All truths are easy to understand once they are discovered; the point is to discover them.

Galileo Galilei, astronomer & physicist

Chapter 3

The Real Numbers

In this chapter we will take a deep dive into structure of the real numbers by building up the multitude of properties you are familiar with by starting with a collection of fundamental axioms. Recall that an axiom is a statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proved. It is worth pointing out that one can carefully construct the real numbers from the natural numbers. However, that will not be the approach we take. Instead, we will simply list the axioms that the real numbers satisfy. Our axioms for the real numbers fall into three categories:

1. **Field Axioms:** These axioms provide the essential properties of arithmetic involving addition and subtraction.

2. **Order Axioms:** These axioms provide the necessary properties of inequalities.

3. **Completeness Axiom:** This axiom ensures that the familiar number line that we use to model the real numbers does not have any holes in it.

Throughout this book, our universe of discourse will be the real numbers. Any time we refer to a generic set, we mean a subset of real numbers. We will often refer to an element in a subset of real numbers as a **point**.

3.1 The Field Axioms

We begin with the Field Axioms.

Axioms 3.1 (Field Axioms)**.** There exist operations $+$ (addition) and $\cdot$ (multiplication) on $\mathbb{R}$ satisfying:

(F1) (Associativity for Addition) For all $a, b, c \in \mathbb{R}$ we have $(a+b)+c = a+(b+c)$;

(F2) (Commutativity for Addition) For all $a, b \in \mathbb{R}$, we have $a+b = b+a$;

(F3) (Additive Identity) There exists $0 \in \mathbb{R}$ such that for all $a \in \mathbb{R}$, $0+a = a$;

(F4) (Additive Inverses) For all $a \in \mathbb{R}$ there exists $-a \in \mathbb{R}$ such that $a+(-a) = 0$;

(F5) (Associativity for Multiplication) For all $a, b, c \in \mathbb{R}$ we have $(ab)c = a(bc)$;

(F6) (Commutativity for Multiplication) For all $a, b \in \mathbb{R}$, we have $ab = ba$;

(F7) (Multiplicative Identity) There exists $1 \in \mathbb{R}$ such that $1 \neq 0$ and for all $a \in \mathbb{R}$, $1a = a$;

(F8) (Multiplicative Inverses) For all $a \in \mathbb{R} \setminus \{0\}$ there exists $a^{-1} \in \mathbb{R}$ such that $aa^{-1} = 1$.

(F9) (Distributive Property) For all $a, b, c \in \mathbb{R}$, $a(b + c) = ab + ac$;

In the language of abstract algebra, Axioms F1–F4 and F5–F8 make each of $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$ an abelian group under addition and multiplication, respectively. Axiom F9 provides a way for the operations of addition and multiplication to interact. Collectively, Axioms F1–F9 make the real numbers a **field**. It follows from the axioms that the elements 0 and 1 of $\mathbb{R}$ are the unique **additive** and **multiplicative identities** in $\mathbb{R}$. For the next proof, suppose 0 and $0'$ are both additive identities in $\mathbb{R}$ and then show that $0 = 0'$. This shows that there can only be one additive identity.

Problem 3.2. Prove that the additive identity of $\mathbb{R}$ is unique.

For the next problem, mimic the approach you used to prove Problem 3.2.

Problem 3.3. Prove that the multiplicative identity of $\mathbb{R}$ is unique.

For every $a \in \mathbb{R}$, the elements $-a$ and a^{-1} (as long as $a \neq 0$) are also the unique **additive** and **multiplicative inverses**, respectively.

Problem 3.4. Prove that every real number has a unique additive inverse.

Problem 3.5. Prove that every nonzero real number has a unique multiplicative inverse.

Since we are taking a formal axiomatic approach to the real numbers, we should make it clear how the natural numbers are embedded in $\mathbb{R}$.

