All truths are easy to understand once they are discovered; the point is to discover them.

Galileo Galilei, astronomer & physicist

Chapter 5

The Real Numbers

In this chapter we will take a deep dive into structure of the real numbers by building up the multitude of properties you are familiar with by starting with a collection of fundamental axioms. Recall that an axiom is a statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proved. It is worth pointing out that one can carefully construct the real numbers from the natural numbers. However, that will not be the approach we take. Instead, we will simply list the axioms that the real numbers satisfy.

5.1 Axioms of the Real Numbers

Our axioms for the real numbers fall into three categories:

1. **Field Axioms:** These axioms provide the essential properties of arithmetic involving addition and subtraction.

2. **Order Axioms:** These axioms provide the necessary properties of inequalities.

3. **Completeness Axiom:** This axiom ensures that the familiar number line that we use to model the real numbers does not have any holes in it.

We begin with the Field Axioms.

Axioms 5.1 (Field Axioms)**.** There exist operations $+$ (addition) and $\cdot$ (multiplication) on $\mathbb{R}$ satisfying:

(F1) (Associativity for Addition) For all $a, b, c \in \mathbb{R}$ we have $(a+b)+c = a+(b+c)$;

(F2) (Commutativity for Addition) For all $a, b \in \mathbb{R}$, we have $a+b = b+a$;

(F3) (Additive Identity) There exists $0 \in \mathbb{R}$ such that for all $a \in \mathbb{R}$, $0+a = a$;

(F4) (Additive Inverses) For all $a \in \mathbb{R}$ there exists $-a \in \mathbb{R}$ such that $a+(-a) = 0$;

(F5) (Associativity for Multiplication) For all $a, b, c \in \mathbb{R}$ we have $(ab)c = a(bc)$;

(F6) (Commutativity for Multiplication) For all $a, b \in \mathbb{R}$, we have $ab = ba$;

(F7) (Multiplicative Identity) There exists $1 \in \mathbb{R}$ such that $1 \neq 0$ and for all $a \in \mathbb{R}$, $1a = a$;

(F8) (Multiplicative Inverses) For all $a \in \mathbb{R} \setminus \{0\}$ there exists $a^{-1} \in \mathbb{R}$ such that $aa^{-1} = 1$.

(F9) (Distributive Property) For all $a, b, c \in \mathbb{R}$, $a(b+c) = ab + ac$;

In the language of abstract algebra, Axioms F1–F4 and F5–F8 make each of $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$ an abelian group under addition and multiplication, respectively. Axiom F9 provides a way for the operations of addition and multiplication to interact. Collectively, Axioms F1–F9 make the real numbers a **field**. It follows from the axioms that the elements 0 and 1 of $\mathbb{R}$ are the unique **additive** and **multiplicative identities** in $\mathbb{R}$. To prove the following theorem, suppose 0 and $0'$ are both additive identities in $\mathbb{R}$ and then show that $0 = 0'$. This shows that there can only be one additive identity.

Theorem 5.2. The additive identity of $\mathbb{R}$ is unique.

To prove the next theorem, mimic the approach you used to prove Theorem 5.2.

Theorem 5.3. The multiplicative identity of $\mathbb{R}$ is unique.

For every $a \in \mathbb{R}$, the elements $-a$ and a^{-1} (as long as $a \neq 0$) are also the unique **additive** and **multiplicative inverses**, respectively.

Theorem 5.4. Every real number has a unique additive inverse.

Theorem 5.5. Every nonzero real number has a unique multiplicative inverse.

Since we are taking a formal axiomatic approach to the real numbers, we should make it clear how the natural numbers are embedded in $\mathbb{R}$.

Definition 5.6. We define the **natural numbers**, denoted by $\mathbb{N}$, to be the smallest subset of $\mathbb{R}$ satisfying:

(a) $1 \in \mathbb{N}$, and

(b) for all $n \in \mathbb{N}$, we have $n + 1 \in \mathbb{N}$.

