Chapter 7

Introduction to Graph Theory

Loosely speaking, a graph is a collection of points called vertices and connecting segments called edges, each of which starts at a vertex, ends at a vertex and contains no other vertices beside these. More formally, we define the term as follows. A **graph** consists of two sets, a nonempty set V of points called **vertices** and a set E whose elements, called **edges**, are multisets of size two from V.

Each edge is associated with either one vertex which serves as both endpoints or two vertices as its endpoints. Technically, each edge is a multiset of the form $\boxed{\{\!\{u, v\}\!\}}$ where $u, v \in V$. We say that u and v are **endpoints** of the edge $\{\!\{u, v\}\!\}$. In an abuse of notation, it is customary to write $\boxed{\{u, v\}}$ even if $u = v$. In fact, we may abbreviate further and denote the edge by $\boxed{uv}$. Note that the order in which the vertices of an edge are listed is irrelevant. That is, $\{\!\{u, v\}\!\} = \{\!\{v, u\}\!\}$, $\{u, v\} = \{v, u\}$, and $uv = vu$. If G is the graph associated with the vertex set V and edge set E, we write $\boxed{G = (V, E)}$. It is worth pointing out that we assumed that V is nonempty, but E is allowed to be empty (i.e., the graph has no edges).

It is customary to represent a graph using visual representations, where each vertex is a dot and each edge is a connecting segment, not necessarily straight.

Problem 7.1. Find at least five different graphs with vertex sets $V = \{a, b, c\}$.

There is a lot of terminology associated to graphs! Here are some of the relevant concepts.

- Vertices u and v of a graph are **adjacent** if they are the endpoints of the same edge.
- If v is an endpoint of the edge e, we say that e is **incident** to v.
- If an edge e is incident to vertices u and v, we say that u and v are **connected** by edge e.
- An edge e that is incident to a single vertex (i.e., $e = uu$ for some $u \in V$) is called a **loop**.
- The **order** of a graph is the number of vertices in the graph. That is, if $G = (V, E)$, then the order of G is $|V|$.

- The **degree** of a vertex v, written $\deg(v)$, is the number of edges incident to v (i.e., the number of edges that have v as an endpoint). Note that a loop contributes 2 to a vertex's degree, one for each of the two ends of the edge. The degree of a vertex v is denoted $\boxed{\deg(v)}$.

Many graphs have similar properties that allow us to categorize them. Here are several families of graphs.

- Complete Graphs. The **complete graph** on $n \geq 1$ vertices, denoted $\boxed{K_n}$, is the graph of order n such that each pair of vertices is connected by exactly one edge, and there are no other edges (i.e., no loops).

- Cycle Graphs. The **cycle graph** on $n \geq 3$ vertices, denoted $\boxed{C_n}$, is the graph such that when the n vertices are suitably labeled $v_1, v_2, \ldots, v_n$, the edges of C_n are $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

- Path Graphs. The **path** on $n \geq 1$ vertices, denoted $\boxed{P_n}$, has a description similar to C_n: for distinct vertices suitably labeled $v_1, v_2, \ldots, v_n$, the edges of P_n are $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$.

- Wheel Graphs. The **wheel graph** on $n \geq 4$ vertices, denoted $\boxed{W_n}$, is the graph C_{n-1} together with one additional vertex that is connected to each of the vertices of C_{n-1}.

- Hypercube Graphs. The **hypercube** of dimension $n \geq 1$, denoted $\boxed{Q_n}$, is the graph whose vertices are labeled with the 2^n bit strings of length n with an edge connecting two vertices if and only if their labels differ in exactly one bit.

Problem 7.2. Draw the first few graphs of each of the graph families above.

Problem 7.3. How many edges do each of the following have?

(a) K_n

(b) C_n

(c) P_n

(d) W_n

(e) Q_n

A **simple graph** is a graph in which each edge connects two distinct vertices and each pair of vertices is connected by at most one edge. Note that the graphs K_n, C_n, P_n, W_n, and Q_n are all simple graphs. A **pseudograph** (or **multigraph**) is like a graph but we allow **multiple edges** between a pair of vertices (i.e., E is a multiset instead of a set).

Problem 7.4. Draw examples of simple graphs, non-simple graphs, and psuedographs on 3 vertices.

