The impediment to action advances action.
What stands in the way becomes the way.

Marcus Aurelius, Roman emperor

Chapter 6

Continuity

In this chapter, we will explore the concept of continuity, which you likely encountered in high school.

6.1 Introduction to Continuity

We begin with the definition of a specific type of function, namely one whose domain and range is a subset of the real numbers.

Definition 6.1. A **real function** is any function $f : A \to \mathbb{R}$ such that A is a nonempty subset of $\mathbb{R}$.

There are many equivalent definitions of continuity for real functions. We will take the following definition as our starting point and then develop several equivalent characterizations of continuity.

Definition 6.2. Suppose f is a real function such that $a \in \mathrm{Dom}(f)$. We say that f is **continuous at** a if for every bounded open interval I containing $f(a)$, there is a bounded open interval J containing a such that if $x \in \mathrm{Dom}(f) \cap J$, then $f(x) \in I$. If f is continuous at every point in $B \subseteq \mathrm{Dom}(f)$, then we say that f is **continuous on** B. If f is continuous on the entire domain, we simply say that f is **continuous**.

Loosely speaking, a real function f is continuous at the point $a \in \mathrm{Dom}(f)$ if we can get $f(x)$ arbitrarily close to $f(a)$ by considering all $x \in \mathrm{Dom}(f)$ sufficiently close to a. The interval I is indicating how close to $f(a)$ we need to be while the interval J is providing the "window" around a needed to guarantee that all points in the window (and in the domain) yield outputs in I. Figure 6.1 illustrates our definition of continuity. Note that in the figure, the point a is fixed while we need to consider all $x \in \mathrm{Dom}(f) \cap J$. The dashed box in the figure has dimensions the length of J by the length of I. Intuitively, the function is continuous at a since given I, we can find J so that the graph of the function never exits the top or bottom of the dashed box.

Perhaps you have encountered the phrase "a function is continuous if you can draw its graph without lifting your pencil." While this description provides some intuition about what continuity of a function means, it is neither accurate nor precise enough to capture the meaning of continuity.

Let's show that our definition of continuity behaves the way we expect.

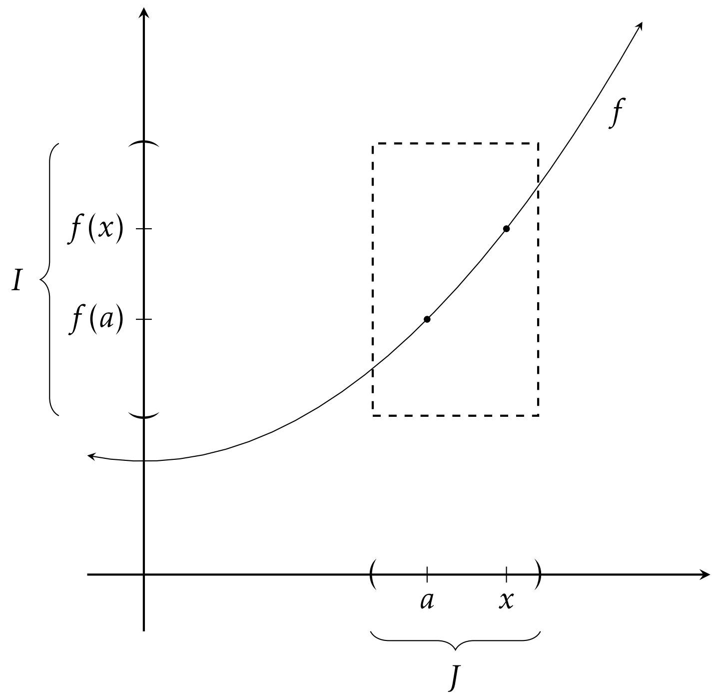

Figure 6.1: Visual representation of continuity of f at a.

Problem 6.3. Prove that each of the following real functions is continuous using Definition 6.2.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x$.

(b) $g : \mathbb{R} \to \mathbb{R}$ defined via $g(x) = 2x$.

(c) $h : \mathbb{R} \to \mathbb{R}$ defined via $h(x) = x + 3$.

Problem 6.4. Prove that every linear function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = mx + b$ is continuous.

After completing the next problem, reflect on the statement "a function is continuous if you can draw its graph without lifting your pencil."

Problem 6.5. Define $f : \mathbb{N} \to \mathbb{R}$ via $f(x) = 1$. Notice the domain! Determine where f is continuous and justify your assertion.

The next problem illustrates that the order in which the sets I and J are considered in the definition of continuity is crucial.

