Supplemental Problems for Exam 1

General Information

Exam 1 focuses on content that we have covered since the first day of class up until the end of class on Wednesday, September 21. In particular, here is a list of topics/concepts that you must have an understanding of or be able to do:

- Be familiar with common identities: algebraic, exponential, logarithmic, and trigonometric
- Understand and be able to identify domain, codomain, range, and default domain.
- Be able to perform function arithmetic and function composition. Be able to state the default domain and range of the resulting function, and be able to evaluate such functions at specific values.
- Understand what an inverse function is, and be able to determine whether a given function has an inverse. If a function has an inverse, you should be able to find it in simple cases.
- Be familiar with the graphs of basic functions:
  - $y = mx + b$
  - $y = x^n$ for positive integer n
  - $y = 1/x^n$ for positive integer n
  - $y = \sqrt[n]{x}$ for positive integer n
  - $y = b^x$ for $b > 0$ and $b \neq 1$ (special case: $y = e^x$)
  - $y = \log_b(x)$ for $b > 0$ and $b \neq 1$ (special case: $y = \ln(x)$)
  - $y = \sin(x)$
  - $y = \cos(x)$
  - $y = \tan(x)$
- Understand and be able to sketch and recognize transformations of function graphs.
- Be able to find an equation and the slope of a line given various types of information.
- You should understand and be able to calculate the average rate of change of a function.
- You should have an intuitive understanding of derivativese based on your knowledge of rate of change, speed, velocity, and slope of tangent lines.
- Understand the basic definition and the notation for the limit of a function at a given point.
- Be able to evaluate limits of functions given graphically or as equations.
- Be able to state and use the Squeeze Theorem.
- Understand what it means for a function to be continuous.
- Understand and be able to apply the limit rules.
- Understand the limit definition of the derivative and be able to apply it to basic functions.
- Be able to give examples and counter examples to demonstrate different properties of functions.
- Call upon your own mental faculties to respond in flexible, thoughtful and creative ways to problems that may seem unfamiliar on first glance.

The problems that follow will provide you with an opportunity to review the relevant topics. However:

This is not a practice test!

It is possible that problems on your exam will resemble problems seen below, but you should not expect exam problems to be identical. This document contains an abundance of problems and it is not the intention that every student will complete every problem. You should complete as many problems in each section below as you think are necessary to solidify your understanding.

In addition, you should review examples done in class, as well as your homework exercises, especially the ones on the Weekly Homework assignments.

Words of Advice

Here are few things to keep in mind when taking your exam:

- Show all work! The thought process and your ability to show how and why you arrived at your answer is more important than the answer itself.
- The exam will be designed so that you can complete it without a calculator. If you find yourself yearning for a calculator, you might be doing something wrong.
- Make sure you have answered the question that you were asked. Also, ask yourself if your answer makes sense.
- If you know you made a mistake, but you can't find it, explain why you think you made a mistake and indicate where the mistake might be. This shows that you have a good understanding of the problem.
- If you write down an "=" sign, then you better be sure that the two expressions on either side are equal. Similarly, if two things are equal and it is necessary that they be equal to make your conclusion, then you better use "=."
- Don't forget to write limits where they are needed.

Rates of Change

1. Suppose a ball is thrown off a 100 foot tall building such that the height of the ball in feet at time t in seconds is given by $h(t) = -16t^2 + 25t + 100$.

   (a) What is average rate of change over the first second of flight?

   (b) How about over $[0, 2]$? Interpret the sign of your answer.

   Solution.

   (a) The first second of flight takes place over the interval $[0, 1]$. So, we get

$$\begin{aligned}
\text{avg}_{[0,1]} &= \frac{h(1) - h(0)}{1 - 0} \\
&= \frac{(-16 + 25 + 100) - 100}{1} \\
&= \frac{9}{1} \\
&= 9 \text{ ft/second.}
\end{aligned}$$

(b) Over the interval $[0,2]$, we get

$$\begin{aligned}\text{avg}_{[0,2]} &= \frac{h(2)-h(0)}{2-0}\\ &= \frac{(-64+50+100)-100}{2}\\ &= \frac{-14}{2}\\ &= -7 \text{ ft/second.}\end{aligned}$$

The negative sign indicates that the ball is falling 7 ft/sec on average between 0 and 2 seconds.

2. The position in meters of a particle moving in a straight line is given for some values of time t in seconds in the following table. What is the average velocity over the first .3 seconds of movement?

t	0	.1	.2	.3	.4
$p(t)$	0	.5	.7	1.2	3

Solution. Over the interval $[0,0.3]$, we get

$$\begin{aligned}\text{avg}_{[0,0.3]} &= \frac{p(0.3)-p(0)}{0.3-0}\\ &= \frac{1.2-0}{0.3}\\ &= 4 \text{ meters/sec.}\end{aligned}$$

Limits

3. True or False? Justify your answer.

(a) If a function f does not have a limit as x approaches a from the left, then f does not have a limit as x approaches a from the right.

(b) If $h(x) \leq f(x) \leq g(x)$ for all real numbers x and $\lim_{x\to a} h(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x)$ also exists.

Solution.

(a) False. The following function does not have a limit as x approaches 0 from the left, but the limit as x approaches 0 is 0.

