Chapter 7

Differentiation

It's time for derivatives!

Definition 7.1. Let $f : A \to \mathbb{R}$ be a function and let $a \in A$ such that f is defined on some open interval I containing a (i.e., $a \in I \subseteq A$). The **derivative** of f at a is defined via

$$\boxed{f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

provided this limit exists. If $f'(a)$ exists, then we say that f is **differentiable** at a. More generally, we say that f is **differentiable** on $B \subseteq A$ if f is differentiable at every point in B. As a special case, f is said to be **differentiable** if it is differentiable at every point in its domain. If f does indeed have a derivative at some points in its domain, then the **derivative** of f is the function denoted by f', such that for each number x at which f is differentiable, $f'(x)$ is the derivative of f at x. We may also write

$$\frac{d}{dx}[f(x)] := f'(x).$$

The lefthand side of the equation above is typically read as, "the derivative of f with respect to x." The notation $f'(x)$ is commonly referred to as "Newton's notation" for the derivative while $\frac{d}{dx}[f(x)]$ is often referred to as "Liebniz's notation".

Note that the definition of derivative automatically excludes the kind of behavior we saw with continuous functions, where a function defined only at a single point was continuous.

Problem 7.2. Find the derivative of $f(x) = x^2 - x + 1$ at $a = 2$.

Problem 7.3. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = c$ for some constant $c \in \mathbb{R}$. Prove that f is differentiable on $\mathbb{R}$ and $f'(x) = 0$ for all $x \in \mathbb{R}$.

Problem 7.4. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = mx + b$ for some constants $m, b \in \mathbb{R}$. Prove that f is differentiable and $f'(x) = m$ for all $x \in \mathbb{R}$.

Problem 7.5. Find and prove a formula for the derivative of $f(x) = ax^2 + bx + c$ for any $a, b, c \in \mathbb{R}$.

Problem 7.6. Explain why any function defined only on $\mathbb{Z}$ cannot have a derivative.

Problem 7.7. If f is differentiable at x and $c \in \mathbb{R}$, prove that the function cf also has a derivative at x and $(cf)'(x) = cf'(x)$.

Problem 7.8. If f and g are differentiable at x, show that the function $f+g$ also has a derivative at x and $(f+g)'(x) = f'(x) + g'(x)$.

The next problem tells us that differentiability implies continuity.

Problem 7.9. Prove that if f has a derivative at $x = a$, then f is also continuous at $x = a$.

The converse of the previous theorem is not true. That is, continuity does not imply differentiability.

Problem 7.10. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = |x|$.

(a) Prove that f is continuous at every point in its domain.

(b) Prove that f is differentiable everywhere except at $x = 0$.

The next problem states the well-known Product and Quotient Rules for Derivatives. You will need to use Problem 7.9 in their proofs.

Problem 7.11. Suppose f and g are differentiable at x. Prove each of the following:

(a) (Product Rule) The function fg is differentiable at x. Moreover, its derivative function is given by
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

(b) (Quotient Rule) The function f/g is differentiable at x provided $g'(x) \neq 0$. Moreover, its derivative function is given by
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Problem 7.12. Define $f : \mathbb{R} \to \mathbb{R}$ via
$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is continuous at $x = 0$, but not differentiable at $x = 0$.

The next problem is sure to make your head hurt.

Problem 7.13. Define $g : \mathbb{R} \to \mathbb{R}$ via
$$g(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{otherwise.} \end{cases}$$

Now, define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2 g(x)$. Determine where f is differentiable.

The next result tells us that if a differentiable function attains a maximum value at some point in an open interval contained in the domain of the function, then the derivative is zero at that point. In a calculus class, we would say that differentiable functions attain local maximums at critical numbers.

Problem 7.14. Let $f : A \to \mathbb{R}$ be a function such that $[a,b] \subseteq A$, $f'(c)$ exists for some $c \in (a,b)$, and $f(c) \geq f(x)$ for all $x \in (a,b)$. Prove that $f'(c) = 0$.

Problem 7.15. Let $f : A \to \mathbb{R}$ be a function such that $f'(c) = 0$ for some $c \in A$. Does this imply that there exists an open interval (a,b) such that either $f(x) \geq f(c)$ or $f(x) \leq f(c)$ for all $x \in (a,b)$? If so, prove it. Otherwise, provide a counterexample.

The next problem asks you to prove a result called Rolle's Theorem.

Problem 7.16 (Rolle's Theorem)**.** Let $f : A \to \mathbb{R}$ be a function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, then prove that there exists a point $c \in (a,b)$ such that $f'(c) = 0$.[1]

We can use Rolle's Theorem to prove the next result, which is the well-known Mean Value Theorem.

Problem 7.17 (Mean Value Theorem)**.** Let $f : A \to \mathbb{R}$ be a function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$ and differentiable on (a,b), then prove that there exists a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$[2]

Problem 7.18. Let $f : A \to \mathbb{R}$ be a function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$ and differentiable on (a,b) such that $f'(x) = 0$ for all $x \in (a,b)$, then prove that f is constant over $[a,b]$.[3]

Problem 7.19. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ such that $[a,b] \subseteq A$. Prove that if $f'(x) = g'(x)$ for all $x \in (a,b)$, then there exists $C \in \mathbb{R}$ such that $f(x) = g(x) + C$.

Problem 7.20. Is the converse of the previous problem true? If so, prove it. Otherwise, provide a counterexample.

[1] *Hint:* First, apply the Extreme Value Theorem to f and $-f$ to conclude that f attains both a maximum and minimum on $[a,b]$. If both the maximum and minimum are attained at the end points of $[a,b]$, then the maximum and minimum are the same and thus the function is constant. What does Problem 7.3 tell us in this case? But what if f is not constant over $[a,b]$? Try using Problem 7.14.

[2] *Hint:* Cleverly define the function $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. Is g continuous on $[a,b]$? Is g differentiable on (a,b)? Can we apply Rolle's Theorem to g using the interval $[a,b]$? What can you conclude? Magic!

[3] *Hint:* Try applying the Mean Value Theorem to $[a,t]$ for every $t \in (a,b)$.