Pure mathematics is the poetry of logical ideas.

Albert Einstein, theoretical physicist

Chapter 2

Mathematics and Logic

Before you get started, make sure you have read Chapter 1, which sets the tone for the work we will begin doing here. In addition, you might find is useful to read Appendix A: Elements of Style for Proofs. As stated at the end of Section 1.5, you are encouraged to review this appendix occasionally as you progress through the book as some guidelines may not initially make sense or seem relevant.

2.1 A Taste of Number Theory

It is important to point out that we are diving in head first here. As we get started, we are going to rely on your intuition and previous experience with proofs. This is intentional. What you will likely encounter is a general sense of what a proof entails, but you may not be able to articulate the finer details that you do and do not comprehend. There are going to be some subtle issues that you will be confronted with and one of our goals will be to elucidate as many of them as possible. We need to calibrate and develop an intellectual need for structure. You are encouraged to just try your hand at writing proofs for the problems in this section without too much concern for whether you are "doing it the right way." In Section 2.2, we will start over and begin to develop a formal foundation for the material in the remainder of the book. Once you have gained more experience and a better understanding of what a proof entails, you should consider returning to this section and reviewing your first attempts at writing proofs. In the meantime, see what you can do!

In this section, we will introduce the basics of a branch of mathematics called **number theory**, which is devoted to studying the properties of the integers. The integers is the set of numbers given by

$$\boxed{\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}}.$$

The collection of positive integers also have a special name. The set of **natural numbers** is given by

$$\boxed{\mathbb{N} := \{1, 2, 3, \ldots\}}.$$

Some mathematicians (set theorists, in particular) include 0 in $\mathbb{N}$, but this will not be our convention. If you look closely at the two sets we defined above, you will notice that we wrote $\boxed{:=}$ instead of $=$. We use $:=$ to mean that the symbol or expression on the left is defined to be equal to the expression on the right. The symbol $\boxed{\mathbb{R}}$ is used to denote the set of all **real numbers**. We will not formally define the real numbers, but instead rely on your prior intuition and understanding.

Because you are so familiar with many of the properties of the integers and real numbers, one of the issues that we will bump into is knowing which facts we can take for granted. As a general rule of thumb, you should attempt to use the definitions provided without relying too much on your prior knowledge. The order in which we develop things is important.

It is common practice in mathematics to use the symbol $\boxed{\in}$ as an abbreviation for the phrase "is an element of" or sometimes simply "in." For example, the mathematical expression "$n \in \mathbb{Z}$" means "n is an element of the integers." However, some care should be taken in how this symbol is used. We will only use the symbol "$\in$" in expressions of the form $\boxed{a \in A}$, where A is a set and a is an element of A. We will write expressions like $\boxed{a, b \in A}$ as shorthand for "$a \in A$ and $b \in A$." We should avoid writing phrases such as "a is a number $\in A$" and "$n \in$ integers".

We will now encounter our very first definition. In mathematics, a **definition** is a precise and unambiguous description of the meaning of a mathematical term. It characterizes the meaning of a word by giving all the properties and only those properties that must be true. Check out Appendix B for a list of other mathematical terms that we should be familiar with.

Definition 2.1. An integer n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$. An integer n is **odd** if $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Notice that we framed the definition of "even" in terms of multiplication as opposed to division. When tackling theorems and problems involving even or odd, be sure to make use of our formal definitions and not some of the well-known divisibility properties. For now, you should avoid arguments that involve statements like, "even numbers have no remainder when divided by two" or "the last digit of an even number is 0, 2, 4, 6, or 8." Also notice that the notions of even and odd apply to zero and negative numbers. In particular, zero is even since $0 = 2 \cdot 0$, where it is worth emphasizing that the occurrence of 0 on the righthand side of equation is an integer. As another example, we see that -1 is odd since $-1 = 2(-1) + 1$. Despite the fact that $-1 = 2(-1/2)$, this does not imply that -1 is also even since $-1/2$ is not an integer. For the remainder of this section, you may assume that every integer is either even or odd but never both.

Our first theorem concerning the integers is stated below. A **theorem** is a mathematical statement that is proved using rigorous mathematical reasoning.

Theorem 2.2. If n is an even integer, then n^2 is an even integer.

One crux in proving the next theorem involves figuring out how to describe an arbitrary pair of consecutive integers.

Theorem 2.3. The sum of two consecutive integers is odd.

One skill we will want to develop is determining whether a given mathematical statement is true or false. In order to verify that a mathematical statement is false, we should provide a specific example where the statement fails. Such an example is called a **counterexample**. Notice that it is sufficient to provide a single example to verify that a general statement is not true. On the other hand, if we want to prove that a general mathematical statement is true, it is usually not sufficient to provide just a single example, or even a hundred examples. Such examples are just evidence that the statement is true.

Problem 2.4. Determine whether each of the following statements is true or false. If a statement is true, prove it. If a statement is false, provide a counterexample.

(a) The product of an odd integer and an even integer is odd.

(b) The product of an odd integer and an odd integer is odd.

(c) The product of an even integer and an even integer is even.

(d) The sum of an even integer and an odd integer is odd.

For the statements that were true in the previous problem, you may cite them later in a future proof as if they are theorems.

Definition 2.5. Given $n, m \in \mathbb{Z}$, we say that n **divides** m, written $\boxed{n|m}$, if there exists $k \in \mathbb{Z}$ such that $m = nk$. If $n|m$, we may also say that m is **divisible by** n or that n is a **factor** of m.

Problem 2.6. For $n, m \in \mathbb{Z}$, how are the following mathematical expressions similar and how are they different? In particular, is each one a sentence or simply a noun?

(a) $n|m$

(b) $\dfrac{m}{n}$

(c) m/n

In this section on number theory, we allow addition, subtraction, and multiplication of integers. In general, we avoid division since an integer divided by an integer may result in a number that is not an integer. The upshot is that we will avoid writing $\frac{m}{n}$. When you feel the urge to divide, switch to an equivalent formulation using multiplication. This will make your life much easier when proving statements involving divisibility.

Theorem 2.7. The sum of any three consecutive integers is always divisible by three.

Problem 2.8. Let $a, b, n, m \in \mathbb{Z}$. Determine whether each of the following statements is true or false. If a statement is true, prove it. If a statement is false, provide a counterexample.

(a) If $a|n$, then $a|mn$.

(b) If 6 divides n, then 2 divides n and 3 divides n.