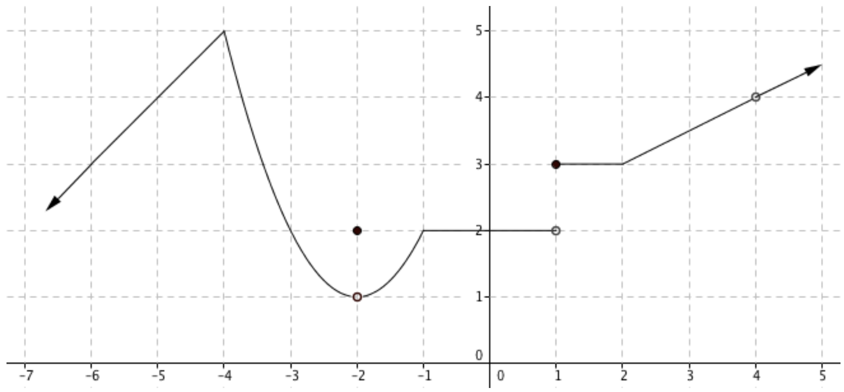
$$f(x) = \begin{cases} x, & x \leq 0 \\ \sin\left(\frac{1}{x}\right), & x > 0 \end{cases}$$

(b) False. Let $h(x) = -1$, $f(x) = \sin(1/x)$, and $g(x) = 1$. Then we have $h(x) \leq f(x) \leq g(x)$ for all real numbers x. Moreover, we have

$$\lim_{x\to 0} h(x) = -1 \quad \text{and} \quad \lim_{x\to 0} g(x) = 1,$$

but the limit of $f(x)$ as x approaches 0 does not exist.

4. Given the graph of f, evaluate each of the following expressions. If an expression does not exist, explain why.



(a) $\lim_{x\to -4^-} f(x)$

(b) $\lim_{x\to -4^+} f(x)$

(c) $\lim_{x\to -4} f(x)$

(d) $f(-4)$

(e) $\lim_{x\to -2^-} f(x)$

(f) $\lim_{x\to -2^+} f(x)$

(g) $\lim_{x\to -2} f(x)$

(h) $f(-2)$

(i) $\lim_{x\to 1^-} f$

(j) $\lim_{x\to 1^+} f(x)$

(k) $\lim_{x\to 1} f(x)$

(l) $f(1)$

(m) $\lim_{x\to 4^-} f(x)$

(n) $\lim_{x\to 4^+} f(x)$

(o) $\lim_{x\to 4} f(x)$

(p) $f(4)$

Solution.

(a) $\lim_{x\to -4^-} f(x) = 5$

(b) $\lim_{x\to -4^+} f(x) = 5$

(c) $\lim_{x\to -4} f(x) = 5$

(d) $f(-4) = 5$

(e) $\lim_{x\to -2^-} f(x) = 1$

(f) $\lim_{x\to -2^+} f(x) = 1$

(g) $\lim_{x\to -2} f(x) = 1$

(h) $f(-2) = 2$

(i) $\lim_{x\to 1^-} f = 2$

(j) $\lim_{x\to 1^+} f(x) = 3$

(k) $\lim_{x\to 1} f(x)$ DNE

(l) $f(1) = 3$

(m) $\lim_{x\to 4^-} f(x) = 4$

(n) $\lim_{x\to 4^+} f(x) = 4$

(o) $\lim_{x\to 4} f(x) = 4$

(p) $f(4)$ undefined

5. Evaluate each of the following limits. If a limit does not exist, specify whether the limit equals ∞, $-\infty$, or simply does not exist (in which case, write DNE).

(a) $\lim_{x\to 2}(x^2+4x-12)$

(b) $\lim_{x\to 2}\dfrac{x^2+4x-12}{x^2+4x+3}$

(c) $\lim_{x\to 2}\dfrac{x^2+4x-12}{x^2-x-2}$

(d) $\lim_{x\to 2}\dfrac{x^2+4x-12}{x^2-4x+4}$

(e) $\lim_{x\to 4}\dfrac{x-4}{\sqrt{x}-2}$

(f) $\lim_{x\to 3}\dfrac{1}{x-3}$

(g) $\lim_{x\to 3}\dfrac{1}{(x-3)^2}$

(h) $\lim_{x\to \pi}\dfrac{x}{\cos(x)}$

(i) $\lim_{x\to 0}\dfrac{\frac{1}{x+4}-\frac{1}{4}}{x}$

(j) $\lim_{x\to 5}\dfrac{|x-5|}{x-5}$

(k) $\lim_{x\to 0^+}\ln(x)$

(l) $\lim_{x\to \infty}\ln(x)$

(m) $\lim_{x\to \infty}\dfrac{4x^2-x+3}{5+x-3x^2}$

(n) $\lim_{x\to \infty}\dfrac{4x^3-x+3}{5+x-3x^2}$

(o) $\lim_{x\to \infty}\dfrac{4x^2-x+3}{5+x-3x^3}$

(p) $\lim_{x\to \infty}\dfrac{\sin(x)}{x}$

(q) $\lim_{x\to \pi/2}\dfrac{\sin(x)}{x}$

(r) $\lim_{x\to 0}\dfrac{\sin(5x)}{7x}$

(s) $\lim_{x\to 0}\dfrac{\sin(2x)-\sin(x)}{x}$

(t) $\lim_{x\to 0}\dfrac{\tan(3x)}{x}$

(u) $\lim_{x\to 0}\dfrac{\tan(4x)}{\tan(5x)}$

(v) $\lim_{x\to 0}\dfrac{\sin^3(x)}{\sin(x^3)}$

(w) $\lim_{x\to 0}\dfrac{1-\cos(x)}{\sqrt{x}}$

(x) $\lim_{x\to 0}\dfrac{1-\cos(x)}{\sin(x)\sin(2x)}$

Solution.

