Example 1.17. How many two-letter words are there using only the letters b, c, d, f?

Solution. We can easily list these, but a different arrangement will illustrate a more powerful technique.

	b	c	d	f
b	bb	bc	bd	bf
c	cb	cc	cd	cf
d	db	dc	dd	df
f	fb	fc	fd	ff

The rectangular table makes it clear that there are 16 two-letter words using the letters b, c, d, and f. Another way to see this is to note that there are 4 choices for the first (left-hand) letter and for each of these there are 4 choices for the right-hand letter, so there are $4 \cdot 4 = 16$ words total. □

Example 1.18. How many "numbered words" are there consisting of two letters from b, c, d, and f, followed by one of the digits 1, 2, 3, 4, and 5?

Solution. A geometric display as in Example 1.17 would seem to require three dimensions, but we can take what we already know and just add a digit to the right:

	1	2	3	4	5
bb	$bb1$	$bb2$	$bb3$	$bb4$	$bb5$
bc	$bc1$	$bc2$	$bc3$	$bc4$	$bc5$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ff	$ff1$	$ff2$	$ff3$	$ff4$	$ff5$

There are 16 rows (each with a word from Example 1.17) and five columns, so there are $16 \cdot 5 = 80$ two-letter words using b, c, d, and f, followed by a digit from 1, 2, 3, 4, and 5. □

If A and B are sets, the **Cartesian product** of A and B, denoted $A \times B$ (read as "A times B" or "A cross B"), is the set of all **ordered pairs** where the first component is from A and the second component is from B. In set-builder notation, we have

$$\boxed{A \times B := \{(a, b) \mid a \in A, b \in B\}}.$$

Similarly, the Cartesian product of sets $A_1, \ldots, A_n$ sets is the collection of n**-tuples** given by

$$\boxed{A_1 \times \cdots \times A_n := \{(a_1, \ldots, a_n) \mid a_j \in A_j \text{ for all } 1 \leq j \leq n\}},$$

where A_i is referred to as the ith **factor** of the Cartesian product. As a special case, the set

$$\underbrace{A \times \cdots \times A}_{n \text{ factors}}$$

is often abbreviated as $\boxed{A^n}$.

Cartesian products are named after French philosopher and mathematician René Descartes (1596–1650).

Example 1.19. If $A = \{a, b, c\}$ and $B = \{☺, ☹\}$, then

$$A \times B = \{(a, ☺), (a, ☹), (b, ☺), (b, ☹), (c, ☺), (c, ☹)\}.$$

Example 1.20. The standard two-dimensional plane $\mathbb{R}^2$ and standard three space $\mathbb{R}^3$ are familiar examples of Cartesian products. In particular, we have

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$$

and

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

Problem 1.21. Consider the sets A and B from Example 1.19.

(a) Find $B \times A$.

(b) Find $B \times B$.

Problem 1.22. Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$, and $C = \{1, 3\}$. Find $A \times B \times C$.

Problem 1.23. If A is a set, then what is $A \times \emptyset$ equal to?

Problem 1.24. If A and B are both finite sets, find a nice formula for $|A \times B|$.

Problem 1.25. Are collections of words or strings that we encountered earlier in the chapter just Cartesian products in disguise? Explain.

Problem 1.26. Is a set of dominoes a Cartesian product in disguise? Explain.

We have been dancing around a fundamental counting principle. Let's state it officially. There are three progressively more general versions, the first of which we already encountered in Problem 1.24. The second version officially follows from the first version by induction, which is a topic we will encounter later. The third version follows from the second by making a careful use of notation to identify the set $\mathcal{O}$ with a Cartesian product.

Theorem 1.27 (The Product Principle)**.** Each of the following are referred to as the **Product Principle**.

