Serre Spectral Sequence!

Spectral Sequence Assoc to any Serre fibration

for our purposes, we only consider fiber bundles!

let
$$\begin{array}{c} F \\ \downarrow \\ E \\ \downarrow \pi \\ B \end{array}$$
be our fibration

and assume B is a CW complex (or has homotopy type)

We get skeletal filtration $\emptyset = B^{(-1)} \subseteq B^{(0)} \subseteq \dots \subseteq B^{(n)} \subseteq \dots$

w/ $B = \bigcup B^{(n)}$

pull this back via π to get $E^{(p)} = \pi^{-1}(B^{(p)})$

a filtration of E!

let's see what filtrations can give us!

Exact Couples

firstly, from the L.E.S. of the pairs (E_p, E_{p-1})

we get

$$\begin{tikzcd}
 & & H_{n-1}(E_{p-2}) \arrow[d, "i^*"] & & \\
H_n(E_p) \arrow[r, "j^*"] & H_n(E_p, E_{p-1}) \arrow[r, "\partial"] & H_{n-1}(E_{p-1}) \arrow[r] \arrow[d, "j^*"] & H_{n-1}(E_p) \arrow[r] & \cdots \\
 & & H_{n-1}(E_{p-1}, E_{p-2}) \arrow[r, "\partial"] & H_{n-2}(E_{p-2}) &
\end{tikzcd}$$

gives "self maps" from $E = \bigoplus_{p,n} H_n(E_p, E_{p-1})$

and $D = \bigoplus_{p,n} H_n(E_p)$

$$\begin{tikzcd}
D \arrow[rr, "i^*"] & & D \arrow[dl, "j^*"] \\
 & E \arrow[ul, "\partial"] &
\end{tikzcd}$$

Called exact couple since it is exact at all places!

we get a self map on E $\quad d = j^* \partial : E \to E$

& $\quad d^2 = (j^* \partial)(j^* \partial) = j^* (\partial j^*) \partial = j^* \circ 0 \circ \partial = 0$

So we can "homology this"! and let

$$E^1 = \ker d / \operatorname{im} d$$

to get exactness again, let $D' = i(D)$.

then $i' = i|_{D'}$

$$j'(i(a)) = [j(a)] \quad \text{(defn by exactness)}$$

$$\partial'([e]) = \partial(e) \quad \text{(well defn by exactness)}$$

if we set $D_{p,q} = H_{p+q}(X_p)$ & $E_{p,q} = H_{p+q}(X_p, X_{p-1})$

$$\begin{aligned} E_{p,q} &= H_{p+q}(X_p, X_{p-1}) \\ &\downarrow \partial \\ D_{p-1,q} &= H_{p+q-1}(X_{p-1}) \\ &\downarrow j \\ E_{p-1,q} &= H_{p+q-1}(X_{p-1}, X_{p-2}) \end{aligned}$$

making

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "\partial'"] & \end{tikzcd}$$

another exact couple

Thus we can repeatedly derive this and get exact couples each time

(Why might we do this? just generalized cellular homology)

$$\begin{aligned} &E^1_{p+1,q} \\ &\downarrow d_2 \\ &E^1_{p,q} \\ &\downarrow d_1 \\ &E^1_{p-1,q} \end{aligned} \qquad E^2_{p,q} = \ker d_1 / \operatorname{im} d_2$$

$$\begin{tikzcd} E^2_{p,q} \arrow[d, "\partial'"] & E^2_{p-2,q+1} \\ D'_{p-1,q} \arrow[r, "\text{lift}"] & D'_{p-2,q+1} \arrow[u, "j'"] \end{tikzcd}$$

differential bi-degree $(-r, r-1)$

Thus, let $A^{(1)}_{p,q} = H_{p+q}(X_p)$

$$E^{(1)}_{p,q} = H_{p+q}(E_p, E_{p-1})$$

So $A^{(n)}_{p,q} = i^{(n-1)} H_{p+q}(X_p)$, $E^{(r)}$ more compli

and $d^{(r)}$ goes from $E^{(r)}_{p,q} \to E^{(r)}_{p-r,q+r-1}$

For the Serre spectral sequence, we will have a first quadrant spectral sequence

Which just means $E^{(1)}_{p,q} = 0$ if $p<0$ or $q<0$

which is also bounded, so $E^{(1)}_{p,\cdot} = 0 \quad \forall p > P \quad P < \infty$

