Homework 12

Combinatorics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on our Discord server. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

New! At the top of each problem, I would like you to list the students that you discussed the problem with. If you worked with the same peers on every problem, then you can simply indicate that once at the top of your assignment.

1. For a finite set U, let $\mathcal{C}[U]$ be the collection of cycles of length $|U|$ we can make out the elements of U. For example, if $U = \{a,b,c\}$, then $\mathcal{C}[U] = \{(a,b,c),(a,c,b)\}$.

   (a) Explain why $S = E \circ \mathcal{C}$, where S-structures on $[n]$ are permutations of length n.

   (b) Using Part (a) and a previous homework problem, find a closed form for the exponential generating function $\mathcal{C}(t)$.

   Note: The number of cycles of length n we can make out of $[n]$ is $(n-1)!$. That is, $|\mathcal{C}[n]| = (n-1)!$. This implies that the coefficient on $t^n/n!$ in your answer for Part (b) is $(n-1)!$. One can also show that $\mathcal{C}^{(n)}(0) = (n-1)!$ (i.e., the nth derivative of $\mathcal{C}(t)$ evaluated at $t = 0$).

2. A **Cayley permutation** is a function $w : [n] \to [n]$ such that $\mathrm{Rng}(w) = [k]$ for some $k \leq n$. Let $\mathrm{Cay}[n]$ denote the collection of Cayley permutations on $[n]$. For $w \in \mathrm{Cay}[n]$, we utilize one-line notation and write $w = w(1)w(2)\cdots w(n)$. That is, a Cayley permutation is a word w of positive integers such that if b appears in w, then all positive integers $a < b$ also appear in w. For example,

$$\mathrm{Cay}[3] = \{111, 112, 121, 122, 123, 132, 211, 212, 213, 221, 231, 312, 321\}.$$

   Let $\mathrm{Bal}[n]$ denote the collection of ordered set partitions of $[n]$ (see Problem 1(c) on Homework 11). Note that Bal is short of "ballot". Find a bijection between $\mathrm{Cay}[n]$ and $\mathrm{Bal}[n]$. It follows that $\mathrm{Cay}(t) = \dfrac{1}{2-e^t}$ by Problem 1(c) on Homework 11.

3. The **adjacent sorting length** of a permutation $w = w(1)\cdots w(n) \in S_n$, denoted $\ell(w)$, is the minimal number of adjacent swaps of positions needed to sort the permutation to the identity. For example, 31542 has adjacent sorting length at most 5 since the permutation can be unscrambled in five moves as follows:

$$31\underline{54}2 \to 314\underline{52} \to 31\underline{42}5 \to \underline{31}245 \to 1\underline{32}45 \to 12345.$$

In fact, $\ell(31542)$ is exactly 5. Each swap of adjacent positions i and $i+1$ corresponds to multiplying the corresponding permutation by $s_i = (i, i+1)$. Prove that $\mathrm{inv}(w) = \ell(w)$ for all $w \in S_n$. *Hint:* Start by proving that applying an adjacent swap to positions of a permutation either increases the number of inversions by one or decreases the number of inversions by one, and then describe an unscrambling algorithm that decreases the number of inversions after each swap. It follows that

$$\sum_{w\in S_n} q^{\ell(w)} = [n]_q!.$$

Note: Even though we have framed things in terms of sorting, we've also proved that the right weak order is ranked by number of inversions. Don't use this fact to do this problem. *Fun Fact:* The adjacent sorting length of a permutation is the same as the Coxeter length in Coxeter groups of type A_{n-1}.

4. The following is a greedy algorithm for sorting a permutation with transpositions. Find the largest value that is out of place, move it to its proper place, and repeat. More precisely, if $w(n) = n$, do nothing and move on to sort $w(1)w(2)\cdots w(n-1)$. Otherwise, if $w(i) = n$ with $i < n$, apply the transposition (i, n) to get $w' = w \circ (i, n)$, so that $w'(n) = n$. Now, we can sort $w'(1)\cdots w'(n-1)$. This algorithm is sometimes called **straight selection sort**. Suppose $t_1, \ldots, t_k$ are the transpositions used in this sorting algorithm for some $w \in S_n$, so that $w \circ t_1 \circ \cdots \circ t_k$ is the identity, where $t_i = (\ell_i, m_i)$ with $\ell_i < m_i$. Define the **sorting index** for w via

$$\mathrm{sor}(w) := \sum_{i=1}^{k} (m_i - \ell_i).$$

Informally, the sorting index measures the "cost" of this algorithm, with transpositions that are far away costing more than elements closer by. Define the **sorting index polynomial** via

$$\mathrm{Sor}_n(q) := \sum_{w\in S_n} q^{\mathrm{sor}(w)}.$$

   (a) Compute sor(3172546).
   (b) Organize all of the permutations in S_4 according to their sorting index and then compute $\mathrm{Sor}_4(q)$.
   (c) Is it true that $\mathrm{sor}(\pi) = \mathrm{inv}(\pi)$ for all $\pi \in S_4$? Justify your answer.
   (d) Prove that $\mathrm{Sor}_n(q) = I_n(q)$. In light of this result, we say that the sorting index has a Mahonian distribution.

5. Given a lattice path p consisting only of North and East steps, define $\mathrm{area}(p)$ to be the total area under the path p (and above the x-axis). Recall the definition of $L(k, n-k)$ that appeared on Part 2 of Exam 1. Note that each path in $L(k, n-k)$ consists of k East steps and $n-k$ North steps for a total of n steps. Prove that

$$\sum_{p\in L(k,n-k)} q^{\mathrm{area}(p)} = \sum_{\substack{w\in S_n \\ \mathrm{Des}(w)\subseteq\{k\}}} q^{\mathrm{inv}(w)}.$$