

# The enumeration of fully commutative arbitrary permutations

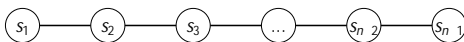
Christopher R. H. Hanusa  
Queens College, CUNY

**Joint work** with Brant C. Jones, James Madison University

# The affine permutations

## (Finite) $n$ -Permutations

$S_n$  has generators  $\{s_1, \dots, s_{n-1}\}$  and braid relations



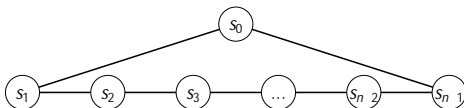
Write elements in **1-line notation** as a permutation of  $\{1, 2, \dots, n\}$ .

Generators transpose a pair of entries:  $s_i : (i \ i+1)$ .

*Example.*  $s_1 s_3 s_4$  is **2 1 4 3**

## Affine $n$ -Permutations

$S_n$  has generators  $\{s_0, s_1, \dots, s_{n-1}\}$  and braid relations



# The affine permutations

## Affine $n$ -Permutations

Write elements in **1-line notation**, as a **permutation of  $\mathbb{Z}$** .

Generators transpose **infinitely many** pairs of entries:

$s_i : (i) \quad (i+1) \dots (n+i) \quad (n+i+1) \dots (2n+i) \quad (2n+i+1) \dots$

In $S_4$ ,	$w(-4)$	$w(-3)$ $w(-2)$ $w(-1)$ $w(0)$	$w(1)$ $w(2)$ $w(3)$ $w(4)$	$w(5)$ $w(6)$ $w(7)$ $w(8)$	$w(9)$
id	4	-3 -2 -1 0	1 2 3 4	5 6 7 8	9
$s_1$	4	-2 -3 -1 0	2 1 3 4	6 5 7 8	10
$s_0$	3	-4 -2 -1 1	0 2 3 5	4 6 7 9	8
$s_1 s_0$	2	-4 -3 -1 2	0 1 3 6	4 5 7 10	8

Translational symmetry:  $w(i+n) = w(i) + n$ .

Therefore,  $w$  is defined by the **window**  $[w(1), w(2), \dots, w(n)]$ .

**Example.** In  $S_4$ ,  $s_1 s_0 = [0, 1, 3, 6]$

# Fully commutative elements

*Definition.* An element in a Coxeter group is **fully commutative** if it has only one reduced expression (up to commutation relations).

**NO BRAIDS ALLOWED!**

*Example.* In  $S_4$ ,  $s_1 s_2 s_3 s_1$  is **not fully commutative** because

$$s_1 s_2 s_3 s_1 \stackrel{\text{OK}}{=} s_1 s_2 s_1 s_3 \stackrel{\text{BAD}}{=} s_2 s_1 s_2 s_3$$

*Question:* **How many** fully commutative elements are there in  $S_n$ ?

*Answer:* Catalan many! (Billey, Jockusch, Stanley, 1993; Knuth, 1973)

$S_1$ : 1. id

$S_2$ : 2. id,  $s_1$

$S_3$ : 5. id,  $s_1$ ,  $s_2$ ,  $s_1 s_2$ ,  $s_2 s_1$

$S_4$ : 14. id,  $s_1$ ,  $s_2$ ,  $s_3$ ,  $s_1 s_2$ ,  $s_2 s_1$ ,  $s_2 s_3$ ,  $s_3 s_2$ ,  $s_1 s_3$ ,  
 $s_1 s_2 s_3$ ,  $s_1 s_3 s_2$ ,  $s_2 s_1 s_3$ ,  $s_3 s_2 s_1$ ,  $s_2 s_1 s_3 s_2$

# Enumerating fully commutative elements

*Question:* **How many** fully commutative elements are there in  $S_n$ ?

*Answer:* Infinitely many! (Even in  $S_3$ .)

$\text{id}, s_1, s_1 s_2, s_1 s_2 s_0, s_1 s_2 s_0 s_1, s_1 s_2 s_0 s_1 s_2, \dots$

Multiplying the generators cyclically does not introduce braids.

