Lecture Notes on the Periodicity Theorem

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March 29, 2023

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1 Introduction

Today we will prove the *Periodicity Theorem*:

Theorem 1. *For X and Y p-local finite CW-Complexes of type n, n finite, (i.e. n is the smallest number such that $\overline{K(n)}_*(X)$ is nontrivial) the following are true:*

1. *There is a self map $f : \Sigma^{d+i}X \to \Sigma^i X$ such that $\overline{K(n)}_*(f)$ is an isomorphism and $\overline{K(m)}_*(f)$ is trivial for $m \neq n$. This is a v_n map.*

2. *For $h : X \to Y$ (assuming the spectra have been suspended enough to be the target of a v_n map) if $f : \Sigma^d X \to X$ and $g : \Sigma^e Y \to Y$ are v_n maps then there exists i, j so that $di = ej$ and*

$$\begin{tikzcd} \Sigma^{di} X \arrow[r, "\Sigma^{di} h"] \arrow[d, "f^i"] & \Sigma^{ej} Y \arrow[d, "g^j"] \\ X \arrow[r, "h"] & Y \end{tikzcd} \tag{1}$$

commutes up to homotopy.

Before we dive into the proof, we give some motivation for why we would want such a theorem to be true. In our understanding, the Periodicity Theorem is useful in the inductive construction of type n Spectra. This is an explication of what appears in lecture 27 of [1].

To begin, we note that finding type 0 and type 1 spectra is not a difficult task. Since $K(0)$ is just rational homology, S the sphere spectrum is a type 0 spectra. Similarly, since $K(1)_*(S) \cong \pi_*(K(1)) \cong \mathbb{F}_p[v_1, v_1^{-1}]$ we can construct a type 1 spectrum by taking the cofiber of $S \xrightarrow{p} S \twoheadrightarrow X$. Since this is a cofiber sequence applying $\overline{K(1)}$ gives $\overline{K(1)}_*(S) \xrightarrow{\cdot p} \overline{K(1)}_*(S) \to \overline{K(1)}_*(X) \to ....$ But since $\overline{K(1)}_*(S)$ is p torsion, the kernel of the map $\overline{K(1)}_*(S) \to \overline{K(1)}_*(X)$ is 0, so $\overline{K(1)}_*(X)$ must be nontrivial. $\overline{K(0)}_*(X)$ is trivial since multiplication by p is a rational isomorphism. Thus X is a type 1 spectrum.

If we have X as a type n spectrum, then, we would hope to mimic the above construction in some way: starting with a self map $f : \Sigma^d X \to X$, we would like for the cofiber $\Sigma^d X \xrightarrow{f} X \to C_f$ to be a type $n+1$ spectrum.

Applying Morava K-theory to the sequences above gives

$$\overline{K(n)}_*(\Sigma^d X) \xrightarrow{\overline{K(n)}_*(f)} \overline{K(n)}_*(X) \to \overline{K(n)}_*(C_f) \to ... \tag{2}$$

And so $\overline{K(n)}_*(C_f)$ is trivial if and only if $\overline{K(n)}_*(f)$ is an isomorphism.

Similarly,

$$\overline{K(n+1)}_*(\Sigma^d X) \xrightarrow{\overline{K(n+1)}_*(f)} \overline{K(n+1)}_*(X) \to \overline{K(n+1)}_*(C_f) \to ... \tag{3}$$

would show that $\overline{K(n+1)}_*(C_f)$ is nontrivial if $\overline{K(n+1)}(f)$ is trivial.

Therefore, the Periodicity Theorem is a very useful tool in building a sequence of spectra, all of increasing types, inductively.

I am not aware of other motivations of the theorem, but if anyone knows of any let me know and I'll add it to this document!

2 Preliminaries

Before we begin the proof, we start with a few preliminary notes.

2.1 Review of Relevant Previous Material

Firstly, we recall the notion of *thick subcategories*.

