Homework 3

Combinatorics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions Discord. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

1. Consider a $1 \times n$ array of the numbers 1 through $n \geq 1$. Suppose we have tiles of size 1×1 and 1×2 such that the tiles cover exactly one and two numbers of our array, respectively. Let F_n denote the number of tilings.
   (a) Prove that $F_n = f_{n+1}$ (where f_n is the nth Fibonacci number given by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$).
   (b) Prove that for $m \geq 3$ and $n \geq 2$, we have $F_{m+n-1} = F_{m-2}F_{n-1} + F_{m-1}F_n$. *Hint:* By definition, the lefthand side counts the number of tilings of an array with $m+n-1$ entries. So, it suffices to show that the righthand side counts the same thing. Number the entries 1 through $m+n-1$, from left to right. Let $\mathcal{S}_m$ be the collection of tilings where there is a 1×2 tile covering the entries labeled by $m-1$ and m, and let $\mathcal{T}_m$ be the collection of tilings where this is not the case.
   (c) What does Part (b) tell us about the Fibonacci sequence?

2. Show that the Fibonacci numbers satisfy the following identity for $n \geq 0$:

$$f_{n+1} = \sum_{k \geq 0} \binom{n-k}{k}.$$

   Hint: There are at least two natural approaches. One method would be using Pascal's Recurrence. A second, more elegant method perhaps, would be to utilize a combinatorial argument with one of the compositions in Problem 5 on Homework 2.

3. Prove that $P(n,n) = P(n,k)P(n-k,n-k)$ by using the meaning of k-permutations and the bijection principle. *Hint:* The product principle gives us $|S_{n,k} \times S_{n-k}| = P(n,k)P(n-k,n-k)$. It suffices to describe a bijection $f : S_n \to S_{n,k} \times S_{n-k}$.

4. Prove that $P(n,k) = P(n-1,k) + kP(n-1,k-1)$ by using the meaning of k-permutations.

5. What are the alternating row sums in Pascal's Triangle? That is, for $n \geq 0$, find a formula for $\sum_{k=0}^{n}(-1)^k\binom{n}{k}$. Instead of using the Binomial Theorem, find a proof that either uses the meaning of $\binom{n}{k}$ or rearrange the sum and use the symmetry theorem (i.e., $\binom{n}{k} = \binom{n}{n-k}$). You might want to consider the cases for n odd versus n even separately.

6. Prove that for any k and m less than or equal to n, we have $\binom{n}{k} = \sum_{j=0}^{k}\binom{n-m}{j}\binom{m}{k-j}$. *Hint:* Split $[n]$ into two piles, say $A_m = \{1,\ldots,m\}$ and $B_m = \{m+1,\ldots,n\}$. For each $j \in \{0,\ldots,k\}$, count number of ways to get a k-subset by selecting the appropriate number from A_m and the appropriate number from B_m.

7. Use the Binomial Theorem to find a formula for each of the following:

   (a) $\displaystyle\sum_{k=0}^{n} 2^k\binom{n}{k}$

   (b) $\displaystyle\sum_{k=0}^{n} (-2)^k\binom{n}{k}$