts of upper and lower Riemann sums.

Contrary to the name, upper and lower Riemann sums are not always Riemann sums.

Problem 9.8. Give an example of an interval $[a,b]$, partition P, and bounded real function f such that $U_p(f)$ is not a Riemann sum.

Problem 9.9. Define $f : [0,1] \to \mathbb{R}$ via

$$f(x) = \begin{cases} 0, & x \in (0,1] \\ 1, & x = 0. \end{cases}$$

(a) Show that $U_P(f) > 0$ for all partitions of $[0,1]$.

(b) Show that for any positive number ε there is a partition P_ε such that $U_{P_\varepsilon}(f) < \varepsilon$.

(c) Fully describe all lower sums of f on $[0,1]$.

For the next problem, it will be useful to recall that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Problem 9.10. Define $f : [0,1] \to \mathbb{R}$ via $f(x) = x$. For each $n \in \mathbb{N}$, let P_n be the regular partition of $[0,1]$ given by $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$.

(a) Compute $U_{P_5}(f)$.

(b) Give a formula for $U_{P_n}(f)$.

(c) Compute $L_{P_5}(f)$.

(d) Give a formula for $L_{P_n}(f)$.

Problem 9.11. Suppose that f is a bounded real function on $[a,b]$ with lower bound m and upper bound M. Show that for any partition P of $[a,b]$, $U_P(f) \leq M(b-a)$ and $L_P(f) \geq m(b-a)$.

Problem 9.12. Suppose that f is a bounded real function on $[a,b]$ and P is a partition of $[a,b]$. Show that $L_P(f) \leq U_P(f)$.

One consequence of Problem 9.11 is that the set of all upper, respectively lower, sums of f over $[a,b]$ is a bounded set. This implies that if f is a bounded real function on $[a,b]$, then the following supremum and infimum exist:

$$\inf\{U_P(f) \mid P \text{ is a partition of } [a,b]\}$$

$$\sup\{L_P(f) \mid P \text{ is a partition of } [a,b]\}$$

This leads to the following definition.

Definition 9.13. Let f be a bounded real function with domain $[a,b]$. The **upper integral** of f from a to b is defined via

$$\boxed{\overline{\int_a^b} f := \inf\{U_P(f) \mid P \text{ is a partition of } [a,b]\}.}$$

Similarly, the **lower integral** of f from a to b is defined via

$$\boxed{\underline{\int_a^b} f := \sup\{L_P(f) \mid P \text{ is a partition of } [a,b]\}.}$$

Problem 9.14. Compute the upper and lower integrals for the function in Problem 9.9.

Problem 9.15. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that $\underline{\int_0^1} f < \overline{\int_0^1} f$.

Definition 9.16. If P and Q are partitions of $[a,b]$ such that $P \subseteq Q$, then we we say that Q is a **refinement** of P, or that Q **refines** P.

Problem 9.17. Let f be a bounded real function with domain $[a,b]$. Prove that if P and Q are partitions of $[a,b]$ such that Q is a refinement of P, then $L_P(f) \leq L_Q(f)$ and $U_P(f) \geq U_Q(f)$.

Problem 9.18. Suppose f is a bounded real function on $[a,b]$. Use the previous problem to prove that

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Problem 9.19. Suppose f is continuous on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and that for some $z \in [a,b]$, $f(z) > 0$. Explain why $\int_a^b f$ exists and then show that $\underline{\int_a^b} f > 0$.

Definition 9.20. Let f be a bounded real function with domain $[a,b]$. We say that f is (**Riemann**) **integrable** on $[a,b]$ if

$$\overline{\int_a^b} f = \underline{\int_a^b} f.$$

If f is integrable on $[a,b]$, then the common value of the upper and lower integrals is called the (**Riemann**) **integral** of f on $[a,b]$, which we denote via

$$\boxed{\int_a^b f} \quad \text{or} \quad \boxed{\int_a^b f(x)\,dx.}$$

Technically, we have defined the **Darboux integral**, with Riemann integrals coming from so-called Riemann sums. The two notions can be proved to be equivalent.

Problem 9.21. Give an example of a function f and an interval $[a,b]$ for which we know $\int_a^b f$ does not exist.

Problem 9.22. Is the function in Problem 9.9 integrable over $[0,1]$? If so, determine the value of the corresponding integral. If not, explain why.

