Homework 5

Combinatorial Game Theory

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on Discord. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

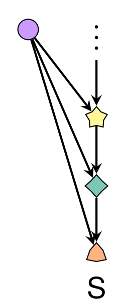
Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

1. We say that a rulegraph R is **simple** if every congruence relation on $V(\mathsf{R})$ is trivial (i.e., every congruence class is a singleton). Prove that a rulegraph R is simple if and only if every option preserving rulegraph map $\alpha : \mathsf{R} \to \mathsf{S}$ is injective.

2. Prove that a rulegraph R is simple if the Opt map is injective (i.e., no two positions have the same option set).

3. Prove that if the quotients $\mathsf{R}/{\sim}$ and $\mathsf{R}/{\approx}$ of the rulegraph R are both simple, then the congruence relations $\sim$ and $\approx$ are the same.

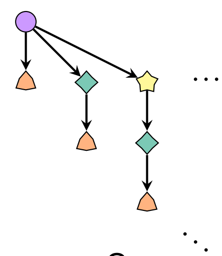
4. Determine whether the following gamegraph is simple. Justify your answer. *Note:* The circular node is intended to be the source for the tower of positions to the right.



5. Is the sum of two simple gamegraphs a simple gamegraph? If so, prove it. Otherwise, provide a counterexample.

6. Given a rulegraph R, sketch the construction of congruence relation $\bowtie$ on $V(\mathsf{R})$ such that $\mathsf{R}/\bowtie$ is simple. You do not need to prove that your construction is a congruence relation. It follows from Problem 3 that $\bowtie$ is unique. We call the rulegraph $\mathsf{R}/\bowtie$ the **minimum quotient** of R. The corresponding quotient map is an example of a surjective option preserving rulegraph map from R to a simple rulegraph. In fact, if $\alpha : \mathsf{R} \to \mathsf{S}$ is a surjective option preserving rulegraph map and S is simple, then S is isomorphic to $\mathsf{R}/\bowtie$.

7. For the gamegraph G given below, use your construction from the previous problem to obtain $\mathsf{G}/\bowtie$.



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