Homework 4

Discrete Mathematics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition.You are highly encouraged to seek out assistance by asking questions in our Discord server. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers. In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. King Arthur and Guinevere are hosting the knights Lancelot, Gawain, Galahad, Percival, Tristan, and Iseult to dinner. The eight of them sit at the Round Table, which consists of eight seats. Rotations of the group do not constitute different seating orders. **No detailed justification required**.

   (a) How many ways can the eight of them be seated around the Round Table?

   (b) How many ways can the eight of them be seated around the Round Table if Arthur never sits next to Guinevere?

2. A history professor has 8 marker pens in her bag, and one day puts them along the bottom of the white board in line. How many ways can this be done if 5 are black and there are 1 each of green, orange, and purple? **No detailed justification required**.

3. A player's tray in Scrabble© spells the word PIKACHU. The player cannot play and decides to discard 3 tiles and pick again. In how many ways can three letters be discarded? *Hint:* We are trying to count the number of subsets of size 3 taken from a larger set of size 7. We will learn a slick way to count this later. In the meantime, here is a method of attack. Recall that the order of the elements in a set does not matter. Temporarily suppose the order did matter. That is, count the number of ways we could choose 3 letters from 7 where the order in which we choose the letters matters. Now, this will be an overcount, so we need to divide by the appropriate number to count the (unordered) subsets. **No detailed justification required**.

4. Recall from Homework 1 that a **set partition** of $[n]$ is a collection of nonempty disjoint subsets (i.e., no overlap amongst the subsets) of $[n]$ whose union is $[n]$. Each subset in the set partition is called a **block**. We define the **Stirling numbers** (of the second kind) via

$$\left\{ {n \atop k} \right\} := \text{number of set partitions of } [n] \text{ with } k \text{ blocks},$$

which we pronounce as "n Stirling k". **Explain** why for each $n \in \mathbb{N}$, we have

$$\left\{ {n \atop 2} \right\} = 2^{n-1} - 1.$$

Hint: Every partition of $[n]$ with two blocks is determined by choosing a subset for one block since the the second block is necessarily the complement. How many subsets of $[n]$ are there? Are any of these bad choices for one of the blocks? Temporarily pretend the order of the two blocks matters and then use the Division Principle to unorder the bocks.

5. Imagine we have $n \geq 1$ distinct balls labeled $1,2,\ldots,n$ and $k \geq 1$ distinct buckets labeled $1,2,\ldots,k$. How many ways can we distribute the n distinct balls into the k distinct buckets? **No detailed justification required**.

6. For $n \in \mathbb{N}$, we define $[n] := \{1,2,\ldots,n\}$. That is, $[n]$ is just clever shorthand for the set containing 1 through n. This notation is meant to resemble interval notation. Now, let n and k be natural numbers. How many functions are there from $[n]$ to $[k]$? That is, count the number of functions of the form $f : [n] \to [k]$. **Briefly explain your answer**.

7. Let's explore a variation of Problem 1.41 from the textbook. It's Halloween and five students arrive at my office begging for candy. I happen to have five pieces of candy. Depending on my mood, I may give away none of the candy, all of the candy, or any amount in between. In class, we assumed that I wouldn't give a student more than one piece of candy. For this problem, let's remove that restriction, so that **a student can receive any amount of candy** (as little as 0 and at most 5). Let's also assume that each piece of candy is distinct. How many different ways can I distribute the candy in this situation? **Briefly explain your answer**.