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as poetry.
>
> Bertrand Russell, philosopher & mathematician

9.2 Properties of Integrals

There are so many facts about integrals, and unfortunately, we do not have time to prove them all! Nonetheless, we will hit some of the key results.

Problem 9.23. Prove that every constant real function is integrable over every interval $[a,b]$.

The following theorem is a useful characterization of when a function is integrable over a closed interval.

Problem 9.24. Suppose f is a bounded real function on $[a,b]$. Then f is (Riemann) integrable if and only if for every $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that $U_P(f) - L_P(f) < \varepsilon$.

It is important to recognize that the previous problem provides us with a technique for determining whether a function is integrable over a closed interval, but does not necessarily help us with determining the value of a particular integral.

Problem 9.25. Define $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x$. Using the tools we currently have at our disposal, prove that f is integrable on $[0,1]$ and compute the value of the integral.

The next set of theorems will vastly expand our repertoire of functions known to be integrable. First, we need a few definitions, which resemble the corresponding concepts we defined for sequences in Chapter 5.

Definition 9.26. A real function f is (strictly) **increasing** if for each pair of points x and y in the domain of f satisfying $x < y$, we have $f(x) < f(y)$. The function is **nondecreasing** if under the same assumptions we have $f(x) \leq f(y)$. The notions of (strictly) **decreasing** and **nonincreasing** are defined analogously. We say that f is a **monotonic** function if f is either nondecreasing or nonincreasing.

Problem 9.27. Prove that if f is a bounded monotonic real function on $[a,b]$, then f is integrable on $[a,b]$.

Problem 9.28. Prove that each of the following exist. Do you know the value of any of these integrals knowing what we know now and perhaps some well-known area formulas?

(a) $\displaystyle\int_1^2 x^2\,dx$

(b) $\displaystyle\int_1^{17} e^{-x}\,dx$

(c) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

(d) $\displaystyle\int_0^1 \sqrt{1+x^4}\,dx$

The next problem tells us that the integral respects scalar multiplication and sums and differences of integrable functions.

Problem 9.29. Suppose f and g are integrable real functions on $[a,b]$ and let $c \in \mathbb{R}$. Prove each of the following:

(a) The function cf is integrable on $[a,b]$ and $\displaystyle\int_a^b cf = c\int_a^b f$.

(b) The function $f+g$ is integrable on $[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. This one is much harder than it looks!

(c) The function $f-g$ is integrable on $[a,b]$ and $\int_a^b (f-g) = \int_a^b f - \int_a^b g$. Consider using parts (a) and (b)

Unfortunately, products of integrable functions are not well behaved.

Problem 9.30. Find two real functions f and g that are integrable on $[0,1]$ such that fg is also integrable on $[0,1]$ but

$$\left(\int_0^1 f\right)\left(\int_0^1 g\right) \neq \int_0^1 fg.$$

Problem 9.31. Prove that if f is integrable on $[a,b]$, then there exists $m, M \in \mathbb{R}$ such that

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Problem 9.32. Assume that $[a,b]$ is a closed interval and suppose f is integrable on $[a,c]$ and $[c,b]$ for $c \in (a,b)$. Show that f is integrable on $[a,b]$ and that

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Problem 9.33. Suppose f is integrable on $[a,b]$. Prove that for every $c \in \mathbb{R}$, the function g defined via $g(x) = f(x-c)$ is integrable on $[a+c, b+c]$ and

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx.$$

Let's turn our attention to continuous functions. Consider using Problem 6.38 when approaching the next problem.

Problem 9.34. Suppose f is continuous on $[a,b]$. Prove that for every $\varepsilon > 0$, there exists a partition $P = \{t_0 = a, t_1, \ldots, t_{n-1}, t_n = b\}$ of $[a,b]$ such that for each $1 \leq i \leq n$, if $u, v \in [t_{i-1}, t_i]$, then $|f(u) - f(v)| < \varepsilon$.

Use the previous problem to tackle the next problem.

Problem 9.35. Prove that if f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Problem 9.36. Is the converse of the previous problem true? If so, prove it. Otherwise, provide a counterexample.

