Chapter 2

Permutations and Combinations

A k-**permutation** of a set A is an injective function $w : [k] \to A$. The set of all k-permutations of A is denoted by $\boxed{S_{A,k}}$. If A happens to be the set $[n]$, we use the notation $\boxed{S_{n,k}}$. And if $n = k$, we write $\boxed{S_n := S_{n,n}}$ and refer to each n-permutation in S_n as a **permutation**. Let $\boxed{P(n,k) := |S_{n,k}|}$. By convention, $\boxed{P(n,0) = 1}$.

We can denote a k-permutation as string $w = w(1)w(2)\cdots w(k)$, where each entry $w(i)$ that appears in the string is unique (since w is an injection). In other words, we can think of a k-permutation as a linear ordered arrangement of k of n objects.

Problem 2.1. Complete the following.

(a) Write down all of the elements in S_3. What is $P(3,3)$?

(b) Write down all of the elements in $S_{4,3}$. What is $P(4,3)$?

Recall that for $n \in \mathbb{N}$, the **factorial** of n is defined $\boxed{n! := n \cdot (n-1) \cdots 2 \cdot 1}$, and we define $\boxed{0! := 1}$ for convenience.

Problem 2.2. Consider the collection of k-permutations in $S_{n,k}$ with $1 \leq k \leq n$. Explain why $P(n,k)$ is equal to the number of nonattacking rook arrangements on an $n \times k$ chess board. *Hint:* Establish a bijection between the collection of nonattacking rook arrangements on an $n \times k$ chess board and the collection of k-permutations.

Theorem 2.3. For $1 \leq k \leq n$, we have

$$\boxed{P(n,k) = n \cdot (n-1) \cdots (n+1-k) = \frac{n!}{(n-k)!}.}$$

Note that as a special case of the formula above, we have $|S_n| = P(n,n) = n!$. For convenience, we can extend the formula above to obtain

$$\boxed{P(0,0) = \frac{0!}{(0-0)!} = 1} \quad \text{and} \quad \boxed{P(n,0) = \frac{n!}{(n-0)!} = 1.}$$

Problem 2.4. How many strings of length three are there using letters from $\{a, b, c, d, e, f, g\}$ if the letters in the string are not repeated?

Problem 2.5. There are 8 finalists at the Olympic Games 100 meters sprint. Assume there are no ties.

(a) How many ways are there for the runners to finish?

(b) How many ways are there for the runners to get gold, silver, bronze?

(c) How many ways are there for the runners to get gold, silver, bronze given that Usain Bolt is sure to get the gold medal?

Problem 2.6. If $1 \leq k \leq n$, prove that $\boxed{P(n,n) = P(n,k)P(n-k,n-k)}$, both using the formula in Theorem 2.3, and separately by using the definition of k-permutations together with the bijection principle.

Problem 2.7. If $1 \leq k \leq n$, prove that $\boxed{P(n,k) = P(n-1,k) + kP(n-1,k-1)}$, both using the formula in Theorem 2.3, and separately by using the definition of k-permutations together with the bijection principle.

Problem 2.8. How many ways can the letters of the word PRESCOTT be arranged?

Problem 2.9. How many ways can the letters of the word POPPY be arranged? Try to solve this problem in two different ways.

Consider a set of n objects that are not necessarily distinct, with p different objects and n objects of type i (for $i = 1, 2, \dots p$), so that $n = n_1 + \cdots + n_p$. An ordered arrangement of these n objects is called a **generalized permutation** and the number of such arrangements is denoted by $\boxed{P(n; n_1, \dots, n_p)}$. For example, the number of words we can make out of the letters of POPPY is $P(5; 3, 1, 1)$.

Theorem 2.10. For $n, n_1, \dots, n_p \in \mathbb{N}$ such that $n = n_1 + \cdots + n_p$, we have

$$\boxed{P(n; n_1, \dots, n_p) = \frac{n!}{n_1! \cdots n_p!}.}$$

Problem 2.11. How many ways can the letters of the word MISSISSIPPI be arranged?

Problem 2.12. In Professor X's class of 9 graduate students she will give two A's, one B, and six C's. How many possible ways are there to do this?

Problem 2.13. Let's revisit Problem 1.14, which involved my walk to get coffee. When we attacked that problem, we did a lot of brute force. Do we now have an easier method?

Problem 2.14. Six friends sit around a circle to play a game. Rotations of the group do not constitute different seating orders.

(a) How many circular seating arrangements are there?

(b) How many circular seating arrangements are there if Sally and Maria always sit next to each other?

The above problem involves what are sometimes called **circular permutations**.

Problem 2.15. How many circular permutations are there involving n objects?

More coming soon...