(c) If ab divides n, then a divides n and b divides n.

A theorem that follows almost immediately from another theorem is called a **corollary**. See if you can prove the next result quickly using a previous result. Be sure to cite the result in your proof.

Corollary 2.9. If $a, n \in \mathbb{Z}$ such that a divides n, then a divides n^2.

The next two theorems are likely familiar to you.

Theorem 2.10. If $a, n \in \mathbb{Z}$ such that a divides n, then a divides $-n$.

Theorem 2.11. If $a, n, m \in \mathbb{Z}$ such that a divides m and a divides n, then a divides $m + n$.

Notice that we have been tinkering with statements of the form "If..., then...". Statements of this form are called **conditional propositions**, which we revisit in the next section. The phrase that occurs after "If" but before "then" is called the **hypothesis** while the phrase that occurs after "then" is called the **conclusion**. For example, in Problem 2.8(a), "$a|n$" is the hypothesis while "$a|mn$" is the conclusion. Note that conditional propositions can also be written in the form "... if ...", where the conclusion is written before "if" and the hypothesis after. For example, we can rewrite Problem 2.8(a) as "$a|mn$ if $a|n$". While the order of the hypothesis and conclusion have been reversed in the sentence, their roles have not.

Whenever we encounter a conditional statement in mathematics, we want to get in the habit of asking ourselves what happens when we swap the roles of the hypothesis and the conclusion. The statement that results from reversing the roles of the hypothesis and conclusion in a conditional statement is called the **converse** of the original statement. For example, the converse of Problem 2.8(a) is "If $a|mn$, then $a|n$", which happens to be false. The converse of Theorem 2.2 is "If n^2 is an even integer, then n is an even integer". While this statement is true it does *not* have the same meaning as Theorem 2.2.

Problem 2.12. Determine whether the converse of each of Corollary 2.9, Theorem 2.10, and Theorem 2.11 is true. That is, for $a, n, m \in \mathbb{Z}$, determine whether each of the following statements is true or false. If a statement is true, prove it. If a statement is false, provide a counterexample.

(a) If a divides n^2, then a divides n. (Converse of Corollary 2.9)

(b) If a divides $-n$, then a divides n. (Converse of Theorem 2.10)

(c) If a divides $m + n$, then a divides m and a divides n. (Converse of Theorem 2.11)

The next theorem is often referred to as the **transitivity of division of integers**.

Theorem 2.13. If $a, b, c \in \mathbb{Z}$ such that a divides b and b divides c, then a divides c.

Once we have proved a few theorems, we should be on the look out to see if we can utilize any of our current results to prove new results. There is no point in reinventing the wheel if we do not have to.

Theorem 2.14. If $a, n, m \in \mathbb{Z}$ such that a divides m and a divides n, then a divides $m - n$.

Theorem 2.15. If $n \in \mathbb{Z}$ such that n is odd, then 8 divides $n^2 - 1$.

> Time spent thinking about a problem is always time well spent. Even if you seem to make no progress at all.
>
> Paul Zeitz, mathematician

2.2 Introduction to Logic

In the previous section, we jumped in head first and attempted to prove several theorems in the context of number theory without a formal understanding of what it was we were doing. Likely, many issues bubbled to the surface. What is a proof? What sorts of statements require proof? What should a proof entail? How should a proof be structured? Let's take a step back and do a more careful examination of what it is we are actually doing. In the the next two sections, we will introduce the basics of **propositional logic**—also referred to as **propositional calculus** or sometimes **zeroth-order logic**.

Definition 2.16. A **proposition** is a sentence that is either true or false but never both. The **truth value** (or **logical value**) of a proposition refers to its attribute of being true or false.

For example, the sentence "All dogs have four legs" is a false proposition. However, the perfectly good sentence "$x = 1$" is *not* a proposition all by itself since we do not actually know what x is.

Problem 2.17. Determine whether each of the following is a proposition. Explain your reasoning.

(a) All cars are red.

(b) Every person whose name begins with J has the name Joe.

(c) $x^2 = 4$.

(d) There exists a real number x such that $x^2 = 4$.

(e) For all real numbers x, $x^2 = 4$.

(f) $\sqrt{2}$ is an irrational number.

(g) p is prime.

(h) Is it raining?

(i) It will rain tomorrow.

(j) Led Zeppelin is the best band of all time.

The last two sentences in the previous problem may stir debate. It is not so important that we come to consensus as to whether either of these two sentences is actually a proposition or not. The good news is that in mathematics we do not encounter statements whose truth value is dependent on either the future or opinion.

Given two propositions, we can form more complicated propositions using **logical connectives**.

Definition 2.18. Let A and B be propositions.

(a) The proposition "**not** A" is true if A is false; expressed symbolically as $\boxed{\neg A}$ and called the **negation** of A.

(b) The proposition "A **and** B" is true if both A and B are true; expressed symbolically as $\boxed{A \wedge B}$ and called the **conjunction** of A and B.

(c) The proposition "A **or** B" is true if at least one of A or B is true; expressed symbolically as $\boxed{A \vee B}$ and called the **disjunction** of A and B.

(d) The proposition "**If** A**, then** B" is true if both A and B are true, or A is false; expressed symbolically as $\boxed{A \implies B}$ and called a **conditional proposition** (or **implication**). In this case, A is called the **hypothesis** and B is called the **conclusion**. Note that $A \implies B$ may also be read as "A implies B", "A only if B", "B if A", or "B whenever A".

(e) The proposition "A **if and only if** B" (alternatively, "A **is necessary and sufficient for** B") is true if both A and B have the same truth value; expressed symbolically as $\boxed{A \iff B}$ and called a **biconditional proposition**. If $A \iff B$ is true, we say that A and B are **logically equivalent**.

Each of the boxed propositions is called a **compound proposition**, where A and B are referred to as the **components** of the compound proposition.

It is worth pointing out that definitions in mathematics are typically written in the form "B if A" (or "B provided that A" or "B whenever A"), where B contains the term or phrase we are defining and A provides the meaning of the concept we are defining. In the case of definitions, we should always interpret "B if A" as describing precisely the collection of "objects" (e.g., numbers, sets, functions, etc.) that should be identified with the term or phrase we defining. That is, if an object does not meet the condition specified in A, then it is never referred to by the term or phrase we are defining. Some authors will write definitions in the form "B if and only if A". However, a definition is not at all the same kind of statement as a usual biconditional since one of the two sides is undefined until the definition is made. A definition is really a statement that the newly defined term or phrase is synonymous with a previously defined concept.

