

BRAID GRAPHS IN SIMPLY-LACED TRIANGLE-FREE COXETER SYSTEMS ARE PARTIAL CUBES

FADI AWIK, JADYN BRELAND, QUENTIN CADMAN, AND DANA C. ERNST

ABSTRACT. In this paper, we study the structure of braid graphs in simply-laced Coxeter systems. We prove that every reduced expression has a unique factorization as a product of so-called links, which in turn induces a decomposition of the braid graph into a box product of the braid graphs for each link factor. When the Coxeter graph has no three-cycles, we use the decomposition to prove that braid graphs are partial cubes, i.e., can be isometrically embedded into a hypercube. For a special class of links, called Fibonacci links, we prove that the corresponding braid graphs are Fibonacci cubes.

1. INTRODUCTION

Every element w of a Coxeter group W can be written as an expression in the generators, and if the number of generators in an expression (including multiplicity) is minimal, we say that the expression is reduced. According to Matsumoto's Theorem [8, Theorem 1.2.2], every pair of reduced expressions for the same group element are related by a sequence of so-called commutation and braid moves. In light of Matsumoto's Theorem, we can define a connected graph on the set of reduced expressions of a given element in a Coxeter group. We define the Matsumoto graph of $w \in W$ to be the graph having vertex set equal to the set of reduced expressions of w , where two vertices are connected by an edge if and only if the corresponding reduced expressions are related by a single commutation or braid move. Bergeron, Ceballos, and Labbé [2] proved that for finite Coxeter groups, every cycle in a Matsumoto graph has even length. In [9], Grinberg and Postnikov extended this result to arbitrary Coxeter systems. Since every cycle in a Matsumoto graph has even length, every Matsumoto graph is always bipartite.

Two reduced expressions for the same Coxeter group element are said to be commutation equivalent if we can obtain one from the other via a sequence of commutation moves. The corresponding equivalence classes are referred to as commutation classes, and have been studied extensively in the literature. In the case of Coxeter systems of type A_n , Elnitsky [6] showed that the set of commutation classes for a given permutation w is in one-to-one correspondence with the set of rhombic tilings of a certain polygon determined by w . Meng [17] studied the number of commutation classes and their relationships via braid moves, and Bédard [1] developed recursive formulas for the number of reduced expressions in each commutation class. According to [12], every finite Coxeter group contains a unique element of maximal length, called the longest element. Determining the number of commutation classes for the longest element in Coxeter systems of type A_n remains an open problem. To our knowledge, this problem was first introduced in 1992 by Knuth in Section 9 of [15] using different terminology. In the paragraph following the proof of Proposition 4.4 of [20], Tenner explicitly states the open problem in terms of commutation classes. Even less is known about the number of commutation classes of the longest elements in other finite Coxeter groups.

Similarly, we define two reduced expressions to be braid equivalent if they are related by a sequence of braid moves, where the corresponding equivalence classes are called braid classes. Braid classes have appeared in the work of Bergeron, Ceballos, and Labbé [2] while Zollinger [23] provided formulas for the cardinality of braid classes in the case of Coxeter systems of type A_n . Fishel et al. [7] provided upper and lower bounds on the number of reduced expressions for a fixed permutation in Coxeter systems of type A_n by studying the commutation classes and braid classes in tandem. However, unlike commutation classes, braid classes have received very little attention. Many natural questions regarding braid classes remain open, even for type A_n . For example, how many braid classes occur for elements of a fixed length? In particular, which elements for a fixed length have maximally many braid classes? And how many braid classes does the

2010 *Mathematics Subject Classification.* 20F55, 05C60, 05E15, 05A05.

Key words and phrases. Coxeter groups, braid class, braid graphs, cubical graphs, partial cubes, Fibonacci cubes.

longest element in the Coxeter system of type A_n have? Answering the latter question might provide insight into the analogous question for commutation classes that was mentioned above.

The relationship among the reduced expressions in a fixed braid class can be encoded in a graph. Define the braid graph for a reduced expression to be the graph with vertex set equal to the corresponding braid class, where two vertices are connected by an edge if and only if the corresponding reduced expressions are related via a single braid move. Note that every braid graph is equal to one of the connected components of the graph obtained by deleting the edges corresponding to commutation moves in the Matsumoto graph for the corresponding group element. The overarching goal of this paper is to understand the structure of braid classes in simply-laced Coxeter systems by studying the combinatorial architecture of the corresponding braid graphs.

We begin by recalling the basic terminology of Coxeter systems and establish our notation in Section 2. In Section 3, for simply-laced Coxeter systems, i.e., all braid moves are of the form $sts \mapsto tst$, we introduce the notion of a braid shadow, which is the location where a braid move may be applied. Loosely speaking, we say that a reduced expression is a link if it consists of a single generator or there is a sequence of overlapping braid shadows that spans the positions of the reduced expression. It turns out that every reduced expression can be written uniquely as a product of maximal links. We refer to this product as the link factorization for the reduced expression. One consequence of this factorization is that every braid graph can be decomposed as the box product of the braid graphs for the corresponding factors (Corollary 3.10). In Section 4, we completely characterize links and braid graphs in Coxeter systems of type A_n . Our notion of link coincides with the definition of string introduced by Zollinger [23] in the type A_n case. Section 5 begins with a few technical lemmas and then concludes with one of our main results (Theorem 5.17), which states that every braid graph in a simply-laced Coxeter system having no three-cycles in its Coxeter graph can be isometrically embedded into a hypercube. Such graphs are called partial cubes. In Section 6, we introduce a special class of links, called Fibonacci links, and then show that the isometric embedding of the corresponding braid graph into the hypercube is an isometry to a Fibonacci cube graph (Theorem 6.10). We conclude with some open questions in Section 7.

2. PRELIMINARIES

A *Coxeter matrix* is an $n \times n$ symmetric matrix $M = (m_{ij})$ with entries $m_{ij} \in \{1, 2, 3, \dots, \infty\}$ such that $m_{ii} = 1$ for all $1 \leq i \leq n$ and $m_{ij} \geq 2$ for $i \neq j$. A *Coxeter system* is a pair (W, S) consisting of a finite set $S = \{s_1, s_2, \dots, s_n\}$ and a group W , called a *Coxeter group*, with presentation

$$W = \langle s_1, s_2, \dots, s_n \mid (s_i s_j)^{m(s_i, s_j)} = e \rangle,$$

where $m(s_i, s_j) := m_{ij}$ for some $n \times n$ Coxeter matrix $M = (m_{ij})$. For $s, t \in S$, the condition $m(s, t) = \infty$ means that there is no relation imposed between s and t . It turns out that the elements of S are distinct as group elements and $m(s, t)$ is the order of st [12]. Since elements of S have order two, the relation $(st)^{m(s, t)} = e$ can be written as

$$\underbrace{sts \dots}_{m(s, t)} = \underbrace{tst \dots}_{m(s, t)}$$

with $m(s, t) \geq 2$ letters. When $m(s, t) = 2$, $st = ts$ is called a *commutation relation* and when $m(s, t) \geq 3$, the corresponding relation is called a *braid relation*. The replacement

$$\underbrace{sts \dots}_{m(s, t)} \mapsto \underbrace{tst \dots}_{m(s, t)}$$

is called a *commutation move* if $m(s, t) = 2$ and a *braid move* if $m(s, t) \geq 3$. We say that a Coxeter system is *simply laced* provided that $m(s, t) \leq 3$ for all $s, t \in S$. Our focus in this paper will be on simply-laced Coxeter systems.

A Coxeter system (W, S) can be encoded by a unique *Coxeter graph* Γ having vertex set S and edges $\{s, t\}$ for each $m(s, t) \geq 3$. Moreover, each edge is labeled with the corresponding $m(s, t)$, although typically the labels of 3 are omitted because they are the most common. In this case, we say that (W, S) , or just W , is of type Γ , and we may denote the Coxeter group as $W(\Gamma)$ and the generating set as $S(\Gamma)$ for emphasis. In the case that Γ has no three-cycles, we say that (W, S) is *triangle free*.

Example 2.1. The Coxeter graphs given in Figure 1 correspond to four common simply-laced Coxeter systems. We summarize the defining relations for the Coxeter systems determined by the graphs in Figures 1(a) and 1(b) below.

- (a) The Coxeter system of type A_n is given by the Coxeter graph in Figure 1(a). In this case, $W(A_n)$ is generated by $S(A_n) = \{s_1, s_2, \dots, s_n\}$ and has defining relations

1. $s_i^2 = e$ for all i ;
2. $s_i s_j = s_j s_i$ when $|i - j| > 1$;
3. $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$.

The Coxeter group $W(A_n)$ is isomorphic to the symmetric group S_{n+1} under the mapping that sends s_i to the adjacent transposition $(i, i + 1)$.

- (b) The Coxeter system of type D_n is given by the graph in Figure 1(b). The Coxeter group $W(D_n)$ has generating set $S(D_n) = \{s_1, s_2, \dots, s_n\}$ and defining relations

1. $s_i^2 = e$ for all i ;
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$ and $i, j \neq 1$;
3. $s_i s_j = s_j s_i$ if $i = 1$ and $j \neq 3$;
4. $s_1 s_3 s_1 = s_3 s_1 s_3$ and $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$.

The Coxeter group $W(D_n)$ is isomorphic to the index two subgroup of the group of signed permutations on n letters having an even number of sign changes.

With the exception of the type \tilde{A}_2 Coxeter system, each of the Coxeter systems described in Figure 1 are triangle free.

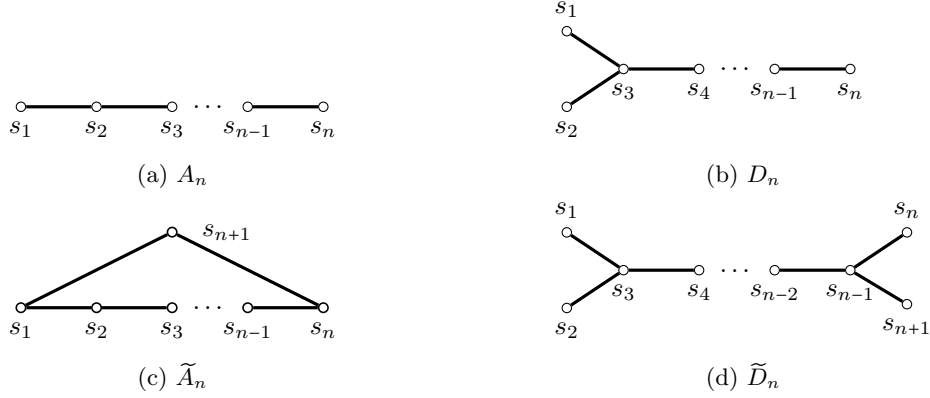


FIGURE 1. Examples of common simply-laced Coxeter graphs.

Given a Coxeter system (W, S) , let S^* denote the free monoid on the alphabet S . An element $\alpha = s_{x_1} s_{x_2} \dots s_{x_m} \in S^*$ is called a *word* while a factor of α is a word of the form $s_{x_i} s_{x_{i+1}} \dots s_{x_{j-1}} s_{x_j}$ for $1 \leq i \leq j \leq m$. If a word $\alpha = s_{x_1} s_{x_2} \dots s_{x_m} \in S^*$ is equal to w when considered as a group element of W , we say that α is an *expression* for w . If m is minimal among all possible expressions for w , we say that α is a *reduced expression* for w , and we call $\ell(w) := m$ the *length* of w . Note that any factor of a reduced expression is also reduced. The set of all reduced expressions for $w \in W$ will be denoted by $\mathcal{R}(w)$. According to [12], every finite Coxeter group contains a unique element of maximal length, called the *longest element*, often denoted by w_0 if the context is clear.

For the remainder of this paper, if we are considering a particular labeling of a Coxeter graph, we will often write i in place of s_i for brevity.

Example 2.2. It is well known that the longest element in $W(A_n)$ is given in one-line notation by

$$w_0 = [n + 1, n, \dots, 2, 1]$$

and has length $\ell(w_0) = \binom{n+1}{2}$. One possible reduced expression for $w_0 \in W(A_n)$ is given by

$$1 \mid 21 \mid 321 \mid \dots \mid n(n-1) \dots 321,$$

where the vertical bars have been placed to aid in recognizing the given pattern. A formula for the number of reduced expressions for $w_0 \in W(A_n)$ is given in [19].

The following theorem, called Matsumoto's Theorem [8, Theorem 1.2.2], characterizes the relationship between reduced expressions for a given group element.

Proposition 2.3 (Matsumoto's Theorem). In a Coxeter system (W, S) , any two reduced expressions for the same group element differ by a sequence of commutation and braid moves.

In light of Matsumoto's Theorem, we can define a graph on the set of reduced expressions of a given element in a Coxeter group. For $w \in W$, define the *Matsumoto graph* $\mathcal{G}(w)$ to be the graph having vertex set equal to $\mathcal{R}(w)$, where two reduced expressions α and β are connected by an edge if and only if α and β are related via a single commutation or braid move. Matsumoto's Theorem implies that $\mathcal{G}(w)$ is connected. The graph obtained by contracting the edges corresponding to commutation moves in the Matsumoto graph for the longest element in a Coxeter systems of type A_n has been studied by several authors, usually in the context of the higher Bruhat order $B(n, 2)$ [16, 22] or rhombic tilings of polygons [6].

Example 2.4. Consider the longest element w_0 in the Coxeter system of type A_3 . It turns out that $\ell(w_0) = 6$ and that there are 16 reduced expressions in $\mathcal{R}(w_0)$. The corresponding Matsumoto graph is given in Figure 2. The 16 reduced expressions are the vertices of $\mathcal{G}(w_0)$ and the edges show how pairs of reduced expressions are related via commutation or braid moves. In order to distinguish between commutation and braid moves, we have colored an edge **orange** if it corresponds to a commutation move and **blue** if it corresponds to a braid move.

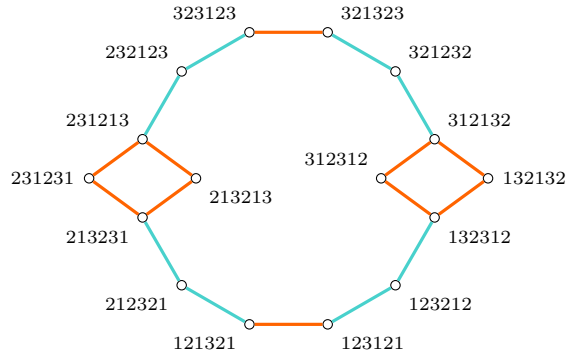


FIGURE 2. Matsumoto graph for the longest element in $W(A_3)$.

We now define two different equivalence relations on the set of reduced expressions for a given element of a Coxeter group. Let (W, S) be a Coxeter system of type Γ and let $w \in W$. For $\alpha, \beta \in \mathcal{R}(w)$, $\alpha \sim_c \beta$ if we can obtain α from β by applying a single commutation move. We define the equivalence relation \sim_c by taking the reflexive and transitive closure of \sim_c . Each equivalence class under \sim_c is called a *commutation class*, denoted $[\alpha]_c$ for $\alpha \in \mathcal{R}(w)$. Two reduced expressions are said to be *commutation equivalent* if they are in the same commutation class.

Similarly, we define $\alpha \sim_b \beta$ if we can obtain α from β by applying a single braid move. The equivalence relation \sim_b is defined by taking the reflexive and transitive closure of \sim_b . Each equivalence class under \sim_b is called a *braid class*, denoted $[\alpha]_b$ for $\alpha \in \mathcal{R}(w)$. Two reduced expressions are said to be *braid equivalent* if they are in the same braid class.

