Homework 10

Combinatorics

You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in his or her own work. In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments with no questions asked**. When doing your homework, I encourage you to consult the Elements of Style for Proofs. Unless otherwise indicated, submit each of the following assignments via BbLearn by the due date. You will need to capture your handwritten work digitally and then upload a PDF to BbLearn. There are many free smartphone apps for doing this. I use TurboScan on my iPhone.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing assignments for this course, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.** Please read NAU's Academic Integrity Policy.

Complete the following problems.

1. Given a lattice path p consisting only of north and east steps, define $\text{area}(p)$ to be the total area under the path p (and above the x-axis). Recall the definition of $L(k, n-k)$ that appeared on Part 2 of Exam 1. Prove that
$$\sum_{p \in L(k,n-k)} q^{\text{area}(p)} = \sum_{\substack{w \in S_n \\ \text{Des}(w) \subseteq \{k\}}} q^{\text{inv}(w)}.$$

2. Suppose $(P, \leq_P)$ and $(Q, \leq_Q)$ are finite posets.
   (a) Define $\leq_{P\times Q}$ on $P \times Q$ via
   $$(x,y) \leq_{P\times Q} (x',y') \text{ if and only if } x \leq_P x' \text{ and } y \leq_Q y'$$
   Prove that $(P \times Q, \leq_{P\times Q})$ is a poset.
   (b) Prove that if P and Q are ranked posets with rank function rk_P and rk_Q, respectively, then $P \times Q$ is a ranked poset with rank function
   $$\text{rk}_{P\times Q}(x,y) := \text{rk}_P(x) + \text{rk}_Q(y).$$

3. Complete Problem 3.1 in textbook.

4. Complete parts 1, 2, and 4 of Problem 3.3 in the textbook.

5. Let Y be the set of all possible Young diagrams (see Problem 3.3 in textbook). The *Young lattice* is the poset $(Y, \leq)$, where $\lambda \leq \mu$ if and only if the Young diagram μ contains the Young diagram λ with their northwest corners aligned. Note that the empty diagram (typically denoted by $\emptyset$) is the Young diagram for the integer 0. This is the unique minimal element in Y.

(a) Prove that Y is a ranked poset with rank function $\mathrm{rk}(\lambda) := |\lambda|$ (where $|\lambda|$ is the number of one-by-one squares in λ).

(b) Prove that Y is a lattice.

(c) Find a "nice" expression for the rank generating function for Y.