Final Exam

Your Name:

Names of Any Collaborators:

Instructions

Answer each of the following questions and then submit your solutions to BbLearn by **5:00pm on Thursday, April 28**. You can either write your solutions on paper and then capture your work digitally or you can write your solutions digitally on a tablet (e.g., iPad). This exam is worth a total of 52 points and is worth 25% of your overall grade.

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts. Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing the following problems, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. Further information:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using.
2. Unless you prove them, you cannot use any results from the course notes/book that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.** To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

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1. (1 points each) For each statement below, determine whether it is TRUE or FALSE. Circle the correct answer. You do not need to justify your answer.

   (a) For all $n > 0$ and $k \geq 0$, $N_{n,k} = N_{n,n-k-1}$.

   TRUE FALSE

   (b) For all $w \in S_n$, $\operatorname{Inv}(w) = \operatorname{Inv}(w^{-1})$.

   TRUE FALSE

   (c) For $n \geq 1$, $\displaystyle\sum_{p \in \mathrm{Dyck}(n)} q^{\mathrm{maj}(p)} = \sum_{p \in \mathrm{Dyck}(n)} q^{\mathrm{pk}(p)}$.

   TRUE FALSE

   (d) For $n \geq 1$, $\displaystyle\sum_{p \in L(n,n)} q^{\mathrm{maj}(p)} = \begin{bmatrix} 2n \\ n \end{bmatrix}$.

   TRUE FALSE

   (e) If $w \in S_n$, then $\mathrm{maj}(w) = \mathrm{inv}(w)$.

   TRUE FALSE

   (f) For $1 \leq k \leq n$, $\left\{ {n \atop k} \right\} = \left\{ {n \atop n-k} \right\}$.

   TRUE FALSE

   (g) For all $0 \leq k \leq n$, $a_{n,k}(1) = \dbinom{n}{k}$, where $a_{n,k}(q)$ is the generating function for paths in $L(k, n-k)$ according to area.

   TRUE FALSE

   (h) For $n \geq 1$, $\displaystyle\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle = \sum_{k=0}^{\binom{n}{2}} I_{n,k}$.

   TRUE FALSE

   (i) If P is a finite poset having unique maximal and unique minimal elements, then P is a lattice.

   TRUE FALSE

   (j) The number of permutations of rank k in $\mathrm{Wk}^{\ell}(S_n)$ is equal to the coefficient on q^k in $[n]_q!$.

   TRUE FALSE

   (k) The number of permutations in S_n with major index k is equal to the coefficient on q^k in $[n]_q!$.

   TRUE FALSE

   (l) The number of subsets of $[n]$ of size k is equal to the coefficient on q^k in the expansion of $(1+q)^n$.

   TRUE FALSE

   (m) Each vertex in the Hasse diagram for the absolute order of S_n has degree $\dbinom{n}{2}$.

   TRUE FALSE

   (n) The number of edges in any path from the minimal element to the maximal element in the Hasse diagram for the noncrossing partition lattice $\mathrm{NC}(n)$ is $n-1$.

   TRUE FALSE

   (o) The number of partitions of rank k in $\mathrm{NC}(n)$ is equal to $N_{n,k}$.

   TRUE FALSE

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(p) If $u, v \in S_n$, then there exists a permutation w such that $\mathrm{Inv}(w) = \mathrm{Inv}(u) \cup \mathrm{Inv}(v)$.

TRUE FALSE

(q) If $([n], \leq)$ is a poset and $i <_P j$, then for all linear extensions w, we have $w^{-1}(i) < w^{-1}(j)$ (where we interpret w as a permutation).

TRUE FALSE

(r) If $(P, \leq)$ is a ranked poset and Q is a subposet of P, then Q is also a ranked poset.

TRUE FALSE

(s) $\displaystyle\sum_{p \in \mathrm{Dyck}(n)} q^{\mathrm{maj}(p)} = \sum_{w \in S_n(231)} q^{\mathrm{maj}(w)}$.

TRUE FALSE

(t) The polynomial $\begin{bmatrix} n \\ k \end{bmatrix}$ evaluated at $q = 1$ is equal to $\binom{n}{k}$.