We can form complicated compound propositions with several components by utilizing logical connectives.

Problem 2.19. Let A represent "6 is an even integer" and B represent "4 divides 6." Express each of the following compound propositions in an ordinary English sentence and then determine its truth value.

(a) $A \wedge B$

(b) $A \vee B$

(c) $\neg A$

(d) $\neg B$

(e) $\neg(A \wedge B)$

(f) $\neg(A \vee B)$

(g) $A \implies B$

Definition 2.20. A **truth table** for a compound proposition is a table that illustrates all possible combinations of truth values for the components of the compound proposition together with the resulting truth value for each combination.

Example 2.21. If A and B are propositions, then the truth table for the compound proposition $A \wedge B$ is given by the following.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Notice that we have columns for each of A and B. The rows for these two columns correspond to all possible combinations of truth values for A and B. The third column yields the truth value of $A \wedge B$ given the possible truth values for A and B.

Each component of a compound proposition has two possible truth values, namely true or false. Thus, if a compound proposition is built from n component propositions, then the truth table will require 2^n rows.

Problem 2.22. Create a truth table for each of the following compound propositions. You should add additional columns to your tables as needed to assist you with intermediate steps. For example, you might need four columns for the third and fourth compound proposition below.

(a) $\neg A$

(b) $A \vee B$

(c) $\neg(A \wedge B)$

(d) $\neg A \wedge \neg B$

Problem 2.23. A coach promises her players, "If we win tonight, then I will buy you pizza tomorrow." Determine the cases in which the players can rightly claim to have been lied to. If the team lost the game and the coach decided to buy them pizza anyway, was she lying?

Problem 2.24. Use Definition 2.18(d) to construct a truth table for $A \implies B$. Compare your truth table with Problem 2.23. The combination you should pay particular attention to is when the hypothesis is false while the conclusion is true.

In accordance with Definition 2.18(d), a conditional proposition $A \implies B$ is only false when the hypothesis is true and the conclusion is false. Perhaps you are bothered by the fact that $A \implies B$ is true when A is false no matter what the truth value of B is. The thing to keep in mind is that the truth value of $A \implies B$ relies on a very specific definition and may not always agree with the colloquial use of "If…, then…" statements that we encounter in everyday language. For example, if someone says, "If you break the rules, then you will be punished", the speaker likely intends the statement to be interpreted as "You will be punished if and only if you break the rules." In logic and mathematics, we aim to remove such ambiguity by explicitly saying exactly what we mean. For our purposes, we should view a conditional proposition as a contract or obligation. If the hypothesis is false and the conclusion is true, the contract is not violated. On the other hand, if the hypothesis is true and the conclusion is false, then the contract is broken.

We can often prove facts concerning logical statements using truth tables. Recall that two propositions P and Q (both of which might be complicated compound propositions) are logically equivalent if $P \iff Q$ is true (see Definition 2.18(e)). This happens when P and Q have the same truth value. We can verify whether P and Q have the same truth value by constructing a truth table that includes columns for each of the components of P and Q, listing all possible combinations of their truth values, together with columns for P and Q that lists their resulting truth values. If the truth values in the columns for P and Q agree, then P and Q are logically equivalent, and otherwise they are not logically equivalent. When constructing truth tables to verify whether P and Q are logically equivalent, you should add any necessary intermediate columns to aid in your "calculations". Use truth tables when attempting to justify the next few problems.

Theorem 2.25. If A is a proposition, then $\neg(\neg A)$ is logically equivalent to A.

The next theorem, referred to as **De Morgan's Law**, provides a method for negating a compound proposition involving a conjunction.

Theorem 2.26 (De Morgan's Law)**.** If A and B are propositions, then $\neg(A \wedge B)$ is logically equivalent to $\neg A \vee \neg B$.

Problem 2.27 (De Morgan's Law)**.** Let A and B be propositions. Conjecture a statement similar to Theorem 2.26 for the proposition $\neg(A \vee B)$ and then prove it. This is also called De Morgan's Law.

We will make use of both versions De Morgan's Law on on a regular basis. Sometimes conjunctions and disjunctions are "buried" in a mathematical statement, which makes negating statements tricky business. Keep this in mind when approaching the next problem.

Problem 2.28. Let x be your favorite real number. Negate each of the following statements. Note that the statement in Part (b) involves a conjunction.

(a) $x < -1$ or $x \geq 3$.

(b) $0 \leq x < 1$.

Theorem 2.29. If A and B are propositions, then $A \iff B$ is logically equivalent to $(A \implies B) \land (B \implies A)$.

Theorem 2.30. If A, B, and C are propositions, then $(A \lor B) \implies C$ is logically equivalent to $(A \implies C) \land (B \implies C)$.

We already introduced the following notion in the discussion following Theorem 2.11

Definition 2.31. If A and B are propositions, then the **converse** of $A \implies B$ is $B \implies A$.

Problem 2.32. Provide an example of a true conditional proposition whose converse is false.

Definition 2.33. If A and B are propositions, then the **inverse** of $A \implies B$ is $\neg A \implies \neg B$.

Problem 2.34. Provide an example of a true conditional proposition whose inverse is false.

Based on Problems 2.32 and 2.34, we can conclude that the converse and inverse of a conditional proposition do not necessarily have the same truth value as the original statement. Moreover, the converse and inverse of a conditional proposition do not necessarily have the same truth value as each other.

Problem 2.35. Provide an example of a conditional proposition whose converse is true but whose inverse is false.

What if we swap the roles of the hypothesis and conclusion of a conditional proposition *and* negate each?

Definition 2.36. If A and B are propositions, then the **contrapositive** of $A \implies B$ is $\neg B \implies \neg A$.

Problem 2.37. Let A and B represent the statements from Problem 2.19. Express each of the following in an ordinary English sentence.

(a) The converse of $A \implies B$.

(b) The contrapositive of $A \implies B$.

Problem 2.38. Find the converse and the contrapositive of the following statement: "If Dana lives in Flagstaff, then Dana lives in Arizona."

Use a truth table to prove the following theorem.

Theorem 2.39. If A and B are propositions, then $A \implies B$ is logically equivalent to its contrapositive.

So far we have discussed how to negate propositions of the form A, $A \wedge B$, and $A \vee B$ for propositions A and B. However, we have yet to discuss how to negate propositions of the form $A \implies B$. Prove the following result with a truth table.