A simple graph $G = (V, E)$ is **bipartite** if there is a partition of V into two nonempty sets V_1, V_2 (i.e., $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, $V_1 \cap V_2 = \emptyset$, and $V_1 \cup V_2 = V$) such that each edge of G connects a vertex in V_1 and a vertex in V_2. The pair (V_1, V_2) is called a **bipartition** of the graph.

Problem 7.5. Provide an example of a bipartite graph with 5 vertices.

The following theorem provides a nice characterization of bipartite graphs.

Theorem 7.6. A graph is bipartite if each vertex can be colored with one of two colors so that each pair of adjacent vertices have different colors.

Problem 7.7. Which complete graphs are bipartite?

Problem 7.8. Which path graphs are bipartite?

Problem 7.9. Which cycle graphs are bipartite?

Problem 7.10. Is Q_3 bipartite?

A bipartite graph with bipartition (V_1, V_2) such that $|V_1| = m$ and $|V_2| = n$ is the **complete bipartite graph** $K_{m,n}$ if it contains each edge $\{u, v\}$ for every pair $u \in V_1$ and $v \in V_2$. Note that $K_{m,n} = K_{n,m}$.

Problem 7.11. Draw $K_{1,1}$, $K_{1,2}$, $K_{2,2}$, $K_{2,3}$, $K_{3,3}$.

The next result is sometimes referred to as the **Handshake Lemma**. Do you see why?

Theorem 7.12 (Degree Sum Formula)**.** In any graph, the sum of the degrees of vertices in the graph is always twice the number of edges. In other words, in a graph $G = (V, E)$,

$$2|E| = \sum_{v \in V} \deg(v).$$

Problem 7.13. At a recent party, 9 people greeted each other by shaking hands. Is it possible that each person shook hands with exactly 7 people at the party?

Sometimes it is convenient to use the term **even vertex** or **odd vertex** to refer to a vertex whose degree is even or odd, respectively.

Problem 7.14. Explain why every graph has an even number of odd vertices.

The **degree sequence** of a graph is the list of the degrees of the vertices of the graph in descending order. A finite list of nonnegative integers in descending order is **graphic** if it is the degree sequence of a simple graph.

Problem 7.15. Find the degree sequences for K_n $(n \geq 1)$, C_n $(n \geq 3)$, P_n $(n \geq 1)$, W_n $(n \geq 4)$, and Q_n $(n \geq 1)$.

Problem 7.16. Which of the following are graphic sequences?

(a) 3332

(b) 3331

(c) 44332

Problem 7.17. Find two different graphs that have 32222111 as their degree sequence.

Theorem 7.18. If $d_1 d_2 \cdots d_n$ is the degree sequence for a graph G of order n, then $\sum_{i=1}^{n} d_i$ must be even.

One consequence of the previous theorem is that any sequence with an odd sum (e.g., 331) is not graphic. It turns out that if a sequence has an even sum, it is the degree sequence of a multigraph. The construction of such a graph is straightforward: connect vertices with odd degrees in pairs, and fill out the remaining even degree counts by self-loops. The question of whether a given degree sequence can be realized by a simple graph is more challenging. This problem is also called **graph realization problem** and can be solved by either the Erdös–Gallai theorem or the Havel–Hakimi algorithm. Unfortunately, this is beyond the scope of this course.

We will now focus on making new graphs from old. Below are several definitions.

- A graph $H = (V_H, E_H)$ is a **subgraph** of a graph $G = (V_G, E_G)$ if $V_H \subseteq V_G$ and $E_H \subseteq E_G$ (i.e., the vertices of H are vertices of G and the edges of H are edges of G). If H is a subgraph of G, we may write $\boxed{H \subseteq G}$.

- A graph $H = (V_H, E_H)$ is an **induced subgraph** of a graph $G = (V_G, E_G)$ if $V_H \subseteq V_G$ and E_H consists of all of the edges in E_G that have both endpoints in E_H. That is, for any two vertices $u, v \in V_H$, u and v are adjacent in H if and only if u and v are adjacent in G.

- If $G = (V, E)$ is a graph and $S \subseteq V$, then the **subgraph of G induced by S**, denoted $\boxed{G[S]}$, is the induced subgraph of G with vertex set S.

- The **union** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$.

- The **complement** of a *simple* graph G of order n is the graph $\boxed{\overline{G}}$ on the same n vertices such for each pair of distinct vertices u and v, $\{u, v\}$ is an edge of $\overline{G}$ if and only if it is not an edge of G.

