Problem Collection for Introduction to Mathematical Reasoning

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Problem 1. Three strangers meet at a taxi stand and decide to share a cab to cut down the cost. Each has a different destination but all are heading in more-or-less the same direction. Bob is traveling 10 miles, Sally is traveling 20 miles, and Mike is traveling 30 miles. If the taxi costs \$2 per mile, how much should each contribute to the total fare? What do you think is the most common answer to this question?

Problem 2. Christine wants to take yoga classes to increase her strength and flexibility. In her neighborhood, there are two yoga studios: Namaste Yoga and Yoga Spirit. At Namaste Yoga, a student's first class costs \$12, and additional classes cost \$10 each. At Yoga Spirit, a student's first class costs \$24, and additional classes cost \$8 each. Because Christine wants to save money, she is interested in comparing the costs of the two studios. For what number of yoga classes do the two studios cost the same amount?

Problem 3. Imagine a hallway with 1000 doors numbered consecutively 1 through 1000. Suppose all of the doors are closed to start with. Then some dude with nothing better to do walks down the hallway and opens all of the doors. Because the dude is still bored, he decides to close every other door starting with door number 2. Then he walks down the hall and changes (i.e., if open, he closes it; if closed, he opens it) every third door starting with door 3. Then he walks down the hall and changes every fourth door starting with door 4. He continues this way, making a total of 1000 passes down the hallway, so that on the 1000th pass, he changes door 1000. At the end of this process, which doors are open and which doors are closed?

Problem 4. The Sunny Day Juice Stand sells freshly squeezed lemonade and orange juice at the farmers' market. The juices are ladled out of large glass jars, each holding exactly the same amount of juice. Linda and Julie set up their stand early one Saturday morning. The first customer of the day ordered orange juice and Linda carefully ladled out 8 ounces into a paper cup. As she was about to hand the cup to the customer, he changed his mind and asked for lemonade instead. Accidentally, Linda dumped the cup of orange juice into the jar of lemonade. She quickly mixed up the juices, ladled out a cup of the mixture (mostly lemonade) and turned to hand it to the customer. "I've decided I don't want anything to drink right now," he said, and frazzled, Linda dumped the cupful of juice mixture into the orange juice jar. Linda's assistant, Julie, watched all of this with amusement. As the man walked away, she wondered aloud, "Now is there more orange juice in the lemonade or more lemonade in the orange juice?"

Problem 5. Imagine you have 25 pebbles, each occupying one square on a 5 by 5 chess board. Tackle each of the following variations of a puzzle.

(a) Variation 1: Suppose that each pebble must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(b) Variation 2: Suppose that all but one pebble (your choice which one) must move to an adjacent square by only moving up, down, left, or right. If this is possible, describe a solution. If this is impossible, explain why.

(c) Variation 3: Consider Variation 1 again, but this time also allow diagonal moves to adjacent squares. If this is possible, describe a solution. If this is impossible, explain why.

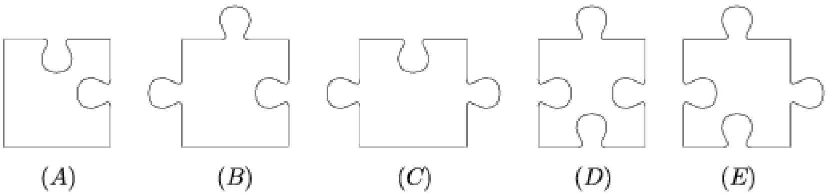
Problem 6. Consider an $n \times n$ chess board and variation 1 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 7. Consider an $n \times n$ chess board and variation 2 of the pebble puzzle from above. For what values of n is the puzzle solvable? For what values of n is the puzzle unsolvable? Justify your answers by either providing a method for a solution or an explanation for why a solution is not possible.

Problem 8. Imagine a 5×5 grid of squares, where each square is occupied by a kangaroo. If each kangaroo can hop only one square to the left, right, up, or down and a square can hold more than one kangaroo, what is the maximum number of unoccupied squares after the kangaroos are done hopping? What about a larger grid?

