Theory: Linear elastoplasicity

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An introduction to the theory applied for elastoplasticity.

1 Governing equations for quasi-static linear elasticity

The strong statement of the balance of linear momentum reads

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{on} \quad \Omega \quad , \tag{1}$$

where $\nabla = \frac{\partial}{\partial x}$ is a differential operator, $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\mathbf{b} = \rho \mathbf{g}$ is the body force density vector. This is expressed in index notation as

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \quad \text{on} \quad \Omega \quad . \tag{2}$$

Pre-multiplying the above by test function $\delta \mathbf{v}$ and integrating over the domain Ω renders

$$- \int\limits_{\Omega} \delta v_i \, \frac{\partial \sigma_{ij}}{\partial x_j} \, dv = \int\limits_{\Omega} \delta v_i \, b_i \, dv \tag{3}$$

that, by using the product rule for derivatives (i.e. integration by parts), becomes

$$\int\limits_{\Omega} \frac{\partial \delta v_i}{\partial x_j} \, \sigma_{ij} \, dv - \int\limits_{\Omega} \frac{\partial}{\partial x_j} \left[\delta v_i \, \sigma_{ij} \right] \, dv = \int\limits_{\Omega} \delta v_i \, b_i \, dv \quad . \tag{4}$$

Finally, by applying divergence theorem to the second term in the above, we attain the weak form of the balance of linear momentum

$$\int\limits_{\Omega} \frac{\partial \delta v_i}{\partial x_j} \, \sigma_{ij} \, dv = \int\limits_{\Omega} \delta v_i \, b_i \, dv + \int\limits_{\partial\Omega} \delta v_i \, \underbrace{\sigma_{ij} \, n_j}_{\bar{t}_i} \, da \quad , \tag{5}$$

wherein $\mathbf{n}$ represents the outward facing normal on $\partial\Omega$, the boundary of the domain, and $\bar{\mathbf{t}}$ the prescribed traction on the Neumann boundary.

2 General framework for elastoplasticity

- Kinematics
  - Small strain tensor

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla\mathbf{u} + [\nabla\mathbf{u}]^T\right] = \nabla\mathbf{u}^{sym} \tag{6}$$

  - Additive volumetric-isochoric split of the strain

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{vol} + \boldsymbol{\varepsilon}^{dev} \quad ; \quad \boldsymbol{\varepsilon}^{vol} = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} \tag{7}$$

  - Additive split of strain into elastic and plastic contributions

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$$

- General free energy function

$$\psi = \psi(\boldsymbol{\varepsilon}^e, \alpha, \boldsymbol{\beta})$$

  - Two new internal variables:
    * α describes the relative increase of the elastic region (isotropic hardening variable)
    * $\boldsymbol{\beta}$ describes the kinematic hardening (a rank-2 tensor; related to the centre point of the elastic region and introduces anisotropy due to plastic flow)
- Principle of irreversibility:
  - Dissipation inequality

$$\begin{aligned}\mathcal{D} &= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi}(\boldsymbol{\varepsilon}^e, \alpha, \boldsymbol{\beta}) \\ &= \boldsymbol{\sigma} : [\dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p] - \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^e} : \dot{\boldsymbol{\varepsilon}}^e - \underbrace{\frac{\partial\psi}{\partial\alpha}}_{R}\,\dot{\alpha} - \underbrace{\frac{\partial\psi}{\partial\boldsymbol{\beta}}}_{\boldsymbol{B}} : \dot{\boldsymbol{\beta}} \\ &= \left[\boldsymbol{\sigma} - \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^e}\right] : \dot{\boldsymbol{\varepsilon}}^e + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + R\dot{\alpha} + \boldsymbol{B} : \dot{\boldsymbol{\beta}} \\ &\geq 0\end{aligned}$$

    * Hardening stress R
    * Back stress $\boldsymbol{B}$
  - Definition of Cauchy stress

$$\boldsymbol{\sigma} := \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}^e} \tag{8}$$

- Reduced dissipation inequality (using definition of Cauchy stress)

$$\mathcal{D}^{red} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + R\dot{\alpha} + \boldsymbol{B} : \dot{\boldsymbol{\beta}} \geq 0$$

