Homework 4

Combinatorics of Genome Rearrangements

You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in his or her own work. In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments with no questions asked**.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing assignments for this course, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.** Please read NAU's Academic Integrity Policy.

Complete the following problems.

1. Consider the permutation $\pi = 6\ 7\ 9\ 10\ 11\ 13\ 8\ 12\ 3\ 1\ 5\ 4\ 2 \in S_{13}$. Generate the strong interval tree for π and the identity in S_{13} and label each of the internal nodes as prime or linear.

2. A permutation $\pi \in S_n$ is called **simple** if the only common intervals between π and the identity in S_n are trivial (i.e., size 1 or n).

   (a) Explain why there are no simple permutations in S_3.

   (b) Find all simple permutations in S_4.

   (c) Prove that if a node in a strong interval tree corresponds to a simple permutation, then it is prime. Is the converse true?

   (d) If a node in a strong interval tree corresponds to a simple permutation, what can you say about the children of this node? Can you make the same conclusion about an arbitrary prime node?

3. Let $\Pi(n)$ denote the set of all set partitions of $[n]$, ordered by reverse refinement.

   (a) Draw the Hasse diagram for $\Pi(n)$.

   (b) Describe the cover relations for $\Pi(n)$.

   (c) Prove that $\Pi(n)$ is a lattice.

   (d) Prove that $\Pi(n)$ is a ranked poset. What is the rank function?

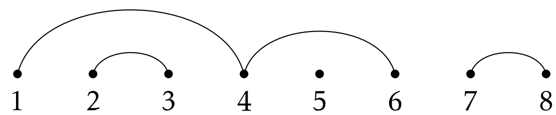
   (e) What is the rank of $\Pi(n)$?

   (f) Enumerate the number of maximal chains in $\Pi(n)$. Part (b) should be helpful.

4. If possible, provide an example of a lattice L together with a subposet Q of L such that Q is not a lattice. If this is not possible, explain why.

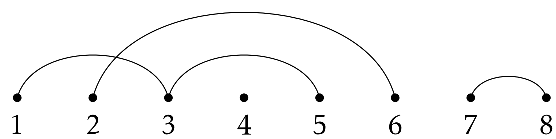
5. If possible, provide an example of a ranked poset P together with a subposet Q of P such that Q is not ranked. If this is not possible, explain why.

6. A partition $p = \{B_1, B_2, \ldots, B_k\} \in \Pi(n)$ is called a **noncrossing partition** of $[n]$ if $\{a,c\} \subseteq B_i$ and $\{b,d\} \subseteq B_j$ with $1 \leq a < b < c < d \leq n$, then $i = j$. That is, two pairs of numbers from distinct blocks cannot be interleaved. Below is an example of a noncrossing matching on $[8]$:



$= \{\{1,4,6\},\{2,3\},\{5\},\{7,8\}\}$

And here is an example of a partition that is not noncrossing:



$= \{\{1,3,5\},\{2,6\},\{4\},\{7,8\}\}$

The visual depictions on the left side in each example above make it easy to tell if a partition is noncrossing or not. A partition is noncrossing if and only if the arcs in the figure do not cross. Let $\mathrm{NC}(n)$ denote the collection of noncrossing partitions in $\Pi(n)$. Since $\mathrm{NC}(n)$ inherits the partial order from $\Pi(n)$, $\mathrm{NC}(n)$ is also a partial order. In particular, we have $p \leq_{\mathrm{NC}} q$ if p refines q as a partition.

(a) Circle the noncrossing partitions in your Hasse diagram for $\Pi(4)$.

(b) If $p, q \in \mathrm{NC}(n)$, it turns out that there is a unique least upper bound of p and q and a unique greatest lower bound of p and q. Explain how to find both the least upper bound of p and q and the greatest lower bound of p and q. This verifies that $\mathrm{NC}(n)$ is a lattice.

(c) Prove that $\mathrm{NC}(n)$ is a ranked poset. What is the rank function?