A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

G.H. Hardy, mathematician

Chapter 4

Standard Topology of the Real Line

In this chapter, we will introduce the notions of open, closed, compact, and connected as they pertain to subsets of the real numbers. These properties form the underpinnings of a branch of mathematics called **topology** (derived from the Greek words *tópos*, meaning 'place, location', and *ology*, meaning 'study of'). Topology, sometimes called "rubber sheet geometry," is concerned with properties of spaces that are invariant under any continuous deformation (e.g., bending, twisting, and stretching like rubber while not allowing tearing apart or gluing together). The fundamental concepts in topology are continuity, compactness, and connectedness, which rely on ideas such as "arbitrary close" and "far apart". These ideas can be made precise using open sets.

Once considered an abstract branch of pure mathematics, topology now has applications in biology, computer science, physics, and robotics. The goal of this chapter is to introduce you to the basics of the set-theoretic definitions used in topology and to provide you with an opportunity to tinker with open and closed subsets of the real numbers. In Chapter 6, we will revisit these concepts when we explore continuous functions.

4.1 Open Sets

Definition 4.1. A set U is called an **open set** if for every $x \in U$, there exists a bounded open interval (a,b) containing x such that $(a,b) \subseteq U$.

It follows immediately from the definition that every open set is a union of bounded open intervals.

Problem 4.2. Determine whether each of the following sets is open. Justify your assertions.

(a) $(1,2)$

(b) $(1,\infty)$

(c) $(1,2) \cup (\pi,5)$

(d) $[1,2]$

(e) $(-\infty,\sqrt{2}]$

(f) $\{4,17,42\}$

(g) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(h) $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

(i) $\mathbb{R}$

(j) $\mathbb{Q}$

(k) $\mathbb{Z}$

(l) $\emptyset$

As expected, every open interval (i.e., intervals of the form (a,b), $(-\infty,b)$, (a,∞), or $(-\infty,\infty)$) is an open set.

Problem 4.3. Prove that every open interval is an open set.

However, it is important to point out that open sets can be more complicated than a single open interval.

Problem 4.4. Provide an example of an open set that is not a single open interval.

Problem 4.5. Prove that if U and V are open sets, then

(a) $U \cup V$ is an open set, and

(b) $U \cap V$ is an open set.

According to the next two problems, the union of arbitrarily many open sets is open while the intersection of a finite number of open sets is open.

Problem 4.6. Prove that if $\{U_\alpha\}_{\alpha\in\Delta}$ is a collection of open sets, then $\bigcup_{\alpha\in\Delta} U_\alpha$ is an open set.

Consider using induction on the next problem.

Problem 4.7. Prove that if $\{U_i\}_{i=1}^n$ is a finite collection of open sets for $n \in \mathbb{N}$, then $\bigcap_{i=1}^n U_i$ is an open set.

Problem 4.8. Explain why we cannot utilize induction to prove that the intersection of infinitely many open sets indexed by the natural numbers is open.

Problem 4.9. Give an example of each of the following.

(a) A collection of open sets $\{U_\alpha\}_{\alpha\in\Delta}$ such that $\bigcap_{\alpha\in\Delta} U_\alpha$ is an open set.

(b) A collection of open sets $\{U_\alpha\}_{\alpha\in\Delta}$ such that $\bigcap_{\alpha\in\Delta} U_\alpha$ is not an open set.

According to the previous problem, the intersection of infinitely many open sets may or may not be open. So, we know that there is no theorem that states that the intersection of arbitrarily many open sets is open. We only know for certain that the intersection of finitely many open sets is open by Problem 4.7.

> Any creative endeavor is built on the ash heap of failure.
>
> Michael Starbird, mathematician

4.2 Accumulation Points and Closed Sets

Definition 4.10. Suppose $A \subseteq \mathbb{R}$. A point $p \in \mathbb{R}$ is an **accumulation point** of A if for every bounded open interval (a,b) containing p, there exists a point $q \in (a,b) \cap A$ such that $q \neq p$.

Notice that if p is an accumulation point of A, then p may or may not be in A. Loosely speaking, p is an accumulation point of a set A if there are points in A arbitrarily close to p. That is, if we zoom in on p, we should always see points in A nearby. If A is a set, the set of accumulation points of A is sometimes denoted by $\boxed{A'}$.

Problem 4.11. Consider the open interval $I = (1,2)$. Prove each of the following.

(a) The points 1 and 2 are accumulation points of I.

(b) If $p \in I$, then p is an accumulation point of I.

(c) If $p < 1$ or $p > 2$, then p is not an accumulation point of I.

Problem 4.12. A point p is an accumulation point of the intervals (a,b), $(a,b]$, $[a,b)$, and $[a,b]$ if and only if $p \in [a,b]$.

Problem 4.13. Prove that the point $p = 0$ is an accumulation point of $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Are there any other accumulation points of A?

Problem 4.14. For each set A, find the set of accumulation points A'. In each case, sketch a proof to justify your assertion.