Definition 1. *Thick Subcategory. A thick subcategory of FH is a subcategory $C \subseteq FH$ that is closed under cofibers and smash products. I.e. if $X \xrightarrow{f} Y \to C_f$ is a cofiber sequence, then it has two out of three w.r.t. membership in C and if $X \vee Y \in C$ then $X, Y \in C$.*

Importantly, we have the following result:

Theorem 2. *Thick Subcategory Theorem. If F is a thick subcategory of $FH_{(p)}$ then either it is the trivial subcategory, the entire category, or it is the subcategory of all p-local spectra $F_{p,n+1}$ such that $v_{n-1}^{-1}\overline{MU}(X) = 0$ or, equivalently, that $\overline{K_{n-1}}(X) = 0$.*

We will also be using the notion of Spanier-Whitehead duality, which we describe below

Theorem 3. *For X a p-local, finite CW-spectrum, there is a unique spectrum DX of the same type, satisfying the following properties.*

1. *$[X, Y] \cong \pi_*(DX \vee Y)$ where the image of the identity map $i : X \to X$ is the unit map $e : S \to DX \vee X$.*
2. *$DDX \cong X$.*
3. *$D(X \vee Y) \cong DX \vee DY$.*

3 The Periodicity Theorem

Let V_n denote the subcategory of p-local finite spectra admitting a v_n map as in the theorem. The periodicity theorem is the assertion that $V_n = F_{p,n}$, so thus we begin by demonstrating some inclusions.

Firstly, $F_{p,n+1} \subseteq V_n$ since if $X \in F_{p,n+1}$ then $\overline{K(n)}(X) = 0$ so the trivial map is v_n map. Then, $V \subseteq F_{p,n}$ since, by contrapositive, if $X \notin F_{p,n}$ then $\overline{K(n-1)}(X) \neq 0$ so $v_{n-1}^{-1}\overline{MU}(X) \neq 0$. We can then use the algebraic machinery from chapter 3, namely 3.3.11 to demonstrate the non-existence of a v_n map.

Thus we have shown that $F_{p,n+1} \subseteq V_n \subseteq F_{p,n}$. The proof of the periodicity theorem will then proceed along the following lines

1. Demonstrate that V_n is a thick subcategory, so that either $V_n = F_{p,n}$ or $V_n = F_{p,n+1}$.

2. Produce a type n spectrum and a v_n map on that spectrum, demonstrating that $V_n = F_{p,n}$.

3.1 V_n is a Thick Subcategory

Theorem 4. *V_n is a thick subcategory. I.e. if $X \vee Y$ carries a v_n map, then X does; and if two out of the three spaces in $X \xrightarrow{f} Y \twoheadrightarrow C_f$ carry a v_n map, then the third does.*

Proof. First, assume we have a v_n map $\Sigma^d(X \vee Y) \xrightarrow{f} X \vee Y$. By Spanier-Whitehead duality, this is adjoint to a map $\hat{f} : S^d \to D(X \vee Y) \vee (X \vee Y) = R$. We use the idempotent $X \vee Y \to X \to X \vee Y$ to transfer information down to X a manner dependent on the following lemma:

Lemma 1. *If $\hat{f}$ is as before, then some iterate $\hat{f}^k$ for $k > 0$ is in the center of $\pi_*(R)$.*

Thus, we replace f with the right iterate so that f commutes with the idempotent above. Therefore the following composition

$$\Sigma^d X \to \Sigma^d(X \vee Y) \xrightarrow{f} X \vee Y \to X \tag{4}$$

is a v_n map, let's call it f'. That $\overline{K(m)}_*(f') = 0$ for $m \neq n$ is obvious, since f itself is a v_n map. It then remains to show that $\overline{K(n)}_*(f')$ is an isomorphism. But since $\overline{K(n)}_*(f)$ is an isomorphism, we can construct it's inverse to get

$$\begin{aligned} &\overline{K(n)}_*(\Sigma^d X) \longrightarrow \overline{K(n)}_*(\Sigma^d(X \vee Y)) \xrightarrow{\overline{K(n)}_*(f)} \overline{K(n)}_*(X \vee Y) \longrightarrow \overline{K(n)}_*(X) \longrightarrow \\ &\longrightarrow \overline{K(n)}_*(X \vee Y) \xrightarrow{\overline{K(n)}_*(f)^{-1}} \overline{K(n)}_*(\Sigma^d(X \vee Y)) \longrightarrow \overline{K(n)}_*(\Sigma^d X) \end{aligned} \tag{5}$$

but we can switch the order of $\overline{K(n)}_*(f)$ and the idempotent, as they already commute up to homotopy to get

$$\overline{K(n)}_*(\Sigma^d X) \longrightarrow \overline{K(n)}_*(\Sigma^d(X \vee Y)) \longrightarrow \overline{K(n)}_*(\Sigma^d X) \longrightarrow \overline{K(n)}_*(\Sigma^d(X \vee Y)) \xrightarrow{\overline{K(n)}_*(f)} \overline{K(n)}_*(X \vee Y) \xrightarrow{\overline{K(n)}_*(f)^{-1}} \overline{K(n)}_*(\Sigma^d(X \vee Y)) \longrightarrow \overline{K(n)}_*(\Sigma^d X) \tag{6}$$

