

# Chapter 6

## Pigeonhole Principle

The **Pigeonhole Principle** is a very natural idea. It says: If a collection of at least  $n + 1$  objects is put into  $n$  boxes, then there is a box with at least two things in it. The Pigeonhole Principle has surprisingly deep applications. We will start with a few examples.

**Example 6.1.** Back in Problem 2.11, we implicitly used the Pigeonhole Principle when we argued that if  $f : A \rightarrow B$  is a function for finite sets  $A$  and  $B$ , then

- (a) If  $f$  is an injection, then  $|A| \leq |B|$ .
- (b) If  $f$  is a surjection, then  $|A| \geq |B|$ .

**Problem 6.2.** A box has blue, green, yellow, red, orange, and white balls. How many must be drawn without looking to be sure of getting at least two of the same color?

**Problem 6.3.** Prove that if seven distinct numbers are selected from  $\{1, 2, \dots, 11\}$ , then some two of these numbers sum to 12.

We would like to generalize the Pigeonhole Principle, but first we need a useful function. The **ceiling function** of a real number  $x$ , written  $\lceil x \rceil$ , is the smallest integer greater than or equal to  $x$ . That is,  $\lceil x \rceil$  is an integer,  $\lceil x \rceil \geq x$ , and there is no other integer between  $\lceil x \rceil$  and  $x$ . You can think of it as the “round-up to an integer” function.

**Example 6.4.** For example,  $\lceil \pi \rceil = 4$ ,  $\lceil -\pi \rceil = -3$ , and  $\lceil 7 \rceil = 7$ .

We can now generalize the Pigeonhole Principle as follows.

**Theorem 6.5** (Generalized Pigeonhole Principle). If  $n$  objects are placed in  $m$  boxes, then there is a box with at least  $\lceil \frac{n}{m} \rceil$  objects.

**Problem 6.6.** If 20 buses seating at most 50 carry 621 passengers to a ball game, then some bus must have at least \_\_\_\_\_ passengers.

**Problem 6.7.** How many balls must be drawn from the box in Problem 6.2 in order to be sure of getting at least 4 of the same color?

**Problem 6.8.** Explain why a list of ten positive integers,  $x_1, x_2, \dots, x_{10}$  must have a sublist in the same order of the original ten whose sum is divisible by 10.