Chapter 3

Sequences and Completeness

Throughout this chapter, our universe of discourse will be the real numbers. Any time we refer to a generic set, we mean a subset of real numbers. We will often refer to an element in a subset of real numbers as a **point**. We begin with a definition.

Definition 3.1. If M is a set, we say that p is an **accumulation point of** M if *every* open interval containing p also contains a point of M different from p.

That is, p is an accumulation point of M if and only if for each open interval O containing p, $(O \setminus \{p\}) \cap M \neq \emptyset$. Notice that if p is an accumulation point of M, then p may or may not be in M.

Problem 3.2. Show that if M is an open interval and $p \in M$, then p is an accumulation point of M.

Problem 3.3. Show that if M is a closed interval and $p \notin M$, then p is not an accumulation point of M.

Problem 3.4. Determine whether the endpoints of an open interval (a, b) are accumulation points of the interval.

It is worth exploring exactly how many points it is possible or impossible for M to have. The next two problems are just a start in investigating that.

Problem 3.5. Show that if M is a set having an accumulation point, then M contains at least two points. Determine whether M must contain at least three points.

Problem 3.6. Show that $\mathbb{Z}$ has no accumulation points.

Problem 3.7. Given sets H and K, determine whether each of the following is true or false. If the statement is true, prove it. Otherwise, provide a counterexample.

(a) If p is an accumulation point of $H \cap K$, then p is an accumulation point of both H and K.

(b) If p is an accumulation point of $H \cup K$, then p is an accumulation point of H or p is an accumulation point of K.

Problem 3.8. Prove that if M is the set of all reciprocals of elements of $\mathbb{N}$, then zero is an accumulation point of M.

We will now begin connecting the concepts of sets to more familiar ones from calculus, beginning with sequences.

Definition 3.9. A **sequence** (of real numbers) is a function a from $\mathbb{N}$ to $\mathbb{R}$.

If $n \in \mathbb{N}$, it is common to write $a_i := a(i)$. We refer to a_i as the ith **term** of the sequence. We will abuse notation and associate a sequence with its list of outputs, namely:

$$\boxed{(a_i)_{i=1}^{\infty} := (a_1, a_2, \ldots)},$$

which we may abbreviate as $\boxed{(a_i)}$.

Example 3.10. Define $a : \mathbb{N} \to \mathbb{R}$ via $a(i) = \frac{1}{2^i}$. Then we have

$$a = \left(\frac{1}{2}, \frac{1}{4}, \ldots\right) = \left(\frac{1}{2^i}\right)_{i=1}^{\infty}.$$

It is important to point out that not every sequence has a description in terms of an algebraic formula. For example, we could form a sequence out of the digits to the right of the decimal in the decimal expansion of π, namely the ith term of the sequence is the ith digit to the right of the decimal. But then there is no nice algebraic formula for describing the ith term of this sequence.

Problem 3.11. Write down several sequences a you are familiar with. If possible, give an algebraic formula for each a_i in terms of i.

Problem 3.12. Give an example of a sequence where the image set of a sequence $\{a_i\}_{i=1}^{\infty}$ is finite. In general, what's the difference between $\{a_i\}_{i=1}^{\infty}$ and $(a_i)_{i=1}^{\infty}$?

There is a deep connection between sequences and accumulation points, which the next few problems will elucidate. First, a definition—one you may have seen in calculus in a different form.

Definition 3.13. We say that the sequence $p = (p_i)_{i=1}^{\infty}$ **converges to the point** x if for every open interval S containing x, there exists an $N \in \mathbb{N}$ such that for all natural numbers $n \geq N$, $p_n \in S$.

Informally, we write $\boxed{p \to x}$ or $\boxed{(p_i) \to x}$ to mean that the sequence p converges to the point x. We simply say that p converges if there exists a point x to which the sequence converges. If a sequence does not converge to *any* point x, then we say it **diverges**.

The first problem about this should be used as a place to test ideas for how to prove convergence. Take a moment to recall all of our axioms and results from Chapter 2—you may need them! As you tackle the next few problems, it might be useful to begin by writing down the first several terms of the sequences.

