Chapter 6

Differentiation

It's time for calculus!

Definition 6.1. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$. For real number D, we say that f has *derivative* D at the point a if the following two conditions hold:

1. The point a is an accumulation point of the domain of f.

2. If S is an open interval containing D, then there is an open interval T containing a such that if $t \in T$, $t \neq a$, and t is in the domain of f, then

$$\frac{f(t) - f(a)}{t - a} \in S.$$

In this case, we say that f is *differentiable* at a. If f does indeed have a derivative at some points in its domain, then the derivative of f is the function denoted by f', such that for each number x at which f is differentiable, $f'(x)$ is the derivative of f at x.

Note that the definition of derivative automatically excludes the kind of behavior we saw with continuous functions, where a function defined only at a single point was continuous.

Exercise 6.2. Explain why any function defined only on $\mathbb{Z}$ cannot have a derivative.

Exercise 6.3. Find and prove a formula for the derivative of $f(x) = 3$.

Problem 6.4. Find and prove a formula for the derivative of $g(x) = 2x - 5$.

The following problem provides an alternative definition for the derivative.

Problem 6.5. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$. Prove that f has derivative D at the point a if and only if the following two conditions hold:

1. The point a is an accumulation point of the domain of f.

2. If $\epsilon > 0$, then there exists $\delta > 0$ such that if t is in the domain of f and $|t - a| < \delta$, then

$$\left| \frac{f(t) - f(a)}{t - a} - D \right| < \epsilon.$$

Problem 6.6. Find the derivative of $h(x) = x^2 - x + 1$ at $x = 2$.

Problem 6.7. Find the derivative of $h(x) = x^2 + ax + b$ for any $a, b \in \mathbb{R}$.

Problem 6.8. If f is differentiable at x and $c \in \mathbb{R}$, show that the function cf also has a derivative at x and $(cf)'(x) = cf'(x)$.

Problem 6.9. If f and g are differentiable at x, show that the function $f + g$ also has a derivative at x and $(f + g)'(x) = f'(x) + g'(x)$.

The next problem tells us that differentiability implies continuity.

Problem 6.10. Show that if f has a derivative at $x = a$, then f is also continuous at $x = a$.

The next problems are the well-known Product and Quotient Rules for Derivatives. You will need to use Problem 6.10 in their proofs.

Problem 6.11. Suppose f and g are differentiable at x. Prove each of the following:

(a) The function fg is differentiable at x. Moreover, its derivative function is given by

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

(b) The function f/g is differentiable at x provided $g'(x) \neq 0$. Moreover, its derivative function is given by

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Definition 6.12. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$. The non-vertical line L is *tangent* to the function f at the point $P = (a, b)$ means that:

1. a is an accumulation point of the domain of f,
2. P is a point of L, and
3. if A and B are non-vertical lines containing P with the line L between them (except at P), then there are two vertical lines H and K with P between them such that if Q is a point of f between H and K which is not P, then Q is between A and B.

If L is tangent to f at P, we say that L is a *tangent line* to f at $x = a$.

In the previous definition we write that we have three distinct lines, A, B, and L with L between A and B (except at P). By this we mean that for any point l on L (except P) there is a point α on A and a point β on B so that either α is below l which is below β or that β is below l which is below α.

Exercise 6.13. Try to draw a picture that captures the definition of tangent line. Your picture should include f, a, $f(a)$, P, L, A, B, H, K, Q, α, and β.

Problem 6.14. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$ such that f has a tangent line at $x = a$. Prove that f does not have two tangent lines at the point $(a, f(a))$.

Problem 6.15. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = |x|$.

(a) Prove that f is continuous on all at all points in its domain.

(b) Prove that f has a (non-vertical) tangent line at all points in its domain except $x = 0$.

Problem 6.16. Use the definition of tangent to show that if f is a function whose domain includes $(-1,1)$, and for each number $x \in (-1,1)$, $-x^2 \leq f(x) \leq x^2$, then the x-axis is tangent to f at the point $(0,0)$.

Problem 6.17. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$. Prove that f has a derivative at $x = a$ if and only if f has a non-vertical tangent line at the point $(a, f(a))$.

The upshot of Problems 6.14 and 6.17 is that derivatives are unique when they exist.

Problem 6.18. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$ and suppose f has a derivative at $x = a$. Explain why $f'(a)$ is the slope of the line tangent to f at the point $(a, f(a))$.

In light of Problem 6.17, if a function f does not a tangent line or has a vertical tangent line at $x = a$, then f is not differentiable at $x = a$. Note that Problem 6.15 shows us that if a function f is continuous at $x = a$ may or may not be differentiable at $x = a$. This problem also illustrates that a function f and its derivative f' might not have the same domain.

More coming soon...