Which just ends up being

$$\overline{K(n)}_*(\Sigma^d X) \longrightarrow \overline{K(n)}_*(\Sigma^d(X \vee Y)) \longrightarrow \overline{K(n)}_*(\Sigma^d X) \longrightarrow \overline{K(n)}_*(\Sigma^d(X \vee Y)) \longrightarrow \overline{K(n)}_*(\Sigma^d X) \tag{7}$$

which is clearly just the identity. Thus $\overline{K(n)}_*(f')$ has an inverse and is thus an isomorphism. Therefore X carries a v_n map, so $X \in V_n$.

Now if we have a cofiber sequence $X \xrightarrow{h} Y \twoheadrightarrow C_h$ with X having a v_n map f and Y having a v_n map g, it would be nice if we were able to get a corresponding 'cofiber sequence' like the following

$$\begin{tikzcd}
\Sigma^d X \arrow[r, "\Sigma^d h"] \arrow[d, "f"] & \Sigma^d Y \arrow[r, two heads] \arrow[d, "g"] & \Sigma^d C_h \arrow[d] \\
X \arrow[r, "h"] & Y \arrow[r, two heads] & C_h
\end{tikzcd} \tag{8}$$

while unfortunately a diagram like 8 does not generally hold, the following lemma provides the best approximation

Lemma 2. *With X and Y as above, there exists integers i, j such that for any $h : X \to Y$ the following diagram commutes:*

$$\begin{tikzcd}
\Sigma^{di} X \arrow[r, "\Sigma^{di} h"] \arrow[d, "f^i"] & \Sigma^{ej} Y \arrow[d, "g^j"] \\
X \arrow[r, "h"] & Y
\end{tikzcd} \tag{9}$$

where $di = ej$.

thus, we can assume we've iterated f and g enough so that $hf \cong gh$ and thus we do get a diagram exactly like Equation 8. We can then apply the five lemma to demonstrate that the map $l : \Sigma^d C_h \to C_h$ induces an isomorphism on $\overline{K(n)}_*$.

However, this does not immediately demonstrate that $\overline{K(m)}_*(l) = 0$, and in fact it will not in general. However, we can demonstrate that $\overline{K(m)}_*(l^2) = 0$ via the following diagram:

$$
\begin{tikzcd}
\dots \arrow[r] \arrow[d] & \overline{K(m)}_*(\Sigma^{2d}Y) \arrow[r] \arrow[d, "\overline{K(m)}_*(g)=0"] & \overline{K(m)}_*(\Sigma^{2d}C_h) \arrow[r, "\overline{K(m)}_*(d)"] \arrow[d, "\overline{K(m)}_*(l)"] & \overline{K(m)}_*(\Sigma^{2d+1}(X)) \arrow[r] \arrow[d, "\overline{K(m)}_*(f)=0"] & \dots \\
\dots \arrow[r] \arrow[d] & \overline{K(m)}_*(\Sigma^{d}Y) \arrow[r, "\overline{K(m)}_*(p)"] \arrow[d, "\overline{K(m)}_*(g)=0"] & \overline{K(m)}_*(\Sigma^{d}C_h) \arrow[r, "\overline{K(m)}_*(d)"] \arrow[d, "\overline{K(m)}_*(l)"] & \overline{K(m)}_*(\Sigma^{d}X) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \overline{K(m)}_*(Y) \arrow[r, "\overline{K(m)}_*(p)"] & \overline{K(m)}_*(C_h) \arrow[r] & \overline{K(m)}_*(\Sigma X) \arrow[r] & \dots
\end{tikzcd}
\tag{10}
$$

For an element $x \in \overline{K(m)}_*(\Sigma^{2d}C_h)$ since $\overline{K(m)}_*(f \circ d) = 0$, we know that $\overline{K(m)}_*(l)(x) \in ker(\overline{K(m)}_*(d))$. Since the rows are exact, we then know that $\exists x' \in \overline{K(m)}_*(\Sigma^d Y)$ such that $\overline{K(m)}_*(l)(x) = \overline{K(m)}_*(p)(x')$. Therefore $\overline{K(m)}_*(l^2)(x) = \overline{K(m)}_*(l \circ p)(x') = \overline{K(m)}_*(p \circ g)(x') = 0$ so $\overline{K(m)}_*(l^2) = 0$.