Problem 3.14. Consider the sequence given by $p_n = \frac{1}{n}$ (remember, $n \in \mathbb{N}$ is part of the definition of a sequence). Show that $p = (p_i)_{i=1}^{\infty}$ converges to 0.

Problem 3.15. Consider the sequence given by $p_n = 1 - \frac{1}{n}$. Show that p converges to 1.

Problem 3.16. Consider the sequence with even terms $p_{2n} = \frac{1}{2n-1}$ and odd terms $p_{2n-1} = \frac{1}{2n}$. Show that p converges to 0.

Problem 3.17. Consider the sequence with odd terms $p_{2n-1} = \frac{1}{2n-1}$ and even terms $p_{2n} = 1 + \frac{1}{2n}$. Determine whether p converges to 0.

The following problem connects accumulation points and sequences. The most profound property of the real numbers is part of this connection, as we shall soon see.

Problem 3.18. Show that if p converges to the point x *and* for each $i \in \mathbb{N}$, $p_i \neq p_{i+1}$, then x is an accumulation point of the image set of $(p_i)_{i=1}^{\infty}$. Why do we need the restriction that $p_i \neq p_{i+1}$? Is this an absolutely necessary restriction for x to be an accumulation point of the image set?

Problem 3.19. Show that the sequence from Problem 3.14 does not converge to a point other than zero.

Problem 3.20. Show that if p converges to the point x and y is a point different from x, then p does *not* converge to y.

We now explore some basic facts concerning the convergence of sequences. In these proofs, you may have to think a little more explicitly about what the intervals around x look like in order to combine sequences. Try doing some examples with explicit numbers in order to get a sense of how to approach the proofs.

Problem 3.21. Show that if c is a real number and $p = (p_i)_{i=1}^{\infty}$ converges to x, then the sequence $cp = (cp_i)_{i=1}^{\infty}$ converges to cx.

Problem 3.22. Show that if $p = (p_i)_{i=1}^{\infty}$ converges to x and $q = (q_i)_{i=1}^{\infty}$ converges to y, then $(p_i + q_i)_{i=1}^{\infty}$ converges to $x + y$.

Products and quotients of sequences behave like you think they will as well, and you can use these facts in the rest of the notes. We will include one special case soon.

Now we introduce a few more definitions that will lead us to one of the key axioms for the real numbers (Completeness Axiom 3.37). We'll continue to see interplay between sequences and sets.

Definition 3.23. We say that a set M is **bounded** if M is a subset of some closed interval.

Definition 3.24. We say that a set M is **bounded above** if there is a point z such that if $x \in M$ then $x \leq z$; such a point is an **upper bound**.

Problem 3.25. The property of a set M being **bounded below** and the notion of a **lower bound** are defined similarly; try defining them.

Problem 3.26. Show that a set (in $\mathbb{R}$) being bounded is the same as it being bounded above and below.

Problem 3.27. Find all upper bounds for $(0,1)$, $[0,1]$, and $(0,1) \cap \mathbb{Q}^C$ (irrationals between 0 and 1).

In the next problem, remember that we 'abuse notation' by using $(p_i)_{i=1}^{\infty}$ to mean more than one mathematical object.

Problem 3.28. If the sequence p converges to the point x, then the image set $(p_i)_{i=1}^{\infty}$ is bounded.

You can use this concept to prove some of the more difficult sequence convergence properties.

Problem 3.29. Show that if q converges to 0 and p converges to x, then $(q_i \cdot p_i)_{i=1}^{\infty}$ (which means what you think it does) converges to 0.

Now we start to approach the heart of why calculus works.

Definition 3.30. We say p is a **supremum** (or **least upper bound**) of a set M if p is an upper bound of M and $p \leq q$ for any other upper bound q of M.

Problem 3.31. Define the **infimum** (or **greatest lower bound**) by analogy.

Problem 3.32. Find the suprema of $(0,1)$, and $(0,1) \cap \mathbb{Q}^C$. If we could apply the definition of supremum to $\emptyset$, what would its supremum be?