Theorem 2.40. If A and B are propositions, then the implication $A \implies B$ is logically equivalent to the disjunction $\neg A \vee B$.

The next result follows quickly from Theorem 2.40 together with De Morgan's Law. You can also verify this result using a truth table.

Corollary 2.41. If A and B are propositions, then $\neg(A \implies B)$ is logically equivalent to $A \wedge \neg B$.

Problem 2.42. Let A and B be the propositions "$\sqrt{2}$ is an irrational number" and "Every rectangle is a trapezoid," respectively.

(a) Express $A \implies B$ as an English sentence involving the disjunction "or."

(b) Express $\neg(A \implies B)$ as an English sentence involving the conjunction "and."

Problem 2.43. It turns out that the proposition "If $.\overline{99} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$, then $.\overline{99} \neq 1$" is false. Write its negation as a conjunction.

Recall that a proposition is exclusively either true or false—it can never be both.

Definition 2.44. A compound proposition that is always false is called a **contradiction**. A compound proposition that is always true is called a **tautology**.

Theorem 2.45. If A is a proposition, then the proposition $\neg A \wedge A$ is a contradiction.

Problem 2.46. Provide an example of a tautology using arbitrary propositions and any of the logical connectives $\neg$, $\wedge$, and $\vee$. Prove that your example is in fact a tautology.

> I didn't want to just know names of things. I remember really wanting to know how it all worked.
>
> Elizabeth Blackburn, biologist

2.3 Techniques for Proving Conditional Propositions

Each of the theorems that we proved in Section 2.1 are examples of conditional propositions. However, some of the statements were disguised as such. For example, Theorem 2.3 states, "The sum of two consecutive integers is odd." We can reword this theorem as, "If $n \in \mathbb{Z}$, then $n+(n+1)$ is odd."

Problem 2.47. Reword Theorem 2.7 so that it explicitly reads as a conditional proposition.

Each of the proofs that you produced in Section 2.1 had the same format, which we refer to as a **direct proof**.

Skeleton Proof 2.48 (Proof of $A \implies B$ by direct proof)**.** If you want to prove the implication $A \implies B$ via a direct proof, then the structure of the proof is as follows.

Proof. *[State any upfront assumptions.]* Assume A.

… *[Use definitions and known results to derive B]* …

Therefore, B. □

Take a few minutes to review the proofs that you wrote in Section 2.1 and see if you can witness the structure of Skeleton Proof 2.48 in your proofs.

The upshot of Theorem 2.39 is that if you want to prove a conditional proposition, you can prove its contrapositive instead. This approach is called a **proof by contraposition**.

Skeleton Proof 2.49 (Proof of $A \implies B$ by contraposition)**.** If you want to prove the implication $A \implies B$ by proving its contrapositive $\neg B \implies \neg A$ instead, then the structure of the proof is as follows.

Proof. *[State any upfront assumptions.]* We will utilize a proof by contraposition. Assume $\neg B$.

… *[Use definitions and known results to derive $\neg A$]* …

Therefore, $\neg A$. We have proved the contrapositive, and hence if A, then B. □

We have introduced the logical symbols $\neg$, $\wedge$, $\vee$, $\implies$, and $\iff$ since it provides a convenient way of discussing the formality of logic. However, when writing mathematical proofs, you should avoid using these symbols.

Problem 2.50. Consider the following statement:

If $x \in \mathbb{Z}$ such that x^2 is odd, then x is odd.

The items below can be assembled to form a proof of this statement, but they are currently out of order. Put them in the proper order.

1. Assume that x is an even integer.

2. We will utilize a proof by contraposition.
3. Thus, x^2 is twice an integer.
4. Since $x = 2k$, we have that $x^2 = (2k)^2 = 4k^2$.
5. Since k is an integer, $2k^2$ is also an integer.
6. By the definition of even, there is an integer k such that $x = 2k$.
7. We have proved the contrapositive, and hence the desired statement is true.
8. Assume $x \in \mathbb{Z}$.
9. By the definition of even integer, x^2 is an even integer.
10. Notice that $x^2 = 2(2k^2)$.

Prove the next two theorems by proving the contrapositive of the given statement.

Theorem 2.51. If $n \in \mathbb{Z}$ such that n^2 is even, then n is even.

Theorem 2.52. If $n, m \in \mathbb{Z}$ such that nm is even, then n is even or m is even.

Suppose that we want to prove some proposition P (which might be something like $A \implies B$ or even more complicated). One approach, called **proof by contradiction**, is to assume $\neg P$ and then logically deduce a contradiction of the form $Q \wedge \neg Q$, where Q is some proposition. Since this is absurd, the assumption $\neg P$ must have been false, so P is true. The tricky part about a proof by contradiction is that it is not usually obvious what the statement Q should be.

Skeleton Proof 2.53 (Proof of P by contradiction)**.** Here is what the general structure for a proof by contradiction looks like if we are trying to prove the proposition P.

Proof. *[State any upfront assumptions.]* For sake of a contradiction, assume $\neg P$.

… *[Use definitions and known results to derive some Q and its negation $\neg Q$.]* …

This is a contradiction. Therefore, P. □

Proof by contradiction can be useful for proving statements of the form $A \implies B$, where $\neg B$ is easier to "get your hands on," because $\neg(A \implies B)$ is logically equivalent to $A \wedge \neg B$ (see Corollary 2.41).

Skeleton Proof 2.54 (Proof of $A \implies B$ by contradiction)**.** If you want to prove the implication $A \implies B$ via a proof by contradiction, then the structure of the proof is as follows.

Proof. *[State any upfront assumptions.]* For sake of a contradiction, assume A and $\neg B$.

… *[Use definitions and known results to derive some Q and its negation $\neg Q$.]* …

This is a contradiction. Therefore, if A, then B. $\square$

Problem 2.55. Assume that $x \in \mathbb{Z}$. Consider the following proposition: If x is odd, then 2 does not divide x.

(a) Prove the contrapositive of this statement.

(b) Prove the statement using a proof by contradiction.

Prove the following theorem via a proof by contradiction. Afterward, consider the difficulties one might encounter when trying to prove the result more directly. The given statement is not true if we replace $\mathbb{N}$ with $\mathbb{Z}$. Do you see why?

Theorem 2.56. Assume that $x, y \in \mathbb{N}$. If x divides y, then $x \leq y$.