Problem 9. What four-digit number reverses its digits when multiplied by 4?

Problem 10. A rectangular puzzle that says "850 pieces" actually consists of 851 pieces. Each piece is identical to one of the 5 samples shown in the diagram. How many pieces of type (E) are there in the puzzle?



Problem 11. Describe where on Earth from which you can travel one mile south, then one mile east, and then one mile north and arrive at your original location. There is more than one such location. Find them all.

Problem 12. You are in a big city where all the streets go in one of two perpendicular directions. You take your car from its parking place and drive on a tour of the city such that you do not pass through the same intersection twice and return back to where you started. If you made 100 left turns, how many right turns did you make?

Problem 13. You are in a big city grid where all the streets go in one of two perpendicular directions and every city block is the same size. Imagine you take your bicycle on a tour of the city, where your tour starts at an intersection, you may retrace part of your path, you may visit the same intersection more than once, and you finish your tour where you started. What can you say about the number of city blocks that you traveled? Note that if you travel along the same city block twice (possibly in opposite directions), this would contribute two to your count. Can you conclude anything about the number of distinct city blocks that you traveled? Justify your answers.

Problem 14. Find the rational number with smallest denominator between 1/3 and 3/8.

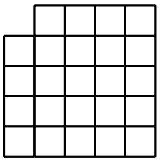
Problem 15. Suppose there are two bags of candy containing 8 pieces and 6 pieces, respectively. You and your friend are going to play a game and the winner gets to eat all of the candy. Here are the rules for the game:

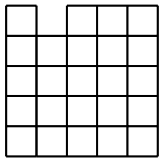
1. You and your friend will alternate removing pieces of candy from the bags. Let's assume that you go first.
2. On each turn, the designated player selects a bag that still has candy in it and then removes at least one piece of candy. The designated player can only remove candy from a single bag and he/she must remove at least one piece.
3. The winner is the one that removes all the candy from the last remaining bag.

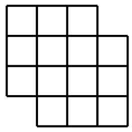
Does one of you have a guaranteed winning strategy? If so, describe that strategy. Can you generalize to handle any number of pieces of candy in either of the two bags?

Problem 16. I have 10 sticks in my bag. The length of each stick is an integer. No matter which 3 sticks I try to use, I cannot make a triangle out of those sticks. What is the minimum length of the longest stick?

Problem 17. Tile the following grids with dominoes. If a tiling is not possible, explain why.

(a) 

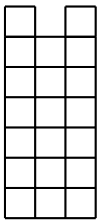
(b) 

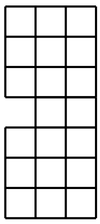
(c) 

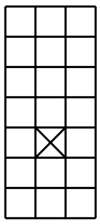
Problem 18. Rufus and Dufus are identical twins. They are each independently given the same 4-digit number. Rufus takes the number and converts it from decimal (base 10) to base 4, and writes down the 6-digit result. Dufus simply writes the first and last digits of the number followed by the number in its entirety. Rufus is shocked to discover that Dufus has written down exactly the same number has him. What was the original number? In other words, if the original number was $xyzw$, which number $xyzw$, when converted from decimal to base 4 becomes $xwxyzw$?

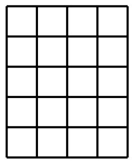
Problem 19. Find all tetrominoes (polyomino with 4 cells). Note that two tetrominoes are considered the same if we can obtain one from the other by rotation or flipping it over. The next problem gives you a hint as to how many there are.

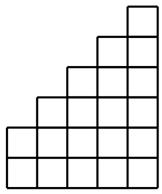
Problem 20. Tile the following grids using every tetromino exactly once. The X in (c) denotes an absence of an available square in the grid. If a tiling is not possible, explain why.