(a) The function is continuous at $x=2$. By substitution, $\lim_{x\to 2}(x^2+4x-12)=2^2+4\cdot 2-12=0$.

(b) The function is continuous at $x=2$. By substitution, $\lim_{x\to 2}\dfrac{x^2+4x-12}{x^2+4x+3}=\dfrac{2^2+4\cdot 2-12}{2^2+4\cdot 2+3}=\dfrac{0}{15}=0$.

(c) The limit has the indeterminate form $0/0$. Factoring the numerator and denominator yields

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-x-2}=\lim_{x\to 2}\frac{(x+6)(x-2)}{(x+1)(x-2)}.$$

Since $x\to 2$, $x\neq 2$ so we can cancel to obtain $\lim_{x\to 2}\dfrac{x+6}{x+1}$, which is a function continuous at $x=2$. By substitution, the limit is $\dfrac{2+6}{2+1}=\dfrac{8}{3}$.

(d) The limit has the indeterminate form $0/0$. Factoring the numerator and denominator yields

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-4x+4}=\lim_{x\to 2}\frac{(x+6)(x-2)}{(x-2)^2}.$$

Since $x\to 2$, $x\neq 2$ so we can cancel to obtain $\lim_{x\to 2}\dfrac{x+6}{x-2}$. Since substitution would give nonzero divided by zero, the one-sided limits are infinite. As $x\to 2^+$, the numerator and denominator are both positive so the limit is ∞; as $x\to 2^-$, the numerator is positive while the denominator is negative, so the limit is $-\infty$. Since the left- and right-side limits are not equal, the two-sided limit that was asked for does not exist.

(e) The limit has the indeterminate form $0/0$. Multiplying by the conjugate, we obtain

$$\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2}=\lim_{x\to 4}\frac{(x-4)(\sqrt{x}+2)}{x-4}.$$

Since $x\to 4$, $x\neq 4$, so we can cancel: $\lim_{x\to 4}(\sqrt{x}+2)=\sqrt{4}+2=4$.

(f) Since substitution would give nonzero divided by zero, the one-sides limits are infinite. As $x \to 3^+$, the numerator and denominator are both positive so the limit is ∞; as $x \to 3^-$, the numerator is positive while the denominator is negative, so the limit is $-\infty$. Since the left- and right-side limits are not equal, the two-sided limit that was asked for does not exist.

(g) Since substitution would give nonzero divided by zero, the one-sided limits are infinite. As $x \to 3^+$, the numerator and denominator are both positive so the limit is ∞; as $x \to 3^-$, the numerator and denominator are still both positive, so the limit is again ∞. Since the left- and right-side limits are both ∞, the two-sided limit that was asked for equals ∞.

(h) The function is continuous at $x = \pi$. By substitution, $\lim\limits_{x\to\pi} \dfrac{x}{\cos(x)} = \dfrac{\pi}{\cos\pi} = \dfrac{\pi}{-1} = -\pi$.

(i) This limit can be computed as follows, with the first four equalities justified by equality of the expression for non-zero x (guaranteed by the limit destination) and the last equality a result of direct substitution into the continuous function $-1/(4(x+4))$.

$$\lim_{x\to 0} \frac{\frac{1}{x+4} - \frac{1}{4}}{x} = \lim_{x\to 0} \frac{\frac{4}{4}\cdot\frac{1}{x+4} - \frac{x+4}{x+4}\cdot\frac{1}{4}}{x} = \lim_{x\to 0} \frac{\frac{4-(x+4)}{4(x+4)}}{x} = \lim_{x\to 0} \frac{\frac{-x}{4(x+4)}}{x} = \lim_{x\to 0} \frac{-1}{4(x+4)} = \frac{-1}{16}.$$

(j) To determine this limit, we examine each of the one-sided limits:

$$\lim_{x\to 5^+} \frac{|x-5|}{x-5} = \lim_{x\to 5^+} \frac{x-5}{x-5} = 1 \quad \text{and} \quad \lim_{x\to 5^-} \frac{|x-5|}{x-5} = \lim_{x\to 5^+} \frac{-(x-5)}{x-5} = -1.$$

Since these limits are not equal, the two-sided limit does not exist.

(k) Examining the graph of $y = \ln(x)$, $\lim\limits_{x\to 0^+} \ln(x) = -\infty$.

(l) Examining the graph of $y = \ln(x)$, $\lim\limits_{x\to\infty} \ln(x) = \infty$.