Oftentimes a conditional proposition can be proved via a direct proof and by using a proof by contradiction. Most mathematicians view a direct proof to be more elegant than a proof by contradiction. When approaching the proof of a conditional proposition, you should strive for a direct proof. In general, if you are attempting to prove $A \implies B$ using a proof by contradiction and you end up with $\neg B$ and B (which yields a contradiction), then this is evidence that a proof by contradiction was unnecessary. On the other hand, if you end up with $\neg Q$ and Q, where Q is not the same as B, then a proof by contradiction is a reasonable approach.

In light of Theorem 2.29, if we want to prove a biconditional of the form $A \iff B$, we need to prove both $A \implies B$ and $B \implies A$. You should always make it clear to the reader when you are proving each implication. One approach is to label each subproof with "($\implies$)" and "($\impliedby$)" (including the parentheses), respectively. Occasionally, you will discover that the proof of one implication is exactly the reverse of the proof of the other implication. If this happens to be the case, you may skip writing two subproofs and simply write a single proof that chains together each step using biconditionals. Such proofs will almost always be shorter, but can be challenging to write in an eloquent way. It is always a safe bet to write a separate subproof for each implication.

When proving each implication of a biconditional, you may choose to utilize a direct proof, a proof by contraposition, or a proof by contradiction. For example, you could prove the first implication using a proof by contradiction and a direct proof for the second implication.

The following theorem provides an opportunity to gain some experience with writing proofs of biconditional statements.

Theorem 2.57. Let $n \in \mathbb{Z}$. Then n is even if and only if 4 divides n^2.

> Making learning easy does not necessarily ease learning.
>
> Manu Kapur, learning scientist

2.4 Introduction to Quantification

In this section and the next, we introduce **first-order logic**—also referred to as **predicate logic**, **quantificational logic**, and **first-order predicate calculus**. The sentence "$x > 0$" is not itself a proposition because its truth value depends on x. In this case, we say that x is a **free variable**. A sentence with at least one free variable is called a **predicate** (or **open sentence**). To turn a predicate into a proposition, we must either substitute values for each free variable or "quantify" the free variables. We will use notation such as $\boxed{P(x)}$ and $\boxed{Q(a,b)}$ to represent predicates with free variables x and a, b, respectively. The letters "P" and "Q" that we used in the previous sentence are not special; we can use any letter or symbol we want. For example, each of the following represents a predicate with the indicated free variables.

- $S(x) :=$ "$x^2 - 4 = 0$"
- $L(a,b) :=$ "$a < b$"
- $F(x,y) :=$ "x is friends with y"

Note that we used quotation marks above to remove some ambiguity. What would $S(x) = x^2 - 4 = 0$ mean? It looks like $S(x)$ equals 0, but actually we want $S(x)$ to represent the whole sentence "$x^2 - 4 = 0$". Also, notice that the order in which we utilize the free variables might matter. For example, compare $L(a,b)$ with $L(b,a)$.

One way we can make propositions out of predicates is by assigning specific values to the free variables. That is, if $P(x)$ is a predicate and x_0 is specific value for x, then $P(x_0)$ is now a proposition that is either true or false.

Problem 2.58. Consider $S(x)$ and $L(a,b)$ as defined above. Determine the truth values of $S(0)$, $S(-2)$, $L(2,1)$, and $L(-3,-2)$. Is $L(2,b)$ a proposition or a predicate?

Besides substituting specific values for free variables in a predicate, we can also make a claim about which values of the free variables apply to the predicate.

Problem 2.59. Both of the following sentences are propositions. Decide whether each is true or false. What would it take to justify your answers?

(a) For all $x \in \mathbb{R}$, $x^2 - 4 = 0$.

(b) There exists $x \in \mathbb{R}$ such that $x^2 - 4 = 0$.

Definition 2.60. "For all" is the **universal quantifier** and "there exists... such that" is the **existential quantifier**.

In mathematics, the phrases "for all", "for any", "for every", and "for each" can be used interchangeably (even though they might convey slightly different meanings in colloquial language). We can replace "there exists... such that" with phrases like "for some" (possibly with some tweaking of the wording of the sentence). It is important to note that the existential quantifier is making a claim about "at least one", *not* "exactly one."

Variables that are quantified with a universal or existential quantifier are said to be **bound**. To be a proposition, *all* variables of a predicate must be bound.

We must take care to specify the collection of acceptable values for the free variables. Consider the sentence "For all x, $x > 0$." Is this sentence true or false? The answer depends on what set the universal quantifier applies to. Certainly, the sentence is false if we apply it for all $x \in \mathbb{Z}$. However, the sentence is true for all $x \in \mathbb{N}$. Context may resolve ambiguities, but otherwise, we must write clearly: "For all $x \in \mathbb{Z}$, $x > 0$" or "For all $x \in \mathbb{N}$, $x > 0$." The collection of intended values for a variable is called the **universe of discourse**.

Problem 2.61. Suppose our universe of discourse is the set of integers.

(a) Provide an example of a predicate $P(x)$ such that "For all x, $P(x)$" is true.

(b) Provide an example of a predicate $Q(x)$ such that "For all x, $Q(x)$" is false while "There exists x such that $Q(x)$" is true.

If a predicate has more than one free variable, then we can build propositions by quantifying each variable. However, *the order of the quantifiers is extremely important!*

Problem 2.62. Let $P(x,y)$ be a predicate with free variables x and y in a universe of discourse U. One way to quantify the variables is "For all $x \in U$, there exists $y \in U$ such that $P(x,y)$." How else can the variables be quantified?

The next problem illustrates that at least some of the possibilities you discovered in the previous problem are *not* equivalent to each other.

Problem 2.63. Suppose the universe of discourse is the set of people and consider the predicate $M(x,y) :=$ "x is married to y". We can interpret the formal statement "For all x, there exists y such that $M(x,y)$" as meaning "Everybody is married to somebody." Interpret the meaning of each of the following statements in a similar way.

(a) For all x, there exists y such that $M(x,y)$.

(b) There exists y such that for all x, $M(x,y)$.

(c) For all x, for all y, $M(x,y)$.

(d) There exists x such that there exists y such that $M(x,y)$.

Problem 2.64. Suppose the universe of discourse is the set of real numbers and consider the predicate $F(x,y) :=$ "$x = y^2$". Interpret the meaning of each of the following statements.

