Exam 2

Your Name:

Names of Any Collaborators:

Instructions

This exam is worth a total of 36 points and worth 20% of your overall grade. The exam is due by midnight on **Saturday, April 18**. When you have finished your exam, **please email me a single PDF**. I'd like you to include the cover page, but if you are unable to do this, recreate a suitable replacement that includes your signature. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems/problems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Problem 2.32, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

 Written by D.C. Ernst

The following problems are related to content from Chapters 4–7 of our course notes. Some of the problems are directly from the notes. Note that you will need to independently digest some new material in Chapter 7. You may freely use any result that appears in any chapter as long as the result appears *before* the problem in question. Moreover, you may use any result that appears below as long as it appears *before* the problem in question.

1. (1 point each) Determine whether each of the following statements is true or false. You do *not* need to justify your answer.
   (a) If M is a connected point set, then M is of the form $[a,b]$, $[a,b)$, $(a,b]$, (a,b), $(-\infty,b]$, $(-\infty,b)$, $[a,\infty)$, (a,∞), or $(-\infty,\infty)$.
   (b) If K is a compact subset of $\mathbb{R}$, then K contains a maximum value and $\sup(K)=\max(K)$.
   (c) Every nonempty non-closed set has at least one accumulation point.
   (d) If f is a continuous function whose domain includes the bounded interval I, then $f(I)$ is a bounded interval.
   (e) If f is a continuous function whose domain includes the closed set C, then $f(C)$ is a closed set.
   (f) If $f:\mathbb{R}\to\mathbb{R}$ is a continuous function and C is a closed set, then the inverse image $f^{-1}(C)$ is a closed set.
   (g) If $f:\mathbb{R}\to\mathbb{R}$ is a continuous function and C is a connected set, then the inverse image $f^{-1}(C)$ is a connected set.
   (h) If $f:X\to Y$ is a continuous bijection for $X,Y\subseteq\mathbb{R}$, then $f^{-1}:Y\to X$ is a continuous bijection.
   (i) If M is a point set such that $\sup(M)$ exists, then for all (a,b) containing $\sup(M)$, there exists $m\in M\cap(a,\sup(M))$.
   (j) If f is differentiable at every point of (a,b) and f is strictly increasing on (a,b), then $f'(x)>0$ for every $x\in(a,b)$.
2. (2 points each) For **three** of the statements above that you indicated were false, provide an appropriate counterexample.
3. (4 points each) Complete **two** of the following.
   (a) Prove that if f is continuous on $[a,b]$, then the image $f([a,b])$ is either a single point or a closed interval.
   (b) Prove that if f is continuous on $[a,b]$ and there exists $c\in(a,b)$ such that $f(c)>0$, then there exists an open interval I containing c such that $f(x)>0$ for all $x\in I$.
   (c) Suppose f and g are functions having domain M and each is continuous at the point $p\in M$. Prove that if h is a function with domain M such that $f(p)=h(p)=g(p)$ and for each number $x\in M$, $f(x)\leq h(x)\leq g(x)$, then h is also continuous at p.
   (d) Prove that if f is a continuous function whose domain includes the closed interval $[a,b]$ and there is a point $c\in[a,b]$ so that $f(c)\geq 0$, then the set $\{x\in[a,b]\mid f(x)\geq 0\}$ is a closed point set.
   (e) Prove that if f is a continuous function whose domain includes a closed interval $[a,b]$ and $p\in[a,b]$, then the set $\{x\in[a,b]\mid f(x)=f(p)\}$ is a closed point set.
4. (4 points) Complete **one** of the following
   (a) Suppose that f is differentiable at every point of $[a,b]$ and suppose that the derivative is never zero on this interval. Prove that f is either strictly increasing or strictly decreasing on $[a,b]$.*

*See Definition 7.26 for definition of strictly increasing and strictly decreasing.

(b) Suppose f is differentiable at every point of (a,b) and suppose $f'(x) > 0$ for all $x \in (a,b)$. Prove that f is strictly increasing on (a,b).

5. (4 points each) Complete **two** of the following. The definition of *integrable* appears in Definition 7.20.

(a) Suppose f is continuous on $[a,b]$ and $f(x) \geq 0$ for all $x \in [a,b]$. Prove that if $\int_a^b f = 0$, then f is the zero function on $[a,b]$.

(b) (Problem 7.19) Suppose f is continuous on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and that for some $z \in [a,b]$, $f(z) > 0$. Explain why $\overline{\int_a^b} f$ exists and then show that $\overline{\int_a^b} f > 0$.

(c) (Problem 7.24) Suppose f is a bounded function on $[a,b]$. Then f is (Riemann) integrable if and only if for every $\epsilon > 0$, there exists a partition P of $[a,b]$ such that $U_P(f) - L_P(f) < \epsilon$.

(d) Define $f : [0,1] \to \mathbb{R}$ via

$$f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Prove that f is integrable over $[0,1]$.

(e) (Problem 7.26) Prove that if f is a bounded monotonic function[†] on $[a,b]$, then f is integrable on $[a,b]$.

[†]See Definition 7.26 for the definition of a monotonic function.