Definition 3.6. We define the **natural numbers**, denoted by $\mathbb{N}$, to be the smallest subset of $\mathbb{R}$ satisfying:

(a) $1 \in \mathbb{N}$, and

(b) for all $n \in \mathbb{N}$, we have $n + 1 \in \mathbb{N}$.

Of course, we use the standard numeral system to represent the natural numbers, so that $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \ldots\}$.

Given the natural numbers, Axiom F3/Problem 3.2 and Axiom F4/Problem 3.4 together with the operation of addition allow us to define the **integers**, denoted by $\mathbb{Z}$, in the obvious way. That is, the integers consist of the natural numbers together with the additive identity and all of the additive inverses of the natural numbers.

We now introduce some common notation that you are likely familiar with. Take a moment to think about why the following is a definition as opposed to an axiom or theorem.

Definition 3.7. For every $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}$, we define the following:

(a) $\boxed{a - b := a + (-b)}$

(b) $\boxed{\dfrac{a}{b} := ab^{-1}}$ (for $b \neq 0$)

(c) $$\boxed{a^n := \begin{cases} \overbrace{aa\cdots a}^{n}, & \text{if } n \in \mathbb{N} \\ 1, & \text{if } n = 0 \text{ and } a \neq 0 \\ \dfrac{1}{a^{-n}}, & \text{if } -n \in \mathbb{N} \text{ and } a \neq 0 \end{cases}}$$

The set of **rational numbers**, denoted by $\mathbb{Q}$, is defined to be the collection of all real numbers having the form given in Part (b) of Definition 3.7. The **irrational numbers** are defined to be $\mathbb{R} \setminus \mathbb{Q}$.

Using the Field Axioms, we can prove each of the following statements.

Problem 3.8. Prove that for all $a, b, c \in \mathbb{R}$, we have the following:

(a) $a = b$ if and only if $a + c = b + c$;

(b) $0a = 0$;

(c) $-a = (-1)a$;

(d) $(-1)^2 = 1$;

(e) $-(-a) = a$;

(f) If $a \neq 0$, then $(a^{-1})^{-1} = a$;

(g) If $a \neq 0$ and $ab = ac$, then $b = c$.

(h) If $ab = 0$, then either $a = 0$ or $b = 0$.

Problem 3.9. Carefully prove that for all $a, b \in \mathbb{R}$, we have $(a + b)(a - b) = a^2 - b^2$. Explicitly cite where you are utilizing the Field Axioms and Problem 3.8.

> Like what you do, and then you will do your best.
>
> ---
>
> Katherine Johnson, mathematician

3.2 The Order Axioms

We now introduce the Order Axioms of the real numbers.

Axioms 3.10 (Order Axioms)**.** For $a, b, c \in \mathbb{R}$, there is a relation $\boxed{<}$ on $\mathbb{R}$ satisfying:

(O1) (Trichotomy Law) If $a \neq b$, then either $a < b$ or $b < a$ but not both;

(O2) (Transitivity) If $a < b$ and $b < c$, then $a < c$;

(O3) If $a < b$, then $a + c < b + c$;

(O4) If $a < b$ and $0 < c$, then $ac < bc$;

Given Axioms O1–O4, we say that the real numbers are a **linearly ordered field**. We call numbers greater than zero **positive** and those greater than or equal to zero **nonnegative**. There are similar definitions for **negative** and **nonpositive**.

Notice that the Order Axioms are phrased in terms of "<". We would also like to be able to utilize ">", "≤", and "≥".

Definition 3.11. For $a, b \in \mathbb{R}$, we define:

(a) $\boxed{a > b}$ if $b < a$;

(b) $\boxed{a \leq b}$ if $a < b$ or $a = b$;

(c) $\boxed{a \geq b}$ if $b \leq a$.

Using the inequalities on the real numbers, we can now define the following special sets.

Definition 3.12. For $a, b \in \mathbb{R}$ with $a < b$, we define the following sets, referred to as **intervals**.

