Homework 8

Combinatorics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on our Discord server. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

New! At the top of each problem, I would like to list the students that you discussed the problem with. If you worked with the same peers on every problem, then you can simply indicate that once at the top of your assignment.

1. Suppose p is any pattern of length three (i.e., $p \in \{123, 132, 213, 231, 312, 321\}$). Prove that the Catalan numbers count the permutations of length n that avoid p. That is, prove $|S_n(p)| = C_n$ for all such p.

2. Determine whether the following statement is true or false. If true, prove it. Otherwise, provide a counterexample.

   Claim: $N_{n,k} = |\{w \in S_n(123) \mid \mathrm{des}(w) = k\}|$

3. Determine whether the following statement is true or false. If true, prove it. Otherwise, provide a counterexample.

   Claim: $\left\{ {n \atop k} \right\} = \left\{ {n \atop n-k-1} \right\}.$

4. Utilize planar binary trees to find a combinatorial proof of the symmetry relationship for Narayana numbers: $N_{n,k} = N_{n-k-1}$.

5. Let $B(n)$ denote the collection of balanced parenthizations of n pairs of parentheses (of a product of length $n+1$). Let f be the map determined by $N \mapsto$ (and $E \mapsto$). Determine whether f induces a well-defined bijection from $\mathrm{Dyck}(n)$ to $B(n)$.

6. Suppose the bivariate generating function $F(t,z)$ satisfies

$$F(t,z) := \sum_{n \geq k \geq 0} a_{n,k} t^k z^n = \frac{1}{1-(1+t)z}.$$

What is $a_{n,k}$?

7. For $n \in \mathbb{N}$, consider $[n]$. Let $E[n]$ denote the collection of sets of size n we can make out of $[n]$. Let $S[n]$ denote the collection of permutations we can make out of $[n]$.

   (a) Find a closed form for the exponential generating function $E(t) := \sum_{n \geq 0} |E[n]| \frac{t^n}{n!}$, where $|F[0]| = 1$ (i.e., the empty set is the only set of size 0).

   (b) Find a closed form for the exponential generating function $S(t) := \sum_{n \geq 0} |S[n]| \frac{t^n}{n!}$, where $|S[0]| = 1$ (i.e., the empty permutation is the only permutation we can make out of the empty set).