Chapter 8

Cardinality

In this chapter, we will explore the notion of cardinality, which formalizes what it means for two sets to be the same "size".

8.1 Introduction to Cardinality

What does it mean for two sets to have the same "size"? If the sets are finite, this is easy: just count how many elements are in each set. Another approach would be to pair up the elements in each set and see if there are any left over. In other words, check to see if there is a one-to-one correspondence (i.e., bijection) between the two sets.

But what if the sets are infinite? For example, consider the set of natural numbers $\mathbb{N}$ and the set of even natural numbers $2\mathbb{N} := \{2n \mid n \in \mathbb{N}\}$. Clearly, $2\mathbb{N}$ is a proper subset of $\mathbb{N}$. Moreover, both sets are infinite. In this case, you might be thinking that $\mathbb{N}$ is "larger than" $2\mathbb{N}$ However, it turns out that there is a one-to-one correspondence between these two sets. In particular, consider the function $f : \mathbb{N} \to 2\mathbb{N}$ defined via $f(n) = 2n$. It is easily verified that f is both one-to-one and onto. In this case, mathematics has determined that the right viewpoint is that $\mathbb{N}$ and $2\mathbb{N}$ do have the same "size". However, it is clear that "size" is a bit too imprecise when it comes to infinite sets. We need something more rigorous.

Definition 8.1. Let A and B be sets. We say that A and B have the same **cardinality** if and only if there exists a bijection between A and B. If A and B have the same cardinality, then we write $\boxed{\operatorname{card}(A) = \operatorname{card}(B)}$.

Note that we have not defined $\operatorname{card}(A)$ by itself. Doing so would not be too difficult for finite sets, but making such a notation precise in general is tricky business. When we write $\operatorname{card}(A) = \operatorname{card}(B)$ (and later $\operatorname{card}(A) \leq \operatorname{card}(B)$ and $\operatorname{card}(A) < \operatorname{card}(B)$), we are asserting the existence of a certain type of function from A to B.

Problem 8.2. Prove each of the following. In each case, you should create a bijection between the two sets. Briefly justify that your functions are in fact bijections.

(a) If $A = \{a, b, c\}$ and $B = \{x, y, z\}$, then $\operatorname{card}(A) = \operatorname{card}(B)$.

(b) If $\mathcal{O}$ is the set of odd natural numbers, then $\text{card}(\mathbb{N}) = \text{card}(\mathcal{O})$.

(c) $\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z})$.

(d) Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Then $\text{card}((a,b)) = \text{card}((c,d))$.[1]

(e) If $R = \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$, then $\text{card}(\mathbb{N}) = \text{card}(R)$.

(f) If $\mathcal{F}$ is the set of functions from $\mathbb{N}$ to $\{0,1\}$, then $\text{card}(\mathcal{F}) = \text{card}(\mathcal{P}(\mathbb{N}))$.[2]

(g) If A is any set, then $\text{card}(A) = \text{card}(A \times \{x\})$.

Theorem 8.3. Let A, B, and C be sets.

(a) $\text{card}(A) = \text{card}(A)$.

(b) If $\text{card}(A) = \text{card}(B)$, then $\text{card}(B) = \text{card}(A)$.

(c) If $\text{card}(A) = \text{card}(B)$ and $\text{card}(B) = \text{card}(C)$, then $\text{card}(A) = \text{card}(C)$.

In light of the previous theorem, the next result should not be surprising.

Corollary 8.4. If X is a set, then "has the same cardinality as" is an equivalence relation on $\mathcal{P}(X)$.

Theorem 8.5. Let A, B, C, and D be sets such that $\text{card}(A) = \text{card}(C)$ and $\text{card}(B) = \text{card}(D)$.

(a) If A and B are disjoint and C and D are disjoint, then $\text{card}(A \cup B) = \text{card}(C \cup D)$.

(b) $\text{card}(A \times B) = \text{card}(C \times D)$.

