Final Exam

Instructions

Answer each of the following questions and then submit your solutions to BbLearn before **10:00am today** (Monday, November 23). You can either write your solutions on paper and then capture your work digitally or you can write your solutions digitally on a tablet (e.g., iPad).

This exam is worth a total of 42 points and is worth 20% of your overall grade. Good luck and have fun!

1. (1 point each) True or False? Circle the correct answer. You do *not* need to justify your answer.

   (a) If $g \in G$ and G is a finite group, then $g^{|G|} = e$.
   True **False**

   (b) If G is a group of order p such that p is prime, then G is cyclic.
   True **False**

   (c) If G is a finite abelian group and $a, b \in G$, then $|ab| = \text{lcm}(|a|, |b|)$.
   True **False**

   (d) If G is group, then $|ab| = |ba|$ for all $a, b \in G$.
   True **False**

   (e) A cyclic group of order 20 contains an element of order 5.
   True **False**

   (f) If G is a finite group of order n and k is a positive integer that divides n, then G has a subgroup of order k.
   True **False**

   (g) The group $\mathbb{Z}_6 \times \mathbb{Z}_8$ is cyclic.
   True **False**

   (h) If H and K are subgroups of a finite group G, then $|H \cap K|$ divides both $|H|$ and $|K|$.
   True **False**

   (i) The symmetric group S_4 has a subgroup isomorphic to $\mathbb{Z}_6$.
   True **False**

   (j) If G_1 and G_2 are groups, then all subgroups of $G_1 \times G_2$ are of the form $H_1 \times H_2$, where $H_1 \leq G_1$ and $H_2 \leq G_2$.
   True **False**

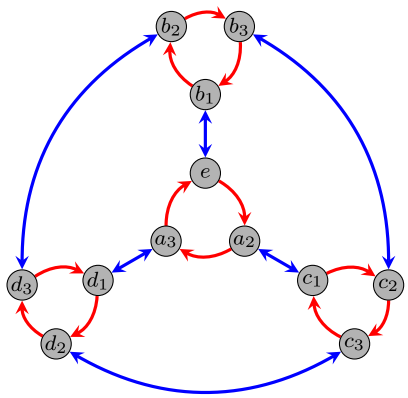
   (k) If $H \leq K \leq G$, then each left coset of H in G is contained in a left coset of K in G.
   True **False**

   (l) If G is a group, then every subgroup of the center of G (i.e., $Z(G)$) is normal in G.
   True **False**

   (m) If $H \trianglelefteq G$ and G/H is cyclic, then G is cyclic.
   True **False**

   (n) If $H \trianglelefteq G$ and G is abelian, then G/H is abelian.
   True **False**

2. (2 points each) The following diagram is the Cayley diagram for the alternating group A_4 using the permutations $(1,2,3)$ and $(1,2)(3,4)$ as generators. You should assume that $e = (1)$ (i.e., the identity in A_4).



   (a) Identify which permutations correspond to the vertices labeled a_2, a_3, b_1, b_2, b_3.

   (b) Perform the quotient process using the subgroup $\langle(1,2,3)\rangle$. Draw the resulting diagram. You do not need to worry about labeling the vertices.

   (c) Does the diagram that you obtained in part (b) correspond to the Cayley diagram for a group? Briefly explain your answer.

   (d) Is the subgroup $\langle(1,2,3)\rangle$ normal in A_4? Briefly justify your answer by either referring to one of your answers in a previous part or by computing cosets.

3. (3 points each) Complete $\boxed{\textbf{two}}$ of the following problems.

   (a) Find the smallest n such that A_n contains an element of order 18.

   (b) Find the largest possible order of an element in $\mathbb{Z}_9 \times \mathbb{Z}_{12}$.

   (c) Explain why the only group homomorphism from D_3 to $\mathbb{Z}_3$ is the trivial homomorphism (i.e., $\varphi(g) = 0$ for all $g \in D_3$).

   (d) Suppose $\varphi : \mathbb{Z}_{20} \to \mathbb{Z}_{20}$ is a group homomorphism such that $\ker(\varphi) = \{0, 5, 10, 15\}$. If $\varphi(13) = 8$, determine all elements that φ maps to 8.

4. Complete $\boxed{\textbf{one}}$ the following. If you choose part (a), you must do both subparts. If you choose part (b), you must do all three subparts.

   (a) (3 points each) Let G be a cyclic group of order n. Fix a positive integer k and define the function $\varphi : G \to G$ via $\varphi(x) = x^k$ for all $x \in G$.

      i. Prove that φ satisfies the homomorphic property.

      ii. Prove that φ is an isomorphism if and only if $\gcd(n, k) = 1$. *Hint:* Suppose g is the generator of G and compute the order of g^k.

   (b) (2 points each) Suppose $\varphi : \mathbb{Z}_{10} \to \mathbb{Z}_6$ satisfies the homomorphic property such that $\varphi(1) = 3$.

      i. List the elements in the range of φ (i.e., $\varphi(\mathbb{Z}_{10})$).

      ii. List the elements in $\ker(\varphi)$.

      iii. What well-known group is $\mathbb{Z}_{10}/\ker(\varphi)$ isomorphic to?

5. (4 points) Prove $\boxed{\textbf{one}}$ of the following theorems. *Overall Hint:* All of these look more terrifying than they are. Each one has a relatively short proof.

   Theorem 1. Assume G is a group and $H \leq G$ such that $|H| = 2$. Then H is a normal subgroup of G if and only if H is contained in the center of G. *Hint:* Suppose $H = \{e, h\}$ and use the definition center together with the definition of normal subgroup (i.e., H normal in G if and only if $gH = Hg$ for all $g \in G$).

   Theorem 2. If G is a finite abelian group of odd order, then the product of all of the elements in G must be equal to the identity in G.

   Theorem 3. If G is a group and H is a subgroup of G such that H is the only subgroup with order $|H|$, then H is a normal subgroup of G. *Hint:* Consider using Theorem 5.35 together with Theorem 3.64.

   Theorem 4. If G is a group such that $N \leq H \leq G$ and $N \trianglelefteq G$, then $H/N := \{hN \mid h \in H\}$ is a subgroup of G/N.* *Hint:* This looks harder than it is. Is N a normal subgroup of H?

6. (4 points) Prove $\boxed{\textbf{one}}$ of the following theorems.

   Theorem 5. As groups, $\mathbb{R}^*/\langle -1 \rangle \cong \mathbb{R}^+$. (Recall that $\mathbb{R}^*$ is the group of nonzero real numbers and $\mathbb{R}^+$ is the group of positive real numbers, both of which are groups under multiplication.) *Hint:* Use the First Isomorphism Theorem.

   Theorem 6. If G and H are groups, then $(G \times H)/(\{e_G\} \times H) \cong G$ (where e_G is the identity in G). *Hint:* Use the First Isomorphism Theorem.

   Theorem 7. As groups, $D_4/\langle r^2 \rangle \cong V_4$. *Hint:* There are a few ways you could approach this problem: (i) use the First Isomorphism Theorem, (ii) draw the Cayley diagram for D_4 using generators r and s, add an extra arrow for r^2 to your diagram, and then perform quotient process, or (iii) consider the possible isomorphism types of $D_4/\langle r^2 \rangle$ and eliminate all except V_4.

*It turns out that every subgroup of G/N is of this form.