Lecture Notes on
Adams: Vector Fields on Spheres

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1 Introduction

In this talk we will determine the maximum number of linearly independent vector fields that can be on a sphere S^n. At the time Adam's published this paper, there was already a construction of $\rho(n)-1$ vector fields on S^{n-1} where, if $n = (2a+1)2^b$ and $b = c+4d$ then $\rho(n) = 2^c + 8d$. We will not start with this construction, but if time permits we will discuss it at the end. The purpose of Adam's paper is the following theorem

Theorem 1.1 There do not exist $\rho(n)$ linearly independent vector fields on S^{n-1}.

Thus the previously known construction is best possible.

While it may seem like the paper doesn't spend much time considering vector fields on spheres, it is due to a reduction that occurs on the end of the paper.

If we assume, so as to reach a contradiction, that we did have $\rho(n)$ linearly independent vector fields on S^{n-1}, then by the work of James all S^{nm-1} would possess a $\rho(n)$-vector field. Correspondingly, the Steifel Bundle $V_{nm,\rho(n)+1}$ (since $S^{nm-1} \subset \mathbb{R}^{nm}$ and we can complete the $\rho(n)$ vector field into an $\rho(n)+1$ vector field by adding the point on the sphere) would have a nonzero section. Again by the work of James, if we take m to be large enough then $V_{nm,\rho(n)+1}$ is approximated by a space $Q_{nm,\rho(n)+1}$ which is a sort of 'stable homotopy dual' (Spanier Whitehead Dual) to a quotient of projective spaces. The existence of the nonzero section implies that this Q space is reducible:

Definition 1.1 – (S-)(Co)Reducible A based space (Y, y_0) with a single n cell is *reducible* if the projection $Y \to Y/Y^{n-1} = S^n$ has a homotopy section. It is S-reducible if the above holds stably. (S-)coreducibility it defined dually

and by the duality above, the projective space quotient $P_{\rho(n)+1-nm+qr,\rho(n)+1} = RP^{qr-nm+\rho(n)}/RP^{qr-nm-1}$ is S-coreducible. If we take q large enough, though, then this would imply that the quotient is coreducible on the nose.

This is where the K theory comes in, as we will use it to show that the quotient cannot be coreducible. This will be performed by examining certain cohomology operations, their effect on projective space quotients, and the fact that the K theory of a coreducible space splits in a manner that the projective space quotients do not.

Theorem 1.2 $RP^{m+\rho(m)}/RP^{m-1}$ is not coreducible, i.e. there is no map $f : RP^{m+\rho(m)}/RP^{m-1} \to S^m$ such that the composite

$$S^m = RP^m/RP^{m-1} \xrightarrow{i} RP^{m+\rho(m)}/RP^{m-1} \xrightarrow{f} S^m$$

is degree 1.

2 K-Theory and Operations

We recall the Grothendieck Ring $K_\Lambda(X)$ from Yigal's talk, where the Λ is just a placeholder for the real, complex, or quaternionic field. We also introduce the analagous construction for representations on a topological group $K'_\Lambda(G)$.

The elements of $K'_\Lambda(G)$ are equivalence classes of representations $\alpha : G \to GL_n(\Lambda)$ where two representations are equivalent if they are equal under a change of basis, modulo the normal subgroup generated by $\{\alpha + \beta\} - \{\alpha\} - \{\beta\}$.

2.1 Compositions

When discussing the various operations one can perform on virtual representations and bundles, Adams makes the following notation a convention: f, g, h will be maps of spaces, ξ, η, ζ will be bundles, κ, λ, μ will be virtual bundles in $K_\Lambda(X)$, α, β, γ will be representations and θ, ϕ, ψ will be virtual representations.

Obviously, if the target of a representation α is the same as the source of a representation β we can compose them to get $\beta \cdot \alpha$. For a bundle with structure group $GL_n(\Lambda)$, we can compose it with a representation $\alpha : GL_n(\Lambda) \to GL_m(\Lambda')$ by changing the transition functions on the same coordinate patches to get $\alpha \cdot \xi$. Finally, $\xi \cdot f = f^*(\xi)$ denotes the pullback.

