Lecture Notes on Adams's *On The Image J(X) IV*

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1 Introduction

To quote the words of Mike Hopkins, the image of the J homomorphism is "represents the only part of the homotopy groups of spheres which is nontrivial but still really understandable." In this talk I'm going to try to discuss two fundamental ideas:

1. One can detect nontriviality of elements in stable homotopy groups by examining their images along different maps of spectra which are easier to understand. Adams achieves this by means of analyzing his d and e invariants.

2 The d and e Invariants

These invariants can be defined for any cohomology theory, so we will work out the general theory here before specifying it.

Given an element $f \in Map(X, Y)$, we can define its k degree as $d(f) = k^*(f) \in Hom(k^*(Y), k^*(X))$ which is a vast generalization of the degree of a map between spheres. The use of this invariant is in its ability to detect wether or not the map is *essential* i.e. if it is not nullhomotopic. For nullhomotopic maps, $k^*(f)$ must be the 0 map, and thus if $d(f) \neq 0$ then f cannot be nullhomotopic.

However, it can often be the case that non-nullhomotopic maps have $d(f) = 0$, for an easy example think about $KU(S^3 \xrightarrow{\nu} S^2)$ which must be zero for degree reasons but it is not nullhomotopic. This problem can be remedied by the e invariant, which is defined whenever $d(f) = 0$ and $d(\Sigma f) = 0$.

If we use f to generate a cofiber sequence

$$X \xrightarrow{f} Y \xhookrightarrow{i} C_f \xrightarrow{g} \Sigma X \xrightarrow{\Sigma f} \Sigma Y ... \tag{1}$$

we can apply k to that diagram

$$k^*(X) \xleftarrow{d(f)} k^*(Y) \xleftarrow{k^*(i)} k^*(C_f) \xleftarrow{k^*(g)} k^*(\Sigma X) \xleftarrow{d(\Sigma f)} k^*(\Sigma Y) ... \tag{2}$$

which then, by our hypotheses, splits as

$$0 \leftarrow k^*(Y) \xleftarrow{k^*(i)} k^*(C_f) \xleftarrow{k^*(g)} k^*(\Sigma X) \leftarrow 0 \tag{3}$$

then by the classification of $Ext^1(M, N)$ as extensions $0 \leftarrow M \leftarrow L \leftarrow N \leftarrow 0$ this given an element $e(f) \in Ext^1(k^*(Y), k^*(\Sigma X))$. However, we need to be careful about which group this Ext is taken over. It turns out that having it just over $k^*(*)$ is often too simple, and the truly powerful results are usually proven using the algebra of operations $k^*(k)$. When we work with K theory soon, we will take our Ext groups to be over the ring of Adams Operations. Thankfully, this can be simplified further in the case of K theory, where we will see that e will actually come from the Chern character.

3 Connecting the e invariant and the Chern character

In order to connect the two, we first need to describe when the d homomorphism vanishes.

Proposition 1. *For $\theta \in \pi_r(S)$ represented by $f : S^{q+r} \to S^q$, $d(\theta)$ is well defined and is guaranteed to be zero unless we use real K theory and $r \equiv 1, 2 \mod 8$*

Proof. For the well definedness, it could be the case that $d(f) : K^*(S^q) \to K^*(S^{q+r})$ depends on $q \mod 2, 8$. However, since $K^*(S^q)$ is a free module over $K^*(*)$ on a generator in $K^0(S^q)$ specifying d on K^0 determines it on all K^t, and by the suspension isomorphism that determines it on all $K^*(S^{q+t}) \to K^*(S^{q+t+r})$.

That the d homomorphism is zero aside from the cases above either boils down to it being impossible on the level of group homomorphisms (where it becomes $\mathbb{Z}/2 \to Z$) or one can demonstrate the impossibility by examining the operations. □

We will now use the Chern character to examine the situation more computationally.