**This is not the right question.**

# Enumerating fully commutative elements

*Question:* **How many** fully commutative elements are there in  $S_n$ , with Coxeter length ?

In  $S_3$ :  $\text{id}$ ,  $s_1$ ,  $s_2$ ,  $s_1s_0$ ,  $s_2s_0$ ,  $s_0s_1$ ,  $s_0s_2$ ,  $s_1s_0s_2$ ,  $s_2s_0s_1$ ,  $s_1s_2s_0$ ,  $s_2s_1s_0$ ,  $\dots$

*Question:* Determine the coefficient of  $q$  in the generating function

$$f_n(q) = \sum_{w \in \mathfrak{S}_n^{FC}} q^{\ell(w)}.$$

$$f_3(q) = 1q^0 + 3q^1 + 6q^2 + 6q^3 + \dots$$

*Answer:* Consult your friendly computer algebra program.

# DdddaaaaAAAAaaaaTTaaaaAA

Brant calls up and says: Hey Chris, look at this data!

$$f_3(q) = 1 + 3q + 6q^2 + 6q^3 + 6q^4 + 6q^5 + \dots$$

$$f_4(q) = 1 + 4q + 10q^2 + 16q^3 + 18q^4 + 16q^5 + 18q^6 + \dots$$

$$f_5(q) = 1 + 5q + 15q^2 + 30q^3 + 45q^4 + 50q^5 + 50q^6 + 50q^7 + 50q^8 + \dots$$

$$f_6(q) = 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 + 150q^7 + 156q^8 + 152q^9 + 156q^{10} + 150q^{11} + 158q^{12} + 150q^{13} + 156q^{14} + 152q^{15} + 156q^{16} + 150q^{17} + 158q^{18} + \dots$$

$$f_7(q) = 1 + 7q + 28q^2 + 77q^3 + 161q^4 + 266q^5 + 364q^6 + 427q^7 + 462q^8 + 483q^9 + 490q^{10} + 490q^{11} + 490q^{12} + 490q^{13} + \dots$$

**Notice:**

The coefficients eventually repeat.

**Goals:** Find a formula for the generating function  $f_n(q)$ .  
Understand this periodicity.

# Pattern Avoidance Characterization

*Key idea:* (Green, 2002)

$w$  is fully commutative

$w$  is 321-avoiding.

*Example.*  $[4, 1, 1, 14]$  is **NOT** fully commutative because:

	$w(-4)$	$w(-3)$	$w(-2)$	$w(-1)$	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$w(4)$	$w(5)$	$w(6)$	$w(7)$	$w(8)$	$w(9)$
$w$	6	-8	-5	-3	10	-4	-1	1	14	0	3	5	18	4



# Game plan

**Goal:** Enumerate 321-avoiding affine permutations  $w$ .

Write  $w = w^0 w$ , where  $w^0 \in S_n/S_n$  and  $w \in S_n$ .

$w^0$  determines the entries;  $w$  determines their order.

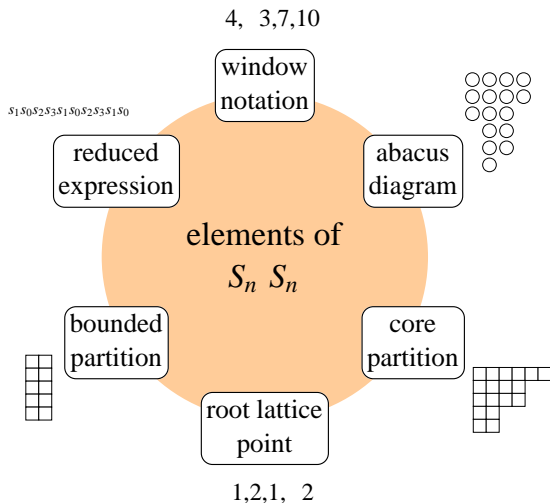
*Example.* For  $w = [\cancel{1}, 11, 20, \cancel{2}, 3, 4, 11, 0] \in S_6$ ,

$w^0 = [\cancel{1}, 11, \cancel{2}, 0, 4, 11, 20]$  and  $w = [1, 3, 6, 4, 5, 2]$ .

