Pass on what you have learned. Strength, mastery. But weakness, folly, failure also. Yes, failure most of all. The greatest teacher, failure is.

Yoda, Jedi master

Chapter 5

Sequences

We will now begin connecting the concepts of sets to more familiar ones from calculus, beginning with sequences.

5.1 Introduction to Sequences

Definition 5.1. A **sequence** (of real numbers) is a function p from $\mathbb{N}$ to $\mathbb{R}$.

If $n \in \mathbb{N}$, it is common to write $p_n := p(n)$. We refer to p_n as the nth **term** of the sequence. We will abuse notation and associate a sequence with its list of outputs, namely:

$$\boxed{(p_n)_{n=1}^{\infty} := (p_1, p_2, \ldots)},$$

which we may abbreviate as $\boxed{(p_n)}$.

Example 5.2. Define $p : \mathbb{N} \to \mathbb{R}$ via $p(n) = \frac{1}{2^n}$. Then we have

$$p = \left(\frac{1}{2}, \frac{1}{4}, \ldots\right) = \left(\frac{1}{2^n}\right)_{n=1}^{\infty}.$$

It is important to point out that not every sequence has a description in terms of an algebraic formula. For example, we could form a sequence out of the digits to the right of the decimal in the decimal expansion of π, namely the nth term of the sequence is the nth digit to the right of the decimal. But then there is no nice algebraic formula for describing the nth term of this sequence.

Problem 5.3. Write down several sequences p you are familiar with. If possible, give an algebraic formula for each p_n in terms of n.

Note that the **image** (or **range**) of a sequence $(p_n)_{n=1}^{\infty}$ is the set $\{p_n\}_{n=1}^{\infty}$.

Problem 5.4. Explain the difference between $\{p_n\}_{n=1}^{\infty}$ and $(p_n)_{n=1}^{\infty}$? Give an example of a sequence whose image set is finite.

There is a deep connection between sequences and accumulation points, which the next few problems will elucidate. First, a definition—one you may have seen in calculus in a different form. When digesting the following definition, try to think about how this definition is capturing the notion that the sequence is getting "closer and closer" to the point that the sequence converges to.

Definition 5.5. We say that the sequence $p = (p_n)_{n=1}^{\infty}$ **converges to the point** x if for every bounded open interval (a, b) containing x, there exists an $N \in \mathbb{N}$ such that for all natural numbers $n \geq N$, $p_n \in (a, b)$.

In the definition above, we sometimes refer to N as the **cutoff point** for the sequence relative to (a, b) and to all p_n with $n \geq N$ as the **tail of the sequence**. It is important to emphasize that the cutoff point and the tail of the sequence depend on the interval (a, b). Informally, we write $\boxed{p \to x}$ or $\boxed{(p_n) \to x}$ to mean that the sequence p converges to the point x. We simply say that p converges if there exists a point x to which the sequence converges. If a sequence does not converge to *any* point x, then we say it **diverges**.

The first problem about this should be used as a place to test ideas for how to prove convergence. As you tackle the next few problems, it might be useful to begin by writing down the first several terms of the sequences.

Problem 5.6. Consider the sequence given by $p_n = \frac{1}{n}$. Prove that $p = (p_n)_{n=1}^{\infty}$ converges to 0.

Problem 5.7. Consider the sequence given by $p_n = 1 - \frac{1}{n}$. Prove that p converges to 1.

Problem 5.8. Consider the sequence with even terms $p_{2n} = \frac{1}{2n-1}$ and odd terms $p_{2n-1} = \frac{1}{2n}$. Prove that p converges to 0.

Problem 5.9. Consider the sequence with odd terms $p_{2n-1} = \frac{1}{2n-1}$ and even terms $p_{2n} = 1 + \frac{1}{2n}$. Determine whether p converges to 0.

The following problem connects accumulation points and sequences.

Problem 5.10. Prove that if p converges to the point x *and* for each $n \in \mathbb{N}$, $p_n \neq p_{n+1}$, then x is an accumulation point of the image set of $(p_n)_{n=1}^{\infty}$. Can we weaken the requirement that $p_n \neq p_{n+1}$ for all $n \in \mathbb{N}$ and still conclude that x is an accumulation point of the image set?

Problem 5.11. Prove that if $(p_n)_{n=1}^{\infty}$ converges to the point x and y is a point different from x, then $(p_n)_{n=1}^{\infty}$ does *not* converge to y.

Problem 5.12. Let $(p_n)_{n=1}^{\infty}$ be a sequence. Prove that if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|p_n - x| < \epsilon$ whenever $n \geq N$, then $(p_n)_{n=1}^{\infty}$ converges to x.

