Final Exam (Part 2)

Your Name:

Names of Any Collaborators:

Instructions

Answer each of the following questions. This part of the Final Exam is worth a total of 16 points and is worth 60% of your overall score on the Final Exam. Your overall score on the Final Exam is worth 25% of your overall grade. This portion of the Final Exam is due by **3pm on Friday, December 13**.

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts. Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing the following problems, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. Further information:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using.
2. Unless you prove them, you cannot use any results from the course notes/book that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.** To convince me that you have read and understand the instructions, sign in the box below.

Signature:

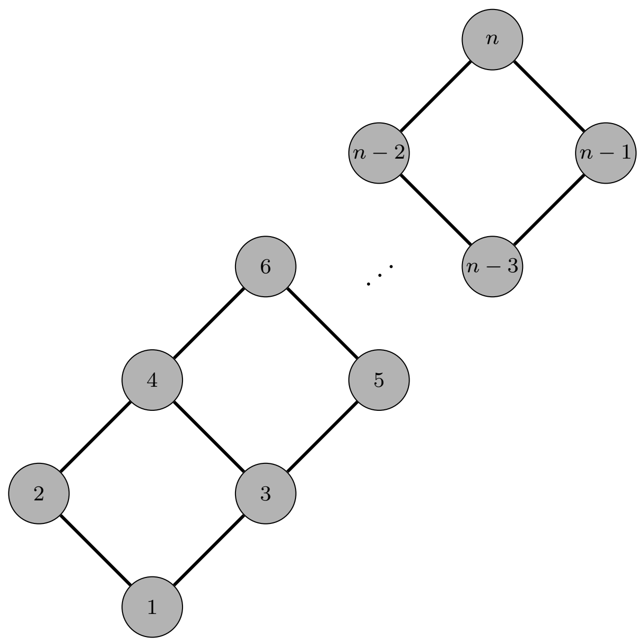
Good luck and have fun!

(4 points each) Complete **four** of the following.

1. Let F be a rule for constructing combinatorial structures subject to the same constraints that we have seen in the past few homework assignments. We define the rule F', called the **derivative of** F, as follows: an F'-structure on a finite set U is an F-structure on $U \sqcup \{\star\}$ (i.e., think of $\star$ as being chosen outside of U). Prove that
$$F'(x) = \frac{d}{dx}\left[F(x)\right].$$

2. Recall that an L-structure on a finite set U is a linear order of U (typically written in one-line notation).
   (a) Prove that that there is a bijection between $L'[n]$ and $(L \cdot L)[n]$.
   (b) Without appealing to Problem 1, independently verify that $L'(x) = \frac{d}{dx}\left[L(x)\right]$.

3. Find the number of linear extensions of the following poset on $[n]$ for n even. Prove that your answer is correct.



4. Consider the following search algorithm for a permutation in 1-line notation. Scan a permutation $w \in S_n$ from left to right. We want to pull out the numbers $1, 2, \ldots, n$ in order. We are only allowed to pull out the numbers in strict succession (e.g., 1 before 2, 2 before 3, etc.). Once we scan through the permutation, we scan through the now shorter permutation, repeating as necessary until we have selected all the numbers. For example, consider $w = 3172546$. Below we have highlighted in bold the numbers selected in each scan:

Scan 1:	3	**1**	7	**2**	5	4	6
Scan 2:	**3**		7		5	**4**	6
Scan 3:			7		**5**		**6**
Scan 4:			**7**				

What we will count here is the number of times a number is *not* selected when scanning. For example, the number 3 was not selected in the first scan while 7 was not selected in the first three scans. In the example above, there are nine instances in which we scanned but did not select a number (3 was overlooked once, 7 was overlooked three times, 5 was overlooked twice, 4 was overlooked once, and 6 was overlooked twice). We call this number the **disorder** of a permutation and denote it by $\operatorname{dis}(w)$. In the previous example, $\operatorname{dis}(w) = 9$. Prove that disorder is Mahonian.

5. Define the nth **q-Catalan number** via
$$C_n[q] := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q.$$
Prove that
$$C_n[q] = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q \begin{bmatrix} 2n \\ n+1 \end{bmatrix}_q$$
and argue that $C_n[q]$ is a polynomial of degree $n(n-1)$.

6. Prove that
$$C_n[q] = \sum_{p \in \operatorname{Dyck}(n)} q^{\operatorname{maj}(p)}.$$
Hint: The general strategy is to mimic and refine the argument we utilized for counting Dyck paths. In that argument, for non-Dyck paths we reflected "tail" of path after the first point below the line $y = x$. For this argument, identify the first minimum valley and consider changing the E step just prior into a N step and keeping everything else the same.

7. A **parking function** of length n is a sequence of positive integers $(a_1, \ldots, a_n)$ such that if $b_1 \leq \cdots \leq b_n$ is an increasing arrangement of $a_1, \ldots, a_n$, then $b_i \leq i$. Here's where the name comes from. Imagine there are n cars that want to park in n spaces on a one-way street. Denote the cars by $C_1, \ldots, C_n$ and let $a_1, \ldots, a_n$ be their preferred parking spaces, respectively. If a car finds its preferred space occupied, it will park in the next available space. All cars will be able to park if and only if $(a_1, \ldots, a_n)$ is a parking function. For example, suppose their are six cars with preferences $(1, 1, 5, 2, 2, 3)$. Then the cars will park as follows:

car	C_1	C_2	C_4	C_5	C_3	C_6
space	1	2	3	4	5	6

On the other hand, if the preferences were $(1, 1, 6, 3, 5, 5)$, the sixth car would be unable to park since when it arrives to park in space five or higher, the only space available is space four. Let $\operatorname{PF}(n)$ denote the set of parking functions of length n. Prove that $|\operatorname{PF}(n)| = (n+1)^{n-1}$.

Hint: First consider "cyclic" parking functions for n cars with $n+1$ parking spaces. That is, suppose we allow each car to have a preference a_i from 1 to $n+1$ and if a car finds the spot they like occupied, they can continue on loop to first available spot. In this case, every list of preferences will result in all the cars parking. How many such lists are there? Now, observe that if we shift all the preferences by one modulo $n+1$, we end up with all the cars in the same spaces relative to one another. We can repeat this shift $n+1$ times and exactly one of these shifts results in space $n+1$ being empty.

8. A parking function $(a_1, \ldots, a_n)$ is called **increasing** if $a_i \leq a_{i+1}$ for $1 \leq i \leq n-1$. Count the number of increasing parking functions of length n.

9. Let $M = \{\{1^{n_1}, 2^{n_2}, \ldots, m^{n_m}\}\}$ be a multiset on $[m]$, where we write k^{n_k} to represent n_k copies of $k \in [m]$. Define inversions, descents, and the major index for permutations of M exactly the way we did for permutations without repetition. Let $P(M)$ denote the set of permutations of M. Prove that
$$\sum_{w \in P(M)} q^{\operatorname{inv}(w)} = \sum_{w \in P(M)} q^{\operatorname{maj}(w)}.$$