- Evolution equations derived by the postulate of maximum dissipation [note: this ensures that hardening occurs when accommodated by the model, thus dissipating more energy than perfect plastic flow]
  - Elastic region is restricted by yield function $\Phi(\boldsymbol{\sigma}, R, \boldsymbol{B})$:

$$\mathbb{E} := \{(\boldsymbol{\sigma}, R, \boldsymbol{B}) \,|\, \Phi(\boldsymbol{\sigma}, R, \boldsymbol{B}) \leq 0\} \tag{9}$$

  - Conditions:
    * Elastic state: $\Phi < 0$
    * Plastic state: $\Phi = 0$
- Define Lagrange multiplier problem to maximise the dissipation, i.e.

$$\text{maximise } \mathcal{D}^{red} \text{ subject to } \Phi(\boldsymbol{\sigma}, R, \boldsymbol{B}) \leq 0$$
$$\mathcal{L}\left(\boldsymbol{\sigma}, R, \boldsymbol{B}, \dot{\lambda}\right) = -\mathcal{D}^{red} + \dot{\lambda}\Phi \quad \rightarrow \quad \text{stationary}$$

    * $\dot{\lambda}$ is known as the consistency parameter
  - Taking directional derivatives, we can identify the following evolution equations for the internal variables:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \boldsymbol{\sigma}} = \mathbf{0} \quad &\Rightarrow \quad \dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial \Psi}{\partial \boldsymbol{\sigma}} \quad \text{(flow rule)} \\
\frac{\partial \mathcal{L}}{\partial R} = 0 \quad &\Rightarrow \quad \dot{\alpha} = \dot{\lambda}\frac{\partial \Psi}{\partial R} \quad \text{(evolution equation for isotropic hardening)} \\
\frac{\partial \mathcal{L}}{\partial \boldsymbol{B}} = \mathbf{0} \quad &\Rightarrow \quad \dot{\boldsymbol{\beta}} = \dot{\lambda}\frac{\partial \Psi}{\partial \boldsymbol{B}} \quad \text{(evolution equation for kinematic hardening)}
\end{aligned}$$

  - Loading / unloading condition:

$$\frac{\partial \Psi}{\partial \boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}} \begin{cases} < 0 & \text{Unloading (elastic)} \\ = 0 & \text{Loading (plastic flow)} \\ > 0 & \text{Loading (plastic hardening)} \end{cases}$$

    * The loading-unloading conditions can be shown to be equivalent to the Karush-Kuhn-Tucker (KKT) conditions (an important statement in optimisation theory):

$$\dot{\lambda} \geq 0 \quad ; \quad \Phi \leq 0 \quad ; \quad \dot{\lambda}\Phi = 0 \tag{10}$$

· Situations and definitions for any $(\boldsymbol{\sigma}, R, \boldsymbol{B}) \in \mathbb{E}$

$$\Phi < 0 \quad \Rightarrow \quad (\boldsymbol{\sigma}, R, \boldsymbol{B}) \in \mathrm{int}(\mathbb{E}) \quad \Rightarrow \quad \dot{\lambda} = 0 \quad \text{(Elastic response)}$$

$$\Phi = 0 \quad \Rightarrow \quad (\boldsymbol{\sigma}, R, \boldsymbol{B}) \in \partial\mathbb{E} \quad \Rightarrow \quad \begin{cases} \dot{\Phi} < 0 & \Rightarrow & \dot{\lambda} = 0 & \text{(Elastic unloading)} \\ \dot{\Phi} = 0 & \text{and} & \dot{\lambda} = 0 & \text{(Neutral loading)} \\ \dot{\Phi} = 0 & \text{and} & \dot{\lambda} > 0 & \text{(Plastic loading)} \end{cases}$$

– Hardening / softening condition:

$$\frac{\partial \Psi}{\partial \alpha}\dot{\alpha} \begin{cases} < 0 & \text{Softening} \\ > 0 & \text{Hardening} \end{cases}$$

- Example of Von-Mises yield criterion
  - Convex yield surface defined by

$$\Phi(\boldsymbol{\sigma}) = \sqrt{\frac{3}{2}}\|\boldsymbol{\sigma}^{dev}\| - \sigma_y \leq 0$$

    * Cylindrical yield-surface as observed in the principal stress space
    * Norm of deviatoric stress contribution:

$$\|\boldsymbol{\sigma}^{dev}\| = \sqrt{\boldsymbol{\sigma}^{dev} : \boldsymbol{\sigma}^{dev}}$$

3 Constitutive law: Linear elasticity

- Linear elasticity (a special case of hyperelasticity)
  - Dissipation (in-)equality

