Chapter 8

Additional Graph Theory

A **digraph** (or **directed graph**) D consists of a set V of vertices and a set E of **directed edges** (or **arrows**), each of which is represented as an ordered pair (u, v), where $u, v \in V$. We say that u is the **initial vertex** and v is the **terminal vertex** of the directed edge (u, v). We write $D = (V, E)$ as we did with undirected graphs. The **indegree** of a vertex v in a digraph, denoted $\boxed{\deg^-(v)}$, is the number of directed edges have v as a terminal vertex while the **outdegree** of v, denoted $\boxed{\deg^+(v)}$, is the number of edges having v as an initial vertex.

As expected, we have the following result that is analogous to the Handshake Lemma (Theorem 7.12).

Theorem 8.1. If $D = (V, E)$ is a digraph, then

$$|E| = \sum_{v\in V} \deg^-(v) = \sum_{v\in V} \deg^+(v).$$

Each graph/digraph is determined by its vertices and the manner in which they are connected by edges, not the way a graph/digraph might be sketched. We can represent a graph in a couple of ways.

The **adjacency list** of a simple graph lists all vertices in one column and all adjacent vertices in second column. For a digraph, the columns contain the initial vertices and the associated terminal vertices.

Problem 8.2. Make up a couple examples to explore adjacency lists for simple graphs and digraphs.

An $m \times n$ **matrix** A is a rectangular array of numbers with m rows and n columns. The entry in the ith row and jth column is indicated by $A_{i,j}$.

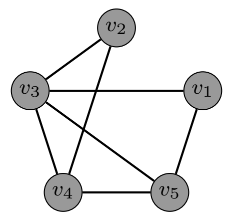
Example 8.3. The example below is a 2×3 matrix:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 10 & 6 & 7 \end{bmatrix}$$

In this example, $A_{1,2} = 3$.

The **adjacency matrix** A of a graph (respectively, digraph) G with vertices listed as $v_1, v_2, \ldots, v_n$ is the $n \times n$ matrix A whose entry $A_{i,j}$ in row i and column j is the number of edges connecting v_i and v_j (respectively, the number of edges from v_i to v_j).

Problem 8.4. Find the adjacency matrix for the following graph.



Problem 8.5. What properties will the adjacency matrix for a simple graph have?

Problem 8.6. Sketch a graph that has the following adjacency matrix.

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 \end{bmatrix}$$

Problem 8.7. Sketch a digraph that has the following adjacency matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 0 \end{bmatrix}$$

Problem 8.8. What will the adjacency matrix for P_n look like, assuming the vertices are taken in the natural order (start at one end of the path and end at the other)? What about C_n? K_n?

Recall that a graph is not determined by a sketch since many sketches give the same graph. It may be hard to recognize from sketches whether two graphs are "essentially" the same even though the vertices may be different points. The notion of isomorphism (same form) gives us a way to deal with this. Two *simple* graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are the **isomorphic**, written $G_1 \cong G_2$, if there is a a bijection $f : V_1 \to V_2$ such that $\{u, v\}$ is an edge in G_1 if and only if $\{f(u), f(v)\}$ is an edge in G_2. The function f is called an **isomorphism**. For digraphs, we require that (u, v) is a directed edge in G_1 if and only if $(f(u), f(v))$ is a directed edge in G_2.

For $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, to show that $G_1 \cong G_2$:

1. State a vertex matching explicitly, and

2. Either

(a) Check adjacency for each pair of vertices in G_1 and the corresponding pair in G_2 (a total of $\binom{|V_1|}{2}$ checks). This could also be as simple as providing sketches for each graph that clearly exhibit the correspondence of vertices and edges.

(b) Demonstrate that the adjacency matrices of G_1 and G_2 are the same using an ordering that is compatible with the vertex matching.

Warning! The second method above usually involves much less writing, but be aware that the adjacency matrices may differ in one ordering but agree with a different ordering.

