Pattern-avoiding Cayley permutations via combinatorial species

UofA Mathematical Physics & Probability Seminar

Dana C. Ernst
Northern Arizona University
Fall 2024

Joint work with G. Cerbai, A. Claesson, H. Golab

Combinatorial species

Big Picture

- The theory of combinatorial species is a method for counting "labeled structures" (e.g., permutations, linear orders, graphs). Introduced by A. Joyal in 1980.
- We have taken a species-first approach to enumerating pattern-avoiding Cayley permutations.

$\mathbb{B}$-species

Definition

A $\mathbb{B}$-species (or simply species) F is a rule that produces

- For all finite sets U, a finite set $F[U]$;
- For all bijections $\sigma : U \to V$, a function $F[\sigma] : F[U] \to F[V]$, where

$$
\begin{tikzcd}
U \arrow[r, "\sigma"] \arrow[rr, bend left, "\tau \circ \sigma"] \arrow[d, "F"'] & V \arrow[r, "\tau"] \arrow[d, "F"] & W \arrow[d, "F"] \\
F[U] \arrow[r, "F[\sigma]"] \arrow[rr, bend right] & F[V] \arrow[r, "F[\tau]"] & F[W]
\end{tikzcd}
$$

$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$$

$$
\begin{tikzcd}
U \arrow[r, "\mathrm{id}_U"] \arrow[d, "F"'] & U \arrow[d, "F"] \\
F[U] \arrow[r] & F[U]
\end{tikzcd}
$$

$$F[\mathrm{id}_U] = \mathrm{id}_{F[U]}$$

$\mathbb{B}$-species (continued)

Comments

- In the language of category theory, a $\mathbb{B}$-species is a functor $F : \mathbb{B} \to \mathbb{B}$, where $\mathbb{B}$ is the category of finite sets with bijective functions as morphisms.
- Each $s \in F[U]$ is called an F-structure on U.
- The function $F[\sigma]$ is called the transport of F-structures (along σ).
- One consequence of the definition is that the number of F-structures on U only depends on $|U|$.
- For the rest of this talk, I will mostly ignore the transport of structure in an attempt to simplify the discussion.

$\mathbb{B}$-species (continued)

Examples

- E: sets;
- E_{even}: sets of even cardinality;
- E_{odd}: sets of odd cardinality;
- X: singletons;
- 1: characteristic of empty set;
- L: linear orders;
- $\mathcal{S}$: permutations;
- $\mathcal{C}$: cyclic permutations;
- Par: set partitions;
- Bal: ballots.

Definition

We define the exponential generating series of the species F to be the formal power series

$$F(x) = \sum_{n \geq 0} |F[n]| \frac{x^n}{n!},$$

where $[n] = \{1, 2, \ldots, n\}$ for $n \geq 0$ with $[0] = \emptyset$.

$\mathbb{B}$-species (continued)

Sets

The species E of sets has E-structures given by

$$E[U] = \{U\}.$$

That is, for any finite set U there is exactly one E-structure, namely the set U itself. This implies that

$$E(x) = \boxed{1}\frac{x^0}{0!} + \boxed{1}\frac{x^1}{1!} + \boxed{1}\frac{x^2}{2!} + \boxed{1}\frac{x^3}{3!} + \boxed{1}\frac{x^4}{4!} + \cdots = e^x.$$

Nonempty Sets

The species E_+ of nonempty sets is given by

$$E_+[U] = \begin{cases} \{U\}, & \text{if } U \neq \emptyset \\ \emptyset, & \text{otherwise.} \end{cases}$$

It follows that

$$E_+(x) = \boxed{0}\frac{x^0}{0!} + \boxed{1}\frac{x^1}{1!} + \boxed{1}\frac{x^2}{2!} + \boxed{1}\frac{x^3}{3!} + \boxed{1}\frac{x^4}{4!} + \cdots = e^x - 1$$

Linear Orders

We identify a linear order of finite set U with a bijection $f : [n] \to U$, where $|U| = n$. Using one-line notation: $f(1) \cdots f(n)$.

