Chapter 3

Permutations

A k-**permutation** of a set A is an injective function $w : [k] \to A$. The set of all k-permutations of A is denoted by $\boxed{S_{A,k}}$. If A happens to be the set $[n]$, we use the notation $\boxed{S_{n,k}}$. And if $n = k$, we write $\boxed{S_n := S_{n,n}}$ and refer to each n-permutation in S_n as a **permutation**. Let $\boxed{P(n,k) := |S_{n,k}|}$. By convention, $\boxed{P(n,0) = 1}$.

We can denote a k-permutation as string $w = w(1)w(2)\cdots w(k)$, where each entry $w(i)$ that appears in the string is unique (since w is an injection). In other words, we can think of a k-permutation as a linear ordered arrangement of k of n objects.

Problem 3.1. Complete the following.

(a) Write down all of the elements in S_3. What is $P(3,3)$?

(b) Write down all of the elements in $S_{4,3}$. What is $P(4,3)$?

Recall that for $n \in \mathbb{N}$, the **factorial** of n is defined $\boxed{n! := n \cdot (n-1) \cdots 2 \cdot 1}$, and we define $\boxed{0! := 1}$ for convenience.

Problem 3.2. Consider the collection of k-permutations in $S_{n,k}$ with $1 \leq k \leq n$. Explain why $P(n,k)$ is equal to the number of nonattacking rook arrangements on an $n \times k$ chess board. *Hint:* Establish a bijection between the collection of nonattacking rook arrangements on an $n \times k$ chess board and the collection of k-permutations.

Theorem 3.3. For $1 \leq k \leq n$, we have

$$\boxed{P(n,k) = n \cdot (n-1) \cdots (n+1-k) = \frac{n!}{(n-k)!}.}$$

Note that as a special case of the formula above, we have $|S_n| = P(n,n) = n!$. For convenience, we can extend the formula above to obtain

$$\boxed{P(0,0) = \frac{0!}{(0-0)!} = 1} \quad \text{and} \quad \boxed{P(n,0) = \frac{n!}{(n-0)!} = 1.}$$

Problem 3.4. How many strings of length three are there using letters from $\{a, b, c, d, e, f, g\}$ if the letters in the string are not repeated?

Problem 3.5. There are 8 finalists at the Olympic Games 100 meters sprint. Assume there are no ties.

(a) How many ways are there for the runners to finish?

(b) How many ways are there for the runners to get gold, silver, bronze?

(c) How many ways are there for the runners to get gold, silver, bronze given that Usain Bolt is sure to get the gold medal?

Problem 3.6. If $1 \leq k \leq n$, prove that $\boxed{P(n, k) = P(n-1, k) + kP(n-1, k-1)}$, both using the formula in Theorem 3.3, and separately using the definition of k-permutations together with the bijection principle.

More coming soon...