Example 2.5. The set of 16 reduced expressions for the longest element in the Coxeter system of type A_3 is partitioned into eight commutation classes and eight braid classes:

$$\begin{aligned}
[232123]_c &= \{232123\} & [123121]_b &= \{123121, 123212, 132312\} \\
[231213]_c &= \{231213, 213213, 213231, 231231\} & [312312]_b &= \{312312\} \\
[321323]_c &= \{321323, 323123\} & [312132]_b &= \{312132, 321232, 321323\} \\
[212321]_c &= \{212321\} & [132132]_b &= \{132132\} \\
[321232]_c &= \{321232\} & [121321]_b &= \{121321, 212321, 213231\} \\
[123123]_c &= \{123121, 121321\} & [213213]_b &= \{213213\} \\
[132312]_c &= \{132312, 132132, 312132, 312312\} & [231213]_b &= \{231213, 232123, 323123\} \\
[123212]_c &= \{123212\} & [231231]_b &= \{231231\}
\end{aligned}$$

In general, it is not the case that the number of commutation classes for a fixed group element is equal to the number of braid classes. Notice that the four braid classes of size 3 correspond to the vertices in the blue connected components of the Matsumoto graph given in Figure 2 while the singleton braid classes correspond to the four vertices that are not incident to any blue edges. A similar structure holds for the commutation classes.

The remainder of this paper will focus exclusively on braid classes in simply-laced Coxeter systems with an aim of describing their combinatorial architecture. We will now write $[\alpha]$ in place of $[\alpha]_b$.

The relationships among the reduced expressions in a fixed braid class are encoded by one of the maximal connected components of the underlying Matsumoto graph consisting only of edges corresponding to braid moves. For example, each braid class in Example 2.5 corresponds to one of the maximal blue connected components in the Matsumoto graph given in Figure 2. Let α be a reduced expression for $w \in W$. We define the *braid graph* for α , denoted $B(\alpha)$, to be the graph with vertex set equal to $[\alpha]$, where $\alpha, \beta \in [\alpha]$ are connected by an edge if and only if α and β are related via a single braid move. Note that we are defining braid graphs with respect to a fixed reduced expression (or equivalence class) as opposed to the corresponding group element. The latter are the graphs that arise from contracting the edges corresponding to braid moves in the Matsumoto graph.

Example 2.6. Below we describe four different braid classes and illustrate their corresponding braid graphs.

- (a) Consider the Coxeter system of type A_4 . The braid class for the reduced expression 1213243 consists of the following reduced expressions:

$$\alpha_1 := 1213243, \alpha_2 := 2123243, \alpha_3 := 2132343, \alpha_4 := 2132434.$$

- (b) In the Coxeter system of type A_6 , the expression 1213243565 is reduced. Its braid class consists of the following reduced expressions:

$$\beta_1 := 1213243565, \beta_2 := 2123243565, \beta_3 := 2132343565, \beta_4 := 2132434565,$$

$$\beta_5 := 1213243656, \beta_6 := 2123243656, \beta_7 := 2132343656, \beta_8 := 2132434656.$$

- (c) Now, consider the Coxeter system of type D_4 . The expression 2321434 is reduced and its braid class consists of the following reduced expressions:

$$\gamma_1 := 2321434, \gamma_2 := 3231434, \gamma_3 := 2321343, \gamma_4 := 3231343, \gamma_5 := 3213143.$$

- (d) Lastly, consider the Coxeter system of type D_5 . The expressions in Part (c) remain reduced. One can extend the braid chain by appending the letters 54 to the right of every element of $[\gamma_1]$. The resulting five expressions are reduced and braid equivalent. However, the expressions $\gamma_1 54$ and $\gamma_2 54$ each provide a new opportunity to apply a braid move. The resulting braid chain consists of the following 7 elements:

$$\delta_1 := 232143454, \delta_2 := 323143454, \delta_3 := 232134354, \delta_4 := 323134354,$$

$$\delta_5 := 321314354, \delta_6 := 232143545, \delta_7 := 323143545.$$

The braid graphs $B(\alpha_1)$, $B(\beta_1)$, $B(\gamma_1)$, and $B(\delta_1)$ are depicted in Figure 3.

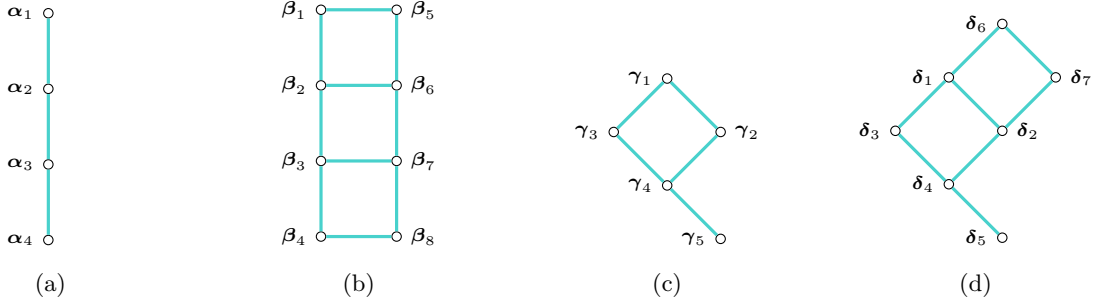


FIGURE 3. Braid graphs generated by various reduced expressions.

3. ARCHITECTURE OF BRAID CLASSES IN SIMPLY-LACED COXETER SYSTEMS

In this section, we introduce the notions of braid shadow and link, thus allowing us to provide a factorization of reduced expressions in simply-laced Coxeter systems into products of maximal links. This in turn yields a decomposition of the braid graph for a reduced expression in simply-laced Coxeter systems into a box product of the braid graphs for the corresponding link factors.

If $i, j \in \mathbb{N}$ with $i \leq j$, then we define the interval $\llbracket i, j \rrbracket := \{i, i+1, \dots, j-1, j\}$. We define the degenerate interval $\llbracket i, i \rrbracket$ to be the singleton set $\{i\}$, which we may write as $\llbracket i \rrbracket$. We will use the intervals $\llbracket i, j \rrbracket$ to represent positions in a reduced expression.

Definition 3.1. Suppose (W, S) is a Coxeter system. If $\alpha = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression for $w \in W$, we define the *local support* of α over $\llbracket i, j \rrbracket$ via

$$\text{supp}_{\llbracket i, j \rrbracket}(\alpha) := \{s_{x_k} \mid k \in \llbracket i, j \rrbracket\}.$$

The *local support* of the braid class $[\alpha]$ over $\llbracket i, j \rrbracket$ is defined by

$$\text{supp}_{\llbracket i, j \rrbracket}([\alpha]) := \bigcup_{\beta \in [\alpha]} \text{supp}_{\llbracket i, j \rrbracket}(\beta).$$

That is, $\text{supp}_{\llbracket i, j \rrbracket}(\alpha)$ is the set consisting of the generators that appear in positions $i, i+1, \dots, j$ of α while $\text{supp}_{\llbracket i, j \rrbracket}([\alpha])$ is the set of generators that appear in positions $i, i+1, \dots, j$ of any reduced expression braid equivalent to α . Also, if $\alpha = s_{x_1} s_{x_2} \cdots s_{x_m}$, we let $\alpha_{\llbracket i, j \rrbracket}$ denote the factor $s_{x_i} s_{x_{i+1}} \cdots s_{x_{j-1}} s_{x_j}$ of α . In the case of the degenerate interval $\llbracket i, i \rrbracket$, we will use the notation $\text{supp}_{\llbracket i \rrbracket}(\alpha)$, $\text{supp}_{\llbracket i \rrbracket}([\alpha])$, and $\alpha_{\llbracket i \rrbracket}$, and we will simply write $\text{supp}(\alpha)$ for the set of generators that appear in α .

Throughout the remainder of this section, we assume that (W, S) is a simply-laced Coxeter system. It is worth pointing out that many of the following results do not hold in arbitrary Coxeter systems.

Definition 3.2. Suppose that (W, S) is a simply-laced Coxeter system. If $\alpha = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression for $w \in W$, then the interval $\llbracket i, i+2 \rrbracket$ is a *braid shadow* for α if $s_{x_i} = s_{x_{i+2}}$ and $m(s_{x_i}, s_{x_{i+1}}) = 3 = m(s_{x_{i+1}}, s_{x_{i+2}})$. The collection of braid shadows for α is denoted by $\mathcal{S}(\alpha)$ and the set of braid shadows for the braid class $[\alpha]$ is given by

$$\mathcal{S}([\alpha]) := \bigcup_{\beta \in [\alpha]} \mathcal{S}(\beta).$$

The cardinality of $\mathcal{S}([\alpha])$ is called the *rank* of α , which we denote by $\text{rank}(\alpha)$.

Note that if $\alpha = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression for $w \in W$ such that $0 \leq m \leq 2$, then $\text{rank}(\alpha) = 0$. Of course, the converse is not always true.

A braid shadow is simply the location in a reduced expression where we have an opportunity to apply a braid move. A reduced expression may have many braid shadows or possibly none at all. The sets $\mathcal{S}(\alpha)$ and $\mathcal{S}([\alpha])$ capture the locations where braid moves can be performed in α and any reduced expression braid equivalent to α , respectively. If $\llbracket i, i+2 \rrbracket$ is a braid shadow for $[\alpha]$, then we may refer to position $i+1$ in any reduced expression in $[\alpha]$ as the *center* of the braid shadow.

Example 3.3. Consider the reduced expressions given in Example 2.6. By inspection, we see that:

- (a) $\mathcal{S}(\alpha_1) = \{\llbracket 1, 3 \rrbracket\}$ and $\mathcal{S}([\alpha_1]) = \{\llbracket 1, 3 \rrbracket, \llbracket 3, 5 \rrbracket, \llbracket 5, 7 \rrbracket\}$,

- (b) $\mathcal{S}(\beta_1) = \{\llbracket 1, 3 \rrbracket, \llbracket 8, 10 \rrbracket\}$ and $\mathcal{S}([\beta_1]) = \{\llbracket 1, 3 \rrbracket, \llbracket 3, 5 \rrbracket, \llbracket 5, 7 \rrbracket, \llbracket 8, 10 \rrbracket\}$,
(c) $\mathcal{S}(\gamma_1) = \{\llbracket 1, 3 \rrbracket, \llbracket 5, 7 \rrbracket\}$ and $\mathcal{S}([\gamma_1]) = \{\llbracket 1, 3 \rrbracket, \llbracket 3, 5 \rrbracket, \llbracket 5, 7 \rrbracket\}$.

The following metric will be useful in the proof of Proposition 3.5.

Definition 3.4. Suppose (W, S) is a simply-laced Coxeter system. If α and β are two braid equivalent reduced expressions for $w \in W$, then the *braid distance* $d(\alpha, \beta)$ between α and β is defined to be the minimum number of braid moves required to transform α into β .

Equivalently, we can interpret the number $d(\alpha, \beta)$ as the length of any minimal path from α to β in the corresponding braid graph. This is a consequence of Matsumoto's Theorem (Proposition 2.3).

Section 2.1 of [7] explicitly states that for Coxeter systems of type A_n , braid shadows for a braid class are either disjoint or overlap by a single position. The next proposition extends this phenomenon to arbitrary simply-laced Coxeter systems.

Proposition 3.5. Suppose (W, S) is a simply-laced Coxeter system. If α is a reduced expression for $w \in W$ with $\llbracket i, i+2 \rrbracket \in \mathcal{S}([\alpha])$, then $\llbracket i+1, i+3 \rrbracket \notin \mathcal{S}([\alpha])$.

Proof. The result is trivial if $\ell(w) \leq 3$. Suppose $\ell(w) = m \geq 4$. For each $i \in \{1, 2, \dots, m-3\}$, define

$$P_i := \{(\beta, \gamma) \mid \beta, \gamma \in [\alpha], \llbracket i, i+2 \rrbracket \in \mathcal{S}(\beta) \text{ and } \llbracket i+1, i+3 \rrbracket \in \mathcal{S}(\gamma)\}$$

and let

$$P := \bigcup_{i=1}^{m-3} P_i.$$

We will prove that $P = \emptyset$. Assume otherwise and choose $(\beta, \gamma) \in P$ such that $d(\beta, \gamma)$ is minimal among all elements of P . Since $(\beta, \gamma) \in P$, there exists $i \in \{1, \dots, m-3\}$ such that $(\beta, \gamma) \in P_i$. This implies that $\llbracket i, i+2 \rrbracket \in \mathcal{S}(\beta)$ while $\llbracket i+1, i+3 \rrbracket \in \mathcal{S}(\gamma)$. Let $\beta_{\llbracket i, i+1 \rrbracket} = sts$ and $\gamma_{\llbracket i+1, i+3 \rrbracket} = uvu$, where $m(s, t) = 3 = m(u, v)$. Suppose $d(\beta, \gamma) = k$ and say $\alpha_0 := \beta, \alpha_1, \dots, \alpha_{k-1}, \alpha_k := \gamma$ is a minimal sequence of braid equivalent reduced expressions, each one braid move apart, that transforms β into γ in k braid moves. Let b_j denote the braid move that transforms α_{j-1} into α_j . We can represent this sequence of moves visually as follows:

$$\underbrace{\cdots \frac{s}{i} \frac{t}{i+1} \frac{s}{i+2} \frac{?}{i+3} \cdots}_{\alpha_0} \xrightarrow{b_1} \cdots \xrightarrow{b_k} \underbrace{\cdots \frac{?}{i} \frac{u}{i+1} \frac{v}{i+2} \frac{u}{i+3} \cdots}_{\alpha_k}$$

Suppose b_1 transforms α_0 into α_1 such that b_1 does not involve position i nor position $i+3$. Then $(\alpha_1, \alpha_k) \in P_i$ while $d(\alpha_1, \alpha_k) = k-1$, a contradiction. So, the opening braid move b_1 must involve either position i or position $i+3$. However, if b_1 acts on positions $\llbracket i-1, i+1 \rrbracket$ or $\llbracket i+1, i+3 \rrbracket$, then α_0 is not reduced. On the other hand, if b_1 transforms α_0 into α_1 by applying the braid move $sts \mapsto tst$ in positions $\llbracket i, i+2 \rrbracket$, then $(\alpha_1, \alpha_k) \in P_i$ while $d(\alpha_1, \alpha_k) = k-1$, again a contradiction. Hence b_1 must act on positions $\llbracket i+2, i+4 \rrbracket$ or positions $\llbracket i-2, i \rrbracket$.

