Chapter 7

Homomorphisms and the Isomorphism Theorems

7.1 Homomorphisms

Let G_1 and G_2 be groups. Recall that $\phi : G_1 \to G_2$ is an isomorphism if and only if ϕ

(a) is one-to-one,

(b) is onto, and

(c) satisfies the homomorphic property.

We say that G_1 is isomorphic to G_2 and write $G_1 \cong G_2$ if such a ϕ exists. Loosely speaking, two groups are isomorphic if they have the "same structure." What if we drop the one-to-one and onto requirement?

Definition 7.1. Let $(G_1, *)$ and $(G_2, \odot)$ be groups. A function $\phi : G_1 \to G_2$ is a **homomorphism** if and only if ϕ satisfies the homomorphic property:

$$\phi(x * y) = \phi(x) \odot \phi(y)$$

for all $x, y \in G_1$. At the risk of introducing ambiguity, we will usually omit making explicit reference to the binary operations and write the homomorphic property as

$$\phi(xy) = \phi(x)\phi(y).$$

Group homomorphisms are analogous to linear transformations on vector spaces that one encounters in linear algebra.

Figure 7.1 captures a visual representation of the homomorphic property. We encountered this same representation in Figure 3.9. If $\phi(x) = x'$, $\phi(y) = y'$, and $\phi(z) = z'$ while $z' = x' \odot y'$, then the only way G_2 may respect the structure of G_1 is for

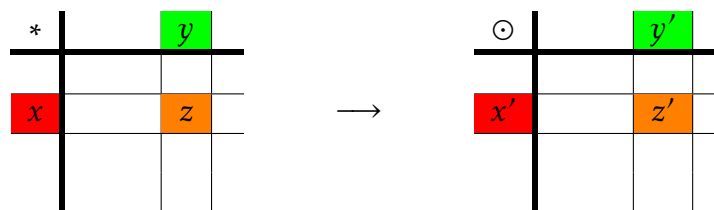$$\phi(x * y) = \phi(z) = z' = x' \odot y' = \phi(x) \odot \phi(y).$$

Figure 7.1

Problem 7.2. Define $\phi : \mathbb{Z}_3 \to D_3$ via $\phi(k) = r^k$. Prove that ϕ is a homomorphism and then determine whether ϕ is one-to-one or onto. Also, try to draw a picture of the homomorphism in terms of Cayley diagrams.

Problem 7.3. Let G and H be groups. Prove that the function $\phi : G \times H \to G$ given by $\phi(g, h) = g$ is a homomorphism. This function is an example of a **projection map**.

There is always at least one homomorphism between two groups.

Theorem 7.4. Let G_1 and G_2 be groups. Define $\phi : G_1 \to G_2$ via $\phi(g) = e_2$ (where e_2 is the identity of G_2). Then ϕ is a homomorphism. This function is often referred to as the **trivial homomorphism** or the **0-map**.

Back in Section 3.3, we encountered several theorems about isomorphisms. However, at the end of that section we remarked that some of those theorems did not require that the function be one-to-one and onto. We collect those results here for convenience.

Theorem 7.5. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism.

(a) If e_1 and e_2 are the identity elements of G_1 and G_2, respectively, then $\phi(e_1) = e_2$.

(b) For all $g \in G_1$, we have $\phi(g^{-1}) = [\phi(g)]^{-1}$.

(c) If $H \le G_1$, then $\phi(H) \le G_2$, where

$$\phi(H) := \{y \in G_2 \mid \text{there exists } h \in H \text{ such that } \phi(h) = y\}.$$

Note that $\phi(H)$ is called the **image** of H. A special case is when $H = G_1$. Notice that ϕ is onto exactly when $\phi(G_1) = G_2$.

The following theorem is a consequence of Lagrange's Theorem.

Theorem 7.6. Let G_1 and G_2 be groups such that G_2 is finite and let $H \le G_1$. If $\phi : G_1 \to G_2$ is a homomorphism, then $|\phi(H)|$ divides $|G_2|$.