These composites are linear in their first factors and thus can be defined on virtual representations/bundles in an associative unital manner. However, these are not linear or even well defined for sums on the second factor for dimensional concerns. Thus if we want to define a cohomology operation we will need to extend our definition.

Since the problem is that the first factor is not compatible with varying dimensions, we will replace the singular representation $\theta : GL_n(\Lambda) \to GL(\Lambda')$ with $\Theta = (\theta_n)$, where each θ_n is an n-dimensional representation.

Apriori these sequences don't need to be linear at all, but for

Definition 2.1 – Additive Sequence A sequence $\Theta = (\theta_n)$ such that, if π, ω are the projections of $GL_n(\Lambda) \times GL_m(\Lambda)$ onto its factors, then $\theta_{n+m}(\pi \oplus \omega) = (\theta_n \cdot \pi) + (\theta_m \cdot \omega)$

When we want our operations to respect the ring structure, we also want multiplicative structure which is guaranteed by

Definition 2.2 – Multiplicative Sequence With $\Theta, \theta_n, \pi, \omega$ as above, we require that $\theta_{nm}(\pi \otimes \omega) = (\theta_n \cdot \pi) \otimes (\theta_m \cdot \omega)$.

Working with sequences with these properties allow us to prove that

Theorem 2.1 For Θ an additive sequence, the map $\kappa \mapsto \Theta \cdot \kappa$ gives rise to a homomorphism $\Theta : K_\Lambda(X) \to K_{\Lambda'}(X)$ such that:

1. $$\begin{array}{ccc} K_\Lambda(Y) & \xrightarrow{\Theta} & K_{\Lambda'}(Y) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ K_\Lambda(X) & \xrightarrow{\Theta} & K_{\Lambda'}(X) \end{array} \tag{1}$$

   commutes for maps $f : X \to Y$ (i.e. Θ is natural)

2. If Θ is multiplicative then the map on K rings preserves products

3. If the virtual degree of θ_1 is 1 then Θ preserves units

For some examples, we have

1. $c_n : GL_n(\mathbb{R}) \to GL_n(\mathbb{C})$ the complexification map, additive and multiplicative
2. $r_n : GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$ the 'real'ification map, additive
3. $t_n : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ the complex conjugation map, additive and multiplicative

These give the first examples of cohomology operations, with all 3 being group homomorphisms, while all but the second are also ring homomorphisms on the corresponding Grothendieck rings. While they are not the exact operation we will use to prove our main theorem, they will play a prominent role.

2.2 The Main Operation

Theorem 2.2 For each $k \in \mathbb{Z}$, and $\Lambda = \mathbb{R}, \mathbb{C}$ there exists a sequence Ψ^k_Λ such that

1. $\psi^k_{\Lambda,n}$ is a virtual representation of $GL_n(\Lambda)$ on Λ of virtual degree n.
2. Ψ^k_Λ is additive and multiplicative
3. $\psi^k_{\Lambda,1}$ is the map $GL_1(\Lambda) \xrightarrow{m \mapsto m^{\otimes k}} GL_1\Lambda$
4. For c the complexification operation, $\Psi^k_\mathbb{C} \cdot c = c \cdot \Psi^k_\mathbb{R}$
5. $\Psi^{kl}_\Lambda = \Psi^k_\Lambda \cdot \Psi^l_\Lambda$
6. For G a topological group, θ a virtual representation of G into Λ, $g \in G$, and χ the character of a representation: $\chi(\Psi^k_\Lambda \cdot \theta)g = \chi(\theta)g^k$
7. $\psi^1_{\Lambda,n}$ is the identity representation, $\psi^0_{\Lambda,n}$ is the trivial representation, and $\psi^{-1}_{\Lambda,n}(M) = (M^\top)^{-1}$

The definition of the $\psi^k_{\Lambda,n}$ is built up from evaluating a polynomial, expressing the kth power sum symmetric function by elementary symmetric polynomials, on certain representations arising from the way an automorphism $M \in GL_n(\Lambda)$ gives rise to an automorphism on the rth exterior power.