(a) $A = \{-17, 1, \pi, 42\}$

(b) $A = [1,3) \cup (3,5)$

(c) $A = [1,3) \cup \{17\}$

(d) $A = \mathbb{R}$

(e) $A = \mathbb{Q}$

(f) $A = \mathbb{Z}$

(g) $M = \displaystyle\bigcup_{n=1}^{\infty}(-1 - \frac{1}{n}, 1 + \frac{1}{n})$

(h) $M = \displaystyle\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n})$

Problem 4.15. Prove that if A is finite, then A has no accumulation points. That is, if A is finite, then $A' = \emptyset$.

Problem 4.16. State the contrapositive of the previous problem.

The upshot of the previous problem is that any set with an accumulation point must be infinite.

Problem 4.17. Determine whether the converse of Problem 4.16 is true or false. Justify your assertion.

Problem 4.18. Provide an example of a set A with exactly two accumulation points.

Problem 4.19. Given sets A and B, determine whether each of the following is true or false. If the statement is true, prove it. Otherwise, provide a counterexample.

(a) If p is an accumulation point of $A \cap B$, then p is an accumulation point of both A and B.

(b) If p is an accumulation point of $A \cup B$, then p is an accumulation point of A or p is an accumulation point of B.

Definition 4.20. A set $A \subseteq \mathbb{R}$ is called **closed** if A contains all of its accumulation points.

That is, a set A is closed if and only if $A' \subseteq A$. Note that if a set A has no accumulation points, then it is vacuously closed.

Problem 4.21. Determine whether each of the sets in Problem 4.2 is closed. Justify your assertions.

The upshot of Parts (i) and (l) of Problems 4.2 and 4.21 is that $\mathbb{R}$ and $\emptyset$ are both open and closed. It turns out that these are the only two subsets of the real numbers with this property. One issue with the terminology that could potentially create confusion is that the open interval $(-\infty, \infty)$ (i.e., the real numbers $\mathbb{R}$) is both an open and a closed set.

Problem 4.22. Provide an example of each of the following. You do not need to prove that your answers are correct.

(a) A set that is open but not closed.

(b) A set that is closed but not open.

(c) A set that neither open nor closed.

Another potentially annoying feature of the terminology illustrated by Problem 4.22 is that if a set is not open, it may or may not be closed. Similarly, if a set is not closed, it may or may not be open. That is, open and closed are not opposites of each other.

The next result justifies referring to $[a,b]$ as a closed interval.

Problem 4.23. Prove that every interval of the form $[a,b]$, $(-\infty, b]$, $[a, \infty)$, or $(-\infty, \infty)$ is a closed set.

Problem 4.24. Prove that if A is finite, then A is a closed set.

Problem 4.25. Prove that $\mathbb{Z}$ is a closed set.

Despite the fact that open and closed are not opposites of each other, there is a nice relationship between open and closed sets in terms of complements.

Problem 4.26. Let $U \subseteq \mathbb{R}$. Prove that U is open if and only if U^C is closed.

Problem 4.27. Prove that if A and B are closed sets, then

(a) $A \cup B$ is a closed set, and

(b) $A \cap B$ is a closed set.

The next two problems are analogous to Problems 4.6 and 4.7.

Problem 4.28. Prove that if $\{A_\alpha\}_{\alpha\in\Delta}$ is a collection of closed sets, then $\bigcap_{\alpha\in\Delta} A_\alpha$ is a closed set.

Problem 4.29. Prove that if $\{A_i\}_{i=1}^n$ is a finite collection of closed sets for $n \in \mathbb{N}$, then $\bigcup_{i=1}^n U_i$ is a closed set.

Problem 4.30. Provide an example of a collection of closed sets $\{A_\alpha\}_{\alpha\in\Delta}$ such that $\bigcup_{\alpha\in\Delta} A_\alpha$ is not a closed set.

Problem 4.31. Determine whether each of the following sets is open, closed, both, or neither.

(a) $V = \displaystyle\bigcup_{n=2}^{\infty}\left(n - \frac{1}{2}, n\right)$

(b) $W = \displaystyle\bigcap_{n=2}^{\infty}\left(n - \frac{1}{2}, n\right)$

(c) $X = \displaystyle\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$

(d) $Y = \displaystyle\bigcap_{n=1}^{\infty}(-n, n)$

(e) $Z = (0,1) \cap \mathbb{Q}$

Problem 4.32. Prove or provide a counterexample: Every non-closed set has at least one accumulation point.

> Getting better is not pretty. To get good we have to be down to struggle, seek out challenges, make some mistakes, to train ugly.
>
> Trevor Ragan, thelearnerlab.com

4.3 Compact and Connected Sets

We now introduce three special classes of subsets of $\mathbb{R}$: compact, connected, and disconnected.

Definition 4.33. A set $K \subseteq \mathbb{R}$ is called **compact** if K is both closed and bounded.

It is important to point out that there is a more general definition of compact in an arbitrary topological space. However, using our notions of open and closed, it is a theorem that a subset of the real line is compact if and only if it is closed and bounded.

Problem 4.34. Determine whether each of the following sets is compact. Briefly justify your assertions.