Determine which  $w^0$  are 321-avoiding.

Determine the ~~note~~  $w$  such that  $w^0 w$  is still 321-avoiding

# Combinatorial interpretations of $S_n/S_n$



# Combinatorial interpretations of $S_n/S_n$

(James and Kerber, 1981)

Given  $w^0 = [w_1, \dots, w_n] \in S_n/S_n$ , we can interpret  $w^0$  as:

## Abacus diagram

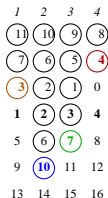
Place integers in  $n$  runners.

Circled: *beads*. Empty: *gaps*

Bijection: Given  $w^0$ , create an abacus where each runner has a lowest bead at  $w_i$

*Example:*

$[4, 3, 7, 10]$

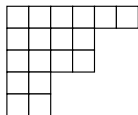


## Core partition

An *n-core* is an integer partition with no  $n$ -ribbons.

Bijection: Read the boundary steps from the abacus:

Bead = vertical; Gap = horiz.



# Normalized abacus and 321-avoiding criterion for $S_n/S_n$

We use a *normalized* abacus diagram;  
shifts all beads so that the ~~last~~ gap is  
in position  $n + 1$ ; this map is invertible.



**Theorem.** (Haglund) Given a normalized abacus for  $w^0 \in S_n/S_n$ ,  
where the last bead occurs in position  $i$ ,

$w^0$  is fully commutative if and only if the lowest beads in runners only occur in  
 $\{1, \dots, n\}$  and  $\{i, i+1, \dots, n\}$ .

**Idea:** Lowest beads in runners entries in base window.

$w(-n+1)$	$w(-n+2)$	...	$w(-1)$	$w(0)$	$w(1)$	$w(2)$	...	$w(n-1)$	$w(n)$	$w(n+1)$	$w(n+2)$	...	$w(2n-1)$	$w(2n)$
lo	lo	...	hi	hi	lo	lo	...	hi	hi	lo	lo	...	hi	hi
lo	lo	med	hi	hi	lo	lo	med	hi	hi	lo	lo	med	hi	hi

# Long versus short elements

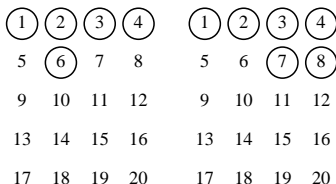
Partition  $S_n$  into long and short elements:

## Short elements

Lowest bead in position  $i \leq 2n$

Finitely many

**Hard to count**



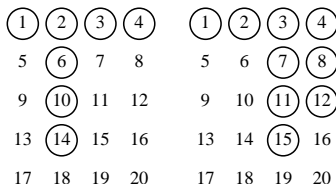
## Long elements

Lowest bead in position  $i > 2n$

Come in infinite families

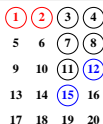
**Easy to count**

*Explain the periodicity*



# Enumerating long elements

For long elements  $w \in S_n$ , the base window for  $w^0$  is  $[a, a, \dots, a, b, b, \dots, b]$  where  $1 \leq a \leq n$ , and  $n+2 \leq b \leq n+1$ .



**Question:** Which permutations  $w \in S_n$  can be multiplied into a  $w^0$ ?

We can not invert any pairs of  $a$ 's nor any pairs of  $b$ 's.  
(Would create a 321-pattern with an adjacent window)

Only possible to *intersperse* the  $a$ 's and the  $b$ 's.

