Supplemental Problems for Exam 1

General Information

Unless your instructor states otherwise, Exam 1 focuses on content in Sections 2.1–2.7, 3.1. In addition, you should be familiar with the review material found in Chapter 1. In particular, here is a list of topics/concepts that you must have an understanding of or be able to do:

- Be familiar with common identities: algebraic, exponential, logarithmic, and trigonometric
- Understand and be able to identify domain, codomain, range, and default domain.
- Be able to perform function arithmetic and function composition. Be able to state the default domain and range of the resulting function, and be able to evaluate such functions at specific values.
- Understand what an inverse function is, and be able to determine whether a given function has an inverse. If a function has an inverse, you should be able to find it in simple cases.
- Be familiar with the graphs of basic functions:
  - $y = mx + b$
  - $y = x^n$ for positive integer n
  - $y = 1/x^n$ for positive integer n
  - $y = \sqrt[n]{x}$ for positive integer n
  - $y = b^x$ for $b > 0$ and $b \neq 1$ (special case: $y = e^x$)
  - $y = \log_b(x)$ for $b > 0$ and $b \neq 1$ (special case: $y = \ln(x)$)
  - $y = \sin(x)$
  - $y = \cos(x)$
  - $y = \tan(x)$
- Understand and be able to sketch and recognize transformations of function graphs.
- Be able to find an equation and the slope of a line given various types of information.
- You should understand and be able to calculate the average rate of change of a function.
- You should have an intuitive understanding of derivativese based on your knowledge of rate of change, speed, velocity, and slope of tangent lines.
- Understand the basic definition and the notation for the limit of a function at a given point.
- Be able to evaluate limits of functions given graphically or as equations.
- Be able to state and use the Squeeze Theorem.
- Understand what it means for a function to be continuous.
- Understand and be able to apply the limit rules.
- Understand the limit definition of the derivative and be able to apply it to basic functions.
- Be able to give examples and counter examples to demonstrate different properties of functions.
- Call upon your own mental faculties to respond in flexible, thoughtful and creative ways to problems that may seem unfamiliar on first glance.

The problems that follow will provide you with an opportunity to review the relevant topics. However:

This is not a practice test!

It is possible that problems on your exam will resemble problems seen below, but you should not expect exam problems to be identical. This document contains an abundance of problems and it is not the intention that every student will complete every problem. You should complete as many problems in each section below as you think are necessary to solidify your understanding.

In addition, you should review exams done in class, as well as your homework exercises, especially the ones on the Weekly Homework assignments.

Words of Advice

Here are few things to keep in mind when taking your exam:

- Show all work! The thought process and your ability to show how and why you arrived at your answer is more important than the answer itself.
- The exam will be designed so that you can complete it without a calculator. If you find yourself yearning for a calculator, you might be doing something wrong.
- Make sure you have answered the question that you were asked. Also, ask yourself if your answer makes sense.
- If you know you made a mistake, but you can't find it, explain why you think you made a mistake and indicate where the mistake might be. This shows that you have a good understanding of the problem.
- If you write down an "=" sign, then you better be sure that the two expressions on either side are equal. Similarly, if two things are equal and it is necessary that they be equal to make your conclusion, then you better use "=."
- Don't forget to write limits where they are needed.

Rates of Change

1. Suppose a ball is thrown off a 100 foot tall building such that the height of the ball in feet at time t in seconds is given by $h(t) = -16t^2 + 25t + 100$.
   (a) What is average rate of change over the first second of flight?
   (b) How about over $[0, 2]$? Interpret the sign of your answer.
2. The position in meters of a particle moving in a straight line is given for some values of time t in seconds in the following table. What is the average velocity over the first .3 seconds of movement?

t	0	.1	.2	.3	.4
$p(t)$	0	.5	.7	1.2	3

Limits

3. True or False? Justify your answer.

   (a) If a function f does not have a limit as x approaches a from the left, then f does not have a limit as x approaches a from the right.

   (b) If $h(x) \leq f(x) \leq g(x)$ for all real numbers x and $\lim_{x\to a} h(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x)$ also exists.