Problem 3.33. Prove that the supremum of a set is unique, if it exists.

Problem 3.34. If M and N are sets with suprema, characterize the supremum of $M \cup N$.

If M and N are sets, define $cM := \{cx \mid x \in M\}$ and $M + N := \{x + y \mid x \in M, y \in N\}$.

Problem 3.35. Assuming M and N have suprema, prove **either** that $\sup(cM)$ is $c\sup(M)$ (given $c > 0$) **or** that $\sup(M+N) = \sup(M) + \sup(N)$.

Problem 3.36. Show that $c\inf(M) = \sup(cM)$ if $c < 0$. What other properties are there relating inf, sup, and c?

The reason the supremum is so important is because of the following fundamental axiom.

Completeness Axiom 3.37. If M is a set that is bounded above, then M has a supremum.

Given the Completeness Axiom, we say that the real numbers satisfy the **least upper bound property**.

Problem 3.38. Explain why that the rational numbers do not satisfy the Completeness Axiom.

Problem 3.39. Find an example of a sequence p such that its image set $\{p_i\}_{i=1}^{\infty}$ is unbounded and hence does not have a supremum.

Problem 3.40. Show that the Completeness Axiom is *not* true if one requires that the supremum be a rational number.

It will be useful in the future to have an equivalent way to formulate completeness in terms of sequences.

Definition 3.41. We say that a sequence p is **nondecreasing** if $p_i \le p_{i+1}$ for all $i \in \mathbb{N}$. The concept of **nonincreasing** is defined similarly.

Problem 3.42. Replace $\le$ above with $<$ to define the notion of (strictly) **increasing**. Find examples of nondecreasing sequences that are not increasing.

Problem 3.43. Prove that if p is a nondecreasing sequence such that the image set $\{p_i\}_{i=1}^{\infty}$ is bounded above, then p converges to some point x.

The previous result is equivalent to the Completeness Axiom. The next problem asks you to verify this, but this is not a result that we need going forward, but rather is an interesting side story.

Problem 3.44. Assuming the result of Problem 3.43, prove the Completeness Axiom.

Why is all this so important? One reason is that we can use the completeness of the reals to *prove* Axiom (07) (sometimes called the Archimedean Principle). It may be thought of as the "real" reason why the following is true, since open intervals can be as small as we need them to be.

Problem 3.45. Using Problem 3.43, show that for any point x, there is an $n \in \mathbb{Z}$ such that $n > x$.[1]

At this point, it not necessary that we complete the following problem, but you might find doing so to be an interesting challenge.

Problem 3.46. Prove that Axiom (O7) follows from the Completeness Axiom.

Problem 3.47. Let M be a nonempty set that is bounded above, with supremum x. Prove that there exists a nondecreasing sequence (p_i) that converges to x, where the image set $\{p_i\} \subseteq M$.

Definition 3.48. A sequence $(b_k)_{k=1}^{\infty}$ is a **subsequence** of $(a_n)_{n=1}^{\infty}$ if there is a sequence of natural numbers $(n_i)_{i=1}^{\infty}$ with $n_i < n_{i+1}$ such that $b_k = a_{n_k}$.

Problem 3.49. Give some examples of subsequences of the sequence from Problem 3.14.

Problem 3.50. Prove that if a sequence converges to x, so does any subsequence of that sequence.

Problem 3.51. Suppose $(a_{n_k})_{k=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$. If a_{n_k} converges to x, does this imply that a_n converges to x? Justify your answer.

[1] *Hint:* Use a proof by contradiction.

Problem 3.52. Provide an example of a sequence (p_i) with image set $\{p_i\} \subseteq \mathbb{N}$ such that *every* sequence of natural numbers is a subsequence of (p_i).

Problem 3.53. Prove that every sequence of real numbers has a nonincreasing or nondecreasing subsequence.

Problem 3.54 (Bolzano–Weierstrass Theorem)**.** Prove that every sequence with bounded image set has a convergent subsequence.