(a) $\boxed{(a, b) := \{x \in \mathbb{R} \mid a < x < b\}}$

(b) $\boxed{[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}}$

(c) $\boxed{[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}}$

(d) $\boxed{(a, \infty) := \{x \in \mathbb{R} \mid a < x\}}$

(e) $\boxed{(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}}$

(f) $\boxed{(-\infty, \infty) := \mathbb{R}}$

We analogously define $\boxed{(a, b]}$, $\boxed{[a, \infty)}$, and $\boxed{(-\infty, b]}$. Intervals of the form (a, b), $(-\infty, b)$, (a, ∞), and $(-\infty, \infty)$ are called **open intervals** while $[a, b]$ is referred to as a **closed interval**. A **bounded interval** is any interval of the form (a, b), $[a, b)$, $(a, b]$, and $[a, b]$. For bounded intervals, a and b are called the **endpoints** of the interval.

We will always assume that any time we write $(a,b),[a,b],(a,b]$, or $[a,b)$ that $a<b$. We will see where the terminology of "open" and "closed" comes from in Chapter 4. Context will help us determine whether (a,b) represents a bounded open interval or an ordered pair.

Using the Order Axioms, we can prove many familiar facts.

Problem 3.13. Prove that for all $a,b\in\mathbb{R}$, if $a,b>0$, then $a+b>0$; and if $a,b<0$, then $a+b<0$.

The next result extends Axiom O3.

Problem 3.14. Prove that for all $a,b,c,d\in\mathbb{R}$, if $a<b$ and $c<d$, then $a+c<b+d$.

Problem 3.15. Prove that for all $a\in\mathbb{R}$, $a>0$ if and only if $-a<0$.

Problem 3.16. Prove that if a, b, c, and d are positive real numbers such that $a<b$ and $c<d$, then $ac<bd$.

Problem 3.17. Prove that for all $a,b\in\mathbb{R}$, we have the following:

(a) $ab>0$ if and only if either $a,b>0$ or $a,b<0$;

(b) $ab<0$ if and only if $a<0<b$ or $b<0<a$.

Problem 3.18. Prove that for all positive real numbers a and b, $a<b$ if and only if $a^2<b^2$.

Consider using three cases when approaching the following proof.

Problem 3.19. Prove that for all $a\in\mathbb{R}$, we have $a^2\geq 0$.

It might come as a surprise that the following result requires proof.

Problem 3.20. Prove that $0<1$.

The previous problem together with Problem 3.15 implies that $-1<0$ as you expect. It also follows from Axiom O3 that for all $n\in\mathbb{Z}$, we have $n<n+1$. We assume that there are no integers between n and $n+1$.

Problem 3.21. Prove that for all $a\in\mathbb{R}$, if $a>0$, then $a^{-1}>0$, and if $a<0$, then $a^{-1}<0$.

Problem 3.22. Prove that for all $a,b\in\mathbb{R}$, if $a<b$, then $-b<-a$.

The last few results allow us to take for granted our usual understanding of which real numbers are positive and which are negative. The next problem yields a result that extends Problem 3.22.

Problem 3.23. Prove that for all $a,b,c\in\mathbb{R}$, if $a<b$ and $c<0$, then $bc<ac$.

> Making learning easy does not necessarily ease learning.
>
> Manu Kapur, learning scientist

3.3 Absolute Value and the Triangle Inequality

There is a special function that we can now introduce.

Definition 3.24. Given $a \in \mathbb{R}$, we define the **absolute value of** a, denoted $|a|$, via

$$\boxed{|a| := \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}}$$

Problem 3.25. Prove that for all $a \in \mathbb{R}$, $|a| \geq 0$ with equality only if $a = 0$.

We can interpret $|a|$ as the distance between a and 0 as depicted in Figure 3.1.

(a) $a > 0$ (b) $a < 0$

Figure 3.1: Visual representation of $|a|$.

Problem 3.26. Prove that for all $a, b \in \mathbb{R}$, we have $|a - b| = |b - a|$.

