

# Chapter 2

## Permutations and Combinations

A  **$k$ -permutation** of a set  $A$  is an injective function  $w : [k] \rightarrow A$ . The set of all  $k$ -permutations of  $A$  is denoted by  $S_{A,k}$ . If  $A$  happens to be the set  $[n]$ , we use the notation  $S_{n,k}$ . And if  $n = k$ , we write  $S_n := S_{n,n}$  and refer to each  $n$ -permutation in  $S_n$  as a **permutation**. Let  $P(n, k) := |S_{n,k}|$ . By convention,  $P(n, 0) = 1$ .

We can denote a  $k$ -permutation as string  $w = w(1)w(2) \cdots w(k)$ , where each entry  $w(i)$  that appears in the string is unique (since  $w$  is an injection). In other words, we can think of a  $k$ -permutation as a linear ordered arrangement of  $k$  of  $n$  objects.

**Problem 2.1.** Complete the following.

- (a) Write down all of the elements in  $S_3$ . What is  $P(3, 3)$ ?
- (b) Write down all of the elements in  $S_{4,3}$ . What is  $P(4, 3)$ ?

Recall that for  $n \in \mathbb{N}$ , the **factorial** of  $n$  is defined  $n! := n \cdot (n-1) \cdots 2 \cdot 1$ , and we define  $0! := 1$  for convenience.

**Problem 2.2.** Consider the collection of  $k$ -permutations in  $S_{n,k}$  with  $1 \leq k \leq n$ . Explain why  $P(n, k)$  is equal to the number of nonattacking rook arrangements on an  $n \times k$  chess board. *Hint:* Establish a bijection between the collection of nonattacking rook arrangements on an  $n \times k$  chess board and the collection of  $k$ -permutations.

**Theorem 2.3.** For  $1 \leq k \leq n$ , we have

$$P(n, k) = n \cdot (n-1) \cdots (n+1-k) = \frac{n!}{(n-k)!}.$$

Note that as a special case of the formula above, we have  $|S_n| = P(n, n) = n!$ . For convenience, we can extend the formula above to obtain

$$P(0, 0) = \frac{0!}{(0-0)!} = 1 \quad \text{and} \quad P(n, 0) = \frac{n!}{(n-0)!} = 1.$$

**Problem 2.4.** How many strings of length three are there using letters from  $\{a, b, c, d, e, f, g\}$  if the letters in the string are not repeated?

**Problem 2.5.** There are 8 finalists at the Olympic Games 100 meters sprint. Assume there are no ties.

- (a) How many ways are there for the runners to finish?
- (b) How many ways are there for the runners to get gold, silver, bronze?
- (c) How many ways are there for the runners to get gold, silver, bronze given that Usain Bolt is sure to get the gold medal?

**Problem 2.6.** If  $1 \leq k \leq n$ , prove that  $P(n, n) = P(n, k)P(n - k, n - k)$ , both using the formula in Theorem 2.3, and separately using a combinatorial argument or by using the definition of  $k$ -permutations together with the bijection principle.

**Problem 2.7.** If  $1 \leq k \leq n$ , prove that  $P(n, k) = P(n - 1, k) + kP(n - 1, k - 1)$ , both using the formula in Theorem 2.3, and separately using a combinatorial argument or by using the definition of  $k$ -permutations together with the bijection principle.

**Problem 2.8.** How many ways can the letters of the word PRESCOTT be arranged?

**Problem 2.9.** How many ways can the letters of the word POPPY be arranged? Try to solve this problem in two different ways.

Consider a set of  $n$  objects that are not necessarily distinct, with  $p$  different objects and  $n_i$  objects of type  $i$  (for  $i = 1, 2, \dots, p$ ), so that  $n = n_1 + \dots + n_p$ . An ordered arrangement of these  $n$  objects is called a **generalized permutation** and the number of such arrangements is denoted by  $P(n; n_1, \dots, n_p)$ . For example, the number of words we can make out of the letters of POPPY is  $P(5; 3, 1, 1)$ .

**Theorem 2.10.** For  $n, n_1, \dots, n_p \in \mathbb{N}$  such that  $n = n_1 + \dots + n_p$ , we have

$$P(n; n_1, \dots, n_p) = \frac{n!}{n_1! \dots n_p!}.$$

**Problem 2.11.** How many ways can the letters of the word MISSISSIPPI be arranged?

**Problem 2.12.** In Professor X's class of 9 graduate students she will give two A's, one B, and six C's. How many possible ways are there to do this?

**Problem 2.13.** Let's revisit Problem 1.14, which involved my walk to get coffee. When we attacked that problem, we did a lot of brute force. Do we now have an easier method?

**Problem 2.14.** Six friends sit around a circle to play a game. Rotations of the group do not constitute different seating orders.

- (a) How many circular seating arrangements are there?

- (b) How many circular seating arrangements are there if Sally and Maria always sit next to each other?

The above problem involves what are sometimes called **circular permutations**.

**Problem 2.15.** How many circular permutations are there involving  $n$  objects?

The notion of  $k$ -permutations captures arrangements of distinct object where order matters. But what should we do if we want to capture a situation where the order of the objects does not matter? Since the order of the objects in a set does not matter, this is the model we should use.