(a) There exists x such that there exists y such that $F(x,y)$.

(b) There exists y such that there exists x such that $F(x,y)$.

(c) For all y, for all x, $F(x,y)$.

There are a couple of key points to keep in mind about quantification. To be a proposition, all variables must be quantified. This can happen in at least two ways:

- The variables are explicitly bound by quantifiers in the same sentence.
- The variables are implicitly bound by preceding sentences or by context. Statements of the form "Let $x = \dots$" and "Assume $x \in \dots$" bind the variable x and remove ambiguity.

Also, the order of the quantification is important. Reversing the order of the quantifiers can substantially change the meaning of a proposition.

Quantification and logical connectives ("and," "or," "If..., then...," and "not") enable complex mathematical statements. For example, if f is a function while c and L are real numbers, then the formal definition of $\lim_{x\to c} f(x) = L$, which you may have encountered in calculus, is:

> For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

In order to study the abstract nature of complicated mathematical statements, it is useful to adopt some notation.

Definition 2.65. The universal quantifier "for all" is denoted $\boxed{\forall}$, and the existential quantifier "there exists... such that" is denoted $\boxed{\exists}$.

Using our abbreviations for the logical connectives and quantifiers, we can symbolically represent mathematical propositions. For example, the (true) proposition "There exists $x \in \mathbb{R}$ such that $x^2 - 1 = 0$" becomes "$(\exists x \in \mathbb{R})(x^2 - 1 = 0)$," while the (false) proposition "For all $x \in \mathbb{N}$, there exists $y \in \mathbb{N}$ such that $y < x$" becomes "$(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y < x)$."

Problem 2.66. Convert the following propositions into statements using only logical and mathematical symbols. Assume that the universe of discourse is the set of real numbers.

(a) There exists x such that $x^2 + 1$ is greater than zero.

(b) There exists a natural number n such that $n^2 = 36$.

(c) For every x, x^2 is greater than or equal to zero.

Problem 2.67. Express the formal definition of a limit (given above Definition 2.65) in logical and mathematical symbols.

If you look closely, many of the theorems that we have encountered up until this point were of the form $A(x) \implies B(x)$, where $A(x)$ and $B(x)$ are predicates. For example, consider Theorem 2.2, which states, "If n is an even integer, then n^2 is an even integer." In this case, "n is an even integer" and "n^2 is an even integer" are both predicates. So, it would be reasonable to assume that the entire theorem statement is a predicate. However, it is standard practice to interpret the sentence $A(x) \implies B(x)$ to mean $(\forall x)(A(x) \implies B(x))$ (where the universe of discourse for x needs to be made clear). We can also retool such statements to "hide" the implication. In particular, $(\forall x)(A(x) \implies B(x))$ has the same meaning as $(\forall x \in U')B(x)$, where U' is the collection of items from the universe of discourse U that makes $A(x)$ true. For example, we could rewrite the statement of Theorem 2.2 as "For every even integer n, n^2 is even."

Problem 2.68. Reword Theorem 2.7 so that it explicitly reads as a universally quantified statement. Compare with Problem 2.47.

Problem 2.69. Find at least two other instances of theorem statements that appeared earlier in the book and are written in the form $A(x) \implies B(x)$. Rewrite each in an equivalent way that makes the universal quantifier explicit while possibly suppressing the implication.

Problem 2.70. Consider the proposition "If $\varepsilon > 0$, then there exists $N \in \mathbb{N}$ such that $1/N < \varepsilon$." Assume the universe of discourse is the set $\mathbb{R}$.

(a) Express the statement in logical and mathematical symbols. Is the statement true?

(b) Reverse the order of the quantifiers to get a new statement. Does the meaning change? If so, how? Is the new statement true?

The symbolic expression $(\forall x)(\forall y)$ can be abbreviated as $\boxed{\forall x, y}$ as long as x and y are elements of the same universe.

Problem 2.71. Express the proposition "For all $x, y \in \mathbb{R}$ with $x < y$, there exists $m \in \mathbb{R}$ such that $x < m < y$" using logical and mathematical symbols.

Problem 2.72. Rewrite each of the following propositions in words and determine whether the proposition is true or false.

(a) $(\forall n \in \mathbb{N})(n^2 \geq 5)$

(b) $(\exists n \in \mathbb{N})(n^2 - 1 = 0)$

(c) $(\exists N \in \mathbb{N})(\forall n > N)(\frac{1}{n} < 0.01)$

(d) $(\forall m, n \in \mathbb{Z})((2|m \wedge 2|n) \implies 2|(m+n))$

(e) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(x - 2y = 0)$

(f) $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})(y \leq x)$

Problem 2.73. Consider the proposition $(\forall x)(\exists y)(xy = 1)$.

(a) Provide an example of a universe of discourse where this proposition is true.

(b) Provide an example of a universe of discourse where this proposition is false.

To whet your appetite for the next section, consider how you might prove a true proposition of the form "For all $x \ldots$." If a proposition is false, then its negation is true. How would you go about negating a statement involving quantifiers?

> Like what you do, and then you will do your best.
>
> Katherine Johnson, mathematician

2.5 More About Quantification

When writing mathematical proofs, we do not explicitly use the symbolic representation of a given statement in terms of quantifiers and logical connectives. Nonetheless, having this notation at our disposal allows us to compartmentalize the abstract nature of mathematical propositions and provides us with a way to talk about the general structure involved in the construction of a proof.

Definition 2.74. Two quantified propositions are **logically equivalent** if they have the same truth value in every universe of discourse.

Problem 2.75. Consider the propositions $(\exists x \in U)(x^2 - 4 = 0)$ and $(\exists x \in U)(x^2 - 2 = 0)$, where U is some universe of discourse.

(a) Do these propositions have the same truth value if the universe of discourse is the set of real numbers?

(b) Provide an example of a universe of discourse such that the propositions yield different truth values.

(c) What can you conclude about the logical equivalence of these propositions?

It is worth pointing out an important distinction. Consider the propositions "All cars are red" and "All natural numbers are positive". Both of these are instances of the **logical form** $(\forall x)P(x)$. It turns out that the first proposition is false and the second is true; however, it does not make sense to attach a truth value to the logical form. A logical form is a blueprint for particular propositions. If we are careful, it makes sense to talk about whether two logical forms are logically equivalent. For example, $(\forall x)(P(x) \implies Q(x))$ is logically equivalent to $(\forall x)(\neg Q(x) \implies \neg P(x))$ since a conditional proposition is logically equivalent to its contrapositive (see Theorem 2.39). For fixed $P(x)$ and $Q(x)$, these two forms will always have the same truth value independent of the universe of discourse. If you change $P(x)$ and $Q(x)$, then the truth value may change, but the two forms will still agree.