Assume that b_1 acts on positions $\llbracket i+2, i+4 \rrbracket$. Then there exists $x \in S$ with $m(s, x) = 3$ such that

$$\underbrace{\cdots \frac{s}{i} \frac{t}{i+1} \frac{s}{i+2} \frac{x}{i+3} \frac{s}{i+4} \cdots}_{\alpha_0} \xrightarrow{b_1} \underbrace{\cdots \frac{s}{i} \frac{t}{i+1} \frac{x}{i+2} \frac{s}{i+3} \frac{x}{i+4} \cdots}_{\alpha_1}$$

This implies that $(\alpha_k, \alpha_1) \in P_{i+1}$ while $d(\alpha_k, \alpha_1) = k-1$, which is a contradiction. We can conclude that b_1 acts on positions $\llbracket i-2, i \rrbracket$. Now, define the subsequences

$$L := \{b_n \mid b_n \text{ acts on } \llbracket j, j+2 \rrbracket \text{ for } j < i-1\} \text{ and } R := \{b_n \mid b_n \text{ acts on } \llbracket j, j+2 \rrbracket \text{ for } j \geq i-1\}.$$

Note that $b_1 \in L$. We will show that $R = \emptyset$. Assume otherwise and let $b_r \in R$. If b_r acts on $\llbracket i-1, i+1 \rrbracket$, then $(\alpha_{r-1}, \alpha_0) \in P_{i-1}$ while $d(\alpha_{r-1}, \alpha_0) = r-1 < k$. Similarly, if b_r acts on $\llbracket i, i+2 \rrbracket$, then $(\alpha_{r-1}, \alpha_k) \in P_i$ while $d(\alpha_{r-1}, \alpha_k) = k-(r-1) < k$. In either case, we contradict the minimality of k . This shows that the positions that the braid moves in L and R act on respectively do not overlap, and so the braid moves in R can be applied in any order relative to the braid moves in L . In particular, the braid moves in R could be applied prior to any of the braid moves in L . But this contradicts the fact that $b_1 \in L$. Therefore, the sequence b_1, \dots, b_k never acts on positions to the right of position i . But this is impossible since these positions are disjoint from $\llbracket i+1, i+3 \rrbracket$, which we must necessarily change to arrive at α_k . We conclude that P is empty, which yields the desired result. \square

Proposition 3.5 motivates the following definition.

Definition 3.6. Suppose (W, S) is a simply-laced Coxeter system. If $\alpha = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression for $w \in W$ with $m \geq 1$, then we say that α is a *link* provided that either $m = 1$ or m is odd and $\mathcal{S}([\alpha]) = \{\llbracket 1, 3 \rrbracket, \llbracket 3, 5 \rrbracket, \dots, \llbracket m-4, m-2 \rrbracket, \llbracket m-2, m \rrbracket\}$. If α is a link, then the corresponding braid class $[\alpha]$ is called a *braid chain*.

Loosely speaking, α is link if there is a sequence of overlapping braid moves that “cover” the positions $1, 2, \dots, m$. Note that if α is a link, then the rank of α is k if and only if α consists of $2k + 1$ letters. Notice that the center of every braid shadow for a braid chain is an even index.

Example 3.7. Consider the reduced expressions given in Example 2.6. Since $\mathcal{S}([\alpha_1]) = \{\llbracket 1, 3 \rrbracket, \llbracket 3, 5 \rrbracket, \llbracket 5, 7 \rrbracket\}$, α_1 is a link and $[\alpha_1]$ is a braid chain. On the other hand, since $\mathcal{S}([\beta_1]) = \{\llbracket 1, 3 \rrbracket, \llbracket 3, 5 \rrbracket, \llbracket 5, 7 \rrbracket, \llbracket 8, 10 \rrbracket\}$, it follows that β_1 is not a link and hence $[\beta_1]$ is not a braid chain. However, it turns out that the factors 1213243 and 565 of β_1 are links in their own right. Lastly, since $\mathcal{S}([\gamma_1]) = \{\llbracket 1, 3 \rrbracket, \llbracket 3, 5 \rrbracket, \llbracket 5, 7 \rrbracket\}$, γ_1 is a link and $[\gamma_1]$ is a braid chain.

Definition 3.8. Suppose (W, S) is a simply-laced Coxeter system. If α is a reduced expression for $w \in W$ with $\ell(w) \geq 1$, then we say that β is a *link factor* of α provided that

- (a) β is a factor of α ,
- (b) β is a link, and
- (c) for every factor γ of α , if β is a factor of γ and γ is a link, then $\beta = \gamma$.

It follows immediately from Definition 3.8 that every reduced expression α for a nonidentity group element can be written uniquely as a product of link factors, say $\alpha_1 \alpha_2 \cdots \alpha_k$, where each α_i is a link factor of α . We refer to this product as the *link factorization* of α . For emphasis, we will often denote such a factorization via $\alpha = \alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_k$. For convenience we say that the link factorization of the identity is product of a single copy of the empty word, but it is important to note that the empty word is not actually a link. The following proposition is a direct consequence of the definitions.

Proposition 3.9. Suppose (W, S) is a simply-laced Coxeter system. If α is a reduced expression for $w \in W$ with link factorization $\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_k$, then

$$[\alpha] = \{\beta_1 \mid \beta_2 \mid \cdots \mid \beta_k : \beta_i \in [\alpha_i] \text{ for } 1 \leq i \leq k\}.$$

Moreover, the cardinality of the braid class for α is given by

$$\text{card}([\alpha]) = \prod_{i=1}^k \text{card}([\alpha_i]),$$

and the rank of α is given by

$$\text{rank}(\alpha) = \sum_{i=1}^k \text{rank}(\alpha_i).$$

Proposition 3.9 implies that the braid graph for any reduced expression for a group element can be decomposed as the box product of the braid graphs for the corresponding link factors in the link factorization. Note that the decomposition is unique if one respects the ordering of the link factors.

Corollary 3.10. Suppose (W, S) is a simply-laced Coxeter system. If α is a reduced expression for $w \in W$ with link factorization $\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_k$, then $B(\alpha) \cong B(\alpha_1) \square B(\alpha_2) \square \cdots \square B(\alpha_k)$.

Proof. An isomorphism of graphs is given by $\beta_1 \mid \beta_2 \mid \cdots \mid \beta_k \mapsto (\beta_1, \beta_2, \dots, \beta_k)$, where each $\beta_j \in [\alpha_j]$. This bijection between vertex sets respects the edges of the corresponding graphs since braid moves on distinct link factors can be applied independently. \square

Example 3.11. Consider the reduced expression $\beta_1 = 1213243565$ defined in Example 2.6. The link factorization for β_1 is $\textcolor{teal}{1213243} \mid \textcolor{violet}{565}$. The decomposition $B(\beta_1) \cong B(\textcolor{teal}{1213243}) \square B(\textcolor{violet}{565})$ guaranteed by Corollary 3.10 is shown in Figure 4. Note that we have utilized colors to help distinguish the link factors.

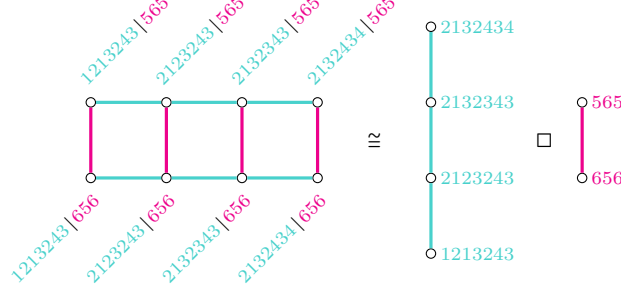


FIGURE 4. Braid graph for the reduced expression from Example 3.11 and its decomposition into a box product of braid graphs for the corresponding link factors.

Example 3.12. Consider the Coxeter system of type D_7 determined by the Coxeter graph in Figure 1(b). The reduced expression 3231343567543231343 has link factorization

$$3231343 \mid 5 \mid 6 \mid 7 \mid 5 \mid 4 \mid 3231343.$$

The braid graphs for the first and last link factors are isomorphic to the braid graph in Figure 3(c). The braid graph for each singleton factor consists of a single vertex. The braid graph for the entire reduced expression and its decomposition are shown in Figure 5.

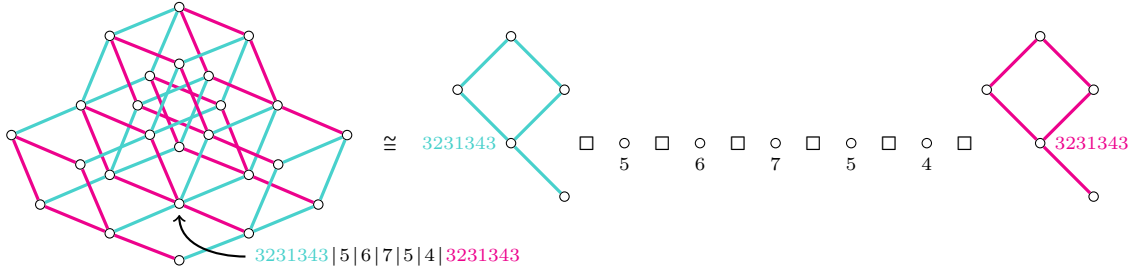


FIGURE 5. Braid graph for the reduced expression from Example 3.12 and its decomposition into a box product of braid graphs for the corresponding link factors.

According to the next proposition, the support of a braid shadow is constant across an entire braid class in simply-laced triangle-free Coxeter systems.

Proposition 3.13. Suppose (W, S) is a simply-laced triangle-free Coxeter system. If α and β are two braid equivalent reduced expressions for $w \in W$ with $\ell(w) \geq 3$, then for all $\llbracket i, i+2 \rrbracket \in \mathcal{S}(\alpha) \cap \mathcal{S}(\beta)$, $\text{supp}_{\llbracket i, i+2 \rrbracket}(\alpha) = \text{supp}_{\llbracket i, i+2 \rrbracket}(\beta)$.

Proof. Let α and β be two braid equivalent reduced expressions for $w \in W$ with $\ell(w) = m \geq 3$. For each $i \in \{1, \dots, m-2\}$, define

$$P_i := \{(\gamma, \delta) \mid \gamma, \delta \in [\alpha], \llbracket i, i+2 \rrbracket \in \mathcal{S}(\gamma) \cap \mathcal{S}(\delta), \text{ and } \text{supp}_{\llbracket i, i+2 \rrbracket}(\gamma) \neq \text{supp}_{\llbracket i, i+2 \rrbracket}(\delta)\},$$

and let

$$P := \bigcup_{i=1}^{m-2} P_i.$$

We will prove that $P = \emptyset$. Suppose otherwise and choose $(\gamma, \delta) \in P$ such that $d(\gamma, \delta)$ is minimal among all elements of P . Then there exists $i \in \{1, \dots, m-2\}$ such that $(\gamma, \delta) \in P_i$, so that $\llbracket i, i+2 \rrbracket \in \mathcal{S}(\gamma) \cap \mathcal{S}(\delta)$ while $\text{supp}_{\llbracket i, i+2 \rrbracket}(\gamma) \neq \text{supp}_{\llbracket i, i+2 \rrbracket}(\delta)$. Let $\gamma_{\llbracket i, i+2 \rrbracket} = sts$ and $\delta_{\llbracket i, i+2 \rrbracket} = uvu$, where $m(s, t) = 3 = m(u, v)$. Suppose $d(\gamma, \delta) = k$ and let

$$\alpha_0 := \gamma, \alpha_1, \dots, \alpha_{k-1}, \alpha_k := \delta$$

be a minimal sequence of braid equivalent reduced expressions that transforms γ into δ in k braid moves such that $d(\alpha_{j-1}, \alpha_j) = 1$ for $1 \leq j \leq k$. It is clear that $k \geq 2$. Let b_j denote the braid move that transforms α_{j-1} into α_j . As in the proof of Proposition 3.5, we represent this sequence of moves visually as follows:

$$\underbrace{\cdots \frac{s}{i} \frac{t}{i+1} \frac{s}{i+2} \cdots}_{\alpha_0} \xrightarrow{b_1} \cdots \xrightarrow{b_k} \underbrace{\cdots \frac{u}{i} \frac{v}{i+1} \frac{u}{i+2} \cdots}_{\alpha_k}$$

Since $\llbracket i, i+2 \rrbracket \in \mathcal{S}(\alpha_0)$, the intervals $\llbracket i-1, i+1 \rrbracket$ and $\llbracket i+1, i+3 \rrbracket$ are not braid shadows for every reduced expression in the sequence $\alpha_0, \alpha_1, \dots, \alpha_{k-1}, \alpha_k$ by Proposition 3.5. By the minimality of k , the interval $\llbracket i, i+2 \rrbracket$ only appears as a braid shadow in α_0 and α_k . That is, $\llbracket i, i+2 \rrbracket \notin \mathcal{S}(\alpha_l)$ for all $1 \leq l \leq k-1$. Together these facts imply that $\text{supp}_{\llbracket i+1 \rrbracket}(\alpha_l) = \{t\}$ for all $0 \leq l \leq k$. In other words, t is fixed in position $i+1$ throughout the entire sequence $\alpha_0, \alpha_1, \dots, \alpha_{k-1}, \alpha_k$. This forces $v = t$, which in turn implies that $m(u, t) = 3$. Again by the minimality of k , it must be the case that b_1 acts on either $\llbracket i-2, i \rrbracket$ or $\llbracket i+2, i+4 \rrbracket$. Without loss of generality, assume that b_1 acts on $\llbracket i+2, i+4 \rrbracket$. Then there exists $x \in S$ with $m(s, x) = 3$ such that $\text{supp}_{\llbracket i+3 \rrbracket}(\alpha_0) = \{x\}$ and $\text{supp}_{\llbracket i+4 \rrbracket}(\alpha_0) = \{s\}$. In summary, we have

$$\underbrace{\cdots \frac{s}{i} \frac{t}{i+1} \frac{s}{i+2} \frac{x}{i+3} \frac{s}{i+4} \cdots}_{\alpha_0} \xrightarrow{b_1} \cdots \underbrace{\frac{s}{i} \frac{t}{i+1} \frac{x}{i+2} \frac{s}{i+3} \frac{x}{i+4} \cdots}_{\alpha_1} \xrightarrow{b_2} \cdots \xrightarrow{b_k} \underbrace{\cdots \frac{u}{i} \frac{t}{i+1} \frac{u}{i+2} \frac{?}{i+3} \frac{?}{i+4} \cdots}_{\alpha_k}$$

Towards a contradiction, suppose $x \neq u$. In order to exchange x with u in position $i+2$, there must exist a reduced expression α_j with $2 \leq j \leq k$ such that $\llbracket i+2, i+4 \rrbracket \in \mathcal{S}(\alpha_j)$ and $\text{supp}_{\llbracket i+2, i+4 \rrbracket}(\alpha_j) \neq \{x, s\}$. Yet if $\text{supp}_{\llbracket i+2, i+4 \rrbracket}(\alpha_j) \neq \{x, s\}$, then $(\alpha_1, \alpha_j) \in P_{i+2}$ while $d(\alpha_1, \alpha_j) = j-1 < k$, a contradiction. Hence $x = u$. This implies that $m(s, u) = 3$, and hence $m(s, u) = m(u, t) = m(t, s) = 3$. But this is contrary to the fact that (W, S) is triangle free. We conclude that $P = \emptyset$, which proves the claim. \square

As the next example illustrates, the previous result is false without the assumption that the Coxeter system is triangle free.

Example 3.14. Consider the Coxeter system of type \tilde{A}_2 , which is determined by the Coxeter graph in Figure 1(c). The expression $\alpha = 1213121$ is a link, and it is easy to see that $\beta = 2123212 \in [\alpha]$. However, $\text{supp}_{\llbracket 3,5 \rrbracket}(\alpha) = \{1, 3\}$ while $\text{supp}_{\llbracket 3,5 \rrbracket}(\beta) = \{2, 3\}$. This shows that Proposition 3.13 is false when the Coxeter graph has a three-cycle.

If one generalizes the notions of braid shadow and link in the natural way, we conjecture that a result analogous to Proposition 3.13 holds in arbitrary Coxeter systems as long as the corresponding Coxeter graph does not contain a three-cycle with edge weights $3, 3, m$, where $m \geq 3$.