How many ways to intersperse  $(k)$   $a$ 's and  $(n-k)$   $b$ 's?  $\binom{n}{k}$

**BUT:** We must also keep track of the *length* of these permutations.

This is counted by the  $q$ -binomial coefficient:  $\binom{n}{k}_q$

$$\binom{n}{k}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}, \text{ where } (q)_n = (1-q)(1-q^2)\cdots(1-q^n)$$

# Enumerating long elements

After we:

Enumerate by length all possible  $w^0$  with  $(k) \leq n$  and  $(n-k) \leq n$

Combine the Coxeter lengths by  $(w) = (w^0) + (w)$ .

Then we get:

**Theorem.** (Hultman) For a fixed  $n \geq 0$ , the generating function by length for **long** fully commutative elements  $w \in S_n^{FC}$  is

$$q^{(w)} = \frac{q^n}{1 - q^n} \sum_{k=1}^{n-1} \binom{n-1}{k} q^k.$$

## Periodicity of fully commutative elements in $S_n$

*Corollary.* (Hsiao) The coefficients of  $f_n(q)$  are eventually periodic with period dividing  $n$ .

**When  $n$  is prime**, the period is 1:  $a_j = \frac{1}{n} \quad \frac{2n}{n} \not\equiv 2$ .

*Proof.* For  $i$  sufficiently large, all elements of length  $i$  are long.  
Our generating function is simply some polynomial over  $(1 - q^n)$ :

$$\frac{q^n}{1 - q^n} = \sum_{k=1}^{\infty} q^{kn} = \frac{P(q)}{1 - q^n} = P(q)(1 + q^n + q^{2n} + \dots)$$

When  $n$  is prime, an extra factor of  $(1 + q + q^2 + \dots + q^{n-1})$  cancels;

$$\frac{1}{1 - q} = \sum_{k=0}^{\infty} q^k$$

As suggested by a referee, we know that  $a_i = P(1) = \frac{1}{n} \sum_{k=1}^n \frac{n!}{k} = \frac{n-1}{2}$ .



# Short elements are hard

For short elements  $w \in S_n$ , the base window for  $w^0$  is  $[a, \dots, a, b, \dots, b, c, \dots, c]$ , and there is more interaction:



No  $a$  can invert with an  $a$  or  $b$ . No  $c$  can invert with a  $b$  or  $c$ .

Count  $w$  where some  $a$  intertwines with some  $c$ .

Count  $w$  w/o intertwining and 0 descents in the  $b$ 's.

Count  $w$  w/o intertwining and 1 descent in the  $b$ 's.

Not so hard to determine the acceptable  $w$  permutations.

Such as  $M \circ X^{L+M+R} \prod_{i=1}^M \frac{1}{i} \prod_{j=1}^L \frac{1}{j} \prod_{k=1}^R \frac{1}{k}$

Count  $w$  w/o intertwining and 2 descents in the  $b$ 's.

Count  $w$  which are  $w$  permutations. (Barcucci et al.)

Solve functional recurrences (Bousquet-Mélou)

Such as  $D(x, q, z, s) =$

$$N(x, q, z, s) + \frac{xqs}{1-qqs} D(x, q, z, 1) + D(x, q, z, qs) + xsD(x, q, z, s)$$

# Future Work

Extend to  $B_n$ ,  $C_n$ , and  $D_n$

Develop combinatorial interpretations (Wait 10 minutes . . .)  
321-avoiding characterization?

**Heap** interpretation of fully commutative elements

Can use Viennot's heaps of pieces theory  
Better bound on periodicity

More combinatorial interpretations for  $W/W$

What do you know?

# Thank you!

Slides available: [people.qc.cuny.edu/chanusa](http://people.qc.cuny.edu/chanusa) > Talks



Christopher R. H. Hanusa and Brant C. Jones.

The enumeration of fully commutative affine permutations

*European Journal of Combinatorics*. Vol 31, 1342–1359. (2010)



Christopher R. H. Hanusa and Brant C. Jones.

Abacus models for parabolic quotients of affine Coxeter groups



Anders Björner and Francesco Brenti.

Combinatorics of Coxeter Groups, Springer, 2005.