Problem 9.37. Suppose f is continuous on $[a,b]$. Prove that

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Definition 9.38. If f is integrable on $[a,b]$, then we define

$$\boxed{\int_b^a f = -\int_a^b f} \quad \text{and} \quad \boxed{\int_a^a f = 0.}$$

The next result is often referred to as the Mean Value Theorem for Integrals. Do you see why?

Problem 9.39 (Mean Value Theorem for Integrals)**.** Suppose f is continuous on $[a,b]$. Prove that there exists $c \in [a,b]$ such that

$$\int_a^b f = f(c)(b-a).$$

Can you draw a picture to capture the essence of this theorem?

Problem 9.40. Suppose f is integrable on $[a,b]$ and define $g : [a,b] \to \mathbb{R}$ via

$$g(x) = \int_a^x f.$$

Prove that g is uniformly continuous on $[a,b]$.

> Mathematics has beauty and romance. It's not a boring place to be, the mathematical world. It's an extraordinary place; it's worth spending time there.
>
> Marcus du Sautoy, mathematician

9.3 Fundamental Theorem of Calculus

The next two problems are the crowning achievement of calculus and of this course. Collectively, these two problems are known as the Fundamental Theorem of Calculus.

Problem 9.41 (Fundamental Theorem of Calculus, Part 1)**.** Suppose f is continuous on $[a,b]$ and define $F : [a,b] \to \mathbb{R}$ via

$$F(x) = \int_a^x f.$$

Prove that for each $c \in (a,b)$, F is differentiable at c and $F'(c) = f(c)$.

It follows from Problem 9.40 that the function F in the previous theorem is continuous, as well.

Problem 9.42 (Fundamental Theorem of Calculus, Part 2)**.** Suppose f is a real function whose domain includes $[a,b]$ such that f is differentiable at each point of $[a,b]$, and the function f' is continuous at each point in $[a,b]$. Prove that

$$\int_a^b f' = f(b) - f(a).$$

These are the keys that made the efforts of Newton and Leibniz into the something one could calculate with. It is worth noting that the previous theorem is true even if f' is just integrable. Truly a great theorem!

It is important to point out that the function we are integrating in Problem 9.42 needs to be continuous. Moreover, this function must be some other function's derivative. Given f' in Problem 9.42, there is an entire family of functions that have the same derivative as f, each differing by a constant, according to Problem 8.22. Each of the functions in this family is referred to as an **antiderivative** of f' and any one of them can be used to compute $\int_a^b f'$ using the Fundamental Theorem of Calculus.

The crux of using the Fundamental Theorem of Calculus boils down to finding an antiderivative of the function you are integrating. Some functions do not have nice antiderivatives! For example, in part (d) of Problem 9.28, we argued that the function given by $f(x) = \sqrt{1+x^4}$ is integrable on $[0,1]$. However, this function does not have an antiderivative that you would recognize. Try asking WolframAlpha for the antiderivative of $f(x) = \sqrt{1+x^4}$ and see what you get.

Most functions you are familiar with are called elementary functions. Loosely speaking, a function is an **elementary function** if it is equal to a sum, product, and/or composition of finitely many polynomials, rational functions, trigonometric functions, exponential functions, and their inverse functions. These are the functions you typically encounter in high school, precalculus, and calculus. However, many functions are not elementary. For example, the function given in Problem 9.15 is not elementary. To complicate matters, many elementary functions do not have elementary antiderivatives. In fact, some rather innocent looking elementary functions do not have elementary antiderivatives. The function from part (d) of Problem 9.28 is such an example. Here are a few more elementary functions that do not have elementary antiderivatives:

- $\sqrt{1-x^4}$
- $\dfrac{1}{\ln(x)}$
- $\sin(x^2)$ and $\cos(x^2)$
- $\dfrac{\sin(x)}{x}$
- $\dfrac{e^x}{x}$
- e^{e^x}

Determining which elementary functions have elementary antiderivatives is not an easy task. The upshot is that utilizing the Fundamental Theorem of Calculus to compute an integral may be difficult for seemingly innocent looking functions.

Problem 9.43. Using Problem 9.42 and your knowledge of antiderivatives from first semester calculus, compute the integrals in parts (a) and (b) of Problem 9.28.

Problem 9.44. According to WolframAlpha,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

Explain why the techniques of this chapter cannot be used to verify this. How one might go about computing this integral? What definitions are needed?