While the sixth condition looks random, is it the condition that helps demonstrate that Ψ^k_Λ is additive and multiplicative.

Thus we have that

Theorem 2.3 The sequence above gives rise to operations $\Psi^k_\Lambda : K_\Lambda(X) \to K_\Lambda(X)$ with the following properties:

1. These operations are natural w.r.t. maps of spaces
2. These operations are ring homomorphisms
3. For ξ a line bundle over X, $\Psi^k_\Lambda \xi = \xi^{\otimes k}$
4. The following diagram
$$\begin{array}{ccc} K_{\mathbb{R}}(X) & \xrightarrow{\Psi^k_{\mathbb{R}}} & K_{\mathbb{R}}(X) \\ \downarrow c & & \downarrow c \\ K_{\mathbb{C}}(X) & \xrightarrow{\Psi^k_{\mathbb{C}}} & K_{\mathbb{C}}(X) \end{array} \tag{2}$$
commutes
5. $\Psi^{kl}_\Lambda(K) = \Psi^k_\Lambda(\Psi^k_\Lambda(K))$
6. If $\kappa \in K_{\mathbb{C}}(X)$ and $ch^q\kappa$ denotes the $2q$th component of the Chern character then $ch^q(\Psi^k_{\mathbb{C}}\kappa) = k^q ch^q(\kappa)$
7. $\Psi^1_\Lambda, \Psi^{-1}_{\mathbb{R}}$ are identities, Ψ^0_Λ assigns to any bundle a trivial bundle of the same dimension and $\Psi^{-1}_{\mathbb{C}}$ is the complex conjugation operation.

Definition 2.3 – Adams Operations The resulting Ψ^k_Λ is the k'th Adams operation on K-Theory.

Parts 1-5 all result from the prior theorem, parts 7 are either obvious or result from the fact that real (resp. complex) bundles have structure group $O(n)$ (resp. $U(n)$) which allow us to determine $(M^\top)^{-1}$ in terms of M.

Part 6 results from the *splitting principle*, i.e. for any bundle $\xi : E \to X$ $\exists Y, f : Y \to X$ so that $f^*(E) \cong \bigoplus_{i=1}^n \lambda_i$ with the λ_i line bundles.

Furthermore this map f induces an injection $i^* : H^{2q}(X;\mathbb{Q}) \to H^{2q}(Y;\mathbb{Q})$ so we can compare the components of the Chern character after applying i^*.

$$i^*ch(\Psi^k_{\mathbb{C}}\xi) = ch(\Psi^k_{\mathbb{C}}(i^*\xi)) =$$
$$ch(\Psi^k_{\mathbb{C}}(\bigoplus_{i=1}^n \lambda_i)) =$$
$$\bigoplus_{i=1}^n ch(\Psi^k_{\mathbb{C}}\lambda_i) = ch(\lambda_i^{\otimes k}) =$$
$$\sum_{i=1}^n e^{-kx_i}$$

while

$$i^*ch(\xi) = ch(\bigoplus_{i=1}^n \lambda_i) = \sum_{i=1}^n e^{-x_i}$$

Thus by comparing the degree $2q$ components we get the result.

3 K-Theory Computations

We begin this section by naming specific elements of various K groups.

Let ξ and η be the canonical real and complex line bundles over $\mathbb{R}P^n$ and $\mathbb{C}P^n$ respectively. We name the following elements

$$\lambda = \xi - \epsilon^1 \in K_{\mathbb{R}}(\mathbb{R}P^n) \tag{3}$$
$$\mu = \eta - \epsilon^1 \in K_{\mathbb{C}}(\mathbb{C}P^n) \tag{4}$$
$$\nu = c\lambda \in K_{\mathbb{C}}(\mathbb{R}P^n) \tag{5}$$

While ν is obviously related to λ we can also relate λ to μ.