The next theorem tells us how to negate logical forms involving quantifiers. Your proof should involve several mini arguments. For example, in Part (a), you will need to proof that if $\neg(\forall x)P(x)$ is true, then $(\exists x)(\neg P(x))$ is also true.

Theorem 2.76. Let $P(x)$ be a predicate in some universe of discourse. Then

(a) $\neg(\forall x)P(x)$ is logically equivalent to $(\exists x)(\neg P(x))$;

(b) $\neg(\exists x)P(x)$ is logically equivalent to $(\forall x)(\neg P(x))$.

Problem 2.77. Negate each of the following sentences. Disregard the truth value and the universe of discourse.

(a) $(\forall x)(x > 3)$

(b) $(\exists x)(x \text{ is prime} \land x \text{ is even})$

(c) All cars are red.

(d) Every Wookiee is named Chewbacca.

(e) Some hippies are Republican.

(f) Some birds are not angry.

(g) Not every video game will rot your brain.

(h) For all $x \in \mathbb{N}$, $x^2 + x + 41$ is prime.

(i) There exists $x \in \mathbb{Z}$ such that $1/x \notin \mathbb{Z}$.

(j) There is no function f such that if f is continuous, then f is not differentiable.

Using Theorem 2.76 and our previous results involving quantification, we can negate complex mathematical propositions by working from left to right. For example, if we negate the false proposition

$$(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y = 0),$$

we obtain the proposition

$$\neg(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(x + y = 0),$$

which is logically equivalent to

$$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x + y \neq 0)$$

and must be true. For a more complicated example, consider the (false) proposition

$$(\forall x)[x > 0 \implies (\exists y)(y < 0 \wedge xy > 0)].$$

Then its negation

$$\neg(\forall x)[x > 0 \implies (\exists y)(y < 0 \wedge xy > 0)]$$

is logically equivalent to

$$(\exists x)[x > 0 \wedge \neg(\exists y)(y < 0 \wedge xy > 0)],$$

which happens to be logically equivalent to

$$(\exists x)[x > 0 \wedge (\forall y)(y \geq 0 \vee xy \leq 0)].$$

Can you identify the theorems that were used in the two examples above?

Problem 2.78. Negate each of the following propositions. Disregard the truth value and the universe of discourse.

(a) $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m < n)$

(b) For every $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $y = x^2$.

(c) For all $y \in \mathbb{R}$, if y is not negative, then there exists $x \in \mathbb{R}$ such that $y = x^2$.

(d) For every $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $y = x^2$.

(e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $y = x^2$.

(f) There exists $y \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $y = x^2$.

(g) $(\forall x, y, z \in \mathbb{Z})((xy \text{ is even} \wedge yz \text{ is even}) \implies xz \text{ is even})$

(h) There exists a married person x such that for all married people y, x is married to y.

Problem 2.79. Consider the following proposition in some universe of discourse.

> "For all goofy wobblers x, there exists a dinglehopper y such that if x is a not a nugget, then y is a doofus."

Find the negation of this proposition so that it includes the phrase "is not a doofus."

Problem 2.80. Consider the following proposition in some universe of discourse.

> "If x and y are both snazzy, then xy is not nifty."

Find the contrapositive of this proposition so that it includes the phrase "not snazzy".

At this point, we should be able to use our understanding of quantification to construct counterexamples to complicated false propositions and proofs of complicated true propositions. Here are some general proof structures for various logical forms.

Skeleton Proof 2.81 (Direct Proof of $(\forall x)P(x)$)**.** Here is the general structure for a direct proof of the proposition $(\forall x)P(x)$. Assume U is the universe of discourse.

> *Proof. [State any upfront assumptions.]* Let $x \in U$.
>
> … *[Use definitions and known results.]* …
>
> Therefore, $P(x)$ is true. Since x was arbitrary, for all x, $P(x)$. □

Combining Skeleton Proof 2.81 with Skeleton Proof 2.48, we obtain the following skeleton proof.

Skeleton Proof 2.82 (Proof of $(\forall x)(P(x) \implies Q(x))$)**.** Below is the general structure for a direct proof of the proposition $(\forall x)(A(x) \implies B(x)$. Assume U is the universe of discourse.

> *Proof. [State any upfront assumptions.]* Let $x \in U$. Assume $A(x)$.
>
> … *[Use definitions and known results to derive $B(x)$]* …
>
> Therefore, $B(x)$. □

Skeleton Proof 2.83 (Proof of $(\forall x)P(x)$ by Contradiction)**.** Here is the general structure for a proof of the proposition $(\forall x)P(x)$ via contradiction. Assume U is the universe of discourse.

Proof. *[State any upfront assumptions.]* For sake of a contradiction, assume that there exists $x \in U$ such that $\neg P(x)$.

… *[Do something to derive a contradiction.]* …

This is a contradiction. Therefore, for all x, $P(x)$ is true. □

Skeleton Proof 2.84 (Direct Proof of $(\exists x)P(x)$)**.** Here is the general structure for a direct proof of the proposition $(\exists x)P(x)$. Assume U is the universe of discourse.

Proof. *[State any upfront assumptions.]* …

… *[Use definitions, axioms, and previous results to deduce that an x exists for which P(x) is true; or if you have an x that works, just verify that it does.]* …

Therefore, there exists $x \in U$ such that $P(x)$. □

Skeleton Proof 2.85 (Proof of $(\exists x)P(x)$ by Contradiction)**.** Below is the general structure for a proof of the proposition $(\exists x)P(x)$ via contradiction. Assume U is the universe of discourse.

Proof. *[State any upfront assumptions.]* For sake of a contradiction, assume that for all $x \in U$, $\neg P(x)$.

… *[Do something to derive a contradiction.]* …

This is a contradiction. Therefore, there exists $x \in U$ such that $P(x)$. □

Note that if $Q(x)$ is a predicate for which $(\forall x)Q(x)$ is false, then a counterexample to this proposition amounts to showing $(\exists x)(\neg Q(x))$, which can be proved by following the structure of Skeleton Proof 2.84.