The species L of linear orders has structures given by

$$L[U] = \{f : [n] \to U \mid f \text{ bijection}\}.$$

Since $|L[U]| = n!$, we obtain

$$L(x) = \boxed{1}\frac{x^0}{0!} + \boxed{1}\frac{x^1}{1!} + \boxed{2}\frac{x^2}{2!} + \boxed{6}\frac{x^3}{3!} + \boxed{24}\frac{x^4}{4!} + \cdots = \frac{1}{1-x}.$$

Permutations

The species $\mathcal{S}$ of permutations has structures defined via

$$\mathcal{S}[U] = \{f : U \to U \mid f \text{ bijection}\}.$$

Not surprisingly, we have

$$\mathcal{S}(x) = \boxed{1}\frac{x^0}{0!} + \boxed{1}\frac{x^1}{1!} + \boxed{2}\frac{x^2}{2!} + \boxed{6}\frac{x^3}{3!} + \boxed{24}\frac{x^4}{4!} + \cdots = \frac{1}{1-x}.$$

Isomorphic species

Definition

Two species F and G are isomorphic if there is a family of bijections

$$\alpha_U : F[U] \to G[U]$$

such that for any bijection $\sigma : U \to V$ between two finite sets, the following diagram commutes:

$$\begin{tikzcd} F[U] \arrow[r, "\alpha_U"] \arrow[d, "{F[\sigma]}"'] & G[U] \arrow[d, "{G[\sigma]}"] \\ F[V] \arrow[r, "\alpha_V"'] & G[V] \end{tikzcd}$$

Isomorphic species (continued)

Comments

- In the language of category theory, F and G are isomorphic if and only if there exists a natural isomorphism between functors F and G.
- We consider two species as equal if they are isomorphic and write $F = G$ for both concepts.
- Clearly, if $F = G$, then $F(x) = G(x)$.
- Converse false for $\mathbb{B}$-species:

$$L \neq \mathcal{S} \text{ despite } L(x) = \mathcal{S}(x).$$

Operations on $\mathbb{B}$-species

We can build new species with operations on previously known species.

Sum of species

For species F and G, an $(F+G)$-structure is either an F-structure or a G-structure:

$$(F+G)[U] = F[U] \sqcup G[U].$$

It is easy to see that

$$(F+G)(x) = F(x) + G(x).$$

Operations on $\mathbb{B}$-species (continued)

Product of species

For species F and G, an $(F \cdot G)$-structure on a finite set U is a pair (s, t) such that s is an F-structure on a subset $U_1 \in U$ and t is a G-structure on $U_2 = U \setminus U_1$:

$$(F \cdot G)[U] = \bigsqcup_{(U_1, U_2)} F[U_1] \times G[U_2],$$

where $U = U_1 \sqcup U_2$. One can show that

$$(F \cdot G)(x) = F(x)G(x).$$

Operations on $\mathbb{B}$-species (continued)

Subsets

The species $\mathcal{P}$ of subsets is given by

$$\mathcal{P} = E \cdot E.$$

We see that

$$\mathcal{P}(x) = \sum_{n \geq 0} 2^n \frac{x^n}{n!} = e^{2x} = e^x e^x = E(x)E(x),$$

as expected.

Composition of species

For species F and G, an $(F \circ G)$-structure is a generalized partition in which each block of a partition carries a G-structure and blocks are structured by F. Formally, if F and G are two species such that $G[\emptyset] = \emptyset$, we define

$$(F \circ G)[U] = \bigsqcup_{\beta=\{B_1,\ldots,B_k\}} F[\beta] \times G[B_1] \times \cdots \times G[B_k],$$

where $\beta = \{B_1, \ldots, B_k\}$ is a partition of U. It turns that

$$(F \circ G)(x) = F(G(x)).$$

Operations on $\mathbb{B}$-species (continued)

Ballots

A ballot (or ordered set partition) on a finite set U is a sequence of sets $(B_1, B_2, \ldots, B_k)$, where each B_i is a nonempty subset (called block) of U, $B_i \cap B_j \neq \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^k B_i = U$.

The species Bal of ballots is given by the composition

$$\mathrm{Bal} = L(E_+).$$

It follows that

$$\mathrm{Bal}(x) = \frac{1}{1-(e^x-1)} = \frac{1}{2-e^x} = \boxed{1}\frac{x^0}{0!} + \boxed{1}\frac{x^1}{1!} + \boxed{3}\frac{x^2}{2!} + \boxed{13}\frac{x^3}{3!} + \cdots$$

The coefficients of this series are known as the Fubini numbers:

$$|\,\mathrm{Bal}[n]| = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}.$$

Operations on $\mathbb{B}$-species (continued)

Derivative of a species

The derivative of a species F, denoted F', is defined via

$$F'[U] = F[U \sqcup \{\star\}].$$

In terms of generating series, we have

$$F'(x) = \frac{d}{dx}[F(x)].$$

Operations on $\mathbb{B}$-species (continued)

