Homework 2

Combinatorics

Let's begin with a few reminders. Homework will consist of a mixture of the following:

- Problems that are modifications of examples we have discussed in class.
- Problems that extend concepts introduced in class.
- Problems that introduce new concepts not yet discussed in class.
- Problems that synthesize multiple concepts that we either introduced in class or in a previous homework problem.

Some homework problems will be straightforward while others are intended to be challenging. You should anticipate not knowing what to do on some of the problems at first glance. You may have several false starts. Some frustration, maybe even a lot of frustration, should be expected. This is part of the natural learning process. On the other hand, it is not my intention to leave you to fend for yourselves. I am here to help and I want to help. You are encouraged to seek assistance from your classmates (while adhering to the **Rules of the Game**) and from me. Please visit office hours and ask questions on our Discord server. I am always willing to give hints/nudges, so please ask.

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on our Discord server. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

1. Complete one of the following.

   (a) Let $f : X \to Y$ be a function. Prove that f is injective if and only if there exists a function $g : Y \to X$ such that $g \circ f = i_X$, where i_X is the identity map on X. Note that the function g is often called a **left inverse** of f.

(b) Let $f : X \to Y$ be a function. Prove that f is surjective if and only if there exists a function $g : Y \to X$ such that $f \circ g = i_Y$, where i_Y is the identity map on Y. Note that the function g is often called a **right inverse** of f.

2. A **composition** of n with k parts is an ordered list of k positive integers whose sum is n, denoted $\alpha = (\alpha_1, \ldots, \alpha_k)$. We say that α_i is the ith part. Prove that the number of compositions of n with k parts is $\binom{n-1}{k-1}$. *Hint:* Consider using a "sticks and stones" model, where the ith part consists of α_i many stones and each part is separated by a stick. For example, the composition $(1,3,2)$ on $n = 6$ corresponds to $\circ \mid \circ\circ\circ \mid \circ\circ$.

3. Prove that the total number of compositions of n is 2^{n-1} without appealing to the previous problem.

4. Use the previous two problems to explain why $\sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}$.

5. How many compositions α of n have the following properties?
   (a) α has parts of size 1 and 2 only.
   (b) α has only odd parts.
   (c) α has all parts greater than 1, except possibly the last entry, e.g., for $n = 9$, $(3,4,2)$ and $(3,3,2,1)$ are acceptable, but $(3,3,1,2)$ is not.

6. Find all 231-avoiding permutations in S_5 (*Hint:* There are 42) and organize them based on the number of maximal decreasing runs.

7. Find all non-crossing partitions on 5 elements and organize them based on the number of blocks.

8. Pick any five 231-avoiding permutations from S_5 and determine which NC-partitions they map to using the bijection that I outlined in class.