(m) We see that

$$\lim_{x\to\infty} \frac{4x^2 - x + 3}{5 + x - 3x^2} = \lim_{x\to\infty} \left(\frac{4x^2 - x + 3}{5 + x - 3x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \right) = \lim_{x\to\infty} \frac{4 - \frac{1}{x} + \frac{3}{x^2}}{\frac{5}{x^2} + \frac{x}{x^2} - 3} = \frac{4}{-3},$$

where the first equality is justified by the unity of $\dfrac{1/x^2}{1/x^2}$ for non-zero x, the second by the distributive property of multiplication, and the third by The Limit Laws. In particular, the limits of $-1/x$, $3/x^2$, $5/x^2$, and x/x^2 at infinity are all 0.

(n) We see that

$$\lim_{x\to\infty} \frac{4x^3 - x + 3}{5 + x - 3x^2} = \lim_{x\to\infty} \left(\frac{4x^3 - x + 3}{5 + x - 3x^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \right) = \lim_{x\to\infty} \frac{4x - \frac{1}{x} + \frac{3}{x^2}}{\frac{5}{x^2} + \frac{x}{x^2} - 3} = \lim_{x\to\infty} \frac{4x}{-3} = -\infty,$$

where the first equality is justified by the unity of $\dfrac{1/x^2}{1/x^2}$ for non-zero x, the second by the distributive property of multiplication, and the third by The Limit Laws. In particular, the limits of $-1/x$, $3/x^2$, $5/x^2$, and x/x^2 at infinity are all 0.

(o) We see that

$$\lim_{x\to\infty} \frac{4x^2 - x + 3}{5 + x - 3x^3} = \lim_{x\to\infty} \left(\frac{4x^2 - x + 3}{5 + x - 3x^3} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \right) = \lim_{x\to\infty} \frac{4 - \frac{1}{x} + \frac{3}{x^2}}{\frac{5}{x^2} + \frac{x}{x^2} - 3x} = \lim_{x\to\infty} \frac{4}{-3x} = 0,$$

where the first equality is justified by the unity of $\dfrac{1/x^2}{1/x^2}$ for non-zero x, the second by the distributive property of multiplication, and the third by The Limit Laws. Also, the limits of $-1/x$, $3/x^2$, $5/x^2$, and x/x^2 at infinity are all 0.

(p) Note that $-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$. Also, $\lim\limits_{x\to\infty} \frac{1}{x} = 0 = \lim\limits_{x\to\infty} -\frac{1}{x}$. Thus, by the Squeeze Theorem, $\lim\limits_{x\to\infty} \frac{\sin(x)}{x} = 0$.

(q) $\lim\limits_{x\to\pi/2} \frac{\sin(x)}{x} = \frac{\sin(\pi/2)}{\pi/2} = \frac{2}{\pi}$.

(r) $\lim\limits_{x\to 0} \frac{\sin(5x)}{7x} = \lim\limits_{x\to 0} \left(\frac{5}{7}\right) \frac{\sin(5x)}{5x} = \left(\frac{5}{7}\right) \lim\limits_{x\to 0} \frac{\sin(5x)}{5x} = \frac{5}{7}$

(s) $\lim\limits_{x\to 0} \dfrac{\sin(2x) - \sin(x)}{x} = \lim\limits_{x\to 0} \dfrac{\sin(2x)}{x} - \lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = \lim\limits_{x\to 0} \dfrac{2x}{x} \cdot \dfrac{\sin(2x)}{2x} - \lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = \lim\limits_{x\to 0} 2 \cdot \dfrac{\sin(2x)}{2x} - \lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 2 \cdot 1 - 1 = 1$

(t) $\lim\limits_{x\to 0} \frac{\tan(3x)}{x} = \lim\limits_{x\to 0} \left(\frac{\sin(3x)}{x} \cdot \frac{1}{\cos(3x)}\right) = \lim\limits_{x\to 0} \left[3\left(\frac{\sin(3x)}{3x}\right)\left(\frac{1}{\cos(3x)}\right)\right] = 3\left(\lim\limits_{x\to 0} \frac{\sin(3x)}{3x}\right)\left(\lim\limits_{x\to 0} \frac{1}{\cos(3x)}\right) = 3$

(u) $\lim\limits_{x\to 0} \frac{\tan(4x)}{\tan(5x)} = \lim\limits_{x\to 0} \left(\frac{\sin(4x)}{\cos(4x)} \cdot \frac{\cos(5x)}{\sin(5x)}\right) = \lim\limits_{x\to 0} \left(\frac{\sin(4x)}{4x} \cdot 4x \cdot \frac{5x}{\sin(5x)} \cdot \frac{1}{5x} \cdot \frac{\cos(5x)}{\cos(4x)}\right) = \frac{4}{5}$

(v) $\lim\limits_{x\to 0} \frac{\sin^3(x)}{\sin(x^3)} = \lim\limits_{x\to 0} \left(\frac{\sin(x)}{x} \cdot \frac{\sin(x)}{x} \cdot \frac{\sin(x)}{x} \cdot \frac{x^3}{\sin(x^3)}\right) = 1$

(w) $\lim\limits_{x\to 0} \frac{1-\cos(x)}{\sqrt{x}} = \lim\limits_{x\to 0} \left(\frac{1-\cos(x)}{x} \cdot \sqrt{x}\right) = \left(\lim\limits_{x\to 0} \frac{1-\cos(x)}{x}\right)\left(\lim\limits_{x\to 0} \sqrt{x}\right) = 0$