The simplest way to show that $G_1 \not\cong G_2$ is to show that a feature preserved under isomorphism (called an **invariant**) holds for one graph but not the other. Here are a few isomorphic invariants:

(a) Order of the graph

(b) Number of edges in the graph

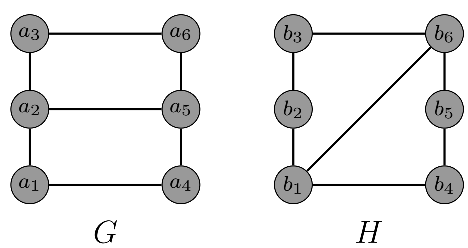
(c) Number of vertices of a given degree

(d) Degree sequence

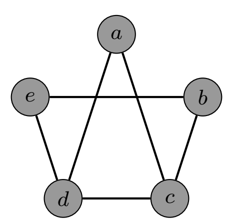
(e) Vertices of degree k and ℓ are adjacent

(f) Subgraph that is isomorphic to C_n or P_n.

Problem 8.9. Determine whether the following graphs are isomorphic.

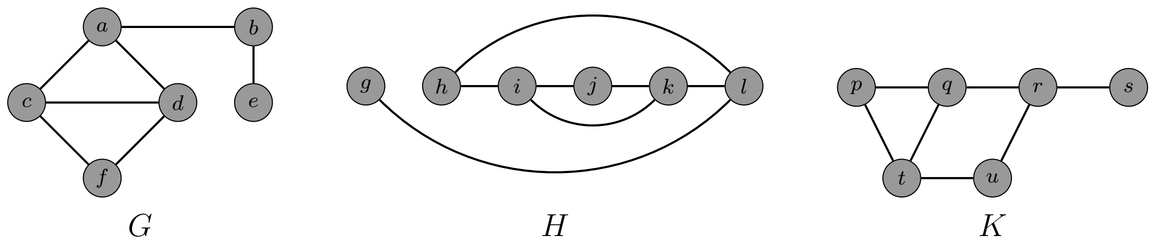


Problem 8.10. Let G be the graph with vertex set $V = \{a, b, c, d, e\}$ and edge set $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, d\}\}$ and let H be the following graph.

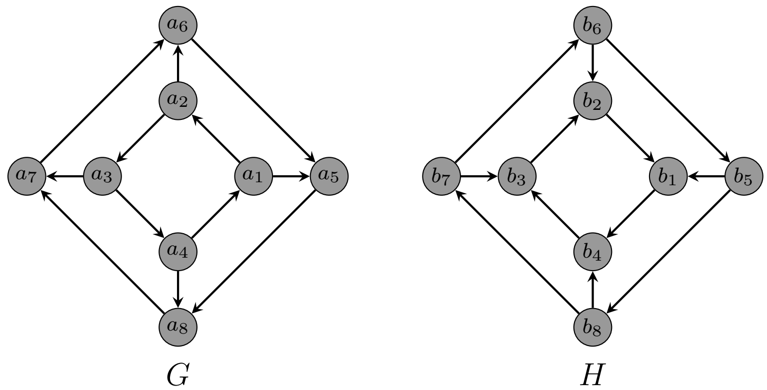


Determine whether G and H are isomorphic.

Problem 8.11. Determine which pairs of the following graphs are isomorphic.



Problem 8.12. Determine whether the following digraphs are isomorphic.



We now introduce several new terms.

- A **walk** in a graph is an alternating sequence of vertices and edges that starts with a vertex and ends with a vertex such that consecutive vertices in the walk are the endpoints of the edge that separates them. In a simple graph, a walk can be specified by a sequence of vertices.
- The **length** of a walk is the number of edges in the walk.
- If the initial and terminal vertices of a walk are the same, then the walk is a **closed walk**.
- A **trail** is a walk with distinct edges (no repeated edges).
- A **circuit** is a closed trail, that is, a closed walk with no repeated edges.
- A **path** is a walk with distinct vertices. This is a subgraph isomorphic to P_n for some n.
- A **cycle** is a closed walk with distinct vertices except the initial and terminal vertices. This corresponds to a subgraph isomorphic to C_n for some n.
- A graph G is **connected** if for each pair of distinct vertices u and v, there is a walk from u to v. A **component** of a graph is a connected subgraph that is not contained in a larger connected subgraph.
- A **cut vertex** of a connected graph G is a vertex which when removed along with all incident edges results in a disconnected graph.

- A **bridge** (or **cut edge**) is an edge of a connected graph which when removed results in a disconnected graph.