Linear orders

We claim that

$$L' = L^2.$$

Combinatorially, each L'-structure is simply an L-structure preceding $\star$ followed by another L-structure. That is, the derivative of a linear order just separates the linear order into two linearly ordered components. For example:

$$41 \star 5263 \mapsto (41, 5263).$$

Consequently, we have

$$L'(x) = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2} = L^2(x).$$

Operations on $\mathbb{B}$-Species (continued)

Pointed species

For a species F, we define the species $F^\bullet$, called F-pointed, via

$$F^\bullet[U] = F[U] \times U.$$

That is, an $F^\bullet$-structure on U is a pair (s, u), where s is an F-structure on U and $u \in U$ is a distinguished element that we can think of as being "pointed at".

The operations of pointing and differentiation are related by

$$F^\bullet = X \cdot F',$$

where $X = E_1$ is the species of singleton sets. Further, we have

$$|F^\bullet[n]| = n|F[n]|.$$

$\mathbb{L}$-Species

Definition

An $\mathbb{L}$-species is similar to a $\mathbb{B}$-species, but where the underlying set is linearly ordered.

Formally, an $\mathbb{L}$-species is a rule F that produces

- For each finite totally ordered set ℓ, a finite set $F[\ell]$;
- For each order preserving bijection $\sigma : \ell_1 \to \ell_2$, a bijection $F[\sigma] : F[\ell_1] \to F[\ell_2]$ such that $F[\sigma \circ \tau] = F[\sigma] \circ F[\tau]$ for all order preserving bijections $\sigma : \ell_1 \to \ell_2$, $\tau : \ell_2 \to \ell_3$, and $F[\mathrm{id}_\ell] = \mathrm{id}_{F[\ell]}$.

$\mathbb{L}$-Species (continued)

Comments

- Any $\mathbb{B}$-species F produces an $\mathbb{L}$-species:

$$F[(U, \preceq)] = F[U],$$

  for any totally ordered set $\ell = (U, \preceq)$, where the transport of structure is obtained by restriction to order-preserving bijections.
- Two $\mathbb{L}$-species F and G are isomorphic if there is a family of bijections

$$\alpha_\ell : F[\ell] \to G[\ell],$$

  for each totally ordered set ℓ that commutes with the transports of structures.

Comments (continued)

- For an $\mathbb{L}$-species F, the associated generating series is defined in the same way as for $\mathbb{B}$-species:

$$F(x) = \sum_{n \geq 0} |F[n]| \frac{x^n}{n!}.$$

- For $\mathbb{L}$-species, since there is a unique order-preserving bijection between any two totally ordered sets with the same cardinality,

$$F = G \text{ iff } F(x) = G(x).$$

- In particular, it is possible for two nonisomorphic $\mathbb{B}$-species to become isomorphic when looked at as $\mathbb{L}$-species. For example, as $\mathbb{L}$-species, $L = \mathcal{S}$.

Operations on $\mathbb{L}$-Species

Operations on $\mathbb{B}$-species can be extended to $\mathbb{L}$-species while new operations also become possible.

Definition

- Derivative: $F'[\ell] = F[\ell \oplus 1]$
- Integral:
$$\left(\int F\right)[\ell] = \begin{cases} \emptyset, & \ell = \emptyset \\ F[\ell \setminus \{\max(\ell)\}], & \ell \neq \emptyset \end{cases}$$
- Ordinal Product: $(F \odot G)[\ell] = \sum_{\ell = \text{prefix} \oplus \text{suffix}} F[\text{prefix}] \times G[\text{suffix}]$
- Convolution: $F * G = F \odot X \odot G$

Operations on $\mathbb{L}$-Species (continued)

Comment

We have the following:

- $F'(x) = \dfrac{d}{dx} F(x)$
- $\left(\int F\right)(x) = \int_0^x F(t)dt$
- $(F * G)(x) = F(x) * G(x) = \int_0^x F(x-t)G(t)dt$

Cayley permutations

Definition

Informally, a Cayley permutation is a word w of positive integers such that if b appears in w, then all positive integers $a < b$ also appear in w.

Formally, we define Cay be the $\mathbb{B}$-species with structures

$$\mathrm{Cay}[U] = \big\{ w : U \to [n] \mid \mathrm{Img}(w) = [k] \text{ for some } k \leq n \big\},$$

where $|U| = n$.

Cayley permutations (continued)

Comment

For $w \in \mathrm{Cay}[n]$, we utilize one-line notation: $w = w_1 w_2 \cdots w_n$. In this case, we say that w is of length n.