> In the broad light of day mathematicians check their equations and their proofs, leaving no stone unturned in their search for rigour. But, at night, under the full moon, they dream, they float among the stars and wonder at the miracle of the heavens. They are inspired. Without dreams there is no art, no mathematics, no life.
>
> Michael Atiyah, mathematician

Appendix A

Elements of Style for Proofs

Mathematics is about discovering proofs and writing them clearly and compellingly. The following guidelines apply whenever you write a proof. Keep these guidelines handy so that you may refer to them as you write your proofs.

1. **The burden of communication lies on you, not on your reader.** It is your job to explain your thoughts; it is not your reader's job to guess them from a few hints. You are trying to convince a skeptical reader who does not believe you, so you need to argue with airtight logic in crystal clear language; otherwise the reader will continue to doubt. If you did not write something on the paper, then (a) you did not communicate it,(b) the reader did not learn it, and (c) the grader has to assume you did not know it in the first place.

2. **Tell the reader what you are proving or citing.** The reader does not necessarily know or remember what "Theorem 2.13" is. Even a professor grading a stack of papers might lose track from time to time. Therefore, the statement you are proving should be on the same page as the beginning of your proof.

   In most proofs you will want to refer to an earlier definition, problem, theorem, or corollary. In this case, you should reference the statement by number, but it is also helpful to the reader to summarize the statement you are citing. For example, you might write something like, "By Theorem 2.3, the sum of two consecutive integers is odd, and so...."

3. **Use English words.** Although there will usually be equations or mathematical statements in your proofs, use English sentences to connect them and display their logical relationships. If you look at proofs in textbooks and research papers, you will see that they consist mostly of English words.

4. **Use complete sentences.** If you wrote a history essay in sentence fragments, the reader would not understand what you meant; likewise in mathematics you must use complete sentences, with verbs, to convey your logical train of thought.

   Some complete sentences can be written purely in mathematical symbols, such as equations (e.g., $a^3 = b^{-1}$), inequalities (e.g., $x < 5$), and other relations (like $5 \mid 10$ or

$7 \in \mathbb{Z}$). These statements usually express a relationship between two mathematical *objects*, like numbers or sets. However, it is considered bad style to begin a sentence with symbols. A common phrase to use to avoid starting a sentence with mathematical symbols is "We see that...".

5. **Show the logical connections among your sentences.** Use phrases like "Therefore", "Thus", "Hence", "Then", "since", "because", "if..., then...", or "if and only if" to connect your sentences.

6. **Know the difference between statements and objects.** A mathematical object is a *thing*, a noun, such as a set, an element, a number, an ordered pair, a vector space, etc. Objects either exist or do not exist. Statements, on the other hand, are mathematical *sentences*: they are either true or false.

   When you see or write a cluster of math symbols, be sure you know whether it is an object (e.g., "$x^2 + 3$") or a statement (e.g., "$x^2 + 3 < 7$"). One way to tell is that every mathematical statement includes a verb, such as $=$, $\leq$, $\in$, "divides", etc.

7. **The symbol "=" means "equals".** Do not write $A = B$ unless you mean that A actually equals B. This guideline seems obvious, but there is a great temptation to be sloppy. In calculus, for example, some people might write $f(x) = x^2 = 2x$ (which is false), when they really mean that "if $f(x) = x^2$, then $f'(x) = 2x$."

8. **Do not interchange $=$ and $\Longrightarrow$.** The equals sign connects two *objects*, as in "$x^2 = b$"; the symbol "$\Longrightarrow$" is an abbreviation for "implies" and connects two *statements*, as in "$a + b = a \Longrightarrow b = 0$." You should avoid using $\Longrightarrow$ in formal write-ups of proofs.

9. **Avoid logical symbols in your proofs.** Similar to $\Longrightarrow$, you should avoid using the logical symbols $\forall, \exists, \vee, \wedge$, and $\Longleftrightarrow$ in your formal write-ups. These symbols are useful for abbreviating in your scratch work.

10. **Say exactly what you mean.** Just as $=$ is sometimes abused, so too people sometimes write $A \in B$ when they mean $A \subseteq B$, or write $a_{ij} \in A$ when they mean that a_{ij} is an entry in matrix A. Mathematics is a very precise language, and there is a way to say exactly what you mean; find it and use it.