Notice the similarity between the definition of the natural numbers presented above and the Axiom of Induction given in Section 4.1. Of course, we use the standard numeral system to represent the natural numbers, so that $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \ldots\}$.

Given the natural numbers, Axiom F3/Theorem 5.2 and Axiom F4/Theorem 5.4 together with the operation of addition allow us to define the **integers**, denoted by $\mathbb{Z}$, in the obvious way. That is, the integers consist of the natural numbers together with the additive identity and all of the additive inverses of the natural numbers.

We now introduce some common notation that you are likely familiar with. Take a moment to think about why the following is a definition as opposed to an axiom or theorem.

Definition 5.7. For every $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}$, we define the following:

(a) $\boxed{a - b := a + (-b)}$

(b) $\boxed{\dfrac{a}{b} := ab^{-1}}$ (for $b \neq 0$)

(c) $$\boxed{a^n := \begin{cases} \overbrace{aa\cdots a}^{n}, & \text{if } n \in \mathbb{N} \\ 1, & \text{if } n = 0 \text{ and } a \neq 0 \\ \dfrac{1}{a^{-n}}, & \text{if } -n \in \mathbb{N} \text{ and } a \neq 0 \end{cases}}$$

The set of **rational numbers**, denoted by $\mathbb{Q}$, is defined to be the collection of all real numbers having the form given in Part (b) of Definition 5.7. The **irrational numbers** are defined to be $\mathbb{R} \setminus \mathbb{Q}$.

Using the Field Axioms, we can prove each of the statements in the following theorem.

Theorem 5.8. For all $a, b, c \in \mathbb{R}$, we have the following:

(a) $a = b$ if and only if $a + c = b + c$;

(b) $0a = 0$;

(c) $-a = (-1)a$;

(d) $(-1)^2 = 1$;

(e) $-(-a) = a$;

(f) If $a \neq 0$, then $(a^{-1})^{-1} = a$;

(g) If $a \neq 0$ and $ab = ac$, then $b = c$.

(h) If $ab = 0$, then either $a = 0$ or $b = 0$.

Carefully prove the next theorem by explicitly citing where you are utilizing the Field Axioms and Theorem 5.8.

Theorem 5.9. For all $a, b \in \mathbb{R}$, we have $(a + b)(a - b) = a^2 - b^2$.

We now introduce the Order Axioms of the real numbers.

Axioms 5.10 (Order Axioms)**.** For $a, b, c \in \mathbb{R}$, there is a relation $\boxed{<}$ on $\mathbb{R}$ satisfying:

(O1) (Trichotomy Law) If $a \neq b$, then either $a < b$ or $b < a$ but not both;

(O2) (Transitivity) If $a < b$ and $b < c$, then $a < c$;

(O3) If $a < b$, then $a + c < b + c$;

(O4) If $a < b$ and $0 < c$, then $ac < bc$;

Given Axioms O1–O4, we say that the real numbers are a **linearly ordered field**. We call numbers greater than zero **positive** and those greater than or equal to zero **nonnegative**. There are similar definitions for **negative** and **nonpositive**.

Notice that the Order Axioms are phrased in terms of "<". We would also like to be able to utilize ">", "≤", and "≥".

Definition 5.11. For $a, b \in \mathbb{R}$, we define:

(a) $\boxed{a > b}$ if $b < a$;

(b) $\boxed{a \leq b}$ if $a < b$ or $a = b$;

(c) $\boxed{a \geq b}$ if $b \leq a$.

Notice that we took the existence of the inequalities "<", ">", "≤", and "≥" on the real numbers for granted when we defined intervals of real numbers in Definition 3.4.

Using the Order Axioms, we can prove many familiar facts.

Theorem 5.12. For all $a, b \in \mathbb{R}$, if $a, b > 0$, then $a + b > 0$; and if $a, b < 0$, then $a + b < 0$.

The next result extends Axiom O3.

