Chapter 3

The Binomial Theorem

Recall that $\binom{n}{k}$ counts the number of subsets of size k taken from a set of size n. Each $\binom{n}{k}$ is called a binomial coefficient, which likely seems like a strange name. Why are these numbers called binomial coefficients? In general a binomial is just a polynomial with two terms. Let's see what we can discover.

We see that

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= xx + xy + yx + yy\\ &= x^2 + 2xy + y^2.\end{aligned}$$

The coefficients in this expansion are $1, 2, 1$. Hey, we saw those row $n = 2$ of Pascal's Triangle! Let's try a larger example. We see that

$$\begin{aligned}(x+y)^3 &= (x+y)^2(x+y)\\ &= (xx + xy + yx + yy)(x+y)\\ &= xxx + xyx + yxx + yyx + xxy + xyy + yxy + yyy\\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

This time the coefficients are $1, 3, 3, 1$, which is the next row of Pascal's Triangle. What's going on here?

Consider the expansion of $(x+y)^n$. The key observation is that *before* commuting factors and collecting like terms, the terms in the expansion consist of *all possible* products where we choose either x or y from each factor. Each such term will consist of n letters. In particular, if there are k copies of x in a term, there will be $n-k$ copies of y (and vice versa). Moreover, every possible configuration of k copies x (i.e., location of the x's in the term *before* doing doing commuting with x and y) will be represented. This means there will be precisely $\binom{n}{k}$ many terms with k copies of x (and $n-k$ copies of y). Thus, when we commute and then collect like terms, the coefficient on $x^k y^{n-k}$ will indeed be $\binom{n}{k}$. This leads us to the following remarkable fact, known as the **Binomial Theorem**.

Theorem 3.1 (The Binomial Theorem)**.** For $n \geq 0$, we have

$$\boxed{(x+y)^n = \sum_{k=0} \binom{n}{k} x^k y^{n-k}.}$$

In light of the Binomial Theorem, the binomial coefficients are the positive integers that occur as coefficients in the expansions of powers of binomials. Said another way, the coefficients in the expansion of $(x+y)^n$ correspond to the entries in the nth row of Pascal's Triangle.

Problem 3.2. Expand each of the following.

(a) $(a+b)^6$

(b) $(2x-3y)^4$

Problem 3.3. What is the coefficient of x^{12} in $(x+2)^{15}$?

Problem 3.4. What is the coefficient of x^6y^6 in $(x^3+2y^2)^5$?

Problem 3.5. In Problem 2.20, we discovered the following combinatorial identity:

$$2^n = \sum_{k=0}^{n} \binom{n}{k} := \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

Prove this fact using the Binomial Theorem.

Problem 3.6. Determine what the sum $\displaystyle\sum_{k=0}^{n} 2^n \binom{n}{k}$ equal to.

Problem 3.7. Prove that the number of subsets of n with an odd number of elements equals the number of subsets with an even number of elements. That is, prove that

$$\sum_{k\geq 0} \binom{n}{2k} = \sum_{k\geq 0} \binom{n}{2k+1}.$$

Problem 3.8. Complete each of the following.

(a) Expand $(1+t)^n$.

(b) Assuming $n \geq 1$, take the derivative with respect to t of each side of the identity you discovered in Part (a).

(c) What happens if $t = 1$?

Problem 3.9. Let $1 \leq k \leq n$ and suppose we have n people. How many ways can we choose a team of k people to play dodgeball where one of the people on the team is designated team captain? *Hint:* Consider cases and then see if a recent problem is helpful.