Exam 1 (Part 2)

Your Name:

Names of Any Collaborators:

Instructions

Answer each of the following questions and then submit your solutions to BbLearn by **11:59pm on Wednesday, September 23**. You can either write your solutions on paper and then capture your work digitally or you can write your solutions digitally on a tablet (e.g., iPad).

This part of Exam 1 is worth a total of 23 points and is worth 30% of your overall score on Exam 1. Your overall score on Exam 1 is worth 20% of your overall grade. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 5.35, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

 Written by D.C. Ernst

1. (3 points) Find all subgroups of D_4. To make my grading easier, please list them by order (smallest to largest).

2. (4 points each) Prove **two** of the following theorems.

   Theorem 1. Suppose G is a group and let $g \in G$ such that $\langle g \rangle$ is finite. If n is the smallest positive integer such that $g^n = e$, then $\langle g \rangle = \{e, g, g^2, \ldots, g^{n-1}\}$ and this set contains n distinct elements.*

   Theorem 2. Suppose G is a group and let $g \in G$ such that $\langle g \rangle$ is finite. If n is the smallest positive integer such that $g^n = e$ and $g^i = g^j$, then n divides $i - j$.†

   Theorem 3. Let G be a group and let $H \leq G$. Define the relation $\sim$ on G via

   $$a \sim b \text{ if and only if } ab^{-1} \in H.$$

   Then $\sim$ is an equivalence relation.‡

   Theorem 4. Let G be a group, $H \leq G$, and $a \in G$. Define the set $Ha := \{ha \mid h \in H\}$ and the function $f : H \to Ha$ via $f(h) = ha$. Then f is one-to-one and onto.

3. (4 points) Let G be a group, $H \leq G$, and let $\sim$ be as in Theorem 3 above. Since $\sim$ is an equivalence relation, for each $a \in G$, we can define the equivalence class $[a] := \{g \in G \mid a \sim g\}$. Note that by MAT 320, an immediate consequence of Theorem 3 is that the set of the equivalence classes partition G (you do not need to prove this). Prove that $[a] = Ha$, where Ha is as in Theorem 4 above.§

4. (2 points) Let $G = D_4$ and let $H = \langle s \rangle$. Find all of the distinct sets Ha as defined in Theorem 4 above. Any observations?

5. (2 point) Prove that if G is a finite group and $H \leq G$, then $|H|$ divides $|G|$.¶

6. If $(G_1, *)$ and $(G_2, \circ)$ are groups, let's call a function $\phi : G_1 \to G_2$ is *wicked awesome* if it satisfies

   $$\phi(a * b) = \phi(a) \circ \phi(b)$$

   for all $a, b \in G_1$. For each of the following, assume ϕ is wicked awesome.

   (a) (2 points) Prove that $\phi(e_1) = e_2$, where e_1 and e_2 are the identities of G_1 and G_2, respectively.

   (b) (2 points) Prove that $\phi(a^{-1}) = [\phi(a)]^{-1}$ for all $a \in G_1$.

*Theorem 2.79 guarantees the existence of such an exponent. To prove this theorem, I suggest you make use of the Division Algorithm, which states that if a is a positive integer and b is any integer, then there exist unique integers q (called the **quotient**) and r (called the **remainder**) such that $b = aq + r$, where $0 \leq r < a$. By the way, the claim that the set contains n distinct elements is not immediate. You need to argue that there are no repeats in the list.

†Try using the Division Algorithm.

‡Recall that an equivalence relation is reflexive, symmetric, and transitive.

§One approach is to do two set containment arguments.

¶The number of points this problem is worth is a hint that you should not do anything too complicated. Use Problem 3 and the fact that the equivalence classes referenced in Problem 3 partition G together with Theorem 4.