Example

$\mathrm{Cay}[1] = \{1\}$;
$\mathrm{Cay}[2] = \{11, 12, 21\}$;
$\mathrm{Cay}[3] = \{111, 112, 121, 122, 123, 132, 211, 212, 213, 221, 231, 312, 321\}$.

Cayley permutations (continued)

For a finite set U, we define $\alpha_U : \mathrm{Cay}[U] \to \mathrm{Bal}[U]$ via

$$\alpha_U(w) = (B_1, B_2, \ldots, B_k), \text{ where } k = |\operatorname{Img}(w)| \text{ and } B_i = w^{-1}(\{i\}).$$

For example:

$$31211245 \in \mathrm{Cay}[8] \mapsto (\{2,4,5\},\{3,6\},\{1\},\{7\},\{8\}) \in \mathrm{Bal}[8].$$

This map is clearly bijective and hence $|\mathrm{Bal}[n]| = |\mathrm{Cay}[n]|$. Moreover, this family of maps is compatible with the corresponding transport of structures.

Proposition

- As $\mathbb{B}$-species, $\mathrm{Cay} = \mathrm{Bal}$.
- $\mathrm{Cay}(x) = \dfrac{1}{2 - e^x}$.
- $|\mathrm{Cay}[n]| = n$th Fubini number.

Pattern avoidance in Cayley permutations

Definition

- If a Cayley permutation $w = w_1 w_2 \cdots w_n \in \mathrm{Cay}[n]$ does not contain a subsequence $w_{i_1} w_{i_2} \cdots w_{i_k}$ that is order isomorphic to a sequence $p = p_1 p_2 \cdots p_k \in \mathrm{Cay}[k]$, we say that w avoids p.
- If P is a finite set such that each $p \in P$ is a Cayley permutation, then we say $w \in \mathrm{Cay}[n]$ avoids P if w avoids every $p \in P$.
- $\mathrm{Cay}(P)[n]$ is the set of Cayley permutations of length n that avoid P. This determines the $\mathbb{L}$-species $\mathrm{Cay}(P)$.

Pattern avoidance in Cayley permutations

Definition

There are 360 Cayley permutations of length 5 that are 112-avoiding.

- $11111, 13221 \in \mathrm{Cay}(112)[5]$
- $12324 \notin \mathrm{Cay}(112)[5]$ since the pattern 112 occurs in positions 2, 4, 5

There are 309 Cayley permutations of length 5 that avoid 111 and 112.

- $13321, 32211 \in \mathrm{Cay}(111, 112)[5]$
- $21232 \notin \mathrm{Cay}(111, 112)[5]$ since it contains the pattern 111 in positions 1, 3, 5, and happens to also contain the pattern 112 in positions 1, 3, 4

Summary for patterns of length 2 and pairs of length 2

<table>
<tr><th>Patterns</th><th>Species</th><th>Series</th><th>Enumeration ($n \geq 1$)</th><th>OEIS</th></tr>
<tr><td>11</td><td>L</td><td>$\frac{1}{1-x}$</td><td>$n!$</td><td>A000142</td></tr>
<tr><td>21</td><td rowspan="2">$1 + \int E^2 = E_{\text{even}} \cdot E$</td><td rowspan="2">$\cosh(x)e^x = \frac{e^{2x}+1}{2}$</td><td rowspan="2">$2^{n-1}$</td><td rowspan="2">A011782</td></tr>
<tr><td>12</td></tr>
<tr><td>11,12</td><td rowspan="3">E</td><td rowspan="3">e^x</td><td rowspan="3">1</td><td rowspan="3">A000012</td></tr>
<tr><td>11,21</td></tr>
<tr><td>12,21</td></tr>
</table>

212-avoiding Cayley permutations

Every nonempty 212-avoiding Cayley permutation is of the form

$$w_1 \cdots w_i m \cdots m w_j \cdots w_n,$$

where $m = \max(w)$. We have three cases:

- Empty permutation
- $\underbrace{\underbrace{w_1 \cdots w_i}_{\mathrm{Cay}(212)} \underbrace{m}_{X} \underbrace{m \cdots m}_{E}}_{\mathrm{Cay}(212) \odot X \odot E}$
- $\underbrace{(w_1 \cdots w_i \overset{\bullet}{w}_{i+1} \cdots w_k,}_{\mathrm{Cay}(212)} \underbrace{m,}_{X} \underbrace{m \cdots m)}_{E} \mapsto \underbrace{w_1 \cdots w_i m \cdots m w_j \cdots w_n}_{\mathrm{Cay}(212)^{\bullet} \odot X \odot E}$

