Homework 9

Discrete Mathematics

Please review the *Rules of the Game* from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition.You are highly encouraged to seek out assistance by asking questions in our Q&A Discussion Board in Canvas. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to three late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. **Unless explicitly stated otherwise, you are expected to justify your answers.** In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. Develop a recurrence relation with initial conditions for the number of ways to climb a staircase of n stairs by taking steps of one, two, or three stairs at a time. How many ways can a person climb a flight of eight stairs under these constraints?

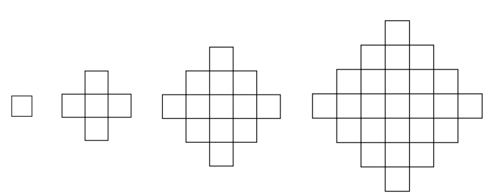
2. If you have enough toothpicks, you can make a large triangular grid. The size 1 grid requires 3 toothpicks, the size 2 grid requires 9 toothpicks. Let t_n be the number of toothpicks required to make a size n triangular grid, so that $t_1 = 3$, $t_2 = 9$, and so on.

   (a) Find a recursive definition for the sequence. Don't forget any necessary initial conditions.

   (b) Is the sequence arithmetic or geometric? If not, is it the sequence of partial sums of an arithmetic or geometric sequence?

   (c) Find a closed formula for the nth term of the sequence.

3. Consider the sequence of figures whose first few terms are as follows.



   Let a_n be the area of the nth figure. Assume we start at $n = 1$, so that $a_1 = 1$, $a_2 = 5$, as so on.

(a) Find a recursive definition for the sequence. Don't forget any necessary initial conditions.

(b) Is the sequence arithmetic or geometric? If not, is it the sequence of partial sums of an arithmetic or geometric sequence?

(c) Find a closed formula for the nth term of the sequence.

4. Solve the recurrence $a_n = 7a_{n-1} - 10a_{n-2}$ with $a_0 = 2$ and $a_1 = 3$.

5. The **Catalan numbers** are defined via $c_0 = 1$ and

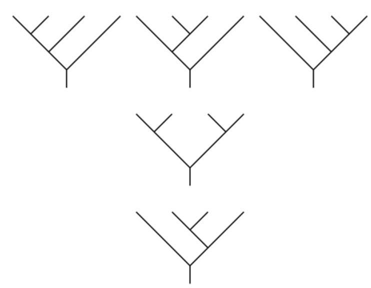
$$c_n = \sum_{i=0} c_i c_{n-1-i}$$

for $n \geq 1$. The equation above is called the **Catalan recurrence**. Using the initial condition and the Catalan recurrence, we can generate the first several terms of the Catalan sequence:

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786$$

There are hundreds of interesting combinatorial objects counted by the Catalan numbers! In the previous homework assignment, you showed that Dyck paths are counted by the Catalan numbers since d_n satisfied the same initial conditions and recurrence as the Catalan numbers.

A **planar binary tree** is a rooted tree such that every internal node has precisely two successors. An internal node is any node that has degree greater than 1. If there are n internal nodes, this means there are $n+1$ leaves. For example, below are all of the planar binary trees with 3 internal nodes.



Notice that two planar binary trees that are symmetric across the vertical midline are considered different. I'm happy to explain this further if the figures and description isn't sufficient. Let p_n denote the number of planar binary trees with n internal nodes. We define $p_0 := 1$. Prove that $b_n = c_n$.

6. Tickets to a show are 50 cents and $2n$ customers stand in a queue at the ticket window. Half of them have \$1 and the others have 50 cents. The cashier starts with no money. How many arrangements of the queue are possible with the proviso that the cashier always be able to make change?