Given two finite sets, it makes sense to say that one set is "larger than" another provided one set contains more elements than the other. We would like to generalize this idea to handle both finite and infinite sets.

Definition 8.6. Let A and B be sets. If there is a one-to-one function (i.e., injection) from A to B, then we say that the **cardinality of** A **is less than or equal to the cardinality of** B. In this case, we write $\boxed{\text{card}(A) \leq \text{card}(B)}$.

Theorem 8.7. Let A, B, and C be sets.

(a) If $A \subseteq B$, then $\text{card}(A) \leq \text{card}(B)$.

(b) If $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$.

(c) If $C \subseteq A$ while $\text{card}(B) = \text{card}(C)$, then $\text{card}(B) \leq \text{card}(A)$.

[1] *Hint:* Try creating a linear function $f : (a,b) \to (c,d)$. Drawing a picture should help.

[2] *Hint:* Define $\phi : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ so that $\phi(f)$ outputs a subset of $\mathbb{N}$ determined by when f outputs a 1.

It might be tempting to think that the existence of a one-to-one function from a set A to a set B that is *not* onto would verify that $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(A) \neq \text{card}(B)$. While this is true for finite sets, it is not true for infinite sets as the next exercise asks you to verify.

Exercise 8.8. Provide an example of sets A and B such that $\text{card}(A) = \text{card}(B)$ despite the fact that there exists a one-to-one function from A to B that is not onto.

Definition 8.9. Let A and B be sets. We write $\boxed{\text{card}(A) < \text{card}(B)}$ provided $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(A) \neq \text{card}(B)$.

It is important to point out that the statements $\text{card}(A) = \text{card}(B)$ and $\text{card}(A) \leq \text{card}(B)$ are symbolic ways of asserting the existence of certain types of functions from A to B. When we write $\text{card}(A) < \text{card}(B)$, we are saying something much stronger than "There exists a function $f : A \to B$ that is one-to-one but not onto." The statement $\text{card}(A) < \text{card}(B)$ is asserting that *every* one-to-one function from A to B is not onto. In general, it is difficult to prove statements like $\text{card}(A) \neq \text{card}(B)$ or $\text{card}(A) < \text{card}(B)$.

8.2 Finite Sets

In the previous section, we used the phrase "finite set" without formally defining it. Let's be a bit more precise.

Definition 8.10. For each $n \in \mathbb{N}$, define $[n] = \{1, 2, \ldots, n\}$.

For example, $[5] = \{1, 2, 3, 4, 5\}$. Notice that our notation looks just like that for the set of relatives given a relation on some set (see Definition 6.32), which is an equivalence class if the relation happens to be an equivalence relation. However, despite the similar notation, these concepts are unrelated. We will have to rely on context to keep them straight.

The next definition should coincide with your intuition about what it means for a set to be finite.

Definition 8.11. A set A is **finite** if and only if $A = \emptyset$ or $\text{card}(A) = \text{card}([n])$ for some $n \in \mathbb{N}$. If $A = \emptyset$, then we say that A has **cardinality** 0 and if $\text{card}(A) = \text{card}([n])$, then we say that A has **cardinality** n.

Let's prove a few results about finite sets.

Theorem 8.12. If A is finite and $\text{card}(A) = \text{card}(B)$, then B is finite.[3]

Theorem 8.13. If A has cardinality $n \in \mathbb{N} \cup \{0\}$ and $x \notin A$, then $A \cup \{x\}$ is finite and has cardinality $n + 1$.

Theorem 8.14. For every $n \in \mathbb{N}$, every subset of $[n]$ is finite.[4]

[3] Don't forget to consider the case when $A = \emptyset$.
[4] *Hint:* Use induction.

Theorem 8.13 shows that adding a single element to a finite set increases the cardinality by 1. As you would expect, removing one element from a finite set decreases the cardinality by 1.

Theorem 8.15. If A has cardinality $n \in \mathbb{N}$, then for all $x \in A$, $A \setminus \{x\}$ is finite and has cardinality $n-1$.