By applying K and the chern character to the cofiber sequence, we get

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K^*(\Sigma S^{q+r}) & \longrightarrow & K^*(S^{q+r} \cup_f S^q) & \longrightarrow & K^*(S^q) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^*(\Sigma S^{q+r};\mathbb{Q}) & \longrightarrow & H^*(S^{q+r} \cup_f S^q;\mathbb{Q}) & \longrightarrow & H^*(S^q;\mathbb{Q}) & \longrightarrow & 0
\end{array}
$$

Now, on the top row, the groups on the extremes are just $\mathbb{Z}$ generated η' and ξ' and there are elements $\eta, \xi \in K^*(S^{q+r} \cup_f S^q)$ which project to η' and ξ' respectively.

Similarly we can do the same for generators of $H^*(-;\mathbb{Z})$ which we call h_{q+r} and h_q respectively. The structure of the Chern character must then be of the form

$$
\begin{aligned}
ch\xi &= a_{11}h_{q+r} + a_{12}h_q \\
ch\eta &= a_{21}h_{q+r} + a_{22}h_q
\end{aligned}
$$

Since η is just the image of the generator of $K^*(\Sigma S^{q+r})$, we can leverage the fact that the chern character is natural to get $ch(\eta) = ch(k^*(g)(\eta')) = H^*(g)(ch(\eta')) = H^*(g)(h_{q+r}) = h_{q+r}$ and similarly $a_{11}h_q = H^*(f)(ch(\xi)) = ch(k^*(f)(\xi)) = ch(\xi') = h_q$ so $ch(\xi) = \lambda h_{q+r} + h_q$. Now the choice of ξ as our generator is non-canonical, so if we replace it with a different generator $\xi + N\eta$ we end up seeing that $ch(\tilde{\xi}) = ch(\xi) + ch(N\eta) = (\lambda + N)h_{q+r} + h_q$. Therefore λ is an invariant of f in the group $\mathbb{Q}/\mathbb{Z}$.

$$
\begin{aligned}
ch\xi &= \lambda h_{q+r} + h_q &\quad (4) \\
ch\eta &= h_{q+r} &\quad (5)
\end{aligned}
$$

We have all of this structure coming from Adams operations, so lets see what additional information they give us about this λ invariant:

We can apply an Adams operation to Equations 4 and 5 to get

$$
\begin{aligned}
ch(\Psi^k\xi) = \Psi^k(ch(\xi)) = \Psi^k(\lambda h_{q+r} + h_q) &= \lambda k^{\frac{q+r}{2}} h_{q+r} + k^{\frac{q}{2}} h_q &\quad (6) \\
ch(\Psi^k\eta) &= k^{\frac{q+r}{2}} h_{q+r} &\quad (7)
\end{aligned}
$$

But Eq 6 can be simplified to

$$
ch(\Psi^k\xi) = \lambda(k^{\frac{q+r}{2}} - k^{\frac{q}{2}})h_{q+r} + \lambda k^{\frac{q}{2}} h_{q+r} + k^{\frac{q}{2}} h_q = ch(\lambda(k^{\frac{q+r}{2}} - k^{\frac{q}{2}})\eta) + ch(k^{\frac{q}{2}}\xi) \quad (8)
$$

which, since the Chern character is injective on torsion free K groups, implies that

$$\Psi^k \xi = \lambda(k^{\frac{q+r}{2}} - k^{\frac{q}{2}})\eta + k^{\frac{q}{2}}\xi \tag{9}$$

and that

$$\Psi^k \eta = k^{\frac{q+r}{2}}\eta \tag{10}$$

Now, since the coefficients in Equations 9 and 10 must be integers, we immediately get that

Proposition 2. *The λ invariant must be of the form $\dfrac{z}{h}$, in reduced form, where z is some integer and h is $gcd(\{k^{\frac{q+r}{2}} - k^{\frac{q}{2}} : k \in \mathbb{Z}\})$*

Now all of this work only relies on the fact we had an extension $0 \to K^*(\Sigma S^{q+r}) \to K^*(S^{q+r} \cup_f S^q) \to K^*(S^q) \to 0$. and that the groups in question had Adams operations. Therefore we immediately get $\theta : Ext^1(K^*(\Sigma S^{q+r}), K^*(S^q)) \to \mathbb{Q}/\mathbb{Z}$ which is just the λ invariant we defined previously.