4. Given the graph of f, evaluate each of the following expressions. If an expression does not exist, explain why.

   (a) $\lim_{x\to -4^-} f(x)$

   (b) $\lim_{x\to -4^+} f(x)$

   (c) $\lim_{x\to -4} f(x)$

   (d) $f(-4)$

   (e) $\lim_{x\to -2^-} f(x)$

   (f) $\lim_{x\to -2^+} f(x)$

   (g) $\lim_{x\to -2} f(x)$

   (h) $f(-2)$

   (i) $\lim_{x\to 1^-} f$

   (j) $\lim_{x\to 1^+} f(x)$

   (k) $\lim_{x\to 1} f(x)$

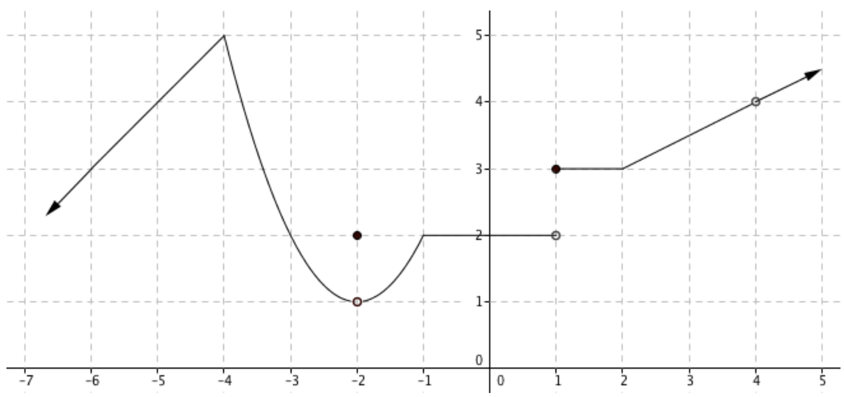
   (l) $f(1)$

   (m) $\lim_{x\to 4^-} f(x)$

   (n) $\lim_{x\to 4^+} f(x)$

   (o) $\lim_{x\to 4} f(x)$

   (p) $f(4)$



5. Evaluate each of the following limits. If a limit does not exist, specify whether the limit equals ∞, $-\infty$, or simply does not exist (in which case, write DNE).

   (a) $\lim_{x\to 2}(x^2+4x-12)$

   (b) $\lim_{x\to 2} \dfrac{x^2+4x-12}{x^2+4x+3}$

   (c) $\lim_{x\to 2} \dfrac{x^2+4x-12}{x^2-x-2}$

   (d) $\lim_{x\to 2} \dfrac{x^2+4x-12}{x^2-4x+4}$

   (e) $\lim_{x\to 4} \dfrac{x-4}{\sqrt{x}-2}$

   (f) $\lim_{x\to 3} \dfrac{1}{x-3}$

   (g) $\lim_{x\to 3} \dfrac{1}{(x-3)^2}$

   (h) $\lim_{x\to \pi} \dfrac{x}{\cos(x)}$

   (i) $\lim_{x\to 0} \dfrac{\frac{1}{x+4}-\frac{1}{4}}{x}$

   (j) $\lim_{x\to 5} \dfrac{|x-5|}{x-5}$

   (k) $\lim_{x\to 0^+} \ln(x)$

   (l) $\lim_{x\to \infty} \ln(x)$

(m) $\lim_{x\to\infty} \frac{4x^2 - x + 3}{5 + x - 3x^2}$

(n) $\lim_{x\to\infty} \frac{4x^3 - x + 3}{5 + x - 3x^2}$

(o) $\lim_{x\to\infty} \frac{4x^2 - x + 3}{5 + x - 3x^3}$

(p) $\lim_{x\to\infty} \frac{\sin(x)}{x}$

(q) $\lim_{x\to\pi/2} \frac{\sin(x)}{x}$

(r) $\lim_{x\to 0} \frac{\sin(5x)}{7x}$

(s) $\lim_{x\to 0} \frac{\sin(2x) - \sin(x)}{x}$

(t) $\lim_{x\to 0} \frac{\tan(3x)}{x}$

(u) $\lim_{x\to 0} \frac{\tan(4x)}{\tan(5x)}$

(v) $\lim_{x\to 0} \frac{\sin^3(x)}{\sin(x^3)}$

(w) $\lim_{x\to 0} \frac{1 - \cos(x)}{\sqrt{x}}$

(x) $\lim_{x\to 0} \frac{1 - \cos(x)}{\sin(x)\sin(2x)}$

6. Sketch the graphs of possible functions g and h such that g satisfies property (a) below and h satisfies property (b) below. (There should be two separate graphs.)

   (a) $\lim_{x\to 0^-} g(x) = -1$ and $\lim_{x\to 0^+} g(x) = 1$

   (b) $\lim_{x\to 0} h(x) \neq h(0)$, where $h(0)$ is defined.

7. Consider the following function.

$$f(x) = \begin{cases} \frac{-1}{x-2}, & x > -1 \\ x^2 + 1, & x \leq -1 \end{cases}$$

   Evaluate each of the following expressions. If an expression does not exist, specify whether it equals ∞, $-\infty$, or simply does not exist (in which case, write DNE). You do *not* need to justify your answers.

   (a) $f(-1)$

   (b) $\lim_{x\to -1^-} f(x)$

   (c) $\lim_{x\to -1^+} f(x)$

   (d) $\lim_{x\to -1} f(x)$

   (e) $f(2)$

   (f) $\lim_{x\to 2^+} f(x)$

8. Sketch the graph of a possible function f that has all properties (a)–(g) listed below.

   (a) The domain of f is $[-1, 2]$

   (b) $f(0) = f(2) = 0$

   (c) $f(-1) = 1$

   (d) $\lim_{x\to 0^-} f(x) = 0$

   (e) $\lim_{x\to 0^+} f(x) = 2$

   (f) $\lim_{x\to 2^-} f(x) = 1$

   (g) $\lim_{x\to -1^+} f(x) = -1$

9. Let f and g be functions such that $\lim_{x\to a} f(x) = -3$ and $\lim_{x\to a} g(x) = 6$. Evaluate the following limits, if they exist.