(x) $\lim\limits_{x\to 0} \frac{1-\cos(x)}{\sin(x)\sin(2x)} = \lim\limits_{x\to 0} \frac{(1-\cos(x))(1+\cos(x))}{\sin(x)\sin(2x)(1+\cos(x))} = \lim\limits_{x\to 0} \frac{1-\cos^2(x)}{\sin(x)\sin(2x)(1+\cos(x))} = \lim\limits_{x\to 0} \frac{\sin^2(x)}{\sin(x)\sin(2x)(1+\cos(x))} = \lim\limits_{x\to 0} \frac{\sin(x)}{\sin(2x)(1+\cos(x))} = \lim\limits_{x\to 0} \left(\frac{\sin(x)}{\sin(2x)(1+\cos(x))} \frac{(2x)x}{(2x)x}\right) = \lim\limits_{x\to 0} \frac{1}{2(1+\cos(x))} = \frac{1}{4}$

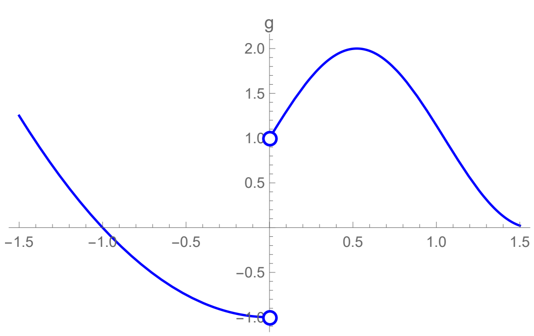
6. Sketch the graphs of possible functions g and h such that g satisfies property (a) below and h satisfies property (b) below. (There should be two separate graphs.)

(a) $\lim\limits_{x\to 0^-} g(x) = -1$ and $\lim\limits_{x\to 0^+} g(x) = 1$

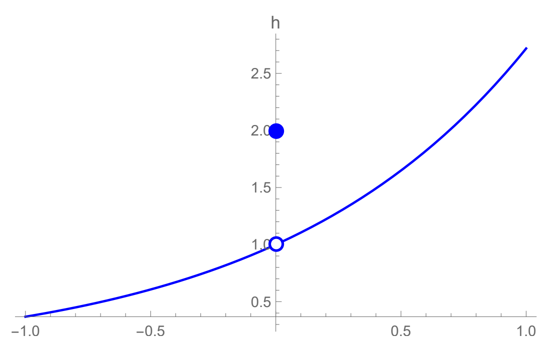
(b) $\lim\limits_{x\to 0} h(x) \neq h(0)$, where $h(0)$ is defined.

Solution.

(a)



(b)



7. Consider the following function.

$$f(x) = \begin{cases} \dfrac{-1}{x-2}, & x > -1 \\ x^2+1, & x \leq -1 \end{cases}$$

Evaluate each of the following expressions. If an expression does not exist, specify whether it equals ∞, $-\infty$, or simply does not exist (in which case, write DNE). You do *not* need to justify your answers.

(a) $f(-1)$

(b) $\lim\limits_{x\to -1^-} f(x)$

(c) $\lim\limits_{x\to -1^+} f(x)$

(d) $\lim\limits_{x\to -1} f(x)$

(e) $f(2)$

(f) $\lim\limits_{x\to 2^+} f(x)$

Solution.

(a) $f(-1) = (-1)^2 + 1 = 2$

(b) $\lim\limits_{x\to -1^-} f(x) = 2$

(c) $\lim\limits_{x\to -1^+} f(x) = \dfrac{-1}{-1-2} = 1/3$

(d) $\lim\limits_{x\to -1} f(x)$ DNE

(e) $f(2)$ DNE

(f) $\lim\limits_{x\to 2^+} f(x) = -\infty$

8. Sketch the graph of a possible function f that has all properties (a)–(g) listed below.

(a) The domain of f is $[-1, 2]$

(b) $f(0) = f(2) = 0$

(c) $f(-1) = 1$

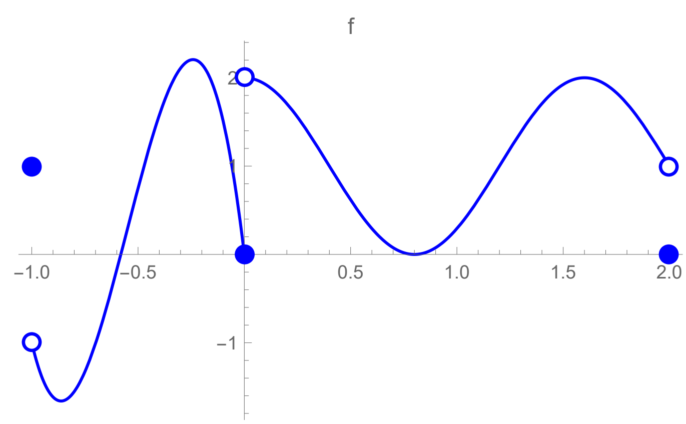
(d) $\lim\limits_{x\to 0^-} f(x) = 0$

(e) $\lim\limits_{x\to 0^+} f(x) = 2$

(f) $\lim\limits_{x\to 2^-} f(x) = 1$

(g) $\lim\limits_{x\to -1^+} f(x) = -1$

Solution.