Given two points a and b, $|a - b|$, and hence $|b - a|$ by the previous problem, is the distance between a and b as shown in Figure 3.2.

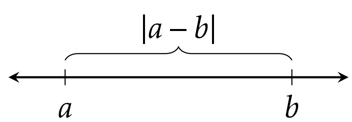

Figure 3.2: Visual representation of $|a - b|$.

Problem 3.27. Prove that for all $a, b \in \mathbb{R}$, $|ab| = |a||b|$.

In the next problem, writing $\pm a \leq b$ is an abbreviation for $a \leq b$ and $-a \leq b$.

Problem 3.28. Prove that for all $a, b \in \mathbb{R}$, if $\pm a \leq b$, then $|a| \leq b$.

Problem 3.29. Prove that for all $a \in \mathbb{R}$, $|a|^2 = a^2$.

Problem 3.30. Prove that for all $a \in \mathbb{R}$, $\pm a \leq |a|$.

Problem 3.31. Prove that for all $a, r \in \mathbb{R}$ with r nonnegative, $|a| \leq r$ if and only if $-r \leq a \leq r$.

The letter r was used in the previous problem because it is the first letter of the word "radius". If r is positive, we can think of the interval $(-r, r)$ as the interior of a one-dimensional circle with radius r centered at 0. Figure 3.3 provides a visual interpretation of Problem 3.31.

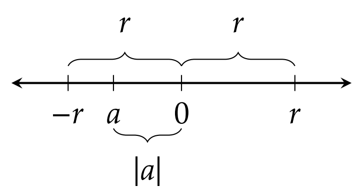

Figure 3.3: Visual representation of $|a| \leq r$.

Problem 3.32. Prove that for all $a, b, r \in \mathbb{R}$ with r nonnegative, $|a-b| \leq r$ if and only if $b - r \leq a \leq b + r$.

Since $|a-b|$ represents the distance between a and b, we can interpret $|a-b| \leq r$ as saying that the distance between a and b is less than or equal to r. In other words, a is within r units of b. See Figure 3.4.

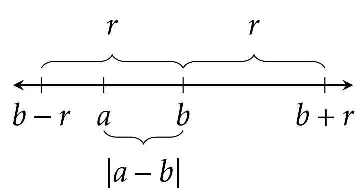

Figure 3.4: Visual representation of $|a-b| \leq r$.

Consider using Problems 3.30 and 3.31 when attacking the next result, which is known as the **Triangle Inequality**. This result can be extremely useful in some contexts.

Problem 3.33 (Triangle Inequality)**.** Prove that for all $a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$.

Figure 3.5 depicts two of the cases for the Triangle Inequality.

(a) $a \geq 0, b \geq 0$

(b) $a < 0, b \geq 0$

Figure 3.5: Visual representation of two of the cases for the Triangle Inequality.

Problem 3.34. Under what conditions do we have equality for the Triangle Inequality?

Where did the Triangle Inequality get its name? Why "Triangle"? For any triangle (including degenerate triangles), the sum of the lengths of any two sides must be greater than or equal to the length of the remaining side. That is, if x, y, and z are the lengths of the sides of the triangle, then $z \leq x + y$, where we have equality only in the degenerate case of a triangle with no area. In linear algebra, the Triangle Inequality is a theorem about lengths of vectors. If **a** and **b** are vectors in $\mathbb{R}^n$, then the Triangle Inequality states that $||\mathbf{a}+\mathbf{b}|| \leq ||\mathbf{a}|| + ||\mathbf{b}||$. Note that $||\mathbf{a}||$ denotes the length of vector **a**. See Figure 3.6. The version of the Triangle Inequality that we presented in Problem 3.33 is precisely the one-dimensional version of the Triangle Inequality in terms of vectors.