11. **Do not utilize anything unproven.** Every statement in your proof should be something you *know* to be true. The reader expects your proof to be a series of statements, each proven by the statements that came before it. If you ever need to write something you do not yet know is true, you *must* preface it with words like "assume," "suppose," or "if" if you are temporarily assuming it, or with words like "we need to show that" or "we claim that" if it is your goal. Otherwise, the reader will think they have missed part of your proof.

12. **Write strings of equalities (or inequalities) in the proper order.** When your reader sees something like

$$A = B \leq C = D,$$

they expect to understand easily why $A = B$, why $B \leq C$, and why $C = D$, and they expect the point of the entire line to be the more complicated fact that $A \leq D$. For example, if you were computing the distance d of the point $(12, 5)$ from the origin, you could write

$$d = \sqrt{12^2 + 5^2} = 13.$$

In this string of equalities, the first equals sign is true by the Pythagorean theorem, the second is just arithmetic, and the conclusion is that the first item equals the last item: $d = 13$.

A common error is to write strings of equations in the wrong order. For example, if you were to write "$\sqrt{12^2 + 5^2} = 13 = d$", your reader would understand the first equals sign, would be baffled as to how we know $d = 13$, and would be utterly perplexed as to why you wanted or needed to go through 13 to prove that $\sqrt{12^2 + 5^2} = d$.

13. **Avoid circularity.** Be sure that no step in your proof makes use of the conclusion!

14. **Do not write the proof backwards.** Beginning students often attempt to write "proofs" like the following, which attempts to prove that $\tan^2(x) = \sec^2(x) - 1$:

$$\begin{aligned}
\tan^2(x) &= \sec^2(x) - 1 \\
\left(\frac{\sin(x)}{\cos(x)}\right)^2 &= \frac{1}{\cos^2(x)} - 1 \\
\frac{\sin^2(x)}{\cos^2(x)} &= \frac{1 - \cos^2(x)}{\cos^2(x)} \\
\sin^2(x) &= 1 - \cos^2(x) \\
\sin^2(x) + \cos^2(x) &= 1 \\
1 &= 1
\end{aligned}$$

Notice what has happened here: the student *started* with the conclusion, and deduced the true statement "$1 = 1$." In other words, they have proved "If $\tan^2(x) = \sec^2(x) - 1$, then $1 = 1$," which is true but highly uninteresting.

Now this is not a bad way of *finding* a proof. Working backwards from your goal often is a good strategy *on your scratch paper*, but when it is time to *write* your proof, you have to start with the hypotheses and work to the conclusion.

Here is an example of a suitable proof for the desired result, where each expression follows from the one immediately proceeding it:

$$\begin{aligned}
\sec^2(x) - 1 &= \frac{1}{\cos^2(x)} - 1 \\
&= \frac{1 - \cos^2(x)}{\cos^2(x)} \\
&= \frac{\sin^2(x)}{\cos^2(x)} \\
&= \left(\frac{\sin(x)}{\cos(x)}\right)^2 \\
&= (\tan(x))^2 \\
&= \tan^2(x).
\end{aligned}$$

15. **Be concise.** Many beginning proof writers err by writing their proofs too short, so that the reader cannot understand their logic. It is nevertheless quite possible to be too wordy, and if you find yourself writing a full-page essay, it is possible that you do not really have a proof, but just some intuition. When you find a way to turn that intuition into a formal proof, it will be much shorter.

16. **Introduce every symbol you use.** If you use the letter "k," the reader should know exactly what k is. Good phrases for introducing symbols include "Let $n \in \mathbb{N}$," "Let k be the least integer such that...," "For every real number a...," and "Suppose $A \subseteq \mathbb{R}$...".

17. **Use appropriate quantifiers (once).** When you introduce a variable $x \in S$, it must be clear to your reader whether you mean "for all $x \in S$" or just "for some $x \in S$." If you just say something like "$y = x^2$ where $x \in S$," the word "where" does not indicate whether you mean "for all" or "some".

    Phrases indicating the quantifier "for all" include "Let $x \in S$"; "for all $x \in S$"; "for every $x \in S$"; "for each $x \in S$"; etc. Phrases indicating the quantifier "some" or "there exists") include "for some $x \in S$"; "there exists an $x \in S$"; "for a suitable choice of $x \in S$"; etc.

    Once you have said "Let $x \in S$," the letter x has its meaning defined. You do not need to say "for all $x \in S$" again, and you definitely should *not* say "let $x \in S$" again.