Lemma 3.1 With $\pi : \mathbb{R}P^{2n-1} = S^{2n-1}/(x \sim -x) \to S^{2n-1}/(x \sim \lambda x, \lambda \in S^1) = \mathbb{C}P^{n-1}$ the projection map, $c\xi \cong \pi^*\eta$

We will first compute $K_{\mathbb{C}}(\mathbb{C}P^n/\mathbb{C}P^n)$, then $K_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^m)$, do some computations, and see why the computation of $K_{\mathbb{R}}(\mathbb{R}P^n/\mathbb{R}P^m)$ is necessary.

Theorem 3.1 – Computation of $K_{\mathbb{C}}(\mathbb{C}P^n/\mathbb{C}P^m)$
$K_{\mathbb{C}}(\mathbb{C}P^n) \cong \mathbb{Z}[\mu]/(\mu^{n+1})$. Moreover, the projection $q : \mathbb{C}P^n \to \mathbb{C}P^n/\mathbb{C}P^n$ induces an isomorphism of $K_{\mathbb{C}}(\mathbb{C}P^n/\mathbb{C}P^m)$ onto the ideal $\langle \mu^{m+1} \rangle$ in $K_{\mathbb{C}}(\mathbb{C}P^n)$. The Adams operations on this ring are determined by $\Psi^k_{\mathbb{C}}(\mu^s) = ((1+\mu)^k - 1)^s$.

Proof. First recall that

$$H^*(\mathbb{C}P^n/\mathbb{C}P^m, \mathbb{Z}) \cong \langle x^{m+1} \rangle \tag{6}$$

where $x \in H^2(\mathbb{C}P^n)$ is a generator.

We then employ the Atiyah-Hirzebruch Spectral Sequence introduced in Yigal's talk (whose E_2 term is $H^p(\mathbb{C}P^n/\mathbb{C}P^m; K^q_{\mathbb{C}}(*))$) which gives which gives

$$
\begin{array}{cc|ccccc}
\circ & 4 & \circ & & \circ & & \circ \\
 & 3 & & & & & \\
\circ & 2 & \circ & & \circ & & \circ \\
 & 1 & & & & & \\
\circ & 0 & \circ & & \circ & & \circ \\
\hline
-2 & -1 & 0 & 1 & 2 & 3 & 4 \\
\circ & -2 & \circ & & \circ & & \circ
\end{array}
$$

All of whose differentials, since the total degree of the differential is odd, are zero. Thus the E_2 page is the E_∞ page

Therefore $K_{\mathbb{C}}(\mathbb{C}P^n/\mathbb{C}P^m) \cong \bigoplus_{i=0}^{\infty} H^i(\mathbb{C}P^n/\mathbb{C}P^m) \cong \mathbb{Z} \oplus \bigoplus_{i=m+1}^{n} \mathbb{Z}$ so it is free on $n-m$ generators (as an abelian group).

Let us assume inductively that $K(\mathbb{C}P^m)$ is generated by μ in the manner described above for $m < n$ (the case $K(\mathbb{C}P^0)$ is trivial). From the LES of a pair we get that

$$0 = K^{-1}(\mathbb{C}P^n) \xrightarrow{\partial} K(\mathbb{C}P^n/\mathbb{C}P^m) \xrightarrow{q^*} K(\mathbb{C}P^n) \xrightarrow{i^*} K(\mathbb{C}P^m) \xrightarrow{\partial} K^1(\mathbb{C}P^m/\mathbb{C}P^n) = 0 \tag{7}$$

Which is just a short exact sequence

$$0 \to K(\mathbb{C}P^n/\mathbb{C}P^m) \to K(\mathbb{C}P^n) \to K(\mathbb{C}P^m) \to 0 \tag{8}$$

The induced inclusion map $i^* : K(\mathbb{C}P^n) \to K(\mathbb{C}P^{n-1})$ is obviously surjective, so we can lift up the basis μ^i to $K(\mathbb{C}P^n)$. Since $ch(\mu) = e^{c_1(\eta)} - 1 = c_1(\eta) + \ldots$, $ch(\mu^n) = c_1(\mu)^n \neq 0$ thus $\mu^n \neq 0 \in K(\mathbb{C}P^n)$.

