Chapter 4

Additional Counting Methods

The **Pigeonhole Principle** is a very natural property. Here it is. If a collection of at least $n+1$ objects is put into n boxes, then there is a box with at least two things in it. The Pigeonhole Principle has surprisingly deep applications. We will start with a few examples.

Example 4.1. Back in Problem 1.47, we implicitly used the Pigeonhole Principle when we argued that if $f : A \to B$ is a function for finite sets A and B, then

(a) If f is an injection, then $|A| \leq |B|$.

(b) If f is a surjection, then $|A| \geq |B|$.

Problem 4.2. A box has blue, green, yellow, red, orange, and white balls. How many must be drawn without looking to be sure of getting at least two of the same color?

Problem 4.3. Explain why any subset of five distinct numbers from $\{1, 3, 5, 7, 9, 11, 13, 15\}$ will contain at least one pair that sums to 17.

We would like to generalize the Pigeonhole Principle, but first we need a useful function. The **ceiling function** of a real number x, written $\boxed{\lceil x \rceil}$, is the smallest integer greater than or equal to x. That is, $\lceil x \rceil$ is an integer, $\lceil x \rceil \geq x$, and there is no other integer between $\lceil x \rceil$ and x. You can think of it as the "round-up to an integer" function.

Example 4.4. For example, $\lceil \pi \rceil = 4$, $\lceil -\pi \rceil = -3$, and $\lceil 7 \rceil = 7$.

We can now generalize the Pigeonhole Principle as follows.

Theorem 4.5 (Generalized Pigeonhole Principle)**.** If n objects are placed in m boxes, then there is a box with at least $\lceil \frac{n}{m} \rceil$ objects.

Problem 4.6. If 20 buses seating at most 50 carry 621 passengers to a ball game, then some bus must have at least __________ passengers.

Problem 4.7. After a passenger train is disabled, buses are called in to transport the passengers. Each bus can hold 36 passengers, and there are a total of 413 passengers. How many buses are needed?

Problem 4.8. How many balls must be drawn from the box in Problem 4.2 in order to be sure of getting at least 4 of the same color?

Problem 4.9. Explain why a list of ten positive integers, $x_1, x_2, \ldots, x_{10}$ must have a sublist in the same order of the original ten whose sum is divisible by 10.

More coming soon...