18. **Use a symbol to mean only one thing.** Once you use the letter x once, its meaning is fixed for the duration of your proof. You cannot use x to mean anything else. There is an exception to this guideline. Sometimes a proof will include multiple subproofs that are distinct from each other. In this case, you can reuse a variable or symbol as long as it is clear to the reader that you have concluded with the previous subproof and have moved onto a new subproof.

19. **Do not "prove by example."** Most problems ask you to prove that something is true "for all"—You *cannot* prove this by giving a single example, or even a hundred. Your

proof will need to be a logical argument that holds for *every example there possibly could be*.

On the other hand, if the claim that you are trying to prove involves the existence of a mathematical object with a particular property, then providing a specific example is sufficient.

20. **Write "Let $x = \ldots$," not "Let $\cdots = x$."** When you have an existing expression, say a^2, and you want to give it a new, simpler name like b, you should write "Let $b = a^2$," which means, "Let the new symbol b mean a^2." This convention makes it clear to the reader that b is the brand-new symbol and a^2 is the old expression he/she already understands.

    If you were to write it backwards, saying "Let $a^2 = b$," then your startled reader would ask, "What if $a^2 \neq b$?"

21. **Make your counterexamples concrete and specific.** Proofs need to be entirely general, but counterexamples should be concrete. When you provide an example or counterexample, make it as specific as possible. For a set, for example, you must specify its elements, and for a function you must specify the corresponding relation (possibly an algebraic rule) and its domain and codomain. Do not say things like "f could be one-to-one but not onto"; instead, provide an actual function f that is one-to-one but not onto.

22. **Do not include examples in proofs.** Including an example very rarely adds anything to your proof. If your logic is sound, then it does not need an example to back it up. If your logic is bad, a dozen examples will not help it (see Guideline 19). There are only two valid reasons to include an example in a proof: if it is a *counterexample* disproving something, or if you are performing complicated manipulations in a general setting and the example is just to help the reader understand what you are saying.

23. **Use scratch paper.** Finding your proof will be a long, potentially messy process, full of false starts and dead ends. Do all that on scratch paper until you find a real proof, and only then break out your clean paper to write your final proof carefully.

    Only sentences that actually contribute to your proof should be part of the proof. Do not just perform a "brain dump," throwing everything you know onto the paper before showing the logical steps that prove the conclusion. *That is what scratch paper is for.*

Appendix B

Fancy Mathematical Terms

Here are some important mathematical terms that you will encounter throughout mathematics.

1. **Definition**—a precise and unambiguous description of the meaning of a mathematical term. It characterizes the meaning of a word by giving all the properties and only those properties that must be true.

2. **Theorem**—a mathematical statement that is proved using rigorous mathematical reasoning. In a mathematical paper, the term theorem is often reserved for the most important results.

3. **Proposition**—a proved and often interesting result, but generally less important than a theorem.

4. **Lemma**—a minor result whose sole purpose is to help in proving a theorem. It is a stepping stone on the path to proving a theorem. Occasionally lemmas can take on a life of their own (Zorn's Lemma, Urysohn's Lemma, Burnside's Lemma, Sperner's Lemma).

5. **Corollary**—a result in which the (usually short) proof relies heavily on a given theorem (we often say that "this is a corollary of Theorem A").

6. **Conjecture**—a statement that is unproved, but is believed to be true (Collatz Conjecture, Goldbach Conjecture, Twin prime Conjecture).

7. **Claim**—an assertion that is then proved. It is often used like an informal lemma.

8. **Counterexample**—a specific example showing that a statement is false.

9. **Axiom/Postulate**—a statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proved (Euclid's five postulates, axioms of ZFC, Peano axioms).

10. **Identity**—a mathematical expression giving the equality of two (often variable) quantities (trigonometric identities, Euler's identity).

11. **Paradox**—a statement that can be shown, using a given set of axioms and definitions, to be both true and false. Paradoxes are often used to show the inconsistencies in a flawed axiomatic theory (e.g., Russell's Paradox). The term paradox is also used informally to describe a surprising or counterintuitive result that follows from a given set of rules (Banach-Tarski Paradox, Alabama Paradox, Gabriel's Horn).