However, since $ch(\mu^{n+1}) = 0$ and ch is injective in this case, ?, $\mu^{n+1} = 0 \in K(\mathbb{C}P^n)$. Finally, since If $\vartheta \in ker(i^*)$ then $\vartheta \in im(q^*)$ we know $i^*(\mu^n) = 0$ but since $K(\mathbb{C}P^n/\mathbb{C}P^{n-1}) \cong \mathbb{Z}$ any class not witnessed by $K(\mathbb{C}P^{n-1})$ must be a multiple of μ^n. Therefore $K(\mathbb{C}P^n) \cong \mathbb{Z}[\mu]/(\mu^{n+1})$ and the short exact sequence identifies $K(\mathbb{C}P^n/\mathbb{C}P^m) = ker(i^*) = \langle \mu^{m+1} \rangle$.

The fact that $\Psi^k_{\mathbb{C}}(\mu^s) = ((1+\mu)^k - 1)^s$ is a simple consequence of the fact that $\mu = \eta - 1$ and $\Psi^k_{\mathbb{C}}(\eta) = \eta^k$. $\square$

Since, generally, $i^*(\mu^{m+1}) = 0$, we denote $\mu^{(m+1)} \in K(\mathbb{C}P^n/\mathbb{C}P^m)$ to be a class that projects to it under q^*. For the real case, since

$$H^k(\mathbb{R}P^n/\mathbb{R}P^m) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, n \text{ odd; or } k = m, \text{ m even} \\ \mathbb{Z}/(2), & \text{if } m < k \leq n, k \equiv 0 \mod 2 \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

the consideration is more complex and we need to split our work up depending on if m is even or odd (n doesn't matter as the cohomology degree would need to be odd, and we won't need to work with that). To compare the two cases, we employ the following lemma

Lemma 3.2 Noting the following diagram:

$$\begin{tikzcd} \mathbb{R}P^{2n+1} \arrow[r, "\pi"] \arrow[d, two heads] & \mathbb{C}P^n \arrow[d, two heads, "q"] \\ \mathbb{R}P^{2n+1}/\mathbb{R}P^{2t} \arrow[r, dashed, "\omega_1"] & \mathbb{C}P^n/\mathbb{C}P^t \end{tikzcd} \tag{10}$$

commutes and sends $\mathbb{R}P^{2t}$ to the basepoint of the quotient, we get the induced map ω_1. We can similarly define an ω_2 by replacing $\mathbb{R}P^{2t}$ with $\mathbb{R}P^{2t+1}$. If we define $\nu_1^{(t+1)}$ to be the pullback of $\omega_1^*\mu^{(t+1)}$ to $\mathbb{R}P^n/\mathbb{R}P^{2t}$ along the inclusion and $\nu_2^{(t+1)}$ similarly then

$$\nu_2^{(t+1)} = i^*\nu_1^{(t+1)}, q_1^*\nu_1^{(t+1)} = q_2^*\nu_2^{(t+1)} = \nu^{t+1}$$

where i^* and q_i^* are maps induced from relevant inclusions and projections. Moreover for $q_3 : \mathbb{R}P^{2n+1}/\mathbb{R}P^{2t+1} \twoheadrightarrow \mathbb{R}P^{2n+1}/\mathbb{R}P^{2t}$, $\nu_1^{(t+1)} = q_3^*\nu_2^{(t+1)}$

Which is true due to relevant commuting topological diagrams inducing maps in K theory.

Theorem 3.2 m Even
If $m = 2t$ is even then $K_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^m) \cong \mathbb{Z}/(2^f)$ where $f = \lfloor \frac{n-m}{2} \rfloor$. Moreover $K_{\mathbb{C}}(\mathbb{R}P^n) \cong \mathbb{Z}[\nu]/(\nu^2 + 2\nu, \nu^{f+1})$ and the quotient map induces an isomorphism of $K_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^m) \cong \langle \nu^{t+1} \rangle$. The Adams operations operate by $\Psi_{\mathbb{C}}^k(\nu_1^(t+1))$ is the identity if k is odd, and 0 otherwise.