9. Let f and g be functions such that $\lim\limits_{x\to a} f(x) = -3$ and $\lim\limits_{x\to a} g(x) = 6$. Evaluate the following limits, if they exist.

(a) $\lim\limits_{x\to a} \dfrac{(g(x))^2}{f(x)+5}$

(b) $\lim\limits_{x\to a} \dfrac{7f(x)}{2f(x)+g(x)}$

(c) $\lim\limits_{x\to a} \sqrt[3]{g(x)+2}$

Solution.

(a) $\lim_{x\to a} \frac{(g(x))^2}{f(x)+5} = \frac{\left(\lim\limits_{x\to a} g(x)\right)^2}{\left(\lim\limits_{x\to a} f(x)\right)+5} = \frac{6^2}{-3+5} = \frac{36}{-3+5} = \frac{36}{2} = 18$

(b) First, notice that

$$\lim_{x\to a}(2f(x)+g(x)) = 0$$

while

$$\lim_{x\to a} 7f(x) = -21.$$

The upshot is that the limit as x approaches a does not exist. We do not have enough information to determine whether the limit is possibly ∞ or $-\infty$.

(c) $\lim_{x\to a}(g(x)+2)^{1/3} = (\lim_{x\to a} g(x)+2)^{1/3} = 8^{1/3} = 2$

10. Let f be defined as follows.

$$f(x) = \begin{cases} 3x & \text{if } x < 0 \\ 3x+4 & \text{if } 0 \le x \le 4 \\ x^2 & \text{if } x > 4 \end{cases}$$

For (a)–(h), evaluate the limits if they exist. If a limit does not exist, specify whether the limit equals ∞, $-\infty$, or simply does not exist (in which case, write DNE). For part (i), answer the question.

(a) $\lim\limits_{x\to 0^+} f(x)$

(b) $\lim\limits_{x\to 0^-} f(x)$

(c) $\lim\limits_{x\to 0} f(x)$

(d) $f(0)$

(e) $\lim\limits_{x\to 4^+} f(x)$

(f) $\lim\limits_{x\to 4^-} f(x)$

(g) $\lim\limits_{x\to 4} f(x)$

(h) $f(4)$

(i) Determine where f is continuous.

Solution.

(a) $\lim\limits_{x\to 0^+} f(x) = 4$

(b) $\lim\limits_{x\to 0^-} f(x) = 0$

(c) $\lim\limits_{x\to 0} f(x)$ does not exist

(d) $f(0) = 4$

(e) $\lim\limits_{x\to 4^+} f(x) = 16$

(f) $\lim\limits_{x\to 4^-} f(x) = 16$

(g) $\lim\limits_{x\to 4} f(x) = 16$

(h) $f(4) = 16$

(i) The function f is not continuous because of a jump discontinuity at $x = 0$. The function is right-continuous, though.

11. If $3x \le f(x) \le x^3+2$ for $0 \le x \le 2$, evaluate $\lim\limits_{x\to 1} f(x)$.

Solution. Since $3x \le f(x) \le x^3+2$ in an open interval containing $x = 1$, we know that $\lim\limits_{x\to 1} 3x \le \lim\limits_{x\to 1} f(x) \le \lim\limits_{x\to 1}(x^3+2)$. Using substitution on the first and last limits we get $3 \le \lim\limits_{x\to 1} f(x) \le 3$, so we must have $\lim\limits_{x\to 1} f(x) = 3$.

12. Use the Squeeze Theorem to prove that $\lim\limits_{x\to 0} x^4\cos(2/x) = 0$.

Solution. Note that: $-1 \leq \cos(2/x) \leq 1$ for all $x \neq 0$ due to the boundedness of sinusoidal functions, $-x^4 \leq x^4\cos(2/x) \leq x^4$ because of the previous inequality and the positivity of x^4 for $x \neq 0$, and $\lim_{x\to 0}(-x^4) = \lim_{x\to 0} x^4 = 0^4 = -0^4$ by the continuity of the polynomials x^4 and $-x^4$. Thus, the hypotheses of the Squeeze Theorem are satisfied (the function whose limit is being considered is bounded around the destination by functions whose limits equate at the destination), and so $\lim_{x\to 0} x^4\cos(2/x) = 0$ as well.

Continuity

13. True or False? If a function is not continuous at $x = a$, then either it is not defined at $x = a$ or it does not have a limit as x approaches a. Justify your answer.

    Solution. False. A counterexample is given in Problem 4. The function is discontinous at $x = -2$ but the limit exists (this is 1) and the function is also defined at that point ($f(-2) = 2$). The problem is that these two values are not the same ($1 \neq 2$).

14. Provide an example of function that is continuous everywhere but does not have a tangent line at $x = 0$. Explain your answer.

    Solution. The function $f(x) = |x|$ is continuous everywhere but is not differentiable at $x = 0$, and hence the graph of $f(x) = |x|$ does not have a tangent line at $x = 0$.