Theorem 5.13. For all $a, b, c, d \in \mathbb{R}$, if $a < b$ and $c < d$, then $a + c < b + d$.

Theorem 5.14. For all $a \in \mathbb{R}$, $a > 0$ if and only if $-a < 0$.

Theorem 5.15. If a, b, c, and d are positive real numbers such that $a < b$ and $c < d$, then $ac < bd$.

Theorem 5.16. For all $a, b \in \mathbb{R}$, we have the following:

(a) $ab > 0$ if and only if either $a, b > 0$ or $a, b < 0$;

(b) $ab < 0$ if and only if $a < 0 < b$ or $b < 0 < a$.

Theorem 5.17. For all positive real numbers a and b, $a < b$ if and only if $a^2 < b^2$.

Consider using three cases when approaching the proof of the following theorem.

Theorem 5.18. For all $a \in \mathbb{R}$, we have $a^2 \geq 0$.

It might come as a surprise that the following result requires proof.

Theorem 5.19. We have $0 < 1$.

The previous theorem together with Theorem 5.14 implies that $-1 < 0$ as you expect. It also follows from Axiom O3 that for all $n \in \mathbb{Z}$, we have $n < n + 1$. We assume that there are no integers between n and $n + 1$.

Theorem 5.20. For all $a \in \mathbb{R}$, if $a > 0$, then $a^{-1} > 0$, and if $a < 0$, then $a^{-1} < 0$.

Theorem 5.21. For all $a, b \in \mathbb{R}$, if $a < b$, then $-b < -a$.

The last few results allow us to take for granted our usual understanding of which real numbers are positive and which are negative. The next theorem yields a result that extends Theorem 5.21.

Theorem 5.22. For all $a, b, c \in \mathbb{R}$, if $a < b$ and $c < 0$, then $bc < ac$.

There is a special function that we can now introduce.

Definition 5.23. Given $a \in \mathbb{R}$, we define the **absolute value of** a, denoted $|a|$, via

$$\boxed{|a| := \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}}$$

Theorem 5.24. For all $a \in \mathbb{R}$, $|a| \geq 0$ with equality only if $a = 0$.

We can interpret $|a|$ as the distance between a and 0 as depicted in Figure 5.1.

(a) $a > 0$ (b) $a < 0$

Figure 5.1: Visual representation of $|a|$.

Theorem 5.25. For all $a, b \in \mathbb{R}$, we have $|a - b| = |b - a|$.

Given two points a and b, $|a - b|$, and hence $|b - a|$ by the previous theorem, is the distance between a and b as shown in Figure 5.2.

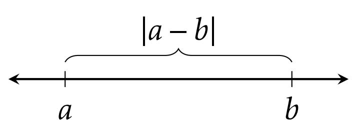

Figure 5.2: Visual representation of $|a - b|$.

Theorem 5.26. For all $a, b \in \mathbb{R}$, $|ab| = |a||b|$.

In the next theorem, writing $\pm a \leq b$ is an abbreviation for $a \leq b$ and $-a \leq b$.

Theorem 5.27. For all $a, b \in \mathbb{R}$, if $\pm a \leq b$, then $|a| \leq b$.

Theorem 5.28. For all $a \in \mathbb{R}$, $|a|^2 = a^2$.

Theorem 5.29. For all $a \in \mathbb{R}$, $\pm a \leq |a|$.

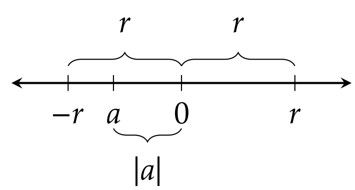

Figure 5.3: Visual representation of $|a| \leq r$.

Theorem 5.30. For all $a, r \in \mathbb{R}$ with r nonnegative, $|a| \leq r$ if and only if $-r \leq a \leq r$.

The letter r was used in the previous theorem because it is the first letter of the word "radius". If r is positive, we can think of the interval $(-r, r)$ as the interior of a one-dimensional circle with radius r centered at 0. Figure 5.3 provides a visual interpretation of Theorem 5.30.