The next theorem tells us that under a homomorphism, the order of the image of an element must divide the order of the pre-image of that element.

Theorem 7.7. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. If $g \in G_1$ such that $|g|$ is finite, then $|\phi(g)|$ divides $|g|$.

Every homomorphism has an important subset of the domain associated with it.

Definition 7.8. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. The **kernel** of ϕ is defined via

$$\ker(\phi) := \{g \in G_1 \mid \phi(g) = e_2\}.$$

The kernel of a homomorphism is analogous to the null space of a linear transformation of vector spaces. Given the definition of the kernel, we see that we can phrase the kernel in terms of a preimage:

$$\ker(\phi) = \phi^{-1}(\{e_2\}).$$

Problem 7.9. Identify the kernel and image for the homomorphism given in Problem 7.2.

Problem 7.10. What is the kernel of a trivial homomorphism (see Theorem 7.4).

Theorem 7.11. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Then $\ker(\phi) \trianglelefteq G_1$.

Theorem 7.12. Let G be a group and let $H \trianglelefteq G$. Then the map $\gamma : G \to G/H$ given by $\gamma(g) = gH$ is a homomorphism with $\ker(\gamma) = H$. This map is called the **canonical projection map**.

The upshot of Theorems 7.11 and 7.12 is that kernels of homomorphisms are always normal and every normal subgroup is the kernel of some homomorphism. It turns out that the kernel can tell us whether ϕ is one-to-one.

The next theorem tells us that two elements in the domain of a group homomorphism map to the same element in the codomain if and only if they are in the same coset of the kernel.

Theorem 7.13. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Then $\phi(a) = \phi(b)$ if and only if $a \in b \ker(\phi)$ if and only if $a \ker(\phi) = b \ker(\phi)$.

Suppose $\phi : G_1 \to G_2$ is a homomorphism. Recall that each of the left cosets of the kernel have the same cardinality by Corollary 5.13. If the kernel has finite order k, then one consequence of Theorem 7.13 is that the homomorphism ϕ is k-to-1. Now, let $g \in G_1$ and suppose $\phi(g) = h$. It follows from the previous theorem that $\phi^{-1}(\{h\}) = g \ker(\phi)$. That is, each left coset of $\ker(\phi)$ is the preimage of some element in G_2 (and vice versa). Figure 7.2 provides a visual representation of this phenomenon in the case when G_1 is finite and ϕ is surjective (and hence G_2 is also finite).

Problem 7.14. Suppose $\phi : \mathbb{Z}_{20} \to \mathbb{Z}_{20}$ is a group homomorphism such that $\ker(\phi) = \{0, 5, 10, 15\}$. If $\phi(13) = 8$, determine all elements that ϕ maps to 8.

The next result is a special case of Theorem 7.13.

Theorem 7.15. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Then ϕ is one-to-one if and only if $\ker(\phi) = \{e_1\}$, where e_1 is the identity in G_1.
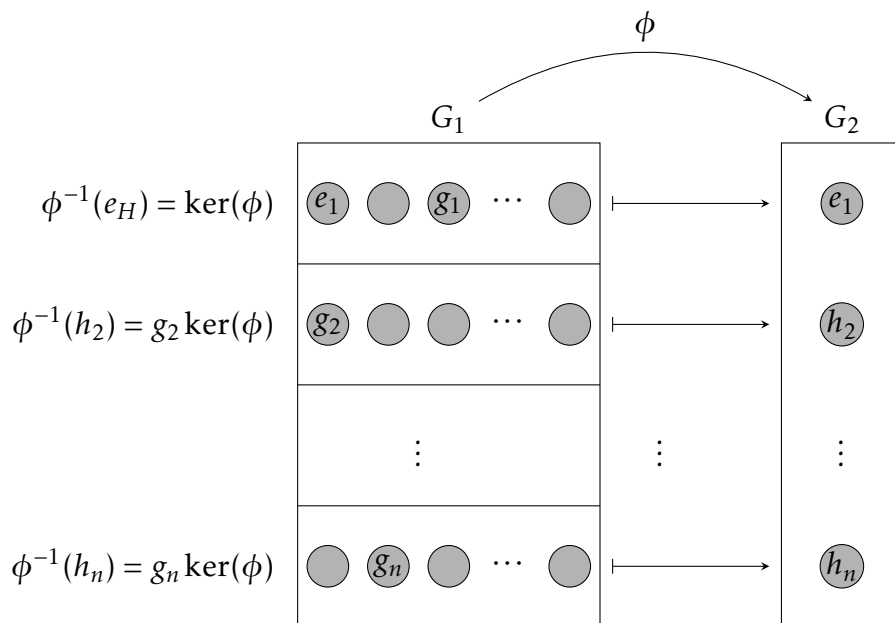
Figure 7.2. Left cosets of $\ker(\phi)$ correspond to preimages of elements in range of ϕ.

Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Given a generating set for G_1, the homomorphism ϕ is uniquely determined by its action on the generating set for G_1. In particular, if you have a word for a group element written in terms of the generators, just apply the homomorphic property to the word to find the image of the corresponding group element.

Problem 7.16. Suppose $\phi : Q_8 \to V_4$ is a group homomorphism satisfying $\phi(i) = h$ and $\phi(j) = v$.

(a) Find $\phi(1)$, $\phi(-1)$, $\phi(k)$, $\phi(-i)$, $\phi(-j)$, and $\phi(-k)$.

(b) Find $\ker(\phi)$.

(c) What well-known group is $Q_8/\ker(\phi)$ isomorphic to?

Problem 7.17. Find a non-trivial homomorphism from $\mathbb{Z}_{10}$ to $\mathbb{Z}_6$.

Problem 7.18. Find all non-trivial homomorphisms from $\mathbb{Z}_3$ to $\mathbb{Z}_6$.

Problem 7.19. Prove that the only homomorphism from D_3 to $\mathbb{Z}_3$ is the trivial homomorphism.

Problem 7.20. Let F be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ and let D be the subset of differentiable functions on $\mathbb{R}$. It turns out that F is a group under addition of functions and D is a subgroup of F (you do not need to prove this). Define $\phi : D \to F$ via $\phi(f) = f'$ (where f' is the derivative of f). Prove that ϕ is a homomorphism. You may recall facts from calculus without proving them. Is ϕ one-to-one? Onto?

7.2 The Isomorphism Theorems

The next theorem is sometimes called the **Fundamental Theorem of Homomorphisms**, but is more commonly is referred to as the **First Isomorphism Theorem**. Recall that if $\phi : G_1 \to G_2$ is a homomorphism, then $\ker(\phi)$ is always a normal subgroup of G_1. This implies that $G_1/\ker(\phi)$ is a well-defined group. The First Isomorphism Theorem states that this quotient group is isomorphic to the image of ϕ. The motivation behind this theorem is baked into Figure 7.2. We can obtain a bijective function ψ from the cosets of the kernel to the image of ϕ by "collapsing" the cosets of the kernel.

Here is an analogy for helping understand the content of First Isomorphism. Suppose we have a collection airplanes filled with passengers and a collection of potential destination cities such that at most one airplane may land at each city. The function ϕ indicates which city each passenger lands at while the function ψ indicates which city each airplane lands at. Moreover, the codomain for the function ψ consists only of the cities that an airplane lands at. Amazingly, this new function $\psi : G_1/\ker(\phi) \to \phi(G_1)$ inherits the homomorphic property from ϕ.

Theorem 7.21 (The First Isomorphism Theorem). Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Then

$$G_1/\ker(\phi) \cong \phi(G_1).$$

If ϕ is onto, then

$$G_1/\ker(\phi) \cong G_2.$$

At this time, you should take a moment to reflect on how the First Isomorphism Theorem and the quotient process we discussed in Section 6.2 are related.