The next result tells us that the cardinality of a proper subset of a finite set is never the same as the cardinality of the original set. It turns out that this theorem does not hold for infinite sets.

Theorem 8.16. Every subset of a finite set is finite. In particular, if A is a finite set, then $\operatorname{card}(B) < \operatorname{card}(A)$ for all proper subsets B of A.

Theorem 8.17. If $A_1, A_2, \ldots, A_k$ is a finite collection of finite sets, then $\displaystyle\bigcup_{i=1}^{k} A_i$ is finite.[5]

The next theorem, called the Pigeonhole Principle, is surprisingly useful. It puts restrictions on when we may have a one-to-one function. The name of the theorem is inspired by the following idea: If n pigeons wish to roost in a house with k pigeonholes and $n > k$, then it must be the case that at least one hole contains more than one pigeon.

Theorem 8.18 (Pigeonhole Principle)**.** If $n, k \in \mathbb{N}$ and $f : [n] \to [k]$ with $n > k$, then f is not one-to-one.[6]

8.3 Infinite Sets

In the previous section, we explored finite sets. Now, let's tinker with infinite sets.

Definition 8.19. A set A is **infinite** if and only if A is not finite.

Let's see if we can utilize this definition to prove that the set of natural numbers is infinite.

Theorem 8.20. The set $\mathbb{N}$ of natural numbers is infinite.[7]

The next theorem is analogous to Theorem 8.12, but for infinite sets. As we shall see later, the converse of this theorem is not generally true.

Theorem 8.21. If A is infinite and $\operatorname{card}(A) = \operatorname{card}(B)$, then B is infinite.[8]

[5] *Hint:* Use induction.
[6] *Hint:* Induct on the number of pigeons. The base case is $n = 2$.
[7] *Hint:* For sake of a contradiction, assume otherwise. Then there exists $n \in \mathbb{N}$ such that $\operatorname{card}([n]) = \operatorname{card}(\mathbb{N})$, which implies that there exists a bijection $f : [n] \to \mathbb{N}$. What can you say about the number $m := \max(f(1), f(2), \ldots, f(n)) + 1$?
[8] *Hint:* Try a proof by contradiction. You should end up composing two bijections, say $f : A \to B$ and $g : B \to [n]$ for some $n \in \mathbb{N}$.

Exercise 8.22. Quickly verify that the following sets are infinite by appealing to Theorem 8.20, Theorem 8.21, and Problem 8.2.

(a) The set of odd natural numbers.

(b) The set of even natural numbers.

(c) The integers.

(d) The set $R = \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$.

(e) The set $\mathbb{N} \times \{x\}$.

Notice that Definition 8.19 tells what infinite sets are not, but it doesn't really tell us what they are. In light of Theorem 8.20, one way of thinking about infinite sets is as follows. Suppose A is some nonempty set. Let's select a random element from A and set it aside. We will call this element the "first" element. Then we select one of the remaining elements and set is aside, as well. This is the "second" element. Imagine we continue this way, choosing a "third" element, and "fourth" element, etc. If the set is infinite, we should never run out of elements to select. Otherwise, we would create a bijection with $[n]$ for some $n \in \mathbb{N}$.

The next problem, sometimes refered to as the Hilbert Hotel[9], illustrates another way to think about infinite sets.

Problem 8.23. The Infinite Hotel has rooms numbered $1, 2, 3, 4, \ldots$. Every room in the Infinite Hotel is currently occupied. Is it possible to make room for one more guest (assuming they want a room all to themselves)? An infinite number of new guests, say $g_1, g_2, g_3, \ldots$, show up in the lobby and each demands a room. Is it possible to make room for all the new guests even in the hotel is already full?

The previous problem verifies that a proper subset of the natural numbers is in bijection with the natural numbers themselves. We also witnessed this in parts (a) and (b) of Exercise 8.22. Notice that Theorem 8.16 forbids this type of behavior for finite sets. It turns out that this phenomenon is true for all infinite sets. The next theorem verifies that that the two viewpoints of infinite sets discussed above are valid.