Similarly,

Theorem 3.3 m Odd
If $m = 2t+1$ is even then $K_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^m) \cong \mathbb{Z} \oplus K_{\mathbb{C}}(\mathbb{R}P^n/\mathbb{R}P^{m+1})$ where the first summand is generated by $\nu_2^{(t+1)}$ and the second summand is the image of the map induced by q_3.

The Adams Operations act by

$$\Psi_{\mathbb{C}}^k \nu_2^{(t+1)} = k^{t+1}\nu_2^{(t+1)} + \begin{cases} \frac{1}{2}k^{t+1}\nu_1^{(t+2)}, & k \equiv 0 \mod 2 \\ \frac{1}{2}(k^{t+1} - 1)\nu_1^{(t+2)}, & k \equiv 1 \mod 2 \end{cases} \tag{11}$$

Proof. We again exploit the Atiyah-Hirzebruch Spectral Sequence. The E_2 page can be described as follows:

$$E_2^{p,q} \cong H^p(\mathbb{R}P^n/\mathbb{R}P^m; K^q(*)) \cong \begin{cases} \mathbb{Z}, & q \text{ even and either } p = n \text{ odd or } p = m+1 \text{ even} \\ \mathbb{Z}/(2), & q \equiv 0 \mod 2 \text{ and } m+1 < p \leq n \text{ even} \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

While, the E_2 page is now not entirely concentrated at even coordinates, the computation of the behaviour on the main diagonal $E_2^{p,-p}$ is not too difficult. It relies on the construction of the AHSS, which gives some information about what you can determine from a given filtration of your space. Importantly, there are always f copies of $\mathbb{Z}/(2)$ on the main diagonal, with a Z term possibly appearing when m is odd. To get the direct sum decomposition is another appeal to the filtration properties.

To demonstrate that $\nu^2 = -2\nu$ and $\nu^{f+1} = 0$ we note that for the real tautological line bundle ξ, $\xi \otimes \xi = \epsilon^2$ since, due to their total Steifel Whitney Classes $(1+x)(1+x) = 1 + 2x + x^2 = 1$ and real line bundles are classified by their total SW class. Then $\xi^2 = (1+\lambda)^2 = 1 + 2\lambda + \lambda^2 = 1$ so $\lambda^2 = -2\lambda$. Upon complexifying we get $\nu^2 = -2\nu$. This fact allows us to solve the group extension problems notes above, showing that the only way the f $\mathbb{Z}/(2)$'s could combine is into $\mathbb{Z}/(2^f)$. Then $2^f\nu = 0$ so $2^f\nu = 2^{f-1}2\nu = -2^{f-1}\nu^2 = -2^{f-2}2\nu \cdot \nu = 2^{f-2}\nu^3 = \ldots = \nu^{f+1} = 0$.

To determine the Adams operations, first note that $\Psi^k_{\mathbb{C}}(\xi)$ is 1 if k is even and ξ if k is odd, so $\Psi^k_{\mathbb{C}}(\nu^s)$ is the identity if k is odd and 0 if k is even. Computing it on $\nu_1^{(t+1)}$ is derived from naturality w.r.t. q^* which is injective.

For the odd m case, we know that $\Psi^k_{\mathbb{C}}(\nu_2^{(t)})$ must be $a\nu_2^{(t)} + b\nu_1^{(t+1)}$ by the direct sum decomposition. Note that the first part is non-torsion while the second is. Thus, for $i : S^{2t} \hookrightarrow \mathbb{R}P^n/\mathbb{R}P^{2t-1}$ (rembering that $K_{\mathbb{C}}(S^{2n}) \cong \mathbb{Z}$) we have that $a\nu_2^{(t)} = i^*\Psi^k_{\mathbb{C}}(\nu_2^{(t)}) = \Psi^k_{\mathbb{C}}(i^*\nu_2^{(t)}) = k^t i^*\nu_2^{(t)} = k^t\nu_2^{(t)}$ where the second to last equality is due to a lemma I did not discuss. b is computed similarly. $\square$

Importantly, we see how a map from/to a sphere can help determine some structure of our K groups. This is why Adams discusses K groups in the first place: the coreducibility question will force some structure on the way in which certain Adams operations act on the K groups of the truncated projective spaces, which he shows are impossible.