Corollary 5.31. For all $a, b, r \in \mathbb{R}$ with r nonnegative, $|a-b| \leq r$ if and only if $b-r \leq a \leq b+r$.

Since $|a-b|$ represents the distance between a and b, we can interpret $|a-b| \leq r$ as saying that the distance between a and b is less than or equal to r. In other words, a is within r units of b. See Figure 5.4.

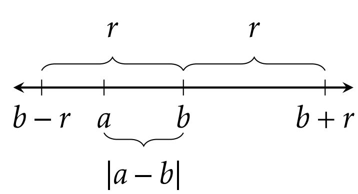

Figure 5.4: Visual representation of $|a-b| \leq r$.

Consider using Theorems 5.29 and 5.30 when attacking the next result, which is known as the **Triangle Inequality**. This result can be extremely useful in some contexts.

Theorem 5.32 (Triangle Inequality)**.** For all $a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$.

Figure 5.5 depicts two of the cases for the Triangle Inequality.

(a) $a \geq 0, b \geq 0$

(b) $a < 0, b \geq 0$

Figure 5.5: Visual representation of two of the cases for the Triangle Inequality.

Problem 5.33. Under what conditions do we have equality for the Triangle Inequality?

Where did the Triangle Inequality get its name? Why "Triangle"? For any triangle (including degenerate triangles), the sum of the lengths of any two sides must be greater than or equal to the length of the remaining side. That is, if x, y, and z are the lengths of the sides of the triangle, then $z \leq x + y$, where we have equality only in the degenerate case of a triangle with no area. In linear algebra, the Triangle Inequality is a theorem about lengths of vectors. If $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^n$, then the Triangle Inequality states that $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$. Note that $\|\mathbf{a}\|$ denotes the length of vector $\mathbf{a}$. See Figure 5.6. The version of the Triangle Inequality that we presented in Theorem 5.32 is precisely the one-dimensional version of the Triangle Inequality in terms of vectors.

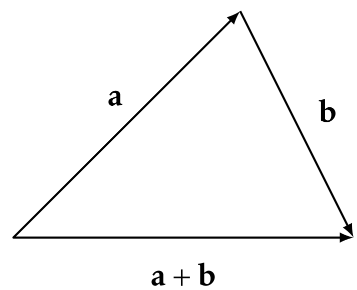

Figure 5.6: Triangle Inequality in terms of vectors.

The next theorem is sometimes called the **Reverse Triangle Inequality**.

Theorem 5.34 (Reverse Triangle Inequality)**.** For all $a, b \in \mathbb{R}$, $|a - b| \geq ||a| - |b||$.

Before we introduce the Completeness Axiom, we need some additional terminology.

Definition 5.35. Let $A \subseteq \mathbb{R}$. A point b is called an **upper bound** of A if for all $a \in A$, $a \leq b$. The set A is said to be **bounded above** if it has an upper bound.

Problem 5.36. The notion of a **lower bound** and the property of a set being **bounded below** are defined similarly. Try defining them.

Problem 5.37. Find all upper bounds and all lower bounds for each of the following sets when they exist.

(a) $\{5, 11, 17, 42, 103\}$

(b) $\mathbb{N}$

(c) $\mathbb{Z}$

(d) $(0, 1]$

(e) $(0, 1] \cap \mathbb{Q}$

(f) $(0, \infty)$

(g) $\{42\}$

(h) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(i) $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

(j) $\emptyset$

Definition 5.38. A set $A \subseteq \mathbb{R}$ is **bounded** if A is bounded above and below.

Notice that a set $A \subseteq \mathbb{R}$ is bounded if and only if it is a subset of some bounded closed interval.

Definition 5.39. Let $A \subseteq \mathbb{R}$. A point p is a **supremum** (or **least upper bound**) of A if p is an upper bound of A and $p \leq b$ for every upper bound b of A. Analogously, a point p is an **infimum** (or **greatest lower bound**) of A if p is a lower bound of A and $p \geq b$ for every lower bound b of A.