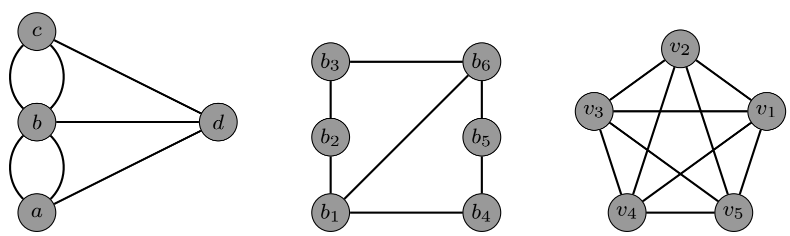
The following theorem likely does not come as a surprise.

Theorem 8.13. A graph G is connected if and only if for each pair of distinct vertices u and v, there is a path from u to v.

A digraph is **strongly connected** if for each pair of distinct vertices u and v there is a (directed) walk from u to v. A digraph is **weakly connected** if the underlying undirected graph in which the direction of edges is removed is connected. Note that a strongly connected digraph will always be weakly connected. A **strongly connected component** of a digraph is a maximal strongly connected subgraph.

We now introduce a couple of important circuits that a graph may or may not possess. An **Euler circuit** in a graph G is a circuit that contains every edge of the graph. An **Euler trail** in a graph is a trail that contains every edge of the graph. Note that an Euler circuit is also an Euler trail as well.

Problem 8.14. Determine whether each of the following graphs has an Euler trail. How about an Euler circuit?



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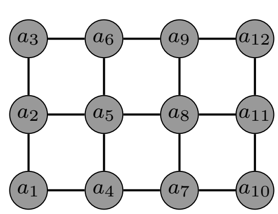
Theorem 8.15. If G is a connected graph of order $n \geq 2$, then G has an Euler circuit if and only if every vertex is even.

Corollary 8.16. If G is a connected graph of order $n \geq 2$, then G has an Euler trail that is not a circuit if and only if G has exactly two odd vertices.

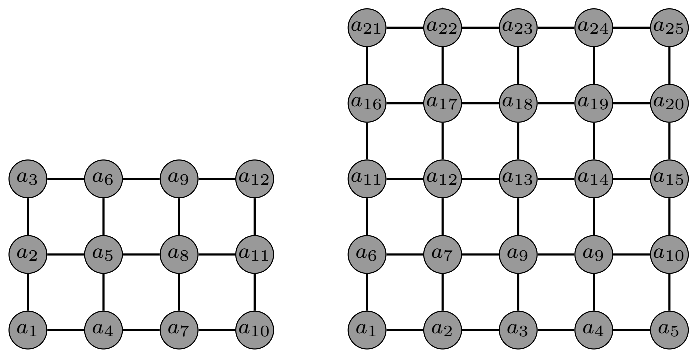
Edges can be added to a connected graph in order to cause it to have an Euler circuit (or trail). In particular, an **Eulerization** of a connected graph is the addition of suitable multiple edges (i.e., duplicate existing edges) to permit an Euler circuit, mimicking what must be done to complete a circuit such as a postal route or other delivery/pick-up route.

Problem 8.17. Eulerize the following graph.



A cycle in a graph that passes through every vertex is a **Hamilton cycle**. This is often called a **Hamilton circuit**. A **Hamilton path** is a path in a graph is a path that includes every vertex.

Problem 8.18. Determine whether each of the following graphs has a Hamilton circuit or a Hamilton path that is not a circuit.



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Unfortunately, unlike the situation for Euler circuits, there is no known simple necessary and sufficient condition for a Hamilton cycle to exist in a graph. We can state some simple cases when one cannot exist, and there are some theorems for the existence of a Hamilton cycle, but these do not cover all possibilities.

Theorem 8.19 (Dirac's Theorem)**.** If G is a simple graph of order $n \geq 3$ in which $\deg(v) \geq n/2$ for each vertex, then G has a Hamilton cycle.

Theorem 8.20 (Ore's Theorem)**.** If G is a simple graph of order $n \geq 3$ in which $\deg(u) + \deg(v) \geq n$ for each pair of vertices u and v, then G has a Hamilton cycle.

Problem 8.21. How many distinct Hamilton cycles does K_n have that start/end at a fixed vertex?

More coming soon...