When a reduced expression has a braid shadow, the collection of generators that may appear at the center of the braid shadow in any braid equivalent reduced expression is completely determined by the support of that braid shadow. The following proposition makes this more precise.

Proposition 3.15. Suppose (W, S) is a simply-laced triangle-free Coxeter system and let α be a link of rank at least one. Then $\llbracket 2i-1, 2i+1 \rrbracket \in \mathcal{S}(\alpha)$ if and only if $\text{supp}_{\llbracket 2i-1, 2i+1 \rrbracket}(\alpha) = \text{supp}_{\llbracket 2i \rrbracket}([\alpha])$.

Proof. To prove the forward implication, let $\llbracket 2i-1, 2i+1 \rrbracket \in \mathcal{S}(\alpha)$ and assume that $\text{supp}_{\llbracket 2i-1, 2i+1 \rrbracket}(\alpha) = \{s, t\}$. It is clear that $\{s, t\} \subseteq \text{supp}_{\llbracket 2i \rrbracket}([\alpha])$. We may assume without loss of generality that $\text{supp}_{\llbracket 2i \rrbracket}(\alpha) = \{s\}$. Let $u \in \text{supp}_{\llbracket 2i \rrbracket}([\alpha])$ and choose $\beta \in [\alpha]$ such that $\text{supp}_{\llbracket 2i \rrbracket}(\beta) = \{u\}$. Then α and β are related by a sequence of braid moves. If no braid move involves position $2i$, then $s = u$. Otherwise, the sequence has a braid move that involves position $2i$. However, by Proposition 3.5, $\llbracket 2i-2, 2i \rrbracket, \llbracket 2i, 2i+2 \rrbracket \notin \mathcal{S}([\alpha])$. Hence there exists $\gamma \in [\alpha]$ such that $\llbracket 2i-1, 2i+1 \rrbracket \in \mathcal{S}(\gamma)$ and $u \in \text{supp}_{\llbracket 2i-1, 2i+1 \rrbracket}(\gamma)$. By Proposition 3.13, $\text{supp}_{\llbracket 2i-1, 2i+1 \rrbracket}(\gamma) = \text{supp}_{\llbracket 2i-1, 2i+1 \rrbracket}(\alpha)$. This shows that $u \in \{s, t\}$, and so $\text{supp}_{\llbracket 2i \rrbracket}([\alpha]) = \{s, t\}$, as desired.

To prove the converse, assume $\text{supp}_{\llbracket 2i-1, 2i+1 \rrbracket}(\alpha) = \text{supp}_{\llbracket 2i \rrbracket}([\alpha])$. Since α is a link, we know $\llbracket 2i-1, 2i+1 \rrbracket \in \mathcal{S}([\alpha])$, and hence we can choose $\beta \in [\alpha]$ such that $\llbracket 2i-1, 2i+1 \rrbracket \in \mathcal{S}(\beta)$. But now we can apply the forward implication to β to conclude that

$$\text{supp}_{\llbracket 2i-1, 2i+1 \rrbracket}(\beta) = \text{supp}_{\llbracket 2i \rrbracket}([\beta]) = \text{supp}_{\llbracket 2i \rrbracket}([\alpha]) = \text{supp}_{\llbracket 2i-1, 2i+1 \rrbracket}(\alpha).$$

This implies that $\llbracket 2i-1, 2i+1 \rrbracket \in \mathcal{S}(\alpha)$. \square

Applying the previous proposition to a pair of overlapping braid shadows yields the following corollary, which says that the supports of overlapping braid shadows intersect at a single element.

Corollary 3.16. Suppose (W, S) is a simply-laced triangle-free Coxeter system. If α is a link of rank at least two, then $\text{card}(\text{supp}_{\llbracket 2i \rrbracket}([\alpha]) \cap \text{supp}_{\llbracket 2i+2 \rrbracket}([\alpha])) = 1$.

If α is a link while (W, S) is not triangle free, then Proposition 3.15 and Corollary 3.16 are false, as illustrated by the next example.

Example 3.17. Consider the link α from Example 3.14. We have $\text{supp}_{\llbracket 3,5 \rrbracket}(\alpha) = \{1, 3\}$ and $\text{supp}_{\llbracket 4 \rrbracket}([\alpha]) = \{1, 2, 3\}$, contrary to Proposition 3.15. We also have $\text{supp}_{\llbracket 6 \rrbracket}([\alpha]) = \{1, 2\}$ so that $\text{supp}_{\llbracket 4 \rrbracket}([\alpha]) \cap \text{supp}_{\llbracket 6 \rrbracket}([\alpha]) = \{1, 2\}$, which clashes with the conclusion of Corollary 3.16. Once again, this shows that the assumption that the Coxeter graph has no three-cycles cannot be discarded.

Remark 3.18. Proposition 3.15 allows us to assume that $\text{supp}_{\llbracket 2i \rrbracket}([\alpha]) = \{s, t\}$ with $m(s, t) = 3$ whenever we have $\llbracket 2i-1, 2i+1 \rrbracket \in \mathcal{S}(\alpha)$ with $\text{supp}_{\llbracket 2i-1, 2i+1 \rrbracket}(\alpha) = \{s, t\}$. Moreover, if additionally we have $\llbracket 2i+1, 2i+3 \rrbracket \in \mathcal{S}(\alpha)$, then we can utilize Corollary 3.16 to conclude that $\text{supp}_{\llbracket 2i+2 \rrbracket}([\alpha]) = \{t, u\}$ with $m(s, t) = 3$. Since (W, S) is simply laced and triangle free, we know $m(s, u) = 2$. Moreover, we can extend Proposition 3.15 and Corollary 3.16 to arbitrary reduced expressions by applying the results to the corresponding link factors. We will frequently use these facts, and we may do so without explicitly mentioning the proposition and corollary.

4. CLASSIFICATION OF LINKS AND BRAID GRAPHS IN COXETER SYSTEMS OF TYPE A_n

Our notions of link and link factorization generalize Zollinger's definitions of string and maximal string decomposition, respectively, for Coxeter systems of type A_n that appear in [23]. Let (l, k, m, ϵ) be a quadruple satisfying:

- (a) l is a positive integer;
- (b) k is a nonnegative integer less than or equal to $l-1$;
- (c) m is a positive integer (not necessarily distinct from l or k) and;
- (d) ϵ is one of $\{+, -, 0\}$;

where $\epsilon = 0$ only when $l \leq 2$. From this quadruple, define $\sigma_{l,k,m,\epsilon}$ with $\epsilon \in \{+, 0\}$ to be the word in the Coxeter system of type A_n , called a *string*, as follows. For $l \in \{1, 2\}$, define

$$\sigma_{1,0,m,0} := s_m, \quad \sigma_{2,0,m,0} := s_m s_{m+1} s_m, \quad \sigma_{2,1,m,0} := s_{m+1} s_m s_{m+1}.$$

When $l \geq 3$, then $\sigma_{l,k,m,+}$ is the first $2l-1$ letters from the following product

$$s_{m+1} s_m s_{m+2} s_{m+1} s_{m+3} \cdots \overline{s_{m+k} s_{m+k-1} s_{m+k} s_{m+k+1} s_{m+k} s_{m+k+2} s_{m+k+1} s_{m+k+3} s_{m+k+2} \cdots}$$

Note that there are two overlapping opportunities to apply a braid move, each of which has been underlined or overlined for emphasis. We define the string $\sigma_{l,k,m,-}$ to be the reverse (i.e., inverse) of $\sigma_{l,l-1-k,m,+}$. Observe that every string consists of an odd number of letters, namely $2l-1$. According to [23], each string is a reduced expression.

Example 4.1. Below are six examples of strings in the Coxeter system of type A_9 :

$$\begin{array}{lll} \sigma_{6,0,4,+} = 45465768798 & \sigma_{6,1,4,+} = 54565768798 & \sigma_{6,2,4,+} = 54656768798 \\ \sigma_{6,3,4,+} = 54657678798 & \sigma_{6,4,4,+} = 54657687898 & \sigma_{6,5,4,+} = 54657687989 \end{array}$$

By applying all possible braid moves, one can verify that each expression above is a link and that these six reduced expressions comprise a single braid class.

The following result is a consequence of Lemmas 1 and 5 in [23] and completely characterizes the links in Coxeter systems of type A_n .

Proposition 4.2. In the Coxeter system of type A_n , a reduced expression is a link if and only if it is a string.

In [23], Zollinger refers to the link factorization of a reduced expression in terms of strings as the *maximal string decomposition*. Given the structure of strings, we can completely describe the corresponding braid graphs with ease. The next result is a reformulation of Lemma 1 from [23] in terms of braid graphs.

Proposition 4.3. For the link $\sigma_{l,k,m,\epsilon}$, we have

$$B(\sigma_{l,k,m,\epsilon}) = \begin{array}{c} \sigma_{l,0,m,\epsilon} \\ \sigma_{l,1,m,\epsilon} \\ \sigma_{l,2,m,\epsilon} \\ \vdots \\ \sigma_{l,l-2,m,\epsilon} \\ \sigma_{l,l-1,m,\epsilon} \end{array}$$

Recall that Corollary 3.10 says that the braid graph for any reduced expression for a group element can be decomposed as the box product of the braid graphs for the corresponding link factors in the link factorization. In light of Proposition 4.3, we can be more explicit for Coxeter systems of type A_n , as the next result indicates. This result is also a reformulation of Corollary 6 in [23] and can be thought of as a classification of braid graphs for reduced expressions in Coxeter systems of type A_n .

Proposition 4.4. If α is a reduced expression for a nonidentity $w \in W(A_n)$ with link factorization $\alpha_1 \mid \alpha_2 \mid \dots \mid \alpha_k$ such that each link α_i has $2l_i - 1$ letters, then

$$B(\alpha) \cong \left(\begin{array}{c} \circ \\ \vdots \\ \circ \end{array} \right) l_1 \square \left(\begin{array}{c} \circ \\ \vdots \\ \circ \end{array} \right) l_2 \square \dots \square \left(\begin{array}{c} \circ \\ \vdots \\ \circ \end{array} \right) l_k ,$$

where the i th link factor in the decomposition is a path graph with l_i vertices.

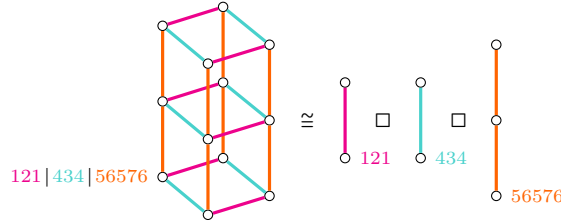


FIGURE 6. Braid graph for the reduced expression from Example 4.5 and its decomposition into a box product of path graphs.

Example 4.5. The word $\alpha = 12143456576$ is a reduced expression for some element in $W(A_7)$. The link factorization for α is $121 \mid 434 \mid 56576$. The resulting braid graph and its decomposition into a box product of path graphs is shown in Figure 6.

5. BRAID GRAPHS AS PARTIAL CUBES

If G is a graph, let $V(G)$ denote the vertex set of G . If $S \subseteq V(G)$ is any subset of vertices of G , then we define the *induced subgraph* $G[S]$ to be the graph whose vertex set is S and whose edge set consists of all of the edges of G that have endpoints in S . An *embedding* of a simple graph G into a simple graph H is an injection $f : V(G) \rightarrow V(H)$ with the property that if u and v are adjacent vertices in G , then $f(u)$ and $f(v)$ are adjacent in H . If in addition, $f(u)$ and $f(v)$ adjacent in H implies u and v adjacent in G , then we say that f is an *induced embedding*. In the graph theory literature, an embedding is often referred to as a monomorphism while an induced embedding is a faithful monomorphism [10]. If f is an induced embedding, then G is isomorphic to the subgraph of H induced by the image of f . That is, $G \cong H[\text{im}(f)]$.

We can view any connected graph G as a metric space by taking the standard geodesic metric. That is, the distance between $u, v \in V(G)$ is defined via

$$d_G(u, v) := \text{length of any minimal path between } u \text{ and } v.$$

An *isometric embedding* of G into H is a function $f : V(G) \rightarrow V(H)$ with the property that $d_G(u, v) = d_H(f(u), f(v))$ for all $u, v \in V(G)$. Since an isometry is injective and two vertices are adjacent if and only

if the distance between them is one, every isometric embedding is also an induced embedding. It is worth mentioning that an induced embedding is not necessarily an isometric embedding. As an example, consider an embedding of a path with four vertices into a cycle with five vertices. This embedding is an induced embedding but is not an isometric embedding.

For a nonnegative integer r we will denote the set of binary strings of length r by $\{0, 1\}^r$. That is,

$$\{0, 1\}^r := \{a_1 a_2 \cdots a_r \mid a_k \in \{0, 1\}\}.$$

The *hypercube graph* of dimension $r \geq 0$, denoted by Q_r , is defined to be the graph whose vertices are elements of $\{0, 1\}^r$ with two binary strings connected by an edge exactly when they differ by a single digit (i.e., the Hamming distance between the two vertices is equal to one). Note that Q_0 consists of a single vertex labeled by the empty string. A *partial cube* is a graph that can be isometrically embedded into a hypercube. The *isometric dimension* of a partial cube is the minimum dimension of a hypercube into which it may be isometrically embedded. That is, the isometric dimension of a partial cube G is the nonnegative integer

$$\dim_I(G) := \min\{m \in \mathbb{N} \cup \{0\} \mid \text{there exists an isometric embedding of } G \text{ into } Q_m\}.$$

The following proposition is a result from [18].

Proposition 5.1. If G_1 and G_2 are partial cubes, then $G_1 \square G_2$ is a partial cube. Moreover, $\dim_I(G_1 \square G_2) = \dim_I(G_1) + \dim_I(G_2)$.

Assume (W, S) is a simply-laced triangle-free Coxeter system and suppose α is a reduced expression for some $w \in W$. The goal of this section is to establish an isometric embedding of $B(\alpha)$ into $Q_{\text{rank}(\alpha)}$ (see Theorem 5.17). The crux is proving that every link can be isometrically embedded in a hypercube whose dimension is at most the rank of the link (see Proposition 5.14). Then we can simply apply the decomposition for reduced expressions given in Corollary 3.10 together with Proposition 5.1 to obtain the result for arbitrary reduced expressions. It also follows that the isometric dimension of every braid graph is bounded above by the number of braid shadows.

Throughout this section, we will utilize many of the properties of braid equivalent reduced expressions and links that were developed in Section 3. We begin with several technical lemmas that will be useful in the the proof of Proposition 5.14. Our first lemma indicates that the left and right ends of a link are fairly rigid in structure.

Lemma 5.2. Suppose (W, S) is a simply-laced triangle-free Coxeter system and let α be a link of rank $r \geq 1$.

- (a) If $\text{supp}_{[2]}([\alpha]) = \{s, t\}$ with $m(s, t) = 3$, then $\alpha_{[1,2]} = st$ or $\alpha_{[1,2]} = ts$.
- (b) If $\text{supp}_{[2r]}([\alpha]) = \{s, t\}$ with $m(s, t) = 3$, then $\alpha_{[2r, 2r+1]} = st$ or $\alpha_{[2r, 2r+1]} = ts$.

