Homework 2

Combinatorics of Genome Rearrangements

You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in his or her own work. In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments with no questions asked**.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing assignments for this course, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.** Please read NAU's Academic Integrity Policy.

Complete the following problems.

1. Define $\sim$ on S_n via $\pi \sim \sigma$ if and only if π and σ have the same cycle type. For example, $\pi = (1,2,3)(4,5,6)(7)$ has the same cycle type as $\sigma = (1,3,5)(4)(2,7,6)$ since each consists of two 3-cycles and one 1-cycle, and so $\pi \sim \sigma$. Prove that $\sim$ is an equivalence relation.

2. For $i < j$, let $\rho_{i,j}$ denote the transposition (i,j) in S_n. Note that in one-line notation, $\rho_{i,j} = [1,2,\ldots,i-1,j,i+1,\ldots,j-1,i,j+1,\ldots,n]$. Let $\pi = [\pi_1,\ldots,\pi_n]$.

   (a) Compute $\pi \circ \rho_{i,j}$. Write your answer in one-line notation. What does $\rho_{i,j}$ do to π when we multiply on the right?

   (b) Compute $\rho_{i,j} \circ \pi$. Write your answer in one-line notation. This one will be a little more challenging to describe. What does $\rho_{i,j}$ do to π when we multiply on the left?

3. Let R be a collection of generators for S_n. Define $d_R : S_n \times S_n \to \mathbb{R}$ via $d_R(\pi,\sigma) = k$ if there exists a sequence of generators $\rho_1,\ldots,\rho_k \in R$ such that

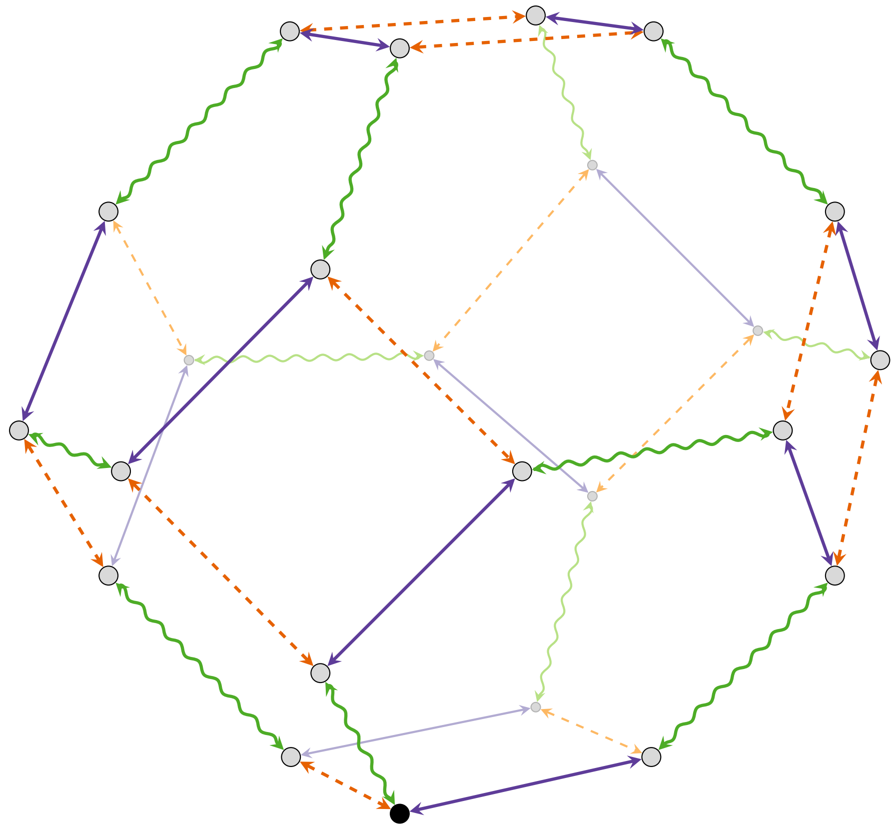
$$\pi \circ \rho_1 \circ \cdots \circ \rho_k = \sigma$$

   and k is minimal among all such sequences. Prove that d is a metric.

4. Let $R = \{(1,2),(2,3),(3,4)\}$ be the collection of adjacent transpositions in S_4, so that R is a generating set for S_4. Consider the figure below and assume the lowest vertex in the figure (shaded in black) corresponds to the identity permutation $[1,2,3,4]$. Each arrow type in the figure corresponds to one of the transpositions in R (you can choose which is which). Label the remaining vertices using one-line notation as follows: A vertex is labeled by σ if there exists a minimal sequence of generators $\rho_1,\ldots,\rho_k$ such that

$$\sigma \circ \rho_1 \circ \cdots \circ \rho_k = [1,2,3,4]$$

   and following the arrows corresponding to $\rho_1,\ldots,\rho_k$ (starting with ρ_1) takes us from the vertex in question to the vertex labeled by $[1,2,3,4]$. Problem 2(a) will be useful. Also, your labeling will not always agree with how you would have labeled this graph in MAT 511.



Notice that following a shortest path of arrows from a vertex σ to $[1,2,3,4]$ provides a method for optimally sorting σ to the identity using adjacent transposition.

5. Given $\pi \in S_n$, an **inversion** in π is a pair (i, j) with $i < j$ and $\pi_i < \pi_j$. In other words, an inversion is a pair of *positions* where the outputs in those positions are out of relative order. The set of all inversions of π is denoted

$$\mathrm{Inv}(\pi) := \{(i,j) \mid i < j \text{ and } \pi_i > \pi_j\}$$

while the number of inversions in π is $\mathrm{inv}(\pi) := |\mathrm{Inv}(w)|$. We say that $\mathrm{inv}(\pi)$ is the **inversion number** of π. For example, if $\pi = [5,2,4,1,3]$, then

$$\mathrm{Inv}(\pi) = \{(1,2),(1,3),(1,4),(1,5),(2,4),(3,4),(3,5),(4,5)\},$$

so that $\mathrm{inv}(\pi) = 8$.

(a) Next to each vertex in the figure in the previous problem, write down the inversion number of the corresponding permutation. Any observations?

(b) Explain why the inversion number of a permutation equals the number of times a pair of strings cross in the corresponding string diagram.

(c) If $\pi \in S_n$, what is the maximum value that $\mathrm{inv}(\pi)$ can take on? Explain your answer.

(d) Find all permutations in S_n that attain the maximum inversion number.

(e) If $\pi \in S_n$, explain why $\mathrm{inv}(\pi) = \mathrm{inv}(\pi^{-1})$.

(f) Is it true that $\mathrm{Inv}(\pi) = \mathrm{Inv}(\pi^{-1})$? If so, prove it. Otherwise, provide a counterexample.