Chapter 4

Standard Topology of the Real Line

In this chapter, we will take a brief tour of the fascinating world of open and closed subsets of the real line.

Definition 4.1. A set M is defined to be an **open** set if for every point $x \in M$ there is an open interval that contains x and is a subset of M.

It is immediate from the definition that open intervals are in fact open sets.

Problem 4.2. Provide several examples of open sets that are not simply open intervals.

Problem 4.3. Show that the intersection of two (and hence, by induction, finitely many) open sets is open, but that the intersection of infinitely many open sets may not be open.

Definition 4.4. A set M is defined to be a **closed** set if every accumulation point of M is contained in M.

If M is a set, the set of accumulation points of M is sometimes denoted by M'. Using this notation, we can say that a set M is closed if and only if $M' \subseteq M$. Note that if a set M has no accumulation points, then it is vacuously closed.

Problem 4.5. Is every closed interval a closed set? Justify your answer.

Problem 4.6. Provide several examples of closed sets that are not closed intervals.

Problem 4.7. Provide an example of a set that is both open and closed.

One annoying feature of the terminology is that if a set is not open, it may or may not be closed. Similarly, if a set is not closed, it may or may not be open. That is, open and closed are not opposites of each other.

Problem 4.8. Provide an example of a set that is neither open nor closed.

Despite the fact that open and closed are not opposites of each other, there is a nice connection involving complements.

Problem 4.9. Prove that if M is a closed set such that $M \neq \mathbb{R}$, then M^c is an open set.

Problem 4.10. Prove that if M is an open set, then M^c is a closed set.

Definition 4.11. A set K is called **compact** if K is both closed and bounded.

It is important to point out that there is a more general definition of compact in an arbitrary topological space. However, using our notions of open and closed, it is a theorem that a subset of the real line is compact if and only if it is closed and bounded.

Problem 4.12. Provide several examples of sets that are compact and some that are not compact. Are finite sets compact?

Problem 4.13. Prove that if K is a compact set, then $\sup(K), \inf(K) \in K$.

The next problem is related to the Bolzano–Weierstrass Theorem.

Problem 4.14. Show that if K is compact, then any sequence with image set in K has a subsequence that converges to a point in K.

Problem 4.15. Come up with examples showing that if M is not closed or not bounded, then there exists a sequence with image set in M that does not have a subsequence converging to a point in M (or possibly not at all).

On the real line, compactness and satisfying the Bolzano–Weierstrass Theorem are equivalent. However, one can concoct examples of other mathematical spaces where they are not the same.