5.2 Properties of Convergent Sequences

We now explore some basic facts concerning the convergence of sequences.

Problem 5.13. Prove that if c is a real number and $(p_n)_{n=1}^{\infty}$ converges to x, then the sequence $(cp_n)_{n=1}^{\infty}$ converges to cx.

Problem 5.14. Prove that if $(p_n)_{n=1}^{\infty}$ converges to x and $(q_n)_{n=1}^{\infty}$ converges to y, then the sequence $(p_n + q_n)_{n=1}^{\infty}$ converges to $x + y$.

Products and quotients of sequences behave like you think they will, as well. We will include one special case soon.

Problem 5.15. If the sequence $(p_n)_{n=1}^{\infty}$ converges to the point x, then the image set $\{p_n\}_{n=1}^{\infty}$ is bounded.

Problem 5.16. Find an example of a sequence $(p_n)_{n=1}^{\infty}$ such that its image set $\{p_n\}_{n=1}^{\infty}$ is unbounded and hence does not have a supremum.

Consider using Problem 5.15 when approaching the next problem.

Problem 5.17. Prove that if $(p_n)_{n=1}^{\infty}$ converges to x and $(q_n)_{n=1}^{\infty}$ converges to 0, then $(p_n \cdot q_n)_{n=1}^{\infty}$ converges to 0.

5.3 Monotone Convergence Theorem

Definition 5.18. We say that a sequence $(p_n)_{n=1}^{\infty}$ is **nondecreasing** if $p_n \leq p_{n+1}$ for all $n \in \mathbb{N}$. The concept of **nonincreasing** is defined similarly. A function that is either nondecreasing or nonincreasing is said to be **monotone**.

Problem 5.19. Replace $\leq$ above with $<$ to define the notion of (strictly) **increasing**. Find examples of nondecreasing sequences that are not increasing. Similarly, define (strictly) **decreasing**.

Problem 5.20 (Monotone Convergence Theorem)**.** Prove that if $(p_n)_{n=1}^{\infty}$ is a nondecreasing sequence such that the image set $\{p_n\}_{n=1}^{\infty}$ is bounded above, then $(p_n)_{n=1}^{\infty}$ converges to some point x.

It turns out that the previous result is equivalent to the Completeness Axiom. The next problem asks you to verify this, but this is not a result that we need going forward, but rather is an interesting side story.

Problem 5.21. Assuming the result of Problem 5.20, prove the Completeness Axiom.

Problem 5.22. Let A be a nonempty set that is bounded above. Prove that there exists a nondecreasing sequence $(p_n)_{n=1}^{\infty}$ that converges to $\sup(A)$, where the image set $\{p_n\} \subseteq A$.

5.4 Subsequences and the Bolzano–Weierstrass Theorem

Definition 5.23. A sequence $(b_k)_{k=1}^{\infty}$ is a **subsequence** of $(a_n)_{n=1}^{\infty}$ if there is a sequence of natural numbers $(n_k)_{k=1}^{\infty}$ with $n_k < n_{k+1}$ such that $b_k = a_{n_k}$.

Problem 5.24. Give some examples of subsequences of the sequence from Problem 5.6.

Problem 5.25. Prove that if a sequence converges to x, so does any subsequence of that sequence.

Problem 5.26. Suppose $(p_{n_k})_{k=1}^{\infty}$ is a subsequence of $(p_n)_{n=1}^{\infty}$. If p_{n_k} converges to x, does this imply that p_n converges to x? Justify your answer.

Problem 5.27. Provide an example of a sequence $(p_n)_{n=1}^{\infty}$ with image set $\{p_n\} \subseteq \mathbb{N}$ such that *every* sequence of natural numbers is a subsequence of (p_n).

Problem 5.28. Prove that every sequence of real numbers has a nonincreasing or nondecreasing subsequence.

Problem 5.29 (Bolzano–Weierstrass Theorem)**.** Prove that every sequence with bounded image set has a convergent subsequence.

The next problem is related to the Bolzano–Weierstrass Theorem.

Problem 5.30. Prove that if K is a nonempty compact subset of $\mathbb{R}$, then any sequence with image set in K has a subsequence that converges to a point in K.

Problem 5.31. Come up with examples showing that if A is not closed or not bounded, then there exists a sequence with image set in A that does not have a subsequence converging to a point in A (or possibly not at all).

On the real line, compactness and satisfying the Bolzano–Weierstrass Theorem are equivalent. However, one can concoct examples of other mathematical spaces where they are not the same.

> It does not matter how slowly you go as long as you do not stop.
>
> Confucius, philosopher