For example, let's see if we can make $\mathbb{R}P^{16+\rho(16)}/\mathbb{R}P^{15}$ coreducible. We would then have composites like

$$S^{16} = \mathbb{R}P^{16}/\mathbb{R}P^{15} \xhookrightarrow{i} \mathbb{R}P^{16+9}/\mathbb{R}P^{16} \xrightarrow{f} S^{16} \tag{13}$$

where f is a homotopy inverse.

We know that

$$K_{\mathbb{C}}(\mathbb{R}P^{25}/\mathbb{R}P^{15}) \cong \mathbb{Z}[\nu_2^{(8)}] \oplus K_{\mathbb{C}}(\mathbb{R}P^{25}/\mathbb{R}P^{16}) \tag{14}$$

$$\cong \mathbb{Z}[\nu_2^{(8)}] \oplus \mathbb{Z}/(2^4)[\nu_1^{(8)}] \tag{15}$$

Let γ be the generator of $K_{\mathbb{C}}(S^{16}) \cong \mathbb{Z}$ so that $i^*(\nu_2^{(8)}) = \gamma$ while $i^*(\nu_1^{(8)}) = 0$ for torsion reasons. Since f is to be a homotopy inverse to i, we would then have that $f^*(\gamma) = \nu_2^{(8)} + N\nu_1^{(8)}$. We want to use our Adams operations to show that we cannot find any N so that this is true.

Let's work with $\Psi^3_{\mathbb{C}}$ to get that

$$\Psi^3_{\mathbb{C}}(f^*\gamma) = \Psi^3_{\mathbb{C}}(\nu_2^{(8)} + N\nu_1^{(8)}) = 3^8\nu_2^{(8)} + \frac{1}{2}(3^8-1)\nu_1^{(8)} + N\nu_1^{(8)} \tag{16}$$

$$= f^*\Psi^3_{\mathbb{C}}(\gamma) = f^*(3^8\gamma) = 3^8(\nu_2^{(8)} + N\nu_1^{(8)}) \tag{17}$$

Where the first equality on Line 17 comes from a lemma I did not cover.

This would mean that $(\frac{1}{2}(3^8-1)+N)\nu_1^{(8)} = 3^8N\nu_1^{(8)}$ which can be rearranged into the condition that

$$(N-\frac{1}{2})(3^8-1) \equiv 0 \mod 2^4 \tag{18}$$

which... unfortunately... it is because $\frac{3^8-1}{2} \equiv 0 \mod 2^4$. However, if we could adjust the degree of the group slightly, so that we were working $\mod 2^5$ instead, then since $\frac{3^8-1}{2} \equiv 16 \mod 2^5$ by valuation reasons, no possible N would work, thus contradicting the coreducibility.

This is why Adams studies the real K theory of the projective spaces, since we have the following theorem (which I will not prove).

Theorem 3.4
If $m \neq -1 \mod 4$ then $K_{\mathbb{R}}(\mathbb{R}P^n/\mathbb{R}P^m) \cong \mathbb{Z}/2^{\phi(n,m)}$ where $\phi(n,m) = |\{m < p \leq n : p \equiv 0,1,2,4 \mod 8\}|$. Similarly to the previous theorem, $K_{\mathbb{R}}(\mathbb{R}P^n)$ is generated by λ with similar relations.

For $m \equiv -1 \mod 4$ $K_{\mathbb{R}}(\mathbb{R}P^n/\mathbb{R}P^m)$ decomposes in the exact same manner as the complex case.

The Adams operators operate on the generators in the exact same way, except every k^t is replaced with a k^{2t} (accounting for the fact that t here accounts for the number of times 4 divides m, instead of 2 like it was in the complex case).

In light of this version, we can replace the example we did above with the analagous computations in real K theory and end up with the equality

$$(N-\frac{1}{2})(3^8-1) \equiv 0 \mod 32 \tag{19}$$

which, as stated above, cannot be satisfied. For all other m, it turns out that $\Psi^3_{\mathbb{R}}$ suffices to demonstrate the contradiction.