With all of this lain out, we state the main theorems:

Theorem 1. *If $r = 4s-1$ then the value of eJ on a suitable generator of $\pi_r(SO)$ is $\dfrac{1}{2}(-1)^{s-1}\dfrac{B_s}{4s}$ whose denominator is $m(2s)$. Thus the element generates a subgroup of $\pi_r(S)$ of order $m(2s)$.*

That these are direct sum factors relies of Adams' $J(X)$ II paper, which is also where the order is proved.

4 Proof of the Main Theorem

The main tool in this theorem is the equivalence between the cofiber $S^q \cup_f S^{q+r}$ where $f = J\phi$ for $\phi \in \pi_r(SO)$ and the Thom Space associated to the real bundle that $\phi : S^r \to SO(q)$ represents. These two spaces turn out to be homotopy equivalent and thus we can use a Thom isomorphism theorem $\phi_K : K^*(S^q) \to K^*(S^q \cup_f S^{q+r})$ and similarly $\phi_H : H^*(S^q) \to H^*(S^q \cup_f S^{q+r})$

Proof. We take $\beta \in \pi_{r+1}(SO)$ where $r = 4s$ such that the bundle β represents is a generator of $K_R(S^{r+1})$. This is equivalent to having $ch_{2s}(c\beta) = a_{4s}h^{4s}$.

We can then construct $\xi = \phi_k 1$ and examine $\phi_H^{-1} ch_R \xi$ which Adams computed to be $1 + \dfrac{1}{2}\alpha_{2s}a_{4s}h^{4s}$. From our notation previously, this implies then that $e_R(J\beta) = \dfrac{1}{2}\alpha_{2s}$. □

5 Periodic Families of Elements

Lemma 1. *For p an odd prime, let $m = p^f$ and $r = (p-1)p^f$. Then there exists an element $\alpha \in \pi^S_{2r-1}$ such that*

1. $m\alpha = 0$
2. $e_c\alpha = \dfrac{-1}{m}$
3. *The Toda bracket $\{m, \alpha, m\}$ is zero modulo $m\pi^S_{2r}$.*

Proof. We can just take α to be any of the elements we have constructed in the image of J with $m(2s)$ torsion. □

We will use these elements to construct 'periodic self maps' as follows

Lemma 2. *With α as above, there exists maps $A : S^{2q+2r-1} \cup_m S^{2q+2r-1} \to S^{2q-1} \cup_m S^{2q-1}$ such that the following diagram commutes up to homotopy and $d_C(A) = 1$.*

$$\begin{tikzcd} S^{2q+2r-1} \cup_m S^{2q+2r+1} \arrow[r, "A"] & S^{2q-1} \cup_M S^{2q-1} \arrow[d, "j"] \\ S^{2q+2r-1} \arrow[u, "i"] \arrow[r, "\alpha"] & S^{2q-1} \end{tikzcd}$$

Proof. The existence of A is guaranteed by the vanishing of the Toda bracket, and the condition on $d_C(A)$ is due to Toda bracket properties that I did not discuss. □

We can then compose this self map A with suspension operators to get $A \circ \Sigma^{2r} A \circ \Sigma^{4r} A \circ \ldots \Sigma^{2r(s-1)} A : \Sigma^{2rs}(S^{2q+1} \cup_m S^{2q+1}) \to S^{2q+1} \cup_m S^{2q+1}$. Then, instead of using a class α to define A, we can use these compositions to get a new homotopy class

$$\begin{tikzcd} S^{2q+2rs-1} \cup_m S^{2q+2rs-1} \arrow[r, "A\circ\Sigma^{2r}A\circ\ldots"] & S^{2q-1} \cup_M S^{2q-1} \arrow[d, "j"] \\ S^{2q+2rs-1} \arrow[u, "i"] \arrow[r, "\alpha_s"] & S^{2q-1} \end{tikzcd}$$

and since $d_C(A) = 1$, we have $d_C(A \circ \Sigma^{2r} A \circ \ldots) = 1$ and thus α_s is an essential map.