Under such conditions

Prop: $E^{\infty}_{p,q} \simeq F^p_n / F^{p-1}_n$ where $n = p+q$ &

$$F^p_n = \mathrm{im}(H_n(X_p) \to H_n(X))$$

proof: firstly $\exists R$ s.t. $\forall r \geq R \quad d^r = 0$

Since our SS is bounded $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$

will have to be zero $\forall r \geq P$

thus the $E^{(r)}$'s converge after a finite numbes of steps

to show they converge to the relevent filtrations consider the unwrapped exact couple

$$E^r_{p+r-1,\, q-r+1} \xrightarrow{\partial^r} A^r_{p+r-2,\, q-r+2} \xrightarrow{i} A^r_{p+r-1,\, q-r+1} \xrightarrow{j^r} E^r_{p,q} \xrightarrow{\partial^r} A^r_{p-1,q} \xrightarrow{i} A^r_{p,\, q-1} \xrightarrow{j^r} E^r_{p-r+1,\, q+r-2}$$

now for p,q fixed & r large

the first & last terms are 0 by boundedness

Similarly since

$$A^r_{p,q} = i^r\left(A^1_{p-r,q+r}\right)$$

the last 2 A terms are also zero

so we have the sequence

$$0 \to A^r_{p+r-2,\, q-r+2} \to A^r_{p+r-1,\, q-r+1} \to E^r_{p,q} \to 0$$

is exact ∯

$$E^\infty_{p,q} \simeq F^p_n / F^{p-1}_n \quad !$$

So the spectral sequence conve to what we call the

associated grades: $gr_n H.(E) = \bigoplus_p F^p_n / F^{p-1}_n$

i.e. $gr_n H.E \simeq \bigoplus_p E^\infty_{p,n-p}$

now if all the terms on the E^∞ page are free abelian.
we will see immediately that we can recover the homology

i.e. to compute $H^n(E)$ we will build it up from the $E^\infty_{p,n-p}$'s

firstly $E^\infty_{0,n} \simeq F^n_0 / F^n_{-1} \simeq F^n_0$

then we have $0 \to F^n_0 \to F^n_1 \to E^\infty_{1,n-1} \to 0$

but as E^∞ is free abelian, the sequence splits ∯

$$F^n_1 \simeq F^n_0 \oplus E^\infty_{1,n-1} \simeq \bigoplus_{p=0}^{1} E^\infty_{p,n-p}$$

Similarly

$$F_p^n \simeq \bigoplus_{i=0}^{p} E^{\infty}_{i,n-i}$$

and since $\quad H_n(E) \simeq \bigcup_{p=0}^{\infty} F_p^n$

$$H_n(E) \simeq \bigoplus_{p=0}^{\infty} E^{\infty}_{p,n-p} \simeq \bigoplus_{p=0}^{n} E^{\infty}_{p,n-p}$$

for non free abelian, we have to worry about extensions

There is also a SS to get cohomology, again coming from

$$\begin{tikzcd}
 & D & E & D & \\
H^n(X_p,X_{p-1}) \arrow[r, "j^*"] & H^n(X_p) \arrow[r, "\delta"] \arrow[d, "i^*"] & H^{n+1}(X_{p+1},X_p) \arrow[r] & H^{n+1}(X_{p+1}) \arrow[r] \arrow[d] & H^{n+2}(X_{p+2},X_{p+1}) \\
H^n(X_{p-1},X_{p-2}) \arrow[r] & H^n(X_{p-1}) \arrow[r] & H^{n+1}(X_p,X_{p-1}) \arrow[r] & H^{n+1}(X_p) \arrow[r] & H^{n+2}(X_{p+1},X_p)
\end{tikzcd}$$

so $\quad D = \bigoplus_{p,n} H^n(X_p)$

$$E = \bigoplus_{p,n} H^n(X_p, X_{p-1})$$

$$\begin{tikzcd}
D \arrow[rr, "i^*"] & & D \arrow[dl, "\delta"] \\
 & E \arrow[ul, "j^*"] &
\end{tikzcd}$$

an exact couple

the filtration now comes from

$$F_p^n := \ker(H^n(X) \xrightarrow{i_p^*} H^n(X_p)) \quad \oint$$

$$E_{p,q}^\infty \approx F_p^n / F_{p+1}^n$$

Serre S.S. time

all the above was completely general for filtered spaces, Serre S.S. provides more structure for computation

as anticipated, let $X_p = \pi^{-1}(B_p) \subseteq E$ be a filtration from this we get our S.S. w/

$$E^1_{p,q} := H^{p+q}(X_p, X_{p-1})$$

obviously $E^1_{p,q} = 0 \quad \forall p < 0$

and if $q < 0$ then since $H^n(X_p, X_{p-1}) = 0 \quad \forall n < p$
(comes from fibration prop., like cellular case)

$E^1_{p,q} = 0$ too. so the Serre S.S. is a first quad.

