Exam 4

Your Name:

Names of Any Collaborators:

Instructions

This exam is worth a total of 32 points and is worth 15% of your overall grade. Your exam is due by midnight on **Monday, April 27**. When you have finished your exam, **please email me a single PDF** (instead of submitting to BbLearn). I'd like you to include the cover page, but if you are unable to do this, recreate a suitable replacement that includes your signature. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems/problems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 2.32, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

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The following problems are related to content from Chapters 7–8 of our textbook. Some of the problems are directly from the book. Note that you will need to independently digest some new material in Chapter 8. You may freely use any result that appears in any chapter as long as the result appears *before* the problem in question.

1. (4 points) Prove **one** of the following.
   (a) (Theorem 7.45) If $f : X \to Y$ is a bijection, then $f^{-1} : Y \to X$ is a bijection and $(f^{-1})^{-1} = f$.
   (b) (Theorem 7.46) If $f : X \to Y$ and $g : Y \to Z$ are both bijections, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

2. (Theorem 7.47) Let $f : X \to Y$ be a function and define $\sim$ on X via $a \sim b$ if and only if $f(a) = f(b)$. Recall that $\sim$ is an equivalence relation by Problem 3(c) on Exam 3. Also, recall that $X/\sim$ is the collection of equivalence classes induced by $\sim$.
   (a) (2 points) Explain why each equivalence class $[a]$ is equal to $f^{-1}(\{f(a)\})$,
   (b) (4 points) Prove that the function $g : X/\sim \to f(X)$ defined via $g([a]) = f(a)$ is a bijection.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.
   (a) (2 points) Prove that $f(0) = 0$.
   (b) (2 points) Prove that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.
   (c) (4 points) Prove that f is on-to-one if and only if $f^{-1}(\{0\}) = \{0\}$.

4. (4 points) Let A be a set. Prove that there exists a one-to-one function $f : \mathbb{N} \to A$ if and only if A can be put in bijection with a proper subset of A.

5. (Problem 8.2, 3 points each) Prove **two** of the following. In each case, you should create a bijection between the two sets. Briefly justify that your functions are in fact bijections.
   (a) $\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z})$.
   (b) Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Then $\text{card}((a, b)) = \text{card}((c, d))$.*
   (c) If $\mathcal{F}$ is the set of functions from $\mathbb{N}$ to $\{0, 1\}$, then $\text{card}(\mathcal{F}) = \text{card}(\mathcal{P}(\mathbb{N}))$.†

6. (Theorem 8.5, 4 points) Let A, B, C, and D be sets such that $\text{card}(A) = \text{card}(C)$ and $\text{card}(B) = \text{card}(D)$. Prove **one** of the following.
   (a) If A and B are disjoint and C and D are disjoint, then $\text{card}(A \cup B) = \text{card}(C \cup D)$.
   (b) $\text{card}(A \times B) = \text{card}(C \times D)$.

* *Hint:* Try creating a linear function $f : (a, b) \to (c, d)$. Drawing a picture should help.
† *Hint:* Define $\phi : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ so that $\phi(f)$ outputs a subset of $\mathbb{N}$ determined by when f outputs a 1.

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