Final Exam

Your Name:

Names of Any Collaborators:

Instructions

This exam is worth a total of 28 points and is worth 15% of your overall grade. Your exam is due by 9AM on **Friday, May 8**. When you have finished your exam, **please email me a single PDF** (instead of submitting to BbLearn). I'd like you to include the cover page, but if you are unable to do this, recreate a suitable replacement that includes your signature. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems/problems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 2.32, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

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The following problems are related to content from Chapter 8 of our textbook. Some of the problems are directly from the book. Note that you will need to independently digest some new material in Chapter 8. You may freely use any result that appears in any chapter as long as the result appears *before* the problem in question. In particular, you can freely use any result in Section 8.2.

1. (4 points) Prove **one** of the following.
   (a) (Theorem 8.32) Let A and B be sets such that A is countable. If $f : A \to B$ is a bijection, then B is countable.
   (b) (Theorem 8.33) Every subset of a countable set is countable.[1]
   (c) (Theorem 8.34) A set is countable if and only if it has the same cardinality of some subset of the natural numbers.
   (d) (Theorem 8.35) If $f : \mathbb{N} \to A$ is an onto function, then A is countable.
2. (4 points) Prove **one** of the following.
   (a) (Theorem 8.36) The set of rational numbers $\mathbb{Q}$ is countable.[2]
   (b) (Theorem 8.37) If A and B are countable sets, then $A \cup B$ is countable.
   (c) (Theorem 8.40) If A and B are countable sets, then $A \times B$ is countable.
   (d) (Theorem 8.41) The set of all finite sequences of 0's and 1's (e.g., 0110010) is countable.
3. (4 points) Prove **one** additional theorem from either Problem 1 or Problem 2 with one caveat: you cannot choose to do both of Problems 2(b) and 2(c).
4. (Problem 8.42) For sake of a contradiction, assume the interval $(0, 1)$ is countable. Then there exists a bijection $f : \mathbb{N} \to (0, 1)$. For each $n \in \mathbb{N}$, its image under f is some number in $(0, 1)$. Let $f(n) := 0.a_{1n}a_{2n}a_{3n}\ldots$, where a_{1n} is the first digit in the decimal form for the image of n, a_{2n} is the second digit, and so on. If $f(n)$ terminates after k digits, then our convention will be to continue the decimal form with 0's. Now, define $b = 0.b_1b_2b_3\ldots$, where

$$b_i = \begin{cases} 2, & \text{if } a_{ii} \neq 2 \\ 3, & \text{if } a_{ii} = 2. \end{cases}$$

   (a) (2 points) Prove that the decimal expansion that defines b above is in standard form.[3]
   (b) (2 points) Prove that for all $n \in \mathbb{N}$, $f(n) \neq b$.
   (c) (2 points) Prove that f is not onto.
   (d) (1 point) Explain why we have a contradiction.
   (e) (1 point) Explain why it follows that the open interval $(0, 1)$ cannot be countable.

   You have just proved that the interval $(0, 1)$ is uncountable!
5. (4 points) Prove **one** of the following.
   (a) (Theorem 8.44) If A and B are sets such that $A \subseteq B$ and A is uncountable, then B is uncountable.[4]

[1] *Hint:* Let A be a countable set. Consider the cases when A is finite versus infinite. The contrapositive of Corollary 8.25 should be useful for the case when A is finite.

[2] *Hint:* Make a table with column headings $0, 1, -1, 2, -2, \ldots$ and row headings $1, 2, 3, 4, 5, \ldots$. If a column has heading m and a row has heading n, then the corresponding entry in the table is given by the fraction m/n. Find a way to zig-zag through the table making sure to hit every entry in the table (not including column and row headings) exactly once. This justifies that there is a bijection between $\mathbb{N}$ and the entries in the table. Do you see why? Now, we aren't done yet because every rational number appears an infinite number of times in the table. Appeal to Theorem 8.33.

[3] See the paragraph above Problem 8.42 for the definition of *standard form*.

[4] *Hint:* Try a proof by contradiction. Take a look at Theorem 8.33.

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(b) (Theorem 8.46) If $f : A \to B$ is a one-to-one function and A is uncountable, then B is uncountable.

Problem 4 together with Problem 5(a) imply that the set $\mathbb{R}$ of real numbers is uncountable. In fact, one can prove that $\text{card}((0,1)) = \text{card}(\mathbb{R})$. One way to do this is toconsider the function $f : (0,1) \to \mathbb{R}$ defined via $f(x) = \tan(\pi x - \frac{\pi}{2})$. It turns out that this function is a bijection from the interval $(0,1)$ to $\mathbb{R}$.

6. (2 points) A reoccurring theme in this semester has been the importance of productive struggle in the learning process. Identify a problem or topic in this course that you found difficult initially, but that you eventually overcame. Describe how your struggle and perseverance were key to conquering this obstacle.

7. (2 points) Describe how your perceptions of learning, especially mathematics, have changed.

8. (2 points, Theorem 8.53) **Bonus Problem!** *This problem is optional, but it's all or nothing. You have to get the problem completely correct to get the points.* Prove that if A is a set, then $\text{card}(A) < \text{card}(\mathcal{P}(A))$.[5]

[5] *Hint:* Mimic Cantor's Diagonalization Argument for showing that the interval $(0,1)$ is uncountable.

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