Derivative at a Point

15. Sketch the graph of each function given below and then sketch the tangent line to that function at the given point.

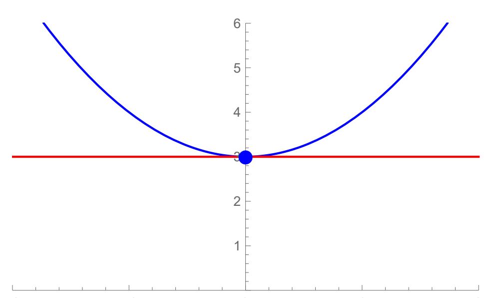
    (a) $f(x) = x^2 + 3$ at $x = 0$

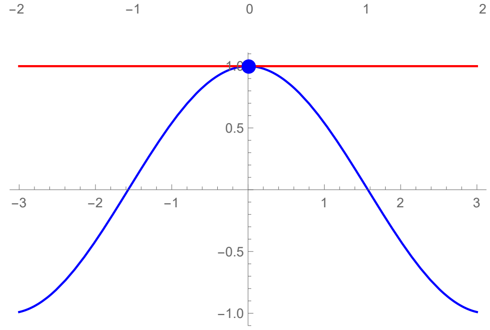
    (b) $g(x) = \cos(x)$ at $x = 0$

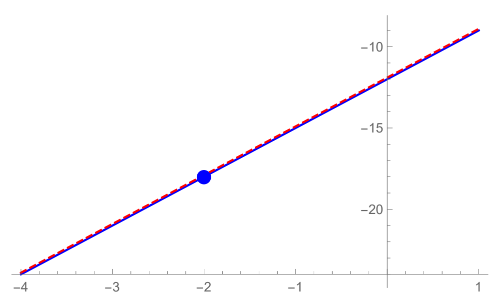
    (c) $h(x) = 3x - 12$ at $x = -2$

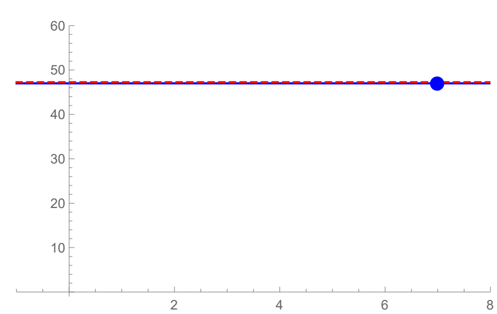
    (d) $f(x) = 47$ at $x = 7$

    Solution.

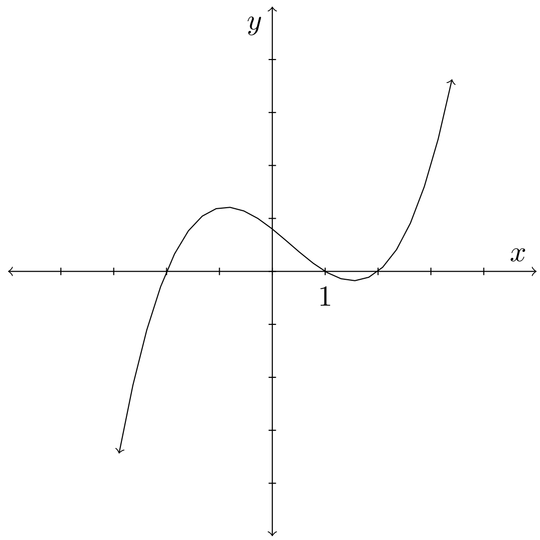
(a) 

(b) 

(c) 

(d) 

16. Consider the graph of the function $y = f(x)$ given in the figure below. Put the following expressions in increasing order: $f'(-2), f'(-1), f'(0), f'(2)$.



Solution. $f'(0) < f'(1) < f'(2) < f'(-2)$

17. Let $f(x) = x^2 - x$.

(a) Find the slope of the tangent line at $x = 2$ using the limit definition.

(b) Find an equation of the tangent line to the graph of f at $x = 2$.

Solution.

(a) We see the difference quotient is:

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\left[(x+h)^2-(x+h)\right]-\left[x^2-x\right]}{h}\\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-x-h-x^2+x}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2-h}{h}\\ &= \lim_{h\to 0}(2x+h-1)\\ &= 2x-1\end{aligned}$$

So the slope at $x = 2$ is $2(2) - 1 = 3$.

(b) We see that $f(2) = 2$. So we find the line of slope $m = 3$ (from part (a)) through the point $(2, 2)$. We get $y = 3x - 4$.

18. For each of the following functions, use the limit definition to find the derivative at the specified x-value.

(a) $f(x) = x^2 + 16x - 57$, $x = 1$

(b) $f(x) = \sqrt{5x+1}$, $x = 3$

(c) $f(x) = \dfrac{1}{x}$, $x = 2$

(d) $f(x) = \dfrac{1}{x^2}$, $x = -1$

Solution.