Problem 6.6. Find an example of a real function f that satisfies each of the following:

(i) $1 \in \text{Dom}(f)$;

(ii) For any open interval J containing 1, there is an open interval I containing $f(1)$ such that if $x \in \text{Dom}(f) \cap J$, then $f(x) \in I$;

(iii) f is not continuous at 1.

The obstacle is the path.

Zen saying, Author Unknown

6.2 Additional Characterizations of Continuity

The next problem tells us that we can reframe continuity in terms of distance.

Problem 6.7. Suppose f is a real function such that $a \in \text{Dom}(f)$. Prove that f is continuous at a if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in \text{Dom}(f)$ and $|x-a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

The previous characterization is typically referred to as the "$\varepsilon - \delta$ definition of continuity", although for us it is a theorem instead of a definition. This characterization is used as the definition of continuity in metric spaces.

Problem 6.8. Draw a figure in the spirit of Figure 6.1 that captures the essence of Problem 6.7

When approaching the next three problems, either utilize Definition 6.2 or Problem 6.7.

Problem 6.9. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Determine where f is continuous and justify your assertion.

Problem 6.10. Define $g : \{0\} \to \mathbb{R}$ via $g(0) = 0$. Prove that g is continuous at 0.

Problem 6.11. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Determine where f is continuous and justify your assertion.

Problem 6.12. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$. Prove that f is continuous.

Problem 6.13. Find a continuous real function f and an open interval I such that the preimage $f^{-1}(I)$ is not an open interval.

Problem 6.14. Suppose f is a real function. Prove that f is continuous if and only if the preimage $f^{-1}(U)$ of every open set U is an open set intersected with the domain of f.

The previous characterization of continuity is often referred to as the "open set definition of continuity" and is the definition used in topology. Since every open set is the union of bounded open intervals (Definition 4.1), the union of open sets is open (Problem 4.6), and preimages respect unions (Problem 2.55), we can strengthen Problem 6.14 into a slightly more useful result.

Problem 6.15. Suppose f is a real function. Prove that f is continuous if and only if the preimage $f^{-1}(I)$ of every bounded open interval I is an open set intersected with the domain of f.

Problem 6.16. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ via $f(x) = \frac{1}{x}$. Determine where f is continuous and justify your assertion.

The previous problem once again calls into question the phrase "a function is continuous if you can draw its graph without lifting your pencil."

It turns out that there is a deep connection between continuity and sequences!

Definition 6.17. Suppose f is a real function such that $a \in \text{Dom}(f)$. We say that f is **sequentially continuous** at a if, for every sequence $(p_n)_{n=1}^{\infty}$ in the domain of f converging to a, the sequence $(f(p_n))_{n=1}^{\infty}$ converges to $f(a)$.

Problem 6.18. Suppose f is a real function such that $a \in \text{Dom}(f)$. Prove that f is continuous at a if and only if f is sequentially continuous at a.

The upshot of the previous problem is that the notions of being *continuous at a point* and *sequentially continuous at a point* are equivalent on the real numbers. However, there are contexts in mathematics where the two are not equivalent. This is a topic in a branch of mathematics called **topology**. If you want to know more, check out the following YouTube video:

https://www.youtube.com/watch?v=sZ5fBHGYurg

The sequential way of thinking of continuity often makes proving some basic facts concerning continuity easier.

At this point, we have four different ways of thinking about continuity.

- Definition 6.2 using open intervals.
- Problem 6.7 using ε and δ.
- Problem 6.14/Problem 6.15 using preimages of open sets.
- Problem 6.18 using sequential continuity.

You should take the time to review each one. For the remainder of the book, feel free to use whichever characterization suits your needs.

Problem 6.19. Suppose f and g are real functions that are continuous at a and let $c \in \mathbb{R}$. Prove that each of the following functions is also continuous at a.

(a) cf

(b) $f+g$

(c) $f-g$

(d) fg

Problem 6.20. Prove that every polynomial is continuous on all of $\mathbb{R}$.

> The most difficult thing is the decision to act.
> The rest is merely tenacity.
>
> Amelia Earhart, aviation pioneer

6.3 Extreme Value Theorem

Problem 6.21. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and consider the closed interval $[a,b]$. Is the image $f([a,b])$ always a closed interval? If so, prove it. Otherwise, provide a counterexample.

Problem 6.22. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and K is a compact set, then the image $f(K)$ is compact.

The next result tells us that continuous functions always attain a maximum value on closed intervals. Of course, we have an analogous result involving minimums.