Our next result tells us that a supremum of a set and an infimum of a set are unique when they exist.

Theorem 5.40. If $A \subseteq \mathbb{R}$ such that the supremum (respectively, infimum) of A exists, then the supremum (respectively, infimum) of A is unique.

In light of the previous theorem, if the supremum of A exists, it is denoted by $\boxed{\sup(A)}$. Similarly, if the infimum of A exists, it is denoted by $\boxed{\inf(A)}$.

Problem 5.41. Find the supremum and the infimum of each of the sets in Problem 5.37 when they exist.

It is important to recognize that the supremum or infimum of a set may or may not be contained in the set. In particular, we have the following theorem concerning suprema and maximums. The analogous result holds for infima and minimums.

Theorem 5.42. Let $A \subseteq \mathbb{R}$. Then A has a maximum if and only if A has a supremum and $\sup(A) \in A$, in which case the $\max(A) = \sup(A)$.

Intuitively, a point is the supremum of a set A if and only if no point smaller than the supremum can be an upper bound of A. The next result makes this more precise.

Theorem 5.43. Let $A \subseteq \mathbb{R}$. An upper bound b is the supremum of A if and only if for every $\varepsilon > 0$, there exists $a \in A$ such that $b - \varepsilon < a$.

Problem 5.44. State and prove the analogous result to Theorem 5.43 involving infimum.

The following axiom states that every nonempty subset of the real numbers that has an upper bound has a least upper bound.

Axiom 5.45 (Completeness Axiom)**.** If A is a nonempty subset of $\mathbb{R}$ that is bounded above, then $\sup(A)$ exists.

Given the Completeness Axiom, we say that the real numbers satisfy the **least upper bound property**. It is worth mentioning that we do not need the Completeness Axiom to conclude that every nonempty subset of the integers that is bounded above has a supremum, as this follows from Theorem 4.39 (a generalized version of the Well-Ordering Principle).

Certainly, the real numbers also satisfy the analogous result involving infimum.

Theorem 5.46. If A is a nonempty subset of $\mathbb{R}$ that is bounded below, then $\inf(A)$ exists.

Our next result, called the **Archimedean Property**, tells us that for every real number, we can always find a natural number that is larger. To prove this theorem, consider a proof by contradiction and then utilize the Completeness Axiom and Theorem 5.43.

Theorem 5.47 (Archimedean Property)**.** For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$.

More generally, we can "squeeze" every real number between a pair of integers. The next result is sometimes referred to at the **Generalized Archimedean Property**.

Theorem 5.48 (Generalized Archimedean Property)**.** For every $x \in \mathbb{R}$, there exists $k, n \in \mathbb{Z}$ such that $k < x < n$.

Theorem 5.49. For any positive real number x, there exists $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < x$.

The next theorem strengthens the Generalized Archimedean Property and says that every real number is either an integer or lies between a pair of consecutive integers. To prove this theorem, let $x \in \mathbb{R}$ and define $L = \{k \in \mathbb{Z} \mid k \leq x\}$. Use the Generalized Archimedean Property to conclude that L is nonempty and then utilize Theorem 4.39.

Theorem 5.50. For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n \leq x < n+1$.

To prove the next theorem, let $a < b$, utilize Theorem 5.49 on $b-a$ to obtain $N \in \mathbb{N}$ such that $\frac{1}{N} < b-a$, and then apply Theorem 5.50 to Na to conclude that there exists $n \in \mathbb{N}$ such that $n \leq Na < n+1$. Lastly, argue that $\frac{n+1}{N}$ is the rational number you seek.

Theorem 5.51. If (a, b) is an open interval, then there exists a rational number p such that $p \in (a, b)$.

