Homework 3

Combinatorics of Genome Rearrangements

You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in his or her own work. In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments with no questions asked**.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing assignments for this course, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.** Please read NAU's Academic Integrity Policy.

Complete the following problems.

1. Let R be the collection of adjacent transpositions in S_n and consider the metric d_R (defined in class and in Problem 3 on Homework 2).

   (a) Prove that d_R is left invariant on S_n.

   (b) Determine whether d_R is right invariant on S_n. Justify your assertion.

   (c) Define $d_R(\pi) := d_R(\pi, e)$. Prove that for all $\pi \in S_n$, $d_R(\pi) = \mathrm{inv}(\pi)$.

   In general, if R is a generating set for $W \leq S_n$, d_R is left invariant on W. You may use this general fact for the rest of the semester.

2. Let R be the collection of all transpositions in S_4 and consider the metric d_R on S_4. Draw the sorting graph for S_4 induced by R. Instead of using the skeleton for S_4 provided in Homework 2, draw your graph so that 1234 is at the bottom and permutations with larger distance from 1234 are higher in your drawing and two permutations with the same distance from 1234 are drawn on the same level. I'm essentially asking you to draw a nice Hasse diagram for S_4 as a ranked poset where the rank is the same as the distance from 1234.

3. For $\pi \in S_n$, define $\mathrm{cyc}(\pi)$ to be the number of cycles (including trivial cycles) in the cycle decomposition of π. For example, $\mathrm{cyc}(31542) = 2$ since 31542 has cycle decomposition $(1,3,5,2)(4)$. Now, let R be the collection of all transpositions in S_n and consider the metric d_R on S_n.

   (a) As in Problem 1(c), define $d_R(\pi) := d_R(\pi, e)$. Prove that $d_R(\pi) = n - \mathrm{cyc}(\pi)$ for all $\pi \in S_n$.

   (b) What is $\max\{d_R(\pi) \mid \pi \in S_n\}$? That is, what is the maximum value that $d_R(\pi)$ takes on?

   (c) What is the diameter of the sorting graph?

   (d) Which permutations attain maximal distance from the identity e?

   (e) How many permutations attain maximal distance from the identity e?

4. Choose any five permutations in S_5 and identify adjacencies, reverse adjacencies, breakpoints, strong breakpoints, and compute the breakpoint distance d_{sb} between each of your permutations and the identity 12345.

5. Is there a relationship between the breakpoints in π, the breakpoints in π^{-1}, or the breakpoints in $\text{rev}(\pi)$ (where $\text{rev}(\pi)$ is the permutation obtained by reversing the order of the one-line notation for π)? Is there is a relationship between $\text{bp}(\pi)$, $\text{bp}(\pi^{-1})$, or $\text{rev}(\pi)$? Justify your assertion. Address the same questions for strong breakpoints.

6. Determine whether $d_{\text{sb}}(\pi, e) = \text{sb}(\pi)$. Justify your assertion.