

# Chapter 6

## Limits

We are now prepared to dig into limits, which you are likely familiar with from calculus. However, chances are that you were never introduced to the formal definition.

**Definition 6.1.** Let  $f : A \rightarrow \mathbb{R}$  be a function, where  $A \subseteq \mathbb{R}$ . The **limit** of  $f$  as  $x$  approaches  $a$  is  $L$  if the following two conditions hold:

1. The point  $a$  is an accumulation point of  $A$ , and
2. For every  $\epsilon > 0$  there exists a  $\delta > 0$  such that if  $x \in A$  and  $0 < |x - a| < \delta$ , then  $|f(x) - L| < \epsilon$ .

Notationally, we write this as

$$\lim_{x \rightarrow a} f(x) = L.$$

It turns out that limits are unique if they exist. You may assume this going forward.

**Problem 6.2.** Why do we require  $0 < |x - a|$  in Definition 6.1?

**Problem 6.3.** Why do you think we require  $a$  to be an accumulation point of the domain of  $f$ ? What happens if  $a \in A$  but  $a$  is not an accumulation point of  $A$  (such points are called **isolated points** of  $A$ )?

**Example 6.4.** It should come as no surprise to you that  $\lim_{x \rightarrow 5} (3x + 2) = 17$ . Let's prove this using Definition 6.1. First, notice that the default domain of  $f(x) = 3x + 2$  is the set of real numbers. So, any  $x$ -value we choose will be in the domain of the function. Now, let  $\epsilon > 0$ . Choose  $\delta = \epsilon/3$ . You'll see in a moment why this is a good choice for  $\delta$ . Suppose  $x \in \mathbb{R}$  such that  $0 < |x - 5| < \delta$ . We see that

$$|(3x + 2) - 17| = |3x - 15| = 3 \cdot |x - 5| < 3 \cdot \delta = 3 \cdot \epsilon/3 = \epsilon.$$

This proves the desired result.

**Example 6.5.** Let's try something a little more difficult. Let's prove that  $\lim_{x \rightarrow 3} x^2 = 9$ . As in the previous example, the default domain of our function is the set of real numbers. Our goal is to prove that for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $x \in \mathbb{R}$  such that

$0 < |x - 3| < \delta$ , then  $|x^2 - 9| < \epsilon$ . Let  $\epsilon > 0$ . We need to figure out what  $\delta$  needs to be. Notice that

$$|x^2 - 9| = |x + 3| \cdot |x - 3|.$$

The quantity  $|x - 3|$  is something we can control with  $\delta$ , but the quantity  $|x + 3|$  seems to be problematic.

To get a handle on what's going on, let's temporarily assume that  $\delta = 1$  and suppose that  $0 < |x - 3| < 1$ . This means that  $x$  is within 1 unit of 3. In other words,  $2 < x < 4$ . But this implies that  $5 < x + 3 < 7$ , which in turn implies that  $|x + 3|$  is bounded above by 7. That is,  $|x + 3| < 7$  when  $0 < |x - 3| < 1$ . It's easy to see that we still have  $|x + 3| < 7$  even if we choose  $\delta$  smaller than 1. That is, we have  $|x + 3| < 7$  when  $0 < |x - 3| < \delta \leq 1$ . Putting this altogether, if we suppose that  $0 < |x - 3| < \delta \leq 1$ , then we can conclude that

$$|x^2 - 9| = |x + 3| \cdot |x - 3| < 7 \cdot |x - 3|.$$

This work informs our choice of  $\delta$ , but remember our scratch work above hinged on knowing that  $\delta \leq 1$ . If  $\epsilon/7 \leq 1$ , we should choose  $\delta = \epsilon/7$ . However, if  $\epsilon/7 > 1$ , the easiest thing to do is to just let  $\delta = 1$ . Let's button it all up.