TRUE FALSE

2. (8 points each) Complete **four** of the following.

(a) Show that every $w \in S_n(123)$ is the "interweaving" (i.e., shuffle) of two decreasing sequences $a_1, a_2, \ldots, a_k$ ($a_m > a_{m+1}$) and $b_1, b_2, \ldots, b_{n-k}$ ($b_m > b_{m+1}$) (where we allow one of the sequences to be empty).

(b) Consider a circle with $2n$ fixed points on the circle. Determine the number of ways of drawing n nonintersecting chords (where each point is connected to exactly one chord).

(c) Recall the definition of parking function given in Problem 3.9. A parking function $(a_1, a_2, \cdots, a_n)$ is called an *increasing* if $a_i \leq a_{i+1}$ for $1 \leq i \leq n-1$. Count the number of increasing parking functions of length n.

(d) Find an explicit bijection between triangulations of a convex $(n+2)$-gon and $PB(n)$.

(e) Recall the definition of integer partition given in Problem 3.4 in the textbook. Let p_n denote the number of integer partitions of n (with $p_0 = 1$). Define the *nugget* of an integer partition λ of n to be the largest $d \times d$ square that exists in the Young diagram for λ. Prove that

$$\sum_{n \geq 0} p_n z^n = \sum_{d \geq 0} \frac{z^{d^2}}{(1-z)^2(1-z^2)^2 \cdots (1-z^d)^2},$$

where $d \times d$ corresponds to the dimensions of the nugget for an integer partition.

(f) Let $u = 12 \cdots k$ and $v = (k+1)(k+2) \cdots n$ for $0 \leq k \leq n$ (those are permutations written in one-line notation). How many elements are in $u \sqcup\!\sqcup\, v$? You can either find a recurrence or a closed form for your enumeration.

(g) Prove that the number of outcomes for a race with n cyclists (with ties allowed) is

$$\sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}.$$

Fun Fact! The number of faces of the permutahedron (polytope we obtain from the weak order) has the same count.

(h) Recall that an *involution* in S_n is a permutation w with the property that $w^{-1} = w$. Let i_n denote the number of involutions in S_n and take $i_0 := 1$. It's clear that $i_1 = 1$ since the identity is an involution. Prove that for $n \geq 2$, we have $i_n = i_{n-1} + (n-1)i_{n-2}$.

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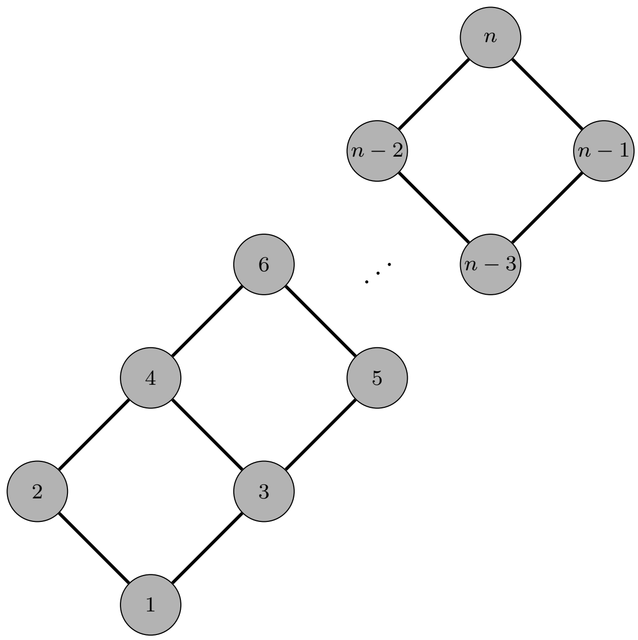
(i) Using the recursion in the previous problem, prove that the exponential generating function for i_n has the following closed form:

$$\sum_{n\geq 0} i_n \frac{z^n}{n!} = e^{z+z^2/2}.$$

You will likely need to solve a separable differential equation along the way.

(j) Find the number of linear extensions of the following poset on $[n]$ for n even.



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