(a) 

(b) 

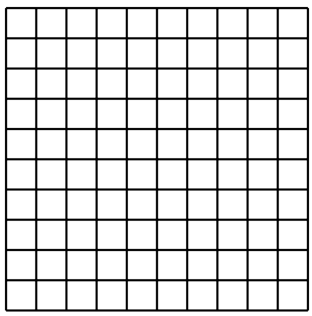
(c) 

(d) 

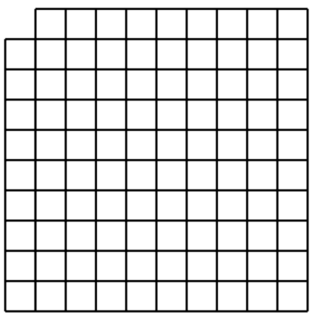
(e) 

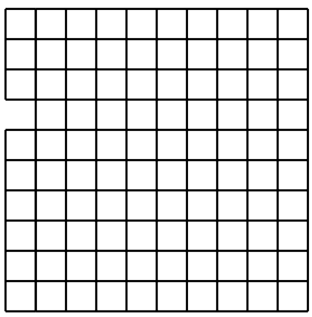
Problem 21. How many factors of 10 are there in 50! (i.e., 50 factorial)?

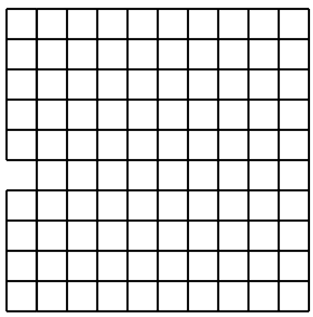
Problem 22. Consider the 10×10 grid of squares below. Show that you can color the squares of the grid with 3 colors so that every consecutive row of 3 squares and every consecutive column of 3 squares uses all 3 colors.



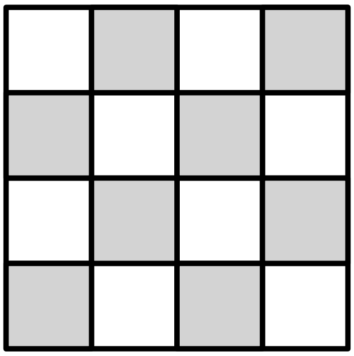
Problem 23. Tile each of the grids below with trominoes that consist of 3 squares in a line. If a tiling is not possible, explain why.

(a) 

(b) 

(c) 

Problem 24. Pennies and Paperclips is a two-player game played on a 4×4 checkerboard as shown below.



One player, "Penny", gets two pennies as her pieces. The other player, "Clip", gets a pile of paperclips as his pieces. Penny places her two pennies on any two different squares on the board. Once the pennies are placed, Clip attempts to cover the remainder of the board with paperclips - with each paperclip being required to cover two vertically or horizontally adjacent squares. Paperclips are not allowed to overlap. If the remainder of the board can be covered with paperclips then Clip is declared the winner. If the remainder of the board cannot be covered with paperclips then Penny is the winner.

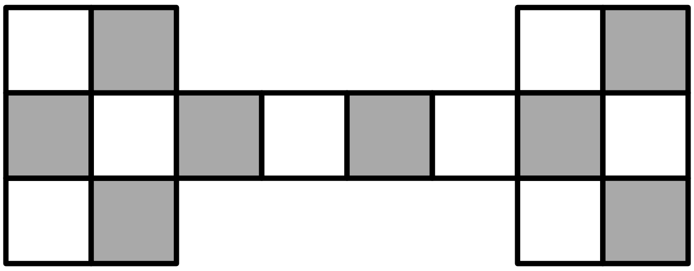
(a) Does either player have a winning strategy? If so, describe the winning strategies.

(b) State and prove a conjecture that determines precisely every situation in which Penny wins based on the placement of the pennies.

(c) State and prove a conjecture that determines precisely every situation in which Clip wins based on the placement of the pennies.

(d) Are there any situations in which neither player wins, or have you characterized all possible outcomes? Explain.

Problem 25. Consider the game Pennies and Paperclips described in the previous problem, but instead of playing on a 4×4 checkerboard, let's play on the following board.



State and prove a conjecture that determines precisely every situation in which Clip wins based on the placement of the pennies.

Problem 26. We call a game board for the Pennies and Paperclips game **fair**, if for each player there is at least one scenario in which they can win.

(a) Is the board from Problem 24 fair?

(b) Is the board from Problem 25 fair?