While this is not terribly useful (E is often complex & working w/ filtration isn't terrible) magic happens on E_2!

Thm: for $F \to E \to B$ a fiber bundle (B path connected)

then (by the above work) we get a S.S. converging to $H^*(E)$ but we also have $E^2_{p,q} = H^p(B; \underline{H^q(F)})$

Pf: the case w/o local system coefs is in Hatcher's Spectral Sequences in Algebraic Topology; working w/ local sys coefs is a similar proof.

Examples:

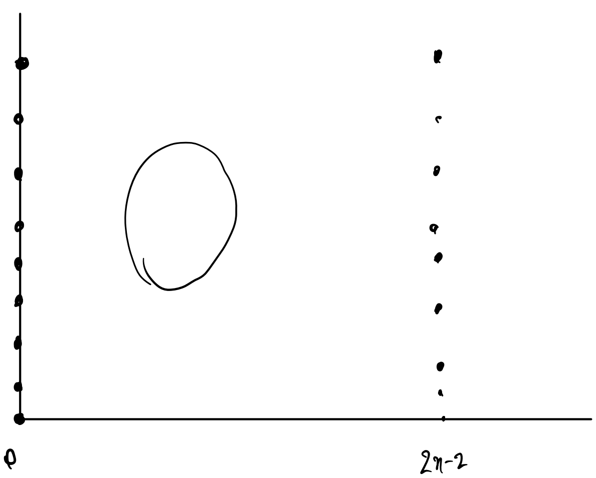
$$\begin{array}{c} SU(n-1) \\ \downarrow \\ SU(n) \\ \downarrow \\ S^{2n-1} \end{array}$$

We will prove $H^*(SU(n)) \simeq \Lambda(x_3, x_5, \ldots, x_{2n-1})$

firstly $H^*(SU(1)) = H^*(\{(1)\}) = 0$ which agrees w/ above

Now by induction, assume $H^*(SU(n-1)) \simeq \Lambda(x_3, x_5, \ldots, x_{2n-3})$

then
$$E^2_{p,q} = H^p(S^{2n-2}; H^q(SU(n-1)))$$
since $H^p(S^{2n-2})$ and $H^q(SU(n-1))$ are free ab.
$$E^2_{p,q} \simeq H^p(S^{2n-2}) \otimes H^q(SU(n-1))$$
so it looks like



the differentials of this S.S. have the form
$$d^r : E^r_{p,q} \to E^r_{p+r,\, q-r+1}$$
so for degree reasons $\forall\ 2 \leq r < 2n-2 \quad d^r = 0$

thus
$$E^{2n-2}_{p,q} = E^2_{p,q}$$

and
$$d^{2n-2} : E^{2n,2}_{p,q} \to E_{p+2n-2,\, q-2n+3}$$

these can only be nonzero if $p=0$

$$d^{2n-2} : E^{2n-2}_{0,q} \to E^{2n-2}_{2n-2,\, q-2n+3}$$

Now $E^{2n-2}_{0q} = E^2_{0,q} = H^0(S^{2n-2}) \otimes H^q(SU(n-1)) = H^q(SU(n-1))$

and $H^q(SU(n-1))$ is generated by $x_3, \dots, x_{2n-3}$ in their respective degrees. $d^{2n-2}(x_i)$ must land in

$$E_{2n-2,\, i-2n+3}$$

but if $3 \leq i \leq 2n-3$

$$i-2n+3 \leq 0 < 3$$

so $d^r(x_i)=0 \quad \forall \; x_i$

For cohomology SS's, we also have a product struct & Liebniz rule (see Hatcher's S26 notes)

so for $x \in E^r_{p,q}$ $\quad y \in E^r_{p',q'}$

$$d^r(xy) = d^r(x)y + (-1)^{p+q} x\, d^r(y)$$

so for any element $x \in H^i(SU(n-1))$ x is a sum of products of the x_i's &

$$d^r\left(\prod_{j=1}^{k} x_{i_j}\right) = d^r(x_{i_1}) \dots + (-1)^{i_j} x_{i_j} \cdot d^r(\dots) = 0 \text{ by induc}$$