Theorem 8.24. Let A be a set. Then the following statements are equivalent.[10]

(i) A is an infinite set.

(ii) There exists a one-to-one function $f : \mathbb{N} \to A$.

(iii) A can be put in bijection with a proper subset of A (i.e., there exists a proper subset B of A such that $\text{card}(B) = \text{card}(A)$).

[9]The Hilbert Hotel is named after mathematician David Hilbert (1862–1942).

[10]*Hint:* Prove (i) if and only if (ii) and (ii) if and only if (iii). For (i) implies (ii), construct f recursively. For (ii) implies (i), try a proof by contradiction. For (ii) implies (iii), let $B = A \setminus \{f(1), f(2), \ldots\}$ and show that A can be put in bijection with $B \cup \{f(2), f(3), \ldots\}$. Lastly, for (iii) implies (ii), suppose $g : A \to C$ is a bijection for some proper subset C of A. Let $a \in A \setminus C$. Define $f : \mathbb{N} \to A$ via $f(n) = g^n(a)$, where g^n means compose g with itself n times.

Corollary 8.25. A set is infinite if and only if it has an infinite subset.

Corollary 8.26. If A is an infinite set, then $\text{card}(\mathbb{N}) \leq \text{card}(A)$.

It is worth mentioning that for the previous theorem, (iii) implies (i) follows immediately from the contrapositive of Theorem 8.16.

Problem 8.27. Find a new proof of Theorem 8.20 that uses (iii) implies (i) from Theorem 8.24.

Exercise 8.28. Quickly verify that the following sets are infinite by appealing to either Theorem 8.24 (use (ii) implies (i)) or Corollary 8.25.

(a) The set of odd natural numbers.

(b) The set of even natural numbers.

(c) The integers.

(d) The set $\mathbb{N} \times \mathbb{N}$.

(e) The set of rational numbers $\mathbb{Q}$.

(f) The set of real numbers $\mathbb{R}$.

(g) The set of perfect squares.

(h) The interval $(0,1)$.

(i) The set of complex numbers $\mathbb{C} := \{a + bi \mid a, b \in \mathbb{R}\}$.

8.4 Countable Sets

Recall that if $A = \emptyset$, then we say that A has cardinality 0. Also, if $\text{card}(A) = \text{card}([n])$ for $n \in \mathbb{N}$, then we say that A has cardinality n. We have a special way of describing sets that are in bijection with the natural numbers.

Definition 8.29. If A is a set such that $\text{card}(A) = \text{card}(\mathbb{N})$, then we say that A is **denumerable** and has **cardinality** $\boldsymbol{\aleph_0}$ (read "aleph naught").

Notice if a set A has cardinality $1, 2, \ldots,$ or $\aleph_0$, we can label the elements in A as "first", "second", and so on. That is, we can "count" the elements in these situations. Certainly, if a set has cardinality 0, counting isn't an issue. This idea leads to the following definition.

Definition 8.30. A set A is called **countable** if and only if A is finite or denumerable. A set is called **uncountable** if and only if it is not countable.

Exercise 8.31. Quickly justify that each of the following sets is countable. Feel free to appeal to previous problems.

(a) The set $A := \{a, b, c\}$

(b) The set of odd natural numbers.

(c) The set of even natural numbers.

(d) The set $R := \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$.

(e) The set of perfect squares.

(f) The integers.

(g) The set $\mathbb{N} \times \{x\}$, where $x \notin \mathbb{N}$.

Theorem 8.32. Let A and B be sets such that A is countable. If $f : A \to B$ is a bijection, then B is countable.

Theorem 8.33. Every subset of a countable set is countable.[11]

Theorem 8.34. A set is countable if and only if it has the same cardinality of some subset of the natural numbers.

Theorem 8.35. If $f : \mathbb{N} \to A$ is an onto function, then A is countable.