Therefore we have produced a v_n map l^2 on C_h, and thus have demonstrated that V_n is a thick subcategory. ☐

However this proof does depend on Lemmas 1 and 2. For a proof of Lemma 1, read the final part of section 6.1 of [2]; it's proof is not terribly illuminating. The proof of Lemma 2 is interesting, though, and we will discuss it here.

3.1.1 Proof of Lemma 2

Ravenel begins with a different lemma,

Lemma 3. *If X has two v_n maps f, g then there are integers i, j so that $f^i \cong g^j$.*

Proof. By Ravenel's Lemma 6.1.1, we can replace f, g with powers so that their effect after applying $\overline{K(m)}_*$ is the same and by applying Lemma 1 we may also assume that they commute. Thus $\overline{K(m)}_*(f - g) = 0$ for all m, which, by the *Nilpotence Theorem*, implies that $(f - g)$ is a nilpotent map. This $\exists i$ so that $(f - g)^{p^i} = 0$.

As f and g commute, we may expand this mod p to get that $f^{p^i} = g^{p^i} \mod p$, and furthermore that $f^{p^{i+k}} = g^{p^{i+k}} \mod p^k$. We then take k to be sufficiently large so that it implies that $f^{p^I} = g^{p^I}$ on the nose, i.e. that $f \cong g$. □

We can then use this lemma to prove Lemma 2.

Proof of Lemma 2. Let $W = DX \vee Y$ so that $h : X \to Y$ is adjoint to $\hat{h} \in \pi_*(W)$. W carries two v_n maps, namely $Df \vee Y$ and $DX \vee g$ which, by the above lemma, are homotopic up to a power; i.e. $Df^i \vee Y \cong DX \vee g^j$. We then note that W is a module spectrum over $DX \vee X$ (with structure map $(DX \vee X) \vee W = DX \vee X \vee DX \vee Y \xrightarrow{DX \vee h \vee DX \vee Y} (DX \vee Y) \vee (DX \vee Y) \xrightarrow{m} DX \vee Y = W$). Then we have that $hf^i \dashv \hat{f}^i\hat{h} = (Df^i \vee Y)\hat{h} \cong (DX \vee g^j)\hat{h} = \hat{g}^j\hat{h} \vdash g^j h$ so $hf^i \cong g^j h$. □

3.2 Construction of a Type n Spectrum with a v_n Map

The plan for this section is to develop some machinery behind *Margolis Homology Groups* and use the *Adams Spectral Sequence* to use properties of those homology groups to guarantee that a given spectrum has a v_n map. Such spectra are called *strongly type n*. This pushes the goalpost further down, as we now need to construct a *strongly type n* spectrum. This will involve the *Smith Construction* which, unfortunately, only produces a *weakly type n* spectrum. That one can convert a weakly type n spectrum into a strongly type n spectrum will not be discussed, but is in the appendices of Ravenel's book (specifically appendix C of [2]).

3.2.1 Margolis Homology Groups and Strongly Type n Spectra

We first recall the structure of the Steenrod Algebra of cohomology operations, as well as the dual algebra of co-operations. For the prime 2 the Dual Steenrod Algebra A_* is $\mathbb{Z}/2[\xi_1, \xi_2, ...]$, and for odd primes it is $\mathbb{Z}/2[\xi_1, ...] \otimes E(\tau_0, \tau_1, \ldots)$ with $|\xi_i| = 2^i - 1$ for $p = 2$ and $2p^i - 2$ for $p \neq 2$ and $|\tau_i| = 2p^i - 1$. We let P^s_t denote the dual of $\xi_t^{p^s}$ w.r.t the monomial basis of A_* and Q_i be dual to τ_i.

