Exam 1 (Part 2)

Your Name:

Names of Any Collaborators:

Instructions

Answer each of the following questions and then submit your solutions by the start of class on **Wednesday, March 2**.

This part of Exam 1 is worth a total of 16 points and is worth 40% of your overall score on Exam 1. Your overall score on Exam 1 is worth 20% of your overall grade. I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts. Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing the following problems, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. Further information:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Problem 3.16, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.** To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

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Note: When proving a result below, you may utilize any result that appears before it on this page (even if it is a result that you chose not to prove). You may also use any result from the book that we have previously discussed.

1. (4 points) Prove **one** of the following theorems.

   Theorem 1. Suppose G is a group and let $g \in G$ such that $\langle g \rangle$ is finite. If n is the smallest positive integer such that $g^n = e$, then $\langle g \rangle = \{e, g, g^2, \ldots, g^{n-1}\}$ and this set contains n distinct elements.*

   Theorem 2. Suppose G is a group and let $g \in G$ such that $\langle g \rangle$ is finite. If n is the smallest positive integer such that $g^n = e$ and $g^i = g^j$, then n divides $i - j$.†

2. (4 points) Prove **one** of the following theorems.

   Theorem 3. Let G be a group and let $H \leq G$. Define the relation $\sim$ on G via

   $$a \sim b \text{ if and only if } ab^{-1} \in H.$$

   Then $\sim$ is an equivalence relation.‡

   Theorem 4. Let G be a group, $H \leq G$, and $a \in G$. Define the set $Ha := \{ha \mid h \in H\}$ and the function $f : H \to Ha$ via $f(h) = ha$. Then f is a bijection.

   Theorem 5. If G is a group, $H \leq G$, and $\sim$ is as in Theorem 3 above, then $[a] = Ha$, where $[a]$ is the equivalence class containing a and Ha is the set in Theorem 4.§

3. (4 points) Prove that if G is a finite group and $H \leq G$, then $|H|$ divides $|G|$.¶

4. (4 points) If $(G_1, *)$ and $(G_2, \circ)$ are groups, let's call a function $\phi : G_1 \to G_2$ **wicked awesome** if it satisfies

   $$\phi(a * b) = \phi(a) \circ \phi(b)$$

   for all $a, b \in G_1$. For each of the following, assume ϕ is wicked awesome. Define $K_\phi := \{a \in G_1 \mid \phi(a) = e_2\}$. Note that K_ϕ is analogous to the null space in linear algebra. Complete **one** of the following.

   (a) Prove that $\phi(e_1) = e_2$, where e_1 and e_2 are the identities of G_1 and G_2, respectively.

   (b) Prove that $\phi(a^{-1}) = [\phi(a)]^{-1}$ for all $a \in G_1$.

   (c) Prove that $K_\phi \leq G_1$.

   (d) Prove that if $K_\phi = \{e_1\}$, then ϕ is an injective function.

*Theorem 2.80 guarantees the existence of such an exponent. To prove this theorem, I suggest you make use of the Division Algorithm, which states that if a is a positive integer and b is any integer, then there exist unique integers q (called the **quotient**) and r (called the **remainder**) such that $b = aq + r$, where $0 \leq r < a$. By the way, the claim that the set contains n distinct elements is not immediate. You need to argue that there are no repeats in the list.

†Try using the Division Algorithm (see previous footnote).

‡Recall that an equivalence relation is reflexive, symmetric, and transitive.

§Since $\sim$ is an equivalence relation, for each $a \in G$, we can define the equivalence class $[a] := \{g \in G \mid a \sim g\}$. Note that by MAT 320, an immediate consequence of Theorem 3 is that the collection of equivalence classes partition G (you do not need to prove this). One approach to proving that $[a] = Ha$ is to do two set containment arguments.

¶You should not do anything too complicated when proving this result. Use Theorem 5 and the fact that the equivalence classes referenced in Theorem 5 partition G together with Theorem 4.

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