Loosely speaking, the next theorem tells us that we can arrange all of the rational numbers then count them "first", "second", "third", etc. Given the fact that between any two distinct rational numbers on the number line, there are an infinite number of other rational numbers (justified by taking repeated midpoints), this may seem counterintuitive.

Theorem 8.36. The set of rational numbers $\mathbb{Q}$ is countable.[12]

Theorem 8.37. If A and B are countable sets, then $A \cup B$ is countable.

We would like to prove a stronger result than the previous theorem. To do so, we need a lemma.

Lemma 8.38. Let $\{A_n\}_{n=1}^{\infty}$ be a (countable) collection of sets. Define $B_1 := A_1$ and for each natural number $n > 1$, define

$$B_n := A_n \setminus \bigcup_{i=1}^{n-1} A_i.$$

Then we we have the following:

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[11] *Hint:* Let A be a countable set. Consider the cases when A is finite versus infinite. The contrapositive of Corollary 8.25 should be useful for the case when A is finite.

[12] *Hint:* Make a table with column headings $0, 1, -1, 2, -2, \ldots$ and row headings $1, 2, 3, 4, 5, \ldots$. If a column has heading m and a row has heading n, then the corresponding entry in the table is given by the fraction m/n. Find a way to zig-zag through the table making sure to hit every entry in the table (not including column and row headings) exactly once. This justifies that there is a bijection between $\mathbb{N}$ and the entries in the table. Do you see why? Now, we aren't done yet because every rational number appears an infinite number of times in the table. Appeal to Theorem 8.33.

(a) The collection $\{B_n\}_{n=1}^{\infty}$ is pairwise disjoint.

(b) $\displaystyle\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$.

Theorem 8.39. Every countable union of countable sets is countable.[13]

Theorem 8.40. If A and B are countable sets, then $A \times B$ is countable.

Theorem 8.41. The set of all finite sequences of 0's and 1's (e.g., 0110010 is a finite sequence of 0's and 1') is countable.

8.5 Uncountable Sets

Recall from Definition 8.30 that a set A is **uncountable** if and only if A is not countable. Since all finite sets are countable, the only way a set could be uncountable is if it is infinite. It follows that a set A is uncountable if and only if there is never a bijection between $\mathbb{N}$ and A. It's not clear that uncountable sets even exist! It turns out that uncountable sets do exist and in this section, we will discover a few of them.

Our first task is to prove that the open interval $(0,1)$ is uncountable. By Exercise 8.22(h), we know that $(0,1)$ is an infinite set, so it is at least plausible that $(0,1)$ is uncountable. The following problem outlines the proof of Theorem 8.43. Our approach is often referred to as **Cantor's Diagonalization Argument**.

Before we get started, recall that every number in $(0,1)$ can be written in decimal form. However, there may be more than one way to write a given number in decimal form. For example, 0.2 equals $0.1\overline{99}$. A number $x = 0.a_1a_2a_3\ldots$ is said to be in **standard form** if and only if there is no k such that for all $i > k$, $a_i = 9$. That is, a decimal expansion is in standard form if and only if the expansion doesn't end with a repeating sequence of 9's. For example, 0.2 is in standard form while $0.1\overline{99}$ is not, even though both represent the same number. It turns out that every real number can be expressed uniquely in standard form.

Problem 8.42. For sake of a contradiction, assume the interval $(0,1)$ is countable. Then there exists a bijection $f : \mathbb{N} \to (0,1)$. For each $n \in \mathbb{N}$, its image under f is some number in $(0,1)$. Let $f(n) := 0.a_{1n}a_{2n}a_{3n}\ldots$, where a_{1n} is the first digit in the decimal form for the image of n, a_{2n} is the second digit, and so on. If $f(n)$ terminates after k digits, then our convention will be to continue the decimal form with 0's. Now, define $b = 0.b_1b_2b_3\ldots$, where

$$b_i = \begin{cases} 2, & \text{if } a_{ii} \neq 2 \\ 3, & \text{if } a_{ii} = 2. \end{cases}$$

(a) Prove that the decimal expansion that defines b above is in standard form.