(a) If A and B are finite sets, then

$$|A \times B| = |A| \cdot |B|.$$

If $A_1, \ldots, A_k$ are finite sets, then

$$|A_1 \times \cdots \times A_k| = |A_1| \cdots |A_k|.$$

(b) If $\mathcal{O}$ is the set of outcomes for a k-step process, where for $1 \leq i \leq k$, there are n_i choices for step i, no matter what earlier choices were made, then

$$|\mathcal{O}| = n_1 n_2 \cdots n_k.$$

The key difference between versions (b) and (c) of the Product Principle is that version (c) does not assume that the set of choices for step i is independent of the previous choices.

Problem 1.28. A fashion-challenged freshman has three pair of pants, five shirts (all t-shirts, but different patterns) and a pair of sandals. How many different ensembles are available for this freshman?

Problem 1.29. Suppose you flip a coin fives times in a row, recording the sequence of heads and tails that you see. How many different sequences of flips are possible?

Problem 1.30. Suppose we roll a six-sided die and then flip a coin. How many distinct outcomes are possible?

Problem 1.31. How many bit strings of length n are there? How many of these start and end with the bit 1?

Problem 1.32. How many subsets does a set with n elements have? Make sure you are taking the case $n = 0$ into account. You should be able to carefully justify your answer by cleverly utilizing the Product Principle.

Problem 1.33. How many four-letter words are there using lower-case English letter consonants (allowing y) that include exactly one occurrence of b?

Problem 1.34. A pass code consists of a string of two or three case-sensitive English letters followed by three digits (i.e., $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$). How many different possible pass codes are there?

Problem 1.35. How many strings of length 6 are there consisting of lower-case English letters subject to the following constraints?

(a) No constraints.

(b) Repetition not allowed.

(c) No instances of letter a.

(d) At least one occurrence of letter a.

Problem 1.36. It's Halloween and five students arrive at my office begging for candy. I happen to have five pieces of candy. Depending on my mood, I may give away none of the candy, all of the candy, or any amount in between. Assuming I don't give any student more than one piece of candy, how many different ways can I distribute the candy? Does it matter if the pieces of candy are identical or not? If so, count both situations.

Let X and Y be two nonempty sets. A **function** f **from** X **to** Y is a subset of $X \times Y$ such that for every $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in f$. The set X is called the **domain** of f and is denoted by $\boxed{\mathrm{Dom}(f)}$. The set Y is called the **codomain** of f and is denoted by $\boxed{\mathrm{Codom}(f)}$ while the subset of the codomain defined via

$$\boxed{\mathrm{Rng}(f) := \{y \in Y \mid \text{there exists } x \text{ such that } (x, y) \in f\}}$$

is called the **range** of f or the **image of** X under f.

There is a variety of notation and terminology associated to functions. We will write $\boxed{f : X \to Y}$ to indicate that f is a function from X to Y. We will make use of statements such as "Let $f : X \to Y$ be the function defined via..." or "Define $f : X \to Y$ via...", where f is understood to be a function in the second statement. Sometimes the word **mapping** (or **map**) is used in place of the word function. If $(x, y) \in f$ for a function f, we often write $\boxed{f(x) = y}$ and say that "f maps x to y" or "f of x equals y". In this case, x may be called an **input** of f and is the **preimage** of y under f while y is called an **output** of f and is the **image** of x under f. Note that the domain of a function is the set of inputs while the range is the set of outputs for the function.

Sometimes we can represent functions visual representations called **function** (or **mapping**) **diagrams**, where the elements of the domain and codomain are indicated by labeled nodes and ordered pairs for the function are indicated by an arrow pointing from the node for input to the node for the output. When drawing function diagrams, it is standard practice to put the elements for the domain on the left and the elements for the codomain on the right, so that all directed edges point from left to right. We may also draw an additional arrow labeled by the name of the function from the domain to the codomain.

Example 1.37. Figure 1.2 depicts a function $f : X \to Y$ for the sets $X = \{a, b, c, d\}$ to $Y = \{1, 2, 3, 4\}$. In this case, we see that $\mathrm{Rng}(f) = \{1, 2, 4\}$. Moreover, we can write things like $f(a) = 2$ and $c \mapsto 4$, and say things like "f maps b to 4" and "the image of d is 1." Note that it is perfectly okay to have both b and c mapped to 4.