Recall that the real numbers consist of rational and irrational numbers. Two examples of an irrational number that you are likely familiar with are π and $\sqrt{2}$. In Section 6.2, we will prove that $\sqrt{2}$ is irrational, but for now we will take this fact for granted. It turns out that $\sqrt{2} \approx 1.41421356237 \in (1, 2)$. This provides an example of an irrational number occurring between a pair of distinct rational numbers. The following theorem is a good challenge to generalize this.

Theorem 5.52. If (a, b) is an open interval, then there exists an irrational number p such that $p \in (a, b)$.

Repeated applications of the previous two theorems implies that every open interval contains infinitely many rational numbers and infinitely many irrational numbers. In light of these two theorems, we say that both the rationals and irrationals are **dense** in the real numbers.

> If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.
>
> John von Neumann, mathematician

5.2 Standard Topology of the Real Line

In this section, we will introduce the notions of open, closed, compact, and connected as they pertain to subsets of the real numbers. These properties form the underpinnings of a branch of mathematics called **topology** (derived from the Greek words *tópos*, meaning 'place, location', and *ology*, meaning 'study of'). Topology, sometimes called "rubber sheet geometry," is concerned with properties of spaces that are invariant under any continuous deformation (e.g., bending, twisting, and stretching like rubber while not allowing tearing apart or gluing together). The fundamental concepts in topology are continuity, compactness, and connectedness, which rely on ideas such as "arbitrary close" and "far apart". These ideas can be made precise using open sets.

Once considered an abstract branch of pure mathematics, topology now has applications in biology, computer science, physics, and robotics. The goal of this section is to introduce you to the basics of the set-theoretic definitions used in topology and to provide you with an opportunity to tinker with open and closed subsets of the real numbers. In Section 8.5, we will revisit these concepts and explore continuous functions.

For this entire section, our universe of discourse is the set of real numbers. You may assume all the usual basic algebraic properties of the real numbers (addition, subtraction, multiplication, division, commutative property, distribution, etc.). We will often refer to an element in a subset of real numbers as a **point**.

Definition 5.53. A set U is called an **open set** if for every $x \in U$, there exists a bounded open interval (a,b) containing x such that $(a,b) \subseteq U$.

It follows immediately from the definition that every open set is a union of bounded open intervals.

Problem 5.54. Determine whether each of the following sets is open. Justify your assertions.

(a) $(1,2)$

(b) $(1,\infty)$

(c) $(1,2)\cup(\pi,5)$

(d) $[1,2]$

(e) $(-\infty,\sqrt{2}]$

(f) $\{4,17,42\}$

(g) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(h) $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

(i) $\mathbb{R}$

(j) $\mathbb{Q}$

(k) $\mathbb{Z}$

(l) $\emptyset$

As expected, every open interval (i.e., intervals of the form interval of the form (a,b), $(-\infty,b)$, (a,∞), or $(-\infty,\infty)$) is an open set.

Theorem 5.55. Every open interval is an open set.

However, it is important to point out that open sets can be more complicated than a single open interval.

Problem 5.56. Provide an example of an open set that is not a single open interval.

Theorem 5.57. If U and V are open sets, then

(a) $U \cup V$ is an open set, and

(b) $U \cap V$ is an open set.

According to the next two theorems, the union of arbitrarily many open sets is open while the intersection of a finite number of open sets is open.

Theorem 5.58. If $\{U_\alpha\}_{\alpha\in\Delta}$ is a collection of open sets, then $\bigcup_{\alpha\in\Delta} U_\alpha$ is an open set.

Consider using induction to prove the next theorem.

Theorem 5.59. If $\{U_i\}_{i=1}^n$ is a finite collection of open sets for $n \in \mathbb{N}$, then $\bigcap_{i=1}^n U_i$ is an open set.

Problem 5.60. Explain why we cannot utilize induction to prove that the intersection of infinitely many open sets indexed by the natural numbers is open.

Problem 5.61. Give an example of each of the following.

(a) A collection of open sets $\{U_\alpha\}_{\alpha\in\Delta}$ such that $\bigcap_{\alpha\in\Delta} U_\alpha$ is an open set.