(a) $f'(1) = \lim_{x\to 1} \dfrac{x^2+16x-57-(1^2+16\cdot 1-57)}{x-1} = \lim_{x\to 1} \dfrac{x^2+16x-17}{x-1} = \lim_{x\to 1} \dfrac{(x+17)(x-1)}{x-1}$
$= \lim_{x\to 1}(x+17) = 1+17 = 18.$

(b) $f'(3) = \lim_{x\to 3} \dfrac{\sqrt{5x+1}-\sqrt{5\cdot 3+1}}{x-3} = \lim_{x\to 3} \dfrac{\sqrt{5x+1}-4}{x-3}\cdot\dfrac{\sqrt{5x+1}+4}{\sqrt{5x+1}+4} = \lim_{x\to 3} \dfrac{5x-15}{(x-3)(\sqrt{5x+1}+4)}$
$= \lim_{x\to 3} \dfrac{5}{\sqrt{5x+1}+4} = \dfrac{5}{8}.$

(c) $f'(2) = \lim_{x\to 2} \dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}\cdot\dfrac{2x}{2x} = \lim x \to 2 \dfrac{2-x}{(x-2)(2x)} = \lim x \to 2 \dfrac{-1}{2x} = \dfrac{-1}{4}.$

(d) $f'(-1) = \lim_{x\to -1} \dfrac{\frac{1}{x^2}-\frac{1}{(-1)^2}}{x+1} = \lim_{x\to -1} \dfrac{\frac{1}{x^2}-1}{x+1}\cdot\dfrac{x^2}{x^2} = \lim_{x\to -1} \dfrac{1-x^2}{x+1} = \lim_{x\to -1} \dfrac{(1+x)(1-x)}{x+1} = \lim_{x\to -1}(1-x)$
$= 1-(-1) = 2.$

19. Suppose the equation of the tangent line to the graph of some function f at $x = 1$ is given by $y = 2x+1$. Find each of the following.

(a) $f'(1)$

(b) $f(1)$

Solution. By the definition of tangent, $f'(1)$ is the slope of the tangent line, which is 2, and $f(1)$ is the value of the tangent line at $x = 1$, which is 3. In summary, $f'(1) = 2$ and $f(1) = 3$.

Proofs

20. If f and g are continuous at $x = c$, prove that $f+g$ is continuous at $x = c$.

Solution. Because f and g are continuous at $x = c$, $\lim_{x\to c} f(x) = f(c)$ and $\lim_{x\to c} g(x) = g(c)$. To determine if $f+g$ is continuous at $x = c$ or not, we need to verify that $\lim_{x\to c}(f+g)(x)$ and $(f+g)(c)$ exist and are equal. By the algebra of functions, $(f+g)(c) = f(c)+g(c)$ which exists indeed. By the Limit Laws, $\lim_{x\to c}(f+g)(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$ since both of the right-hand limits exist. Because these right hand limits sum to $f(c)+g(c)$ due to continuity, we have shown that $f+g$ is continuous at $x = c$.

21. Using the fact that $\cos(\theta) \le \dfrac{\sin(\theta)}{\theta} \le 1$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $\theta \ne 0$, prove:

(a) $\lim_{\theta\to 0} \dfrac{\sin(\theta)}{\theta} = 1$, and

(b) $\lim_{\theta\to 0} \dfrac{1-\cos(\theta)}{\theta} = 0$.

Solution.

(a) We are given that $\cos(\theta) \le \dfrac{\sin(\theta)}{\theta} \le 1$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $\theta \ne 0$. Also, by continuity, we know that $\lim_{\theta\to 0}\cos(\theta) = 1$ and $\lim_{\theta\to 0} 1 = 1$. Thus, by the Squeeze Theorem we have that

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1.$$

(b) We see that

$$\begin{aligned}
\lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta} &= \lim_{\theta\to 0}\frac{1-\cos(\theta)}{\theta}\cdot\frac{1+\cos(\theta)}{1+\cos(\theta)} \\
&= \lim_{\theta\to 0}\left(\frac{1}{1+\cos(\theta)}\cdot\frac{1-\cos^2(\theta)}{\theta}\right) \\
&= \lim_{\theta\to 0}\left(\frac{1}{1+\cos(\theta)}\cdot\frac{\sin^2(\theta)}{\theta}\right) \\
&= \lim_{\theta\to 0}\left(\frac{1}{1+\cos(\theta)}\cdot\frac{\sin(\theta)}{\theta}\cdot\sin(\theta)\right) \\
&= \frac{1}{2}\cdot 1\cdot 0 && \text{(by continuity and part (a))} \\
&= 0.
\end{aligned}$$

22. If $f(x) = mx + b$ is a linear function, use the limit definition of the derivative to prove that $f'(x) = m$. Can you also justify this fact by appealing to the graph of f?

Solution. Using the limit definition of the derivative, we see that

$$f'(x) = \lim_{h\to 0}\frac{m(x+h)+b-(mx+b)}{h} = \lim_{h\to 0}\frac{mx+mh+b-mx-b}{h} = \lim_{h\to 0}\frac{mh}{h} = \lim_{h\to 0} m = m.$$

Since the graph of $f(x) = mx + b$ is a line, every tangent line to the graph of this function will coincide with the line. Hence very tangent line to the graph of $f(x) = mx + b$ will have slope m.

23. If $f(x) = c$ is a constant function, use the limit definition of the derivative to prove that $f'(x) = 0$. Can you also justify this fact by appealing to the graph of f?

Solution. Using the limit definition of the derivative, we see that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to o}\frac{0}{h} = \lim_{h\to 0} 0 = 0.$$

The graph of f is a horizontal line, and so every tangent line to the graph of this function will coincide with the line. Since the line is horizontal, every tangent line has slope 0.