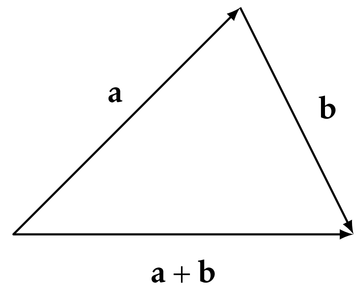

Figure 3.6: Triangle Inequality in terms of vectors.

The next result is sometimes called the **Reverse Triangle Inequality**.

Problem 3.35 (Reverse Triangle Inequality)**.** Prove that for all $a, b \in \mathbb{R}$, $|a-b| \geq ||a|-|b||$.

> I didn't want to just know names of things. I remember really wanting to know how it all worked.
>
> Elizabeth Blackburn, biologist

3.4 Suprema, Infima, and the Completeness Axiom

Before we introduce the Completeness Axiom, we need some additional terminology.

Definition 3.36. Let $A \subseteq \mathbb{R}$. A point b is called an **upper bound** of A if for all $a \in A$, $a \leq b$. The set A is said to be **bounded above** if it has an upper bound.

Problem 3.37. The notion of a **lower bound** and the property of a set being **bounded below** are defined similarly. Try defining them.

Problem 3.38. Find all upper bounds and all lower bounds for each of the following sets when they exist.

(a) $\{5, 11, 17, 42, 103\}$

(b) $\mathbb{N}$

(c) $\mathbb{Z}$

(d) $(0, 1]$

(e) $(0,1] \cap \mathbb{Q}$

(f) $(0,\infty)$

(g) $\{42\}$

(h) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(i) $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

(j) $\emptyset$

Definition 3.39. A set $A \subseteq \mathbb{R}$ is **bounded** if A is bounded above and below.

Notice that a set $A \subseteq \mathbb{R}$ is bounded if and only if it is a subset of some bounded closed interval.

Definition 3.40. Let $A \subseteq \mathbb{R}$. A point p is a **supremum** (or **least upper bound**) of A if p is an upper bound of A and $p \leq b$ for every upper bound b of A. Analogously, a point p is an **infimum** (or **greatest lower bound**) of A if p is a lower bound of A and $p \geq b$ for every lower bound b of A.

Our next result tells us that a supremum of a set and an infimum of a set are unique when they exist.

Problem 3.41. Prove that if $A \subseteq \mathbb{R}$ such that the supremum (respectively, infimum) of A exists, then the supremum (respectively, infimum) of A is unique.

In light of the previous problem, if the supremum of A exists, it is denoted by $\boxed{\sup(A)}$. Similarly, if the infimum of A exists, it is denoted by $\boxed{\inf(A)}$.

Problem 3.42. Find the supremum and the infimum of each of the sets in Problem 3.38 when they exist.

It is important to recognize that the supremum or infimum of a set may or may not be contained in the set. In particular, we have the following theorem concerning suprema and maximums. The analogous result holds for infima and minimums.

Problem 3.43. Let $A \subseteq \mathbb{R}$. Prove that A has a maximum if and only if A has a supremum and $\sup(A) \in A$, in which case the $\max(A) = \sup(A)$.

Intuitively, a point is the supremum of a set A if and only if no point smaller than the supremum can be an upper bound of A. The next result makes this more precise.

Problem 3.44. Let $A \subseteq \mathbb{R}$. Prove that an upper bound b is the supremum of A if and only if for every $\varepsilon > 0$, there exists $a \in A$ such that $b - \varepsilon < a$.

Problem 3.45. State and prove the analogous result to Problem 3.44 involving infimum.

The following axiom states that every nonempty subset of the real numbers that has an upper bound has a least upper bound.

Axiom 3.46 (Completeness Axiom)**.** If A is a nonempty subset of $\mathbb{R}$ that is bounded above, then $\sup(A)$ exists.

Given the Completeness Axiom, we say that the real numbers satisfy the **least upper bound property**. It is worth mentioning that we do not need the Completeness Axiom to conclude that every nonempty subset of the integers that is bounded above has a supremum, as this follows from a generalized version of the Well-Ordering Principle (see Problem 2.25).