If  $A$  is a set and  $B \subseteq A$  with  $|B| = k$ , we refer to  $B$  as a  **$k$ -subset** of  $A$ . The collection of all  $k$ -subsets of  $A$  is defined via

$$\binom{A}{k} := \{B \subseteq A \mid |B| = k\}.$$

The **binomial coefficient** is defined via

$$\binom{n}{k} := \text{number of } k\text{-subsets of an } n\text{-element set.}$$

We read  $\binom{n}{k}$  as “ $n$  choose  $k$ ”. In particular, if  $|A| = n$ , then  $|\binom{A}{k}| = \binom{n}{k}$ . Alternate notations for binomial coefficients include  $C(n, k)$  and  ${}_nC_k$ . We will see later why  $\binom{n}{k}$  is referred to as a binomial coefficient.

**Example 2.16.** If  $A = \{a, b, c, d\}$ , then

$$\binom{A}{2} = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\},$$

which implies that  $\binom{4}{2} = 6$ .

**Problem 2.17.** For any  $A$ , including the empty set, what is  $\binom{A}{0}$ ? For  $n \geq 0$ , what is  $\binom{n}{0}$  equal to?

**Problem 2.18.** For  $n \geq 0$ , what is  $\binom{n}{n}$  equal to?

If we let  $n$  and  $k$  vary, we can organize the binomial coefficients in a triangular array, often referred to as **Pascal’s Triangle**. See Table 2.1.

**Problem 2.19.** Suppose you have a pool of 6 applicants for a job opening. Let’s assume you believe the values in Table 2.1.

- (a) How many ways can you choose 3 of the 6 applicants to interview?
- (b) How many ways can you hire 3 of the 6 applicants for 3 distinct jobs?

$n \setminus k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Table 2.1: Pascal's Triangle of binomial coefficients.

**Problem 2.20.** What are the row sums in Pascal's Triangle? That is, what is the following sum equal to for any  $n \geq 0$ ?

$$\sum_{k=0}^n \binom{n}{k} := \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

**Problem 2.21.** Using the meaning of  $k$ -subset and  $k$ -permutation, explain why

$$P(n, k) = \binom{n}{k} \cdot k!.$$

Using the previous problem, we immediately get the following handy formula for computing binomial coefficients.

**Theorem 2.22.** For  $0 \leq k \leq n$ , we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Notice that the previous formula is equal to  $\frac{P(n,k)}{k!}$ . The numerator is counting how many distinct arrangements (order matters) there are of  $k$  objects taken from  $n$  objects and the denominator is essentially unordering arrangements that consist of the same objects.

**Problem 2.23.** A state senate consists of 19 Republicans and 14 Democrats. In how many ways can a committee be chosen if:

- (a) The committee contains 6 senators without regard to party?
- (b) The committee contains 3 Republicans and 3 Democrats?

**Problem 2.24.** How many bit strings of length 10 have exactly three 1's?

**Problem 2.25.** How many bit strings of length 6 have an odd number of 0's?

**Problem 2.26.** As we noted earlier, we did quite a bit of brute force to determine how many paths I could take to get coffee in Problem 1.14. Find a solution that utilizes binomial coefficients.

**Problem 2.27.** How many strings of 10 lower-case English letters have exactly two  $g$ 's and exactly three  $v$ 's?

**Problem 2.28.** Assume  $1 \leq k \leq n$ .

- (a) Using the definition of  $\binom{n}{k}$  in terms of  $k$ -subsets (as opposed to the formula in Theorem 2.22), explain why

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

This identity is often called **Pascal's Identity** (or **Pascal's Recurrence**).

- (b) Connect the formula above with Problem 1.14 involving my walk to get coffee.

**Problem 2.29.** Assume  $1 \leq k \leq n$ . It turns out that

$$\binom{n}{k} = \binom{n}{n-k}.$$

- (a) Prove the identity above using the formula for  $\binom{n}{k}$  given in Theorem 2.22.  
 (b) Explain why the identity is true by using the definition of  $\binom{n}{k}$  in terms of  $k$ -subsets.

The upshot is that each row of Pascal's Triangle is a palindrome.

**Problem 2.30.** Explain why  $1 + 2 + \cdots + n = \binom{n+1}{2}$ .

By the way, the number defined by  $T_n := 1 + 2 + \cdots + n$  is called the  $n$ th **Triangular number** (due to the shape we get by representing each number in the sum by a stack of balls).

**Problem 2.31.** Consider the linear equation  $x_1 + x_2 + x_3 = 11$ . How many *integer* solutions are there if:

- (a)  $x_1, x_2, x_3 \geq 0$ ?  
 (b)  $x_1, x_2, x_3 > 0$ ? *Note:* ChatGPT did not get correct answer for me!  
 (c)  $x_1 \geq 1, x_2 \geq 0, x_3 \geq 2$ ?

**Problem 2.32.** How many ways can you distribute 5 identical lollipops to 6 kids?

These last two problems illustrate a technique known as **stars and bars**. In general,  $n$  stars tally the number of objects and  $r - 1$  bars separate them into  $r$  distinct categories.

**Theorem 2.33.** The number of possible collections of  $n$  objects of  $r$  different types is

$$\binom{n+r-1}{r-1} = \binom{n+r-1}{n}.$$

**Problem 2.34.** Zittles come in 5 colors: green, yellow, red, orange, and purple. How many different collections of 32 Zittles are possible?