Homework 4

Combinatorics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on our Discord server. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

1. For $n \in \mathbb{N}$, a **set partition** of $[n]$ is a collection of nonempty disjoint subsets of $[n]$ whose union is $[n]$. Each subset in the set partition is called a **block**. We define the **Stirling numbers** (of the second kind) via

$$\left\{ {n \atop k} \right\} := \text{number of set partitions of } [n] \text{ with } k \text{ blocks.}$$

I usually pronounce this as "n Stirling k". For example, there are 7 set partitions of $[4]$ with 2 blocks, namely:

$$\{\{1,2,3\},\{4\}\}, \{\{1,2,4\},\{3\}\}, \{\{1,3,4\},\{2\}\}, \{\{2,3,4\},\{1\}\}, \{\{1,2\},\{3,4\}\}, \{\{1,3\},\{2,4\}\}, \{\{1,4\},\{2,3\}\}.$$

Hence $\left\{ {4 \atop 2} \right\} = 7$. Notice that the Stirling numbers form an array like Pascal's triangle.

   (a) Explain why $\left\{ {n \atop 1} \right\} = 1 = \left\{ {n \atop n} \right\}$.

   (b) Prove that the Stirling numbers satisfy the following "Pascal-like" recurrence for $1 \leq k \leq n$:

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}.$$

   (c) Prove that $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$.

(d) Prove that $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$.

2. We define the nth **Bell number** B_n to be the total number of set partitions of $[n]$. Notice that the nth Bell number is the sum of Stirling numbers where n is fixed and k varies from 1 to n. That is,

$$B_n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\}.$$

This says the sum of the entries in the nth row of the array for the Stirling numbers is the nth Bell number. This is analogous to how the sum of the nth row in Pascal's triangle is a power of 2. If we take $B_0 = 1$, prove that for $n \geq 1$, we have

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k.$$

Hint: There is a block containing 1. Consider cases based on how many elements this block contains. Let k be the number of elements that are not in the block containing 1.

3. It's Halloween and $k \in \mathbb{N}$ students arrive at my office begging for candy. I happen to have n pieces of candy with $1 \leq k \leq n$. Depending on my mood, I may give away none of the candy, all of the candy, or any amount in between. Answer each of the following.

(a) How many ways can I distribute the candy if each piece of candy is distinct (e.g., Snickers, Milkyway, etc)?

(b) How many ways can I distribute the candy if all the candy is identical?

(c) If each piece of candy is distinct and I give each student at most one piece of candy, how many ways can I distribute the candy?

(d) If all of the candy is identical and I give each student at most one piece of candy, how many ways can I distribute the candy?

(e) If each piece of candy is distinct and I give each student at least one piece of candy, how many ways can I distribute the candy?

(f) If all of the candy is identical and I give each student at least one piece of candy, how many ways can I distribute the candy?

4. Find the power series expansions for $\sin(t)$ and $\cos(t)$. It is not important whether you do this by hand (using Taylor's Theorem) or just look them up.

(a) What sequences have these as ordinary generating functions?

(b) What sequences have these as exponential generating functions?

5. Find the sequences that are defined by the following (ordinary) generating functions.

(a) $\dfrac{1}{1-2t}$

(b) $\dfrac{1}{1-5t+6t^2}$

Hint: One approach for part (b) is to use a partial fraction decomposition. Alternatively, you could use the formula for the product of two sums together with the following fact that you may freely use:

$$(x-y)(x^k + x^{k-1}y + x^{k-2}y^2 + \cdots + xy^{k-1} + y^k) = x^{k+1} - y^{k+1}.$$