(c) Are there game boards that are not fair? That is, are there game boards on which one player can never win? If so, provide such a board and explain why it must be unfair. If not, explain why no such board exists.

(d) Can you create a fair board in which your conjecture from Problem 24(c) does not always hold?

Problem 27. There is a plate of 40 cookies. You and your friend are going to take turns taking either 1 or 2 cookies from the plate. However, it is a faux pas to take the last cookie, so you want to make sure that you do not take the last cookie. How can you guarantee that you will never be the one taking the last cookie? What about n cookies?

Problem 28. The Sylver Coinage Game is a game in which 2 players alternately name positive integers that are not the sum of nonnegative multiples of previously named integers. The person who names 1 is the loser! Here is a sample game between A and B:

1. A opens with 5. Now neither player can name $5, 10, 15, \ldots$
2. B names 4. Now neither player can name 4, 5, 8, 9, 10, or any number greater than 11.
3. A names 11. Now the only remaining numbers are 1, 2, 3, 6, and 7.
4. B names 6. Now the only remaining numbers are 1, 2, 3, and 7.
5. A names 7. Now the only remaining numbers are 1, 2, and 3.
6. B names 2. Now the only remaining numbers are 1 and 3.
7. A names 3, leaving only 1.
8. B is forced to name 1 and loses.

If player A names 3, can you find a strategy that guarantees that the second player wins? If so, describe the strategy? If such a strategy is not possible, then explain why?

Problem 29. Find all distinct pairs of numbers with largest gcd between and including 51 and 100. By distinct pair, we mean that you cannot choose the same number twice. Note that gcd is short for greatest common divisor. For example, $\gcd(14, 20) = 2$.

Problem 30. Four red ants and two black ants are walking along the edge of a one meter stick. The four red ants, called Albert, Bart, Debbie, and Edith, are all walking from left to right, and the two black ants, Cindy and Fred, are walking from right to left. The ants always walk at exactly one centimeter per second. Whenever they bump into another ant, they immediately turn around and walk in the other direction. And whenever they get to the end of a stick, they fall off. Albert starts at the left hand end of the stick, while Bart starts 20.2 cm from the left, Debbie is at 38.7cm, Edith is at 64.9cm and Fred is at 81.8cm. Cindy's position is not known—all we know is that he starts somewhere between Bart and Debbie. Which ant is the last to fall off the stick? And how long will it be before he or she does fall off?

Problem 31. A certain fast-food chain sells a product called "nuggets" in boxes of 6, 9, and 20. A number n is called *nuggetable* if one can buy exactly n nuggets by buying some number of boxes. For example, 21 is nuggetable since you can buy two boxes of six and one box of nine to get 21. Here are the first few nuggetable numbers:

$$6, 9, 12, 15, 18, 20, 21, 24, 26, 27, \ldots$$

and here are the first few non-nuggetable numbers:

$$1, 2, 3, 4, 5, 7, 8, 10, 11, 13, \ldots$$

What is the largest non-nuggetable number?

Problem 32. Take 15 poker chips or coins, divide into any number of piles with any number of chips in each pile. Arrange piles in adjacent columns. Take the top chip off every column and make a new column to the left. Repeat forever. What happens? Make conjectures about what happens when we change the number of chips.

Problem 33. The nth *triangular number* is defined via $t_n := 1+2+\cdots+n$. For example, $t_4 = 1+2+3+4 = 10$. Find a visual proof of the following fact. By "visual proof" we mean a sufficiently general picture that is convincing enough to justify the claim.

For all $n \in \mathbb{N}$, $t_n = \dfrac{n(n+1)}{2}$.

Problem 34. Let t_n denote the nth triangular number. Find both an algebraic proof and a visual proof of the following fact.

For all $n \in \mathbb{N}$, $t_n + t_{n+1} = (n+1)^2$.

Problem 35. Find a visual proof of the following fact. *Warning:* This problem is not about triangular numbers.

For $n \in \mathbb{N}$, $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

Problem 36. Suppose someone draws 20 distinct random lines in the plane. What is the maximum number of intersections of these lines?