So $d^r = 0$ $\forall$ $r \geq 2$ so $E^2 = E^\infty$!

thus $H^*(SU(n)) \simeq \Lambda(x_3, \dots, x_{2n-1})$

where the final generator comes from $E^2_{2n-2,0} = H^{2n-2}(S^{2n-2}) \otimes H^0(SU(n-1))$
$= \mathbb{Z}\, x_{2n-1} \otimes 1$

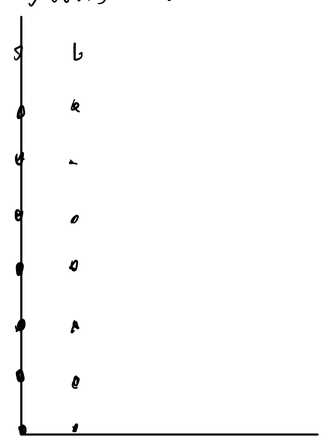
Similarly, for

$$\begin{array}{c} SU(n) \\ \downarrow \\ U(n) \\ \downarrow \\ S^1 \end{array}$$

we have $E^2_{p,q} \simeq H^p(S^1; H^q(SU(n)))$

$\simeq H^p(S^1) \otimes H^q(SU(n))$

Which looks like



$d^2 = 0$ by degree reasons

So $H^*(U(n)) \simeq \Lambda(x_1, \dots, x_{2n-1})$ $\quad E^2_{1,0} = \mathbb{Z}(x_1 \otimes 1)$

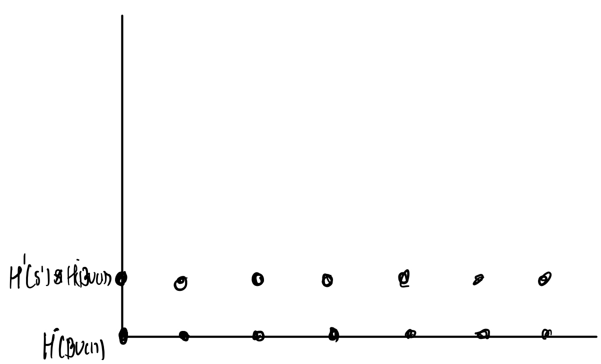
$$H^*(BU(n)) \simeq \mathbb{Z}[c_2, \dots, c_{2n}]$$

firstly we have

$$\begin{array}{c} U(1) \simeq S^1 \\ \downarrow \\ EU(1) \simeq S^\infty \\ \downarrow \\ BU(1) \simeq \mathbb{C}P^\infty \end{array}$$

so $E^2_{p,q} \simeq H^p(BU(1)) \otimes H^q(U(1))$

$\simeq H^p(BU(1)) \otimes H^q(S^1)$



obviously $d^r = 0$ $\forall r \geq 3$ since it would have bidegree $(r, -r+1)$ & $-r+1 \leq -2$

We know $H^*(EU(1)) \simeq \mathbb{Z}$ so

$E^2_{1,0} = H^1(BU(1)) = 0$ & $d^2 : E^2_{p,1} \to E^2_{p+2,0}$

must be an iso so that $E^3_{p,q} = 0$ $\forall (p,q) \neq 0$

let $E^2_{1,0} = \mathbb{Z}[x]$ & $u = d^2(x)$

Since d^2 is an iso $H^2(BU(1)) \simeq \mathbb{Z}[x]$

then $d^2(x \otimes u) = d^2(x)u + (-1)^1 \cancel{x \cdot d^2(u)}$

$= u^2$ is a generator of $H^4(BU(1))$

assuming u^n is a generator of $H^{2n}(BU(1))$

$d(x \otimes u^n) = u^{n+1}$ shows $H^{2n+2}(BU(1)) \cong \mathbb{Z}[u^{n+1}]$

therefore $H^*(BU(1)) \simeq \mathbb{Z}[u]$ $|u|=2$

we have the standard "classifying bundle"

$$\begin{tikzcd}
U(n-1) \arrow[r] \arrow[d] \arrow[dr] & U(n) \arrow[d] \\
EU(n-1) \arrow[d] & EU(n) \arrow[d] \\
BU(n-1) & BU(n)
\end{tikzcd}$$