Proof. Suppose α is a link such that $\text{supp}_{[2]}([\alpha]) = \{s, t\}$ with $m(s, t) = 3$ and let $\alpha_{[1,2]} = us$. Note that $u \neq s$ since α is reduced. Now, consider the set

$$X = \{\beta \in [\alpha] \mid [1, 3] \in \mathcal{S}(\beta) \text{ and } u \in \text{supp}_{[1,3]}(\beta)\}.$$

Since α is a link, X is nonempty. Choose any $\beta \in X$. Then $\text{supp}_{[1,3]}(\beta) = \{s, t\}$ by Proposition 3.15, and so $u = t$. This proves Part (a). Part (b) follows from a symmetric argument. \square

The next lemma states that the support of the common position of two overlapping braid shadows has cardinality three and places restrictions on the local structure of overlapping braid shadows.

Lemma 5.3. Suppose (W, S) is a simply-laced triangle-free Coxeter system. If α is a link of rank $r \geq 2$ such that for $1 \leq i \leq r-1$, $\text{supp}_{[2i]}([\alpha]) = \{s, t\}$ and $\text{supp}_{[2i+2]}([\alpha]) = \{t, u\}$ with $m(s, t) = 3 = m(t, u)$ according to Remark 3.18, then $\text{supp}_{[2i+1]}([\alpha]) = \{s, t, u\}$, $\alpha_{[2i]} \neq \alpha_{[2i+2]}$, and $\alpha_{[2i+1]} \in \{s, t, u\} \setminus \{\alpha_{[2i]}, \alpha_{[2i+2]}\}$.

Proof. Let α be a link of rank $r \geq 2$ such that $\text{supp}_{[2i]}([\alpha]) = \{s, t\}$ and $\text{supp}_{[2i+2]}([\alpha]) = \{t, u\}$ with $m(s, t) = 3 = m(t, u)$. It is clear that $\{s, t, u\} \subseteq \text{supp}_{[2i+1]}([\alpha])$. Now, let $v \in \text{supp}_{[2i+1]}([\alpha])$ and consider the set

$$X = \{\beta \in [\alpha] \mid \text{supp}_{[2i+1]}(\beta) = \{v\}\}.$$

Since $\text{supp}_{[2i+1]}([\alpha]) \neq \{v\}$, there exists $\beta \in X$ such that either $[2i-1, 2i+1] \in \mathcal{S}(\beta)$ or $[2i+1, 2i+3] \in \mathcal{S}(\beta)$. Then $v \in \{s, t\}$ or $v \in \{t, u\}$ according to Proposition 3.13. This proves the first claim. If $\alpha_{[2i]} = \alpha_{[2i+2]}$, then

it must be the case that t occupies positions $2i$ and $2i+2$ in α . But this would imply that $\alpha_{[2i+1]} \in \{s, u\}$, which violates Proposition 3.5 in either case. Thus, $\alpha_{[2i]} \neq \alpha_{[2i+2]}$. It follows that $\alpha_{[2i+1]} \in \{s, t, u\} \setminus \{\alpha_{[2i]}, \alpha_{[2i+2]}\}$, otherwise α would not be reduced. \square

One consequence of Lemma 5.3 is that for any two overlapping braid shadows in a braid chain $[\alpha]$, there are three possible forms that $\alpha_{[2i, 2i+2]}$ may take:

$$\begin{aligned} \text{(a)} \quad & \underbrace{\cdots \frac{?}{2i-1} \frac{s}{2i} \frac{u}{2i+1} \frac{t}{2i+2} \frac{?}{2i+3} \cdots}_{\alpha} \\ \text{(b)} \quad & \underbrace{\cdots \frac{?}{2i-1} \frac{s}{2i} \frac{\alpha}{2i+1} \frac{u}{2i+2} \frac{?}{2i+3} \cdots}_{\alpha} \\ \text{(c)} \quad & \underbrace{\cdots \frac{?}{2i-1} \frac{t}{2i} \frac{\alpha}{2i+1} \frac{u}{2i+2} \frac{?}{2i+3} \cdots}_{\alpha} \end{aligned}$$

where $m(s, t) = 3 = m(t, u)$ and $m(s, u) = 2$. Note that position $2i+1$ is the location where the two braid shadows in $[\alpha]$ overlap.

The next lemma states that for every link α and every pair of overlapping braid shadows for $[\alpha]$, there exists a link in $[\alpha]$ such that overlapping braid shadows occur simultaneously.

Lemma 5.4. If (W, S) is a simply-laced triangle-free Coxeter system and α is a link of rank $r \geq 2$, then for all $1 \leq i \leq r-1$ there exists $\sigma \in [\alpha]$ with the property that $[2i-1, 2i+1], [2i+1, 2i+3] \in \mathcal{S}(\sigma)$.

Proof. This result follows from Lemma 5.3. \square

Equivalently, the previous lemma states that if (W, S) is a simply-laced triangle-free Coxeter system and α is a link of rank at least two such that $\text{supp}_{[2i]}([\alpha]) = \{s, t\}$ and $\text{supp}_{[2i+2]}([\alpha]) = \{t, u\}$ with $m(s, t) = 3 = m(t, u)$, then there exists a link $\sigma \in [\alpha]$ with the property that $\sigma_{[2i-1, 2i+3]} = tstut$.

Given a reduced expression α consisting of at least two letters, let $\hat{\alpha}$ be the reduced expression obtained from α by deleting the rightmost two letters. If α is a link, we should not expect $\hat{\alpha}$ to be a link. However, the next lemma indicates that we will obtain a link if we delete the rightmost two letters from a carefully chosen link. Certainly, we also have a “left-handed” version of this lemma. There is likely a generalized version of this lemma where we delete all the letters on either side of the position where two braid shadows overlap, however, this stronger version is not needed for our purposes.

Lemma 5.5. Suppose (W, S) is a simply-laced triangle-free Coxeter system and α is a link of rank $r \geq 2$ and choose $\sigma \in [\alpha]$ such that $[2r-3, 2r-1], [2r-1, 2r+1] \in \mathcal{S}(\sigma)$ according to Lemma 5.4. Then $\hat{\sigma}$ is a link of rank $r-1$, and if $\beta \in [\alpha]$ such that $\text{supp}_{[2r]}(\beta) = \text{supp}_{[2r]}(\sigma)$, then $\hat{\beta} \in [\hat{\sigma}]$. Moreover, every element of $[\hat{\sigma}]$ is of the form $\hat{\beta}$ for some $\beta \in [\alpha]$ satisfying $\text{supp}_{[2r]}(\beta) = \text{supp}_{[2r]}(\sigma)$.

Proof. The fact that $\hat{\sigma}$ is a link of rank $r-1$ is easily seen. Let $\beta \in [\alpha]$ such that $\text{supp}_{[2r]}(\beta) = \text{supp}_{[2r]}(\sigma)$. It follows from Lemma 5.2 that $\beta_{[2r, 2r+1]} = \sigma_{[2r, 2r+1]}$. Any minimal sequence of braid moves that transforms σ into β does not involve the generators appearing in the rightmost braid shadow, and hence the same sequence of braid moves will transform $\hat{\sigma}$ into $\hat{\beta}$. Thus, $\hat{\beta} \in [\hat{\sigma}]$. On the other hand, suppose that $\sigma_{[2r, 2r+1]} = ut$ and let $\gamma \in [\hat{\sigma}]$. Then there is a sequence of braid moves that transforms $\hat{\sigma}$ into γ . Set $\beta := \gamma ut$ so that $\gamma = \hat{\beta}$. When applied to σ instead of $\hat{\sigma}$, this sequence of braid moves transforms σ into β . It follows from Proposition 2.3 that β is reduced and hence $\beta \in [\alpha]$. Clearly, $\text{supp}_{[2r]}(\beta) = \{u\} = \text{supp}_{[2r]}(\sigma)$, which completes the proof. \square

Our next lemma has the important conclusion that every link is uniquely determined by the generators appearing at the even indices of the word. Moreover, every factor that begins and ends at an even index is uniquely determined by the even-index generators that appear in the factor.

Lemma 5.6. Suppose (W, S) is a simply-laced triangle-free Coxeter system and let α and β be two braid equivalent links of rank $r \geq 1$. Then $\alpha = \beta$ if and only if $\alpha_{[2i]} = \beta_{[2i]}$ for all $1 \leq i \leq r$.

Proof. The forward implication is immediate. Conversely, assume that $\alpha_{[2i]} = \beta_{[2i]}$ for all $1 \leq i \leq r$. The fact that $\alpha_{[2i+1]} = \beta_{[2i+1]}$ for all $1 \leq i \leq r-1$ follows from Lemma 5.3. Thus, we have $\alpha_{[2,2r]} = \beta_{[2,2r]}$. Applying Lemma 5.2 then yields the desired conclusion. \square

The following notation will be useful in the next lemma. If α is a link such that $[2i-1, 2i+1] \in \mathcal{S}(\alpha)$, then let $b_{[2i-1, 2i+1]}(\alpha)$ denote the link obtained from α by applying the braid move in positions $[2i-1, 2i+1]$. We certainly have $b_{[2i-1, 2i+1]}(\alpha) \in [\alpha]$ with $d(b_{[2i-1, 2i+1]}(\alpha), \alpha) = 1$. Suppose α is a link of rank $r \geq 1$. Given any $\gamma \in [\alpha]$, we associate subsets $X_\gamma, Y_\gamma \subseteq [\alpha]$ defined as follows:

$$X_\gamma := \{\beta \in [\alpha] \mid \text{supp}_{[2r]}(\beta) = \text{supp}_{[2r]}(\gamma)\} \quad \text{and} \quad Y_\gamma := \{\beta \in [\alpha] \mid \text{supp}_{[2r]}(\beta) \neq \text{supp}_{[2r]}(\gamma)\}.$$

The set X_γ is simply the collection of links in $[\alpha]$ that share the same letter as γ in the second position from the right while Y_γ is the complement of X_γ relative to $[\alpha]$. In particular, if $\text{supp}_{[2r]}([\alpha]) = \{s, t\}$ and $\text{supp}_{[2r]}(\gamma) = \{s\}$, then every link in X_γ has s in position $2r$ while every link in Y_γ has t in position $2r$. It follows from Lemma 5.2 that if $\beta \in X_\gamma$, then $\beta_{[2r, 2r+1]} = \gamma_{[2r, 2r+1]}$. That is, every pair of links in X_γ agree on the last two letters. Similarly, every pair of links in Y_γ agree on the last two letters.

Example 5.7. Consider the links presented in Parts (a) and (c) of Example 2.6. We see that $X_{\alpha_3} = \{\alpha_1, \alpha_2, \alpha_3\}$ and $Y_{\alpha_3} = \{\alpha_4\}$ while $X_{\gamma_4} = \{\gamma_3, \gamma_4, \gamma_5\}$ and $Y_{\gamma_4} = \{\gamma_1, \gamma_2\}$.

Lemma 5.8. Suppose (W, S) is a simply-laced triangle-free Coxeter system and α is a link of rank $r \geq 2$. Choose $\sigma \in [\alpha]$ such that $[2r-3, 2r-1], [2r-1, 2r+1] \in \mathcal{S}(\sigma)$ according to Lemma 5.4 and let X_σ and Y_σ be defined as above. Then

- (a) $\{X_\sigma, Y_\sigma\}$ is a partition of $[\alpha]$.
- (b) If $\beta \in X_\sigma$, then $\tilde{\beta} \in [\hat{\sigma}]$. Moreover, every element of $[\hat{\sigma}]$ is of the form $\tilde{\beta}$ for some $\beta \in X_\sigma$.
- (c) If $\beta \in Y_\sigma$, then $[2r-1, 2r+1] \in \mathcal{S}(\beta)$ and $(b_{[2r-1, 2r+1]}(\beta))_{[1, 2r-1]} \in [\hat{\sigma}]$.

Proof. Certainly, X_σ and Y_σ are disjoint and $X_\sigma \cup Y_\sigma = [\alpha]$. Moreover, X_σ is nonempty since $\sigma \in X_\sigma$. Applying the braid move occurring in the rightmost braid shadow in σ results in a link in Y_σ , and so Y_σ is also nonempty. Thus, $\{X_\sigma, Y_\sigma\}$ is a partition of $[\alpha]$, which verifies Part (a). Part (b) follows immediately from Lemma 5.5.

Using Lemma 5.4, we can write $\sigma_{[2r-3, 2r+1]} = tstut$, where $\text{supp}_{[2r-2]}([\alpha]) = \{s, t\}$ and $\text{supp}_{[2r]}([\alpha]) = \{t, u\}$ with $m(s, t) = 3 = m(t, u)$. Let $\beta \in Y_\sigma$, so that $\beta_{[2r, 2r+1]} = tu$ by Lemma 5.2. By applying Lemma 5.3 with $i = r-1$, we have $\text{supp}_{[2r-1]}([\alpha]) = \{s, t, u\}$. By considering all possibilities for $\beta_{[2r-2, 2r+1]}$ and using the fact that β is reduced while $[2r-2, 2r] \notin \mathcal{S}([\alpha])$, we can conclude that $\beta_{[2r-2, 2r+1]} = sutu$. Since $m(t, u) = 3$, it follows that $[2r-1, 2r+1] \in \mathcal{S}(\beta)$. But then $b_{[2r-1, 2r+1]}(\beta) \in X_\sigma$, and hence $(b_{[2r-1, 2r+1]}(\beta))_{[1, 2r-1]} \in [\hat{\sigma}]$ by Part (b). This verifies Part (c). \square

Suppose α is a link of rank $r \geq 2$ and choose $\sigma \in [\alpha]$ such that $[2r-3, 2r-1], [2r-1, 2r+1] \in \mathcal{S}(\sigma)$ according to Lemma 5.4. In light of Part (c) of Lemma 5.8, for $\beta \in Y_\sigma$, define $\tilde{\beta} := (b_{[2r-1, 2r+1]}(\beta))_{[1, 2r-1]}$. That is, for $\beta \in Y_\sigma$, $\tilde{\beta}$ is the link we obtain in $[\hat{\sigma}]$ by first applying the braid move available in the rightmost braid shadow of β and then deleting the two rightmost letters of the resulting expression.

An important corollary of the preceding lemma is that the braid graph for a link α is obtained by gluing together two induced subgraphs, one of which is the image of an isometric embedding of $B(\hat{\sigma})$ into $B(\alpha)$. In order to state this precisely, we need the following definition.

Definition 5.9. Suppose (W, S) is a simply-laced triangle-free Coxeter system and α is a link of rank $r \geq 2$. Choose $\sigma \in [\alpha]$ such that $[2r-3, 2r-1], [2r-1, 2r+1] \in \mathcal{S}(\sigma)$ according to Lemma 5.4, and suppose that $\sigma_{[2r-3, 2r+1]} = tstut$, where $m(s, t) = 3 = m(t, u)$. Let $\Omega_\alpha^\sigma : [\hat{\sigma}] \rightarrow [\alpha]$ be the function that conjoins the letters ut on the right of each element of $[\hat{\sigma}]$.

Note that the function Ω_α^σ is well defined by Lemma 5.8.

Corollary 5.10. Suppose (W, S) is a simply-laced triangle-free Coxeter system and α is a link of rank $r \geq 2$ and assume the notation of Definition 5.9. Then

- (a) The function $\Omega_\alpha^\sigma : [\hat{\sigma}] \rightarrow [\alpha]$ is an isometric embedding from $B(\hat{\sigma})$ to $B(\alpha)$. In particular, $\text{im}(\Omega_\alpha^\sigma) = X_\sigma$ and $B(\hat{\sigma})$ is isomorphic to the induced subgraph $B(\alpha)[X_\sigma]$.
- (b) The induced subgraph $B(\alpha)[Y_\sigma]$ is an isometric subgraph of $B(\alpha)$.