(b) A collection of open sets $\{U_\alpha\}_{\alpha\in\Delta}$ such that $\bigcap_{\alpha\in\Delta} U_\alpha$ is not an open set.

According to the previous problem, the intersection of infinitely many open sets may or may not be open. So, we know that there is no theorem that states that the intersection of arbitrarily many open sets is open. We only know for certain that the intersection of finitely many open sets is open by Theorem 5.59.

Definition 5.62. Suppose $A \subseteq \mathbb{R}$. A point $p \in \mathbb{R}$ is an **accumulation point** of A if for every bounded open interval (a,b) containing p, there exists a point $q \in (a,b) \cap A$ such that $q \neq p$.

Notice that if p is an accumulation point of A, then p may or may not be in A. Loosely speaking, p is an accumulation point of a set A if there are points in A arbitrarily close to p. That is, if we zoom in on p, we should always see points in A nearby.

Problem 5.63. Consider the open interval $I = (1, 2)$. Prove each of the following.

(a) The points 1 and 2 are accumulation points of I.

(b) If $p \in I$, then p is an accumulation point of I.

(c) If $p < 1$ or $p > 2$, then p is not an accumulation point of I.

Theorem 5.64. A point p is an accumulation point of the intervals (a, b), $(a, b]$, $[a, b)$, and $[a, b]$ if and only if $p \in [a, b]$.

Problem 5.65. Prove that the point $p = 0$ is an accumulation point of $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Are there any other accumulation points of A?

Problem 5.66. Provide an example of a set A with exactly two accumulation points.

Consider using Theorems 5.51 and 5.52 when proving the next result.

Theorem 5.67. If $p \in \mathbb{R}$, then p is an accumulation point of $\mathbb{Q}$.

Definition 5.68. A set $A \subseteq \mathbb{R}$ is called **closed** if A contains all of its accumulation points.

Problem 5.69. Determine whether each of the sets in Problem 5.54 is closed. Justify your assertions.

The upshot of Parts (i) and (l) of Problems 5.54 and 5.69 is that $\mathbb{R}$ and $\emptyset$ are both open and closed. It turns out that these are the only two subsets of the real numbers with this property. One issue with the terminology that could potentially create confusion is that the open interval $(-\infty, \infty)$ is both an open and a closed set.

Problem 5.70. Provide an example of each of the following. You do not need to prove that your answers are correct.

(a) A set that is open but not closed.

(b) A set that is closed but not open.

(c) A set that neither open nor closed.

Another potentially annoying feature of the terminology illustrated by Problem 5.70 is that if a set is not open, it may or may not be closed. Similarly, if a set is not closed, it may or may not be open. That is, open and closed are not opposites of each other.

The next result justifies referring to $[a, b]$ as a closed interval.

Theorem 5.71. Every interval of the form $[a, b]$, $(-\infty, b]$, $[a, \infty)$, or $(-\infty, \infty)$ is a closed set.

Theorem 5.72. Every finite subset of $\mathbb{R}$ is closed.

Despite the fact that open and closed are not opposites of each other, there is a nice relationship between open and closed sets in terms of complements.

Theorem 5.73. Let $U \subseteq \mathbb{R}$. Then U is open if and only if U^C is closed.

Theorem 5.74. If A and B are closed sets, then

(a) $A \cup B$ is a closed set, and

(b) $A \cap B$ is a closed set.

The next two theorems are analogous to Theorems 5.58 and 5.59.

Theorem 5.75. If $\{A_\alpha\}_{\alpha\in\Delta}$ is a collection of closed sets, then $\bigcap_{\alpha\in\Delta} A_\alpha$ is a closed set.

Theorem 5.76. If $\{A_i\}_{i=1}^{n}$ is a finite collection of closed sets for $n \in \mathbb{N}$, then $\bigcup_{i=1}^{n} U_i$ is a closed set.

