Homework 2

Discrete Mathematics

Let's begin with a few reminders. Homework will consist of a mixture of the following:

- Problems that are modifications of examples we have discussed in class.
- Problems that extend concepts introduced in class.
- Problems that introduce new concepts not yet discussed in class.
- Problems that synthesize multiple concepts that we either introduced in class or in a previous homework problem.

Some homework problems will be straightforward while others are intended to be challenging. You should anticipate not knowing what to do on some of the problems at first glance. You may have several false starts. Some frustration, maybe even a lot of frustration, should be expected. This is part of the natural learning process. On the other hand, it is not my intention to leave you to fend for yourselves. I am here to help and I want to help. You are encouraged to seek assistance from your classmates (while adhering to the **Rules of the Game**) and from me. Please visit office hours and ask questions on our Discord server. I am always willing to give hints/nudges, so please ask.

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition.You are highly encouraged to seek out assistance by asking questions in our Q&A Discussion Board in Canvas. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers. In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. Consider the collection of bit strings (finite ordered lists of 0's and 1's) of length 7. Answer each of the following. **No detailed justification required**.

(a) How many bit strings of length 7 have exactly 4 consecutive 1s? *Hint:* There might be more than 4 occurrences of 1, but we can't have more than 4 consecutive 1s. Consider using a brute-force case analysis.

(b) How many bit strings of length 7 start and end with repeated symbols (e.g., 0010011, 1111000)?

2. The game of chess is played on an 8×8 grid. For this problem, we'll consider chessboards with arbitrary dimensions, say $k \times n$ (k rows, n columns) with $n \leq k$. A rook is a castle-shaped piece that can move horizontally or vertically any number of squares. In a typical chess game, there are two black rooks and two white rooks. For this problem, we will assume we have n rooks, where all the rooks are the same color. We say that an arrangement of rooks on an $k \times n$ chessboard is **non-attacking** if no two rooks lie in the same row or column. Answer each of the following. **No detailed justification required**.

   (a) How many different non-attacking rook arrangements are there on an 8×8 chessboard involving 8 rooks?

   (b) How many different non-attacking rook arrangements are there on an 8×8 chessboard involving 8 rooks if there is a rook in the bottom left corner?

   (c) How many different non-attacking rook arrangements are there on an $n \times n$ chessboard involving n rooks?

   (d) How many different non-attacking rook arrangements are there on an 8×6 chessboard involving 6 rooks?

   (e) How many different non-attacking rook arrangements are there on an $k \times n$ (with $n \leq k$) chessboard involving n rooks?

3. A **hexadecimal** is a string consisting of the symbols $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F$. Answer each of the following. **No detailed justification required**.

   (a) How many hexadecimals of length 5 are there?

   (b) How many hexadecimals of length 5 use only the letters A, B, C, D, E, F?

   (c) How many hexadecimals of length 5 use only the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$?

   (d) How many hexadecimals of length 5 contain *both* numbers and letters? *Hint:* Consider using subtraction.

4. I'm in the mood for more coffee! Assume I walk to the same coffee shop as in Problem 1.15 subject to the same constraints (coffee shop is 4 blocks East and 3 blocks North from start). However, this time there is construction at the intersection that is 3 blocks East and 2 blocks North from my starting location. Unfortunately, I need to avoid this intersection on my walk. How many routes can I take to get coffee if I avoid this intersection? For this problem, you should only rely on techniques we have already discussed. You will need to do some brute force. **Explain your solution.** *Hint:* Consider using the answer for Problem 1.15 together with subtraction.

5. For convenience, define $[n] := \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. For example, $[4] = \{1, 2, 3, 4\}$. A **set partition** of $[n]$ is a collection of nonempty disjoint subsets (i.e., no overlap amongst the subsets) of $[n]$ whose union is $[n]$. Each subset in the set partition is called a **block**. For

example, there are 7 set partitions of [4] with 2 blocks, namely:

$$\begin{array}{ll} \{\{1,2,3\},\{4\}\} & \{\{1,2,4\},\{3\}\} \\ \{\{1,3,4\},\{2\}\} & \{\{2,3,4\},\{1\}\} \\ \{\{1,2\},\{3,4\}\} & \{\{1,3\},\{2,4\}\} \\ \{\{1,4\},\{2,3\}\} & \end{array}$$

We define the **Stirling numbers** (of the second kind) via

$$\left\{ n \atop k \right\} := \text{number of set partitions of } [n] \text{ with } k \text{ blocks.}$$

I usually pronounce this as "n Stirling k". Based on the information above, we know $\left\{ 4 \atop 2 \right\} = 7$. Answer each of the following.

(a) Compute $\left\{ 4 \atop 3 \right\}$ via brute-force. That is, write down all of the set partitions of [4] with 3 blocks and then count how many you have. **No detailed justification required**.

(b) Compute $\left\{ 4 \atop 1 \right\}$ via brute-force. **No detailed justification required**.

(c) Compute $\left\{ 4 \atop 4 \right\}$ via brute-force. **No detailed justification required**.

(d) Explain why $\left\{ n \atop 1 \right\} = 1 = \left\{ n \atop n \right\}$ for every $n \in \mathbb{N}$. **Justify your solution.**

(e) We define the nth **Bell number** B_n to be the total number of set partitions of $[n]$ into any number of blocks. Explain why

$$B_n = \left\{ n \atop 1 \right\} + \left\{ n \atop 2 \right\} + \cdots + \left\{ n \atop n \right\} = \sum_{k=1}^{n} \left\{ n \atop k \right\}.$$

Justify your solution. *Note:* You don't need to worry about justifying the expression on the righthand side. That is simply meant to be a reminder about how "Sigma notation" works.

(f) Find B_4 using some of your previous answers.