Homework 1

Discrete Mathematics

Please review the *Rules of the Game* from the syllabus. You are highly encouraged to seek out assistance by asking questions in our Q&A Discussion Board in Canvas. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.** In general, late homework will not be accepted. However, you are allowed to turn in **up to three late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers. In many problems this means that you should use words to describe what you are doing and why. In other problem, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. How many dots are there on a set of dominoes (see Problem 1.13) that includes double blank through double n? *Note:* It's expected that your answer won't be pretty.

2. I'm in the mood for more coffee! Assume I walk to the same coffee shop as in Problem 1.14 subject to the same constraints (coffee shop is 4 blocks East and 3 blocks North from start). However, this time there is construction at the intersection that is 3 blocks East and 2 blocks North from my starting location. Unfortunately, I need to avoid this intersection on my walk. How many routes can I take to get coffee if I avoid this intersection?

3. Consider the collection of bit strings of length 7.

   (a) How many bit strings of length 7 have exactly 4 consecutive 1s? *Hint:* Consider using a brute-force case analysis.

   (b) How many bit strings of length 7 start and end with repeated symbols (e.g., 0010011, 1111000)?

4. A **hexadecimal** is a string consisting of the symbols $0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F$.

   (a) How many hexadecimals of length 5 are there?

   (b) How many hexadecimals of length 5 use only the letters A,B,C,D,E,F?

   (c) How many hexadecimals of length 5 use only the digits $0,1,2,3,4,5,6,7,8,9$?

   (d) How many hexadecimals of length 5 contain *both* numbers and letters?

   (e) How many hexadecimals of length 5 have at least one repeated symbol?

   (f) How many hexadecimals of length 5 have at least four consecutive 6's?

5. The game of chess is played on an 8×8 grid. For this problem, we'll consider chessboards with arbitrary dimension, say $m \times n$ with $m \geq n$. A rook is a castle-shaped piece that can move horizontally or vertically any number of squares. In a typical chess game, there are two black rooks and two white rooks. For this problem, we will assume we have n rooks all the rooks are the same color. We say that an arrangement of rooks on an $m \times n$ chessboard is **non-attacking** if no two rooks lie in the same row or column.

   (a) How many different non-attacking rook arrangements are there on an 8×8 chessboard involving 8 rooks?

(b) How many different non-attacking rook arrangements are there on an 8×8 chessboard involving 8 rooks if there is a rook in the bottom left corner?

(c) How many different non-attacking rook arrangements are there on an 8×8 chessboard involving 8 rooks if there is a rook in the second column, third row? *Note:* It doesn't matter how you label your rows and columns; just pick a convention and stick to it.

(d) How many different non-attacking rook arrangements are there on an $n \times n$ chessboard involving n rooks?

(e) How many different non-attacking rook arrangements are there on an 8×6 chessboard involving 6 rooks?

(f) How many different non-attacking rook arrangements are there on an $m \times n$ (with $m \geq n$) chessboard involving n rooks?

6. For convenience, define $[n] := \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. For example, $[4] = \{1, 2, 3, 4\}$. A **set partition** of $[n]$ is a collection of nonempty disjoint subsets of $[n]$ whose union is $[n]$. Each subset in the set partition is called a **block**. For example, there are 7 set partitions of [4] with 2 blocks, namely:

$$\begin{array}{ll}
\{\{1,2,3\},\{4\}\} & \{\{1,2,4\},\{3\}\} \\
\{\{1,3,4\},\{2\}\} & \{\{2,3,4\},\{1\}\} \\
\{\{1,2\},\{3,4\}\} & \{\{1,3\},\{2,4\}\} \\
\{\{1,4\},\{2,3\}\} &
\end{array}$$

We define the **Stirling numbers** (of the second kind) via

$$\left\{ {n \atop k} \right\} := \text{number of set partitions of } [n] \text{ with } k \text{ blocks.}$$

I usually pronounce this as "n Stirling k". Based on the information above, we know $\left\{ {4 \atop 2} \right\} = 7$.

(a) Compute $\left\{ {4 \atop 3} \right\}$ via brute-force.

(b) Explain why $\left\{ {n \atop 1} \right\} = 1 = \left\{ {n \atop n} \right\}$.

(c) Explain why the Stirling numbers satisfy the following for $1 \leq k \leq n$:

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}.$$

Hint: Consider two disjoint sets, namely, set partitions of $[n]$ with k blocks where the element n is in a block by itself, and set partitions of $[n]$ with k blocks where the element n is not in a block by itself. To count one of the sets, you'll need to use the Product Principle.