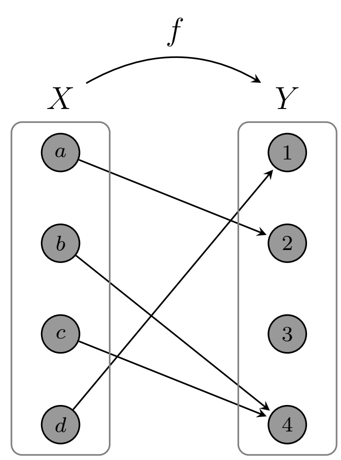

Figure 1.2: Function diagram for a function from $X = \{a, b, c, d, \}$ to $Y = \{1, 2, 3, 4\}$.

Sometimes we can define a function using a formula. For example, we can write $f(x) = x^2 - 1$ to mean that each x in the domain of f maps to $x^2 - 1$ in the codomain. However, notice that providing only a formula is ambiguous! A function is determined by its domain, codomain, and the correspondence between these two sets. If we only provide a description for the correspondence, it is not clear what the domain and codomain are. Two functions that are defined by the same formula, but have different domains or codomains are *not* equal. It is important to point out that not every function can be described using a formula! Despite your prior experience, functions that can be represented succinctly using a formula are rare.

Example 1.38. The function $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x^2 - 1$ is not equal to the function $g : \mathbb{N} \to \mathbb{R}$ defined by $g(x) = x^2 - 1$ since the two functions do not have the same domain.

Problem 1.39. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs that represents the assignment, or write a formula as long as the domain and codomain are clear.

(a) A function f from a set with 4 elements to a set with 3 elements such that $\mathrm{Rng}(f) = \mathrm{Codom}(f)$.

(b) A function g from a set with 4 elements to a set with 3 elements such that $\mathrm{Rng}(g)$ is strictly smaller than $\mathrm{Codom}(g)$.

Problem 1.40. In high school you may have been told that a graph represents a function if it passes the **vertical line test**. Carefully state what the vertical line test says and then explain why it works.

A **piecewise-defined function** (or **piecewise function**) is a function defined by specifying its output on a partition (i.e., "disjoint chunks") of the domain. Note that "piecewise" is a way of expressing the function, rather than a property of the function itself.

Example 1.41. The function $f : \mathbb{R} \to \mathbb{R}$ defined via

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x \geq 0, \\ 17, & \text{if } -2 \leq x < 0, \\ -x, & \text{if } x < -2 \end{cases}$$

is an example of a piecewise-defined function.

Problem 1.42. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ via $f(x) = \frac{|x|}{x}$. Express f as a piecewise function.

Problem 1.43. Let $n \in \mathbb{N}$. Count all functions $f : \{1, 2, \ldots, n\} \to \{0, 1\}$. How is this problem related to Problem 1.31? How is this problem related to Problem 1.32?

Let $f : X \to Y$ be a function. We define the following.

(a) The function f is said to be **injective** (or **one-to-one**) if for all $y \in \mathrm{Rng}(f)$, there is a unique $x \in X$ such that $y = f(x)$. Said another way, f is one-to-one provided $f(x) = f(y)$ implies that $x = y$, or equivalently $x \neq y$ (in X) implies $f(x) \neq f(y)$. That is, different inputs produce different outputs.

(b) The function f is said to be **surjective** (or **onto**) if for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$.

(c) If f is both injective and surjective, we say that f is **bijective**.

Problem 1.44. Assume that X and Y are finite sets. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs, or write a formula as long as the domain and codomain are clear.

(a) A function $f : X \to Y$ that is injective but not surjective.

(b) A function $f : X \to Y$ that is surjective but not injective.

(c) A function $f : X \to Y$ that is a bijection.

(d) A function $f : X \to Y$ that is neither injective nor surjective.