[13] *Hint:* A countable union is a union of countably many sets. Recall that a countable set may be finite or infinite. Consider three cases: (1) finite union of countable sets (use induction with base case $n = 2$), (2) countably infinite union of finite sets, (3) countably infinite union of countably infinite sets.

(b) Prove that for all $n \in \mathbb{N}$, $f(n) \neq b$.

(c) Prove that f is not onto.

(d) Explain why we have a contradiction.

(e) Explain why it follows that the open interval $(0,1)$ cannot be countable.

The steps above prove the following theorem.

Theorem 8.43. The open interval $(0,1)$ is uncountable.

Loosely speaking, what Theorem 8.43 says is that the open interval $(0,1)$ is "bigger" in terms of the number of elements it contains than the natural numbers and even the rational numbers. This shows that there are infinite sets of different sizes!

One consequence of Theorem 8.43 is that we know there is at least one uncountable set. The next three results are useful for finding other uncountable sets.

Theorem 8.44. If A and B are sets such that $A \subseteq B$ and A is uncountable, then B is uncountable.[14]

Corollary 8.45. If A and B are sets such that A is uncountable and B is countable, then $A \setminus B$ is uncountable.

Theorem 8.46. If $f : A \to B$ is a one-to-one function and A is uncountable, then B is uncountable.

Theorem 8.47. The set $\mathbb{R}$ of real numbers is uncountable. Moreover, $\operatorname{card}((0,1)) = \operatorname{card}(\mathbb{R})$.[15]

Theorem 8.48. If $a, b \in \mathbb{R}$ with $a < b$, then (a,b), $[a,b]$, $(a,b]$, and $[a,b)$ are all uncountable.

Theorem 8.49. The set of irrational numbers is uncountable.

Theorem 8.50. The set $\mathbb{C}$ of complex numbers is uncountable.

Problem 8.51. Determine whether each of the following statements is true or false. If a statement is true, prove it. If a statement is false, provide a counterexample.

(a) If A and B are sets such that A is uncountable, then $A \cup B$ is uncountable.

(b) If A and B are sets such that A is uncountable, then $A \cap B$ is uncountable.

(c) If A and B are sets such that A is uncountable, then $A \times B$ is uncountable.

(d) If A and B are sets such that A is uncountable, then $A \setminus B$ is uncountable.

Problem 8.52. Let S be the set of infinite sequences of 0's and 1's. Determine whether S is countable or uncountable and prove that your answer is correct.

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[14]*Hint:* Try a proof by contradiction. Take a look at Theorem 8.33.

[15]*Hint:* To show that $\mathbb{R}$ is uncountable, appeal to Theorem 8.44. To show that $\operatorname{card}((0,1)) = \operatorname{card}(\mathbb{R})$, consider the function $f : (0,1) \to \mathbb{R}$ defined via $f(x) = \tan(\pi x - \frac{\pi}{2})$. It is worth pointing out that proving $\operatorname{card}((0,1)) = \operatorname{card}(\mathbb{R})$ automatically proves that $\mathbb{R}$ is uncountable.

It turns out that the two uncountable sets may or may not have the same cardinality. Perhaps surprisingly, there are sets that are even "bigger" than the set of real numbers. Given any set, we can always increase the cardinality by considering its power set.

Theorem 8.53. If A is a set, then $\operatorname{card}(A) < \operatorname{card}(\mathcal{P}(A))$.[16]

Recall that cardinality provides a way for talking about "how big" a set is. The fact that the natural numbers and the real numbers have different cardinality (one countable, the other uncountable), tells us that there are at least two different "sizes of infinity". Theorem 8.53 tells us that there are infinitely many "sizes of infinity."

Theorem 8.54. Consider the set S from Problem 8.52. Then $\operatorname{card}(\mathcal{P}(\mathbb{N})) = \operatorname{card}(S)$.

[16] *Hint:* Mimic Cantor's Diagonalization Argument for showing that the interval $(0,1)$ is uncountable.