Problem 5.77. Provide an example of a collection of closed sets $\{A_\alpha\}_{\alpha\in\Delta}$ such that $\bigcup_{\alpha\in\Delta} A_\alpha$ is not a closed set.

Problem 5.78. Determine whether each of the following sets is open, closed, both, or neither.

(a) $V = \displaystyle\bigcup_{n=2}^{\infty}\left(n - \frac{1}{2}, n\right)$

(b) $W = \displaystyle\bigcap_{n=2}^{\infty}\left(n - \frac{1}{2}, n\right)$

(c) $X = \displaystyle\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$

(d) $Y = \displaystyle\bigcap_{n=1}^{\infty}(-n, n)$

(e) $Z = (0,1) \cap \mathbb{Q}$

Problem 5.79. Prove or provide a counterexample: Every non-closed set has at least one accumulation point.

We now introduce three special classes of subsets of $\mathbb{R}$: compact, connected, and disconnected.

Definition 5.80. A set $K \subseteq \mathbb{R}$ is called **compact** if K is both closed and bounded.

It is important to point out that there is a more general definition of compact in an arbitrary topological space. However, using our notions of open and closed, it is a theorem that a subset of the real line is compact if and only if it is closed and bounded.

Problem 5.81. Determine whether each of the following sets is compact. Briefly justify your assertions.

(a) $[0,1)\cup[2,3]$

(b) $[0,1)\cup(1,2]$

(c) $[0,1)\cup[1,2]$

(d) $\mathbb{R}$

(e) $\mathbb{Q}$

(f) $\mathbb{R}\setminus\mathbb{Q}$

(g) $\mathbb{Z}$

(h) $\{\frac{1}{n}\mid n\in\mathbb{N}\}$

(i) $[0,1]\cup\{1+\frac{1}{n}\mid n\in\mathbb{N}\}$

(j) $\{17,42\}$

(k) $\{17\}$

(l) $\emptyset$

Problem 5.82. Is every finite set compact? Justify your assertion.

The next theorem says that every nonempty compact set contains its greatest lower bound and its least upper bound. That is, every nonempty compact set attains a minimum and a maximum value.

Theorem 5.83. If K is a nonempty compact subset of $\mathbb{R}$, then $\sup(K),\inf(K)\in K$.

Definition 5.84. A set $A\subseteq\mathbb{R}$ is **disconnected** if there exists two disjoint open sets U_1 and U_2 such that $A\cap U_1$ and $A\cap U_2$ are nonempty but $A\subseteq U_1\cup U_2$ (equivalently, $A=(A\cap U_1)\cup(A\cap U_2)$). If a set is not disconnected, then we say that it is **connected**.

In other words, a set is disconnected if it can be partitioned into two nonempty subsets such that each subset does not contain points of the other and does not contain any accumulation points of the other. Showing that a set is disconnected is generally easier than showing a set is connected. To prove that a set is disconnected, you simply need to exhibit two open sets with the necessary properties. However, to prove that a set is connected, you need to prove that no such pair of open sets exists.

Problem 5.85. Determine whether each of the sets in Problem 5.81 is is connected or disconnected. Briefly justify your assertions.

Theorem 5.86. If $a\in\mathbb{R}$, then $\{a\}$ is connected.

The proof of the next theorem is harder than you might expect. Consider a proof by contradiction and try to make use of the Completeness Axiom.

Theorem 5.87. Every closed interval $[a,b]$ is connected.

It turns out that every connected set in $\mathbb{R}$ is either a singleton or an interval. We have not officially proved this claim, but we do have the tools to do so. Feel free to try your hand at proving this fact.

> If you learn how to learn, it's the ultimate meta skill and I believe you can learn how to be healthy, you can learn how to be fit, you can learn how to be happy, you can learn how to have good relationships, you can learn how to be successful. These are all things that can be learned. So if you can learn that is a trump card, it's an ace, it's a joker, it's a wild card. You can trade it for any other skill.
>
> Naval Ravikant, entrepreneur & investor