These elements, as they belong to A can act on any A-module M, thus allowing us to define the Margolis Homology groups $H_*(M; P^s_t)$, $H_*(M; Q_i)$ in a manner described in Ravenel ([2]Definition 6.2.1). The details are not important, as the intricacies are handled in a result to which we appeal.

Importantly, though, spectra whose Margolis Homology Groups satisfy certain conditions are the strongly type n spectra which we define below

Definition 2. *Strongly Type n Spectrum. A p-local finite CW-complex Y is strongly of type n if*

- *For $p = 2$ $H_*(Y; P^s_t) := H_*(H^*(Y); P^s_t)$ vanishes for $s + t \leq n + 1$ and $(s, t) \neq (0, n + 1)$. For $p > 2$, we require that $H_*(Y; P^s_t)$ vanish for $s + t \leq n$ and $H_*(Y; Q_i)$ vanishes for $i < n$.*

- *Q_n acts trivially on $H^*(Y)$*

- *$H^*(Y)$ and $K(n)_*(Y)$ have the same rank.*

Spectra satisfying these conditions are what we need to provide the periodicity theorem as:

Theorem 5. *If Y is strongly type n then it carries a v_n map.*

3.2.2 Proof of Theorem 5

For a v_n map $f : \Sigma^d Y \to Y$, we have that its Spanier-Whitehead adjoint is $\hat{f} \in \pi_d(R), R = DY \vee DY$. Thus to demonstrate the existence of a v_n map, we study $\pi_*(R)$ - a task very suitable for the *Adams Spectral Sequence*.

In this case, the E_2 term of the AdSS is of the form $E_2^{s,t} = Ext_A^{s,t}(H^*(R), \mathbb{Z}/p)$ and the differentials have the signature $d_r : E_r^{s,t} \to E_{r+1}^{s+r,t+r-1}$. The result to which we appeal is the following:

Lemma 4. *If a spectrum Y is strongly type n then for $R = DY \vee Y$ the E_2 Ext terms of the Adams Spectral Sequence vanishes above a line of slope $\dfrac{1}{|v_n|}$. I.e. the group $Ext_A^{s,t}(H^*(R), \mathbb{Z}/p)$ vanishes for $s > c + \dfrac{1}{2p^n - 2}(t - s)$.*

Implicitly we are visualizing the AdSS as a spectral sequence with x coordinate $(t - s)$ representing the topological dimension and y coordinate s. While this result helps us understand a large region of the Adams spectral sequence, the following result will help us to understand a different part in preparation for further results. It relies on a filtration of the Steenrod Algebra where for $p = 2$ we have $A_N = \{Sq^1, Sq^2, \ldots, Sq^N\} \subset A$ and for $p > 2$ we have $A_N = \{\beta, \mathcal{P}^{p^0}, \mathcal{P}^{p^1}, \ldots, \mathcal{P}^{p^{N-1}}\}$ the elements of which are all generators at their respective primes. The following lemma is a way to formalize how $Ext_{A_N}(H^*(Y), \mathbb{Z}/p)$ approximates $Ext_A(H^*(Y), \mathbb{Z}/p)$.

Lemma 5. *For a strongly type n p-local finite CW-spectrum R, like the one above, and $\forall N > n, \exists k_N > 0$ such that the map $Ext_A^{s,t}(H^*(R), \mathbb{Z}/p) \xrightarrow{\phi} Ext_{A_N}(H^*(R), \mathbb{Z}/p)$ is an isomorphism if $s > \dfrac{1}{2p^n - 2}(t - s) - k_N$. Moreover $\lim_{N \to \infty} k_N = \infty$.*

These approximations allow us to build the following diagram:

$$\begin{tikzcd}
Ext_A(\mathbb{Z}/p, \mathbb{Z}/p) \arrow[r, "i"] \arrow[d, "\phi"] & Ext_A(H^*(R), \mathbb{Z}/p) \arrow[d, "\phi"] \\
Ext_{A_N}(\mathbb{Z}/p, \mathbb{Z}/p) \arrow[r, "i"] \arrow[d, "\lambda"] & Ext_{A_N}(H^*(R), \mathbb{Z}/p) \arrow[d, "\lambda"] \\
Ext_{E(Q_n)}(\mathbb{Z}/p, \mathbb{Z}/p) \arrow[r, "i"] \arrow[d, "\cong"] & Ext_{E(Q_n)}(H^*(R), \mathbb{Z}/p) \arrow[d, "\cong"] \\
P(v_n) \arrow[r, "i"] \arrow[d] & P(v_n) \otimes H^*(R) \arrow[d] \\
K(n)_* \arrow[r, "i"] & K(n)_*(R)
\end{tikzcd} \tag{11}$$