It is important to point out that sometimes we will have to combine various proof techniques in a single proof. For example, if you wanted to prove a proposition of the form $(\forall x)(P(x) \implies Q(x))$ by contradiction, we would start by assuming that there exists x in the universe of discourse such that $P(x)$ and $\neg Q(x)$.

Problem 2.86. Determine whether each of the following statements is true or false. If the statement is true, prove it. If the statement is false, provide a counterexample.

(a) For all $n \in \mathbb{N}$, $n^2 \geq 5$.

(b) There exists $n \in \mathbb{N}$ such that $n^2 - 1 = 0$.

(c) There exists $x \in \mathbb{N}$ such that for all $y \in \mathbb{N}$, $y \leq x$.

(d) For all $x \in \mathbb{Z}$, $x^3 \geq x$.

(e) For all $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$ such that $n + m = 0$.

(f) There exists integers a and b such that $2a + 7b = 1$.

(g) There do not exist integers m and n such that $2m + 4n = 7$.

(h) For all $a, b, c \in \mathbb{Z}$, if a divides bc, then either a divides b or a divides c.

(i) For all $a, b \in \mathbb{Z}$, if ab is even, then either a or b is even.

Problem 2.87. Explain why the following "proof" is not a valid argument.

> **Claim.** For all $x, y \in \mathbb{Z}$, if x and y are even, then $x + y$ is even.
>
> *"Proof."* Suppose $x, y \in \mathbb{Z}$ such that x and y are even. For sake of a contradiction, assume that $x+y$ is odd. Then there exists $k \in \mathbb{Z}$ such that $x+y = 2k+1$. This implies that $(x+y)-2k = 1$. We see that the left side of the equation is even because it is the difference of even numbers. However, the right side is odd. Since an even number cannot equal an odd number, we have a contradiction. Therefore, $x + y$ is even. □

Sometimes it is useful to split the universe of discourse into multiple collections to deal with separately. When doing this, it is important to make sure that your cases are exhaustive (i.e., every possible element of the universe of discourse has been accounted for). Ideally, your cases will also be disjoint (i.e., you have not considered the same element more than once). For example, if our universe of discourse is the set of integers, we can separately consider even versus odd integers. If our universe of discourse is the set of real numbers, we might want to consider rational versus irrational numbers, or possibly negative versus zero versus and positive. Attacking a proof in this way, is often referred to as a **proof by cases** (or **proof by exhaustion**). A proof by cases may also be useful when dealing with hypotheses involving "or". Note that the use of a proof by cases is justified by Theorem 2.30.

If you decide to approach a proof using cases, be sure to inform the reader that you are doing so and organize your proof in a sensible way. Note that doing an analysis of cases should be avoided if possible. For example, while it is valid to separately consider the cases of whether a is an even integer versus odd integer in the proof of Theorem 2.11, it is completely unnecessary. To prove the next theorem, you might want to consider two cases.

Theorem 2.88. For all $n \in \mathbb{Z}$, $3n^2 + n + 14$ is even.

Prove the following theorem by proving the contrapositive using two cases.

Theorem 2.89. For all $n, m \in \mathbb{Z}$, if nm is odd, then n is odd and m is odd.

When proving the previous theorem, you likely experienced some dèjá vu. You should have assumed "n is even or m is even" at some point in your proof. The first case is "n is even" while the second case is "m is even." (Note that you do not need to handle the case when both n and m are even since the two individual cases already yield the desired result.) The proofs for both cases are identical except the roles of n and m are interchanged. In instances such as this, mathematicians have a shortcut. Instead of writing two essentially identical proofs for each case, you can simply handle one of the cases and indicate that the remaining case follows from a nearly identical proof. The quickest way to do this is to use the phrase, "Without loss of generality, assume...". For example, here is a proof of Theorem 2.89 that utilizes this approach.

Proof of Theorem 2.89. We will prove the contrapositive. Let $n, m \in \mathbb{Z}$ and assume n is even or m is even. Without loss of generality, assume n is even. Then there exists $k \in \mathbb{Z}$ such that $n = 2k$. We see that

$$nm = (2k)m = 2(km).$$

Since both k and m are integers, km is an integer. This shows that nm is even. We have proved the contrapositive, and hence for all $n, m \in \mathbb{Z}$, if nm is odd, then n is odd and m is odd. □

Note that it would not be appropriate to utilize the "without loss of generality" approach to combine the two cases in the proof of Theorem 2.88 since the proof of the second case is not as simple as swapping the roles of symbols in the proof of the first case.

There are times when a theorem will make a claim about the *uniqueness* of a particular mathematical object. For example, in Section 5.1, you will be asked to prove that both the additive and multiplicative identities (i.e, 0 and 1) are unique (see Theorems 5.2 and 5.3). As another example, the Fundamental Theorem of Arithmetic (see Theorem 6.17) states that every natural number greater than 1 can be expressed uniquely (up to the order in which they appear) as the product of one or more primes. The typical approach to proving uniqueness is to suppose that there are potentially two objects with the desired property and then show that these objects are actually equal. Whether you approach this as a proof by contradiction is a matter of taste. It is common to use $\boxed{\exists!}$ as a symbolic abbreviation for "there exists a unique... such that".

Skeleton Proof 2.90 (Direct Proof of $(\exists! x)P(x)$)**.** Here is the general structure for a direct proof of the proposition $(\exists! x)P(x)$. Assume U is the universe of discourse.

> *Proof.* *[State any upfront assumptions.]* ...
>
> ... *[Use definitions, axioms, and previous results to deduce that an x exists for which $P(x)$ is true; or if you have an x that works, just verify that it does.]* ...
>
> Therefore, there exists $x \in U$ such that $P(x)$. Now, suppose $x_1, x_2 \in U$ such that $P(x_1)$ and $P(x_2)$.
>
> ... *[Prove that $x_1 = x_2$.]* ...
>
> This implies that there exists a unique x such that $P(x)$. □

The next theorem provides an opportunity to practice proving uniqueness.

Theorem 2.91. If $c, a, r \in \mathbb{R}$ such that $c \neq 0$ and $r \neq a/c$, then there exists a unique $x \in \mathbb{R}$ such that $(ax+1)/(cx) = r$.

> With two published novels and a file full of ideas for others, the only thing I know about writing is this: it only happens when you sit down and do it. Studying good writing is important, reading good writing is important, talking to other writers is important, but the only way you can produce good writing is to write.
>
> Jamie Beth Cohen, novelist