Homework 11

Combinatorics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on our Discord server. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

New! At the top of each problem, I would like you to list the students that you discussed the problem with. If you worked with the same peers on every problem, then you can simply indicate that once at the top of your assignment.

1. Let F and G be two rules for constructing combinatorial structures subject to the same constraints that we've seen in the past two homework assignments. Further let's suppose that $G[\emptyset] = \emptyset$ (i.e., there are no G-structures on the empty set). Using F and G, we can construct new structures by "composing" F and G. We define the rule $(F \circ G)$ as follows: An $(F \circ G)$-structure on a finite set U is a triplet $s = (\pi, \phi, \gamma)$, where

   - π is a partition on U;
   - ϕ is an F-structure on the set of blocks of π;
   - $\gamma = (\gamma_B)_{B \in \pi}$, where for each block B of π, γ_B is a G-structure on B.

   In other words, the any finite set U, we have

   $$(F \circ G)[U] = \bigsqcup_{\pi \text{ partition of } U} F[\pi] \times \prod_{B \in \pi} G[B].$$

   This looks more complicated than it is. An $(F \circ G)$-structure of U simply means we put G-structures on each of the blocks of a partition of U and then put an F-structure on the collection of blocks.

   (a) Prove that the corresponding exponential generating function satisfies $(F \circ G)(t) = F(G(t))$.

(b) Use the previous part to find a closed form for the exponential generating function for the number of set partitions of $[n]$. Compare to your answer and work to Problem 3 on Homework 7.

(c) An **ordered set partition** of $[n]$ is an ordered list of the form $(B_1, B_2, \ldots, B_k)$, where $\{B_1, B_2, \ldots, B_k\}$ is a set partition of $[n]$. Find a closed form for the exponential generating function for the number of ordered set partitions of $[n]$.

2. Let $\Pi(n)$ denote the poset of all set partitions of $[n]$ ordered by reverse refinement.

   (a) Draw the Hasse diagrams for $\Pi(4)$, highlighting NC(4) as a sub-poset.

   (b) Prove that $\Pi(n)$ is a lattice.

   (c) Enumerate the number of maximal chains in $\Pi(n)$.

3. Prove that the interval below any n-cycle in the absolute order on S_n is isomorphic (as posets) to any other. *Hint:* You may take for granted the following two facts regarding the symmetric groups: (1) Conjugation preserves cycle type, and (2) Given two permutations u and v with the same cycle type, there exists $w \in S_n$ such that $u = wvw^{-1}$. Show that conjugation by w takes cover relations to cover relations in the respective posets.

4. Let $p_{n,k}$ denote the number of integer partitions of n into k parts. Refine the rank generating function for the Young lattice $\mathcal{Y}$ (see previous homework) to obtain an expression for the following generating function:
$$\sum_{n,k\geq 0} p_{n,k} t^n z^k.$$

5. The *conjugate* of an integer partition λ is the partition λ' with Young diagram obtained by transposing (i.e., swap rows and columns) the Young diagram for λ. Let q_n denote the number of integer partitions that are *self-conjugate* (i.e., λ is equal to its conjugate). Prove that
$$\sum_{n\geq 0} q_n z^n = \prod_{i\geq 1} (1 + z^{2i-1}).$$
Hint: Show that q_n also counts the number of partitions of n into distinct odd parts.