Problem 1.45. Determine whether each of the following functions is injective, surjective, both, or neither. Justify your answer.

(a) Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$

(b) Define $g : \mathbb{R} \to [0, \infty)$ via $g(x) = x^2$

(c) Define $h : \mathbb{R} \to \mathbb{R}$ via $h(x) = x^3$

(d) Define $k : \mathbb{R} \to \mathbb{R}$ via $k(x) = x^3 - x$

(e) Define $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via $c(x, y) = x^2 + y^2$

(f) Define $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ via $f(n) = (n, n)$

(g) Define $g : \mathbb{Z} \to \mathbb{Z}$ via

$$g(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

(h) Define $\ell : \mathbb{Z} \to \mathbb{N}$ via

$$\ell(n) = \begin{cases} 2n + 1, & \text{if } n \geq 0 \\ -2n, & \text{if } n < 0 \end{cases}$$

Problem 1.46. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is ____________ if and only if every horizontal line hits the graph of f *at most once*.

This statement is often called the **horizontal line test**. Explain why the horizontal line test is true. What kind of theorems to we get if we replace "at most once" with either "at least once" or "exactly once"?

Problem 1.47. Suppose $f : A \to B$ is a function for finite sets A and B. Explain why each of the following statements is true.

(a) If f is an injection, then $|A| \leq |B|$.

(b) If f is a surjection, then $|A| \geq |B|$.

(c) If f is a bijection, then $|A| = |B|$.

Problem 1.48. Let A and B be nonempty finite sets with $|A| = m$ and $|B| = n$.

(a) How many different functions are there from A to B?

(b) If $m \leq n$, how many injective functions are there from A to B?

(c) If $m = n$, how many bijective functions are there from A to B?

(d) If $m \geq n$, do you think it would be challenging to count the number of surjective functions from A to B?

The next problem illustrates an important counting technique, which we refer to as the **Bijection Principle**: If we know $|B| = n$ and we know there is a bijection between A and B, we can conclude that $|A| = n$, as well.

Problem 1.49. Let A denote the set of ways to distribute candy in Problem 1.36 and let B denote the set of sequence of coin flips in Problem 1.29. Find a bijection $f : A \to B$.

Problem 1.50. Utilize a bijection to connect Problems 1.31 and 1.43. Utilize a bijection to connect Problems 1.32 and 1.43.

The next problem illustrates that some care must be taken when attempting to define a function.

Problem 1.51. For each of the following, explain why the given description does not define a function.

(a) Define $f : \{1, 2, 3\} \to \{1, 2, 3\}$ via $f(a) = a - 1$.

(b) Define $g : \mathbb{N} \to \mathbb{Q}$ via $g(n) = \frac{n}{n-1}$.

(c) Let $A_1 = \{1, 2, 3\}$ and $A_2 = \{3, 4, 5\}$. Define $h : A_1 \cup A_2 \to \{1, 2\}$ via

$$h(x) = \begin{cases} 1, & \text{if } x \in A_1 \\ 2, & \text{if } x \in A_2. \end{cases}$$

(d) Define $s : \mathbb{Q} \to \mathbb{Z}$ via $s(a/b) = a + b$.

In mathematics, we say that an expression is **well defined** (or **unambiguous**) if its definition yields a unique interpretation. Otherwise, we say that the expression is not well defined (or is **ambiguous**). For example, if $a, b, c \in \mathbb{R}$, then the expression abc is well defined since it does not matter if we interpret this as $(ab)c$ or $a(bc)$ since the real numbers are associative under multiplication.

When we attempt to define a function, it may not be clear without doing some work that our definition really does yield a function. If there is some potential ambiguity in the definition of a function that ends up not causing any issues, we say that the function is well defined. However, this phrase is a bit of misnomer since all functions are well defined. The issue of whether a description for a proposed function is well defined often arises when defining things in terms of how an element of the domain is written. For example, the descriptions given in Parts (c) and (d) of Problem 1.51 are not well defined.