Problem 37. Let t_n denote the nth triangular number. Find an algebraic and a visual proof of the following fact.

For all $a, b \in \mathbb{N}$, $t_{ab} = t_a t_b + t_{a-1} t_{b-1}$.

Problem 38. Consider a room with 25 people. If every person shakes hands with every other person (only once), then how many handshakes occurred? How about n people?

Problem 39. How many ways can 110 be written as the sum of 14 different positive integers? *Hint:* First, figure out what the largest possible integer could be in the sum. Note that the largest integer in the sum will be maximized when the other 13 numbers are as small as possible. Finish off the problem by doing an analysis of cases.

Problem 40. In the game Turnaround, you are given a permutation of the numbers from 1 to n. Your goal is to get them in the natural order $12 \cdots n$. At each stage, your only option is to reverse the order of the first k places (you get to pick k at each stage). For instance, given 6375142, you could reverse the first four to get 5736142 and then reverse the first six to get 4163752. Solve the following sequence in as few moves as possible: 352614.

Problem 41. A signed permutation of the numbers 1 through n is a fixed arrangement of the numbers 1 through n, where each number can be either be positive or negative. For example, $(-2, 1, -4, 5, 3)$ is a signed permutation of the numbers 1 through 5. In this case, think of positive numbers as being right-side-up and negative numbers as being upside-down. A *reversal* of a signed permutation is the act of performing a 180-degree rotation to some consecutive subsequence of the permutation. That is, a reversal swaps the order of a subsequence of numbers while changing the sign of each number in the subsequence. Performing a reversal to a signed permutation results in a new signed permutation. For example, if we perform a reversal on the second, third, and fourth entries in $(-2, 1, -4, 5, 3)$, we obtain $(-2, -5, 4, -1, 3)$. The *reversal distance* of a signed permutation of 1 through n is the minimum number of reversals required to transform the given signed permutation into $(1, 2, \ldots, n)$. It turns out that the reversal distance of $(3, 1, 6, 5, -2, 4)$ is 5. Find a sequence of 5 reversals that transforms $(3, 1, 6, 5, -2, 4)$ into $(1, 2, 3, 4, 5, 6)$.

Problem 42. A soul swapping machine swaps the souls inside two bodies placed in the machine. Soon after the invention of the machine an unforeseen limitation is discovered: swapping only works on a pair of bodies once. Souls get more and more homesick as they spend time in another body and if a soul is not returned to its original body after a few days, it will kill its current host.

(a) Suppose Tom and Jerry swap souls and Garfield and Odie swap souls. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

(b) Suppose Batman and Robin swap souls and then Robin's body and Flash utilize the machine. Argue that it is not possible to return the swapped souls to their original bodies using only Batman, Robin, and Flash.

(c) Consider the scenario of the previous problem. Suppose Wonder Woman and Superman are now available to sit in the machine after Batman, Robin, and Flash have already swapped souls. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

(d) Now, suppose the soul swapping machine is used by the following pair of bodies (in the order listed): Adam and Alicia, Alicia and Gwen, Gwen and Blake. In addition, Pharrell and Miley are standing nearby. Is it possible to return the swapped souls back to their original bodies? If so, find a solution that minimizes the number of times the soul swapping machine must be used.

Problem 43. Four prisoners are making plans to escape from jail. Their current plan requires them to cross a narrow bridge in the dark that has no handrail. In order to successfully cross the bridge, they must use a flashlight. However, they only have a single flashlight. To complicate matters, at most two people can be on the bridge at the same time. So, they will need to make multiple trips across the bridge, returning the flashlight back to the first side of the bridge by having someone walk it back. Unfortunately, they can't throw the flashlight. It takes 1, 2, 5, and 10 minutes for prisoner A, prisoner B, prisoner C, and prisoner D to cross the bridge and when two prisoners are walking together with the flashlight, it takes the time of the slower prisoner. What is the minimum total amount of time it takes all four prisoners to get across the bridge?