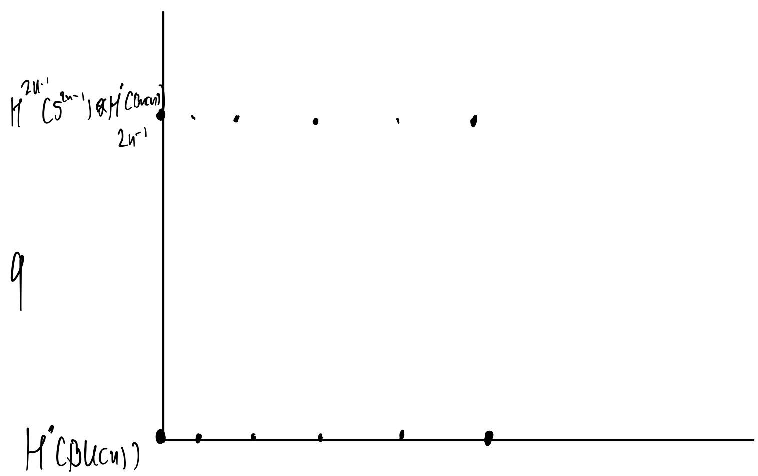
Which we can modify to get

$$\begin{tikzcd}
S^{2n-1} \arrow[r] & BU(n-1) \arrow[d] \\
& BU(n)
\end{tikzcd}$$

Since an $n-1$ plane can be thought of as an n plane Λ and a unit vector $u \in \Lambda$ giving $u^\perp \subseteq \Lambda$

then we have

$$E^2_{p,q} = H^p(BU(n)) \otimes H^q(S^{2n-1})$$



$$d^r : E^r_{p,2n-1} \longrightarrow E^r_{p+r,\, 2n-1-r+1}$$

is nontrivial only when $r = 2n$

so $\quad E^r_{p,q} = E^2_{p,q} \quad \forall\ 2 \leq r \leq 2n$

and $\quad E^{2n+1}_{p,q} = E^\infty_{p,q}$

Assume inductively that $\quad H^*(BU(n)) \cong \mathbb{Z}[c_2, \dots, c_{2n-2}]$

by degree reasons we have that

$$H^p(BU(n-1)) \cong E^\infty_{p,0} = E^2_{p,0} \cong H^p(BU(n)) \quad \forall p < 2n$$

for $p = 2n$ we have

$$d^{2n}_{0,2n-1} : E^2_{0,2n-1} \to E^2_{2n,0} = H^{2n}(BU(n))$$
$$\parallel$$
$$H^{2n-1}(S^{2n-1}) = \mathbb{Z}[x]$$

so $\ker(d^{2n}_{0,2n-1}) \oplus H^{2n}(BU(n)) / \operatorname{im}(d^{2n}_{0,2n-1}) \cong H^{2n}(BU(n-1))$

however since we know all the lower degree pieces of $H^{\cdot}(BU(n))$ agree w/ those of $H^{\cdot}(BU(n-1))$ &
the latter does not have a degree $2n$ generator
Also, can realize d^{2n} as map giving Thom class of the bundle so d^{2n} cannot be zero.

$d^{2n}_{0,2n-1}$ must not be zero so

$\ker(d^{2n}_{0,2n-1}) = 0$ &

$$H^{2n}(BU(n)) \cong H^{2n}(BU(n-1)) \oplus \mathbb{Z}[d^{2n}(x)]$$

let $c_{2n} = d^{2n}(x)$

so for $y \in H^{\cdot}(BU(n))$ $\quad d^{2n}(x \otimes y) = d^{2n}(x)\,y + (-1)^{2n-1} x\, \cancel{d^{2n}(y)}$

$$= c_{2n}\, y$$

so we get a sequence

$$
\begin{tikzcd}
0 \arrow[r] & H^p(BU(n)) \arrow[r, "- \smile c_{2n}"] \arrow[dr, "x \otimes -"'] & H^{p+2n}(BU(n)) \arrow[r, "\pi"] & H^{p+2n}(BU(n-1)) \arrow[r] & 0 \\
& & H^{2n-1}(S^{2n-1}) \otimes H^p(BU(n)) \arrow[u, "d^{2n}"'] & &
\end{tikzcd}
$$

where π is the map coming from

$$\cancel{\ker(d^{2n}_{p,2n-1})} \oplus H^{2n+p}(BU(n)) / \operatorname{im}(d^{2n}_{p,2n-1}) \simeq H^{p+2n}(BU(n-1))$$

thus $\quad H^*(BU(n)) \simeq H^*(BU(n-1))[c_{2n}] \simeq \mathbb{Z}[c_2, \ldots, c_{2n}]$!