## Chapter 7

## Principle of Inclusion and Exclusion

We now introduce a concept known as the **Principal of Inclusion and Exclusion**. Recall Theorem 1.14, which states that if A and B are sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

**Problem 7.1.** How many integers between 1 and 881 inclusively are divisible by 3 or 5?

But what do we do if we have more than two sets? Let's first examine the situation with three sets.

**Problem 7.2.** If A, B, and C are sets, then find a formula in the same vein as Theorem 1.14 for  $|A \cup B \cup C|$ .

The upshot is that we add "singles" subtract "doubles" and add "triples".

**Problem 7.3.** In the Natteranian township, 750 of the residents have a smart phone, 620 have a laptop computer, 480 have a desktop computer, 420 have both a laptop and a smart phone, 390 have both a smart phone and a desktop, 212 have both a laptop and a desktop computer and 164 have all three items.

- (a) How many residents have at least one of the three items?
- (b) How many residents do not have desktop computer?
- (c) How many residents have a smart phone or a laptop?
- (d) How many have a smart phone or a laptop but not a desktop?

We can generalize to any finite number of sets.

**Theorem 7.4** (Principal of Inclusion and Exclusion). The number of elements in the union of finite sets  $A_1, A_2, \ldots, A_n$  is

$$|A_1 \cup \dots \cup A_n| = \sum_i |A_i| - \sum_{i < j} |A_i \cap A_j| + \sum_{i < j < k} |A_i \cap A_j \cap A_k| - \dots + (-1)^{n+1} |A_1 \cap A_2 \dots \cap A_n|.$$

**Problem 7.5.** How many nonnegative integer solutions does the equation  $x_1 + x_2 + x_3 + x_4 = 25$  have such that  $x_1 < 7$ ,  $x_2 < 5$ , and  $x_4 < 8$ ?

We now discuss one important application of the Principle of Inclusion and Exclusion. Formally, a **derangement** is a permutation  $w : [n] \to [n]$  such that  $w(i) \neq i$  for all  $1 \leq i \leq n$  (i.e., w has no fixed points). That is, a derangement is a special rearrangement of objects such that none is in its original spot.

**Problem 7.6.** How many derangements of CAT are there?

Let  $d_n$  denote the number of derangements of [n]. We set  $d_0 := 1$ .

**Problem 7.7.** For  $1 \le i \le n$ , let  $F_i$  be the set of permutations that fix i.

(a) Explain why

$$d_n = |F_1^c \cap \dots \cap F_n^c|.$$

(b) Explain why

$$d_n = n! - \sum_i |F_i| + \sum_{i < j} |F_i \cap F_j| - \sum_{i < j < k} |F_i \cap F_j \cap F_k| + \dots + (-1)^n |F_1 \cap \dots \cap F_n|.$$

(c) Explain why the number of derangements of [n] is

$$d_n = n! \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right) = n! \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

**Problem 7.8.** Using the previous problem, verify that we got the right answer to Problem 7.6.

**Problem 7.9.** If 7 hats are left at the hat-check window, in how many ways can they be returned so that no one gets the correct hat?

Now, just for funsies...from second semester calculus, we know

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

which implies that

$$e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$

Using Part (c) of Problem 7.7, we see that

$$\lim_{n \to \infty} \frac{d_n}{n!} = \lim_{n \to \infty} \sum_{k=0}^n \frac{(-1)^k}{k!} = \frac{1}{e} \approx 0.367879.$$

In other words, when n is large, the probably of selecting a derangement at random from the collection of permutations of n is approximately 1/e. As n increases, the approximation improves. Boom.