Problem 7.19. Make up a few examples of graphs and explore the concepts above.

Problem 7.20. Find a subgraph of C_4 that is not an induced subgraph of C_4.

Problem 7.21. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) Any subgraph of a complete graph is also complete.

(b) Any induced subgraph of a complete graph is also complete.

(c) Any subgraph of a bipartite graph is bipartite.

If $G = (V, E)$ is a graph and $S \subseteq V$, the **neighborhood** of S, denoted $N(S)$ is the set of all vertices in V adjacent to at least one member of S. Of course, we can consider the neighborhood of a single vertex v, which is denoted $\boxed{N(v)}$. A neighborhood of a single vertex does not include itself, and is more specifically the **open neighborhood** of v. It is also possible to define a neighborhood in which v itself is also included. This is called the **closed neighborhood** of v,is sometimes denoted by $\boxed{N[v]}$. Otherwise stated otherwise, a neighborhood is assumed to be open.

Problem 7.22. Make up a few examples of graphs and explore the concept of neighborhood.

Problem 7.23. Consider the graph $K_{3,5}$ with bipartition V_1 and V_2, where $|V_1| = 3$ and $|V_2| = 5$.

(a) For $v \in V_1$, what is $N(v)$?

(b) What is $N(V_1)$?

(c) What is $N(V_2)$?

A **matching** (or **independent edge set**) in a graph $G = (V, E)$ is a subset of edges $M \subseteq E$ without common vertices. That is, a subset of the edges is a matching if each vertex appears as an endpoint in at most one edge of that matching. If M is a matching, a vertex is said to be **matched** if it is an endpoint of one of the edges in M. Otherwise, the vertex is called **unmatched**. We say that M covers a subset $S \subseteq V$ if every vertex of S is matched by M.

We now explore a type of matching problem. Suppose we have a bipartite graph $G = (V, E)$ with a bipartition (V_1, V_2). We want to match up each element $v_1 \in V_1$ with exactly one element $v_2 \in V_2$ that is adjacent to it in G and that is not matched to any other element of V_1.

A **total matching** from V_1 to V_2 is matching that covers V_1. In other words, for each $v_1 \in V_1$, there is a unique edge $m \in M$ and a unique $v_2 \in V_2$ such that m is incident to v_1 and v_2. We can think of M as specifying an injective function from V_1 to V_2. If every vertex in V_2 is also matched (i.e., V_2 is covered), then the matching is called a **perfect matching**.

Finding a matching in a bipartite graph can be treated as a network flow problem.

Problem 7.24. Find an example of a bipartite graph G with bipartition (V_1, V_2) that has a total matching from V_1 to V_2.

Problem 7.25. Find an example of a bipartite graph G with bipartition (V_1, V_2) that does not have a total matching from V_1 to V_2.

Problem 7.26. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If a bipartite graph $G = (V, E)$ has a perfect matching, then $|V|$ is even.

(b) If $G = (V, E)$ is a bipartite graph such that $|V|$ is even, then G has a perfect matching.

In order for there to be a total matching from V_1 to V_2, we need $\deg(v_1) > 0$ for each $v_1 \in V_1$. However, that is not enough. Phillip Hall (1904–1982) discovered the following condition, known as **Hall's Marriage Theorem**, needed for a total matching.

Theorem 7.27 (Hall's Marriage Theorem)**.** A bipartite graph with bipartition (V_1, V_2) has a total matching from V_1 to V_2 if and only if $|N(S)| \geq |S|$ for all subsets $S \subseteq V_1$. In other words, every subset S of V_1 must have sufficiently many neighbors in V_2.

Hall actually stated and proved a more general theorem, but we have given its formulation in the context of graph theory.

A graph $G = (V, E)$ is called ***k*-regular** if $\deg(v) = k$ for every $v \in V$.

Example 7.28. The cycle graph C_n is 2-regular, the complete graph K_n is $(n-1)$-regular, the hypercube graph Q_n is n-regular, and the complete bipartite graph $K_{n,n}$ is n-regular.

Problem 7.29. Prove that if G is a k-regular bipartite graph with bipartition (V_1, V_2), then $|V_1| = |V_2|$.

Problem 7.30. Prove that if G is a k-regular bipartite graph with $k > 1$, then G has a perfect matching.