Chapter 7

Introduction to Graph Theory

Loosely speaking, a graph is a collection of points called vertices and connecting segments called edges, each of which starts at a vertex, ends at a vertex and contains no other vertices beside these. More formally, we define the term as follows. A **graph** consists of two sets, a nonempty set V of points called **vertices** and a set E whose elements, called **edges**, are are multisets of size two from V.

Each edge is associated with either one vertex which serves as both endpoints or two vertices as its endpoints. Technically, each edge is a multiset of the form $\boxed{\{\!\{u, v\}\!\}}$ where $u, v \in V$. We say that u and v are **endpoints** of the edge $\{\!\{u, v\}\!\}$. In an abuse of notation, it is customary to write $\boxed{\{u, v\}}$ even if $u = v$. In fact, we may abbreviate further and denote the edge by $\boxed{uv}$. Note that the order in which the vertices of an edge are listed is irrelevant. That is, $\{\!\{u, v\}\!\} = \{\!\{v, u\}\!\}$, $\{u, v\} = \{v, u\}$, and $uv = vu$. If G is the graph associated with the vertex set V and edge set E, we write $\boxed{G = (V, E)}$. It is worth pointing out that we assumed that V is nonempty, but E is allowed to be empty (i.e., the graph has no edges).

It is customary to represent a graph using visual representations, where each vertex is a dot and each edge is a connecting segment, not necessarily straight.

Problem 7.1. Find at least five different graphs with vertex sets $V = \{a, b, c\}$.

There is a lot of terminology associated to graphs! Here are some of the relevant concepts.

- Vertices u and v of a graph are **adjacent** if they are the endpoints of the same edge.
- If v is an endpoint of the edge e, we say that e is **incident** to v.
- If an edge e is incident to vertices u and v, we say that u and v are **connected** by edge e.
- An edge e that is incident to a single vertex (i.e., $e = uu$ for some $u \in V$) is called a **loop**.
- The **order** of a graph is the number of vertices in the graph. That is, if $G = (V, E)$, then the order of G is $|V|$.
- The **degree** of a vertex v, written $\deg(v)$, is the number of edges incident to v (i.e., the number of edges that have v as an endpoint).

Many graphs have similar properties that allow us to categorize them. Here are several families of graphs.

- Complete Graphs. The **complete graph** on $n \geq 1$ vertices, denoted $\boxed{K_n}$, is the graph of order n such that each pair of vertices is connected by exactly one edge, and there are no other edges (i.e., no loops).
- Cycle Graphs. The **cycle graph** on $n \geq 3$ vertices, denoted $\boxed{C_n}$, is the graph such that when the n vertices are suitably labeled $v_1, v_2, \ldots, v_n$, the edges of C_n are $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.
- Path Graphs. The **path** on $n \geq 1$ vertices, denoted $\boxed{P_n}$, has a description similar to C_n: for distinct vertices suitably labeled $v_1, v_2, \ldots, v_n$, the edges of P_n are $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$.
- Wheel Graphs. The **wheel graph** on $n \geq 4$ vertices, denoted $\boxed{W_n}$, is the graph C_{n-1} together with one additional vertex that is connected to each of the vertices of C_{n-1}.
- Hypercube Graphs. The **hypercube** of dimension $n \geq 1$, denoted $\boxed{Q_n}$, is the graph whose vertices are labeled with the 2^n bit strings of length n with an edge connecting two vertices if and only if their labels differ in exactly one bit.

Problem 7.2. Draw the first few graphs of each of the graph families above.

Problem 7.3. How many edges do each of the following have?

(a) K_n

(b) C_n

(c) P_n

(d) W_n

(e) Q_n

A **simple graph** is a graph in which each edge connects two distinct vertices and each pair of vertices is connected by at most one edge. Note that the graphs K_n, C_n, P_n, W_n, and Q_n are all simple graphs. A **pseudograph** (or **multigraph**) is like a graph but we allow **multiple edges** between a pair of vertices (i.e., E is a multiset instead of a set).

Problem 7.4. Draw examples of simple graphs, non-simple graphs, and psuedographs on 3 vertices.

A simple graph $G = (V, E)$ is **bipartite** if there is a partition of V into two nonempty sets V_1, V_2 (i.e., $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, $V_1 \cap V_2 = \emptyset$, and $V_1 \cup V_2 = V$) such that each edge of G connects a vertex in V_1 and a vertex in V_2. The pair (V_1, V_2) is called a **bipartition** of the graph.

Problem 7.5. Provide an example of a bipartite graph with 5 vertices.

The following theorem provides a nice characterization of bipartite graphs.

Theorem 7.6. A graph is bipartite if each vertex can be colored with one of two colors so that each pair of adjacent vertices have different colors.

Problem 7.7. Which complete graphs are bipartite?

Problem 7.8. Which path graphs are bipartite?

Problem 7.9. Which cycle graphs are bipartite?

Problem 7.10. Is Q_3 bipartite?

A bipartite graph with bipartition (V_1, V_2) such that $|V_1| = m$ and $|V_2| = n$ is the **complete bipartite graph** $K_{m,n}$ if it contains each edge $\{u, v\}$ for every pair $u \in V_1$ and $v \in V_2$. Note that $K_{m,n} = K_{n,m}$.

Problem 7.11. Draw $K_{1,1}$, $K_{1,2}$, $K_{2,2}$, $K_{2,3}$, $K_{3,3}$.

More coming soon...