Homework 9

Combinatorial Game Theory

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on Discord. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Let's establish the following notation:

- $\mathsf{SSC}\,1(n)$:= Simplified Sylver Coinage Version 1 played on $[n]$;
- $\mathsf{SSC}\,2(n)$:= Simplified Sylver Coinage Version 2 played on $[n]$.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

1. Consider $\mathsf{SSC}\,1(n)$ for $n \geq 2$.
   (a) Prove that if $\text{nim}(\mathsf{SSC}\,1(n)) = 0$, then $\text{nim}(\mathsf{SSC}\,1(n+1)) \neq 0$.
   (b) Draw the minimum quotient gamegraph for $\mathsf{SSC}\,1(5)$ and label each congruence class with the appropriate nim-value.
   (c) Draw the minimum quotient gamegraph for for either $\mathsf{SSC}\,1(4)$ or $\mathsf{SSC}\,1(6)$ and label each congruence class with the appropriate nim-value.
   (d) Any conjectures regarding the outcome of $\mathsf{SSC}\,1(n)$? Explain why you believe your conjecture.
   (e) Any conjectures regarding the spectrum of nim-values for $\mathsf{SSC}\,1(n)$? And/or any conjectures for what the nim-value of $\mathsf{SSC}\,1(n)$ might be?
   (f) (Optional) Any other observations not mentioned elsewhere?

2. Consider $\mathsf{SSC}\,2(n)$ for $n \geq 2$.
   (a) Describe the terminal positions of $\mathsf{SSC}\,2(n)$.

(b) (Temporary note: I might be confused about this. . . maybe the only *maximal* nongenerating set is $\{2, 3, \ldots, n\}$.) Explain why the collection of terminal positions of $\mathsf{SSC}\,2(n)$ does not form a Sperner family[1]. *Note:* The upshot is that unlike what I suggested in class, we cannot actually apply structure theory to $\mathsf{SSC}\,2(n)$ in the hopes of computing the nim-value of the game. Poop. This was discovered by Savannah, Ruth, and Hannah G.

(c) Draw the minimum quotient gamegraph for $\mathsf{SSC}\,2(5)$ and label each congruence class with the appropriate nim-value.

(d) Draw the minimum quotient gamegraph for $\mathsf{SSC}\,2(6)$ and label each congruence class with the appropriate nim-value.

(e) Determine the outcome for $\mathsf{SSC}\,2(n)$ by describing a winning strategy for the appropriate player.

(f) For which n do we have $\text{nim}(\mathsf{SSC}\,2(n)) = 0$? For which n do we have $\text{nim}(\mathsf{SSC}\,2(n)) \neq 0$?

(g) Any conjectures regarding the spectrum of nim-values for $\mathsf{SSC}\,2(n)$? And/or any conjectures for what the nim-value of $\mathsf{SSC}\,2(n)$ might be?

(h) (Optional) Any other observations not mentioned elsewhere?

[1] Dr. Sieben introduced this term, but if you don't have it written down, then feel free to Google it.