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi}\,(\boldsymbol{\varepsilon}) = 0 \quad \Rightarrow \quad \boldsymbol{\sigma} = \frac{\partial \psi\,(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \tag{11}$$

  - Free/Strain energy function (Hooke's law)

$$\psi(\boldsymbol{\varepsilon}) = \frac{\lambda}{2}\,[\mathrm{tr}(\boldsymbol{\varepsilon})]^2 + \mu\ \mathrm{tr}(\boldsymbol{\varepsilon}^2) \tag{12}$$

  - Cauchy stress

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \lambda\ \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon} \tag{13}$$

  - Elastic tangent

$$\mathbb{C} = \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon}^2} : \dot{\boldsymbol{\varepsilon}} = \lambda\mathbf{I} \otimes \mathbf{I} + 2\mu\ \mathbb{I}^{sym} \tag{14}$$

    * with the linear relationship

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} \tag{15}$$

4 Constitutive law: Elastoplasticity

The linear elastoplastic constitutive laws and framework described here are described in detail by [1, 2].

4.1 Associated Von Mises elastoplasticity with linear isotropic and kinematic hardening

- Kinematics

$$\varepsilon = \varepsilon^e + \varepsilon^p$$

- Free energy (additive decomposition into elastic and plastic parts)

$$\begin{aligned}
\psi(\boldsymbol{\varepsilon}^e, \alpha, \boldsymbol{\beta}) &= \psi^e(\boldsymbol{\varepsilon}^e) + \psi^p(\alpha, \boldsymbol{\beta}) \\
\psi^e(\boldsymbol{\varepsilon}^e) &= \frac{\lambda}{2}\left[\mathrm{tr}(\varepsilon^e)\right]^2 + \mu \mathrm{tr}\left(\left[\varepsilon^e\right]^2\right) \\
\psi^p(\alpha, \boldsymbol{\beta}) &= \frac{1}{2} k r \alpha^2 + \frac{1}{2} k \left[1 - r\right] \|\boldsymbol{\beta}\|^2 \quad , \quad r \in [0, 1]
\end{aligned}$$

- Stresses

$$\begin{aligned}
\boldsymbol{\sigma} &= \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e} = \lambda \ \mathrm{tr}(\boldsymbol{\varepsilon}^e)\mathbf{1} + 2\mu\boldsymbol{\varepsilon}^e \\
R &= -\frac{\partial \psi}{\partial \alpha} = -kr\alpha \\
\boldsymbol{B} &= -\frac{\partial \psi}{\partial \boldsymbol{\beta}} = -k\left[1 - r\right]\boldsymbol{\beta}
\end{aligned}$$

- Yield function (Von Mises)

$$\Phi\left(\boldsymbol{\sigma}, R, \boldsymbol{B}\right) = \|\boldsymbol{\sigma}^{dev} + \boldsymbol{B}\| - \sqrt{\frac{2}{3}}\left[\sigma_y - R\right] \leq 0$$

- Evolution equations

$$\begin{aligned}
\dot{\boldsymbol{\varepsilon}}^p &= \dot{\lambda}\frac{\partial \Phi}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\frac{\boldsymbol{\sigma}^{dev} + \boldsymbol{B}}{\|\boldsymbol{\sigma}^{dev} + \boldsymbol{B}\|} = \dot{\lambda}\boldsymbol{n} \\
\dot{\alpha} &= \dot{\lambda}\frac{\partial \Phi}{\partial R} = \sqrt{\frac{2}{3}}\dot{\lambda} \\
\dot{\boldsymbol{\beta}} &= \dot{\lambda}\frac{\partial \Phi}{\partial \boldsymbol{B}} = \dot{\lambda}\frac{\boldsymbol{\sigma}^{dev} + \boldsymbol{B}}{\|\boldsymbol{\sigma}^{dev} + \boldsymbol{B}\|} = \dot{\lambda}\boldsymbol{n}
\end{aligned}$$

  - Must satisfy KTT conditions

$$\dot{\lambda} \geq 0 \quad ; \quad \Phi \leq 0 \quad ; \quad \dot{\lambda}\Phi = 0$$