Let  $\epsilon > 0$ . Choose  $\delta = \min\{1, \epsilon/7\}$  and suppose  $0 < |x - 3| < \delta$ . We see that

$$|x^2 - 9| = |x + 3| \cdot |x - 3| < 7 \cdot |x - 3| < 7 \cdot \delta \leq \epsilon$$

since

$$7 \cdot \delta = \begin{cases} 7, & \text{if } \epsilon > 7 \\ 7 \cdot \epsilon/7, & \text{if } \epsilon \leq 7. \end{cases}$$

Therefore,  $\lim_{x \rightarrow 3} x^2 = 9$ , as expected.

**Problem 6.6.** Prove that  $\lim_{x \rightarrow 1} (17x - 42) = -25$  using Definition 6.1.

**Problem 6.7.** Prove that  $\lim_{x \rightarrow 2} x^3 = 8$  using Definition 6.1.

**Problem 6.8.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  via

$$f(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 17, & \text{if } x = 0. \end{cases}$$

Using Definition 6.1, prove that  $\lim_{x \rightarrow 0} f(x) = 0$ .

**Problem 6.9.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  via

$$f(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ -1, & \text{if } x > 0. \end{cases}$$

Using Definition 6.1, prove that  $\lim_{x \rightarrow 0} f(x)$  does not exist.

**Problem 6.10.** Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Using Definition 6.1, prove that  $\lim_{x \rightarrow a} f(x)$  does not exist for all  $a \in \mathbb{R}$ .

**Problem 6.11.** Let  $f : A \rightarrow \mathbb{R}$  be a function. Prove that if  $\lim_{x \rightarrow a} f(x)$  exists, then the limit is unique.

The  $\epsilon - \delta$  approach to a function  $f$  being continuous at  $x = a$  (see Problem 5.5) looks awfully similar to the definition of the limit of  $f$  as  $x$  approaches  $a$ . Let's explore this a bit.

**Problem 6.12.** Explain the similarities and differences between the definitions of continuity at  $x = a$  versus the limit as  $x$  approaches  $a$ . State a theorem about continuity involving limits. You will have to make a special statement about isolated points of the domain.

Perhaps not surprisingly, there is a nice connection between limits and sequences.

**Problem 6.13.** Let  $f : A \rightarrow \mathbb{R}$  be a function and let  $a$  be an accumulation point of  $A$ . Then  $\lim_{x \rightarrow a} f(x)$  exists if and only if for every sequence  $(x_n)$  in  $A \setminus \{a\}$  converging to  $a$ , the sequence  $(f(x_n))$  converges, in which case,  $\lim_{x \rightarrow a} f(x)$  equals the limit of the sequence  $(f(x_n))$ . This is often written as

$$\lim_{x \rightarrow a} f(x) = \lim_{n \rightarrow \infty} f(x_n).$$

In order for limits to be a useful tool, we need to prove a few important facts.

**Problem 6.14** (Limit Laws). Let  $f : A \rightarrow \mathbb{R}$  and  $g : B \rightarrow \mathbb{R}$  be functions. Prove each of the following using Definition 6.1.

(a) If  $c \in \mathbb{R}$ , then  $\lim_{x \rightarrow a} c = c$ .

(b) If  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  both exist, then

$$\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x).$$

(c) If  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  both exist, then

$$\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x).$$

(d) If  $c \in \mathbb{R}$  and  $\lim_{x \rightarrow a} f(x)$  exists, then

$$\lim_{x \rightarrow a} (c \cdot f(x)) = c \cdot \lim_{x \rightarrow a} f(x).$$

(e) If  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  both exist and  $\lim_{x \rightarrow a} g(x) \neq 0$ , then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}.$$

(f) If  $f$  is continuous at  $b$  and  $\lim_{x \rightarrow a} g(x) = b$ , then

$$\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x)) = f(b).$$

The next problem is extremely useful. It allows us to simplify our calculations when computing limits.

**Problem 6.15.** Let  $f : A \rightarrow \mathbb{R}$  and  $g : A \rightarrow \mathbb{R}$  be functions and let  $a$  be an accumulation point of  $A$ . If there exists an open interval  $S$  such that  $f(x) = g(x)$  for all  $x \in (S \cap A) \setminus \{a\}$ , then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

provided one of the limits exists.