Certainly, the real numbers also satisfy the analogous result involving infimum.

Problem 3.47. Prove that if A is a nonempty subset of $\mathbb{R}$ that is bounded below, then $\inf(A)$ exists.

Problem 3.48. If A and B are each bounded above, characterize the supremum of each of the following sets.

(a) $A \cup B$

(b) $A \cap B$

What are the analogous results involving infimum?

If A and B are sets, define $\boxed{A + B := \{a + b \mid a \in A, b \in B\}}$.

Problem 3.49. Prove each of the following.

(a) If A and B are each bounded above, then $\sup(A+B) = \sup(A) + \sup(B)$.

(b) If A and B are each bounded below, then $\inf(A+B) = \inf(A) + \inf(B)$.

For a set A and $c \in \mathbb{R}$, define $\boxed{cA := \{ca \mid a \in A\}}$.

Problem 3.50. Let A be a set and $c \in \mathbb{R}$. Prove each of the following. Assuming M and N have suprema, prove one of the following.

(a) If $c > 0$ and A is bounded above, then $\sup(cA) = c\sup(A)$.

(b) If $c < 0$ and A is bounded below, then $c\inf(A) = \sup(cA)$.

What other properties are there relating inf, sup, and c?

> Time spent thinking about a problem is always time well spent. Even if you seem to make no progress at all.
>
> Paul Zeitz, mathematician

3.5 The Archimedean Property

Our next result, called the **Archimedean Property**, tells us that for every real number, we can always find a natural number that is larger. To prove this theorem, consider a proof by contradiction and then utilize the Completeness Axiom and Problem 3.44.

Problem 3.51 (Archimedean Property)**.** Prove that for every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$.

More generally, we can "squeeze" every real number between a pair of integers. The next result is sometimes referred to at the **Generalized Archimedean Property**.

Problem 3.52 (Generalized Archimedean Property)**.** Prove that for every $x \in \mathbb{R}$, there exists $k, n \in \mathbb{Z}$ such that $k < x < n$.

Problem 3.53. Prove that for any positive real number x, there exists $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < x$.

The next problem strengthens the Generalized Archimedean Property and says that every real number is either an integer or lies between a pair of consecutive integers. To tackle the next problem, let $x \in \mathbb{R}$ and define $L = \{k \in \mathbb{Z} \mid k \leq x\}$. Use the Generalized Archimedean Property to conclude that L is nonempty and then utilize Problem 2.25.

Problem 3.54. Prove that for every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n \leq x < n+1$.

For the next proof, let $a < b$, utilize Problem 3.53 on $b - a$ to obtain $N \in \mathbb{N}$ such that $\frac{1}{N} < b-a$, and then apply Problem 3.54 to Na to conclude that there exists $n \in \mathbb{N}$ such that $n \leq Na < n+1$. Lastly, argue that $\frac{n+1}{N}$ is the rational number you seek.

Problem 3.55. Prove that if (a, b) is an open interval, then there exists a rational number p such that $p \in (a, b)$.

Recall that the real numbers consist of rational and irrational numbers. Two examples of an irrational number that you are likely familiar with are π and $\sqrt{2}$. In Section **??**, we will prove that $\sqrt{2}$ is irrational, but for now we will take this fact for granted. It turns out that $\sqrt{2} \approx 1.41421356237 \in (1, 2)$. This provides an example of an irrational number occurring between a pair of distinct rational numbers. The following problem is a good challenge to generalize this.

Problem 3.56. Prove that if (a, b) is an open interval, then there exists an irrational number p such that $p \in (a, b)$.

Repeated applications of the previous two problems implies that every open interval contains infinitely many rational numbers and infinitely many irrational numbers. In light of these two problems, we say that both the rationals and irrationals are **dense** in the real numbers.

> If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.
>
> John von Neumann, mathematician