Chapter 7

Integration

Unlike with differentiation, we will need a number of auxiliary definitions for beginning integration.

Definition 7.1. A set of points $P = \{t_0, t_1, \ldots, t_n\}$ is a *partition* of the closed interval $[a, b]$ if $a = t_0 < t_1 < \cdots t_{n-1} < t_n = b$. If $t_i - t_{i-1} = \frac{b-a}{n}$ for all i, we say that the partition is a *regular partition* of $[a, b]$. In this case, we may use the notation $\Delta t := t_i - t_{i-1}$.

Exercise 7.2. Give some partitions, regular and not regular, of $[0, 1]$, $[2, 4]$, and $[-1, 0]$.

Definition 7.3. We say that a function is *bounded* if it has bounded image set.

Important! For the next four definitions, we assume that f is a bounded function with domain the closed interval $[a, b]$.

Definition 7.4. Let f be a bounded function with domain $[a, b]$ and let $\{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. We say that any sum S of the form

$$S = \sum_{i=1}^{n} f(x_i)(t_i - t_{i-1}),$$

where $x_i \in [t_{i-1}, t_i]$ is a *Riemann sum* for f on $[a, b]$.

Definition 7.5. Let f be a bounded function with domain $[a, b]$ and let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. For each $i \in \{1, 2, \ldots, n\}$, define $M_i := \sup\{f(x) \mid x \in [t_{i-1}, t_i]\}$. We say that the sum

$$U_P(f) := \sum_{i=1}^{n} M_i(t_i - t_{i-1}),$$

is the *upper Riemann sum* for f with partition P.

Definition 7.6. Let f be a bounded function with domain $[a, b]$ and let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. For each $i \in \{1, 2, \ldots, n\}$, define $m_i := \inf\{f(x) \mid x \in [t_{i-1}, t_i]\}$. We say that the sum

$$L_P(f) := \sum_{i=1}^{n} m_i(t_i - t_{i-1}),$$

is the *lower Riemann sum* for f with partition P.

Exercise 7.7. Draw pictures that capture the concepts of upper and lower Riemann sums.

Contrary to the name, upper and lower Riemann sums are not always Riemann sums.

Problem 7.8. Give an example of an interval $[a,b]$, partition P, and bounded function f such that $U_p(f)$ is not a Riemann sum.

Problem 7.9. Define $f:[0,1]\to\mathbb{R}$ via

$$f(x)=\begin{cases}0, & x\in(0,1]\\ 1, & x=0.\end{cases}$$

(a) Show that $U_P(f)>0$ for all partitions of $[0,1]$.

(b) Show that for any positive number ϵ there is a partition P_ϵ such that $U_{P_\epsilon}(f)<\epsilon$.

(c) Fully describe all lower sums of f on $[0,1]$.

Problem 7.10. Define $f:[0,1]\to\mathbb{R}$ via $f(x)=x$. For each $n\in\mathbb{N}$, let P_n be the regular partition of $[0,1]$ given by $\left\{0,\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n},1\right\}$.

(a) Compute $U_{P_5}(f)$.

(b) Give a formula for $U_{P_n}(f)$.[1]

(c) Compute $L_{P_5}(f)$.

(d) Give a formula for $L_{P_n}(f)$.

Problem 7.11. Suppose that f is a bounded function on $[a,b]$ with lower bound m and upper bound M. Show that for any partition P of $[a,b]$, $U_P(f)\le M(b-a)$ and $L_P(f)\ge m(b-a)$.

Problem 7.12. Suppose that f is a bounded function on $[a,b]$ and P is a partition of $[a,b]$. Show that $L_P(f)\le U_P(f)$.

One consequence of Problem 7.11 is that the set of all upper, respectively lower, sums of f over $[a,b]$ is a bounded point set. This implies that if f is a bounded function on $[a,b]$, then the following supremum and infinum exist:

$$\inf\{U_P(f)\mid P \text{ is a partition of } [a,b]\}$$

$$\sup\{L_P(f)\mid P \text{ is a partition of } [a,b]\}$$

This leads to the following definition.

[1] Recall that the sum $\sum_{i=1}^{k} i=\frac{k(k+1)}{2}$.

Definition 7.13. Let f be a bounded function with domain $[a,b]$. The *upper integral* of f from a to b is defined via

$$\overline{\int_a^b} f := \inf\{U_P(f) \mid P \text{ is a partition of } [a,b]\}.$$

Similarly, the *lower integral* of f from a to b is defined via

$$\underline{\int_a^b} f := \sup\{L_P(f) \mid P \text{ is a partition of } [a,b]\}.$$

Problem 7.14. Compute the upper and lower integrals for the function in Problem 7.9.

Problem 7.15. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that $\underline{\int_0^1} f < \overline{\int_0^1} f$.

Definition 7.16. If P and Q are partitions of $[a,b]$ such that $P \subseteq Q$, then we we say that Q is a *refinement* of P, or that Q *refines* P.

Problem 7.17. Let f be a bounded function with domain $[a,b]$. Prove that if P and Q are partitions of $[a,b]$ such that Q is a refinement of P, then $L_P(f) \leq L_Q(f)$ and $U_P(f) \geq U_Q(f)$.

Problem 7.18. Suppose f is a bounded function on $[a,b]$. Use the previous problem to prove that

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Problem 7.19. Suppose f is continuous on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and that for some $z \in [a,b]$, $f(z) > 0$. Explain why $\int_a^b f$ exists and then show that $\int_a^b f > 0$.

Definition 7.20. Let f be a bounded function with domain $[a,b]$. We say that f is (*Riemann*) *integrable* on $[a,b]$ if

$$\overline{\int_a^b} f = \underline{\int_a^b} f.$$

If f is integrable on $[a,b]$, then the common value of the upper and lower integrals is called the (*Riemann*) *integral* of f on $[a,b]$, which we denote via

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)\, dx.$$

Technically, we have defined the *Darboux integral*, with Riemann integrals coming from so-called Riemann sums. The two notions can be proved equivalent.

Problem 7.21. Give an example of a function f and an interval $[a,b]$ for which we know $\int_a^b f$ does not exist.

Problem 7.22. Is the function in Problem 7.9 integrable over $[0,1]$? If so, determine the value of the corresponding integral. If not, explain why.

Problem 7.23. Prove that every constant function is integrable over every interval $[a,b]$.

More coming soon...