Problem 44. You need to pack several items into your shopping bag without squashing anything. The items are to be placed one on top of the other. Each item has a weight and a strength, defined as the maximum weight that can be placed above that item without it being squashed. A packing order is safe if no item in the bag is squashed, that is, if, for each item, that item's strength is at least the combined weight of what's placed above that item. For example, here are three items and a packing order:

Ordering	Item	Weight	Strength
Top	Apples	5	6
Middle	Bread	4	4
Bottom	Carrots	12	9

This packing is not safe. The bread is squashed because the weight above it, 5, is greater than its strength, 4.

(a) Find all safe orderings of the three items above.

(b) Consider packing items in weight order, with the heaviest at the bottom. Show by giving an example (i.e., invent some items and give them weights and strengths of your choosing) that this strategy might not produce a safe packing order, even if one exists.

(c) Consider packing items in strength order, with the strongest at the bottom. Show by giving an example (i.e., invent some items and give them weights and strengths of your choosing) that this strategy might not produce a safe packing order, even if one exists.

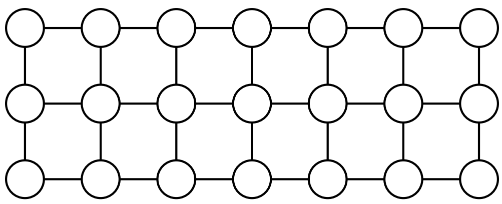
(d) Consider we have a safe packing order in our bag. Assume that item j sits directly on item i. Suppose also that

$$(\text{weight of } j)-(\text{strength of } i) \geq (\text{weight of } i)-(\text{strength of } j).$$

Show that if we swap items i and j we still have a safe packing order.

(e) Suggest a practical method of producing a safe packing order if one exists. Explain your answer.

Problem 45. In the lattice below, we color 11 vertices points black. Prove that no matter which 11 are colored black, we always have a rectangle with black corners.



Problem 46. Each point of the plane is colored red or blue. Show that there is a rectangle whose corners are all the same color.

Problem 47. Our space ship is at a Star Base with coordinates $(1,2)$. Our hyper drive allows us to jump from coordinates (a,b) to either coordinates $(a,a+b)$ or to coordinates $(a+b,b)$. How can we reach the impending enemy attack at coordinates $(8,13)$?

Problem 48. Consider our Star Base from the previous problem. Recall that our hyper drive allows us to jump from coordinates (a,b) to either coordinates $(a,a+b)$ or to coordinates $(a+b,b)$. If we start at $(1,0)$, which points in the plane can we get to by using our hyper drive? Justify your answer.

Problem 49. Suppose you randomly cut a stick into 3 pieces. What is the probability that you can form a triangle out of these 3 pieces?

Problem 50. Suppose you randomly pick 3 distinct points on a circle. What is the probability that the center of the circle lies in the interior of the triangle formed by these 3 points?

Problem 51. You have 14 coins, dated 1901 through 1914. Seven of these coins are real and weigh 1.000 ounce each. The other seven are counterfeit and weigh 0.999 ounces each. You do not know which coins are real or counterfeit. You also cannot tell which coins are real by look or feel. Fortunately for you, Zoltar the Fortune-Weighing Robot is capable of making very precise measurements. You may place any number of coins in each of Zoltar's two hands and Zoltar will do the following:

- If the weights in each hand are equal, Zoltar tells you so and returns all of the coins.
- If the weight in one hand is heavier than the weight in the other, then Zoltar takes one coin, at random, from the heavier hand as tribute. Then Zoltar tells you which hand was heavier, and returns the remaining coins to you.

Your objective is to identify a single real coin that Zoltar has not taken as tribute.

Problem 52. Welcome to Circle-Dot[1]. We'll approach Circle-Dot as a game, where the object of the game is to construct a word made entirely of $\circ$'s and $\bullet$'s. Circle-Dot begins with two words; called axioms. Using the two axioms and three rules of inference, we can create new Circle-Dot words, which are theorems in the Circle-Dot System. The process of creating Circle-Dot words using the axioms and rules of inference are proofs in the system.

On each of your "turns" in the game you can apply one of the 5 available axioms or rules to your current list of constructed Circle-Dot words to produce a new word. Also, once you have produced a new word, you can use this theorem in future "games."