(a) $[0,1) \cup [2,3]$

(b) $[0,1) \cup (1,2]$

(c) $[0,1) \cup [1,2]$

(d) $\mathbb{R}$

(e) $\mathbb{Q}$

(f) $\mathbb{R} \setminus \mathbb{Q}$

(g) $\mathbb{Z}$

(h) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(i) $[0,1] \cup \{1 + \frac{1}{n} \mid n \in \mathbb{N}\}$

(j) $\{17, 42\}$

(k) $\{17\}$

(l) $\emptyset$

Problem 4.35. Is every finite set compact? Justify your assertion.

The next problem says that every nonempty compact set contains its greatest lower bound and its least upper bound. That is, every nonempty compact set attains a minimum and a maximum value.

Problem 4.36. Prove that if K is a nonempty compact subset of $\mathbb{R}$, then $\sup(K), \inf(K) \in K$.

Definition 4.37. Let A be a subset of real numbers and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Delta}$ be a collection of open subsets of $\mathbb{R}$. Then $\mathcal{U}$ is an **open cover** of A if $A \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. An **open subcover** $\mathcal{V}$ of an open cover $\mathcal{U}$ of A is a subcollection of $\mathcal{U}$ whose elements form an open cover of A. If $\mathcal{V}$ is an open subcover of $\mathcal{U}$ consisting of a finite number of open sets, then we say that $\mathcal{V}$ is a **finite open subcover**.

For example, the collections $\mathcal{U}_1 = \{(-n, n) \mid n \in \mathbb{N}\}$ and $\mathcal{U}_2 = \{(n, n+2) \mid n \in \mathbb{Z}\}$ are each open subcovers of the real numbers. The collection $\mathcal{V}_1 = \{(-n, n) \mid n \in \mathbb{N} \text{ and } n \geq 17\}$ is an example of an open subcover of $\mathcal{U}_1$. Does $\mathcal{U}_1$ have an example of a finite open subcover? Does $\mathcal{U}_2$ have an example of a proper open subcover (i.e., not the entire collection)?

Problem 4.38. Let $A = [0,1]$, $\mathcal{U}_1 = \{(-\frac{n}{2}, n2)\}$, and $\mathcal{U}_1 = \{(-\frac{1}{n}, 1 + \frac{1}{n})\}$. It turns out that both $\mathcal{U}_1$ and $\mathcal{U}_2$ are open covers of A.

(a) If possible, find a finite open subcover of $\mathcal{U}_1$. If this is not possible, explain why.

(b) If possible, find a finite open subcover of $\mathcal{U}_2$. If this is not possible, explain why.

Problem 4.39. Let $A = (0,1)$, $\mathcal{U}_1 = \{(-\frac{n}{2}, n2)\}$, and $\mathcal{U}_1 = \{(\frac{1}{n}, 1)\}$. It turns out that both $\mathcal{U}_1$ and $\mathcal{U}_2$ are open covers of A.

(a) If possible, find a finite open subcover of $\mathcal{U}_1$. If this is not possible, explain why.

(b) If possible, find a finite open subcover of $\mathcal{U}_2$. If this is not possible, explain why.

Problem 4.40. Prove that a set A is compact if and only if every open cover $\mathcal{U}$ of A has a finite open subcover.

The characterization of compact in given in the previous problem is actually the standard definition of compact used in topology. If we had used this as our definition, we could have proved that a set is compact if and only if it is closed and bounded. This theorem is known as the **Heine–Borel Theorem**. Instead we took this as our definition to simplify things slightly.

Definition 4.41. A set $A \subseteq \mathbb{R}$ is **disconnected** if there exists two disjoint open sets U_1 and U_2 such that $A \cap U_1$ and $A \cap U_2$ are nonempty but $A \subseteq U_1 \cup U_2$ (equivalently, $A = (A \cap U_1) \cup (A \cap U_2)$). If a set is not disconnected, then we say that it is **connected**.

In other words, a set is disconnected if it can be partitioned into two nonempty subsets such that each subset does not contain points of the other and does not contain any accumulation points of the other. Showing that a set is disconnected is generally easier than showing a set is connected. To prove that a set is disconnected, you simply need to exhibit two open sets with the necessary properties. However, to prove that a set is connected, you need to prove that no such pair of open sets exists.

Problem 4.42. Determine whether each of the sets in Problem 4.34 is is connected or disconnected. Briefly justify your assertions.

Problem 4.43. Prove that if $a \in \mathbb{R}$, then $\{a\}$ is connected.

The next proof is harder than you might expect. Consider a proof by contradiction and try to make use of the Completeness Axiom.

Problem 4.44. Prove that every closed interval $[a, b]$ is connected.

It turns out that every connected set in $\mathbb{R}$ is either a singleton or an interval. We have not officially proved this claim, but we do have the tools to do so. Feel free to try your hand at proving this fact.

> If you learn how to learn, it's the ultimate meta skill and I believe you can learn how to be healthy, you can learn how to be fit, you can learn how to be happy, you can learn how to have good relationships, you can learn how to be successful. These are all things that can be learned. So if you can learn that is a trump card, it's an ace, it's a joker, it's a wild card. You can trade it for any other skill.
>
> Naval Ravikant, entrepreneur & investor