(c) If $\beta \in X_\sigma$ and $\gamma \in Y_\sigma$, then $d_{B(\alpha)}(\beta, \gamma) = d_{B(\alpha)}(\beta, b_{[2r-1, 2r+1]}(\gamma)) + 1$.

Proof. Using Lemma 5.8, choose $\hat{\beta}, \hat{\gamma} \in [\hat{\sigma}]$ for $\beta, \gamma \in [\alpha]$. Then clearly

$$d_{B(\hat{\sigma})}(\hat{\beta}, \hat{\gamma}) = d_{B(\alpha)}(\hat{\beta}ut, \hat{\gamma}ut) = d_{B(\alpha)}(\Omega_\alpha^\sigma(\hat{\beta}), \Omega_\alpha^\sigma(\hat{\gamma})).$$

The statement $\text{im}(\Omega_\alpha^\sigma) = X_\sigma$ follows immediately from Lemma 5.8(b). The image of an isometric embedding is the subgraph induced by the image, and so $B(\hat{\sigma}) \cong B(\alpha)[X_\sigma]$. This verifies Part (a).

Now, let $\beta, \gamma \in Y_\sigma$. Any minimal sequence of braid moves that transforms β into γ does not involve the braid shadow $[2r-1, 2r+1]$. Therefore, every shortest path between β and γ is contained in $B(\alpha)[Y_\sigma]$. Thus, $B(\alpha)[Y_\sigma]$ is convex and hence an isometric subgraph of $B(\alpha)$. This proves Part (b).

Finally, let $\beta \in X_\sigma$ and $\gamma \in Y_\sigma$ and choose a minimal sequence of braid moves b_1, b_2, \dots, b_k that transforms β into $b_{[2r-1, 2r+1]}(\gamma)$. Note that the latter element is a member of X_σ by Lemma 5.8(c). Then $b_1, b_2, \dots, b_k, b_{[2r-1, 2r+1]}$ is a minimal sequence of braid moves that transforms β into γ , and hence

$$d_{B(\alpha)}(\beta, \gamma) = k + 1 = d_{B(\alpha)}(\beta, b_{[2r-1, 2r+1]}(\gamma)) + 1.$$

This completes the proof of Part (c). \square

Corollary 5.10 explicitly states that the induced subgraph $B(\alpha)[X_\sigma]$ is the braid graph for some link. We conjecture that the induced subgraph $B(\alpha)[Y_\sigma]$ is also the braid graph for some link. We prove this conjecture for Fibonacci links in Section 6.

Example 5.11. Consider the link $\alpha = 32313435464$ in the simply-laced triangle-free Coxeter system of type \tilde{D}_5 , whose Coxeter graph is shown in Figure 1. One possible choice for a link satisfying Definition 5.9 is $\sigma = 32314345464$. The braid graph for α is given in Figure 7. We have highlighted $B(\alpha)[X_\sigma]$ and $B(\alpha)[Y_\sigma]$ in green and magenta, respectively. In this case, $\hat{\sigma} = 323143454$, and in agreement with Corollary 5.10, we have $B(\hat{\sigma}) \cong B(\alpha)[X_\sigma]$. We encountered the braid graph for $B(\hat{\sigma})$ in Example 2.6(d). Moreover, we see that $B(\alpha)[Y_\sigma]$ happens to be isomorphic to the braid graph for 3231343, which is the link we obtain by deleting the last four letters from $b_{[9, 11]}(\alpha) = 32313435646$. This is the braid graph for γ_4 that we saw in Example 2.6(c). Each of the edges joining $B(\alpha)[X_\sigma]$ and $B(\alpha)[Y_\sigma]$ correspond to the braid move applied in the rightmost braid shadow.

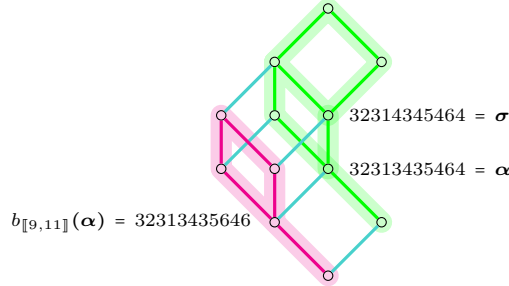


FIGURE 7. Braid graph for the reduced expression in Example 5.11 together with a partition of the vertices according to Lemma 5.8.

Recall that the crux of this section is to prove that every link can be isometrically embedded in a hypercube whose dimension is at most the rank of the link. We utilize the following definition for this purpose.

Definition 5.12. Suppose (W, S) is a simply-laced triangle-free Coxeter system and let α be a link of rank $r \geq 0$. For $r \geq 1$, define $\Phi_\alpha : [\alpha] \rightarrow \{0, 1\}^r$ via $\Phi_\alpha(\beta) = a_1 a_2 \dots a_r$, where

$$a_k = \begin{cases} 0, & \text{if } \text{supp}_{[2k]}(\beta) = \text{supp}_{[2k]}(\alpha) \\ 1, & \text{if } \text{supp}_{[2k]}(\beta) \neq \text{supp}_{[2k]}(\alpha). \end{cases}$$

In the case that $r = 0$, the unique element (consisting of a single letter) in the braid class is mapped to the empty binary string.

It is worth pointing out that the definition of the map Φ_α depends on the choice of link. Choosing a different representative of $[\alpha]$ will necessarily result in a different mapping. However, any two such embeddings differ only by an automorphism of the hypercube, cf. Lemma 5.13.

Lemma 5.13. Suppose (W, S) is a simply-laced triangle-free Coxeter system. If α and β are braid equivalent links of rank $r \geq 0$, then there is an automorphism $F : \{0, 1\}^r \rightarrow \{0, 1\}^r$ of the hypercube Q_r satisfying $F \circ \Phi_\alpha = \Phi_\beta$. In particular, Φ_α is an isometric embedding if and only if Φ_β is also an isometric embedding.

Proof. Define $F : \{0, 1\}^r \rightarrow \{0, 1\}^r$ via $F(\mathbf{a}) = \mathbf{a} + \Phi_\beta(\alpha)$, where $+$ denotes the bitwise XOR operation. It is clear that F is an automorphism of Q_r . Now, let $\gamma \in [\alpha]$. We need to prove that $\Phi_\beta(\gamma) = \Phi_\alpha(\gamma) + \Phi_\beta(\alpha)$. Write $\Phi_\beta(\gamma) = a_1 a_2 \dots a_r$, $\Phi_\alpha(\gamma) = b_1 b_2 \dots b_r$, and $\Phi_\beta(\alpha) = c_1 c_2 \dots c_r$. By definition of the bitwise XOR operation, it suffices to show that $b_i + c_i \equiv a_i \pmod{2}$ for all $i \in \{1, \dots, r\}$. We proceed on a case-by-case basis.

First, suppose $b_i + c_i \equiv 0 \pmod{2}$. Then $b_i = c_i$. If $b_i = 0 = c_i$, then $\text{supp}_{[2i]}(\gamma) = \text{supp}_{[2i]}(\alpha)$ and $\text{supp}_{[2i]}(\alpha) = \text{supp}_{[2i]}(\beta)$ by definitions of Φ_α and Φ_β , respectively. Hence $\text{supp}_{[2i]}(\gamma) = \text{supp}_{[2i]}(\beta)$, so that $a_i = 0$, as well. On the other hand, if $b_i = 1 = c_i$, then $\text{supp}_{[2i]}(\gamma) \neq \text{supp}_{[2i]}(\alpha)$ and $\text{supp}_{[2i]}(\alpha) \neq \text{supp}_{[2i]}(\beta)$. It follows that $\text{supp}_{[2i]}(\gamma) = \text{supp}_{[2i]}(\beta)$, and so $a_i = 0$.

Next, suppose that $b_i + c_i \equiv 1 \pmod{2}$, so that $b_i \neq c_i$. If $b_i = 0$ while $c_i = 1$, then $\text{supp}_{[2i]}(\gamma) = \text{supp}_{[2i]}(\alpha)$ while $\text{supp}_{[2i]}(\alpha) \neq \text{supp}_{[2i]}(\beta)$. This implies that $\text{supp}_{[2i]}(\gamma) \neq \text{supp}_{[2i]}(\beta)$, and so $a_i = 1$. Similarly, if $b_i = 1$ while $c_i = 0$, then $\text{supp}_{[2i]}(\gamma) \neq \text{supp}_{[2i]}(\alpha)$ while $\text{supp}_{[2i]}(\alpha) = \text{supp}_{[2i]}(\beta)$. Thus, $\text{supp}_{[2i]}(\gamma) \neq \text{supp}_{[2i]}(\beta)$, so that $a_i = 1$.

This proves that F is an automorphism of Q_r having the desired property. The second claim follows immediately. \square

We now prove the following proposition, which shows that the braid graph for every link is a partial cube with isometric dimension at most the rank of the link.

Proposition 5.14. If (W, S) is a simply-laced triangle-free Coxeter system and α is a link of rank $r \geq 0$, then the map Φ_α given in Definition 5.12 is an isometric embedding of $B(\alpha)$ into Q_r . In particular, the braid graph for a link is a partial cube with isometric dimension at most r .

Proof. We proceed by induction on r . The result is immediate if $r \in \{0, 1\}$. Now, suppose that $r \geq 2$ and assume that the map $\Phi_\beta : [\beta] \rightarrow \{0, 1\}^{r-1}$ is an isometric embedding of $B(\beta)$ into Q_{r-1} for every link β of rank $r - 1$. Let α be a link of rank r . Choose $\sigma \in [\alpha]$ such that $[2r - 3, 2r - 1], [2r - 1, 2r + 1] \in \mathcal{S}(\sigma)$ according to Lemma 5.4. Then by Lemma 5.5, $\hat{\sigma}$ is a link of rank $r - 1$. Applying the inductive hypothesis allows us to conclude that $\Phi_{\hat{\sigma}} : [\hat{\sigma}] \rightarrow \{0, 1\}^{r-1}$ is an isometric embedding of $B(\hat{\sigma})$ into Q_{r-1} . Now, utilizing Lemma 5.8, define $\Phi : [\alpha] \rightarrow \{0, 1\}^r$ via

$$\Phi(\beta) := \begin{cases} \Phi_{\hat{\sigma}}(\hat{\beta})0, & \text{if } \beta \in X_\sigma \\ \Phi_{\hat{\sigma}}(\hat{\beta})1, & \text{if } \beta \in Y_\sigma. \end{cases}$$

Note that the map Φ is appending a 0 or 1 to the appropriate binary string depending on whether the input is in X_σ or Y_σ , respectively. It is not hard to see that $\Phi = \Phi_\sigma$. According to Lemma 5.13, it suffices to prove that Φ is an isometric embedding. Suppose $\text{supp}_{[2r]}([\alpha]) = \{s, t\}$ and $\sigma_{[2r]} = s$, where $m(s, t) = 3$. There are three cases to consider.

First, suppose $\beta, \gamma \in X_\sigma$. Then $\beta = \hat{\beta}st$ and $\gamma = \hat{\gamma}st$ according to Lemma 5.2(b). Then we have

$$\begin{aligned} d_{B(\alpha)}(\beta, \gamma) &= d_{B(\alpha)}(\hat{\beta}st, \hat{\gamma}st) \\ &= d_{B(\hat{\sigma})}(\hat{\beta}, \hat{\gamma}) && \text{(Corollary 5.10(a))} \\ &= d_{Q_{r-1}}(\Phi_{\hat{\sigma}}(\hat{\beta}), \Phi_{\hat{\sigma}}(\hat{\gamma})) && \text{(inductive hypothesis)} \\ &= d_{Q_r}(\Phi_{\hat{\sigma}}(\hat{\beta})0, \Phi_{\hat{\sigma}}(\hat{\gamma})0) \\ &= d_{Q_r}(\Phi_\sigma(\beta), \Phi_\sigma(\gamma)). \end{aligned}$$

The case where $\beta, \gamma \in Y_\sigma$ is similar, except Corollary 5.10(b) is used instead of Corollary 5.10(a) and we append a 1 in place of a 0. Lastly, suppose that $\beta \in X_\sigma$ while $\gamma \in Y_\sigma$. Then $\beta = \hat{\beta}st$ and $b_{[2r-1, 2r+1]}(\gamma) = \tilde{\gamma}st$.

Thus, we have

$$\begin{aligned}
 d_{B(\alpha)}(\beta, \gamma) &= d_{B(\alpha)}(\beta, b_{\llbracket 2r-1, 2r+1 \rrbracket}(\gamma)) + 1 && \text{(Corollary 5.10(c))} \\
 &= d_{B(\alpha)}(\hat{\beta}_{st}, \hat{\gamma}_{st}) + 1 \\
 &= d_{B(\hat{\sigma})}(\hat{\beta}, \hat{\gamma}) + 1 && \text{(Corollary 5.10(a))} \\
 &= d_{Q_{r-1}}(\Phi_{\hat{\sigma}}(\hat{\beta}), \Phi_{\hat{\sigma}}(\hat{\gamma})) + 1 && \text{(inductive hypothesis)} \\
 &= d_{Q_r}(\Phi_{\hat{\sigma}}(\hat{\beta})0, \Phi_{\hat{\sigma}}(\hat{\gamma})1) \\
 &= d_{Q_r}(\Phi_{\sigma}(\beta), \Phi_{\sigma}(\gamma)).
 \end{aligned}$$

This verifies that Φ is an isometric embedding. It follows that $B(\alpha)$ is a partial cube with isometric dimension at most $\text{rank}(\alpha)$. \square

Example 5.15. Consider the reduced expressions $\gamma_1, \dots, \gamma_5$ given in Example 2.6(c). The braid graph $B(\gamma_1)$ was shown in Figure 3(c). In Example 3.7, we showed that γ_1 is a link of rank 3. Proposition 5.14 guarantees that there are at least five distinct isometric embeddings of $B(\gamma_1)$ into Q_3 , one for each of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, and γ_5 . One possible embedding, using γ_4 , is shown in Figure 8. It is clear in this example that $\dim_I(B(\gamma_1)) = 3 = \text{rank}(\gamma_1)$ since Q_2 has only four vertices.

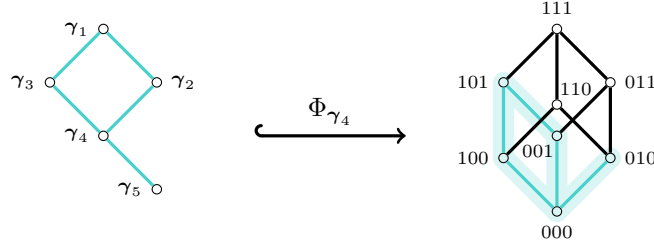


FIGURE 8. An isometric embedding of $B(\gamma_1)$ into Q_3 for Example 5.15.

We make the following conjecture regarding the isometric dimension of the braid graph for a link.

Conjecture 5.16. If (W, S) is a simply-laced triangle-free Coxeter system and α is a link, then $\dim_I(B(\alpha)) = \text{rank}(\alpha)$.

Given any reduced expression in a simply-laced triangle-free Coxeter system, we can apply Corollary 3.10, Proposition 5.14, and Proposition 5.1 to immediately obtain the following theorem, which is the main result of this paper. Note that one can acquire the appropriate labeling in terms of binary strings by applying Proposition 5.14 to each link factor and then concatenating the resulting binary strings.