Problem 7.22. Let $\phi : Q_8 \to V_4$ be the homomorphism described in Problem 7.16. Use the First Isomorphism Theorem to prove that $Q_8/\langle -1 \rangle \cong V_4$.

Problem 7.23. For $n \geq 2$, define $\phi : S_n \to \mathbb{Z}_2$ via

$$\phi(\sigma) = \begin{cases} 0, & \sigma \text{ even} \\ 1, & \sigma \text{ odd.} \end{cases}$$

Use the First Isomorphism Theorem to prove that $S_n/A_n \cong \mathbb{Z}_2$.

Problem 7.24. Use the First Isomorphism Theorem to prove that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}_6$. Attempt to draw a picture of this using Cayley diagrams.

Problem 7.25. Use the First Isomorphism Theorem to prove that $(\mathbb{Z}_4 \times \mathbb{Z}_2)/(\{0\} \times \mathbb{Z}_2) \cong \mathbb{Z}_4$.

Problem 7.26. Define $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ (i.e, the set of complex numbers without zero). It turns out that $\mathbb{C}^*$ is a group under multiplication, where 1 is the identity of the group. You can take this for granted. Define $\phi : \mathbb{R} \to \mathbb{C}^*$ via $\phi(t) = \cos(2\pi t) + i \sin(2\pi t)$, where $i^2 = -1$. It is not too difficult to see that the image of ϕ is a circle of radius 1 centered at the origin in the complex plane. If ϕ is a group homomorphism, then Theorem 7.5(c) implies that this circle is a subgroup of $\mathbb{C}^*$.

(a) Prove that ϕ is a group homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{C}^*, \cdot)$. The following trigonometric identities might come in handy:

- $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
- $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.

(b) Prove that $\ker(\phi) = \mathbb{Z}$.

(c) In light of the First Isomorphism Theorem, what conclusion can you make? Be as specific as possible.

Problem 7.27. Use the First Isomorphism Theorem to prove that $\mathbb{Q}*/\langle -1 \rangle \cong \mathbb{Q}$. Reviewing Problem 6.49 may be helpful.

The next theorem is a generalization of Theorem 7.7 and follows from the First Isomorphism Theorem together with Lagrange's Theorem.

Theorem 7.28. Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. If G_1 is finite, then $|\phi(G_1)|$ divides $|G_1|$.

We finish the chapter by listing a few of the remaining isomorphism theorems.

Theorem 7.29 (The Second Isomorphism Theorem). Let G be a group with $H \leq G$ and $N \trianglelefteq G$. Then

(a) $HN := \{hn \mid h \in H, n \in N\} \leq G$;

(b) $N \trianglelefteq HN$;

(c) $H \cap N \trianglelefteq H$;

(d) $H/(H \cap N) \cong HN/N$.

Theorem 7.30 (The Third Isomorphism Theorem). Let G be a group with $H, K \trianglelefteq G$ and $K \leq H$. Then $H/K \trianglelefteq G/K$ and

$$G/H \cong (G/K)/(H/K).$$

The last isomorphism theorem is sometimes called the *Lattice Isomorphism Theorem*.

Theorem 7.31 (The Fourth Isomorphism Theorem). Let G be a group with $N \trianglelefteq G$. Then there is a bijection from the set of subgroups of G that contain N onto the set of subgroups of G/N. In particular, every subgroup G is of the form H/N for some subgroup H of G containing N (namely, its preimage in G under the canonical projection homomorphism from G to G/N.) This bijection has the following properties: for all $H, K \leq G$ with $N \leq H$ and $N \subseteq K$, we have

(a) $H \leq K$ if and only if $H/N \leq K/N$

(b) If $H \leq K$, then $[K : H] = [K/N : H/N]$

(c) $\langle H, K \rangle /N = \langle H/N, K/N \rangle$

(d) $(H \cap K)/N = H/N \cap K/N$

(e) $H \trianglelefteq G$ if and only if $H/N \trianglelefteq G/N$.