   (a) $\lim_{x\to a} \frac{(g(x))^2}{f(x) + 5}$

   (b) $\lim_{x\to a} \frac{7f(x)}{2f(x) + g(x)}$

   (c) $\lim_{x\to a} \sqrt[3]{g(x) + 2}$

10. Let f be defined as follows.

$$f(x) = \begin{cases} 3x & \text{if } x < 0 \\ 3x + 4 & \text{if } 0 \leq x \leq 4 \\ x^2 & \text{if } x > 4 \end{cases}$$

   For (a)–(h), evaluate the limits if they exist. If a limit does not exist, specify whether the limit equals ∞, $-\infty$, or simply does not exist (in which case, write DNE). For part (i), answer the question.

(a) $\lim_{x \to 0^+} f(x)$

(b) $\lim_{x \to 0^-} f(x)$

(c) $\lim_{x \to 0} f(x)$

(d) $f(0)$

(e) $\lim_{x \to 4^+} f(x)$

(f) $\lim_{x \to 4^-} f(x)$

(g) $\lim_{x \to 4} f(x)$

(h) $f(4)$

(i) Determine where f is continuous.

11. If $3x \leq f(x) \leq x^3 + 2$ for $0 \leq x \leq 2$, evaluate $\lim_{x \to 1} f(x)$.

12. Use the Squeeze Theorem to prove that $\lim_{x \to 0} x^4 \cos(2/x) = 0$.

Continuity

13. True or False? If a function is not continuous at $x = a$, then either it is not defined at $x = a$ or it does not have a limit as x approaches a. Justify your answer.

14. Provide an example of function that is continuous everywhere but does not have a tangent line at $x = 0$. Explain your answer.

Derivative at a Point

15. Sketch the graph of each function given below and then sketch the tangent line to that function at the given point.

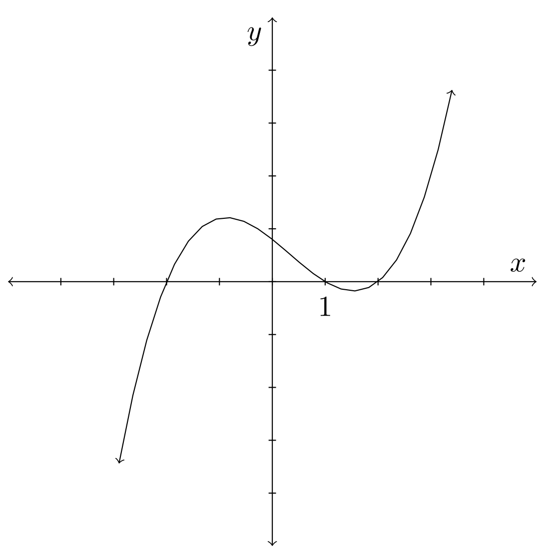
    (a) $f(x) = x^2 + 3$ at $x = 0$

    (b) $g(x) = \cos(x)$ at $x = 0$

    (c) $h(x) = 3x - 12$ at $x = -2$

    (d) $f(x) = 47$ at $x = 7$

16. Consider the graph of the function $y = f(x)$ given in the figure below. Put the following expressions in increasing order: $f'(-2), f'(-1), f'(0), f'(2)$.



17. Let $f(x) = x^2 - x$.

(a) Find the slope of the tangent line at $x = 2$ using the limit definition.

(b) Find an equation of the tangent line to the graph of f at $x = 2$.

18. For each of the following functions, use the limit definition to find the derivative at the specified x-value.

(a) $f(x) = x^2 + 16x - 57$, $x = 1$

(b) $f(x) = \sqrt{5x+1}$, $x = 3$

(c) $f(x) = \dfrac{1}{x}$, $x = 2$

(d) $f(x) = \dfrac{1}{x^2}$, $x = -1$

19. Suppose the equation of the tangent line to the graph of some function f at $x = 1$ is given by $y = 2x + 1$. Find each of the following.

(a) $f'(1)$

(b) $f(1)$

Proofs

20. If f and g are continuous at $x = c$, prove that $f + g$ is continuous at $x = c$.

21. Using the fact that $\cos(\theta) \leq \dfrac{\sin(\theta)}{\theta} \leq 1$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $\theta \neq 0$, prove:

(a) $\displaystyle\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$, and

(b) $\displaystyle\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta} = 0$.

22. If $f(x) = mx + b$ is a linear function, use the limit definition of the derivative to prove that $f'(x) = m$. Can you also justify this fact by appealing to the graph of f?

23. If $f(x) = c$ is a constant function, use the limit definition of the derivative to prove that $f'(x) = 0$. Can you also justify this fact by appealing to the graph of f?