Where the left isomorphism is, apparently, a standard calculation. (Note, I'm unsure of what $P(v_n)$ means...) The right isomorphism is because of Q_n trivially acting on $H^*(Y)$ and resulting from the left isomorphism. Finally, the condition that the rank of $H^*(Y)$ and $K(n)_*(Y)$ are the same demonstrates that the bottom right map is an injection.

The crux of the argument, in our opinion, is the following lemma below which may or may not be miswritten in Ravenel ([2]Lemma 6.3.3):

Lemma 6. *For all $N \geq n$ there is an integer $t > 0$ so that $v_n \in im(\lambda)$. I.e. $\exists x \in Ext_{A_N}(\mathbb{Z}/p, \mathbb{Z}/p)$ such that $\lambda(x) = v_n$. NOTE: In Ravenel this appears as $x \in Ext_{A_N}(H^*(Y), \mathbb{Z}/p)$ but we believe this is a typo.*

To this end, x represents a 'universal' v_n map which we will attempt to explicate in the rest of this section. This element x and its image under i lie under the vanishing line provided by Lemma 4, and moreover lie on a line of that slope *through the origin*, necessarily above the line in Lemma 5. Thus we are in the range where ϕ is an isomorphism, so $y = (\phi^{-1} \circ i)(x) \in Ext_A(H^*(Y), \mathbb{Z}/p)$ is a class in the E_2 term of the AdSS associated to R. Our goal is now to demonstrate that this is a permanent cycle and that it gives rise to a v_n map. This is the sense in which x is a 'universal' v_n map.

Lemma 7. *Some power y^{p^i} is a permanent cycle in the AdSS.*

Proof. Since $\phi(y) = i(x)$ is the image of a class in the Adams term for the stable homotopy groups of spheres, y is a central element in Ext_{A_N} and thus commutes with all elements above the line in Lemma 5.

If y itself it not a permanent cycle, then $\exists r > 0$ so that $d_r(y) = u \neq 0$. Since the differentials in the AdSS go up and to the left, u is above the line in Lemma 5 as y is. Thus y commutes with u and so $d_r(y^p) = py^{p-1}u = 0$. If y^p is also not a permanent cycle, we find another $r_1 > r$ so that $d_{r_1}(y^p) = u_1 \neq 0$. However, we once again have that y^p and u_1 commute, meaning that $d_{r_1}(y^{p^2}) = d_{r_1}((y^p)^p) = p(y^p)^{p-1}u_1 = 0$.

If we continue in this manner, we get a sequence of integers $r < r_1 < r_2 < ...$ so that $d_{r_i}(y^{p^i}) = u_i$ and that $d_{r_i}((y^{p^i})^p) = d_{r_i}(y^{p^{i+1}}) = py^{p^i}u_i = 0$. But these u_i will at some point go above the vanishing line guaranteed by Lemma 4, so that necessarily $d_{r_I}(y^{p^I}) = 0$ giving that some y^{p^I} is a permanent cycle. □

Because of how we constructed y, and the commutativity of Diagram 11, we know that y^{p^i} must project to the same element x^{p^i} projects to in $\overline{K(n)}_*(R)$, which is some power of v_n. Thus the image of y^{p^i} in $\pi_*(R)$ is our desired v_n map.

3.2.3 Explicit Constructions

Unfortunately, I both don't understand the Smith construction part well enough, don't consider the remaining parts to be too elucidating, and find that Ravenel obscures too much of the details to the appendices to give a full description of the construction.

Suffice it to say that Ravenel weakens the assumptions that are a part of the strongly type n spectra to a weakly type n spectra, and proceeds to construct a weakly type n spectrum build from the classifying spaces of $\mathbb{Z}/p$. While this is only a weakly type n spectra, the construction of Smith provides a way to take that weakly type n spectrum and modify it into being a strongly type n spectrum.

4 Bibliography

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