Exam 1 (Part 1)

Your Name:

Instructions

Answer each of the following questions. This part of Exam 1 is worth a total of 41 points and is worth 50% of your overall score on Exam 1. Part 2 of the exam is due by the beginning of class on Monday, October 16. Your overall score on Exam 1 is worth 20% of your overall grade. Good luck and have fun!

1. (1 point each) For each statement below, determine whether it is TRUE or FALSE. Circle your answer. You do not need to justify your answer.

   (a) For all $n, m, a \in \mathbb{Z}$, if a divides $m+n$, then a divides m and a divides n.

   TRUE FALSE

   (b) For all $a, b, m \in \mathbb{Z}$, if ab divides m, then a divides m and b divides m.

   TRUE FALSE

   (c) The sum of any four consecutive integers is divisible by four.

   TRUE FALSE

   (d) For all $x, y, z \in \mathbb{Z}$, if $x+y$ and $y+z$ are both even, then $x+z$ is even.

   TRUE FALSE

   (e) $\emptyset \subseteq \{\emptyset\}$

   TRUE FALSE

   (f) $\emptyset \in \{\emptyset\}$

   TRUE FALSE

   (g) $\{\{x\}\} \subseteq \{x, \{x\}\}$

   TRUE FALSE

   (h) If A is a set such that $A \neq \emptyset$, then A has at least two distinct subsets.

   TRUE FALSE

2. (2 points each) Consider the following sentence.

   $(\forall x)(\exists y)(xy = 1)$

   (a) Provide an example of a universe of discourse where this sentence is *true*.

   (b) Provide an example of a universe of discourse where this sentence is *false*.

3. (2 points each) Consider the following statement.

   For all $n, m, a \in \mathbb{Z}$, if a divides mn, then a divides m or a divides n.

   (a) It turns out that the given statement is *false*. Find its negation so that it includes the phrase "a does not divide n."

   (b) Justify that the original given statement is false. Or equivalently, justify that your statement in Part (a) is true.

4. (2 points each) Consider the following statement.

   For all sets A and B in some universe of discourse U, if there does not exist an element $x \in U$ such that $x \in A$ and $x \notin B$, then $A = B$.

   (a) It turns out that the given statement is *false*. Find its negation so that it includes the phrase "$A \neq B$". You can either negate the original statement or your answer in Part (a).

   (b) Justify that the original given statement is false. Or equivalently, justify that your statement in Part (b) is true.

5. (1 point each) Suppose the universe of discourse is the set of all people. Consider the predicate

$$S(x, y) := \text{“}x \text{ smurfs } y\text{”}.$$

   For each of the statements (a)–(e) on the left, find an equivalent symbolic proposition chosen from the list (i)–(vii) on the right. Note that not every statement on the right will get used. You do *not* need to justify your answer.

   (a) Somebody smurfs everybody. *Answer:* ____________

   (b) Everybody smurfs somebody. *Answer:* ____________

   (c) Everybody smurfs everybody. *Answer:* ____________

   (d) No one smurfs anyone. *Answer:* ____________

   (e) Someone doesn't smurf anyone. *Answer:* ____________

   (i) $(\forall x)(\forall y)(S(x, y))$

   (ii) $(\forall x)(\exists y)(S(x, y))$

   (iii) $(\exists x)(\forall y)(S(x, y))$

   (iv) $\neg(\forall x)(\forall y)(S(x, y))$

   (v) $(\exists x)(\forall y)(\neg S(x, y))$

   (vi) $(\forall x)(\forall y)(\neg S(x, y))$

   (vii) $(\exists x)(\exists y)(S(x, y))$

6. (4 points each) Prove $\boxed{\textbf{two}}$ of the following theorems.

   Theorem 1. If n is an odd integer, then 8 divides $n^2 - 1$.

   Theorem 2. Let $n \in \mathbb{Z}$. Then n is even if and only if 4 divides n^2.

   Theorem 3. For all $n \in \mathbb{Z}$, $3n^2 + n + 14$ is even.

   Theorem 4. Suppose that A, B, and C are sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

7. (4 points) Prove $\boxed{\textbf{one}}$ of the following theorems.

Theorem 5. For all $a, b, c, x, y \in \mathbb{Z}$, if a divides b and a divides c, then a divides $bx + cy$.

Theorem 6. For all $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$, if a divides b, then a^n divides b^n.

For the remaining problem, you will need to digest the following definitions. Let A and B be sets in some universe of discourse U.

- The **union** of the sets A and B is $A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\}$.
- The **intersection** of the sets A and B is $A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\}$.
- The **complement of** A (relative to U) is the set $A^c := \{x \in U \mid x \notin A\}$.

8. (4 points) Prove **one** of the following theorems.

   Theorem 7. If A and B be sets such that $A \subseteq B$, then $B^c \subseteq A^c$.

   Theorem 8. Let A and B be sets in a universe U. If $A \cup B^c = U$, then $B \subseteq A$.