Problem 6.23 (Extreme Value Theorem)**.** Suppose f is a real function and let $I = [a,b]$ be a closed interval. Prove that if f is continuous on I, then there exists $x_M \in I$ such that $f(x_M) \geq f(x)$ for all $x \in I$.

Problem 6.24. Is the hypothesis that I is closed needed in the Extreme Value Theorem? Justify your assertion.

Problem 6.25. Is the converse of the Extreme Value Theorem true? That is, if a function attains a maximum value over a closed interval, does that imply that the function is continuous. If so, prove it. Otherwise, provide a counterexample.

Problem 6.26. Let $f : [0,1] \to \mathbb{R}$ and assume that the image $f([0,1])$ has a supremum. Prove that there is a sequence of points $(p_n)_{n=1}^{\infty}$ in $[0,1]$ such that $(f(p_n))_{n=1}^{\infty}$ converges to that supremum. Does this show that f is continuous on $[0,1]$?

> God created infinity, and man, unable to understand infinity, had to invent finite sets.
>
> Gian-Carlo Rota, mathematician & philosopher

6.4 Intermediate Value Theorem

The next problem is analogous to Problem 6.22. It also likely captures your intuition about continuity from high school and calculus.

Problem 6.27. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and C is a connected set, then the image $f(C)$ is connected.

The next result is a special case of the well-known **Intermediate Value Theorem**, which states that if f is a continuous real function whose domain contains the bounded closed interval $[a,b]$, then f attains every value between $f(a)$ and $f(b)$ at some point within the interval $[a,b]$. To prove the special case, utilize Problems 4.44 and Problem 6.27 together with a proof by contradiction.

Problem 6.28. Suppose f is a real function and let $I = [a,b]$ be a closed interval. Prove that if f is continuous on I such that $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, then there exists $r \in I$ such that $f(r) = 0$.

If we generalize the previous result, we obtain the Intermediate Value Theorem.

Problem 6.29 (Intermediate Value Theorem)**.** Suppose f is a real function and let $I = [a,b]$ be a closed interval. Prove that if f is continuous on I such that $f(a) < c < f(b)$ or $f(a) > c > f(b)$, then there exists $r \in I$ such that $f(r) = c$.

Problem 6.30. Is the converse of the Intermediate Value Theorem true? If so, prove it. Otherwise, provide a counterexample.

Problem 6.31. Let $f : [0,1] \to \mathbb{R}$ be a function such that $f(0) = -1$, $f(1) = 1$, and $f([0,1]) = \{-1, 1\}$. Prove that there exists $a \in [0,1]$ such that f is not continuous at a.

> You will become clever through your mistakes.
>
> German Proverb

6.5 Uniform Continuity

In $\varepsilon - \delta$ characterization of continuity at a given in Problem 6.7, each choice of δ depends on both ε and a. In the next definition, the choice of δ only depends on ε and is independent of $a \in \mathrm{Dom}(f)$.

Definition 6.32. Suppose f is a real function. We say that f is **uniformly continuous** if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \operatorname{Dom}(f)$ and $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Continuity itself is a *local property* of a real function. That is, a real function f is either continuous or not at a particular point. Whether f is continuous at a particular point can be determined by looking only at the values of the function "near" that specific point. In this case, the choice of δ may depend on the specific point in question. When we speak of a function being continuous on a set, we mean that it is continuous at each point of the set. In contrast, uniform continuity is a *global property* of f in the sense that the definition refers to pairs of points rather than individual points, and the choice of δ only depends on ϵ.

Loosely speaking, a function is uniformly continuous if and only if there is a one-size fits all δ for each ϵ. In other words, a function is uniformly continuous if for every $x \in \operatorname{Dom}(f)$, the graph of f never exits the top or bottom of the rectangle of dimensions 2δ by 2ε centered at $(x, f(x))$ on the graph of f.

Problem 6.33. Let f be a real function. Prove that if f is uniformly continuous, then f is continuous.

Problem 6.34. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = 5x - 3$. Prove that f is uniformly continuous.

Problem 6.35. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$. Prove that f is not uniformly continuous.

Problem 6.36. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = \sqrt{x}$. Prove that f is uniformly continuous.

Problem 6.37. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ via $f(x) = \frac{1}{x}$. Determine whether f is uniformly continuous.

The next result tells us that every function that is continuous on a compact set is uniformly continuous.

Problem 6.38. Let $f : K \to \mathbb{R}$ be a continuous real function such that K is compact. Prove that f is uniformly continuous.

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.
>
> Eugene Paul Wigner, theoretical physicist