Homework 10

Combinatorics

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on our Discord server. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

New! At the top of each problem, I would like you to list the students that you discussed the problem with. If you worked with the same peers on every problem, then you can simply indicate that once at the top of your assignment.

1. Recall that $S[n]$ denotes the collection of permutations of $[n]$. Formally, a **derangement** is a permutation $w : [n] \to [n]$ such that $w(i) \neq i$ for all $1 \leq i \leq n$ (i.e., w has no fixed points). That is, a derangement is a special rearrangement of objects such that none is in its original spot. Let $\mathrm{Der}[n]$ denote the collection of derangements of $[n]$.
   (a) Explain why $|S[n]| = |(E \cdot \mathrm{Der})[n]|$.
   (b) Using Part (a) and a result from the previous homework to find a closed form for $\mathrm{Der}(x)$.
   (c) Using your answer from Part (b) and something like WolframAlpha or Sage, find $|\mathrm{Der}[15]|$. Indicate what you typed in order to get the desired answer.
2. Let P be a labeled poset consisting of the disjoint union of the chains $1 <_P 2 <_P \cdots <_P k$ and $k+1 <_P k+2 <_P \cdots <_P n$ for some k. Characterize the set of linear extensions of P. *Hint:* For $w \in \mathcal{L}(P)$, consider $\mathrm{Des}(w^{-1})$.
3. A **partition** of a positive integer n is a weakly decreasing sequence of nonnegative integers whose sum is n, i.e., $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n if $\lambda_1 \geq \cdots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$. We often draw partitions as a collection of n boxes that are upper- and left-justified, so that the number of boxes in row i is λ_i (we number rows top to bottom). Such a picture is called a **Young diagram**. Let $\mathcal{Y}$ be the set of all possible Young diagrams. The **Young lattice** is the poset $(\mathcal{Y}, \leq)$, where $\lambda \leq \mu$ if and only if the Young diagram μ contains the Young diagram λ

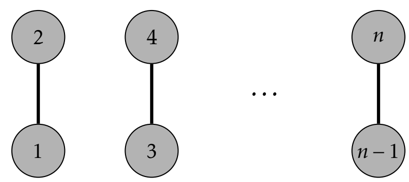
with their northwest corners aligned. Note that the empty diagram (typically denoted by $\emptyset$) is the Young diagram for the integer 0. This is the unique minimal element in $\mathcal{Y}$.

(a) Prove that $\mathcal{Y}$ is a ranked poset with rank function $\mathrm{rk}(\lambda) := |\lambda|$, where $|\lambda|$ is the number of one-by-one squares in λ.

(b) Prove that $\mathcal{Y}$ is a lattice.

(c) Find a "nice" expression for the rank generating function for $\mathcal{Y}$.

4. Find a reasonable formula for the number of linear extensions of the following poset on $[n]$ for n even.



5. Suppose P is an antichain on $[n]$ (i.e., there are no relations among $\{1, 2, \ldots, n\}$).

(a) Prove that $\Omega(P; k) = k^n$.

(b) Prove that the order polynomial generating function for P is given by

$$H(P;t) = \frac{tS_n(t)}{(1-t)^{n+1}},$$

where $S_n(t)$ is the nth Eulerian polynomial.

(c) Use parts (a) and (b) to derive the Carlitz identity.