Theorem 5.17. If (W, S) is a simply-laced triangle-free Coxeter system and α is a reduced expression for some $w \in W$, then there is an isometric embedding of $B(\alpha)$ into $Q_{\text{rank}(\alpha)}$. In particular, $B(\alpha)$ is a partial cube with isometric dimension at most $\text{rank}(\alpha)$.

Since the number of vertices in $Q_{\text{rank}(\alpha)}$ is $2^{\text{rank}(\alpha)}$, we obtain the follow corollary.

Corollary 5.18. If (W, S) is a simply-laced triangle-free Coxeter system and α is a reduced expression for some $w \in W$, then $\text{card}([\alpha]) \leq 2^{\text{rank}(\alpha)}$. Moreover, if α is a reduced expression with link factorization $\alpha_1 \mid \alpha_2 \mid \dots \mid \alpha_k$, then the bound is achieved precisely when $\text{rank}(\alpha_i) \leq 1$ for every i .

If Conjecture 5.16 is true, then we immediately obtain the following conjecture by applying Proposition 5.1.

Conjecture 5.19. If (W, S) is simply-laced triangle-free Coxeter system and α is a reduced expression with link factorization $\alpha_1 \mid \alpha_2 \mid \dots \mid \alpha_k$, then

$$\dim_I(B(\alpha)) = \sum_{i=1}^k \text{rank}(\alpha_i).$$

If (W, S) is a simply-laced Coxeter system and $w \in W$ such that $\ell(w) = m$, then the maximum number of braid shadows that any reduced expression for w can have is $\lfloor \frac{m-1}{2} \rfloor$. For simply-laced Coxeter systems, it is clear that as the length increases, so too does the number of possible braid shadows. For finite Coxeter groups, the length function is bounded above by the length of the longest element. It turns out that all of the finite simply-laced Coxeter systems are triangle free [12]. Moreover, every finite simply-laced Coxeter system is isomorphic to a direct product of some combination of Coxeter systems of types A_n ($n \geq 1$), D_n ($n \geq 4$), E_6 , E_7 , and E_8 . For each of these groups, we can utilize Corollary 5.18 to obtain an upper bound on the cardinality of any braid class in that group.

Since the length of the longest element in the Coxeter system of type A_n is $\frac{(n+1)n}{2}$, the maximum number of braid shadows that a reduced expression for an element in $W(A_n)$ could have is $\lfloor \frac{n^2+n-2}{4} \rfloor$. This bound is attained at least when $n = 1, 2, 3$. Nonetheless, Corollary 5.18 implies that if α is a reduced expression for some element in $W(A_n)$, then $\text{card}([\alpha]) \leq 2^{\lfloor \frac{n^2+n-2}{4} \rfloor}$. For example, if $n = 3$, then the cardinality of every braid class in $W(A_3)$ must be less than or equal to 4. However, the maximum size of a braid class in $W(A_3)$ is 3. As n increases, our bound deteriorates. In [23], Zollinger establishes sharp upper bounds on the cardinality of a braid class for a fixed length across all Coxeter systems of type A_n .

In the case of type D_n , the longest element has length $n^2 - n$, and so the maximum number of braid shadows that a reduced expression for an element in $W(D_n)$ could have is $\lfloor \frac{n^2-n-1}{2} \rfloor$. Thus, if α is a reduced expression for some element in $W(D_n)$, then $\text{card}([\alpha]) \leq 2^{\lfloor \frac{n^2-n-1}{2} \rfloor}$.

6. FIBONACCI LINKS AND FIBONACCI CUBES

We now introduce a special class of links with connections to the Fibonacci numbers. As usual, we define the Fibonacci numbers via $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$.

Definition 6.1. Suppose (W, S) is a simply-laced triangle-free Coxeter system. If φ is a link with the property that $\mathcal{S}([\varphi]) = \mathcal{S}(\varphi)$, then we will refer to φ as a *Fibonacci link* and the corresponding braid class $[\varphi]$ will be referred to as a *Fibonacci chain*.

By invoking Lemma 5.5, we see that if φ is a Fibonacci link of rank $r \geq 2$, then $\hat{\varphi}$ and $\hat{\hat{\varphi}}$ are Fibonacci links of ranks $r - 1$ and $r - 2$, respectively. The following proposition describes the connection between a Fibonacci chain and the Coxeter graph. Recall that the *star graph* S_k with one internal node and k leaves is defined to be the complete bipartite graph $K_{1,k}$.

Proposition 6.2. Suppose (W, S) is a simply-laced triangle-free Coxeter system of type Γ and let α be a link of rank at least one. Then α is braid equivalent to a Fibonacci link if and only if the induced subgraph $\Gamma[\text{supp}(\alpha)]$ is a star graph.

Proof. Suppose that $\varphi \in [\alpha]$ is a Fibonacci link and write $\varphi = s_{x_1} s_{x_2} \cdots s_{x_{2r+1}}$. By definition, $\llbracket 2i - 1, 2i + 1 \rrbracket \in \mathcal{S}(\varphi)$ for all $i \in \{1, 2, \dots, r\}$, which implies that the letters in each factor $\varphi_{\llbracket 2i-1, 2i+1 \rrbracket}$ satisfy $s_{x_{2i-1}} = s_{x_{2i+1}}$ and $m(s_{x_{2i-1}}, s_{x_{2i}}) = 3 = m(s_{x_{2i}}, s_{x_{2i+1}})$. If we write $s := s_{x_1}$, then we have $s = s_{x_{2i+1}}$ for all $i \in \{1, 2, \dots, r\}$, so that $\text{supp}(\alpha) = \text{supp}(\varphi) = \{s, s_{x_2}, s_{x_4}, \dots, s_{x_{2r}}\}$, possibly with repeats among $s_{x_2}, s_{x_4}, \dots, s_{x_{2r}}$. Notice that $m(s_{x_{2k}}, s_{x_{2j}}) = 2$ whenever $s_{x_{2k}} \neq s_{x_{2j}}$ since $m(s, s_{x_{2k}}) = 3 = m(s, s_{x_{2j}})$ and Γ has no three-cycles. Thus, the subgraph of Γ induced by $\text{supp}(\alpha)$ is a star graph with $\text{card}(\text{supp}(\alpha)) - 1 \leq r$ leaves.

We will prove the converse by induction. If α is a link of rank 1, then $\alpha = sts$ for some $s, t \in S$ with $m(s, t) = 3$, in which case, α is a Fibonacci link. Now, assume that the statement holds for all links of rank $r \geq 1$ and let α be a link of rank $r + 1$ such that the induced subgraph $\Gamma[\text{supp}(\alpha)]$ is a star graph. Then we may assume that $\text{supp}(\alpha) = \{s, t_1, t_2, \dots, t_k\}$ for some k , where $m(s, t_i) = 3$ for all i and $m(t_i, t_j) = 2$ for all $i \neq j$. According to Lemma 5.4, choose a link $\sigma \in [\alpha]$ with the property that $\llbracket 2r - 1, 2r + 1 \rrbracket, \llbracket 2r + 1, 2r + 3 \rrbracket \in \mathcal{S}(\sigma)$. Suppose $\sigma_{\llbracket 2r-1, 2r+1 \rrbracket} = st_k s$. By Lemma 5.5, $\hat{\sigma}$ is a link of rank r with $\text{supp}(\hat{\sigma}) \subseteq \text{supp}(\sigma)$, so that $\Gamma[\text{supp}(\hat{\sigma})]$ is also a star graph. By the inductive hypothesis, there exists a Fibonacci link $\psi \in [\hat{\sigma}]$. But then by Lemma 5.8(b), there exists $\varphi \in X_\sigma$ such that $\hat{\varphi} = \psi$. This implies that $\varphi = \psi t_k s$, and since ψ is a Fibonacci link ending in s , it follows that φ is a Fibonacci link braid equivalent to α . \square

Implicit in the proof of the preceding proposition is the fact that every Fibonacci link of rank $r \geq 1$ can be written in the form

$$\varphi = st_1 st_2 \cdots st_{r-1} st_r s$$

for some $s, t_1, t_2, \dots, t_r \in S$, possibly with repeats among t_1, t_2, \dots, t_r , where $m(s, t_i) = 3$ for all $i \in \{1, 2, \dots, r\}$ and $m(t_i, t_j) = 2$ for $t_i \neq t_j$. We will use this fact frequently in what follows.

Example 6.3. Not every link is a Fibonacci link. For instance, in the Coxeter system of type A_4 , the reduced expression 1213243 is a link, but not a Fibonacci link. Similarly, in the Coxeter system of type D_5 , the reduced expression 4534313234313 is a link, but it is not a Fibonacci link. Moreover, neither link is braid equivalent to a Fibonacci link according to Proposition 6.2.

Example 6.4. In the Coxeter system of type A_n , there are no Fibonacci links of rank 3 or larger. This follows from Proposition 4.3 or Lemma 1 from [23]. The Fibonacci links of rank 0 are precisely the reduced expressions consisting of a single generator. The Fibonacci links of ranks 1 and 2 are of the form sts and $tstut$, respectively, where $m(s, t) = 3 = m(t, u)$ and $m(s, u) = 2$.

Example 6.5. It is more difficult to write down all the Fibonacci links in the Coxeter system of type D_n . The following reduced expressions are Fibonacci links of ranks 1 through 5:

$$343, 34313, 3431323, 343132343, 34313234313.$$

In fact, it is not possible to find a Fibonacci link of rank greater than 5 in type D_n . This fact is obvious in type D_4 since the longest element has length 12, but remains true even for large n . Notice that Proposition 6.2 implies that every link in type D_4 is braid equivalent to a Fibonacci link since the Coxeter graph of type D_4 is a star graph.

In his MS thesis [4], Cadman proved that Fibonacci links of arbitrary rank occur in any simply-laced triangle-free Coxeter system whose corresponding Coxeter graph contains the Coxeter graph of type \tilde{D}_4 as a subgraph. The crux of Cadman's construction is proving that the proposed class of expressions is reduced. His argument relied heavily on technical results involving root sequences.

The following definition is similar to Definition 5.9.

Definition 6.6. Suppose (W, S) is a simply-laced triangle-free Coxeter system and $\varphi = st_1st_2 \cdots st_{r-1}st_rs$ is a Fibonacci link of rank $r \geq 2$. Denote by $\Sigma : [\hat{\varphi}] \rightarrow [\varphi]$ the function that conjoins to each element of $[\hat{\varphi}]$ the letters $t_{r-1}t_rst_r$ on the right.

Note that the function $\Sigma : [\hat{\varphi}] \rightarrow [\varphi]$ is well defined by two applications of Lemma 5.8. Each Fibonacci link carries with it rich recursive properties that manifest in the structure of the braid class and the corresponding braid graph. The next result summarizes these properties.

Proposition 6.7. If (W, S) is a simply-laced triangle-free Coxeter system and $\varphi = st_1st_2 \cdots st_{r-1}st_rs$ is a Fibonacci link of rank $r \geq 2$, then the function $\Sigma : [\hat{\varphi}] \rightarrow [\varphi]$ is an isometric embedding from $B(\hat{\varphi})$ into $B(\varphi)$. In particular, $\text{im}(\Sigma) = Y_\varphi$ and $B(\hat{\varphi})$ is isomorphic to the induced subgraph $B(\varphi)[Y_\varphi]$.

Proof. Using Corollary 5.10(b), it suffices to prove that $\text{im}(\Sigma) = Y_\varphi$. It is clear from the definitions that $\text{im}(\Sigma) \subseteq Y_\varphi$. Let $\beta \in Y_\varphi$. Then $\llbracket 2r-1, 2r+1 \rrbracket \in S(\beta)$ and $\tilde{\beta} \in [\hat{\varphi}]$ according to Lemma 5.8(c). Moreover, the proof of Lemma 5.8(c) shows that $\beta_{\llbracket 2r-2 \rrbracket} = \varphi_{\llbracket 2r-2 \rrbracket}$ so that $\tilde{\beta} \in X_{\hat{\varphi}}$. Hence, $\tilde{\beta} \in [\hat{\varphi}]$. But $\tilde{\beta}$ is obtained from β by deleting the rightmost four letters so that $\Sigma(\tilde{\beta}) = \beta$. \square

The preceding proposition is a strengthening of Corollary 5.10(b) for Fibonacci links. By combining Corollary 5.10 with Proposition 6.7, we can conclude that the braid graph for a Fibonacci link φ of rank $r \geq 2$ is, in some sense, a gluing together of the braid graphs for a Fibonacci link of rank $r-1$ and a Fibonacci link of rank $r-2$. The gluing of these two isometric subgraphs corresponds to applying the braid move in the braid shadow $\llbracket 2r-1, 2r+1 \rrbracket$ to any element of Y_φ . A concrete example is given below.

Example 6.8. Consider the Coxeter system of type D_4 . The reduced expression $\varphi = 343132343$ is a Fibonacci link of rank 4. The partition of $[\varphi]$ from Lemma 5.8 is given by

$$X_\varphi = \{343132343, 434132343, 434123243, 341312343, 343123243\}$$

and

$$Y_\varphi = \{343132434, 434132434, 341312434\}.$$

The first block has $F_5 = 5$ elements and is in bijection with $[\hat{\varphi}]$, while the second block has $F_4 = 3$ elements and is in bijection with $[\hat{\varphi}]$. The braid graph for φ is shown in Figure 9. The isometric subgraphs $B(\varphi)[X_\varphi]$

and $B(\varphi)[Y_\varphi]$ are highlighted in green and magenta, and are isomorphic to $B(\hat{\varphi})$ and $B(\hat{\hat{\varphi}})$, respectively. Similar to Example 5.11, each of the edges joining $B(\varphi)[X_\varphi]$ and $B(\varphi)[Y_\varphi]$ correspond to the braid move applied in the rightmost braid shadow.

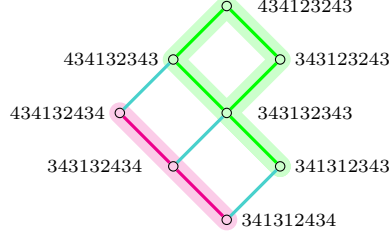


FIGURE 9. Braid graph for the Fibonacci link in Example 6.8 together with a partition of the vertices according to Lemma 5.8.

Our next objective is to prove that Fibonacci chains contain a Fibonacci number of links.

Proposition 6.9. Suppose (W, S) is a simply-laced triangle-free Coxeter system. If φ is a Fibonacci link of rank $r \geq 0$, then $\text{card}([\varphi]) = F_{r+2}$.

Proof. The proof is by induction on r . The case with $r = 0$ is immediate. If $r = 1$, then $\varphi = sts$ for some $s, t \in S$ with $m(s, t) = 3$. In this case, $\text{card}([\varphi]) = \text{card}(\{sts, tst\}) = 2 = F_3$. Now, let $r \geq 1$ and assume that every Fibonacci link of rank $k \leq r$ has F_{k+2} elements in its braid class. Let φ be a Fibonacci link of rank $r + 1$. Induction together with Lemma 5.8(a) and Proposition 6.7 implies that

$$\text{card}([\varphi]) = \text{card}([\hat{\varphi}]) + \text{card}([\hat{\hat{\varphi}}]) = F_{r+1} + F_{r+2} = F_{r+3},$$

which proves the claim. \square

Next, we show that the braid graph for a Fibonacci link of rank r is isomorphic to a well-known isometric subgraph of Q_r . We define the *Fibonacci cube* of order r as the subgraph $\mathcal{F}_r := Q_r[V_r]$ induced by set of vertices

$$V_r = \{a_1 a_2 \cdots a_r \in \{0, 1\}^r \mid a_i a_{i+1} = 0, 1 \leq i \leq r-1\}.$$

That is, V_r is the collection of length r binary strings that do not contain the consecutive substring 11. Fibonacci cubes have been studied extensively in the literature and were introduced as a model for interconnection networks [5, 11]. According to [14], the Fibonacci cube \mathcal{F}_r is a partial cube consisting of F_{r+2} many vertices.

