Chapter 7

Integration

Unlike with differentiation, we will need a number of auxiliary definitions for beginning integration.

Definition 7.1. A set of points $P = \{t_0, t_1, \ldots, t_n\}$ is a *partition* of the closed interval $[a,b]$ if $a = t_0 < t_1 < \cdots t_{n-1} < t_n = b$. If $t_i - t_{i-1} = \frac{b-a}{n}$ for all i, we say that the partition is a *regular partition* of $[a,b]$. In this case, we may use the notation $\Delta t := t_i - t_{i-1}$.

Exercise 7.2. Give some partitions, regular and not regular, of $[0,1]$, $[2,4]$, and $[-1,0]$.

Definition 7.3. We say that a function is *bounded* if it has bounded image set.

Important! For the next four definitions, we assume that f is a bounded function with domain the closed interval $[a,b]$.

Definition 7.4. Let f be a bounded function with domain $[a,b]$ and let $\{t_0, t_1, \ldots, t_n\}$ be a partition of $[a,b]$. We say that any sum S of the form

$$S = \sum_{i=1}^{n} f(x_i)(t_i - t_{i-1}),$$

where $x_i \in [t_{i-1}, t_i]$ is a *Riemann sum* for f on $[a,b]$.

Definition 7.5. Let f be a bounded function with domain $[a,b]$ and let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a,b]$. For each $i \in \{1, 2, \ldots, n\}$, define $M_i := \sup\{f(x) \mid x \in [t_{i-1}, t_i]\}$. We say that the sum

$$U_P(f) := \sum_{i=1}^{n} M_i(t_i - t_{i-1}),$$

is the *upper Riemann sum* for f with partition P.

Definition 7.6. Let f be a bounded function with domain $[a,b]$ and let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a,b]$. For each $i \in \{1, 2, \ldots, n\}$, define $m_i := \inf\{f(x) \mid x \in [t_{i-1}, t_i]\}$. We say that the sum

$$L_P(f) := \sum_{i=1}^{n} m_i(t_i - t_{i-1}),$$

is the *lower Riemann sum* for f with partition P.

Exercise 7.7. Draw pictures that capture the concepts of upper and lower Riemann sums.

Contrary to the name, upper and lower Riemann sums are not always Riemann sums.

Problem 7.8. Give an example of an interval $[a,b]$, partition P, and bounded function f such that $U_p(f)$ is not a Riemann sum.

Problem 7.9. Define $f:[0,1]\to\mathbb{R}$ via

$$f(x)=\begin{cases}0, & x\in(0,1]\\ 1, & x=0.\end{cases}$$

(a) Show that $U_P(f)>0$ for all partitions of $[0,1]$.

(b) Show that for any positive number ϵ there is a partition P_ϵ such that $U_{P_\epsilon}(f)<\epsilon$.

(c) Fully describe all lower sums of f on $[0,1]$.

Problem 7.10. Define $f:[0,1]\to\mathbb{R}$ via $f(x)=x$. For each $n\in\mathbb{N}$, let P_n be the regular partition of $[0,1]$ given by $\left\{0,\frac{1}{n},\frac{2}{n},\ldots,\frac{n-1}{n},1\right\}$.

(a) Compute $U_{P_5}(f)$.

(b) Give a formula for $U_{P_n}(f)$.[1]

(c) Compute $L_{P_5}(f)$.

(d) Give a formula for $L_{P_n}(f)$.

Problem 7.11. Suppose that f is a bounded function on $[a,b]$ with lower bound m and upper bound M. Show that for any partition P of $[a,b]$, $U_P(f)\le M(b-a)$ and $L_P(f)\ge m(b-a)$.

Problem 7.12. Suppose that f is a bounded function on $[a,b]$ and P is a partition of $[a,b]$. Show that $L_P(f)\le U_P(f)$.

Note that as a result of the previous problems that the set of all upper and lower sums of f is a bounded point set.

More coming soon...

[1]Recall that the sum $\sum_{i=1}^{k} i=\frac{k(k+1)}{2}$.