Below are the axioms for Circle-Dot. Note that $\circ$ and $\bullet$ are valid symbols in the system while w and v are variables that stand for any sequence of $\circ$'s and $\bullet$'s.

Axiom A. $\circ\bullet$

Axiom B. $\bullet\circ$

At any time in your proof, you may quote an axiom. Below are the rules for generating new statements from known statements.

Rule 1. Given wv and vw, conclude w

Rule 2. Given w and v, conclude $w \bullet v$

Rule 3. Given $wv\bullet$, conclude $w\circ$

As an example, let's try to prove the following theorem.

Theorem C. $\bullet$ (just a single dot)

At the moment, the only tools we have for getting started are the axioms. As we prove theorems, we'll be able to incorporate them into our proofs, as well. To get started, let's apply Axiom A and see what that gets us. Applying Axiom A, we get $\circ\bullet$. Looking at Rules 2 and 3, it should be moderately clear that they won't help us get a single dot. So, perhaps Rule 1 will be useful, but to use it, we see that we need to have wv and vw. Applying Axiom B, we get $\bullet\circ$. Now, if we let $w = \bullet$ and $v = \circ$, then $wv = \bullet\circ$ and $vw = \circ\bullet$. Applying Rule 1, we can conclude that $\bullet$ holds. Putting this altogether, we can write something like the following.

Proof of Theorem C.

1. $\circ\bullet$ by Axiom A
2. $\bullet\circ$ by Axiom B

[1] The Circle-Dot System was developed by Ken Monks from the University of Scranton.

3. • by Rule 1 (using lines 2 and 1)

Now, try proving the following theorems.

Theorem D. ∘

Theorem E. • • •

Theorem F. • • ∘

Theorem G. • ∘ ∘

Theorem H. ∘ • • ∘

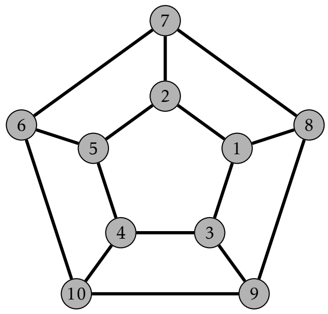
Theorem I. ∘ ∘ ∘ ∘

Theorem J. • ∘ •

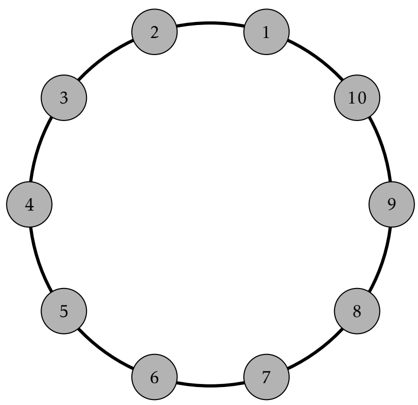
Theorem K. • ∘ ∘ ∘

Make a conjecture about which sequences of ∘'s and •'s are theorems in the Circle-Dot system.

Problem 53. The graph depicted below is an example of a Hastings Helm. Notice that we have labeled the 10 vertices of the graph with the natural numbers 1 through 10. Two vertices are said to be *adjacent* if they are joined by an edge. For example, the vertex currently labeled by 4 is adjacent to the vertices labeled by 3, 5, and 10. Is it possible to relabel the vertices so that the labels of adjacent vertices have no factors other than 1 in common? Notice that since the vertices currently labeled by 3 and 9 are adjacent and have a factor of 3 in common, the current labeling will not do the job. If you can find an appropriate labeling, then show it. If no such labeling exists, then explain why.



Problem 54. Ten people form a circle. Each picks a number and tells it to the two neighbors adjacent to him/her in the circle. Then each person computes and announces the average of the numbers of his/her two neighbors. The figure shows the average announced by each person. What is the number picked by the person who announced 6?



Problem 55. Consider a tournament with 30 teams. If every team plays every other team, how many games were played?

Problem 56. Find a solution to the equation $28x + 30y + 31z = 365$, where x, y, and z are positive whole numbers.