This yields the functional species equation

$$\mathrm{Cay}(212) = 1 + \mathrm{Cay}(212) * E + \mathrm{Cay}(212)^{\bullet} * E$$

Using the convolution product in terms of the integral, we obtain

$$\mathrm{Cay}(212)(x) = \frac{x^2 - 2x + 2}{2(x-1)^2} = \boxed{1}\frac{x^0}{0!} + \boxed{1}\frac{x^1}{1!} + \boxed{3}\frac{x^2}{2!} + \boxed{12}\frac{x^3}{3!} + \boxed{60}\frac{x^4}{4!} + \cdots$$

Summary for patterns of length 3

Pattern	Species	Series	Enumeration ($n \geq 1$)	OEIS
111	$L(E_1 + E_2)$	$\dfrac{2}{2-2x-x^2}$	$n! \cdot \dfrac{(1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}}{2^{n+1}\sqrt{3}}$	A080599
212 121 112 211 221 122	Alt'	$\dfrac{x^2-2x+2}{2(x-1)^2}$	$\dfrac{(n+1)!}{2}$	A001710
123 321 213 312 231 132	?	?	$\sum_{j=0}^{n} (-1)^j 2^{n-j-1} \binom{n-j}{j} C_{n-j}$	A226316

$(112, 221)$-avoiding Cayley permutations

Every 112-avoiding and 221-avoiding Cayley permutation w is either a linear order or there is a repeated element, in which case

$$w = \underbrace{\underbrace{w_1 \cdots w_k}_{L^\bullet} \; \underbrace{a}_{X} \; \underbrace{aa \cdots a}_{E}}_{L^\bullet \odot X \odot E},$$

where the first repetition is equal to the pointed entry of $L^\bullet$; else an occurrence of 112 or 221 would be created. It follows that

$$\mathrm{Cay}(112, 221) = L + L^\bullet * E = L + (L - E) = 2L - E.$$

It follows that

$$\mathrm{Cay}(112, 221)(x) = \frac{2}{1-x} - e^x = \boxed{1}\frac{x^0}{0!} + \boxed{1}\frac{x^1}{1!} + \boxed{3}\frac{x^2}{2!} + \boxed{11}\frac{x^3}{3!} + \boxed{47}\frac{x^4}{4!} + \cdots$$

Summary for pairs of patterns of length 3

Patterns	Species	Series	Enumeration ($n \geq 1$)	OEIS
112,221 211,122	$2L - E$	$\dfrac{2}{1-x} - e^x$	$2n! - 1$	A020543
123,321	$1 + \int E \cdot \mathrm{Prim}(123, 321)'$	$6e^{2x} - (3x^2 + 4x + 7)e^x + 2$	$\begin{cases} 1, & n = 1 \\ 6 \cdot 2^n - 3n^2 + n - 7, & n \geq 2 \end{cases}$	
112,213 211,312 221,231 122,132	Species equation?	Involves Bessel function	Little Schröder #'s	A001003

Summary for pairs of patterns of length 3 (continued)

Patterns	Species	Series	Enumeration ($n \geq 1$)	OEIS
111,112				
111,211				
111,221	$L \cdot \mathrm{Der}$	$\dfrac{e^{-x}}{(1-x)^2}$	$\dfrac{!n}{n+1}$	A000255
111,122				
111,121				
111,212				
221,212				
122,212				
112,121	$1 + \int E \cdot L^2$	$1 + \int_0^x \dfrac{e^t}{(1-t)^2}\,dt$	?	A001339
211,121				
121,212				

Summary for pairs of patterns of length 3 (continued)

- There are $\binom{13}{2} = 78$ pairs of distinct patterns of length 3 that fall into 25 symmetry classes (each of size 1, 2, or 4).
- We have species and/or EGF for 7 symmetry classes (which merge into 5 Wilf classes).
- Another team of researchers (Bean et al.) have developed tools for finding OGFs for 12 symmetry classes (10 Wilf classes?). One of our symmetry classes appears to be in same Wilf class as two of their symmetry classes.
- We also know that two additional symmetry classes (one of which is tackled by Bean et al.) merge into a single Wilf class.
- 5 symmetry classes (5 distinct Wilf classes) unresolved, none of which have OEIS hits.
- Data suggests that there are 18 Wilf classes.