Theorem 6.10. Suppose (W, S) is a simply-laced triangle-free Coxeter system. If φ is a Fibonacci link of rank r , then the braid graph $B(\varphi)$ is isomorphic to the Fibonacci cube \mathcal{F}_r .

Proof. According to Proposition 5.14, the map $\Phi_\varphi : [\varphi] \rightarrow \{0, 1\}^r$ is an isometric embedding of $B(\varphi)$ into Q_r . Thus, it suffices to show that $\text{im}(\Phi_\varphi) = V_r$. Using Proposition 6.9, we already know that $\text{card}([\varphi]) = F_{r+2} = \text{card}(V_r)$. Therefore, we only need to show that $\text{im}(\Phi_\varphi) \subseteq V_r$. We proceed by induction on r . The claim is trivial if $r = 0$ or $r = 1$. Suppose that $r \geq 2$ and assume the statement is true for every Fibonacci link of rank $k \leq r$. Let φ be a link of rank $r + 1$ and let $a_1 a_2 \cdots a_r a_{r+1} \in \text{im}(\Phi_\varphi)$. Choose $\beta \in [\varphi]$ such that $\Phi_\varphi(\beta) = a_1 a_2 \cdots a_r a_{r+1}$. Recall from the proof of Proposition 5.14 that $\Phi_\varphi(\beta) = \Phi_{\hat{\varphi}}(\hat{\beta})0$ if $\beta \in X_\varphi$ and $\Phi_\varphi(\beta) = \Phi_{\hat{\varphi}}(\hat{\beta})1$ if $\beta \in Y_\varphi$. In fact, in the latter case it is easy to see that $\text{supp}_{[2r-2]}(\beta) = \text{supp}_{[2r-2]}(\varphi)$ since $[2r-3, 2r-1], [2r-1, 2r+1] \in \mathcal{S}(\varphi)$ while β and φ differ by a single braid move in the braid shadow $[2r-1, 2r+1]$. Thus, $\Phi_\varphi(\beta) = \Phi_{\hat{\varphi}}(\hat{\beta})01$ if $\beta \in Y_\varphi$. Since $\hat{\varphi}$ and $\hat{\hat{\varphi}}$ are links of rank r and $r-1$, respectively, the inductive hypothesis implies that $\Phi_{\hat{\varphi}}(\hat{\beta}) \in V_r$ or $\Phi_{\hat{\varphi}}(\hat{\beta}) \in V_{r-1}$, depending on whether $\beta \in X_\varphi$ or $\beta \in Y_\varphi$. Then it is clear in either case that $\Phi_\varphi(\beta) \in V_{r+1}$ since either $\Phi_\varphi(\beta) = \Phi_{\hat{\varphi}}(\hat{\beta})0$ or $\Phi_\varphi(\beta) = \Phi_{\hat{\varphi}}(\hat{\beta})01$. This completes the proof. \square

The previous theorem together with [3, Proposition 6.1] implies that Conjecture 5.16 is settled for Fibonacci links.

Corollary 6.11. If (W, S) is a simply-laced triangle-free Coxeter system and φ is a Fibonacci link, then $\dim_I(B(\varphi)) = \text{rank}(\varphi)$.

According to [14, Corollary 4.2], a Fibonacci cube of dimension $r \geq 2$ has a unique vertex of degree r . Certainly this vertex attains the maximum degree in \mathcal{F}_r since every vertex of Q_r has degree r . If φ is a Fibonacci link of rank $r \geq 2$, it must be the case that the vertex corresponding to φ is the vertex that attains the maximum degree r . This yields the following corollary.

Corollary 6.12. If (W, S) is a simply-laced triangle-free Coxeter system and φ is a Fibonacci link of rank at least two, then φ is the unique Fibonacci link in the Fibonacci chain $[\varphi]$.

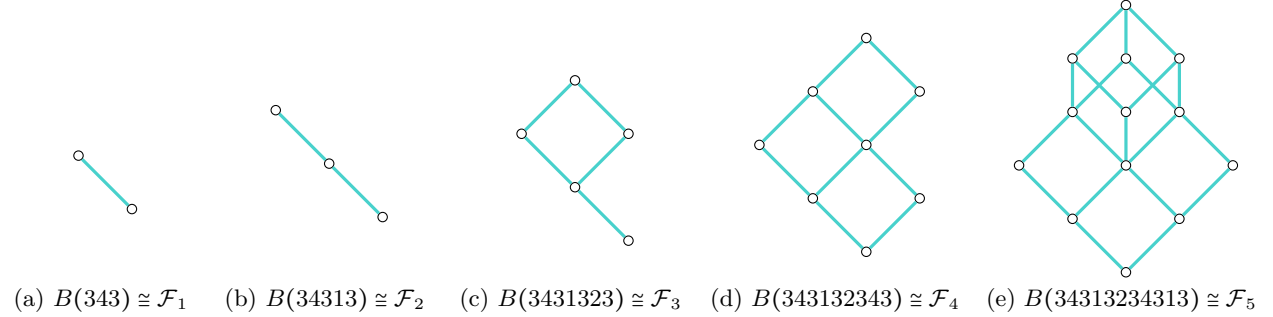


FIGURE 10. Braid graphs for several Fibonacci links in the Coxeter system of type D_4 .

Example 6.13. Consider the Fibonacci links in the Coxeter system of type D_4 from Example 6.4:

$$343, 34313, 3431323, 343132343, 34313234313.$$

According to Theorem 6.10, the corresponding braid graphs are Fibonacci cubes. Each braid graph is depicted in Figure 10. Note that for rank 2 and larger, the Fibonacci link always corresponds to the vertex of highest degree.

The Fibonacci cube is not the only interesting graph that arises as the braid graph for a reduced expression. For instance, the *matchable Lucas cubes* introduced in [21] can be found in the Coxeter system of type D_5 as seen in the following example. The matchable Lucas cubes are very similar to the Fibonacci cubes, for example, the number of vertices in the n th matchable Lucas cube is equal to the Lucas number L_n defined by $L_n = L_{n-1} + L_{n-2}$ with initial conditions $L_0 = 2$ and $L_1 = 1$.

Example 6.14. The braid graph for the reduced expression 4534313234313 in the Coxeter system of type D_5 from Example 6.3 is depicted in Figure 11. It turns out that this graph is isomorphic to the 6th matchable Lucas cube as introduced in [21]. In fact, it is not hard to see that the braid graph for 4534313234313 is a gluing together of the braid graphs for the reduced expressions 453431323 and 45343132343. The braid graph for 453431323 is highlighted in **magenta** and is isomorphic to the 4th matchable Lucas cube, while the braid graph for 45343132343 is highlighted in **green** and is isomorphic to the 5th matchable Lucas cube. Each of the edges joining the braid graphs highlighted in **magenta** and **green** correspond to the braid move applied in rightmost braid shadow. Compare this example with Examples 5.11 and 6.8.

7. CLOSING

There are several open problems and potential natural generalizations that arise from this paper that we outline below.

- (1) Theorem 5.17 (and its counterpart for links, Proposition 5.14) states that if (W, S) is a simply-laced triangle-free Coxeter system and α is a reduced expression for some $w \in W$, $B(\alpha)$ is a partial cube with isometric dimension at most $\text{rank}(\alpha)$. We conjecture that the isometric dimension is exactly $\text{rank}(\alpha)$ (see Conjectures 5.16 and 5.19).

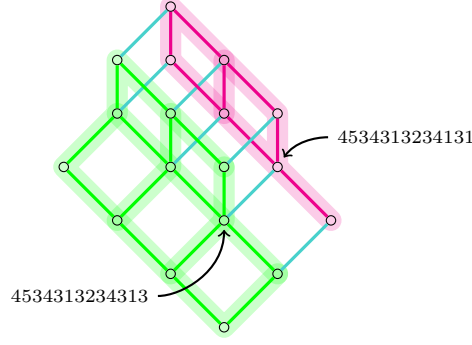


FIGURE 11. Braid graph for the reduced expression in Example 6.14.

- (2) There is an intimate connection between partial cubes and the so-called Djoković–Winkler relation on the collection of edges (see [18]). It turns out that since braid graphs for reduced expressions in simply-laced triangle-free Coxeter systems are partial cubes, the Djoković–Winkler relation on set of edges of the braid graph is an equivalence relation. We conjecture that the corresponding equivalence classes are the collections of edges that correspond to specific braid shadows. If this is true, it would settle the conjecture in the previous item.
- (3) Suppose (W, S) is a simply-laced triangle-free Coxeter system and α is a link of rank $r \geq 2$ and assume the notation of Definition 5.9. In Corollary 5.10, we state that the induced subgraph $B(\alpha)[Y_\sigma]$ is an isometric subgraph of $B(\alpha)$. In the spirit of Proposition 6.7, we conjecture that $B(\alpha)[Y_\sigma]$ is also a braid graph for some link derived from a link in Y_σ .
- (4) Can we strengthen the conclusion of Theorem 5.17? We conjecture that braid graphs in simply-laced triangle-free Coxeter systems are median graphs. Since Fibonacci cubes are median graphs (see [13, Theorem 1]), this conjecture is true for Fibonacci links. In fact, every braid graph exhibited in this paper is a median graph.
- (5) We conjecture that the conclusion of Theorem 5.17 holds for all simply-laced Coxeter systems, not just those that are triangle-free.
- (6) In Section 4, we provide a classification of braid graphs in Coxeter systems of type A_n . It would be interesting to provide an analogous classification of braid graphs in other simply-laced arbitrary Coxeter system (e.g., types D_n , \tilde{D}_n , and \tilde{A}_n).
- (7) Given a link in a simply-laced triangle-free Coxeter system, what is the image of the corresponding braid class under the map described in Definition 5.12?
- (8) Call a subset $X \subseteq \{0, 1\}^n$ *admissible* if there exists a simply-laced triangle-free Coxeter system (W, S) and reduced expression α for $w \in W$ such that the induced subgraph $Q_n[X]$ is isomorphic to $B(\alpha)$. Are there necessary and sufficient conditions for X to be admissible?
- (9) Extend our notion of braid shadow to non-simply-laced Coxeter systems and prove analogous results to those appearing in Sections 3, 5, and 6. For example, if one generalizes the notions of braid shadow and link in the natural way, we conjecture that a result analogous to Proposition 3.13 holds in arbitrary Coxeter systems as long as the corresponding Coxeter graph does not contain a three-cycle with edge weights $3, 3, m$, where $m \geq 3$.

ACKNOWLEDGEMENTS

We are extremely grateful to Hugh Denoncourt for providing valuable insight on this project and for assisting with implementing Python code, which we used to gather data and test conjectures. We are also grateful to the referees for several useful suggestions.

REFERENCES

- [1] R. Bedard. On Commutation Classes of Reduced Words in Weyl Groups. *European J. Combin.*, 20, 1999.
- [2] N. Bergeron, C. Ceballos, and J.P. Labbé. Fan realizations of subword complexes and multi-associahedra via Gale duality. *Discrete Comput. Geom.*, 54(1):195–231, 2015.
- [3] S. Cabello, D. Eppstein, and S. Klavžar. Structure of Fibonacci cubes: a survey. *Electron. J. Combin.*, 18(1), 2011.

- [4] Q. Cadman. *Structure of braid graphs in simply-laced Coxeter systems*. MS Thesis, Northern Arizona University, 2021.
- [5] B. Cong, S. Zheng, and S. Sharma. On simulations of linear arrays, rings and 2d meshes on Fibonacci cube networks. In *Proceedings of the 7th International Parallel Processing Symposium*, pages 747–751. 1993.
- [6] S. Elnitsky. Rhombic Tilings of Polygons and Classes of Reduced Words in Coxeter Groups. *J. Combin. Theory, Ser. A*, 77(2), 1997.
- [7] S. Fishel, E. Milićević, R. Patrias, and B.E. Tenner. Enumerations relating braid and commutation classes. *European J. Combin.*, 74, 2018.
- [8] M. Geck and G. Pfeiffer. *Characters of finite Coxeter groups and Iwahori–Hecke algebras*. 2000.
- [9] D. Grinberg and A. Postnikov. Proof of a conjecture of Bergeron, Ceballos and Labbé. *New York Journal of Mathematics*, 23:1581–1610, 2017.
- [10] G. Hahn and C. Tardif. Graph homomorphisms: structure and symmetry. In G. Hahn and G. Sabidussi, editors, *Graph Symmetry: Algebraic Methods and Applications*, pages 107–166. Springer Netherlands, 1997.
- [11] W.-J. Hsu. Fibonacci cubes—a new interconnection topology. *IEEE Transactions on Parallel and Distributed Systems*, 4(1):3–12, 1993.
- [12] J.E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, Cambridge, 1990.
- [13] S. Klavžar. On median nature and enumerative properties of Fibonacci-like cubes. *Discrete Math.*, 299(1–3):145–153, 2005.
- [14] S. Klavžar. The Fibonacci dimension of a graph. *Journal of Combinatorial Optimization*, 25(4):505–522, 2013.
- [15] D. Knuth. *Axioms and Hulls*. Springer-Verlag, Berlin, 1992.
- [16] Yu.I. Manin and V.V. Shekhtman. Arrangements of hyperplanes, higher braid groups and higher Bruhat orders. In *Explicit universal deformations of Galois representations*, pages 289–308. Academic Press, Boston, 1989.
- [17] D. Meng. Reduced decompositions and commutation classes. [arXiv:1009.0886](https://arxiv.org/abs/1009.0886), 2010.
- [18] S. Ovchinnikov. Partial cubes: structures, characterizations, and constructions. *Discrete Mathematics*, 308(23):5597–5621, 2008.
- [19] R.P. Stanley. On the number of reduced decompositions of elements of Coxeter groups. *European J. Combin.*, 5(4), 1984.
- [20] B.E. Tenner. Reduced decompositions and permutation patterns. *J. Algebraic Combin.*, 24(3):263–284, 2006.
- [21] Xu Wang, Xuxu Zhao, and Haiyuan Yao. Structure and enumeration results of matchable Lucas cubes. *Discrete Applied Mathematics*, 277:263–279, 2020.
- [22] G.M. Ziegler. Higher Bruhat orders and cyclic hyperplane arrangements. *Topology*, 32(2), 1993.
- [23] D.M. Zollinger. *Equivalence classes of reduced words*. MS Thesis, University of Minnesota, 1994.

DEPARTMENT OF MATHEMATICS AND STATISTICS, NORTHERN ARIZONA UNIVERSITY PO BOX 5717, FLAGSTAFF, AZ 86011
 Email address: faa234@nau.edu, qdc4@nau.edu, Dana.Ernst@nau.edu

MATHEMATICS DEPARTMENT, UNIVERSITY OF CALIFORNIA SANTA CRUZ, 1156 HIGH STREET, SANTA CRUZ, CA 95064
 Email address: jbreland@ucsc.edu