Homework 6

Combinatorics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on our Discord server. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

1. The number of **readings** of a permutation w, denoted $\text{read}(w)$, is the number of times one must scan the one-line notation for w from left to right to find the numbers $1, 2, \ldots, n$ in order. For example, with $w = 1374265$, we encounter 1 and 2 on the first read; 3, 4, and 5 on the second read; 6 on the third read; and finally 7 on the fourth read. So, $\text{read}(w) = 4$. Prove that the number of readings of permutations in S_n is an Eulerian statistic.

2. Use Worpitzsky's identity to find a closed form for $\left\langle {n \atop 3} \right\rangle$.

3. Recall our definition of a barred permutation on n on Homework 5.

   (a) Let $n \geq 1$ and $k > 0$. Explain why the number of configurations of n labeled balls into $k + 1$ labeled boxes is $(k + 1)^n$.

   (b) Using part (a) above together with Homework 5, prove that

   $$\frac{S_n(t)}{(1 - t)^{n+1}} = \sum_{k \geq 0} (k + 1)^n t^k,$$

   where $S_n(t)$ is the nth Eulerian polynomial. This proves the Carlitz identity.

4. For a permutation w, an **inversion** is a pair $i < j$ such that $w(i) > w(j)$. The set of all inversions for w is denoted by

   $$\text{Inv}(w) := \{1 \leq i < j \leq n \mid w(i) > w(j)\}$$

and $\mathrm{inv}(w) := |\mathrm{Inv}(w)|$. Let $I_{n,k}$ denote the number of permutations in S_n with k inversions. Notice that the notation for $I_{n,k}$ is a departure from our usual style. In particular, we are using a capital letter to represent a number, not a set. The numbers $I_{n,k}$ are often called the **Mahonian numbers** (named after Percy MacMahon). As expected the Mahonian numbers form a triangular array. Let $I_n(t)$ denote the generating function for the Mahonian numbers with fixed n. That is, for each n, we define

$$I_n(t) := \sum_{k \geq 0} I_{n,k} t^k = \sum_{w \in S_n} t^{\mathrm{inv}(w)}.$$

Note: In light of part (b) below, the sum above is actually a finite sum (with an infinite number of zeros added on).

(a) Explain why $\mathrm{des}(w) \leq \mathrm{inv}(w)$ for all $w \in S_n$.

(b) Explain why $0 \leq \mathrm{inv}(w) \leq \binom{n}{2}$ for all $w \in S_n$.

(c) Prove that $\mathrm{inv}(w) = \mathrm{inv}(w^{-1})$ for all $w \in S_n$. Is $\mathrm{Inv}(w)$ always equal to $\mathrm{Inv}(w^{-1})$?

(d) What is the meaning of $I_n(1)$ and what is its value?

(e) Prove that $I_n(t) = (1 + t + \cdots + t^{n-1}) I_{n-1}(t)$.

(f) Explain why $I_n(t) = \prod_{i=1}^{n} (1 + t + \cdots + t^{i-1})$. *Food for thought:* This product is sometimes called a t-**factorial**. Do you see why? Perhaps compare with your answer for part (d).

(g) Explain why $I_n(t) = \dfrac{\prod_{i=1}^{n}(1 - t^i)}{(1 - t)^n}$. *Food for thought:* Why might you be bothered by this formula in light of part (d)?