Appendix C

Definitions in Mathematics

It is difficult to overstate the importance of definitions in mathematics. Definitions play a different role in mathematics than they do in everyday life.

Suppose you give your friend a piece of paper containing the definition of the rarely-used word **rodomontade**. According to the Oxford English Dictionary[1] (OED) it is:

> A vainglorious brag or boast; an extravagantly boastful, arrogant, or bombastic speech or piece of writing; an arrogant act.

Give your friend some time to study the definition. Then take away the paper. Ten minutes later ask her to define rodomontade. Most likely she will be able to give a reasonably accurate definition. Maybe she'd say something like, "It is a speech or act or piece of writing created by a pompous or egotistical person who wants to show off how great they are." It is unlikely that she will have quoted the OED word-for-word. In everyday English that is fine—you would probably agree that your friend knows the meaning of the rodomontade. This is because most definitions are *descriptive*. They describe the common usage of a word.

Let us take a mathematical example. The OED[2] gives this definition of **continuous**.

> Characterized by continuity; extending in space without interruption of substance; having no interstices or breaks; having its parts in immediate connection; connected, unbroken.

Likewise, we often hear calculus students speak of a continuous function as one whose graph can be drawn "without picking up the pencil." This definition is descriptive. However, as we learned in calculus, the picking-up-the-pencil description is not a perfect description of continuous functions. This is not a mathematical definition.

Mathematical definitions are *prescriptive*. The definition must prescribe the exact and correct meaning of a word. Contrast the OED's descriptive definition of continuous with the definition of continuous found in a real analysis textbook.

> A function $f : A \to \mathbb{R}$ is **continuous at a point** $c \in A$ if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x-c| < \delta$ (and $x \in A$) it follows that $|f(x)-f(c)| < \varepsilon$. If f

[1] http://www.oed.com/view/Entry/166837
[2] http://www.oed.com/view/Entry/40280

is continuous at every point in the domain A, then we say that f is **continuous on** A.[3]

In mathematics there is very little freedom in definitions. Mathematics is a deductive theory; it is impossible to state and prove theorems without clear definitions of the mathematical terms. The definition of a term must completely, accurately, and unambiguously describe the term. Each word is chosen very carefully and the order of the words is critical. In the definition of continuity changing "there exists" to "for all," changing the orders of quantifiers, changing $<$ to $\leq$ or $>$, or changing $\mathbb{R}$ to $\mathbb{Z}$ would completely change the meaning of the definition.

What does this mean for you, the student? Our recommendation is that at this stage you memorize the definitions word-for-word. It is the safest way to guarantee that you have it correct. As you gain confidence and familiarity with the subject you may be ready to modify the wording. You may want to change "for all" to "given any" or you may want to change $|x-c| < \delta$ to $-\delta < x - c < \delta$ or to "the distance between x and c is less than δ."

Of course, memorization is not enough; you must have a conceptual understanding of the term, you must see how the formal definition matches up with your conceptual understanding, and you must know how to work with the definition. It is perhaps with the first of these that descriptive definitions are useful. They are useful for building intuition and for painting the "big picture." Only after days (weeks, months, years?) of experience does one get an intuitive feel for the epsilon-delta definition of continuity; most mathematicians have the "picking-up-the-pencil" definitions in their head. This is fine as long as we know that it is imperfect, and that when we prove theorems about continuous functions in mathematics we use the mathematical definition.

We end this discussion with an amusing real-life example in which a descriptive definition was not sufficient. In 2003 the German version of the game show *Who wants to be a millionaire?* contained the following question: "Every rectangle is: (a) a rhombus, (b) a trapezoid, (c) a square, (d) a parallelogram."

The confused contestant decided to skip the question and left with €4000. Afterward the show received letters from irate viewers. Why were the contestant and the viewers upset with this problem? Clearly a rectangle is a parallelogram, so (d) is the answer. But what about (b)? Is a rectangle a trapezoid? We would describe a trapezoid as a quadrilateral with a pair of parallel sides. But this leaves open the question: can a trapezoid have *two* pairs of parallel sides or must there only be *one* pair? The viewers said two pairs is allowed, the producers of the television show said it is not. This is a case in which a clear, precise, mathematical definition is required.

[3] This definition is taken from page 109 of Stephen Abbott's *Understanding Analysis*, but the definition would be essentially the same in any modern real analysis textbook.