- Principle of irreversibility

$$
\begin{aligned}
\mathcal{D}^{red} &= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + R\dot{\alpha} + \boldsymbol{B} : \dot{\boldsymbol{\beta}} \\
&= \boldsymbol{\sigma} : \dot{\lambda}\boldsymbol{n} + R\sqrt{\frac{2}{3}}\dot{\lambda} + \boldsymbol{B} : \dot{\lambda}\boldsymbol{n} \\
&= \dot{\lambda}\left[[\boldsymbol{\sigma} + \boldsymbol{B}] : \boldsymbol{n} + R\sqrt{\frac{2}{3}} \right] \\
&= \dot{\lambda}\left[\left[\boldsymbol{\sigma}^{dev} + \boldsymbol{B}\right] : \frac{\boldsymbol{\sigma}^{dev} + \boldsymbol{B}}{\|\boldsymbol{\sigma}^{dev} + \boldsymbol{B}\|} + R\sqrt{\frac{2}{3}} \right] \\
&= \dot{\lambda}\left[\ \|\boldsymbol{\sigma}^{dev} + \boldsymbol{B}\| + R\sqrt{\frac{2}{3}} \right] \\
&\geq 0
\end{aligned}
$$

  - If $\Phi < 0 \quad \Rightarrow \quad \dot{\lambda} = 0$
  - If $\Phi = 0 \quad \Rightarrow \quad \dot{\lambda} > 0$ and $\|\boldsymbol{\sigma}^{dev} + \boldsymbol{B}\| + R\sqrt{\frac{2}{3}} = \sqrt{\frac{2}{3}}\sigma_y \geq 0$

- Consistency condition (evolution of yield surface)

$$
\begin{aligned}
\dot{\Phi} &= \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial \Phi}{\partial R}\dot{R} + \frac{\partial \Phi}{\partial \boldsymbol{B}} : \dot{\boldsymbol{B}} \\
&= \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon}^{\boldsymbol{e}} \otimes \partial \boldsymbol{\varepsilon}^{\boldsymbol{e}}} : \underbrace{\left[\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}^{\boldsymbol{p}}}\right]}_{\dot{\boldsymbol{\varepsilon}^{\boldsymbol{e}}}} - \frac{\partial \Phi}{\partial R}\frac{\partial^2 \psi}{\partial \alpha \partial \alpha}\dot{\alpha} - \frac{\partial \Phi}{\partial \boldsymbol{B}} : \frac{\partial^2 \psi}{\partial \boldsymbol{\beta} \otimes \partial \boldsymbol{\beta}} : \dot{\boldsymbol{\beta}} \\
&= \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon}^{\boldsymbol{e}} \otimes \partial \boldsymbol{\varepsilon}^{\boldsymbol{e}}} : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon}^{\boldsymbol{e}} \otimes \partial \boldsymbol{\varepsilon}^{\boldsymbol{e}}} : \frac{\partial \Phi}{\partial \boldsymbol{\sigma}}\dot{\lambda} - \frac{\partial \Phi}{\partial R}\frac{\partial^2 \psi}{\partial \alpha \partial \alpha}\frac{\partial \Phi}{\partial R}\dot{\lambda} - \frac{\partial \Phi}{\partial \boldsymbol{B}} : \frac{\partial^2 \psi}{\partial \boldsymbol{\beta} \otimes \partial \boldsymbol{\beta}} : \frac{\partial \Phi}{\partial \boldsymbol{B}}\dot{\lambda} \\
&= \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon}^{\boldsymbol{e}} \otimes \partial \boldsymbol{\varepsilon}^{\boldsymbol{e}}} : \dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\left[\frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon}^{\boldsymbol{e}} \otimes \partial \boldsymbol{\varepsilon}^{\boldsymbol{e}}} : \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} + \frac{\partial \Phi}{\partial R}\frac{\partial^2 \psi}{\partial \alpha \partial \alpha}\frac{\partial \Phi}{\partial R} + \frac{\partial \Phi}{\partial \boldsymbol{B}} : \frac{\partial^2 \psi}{\partial \boldsymbol{\beta} \otimes \partial \boldsymbol{\beta}} : \frac{\partial \Phi}{\partial \boldsymbol{B}} \right] \\
&= 0 \quad \text{(stay on yield surface during plastic deformation)}
\end{aligned}
$$

$$
\Rightarrow \quad \dot{\lambda} = \frac{\dfrac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \dfrac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon}^{\boldsymbol{e}} \otimes \partial \boldsymbol{\varepsilon}^{\boldsymbol{e}}} : \dot{\boldsymbol{\varepsilon}}}{\dfrac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \dfrac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon}^{\boldsymbol{e}} \otimes \partial \boldsymbol{\varepsilon}^{\boldsymbol{e}}} : \dfrac{\partial \Phi}{\partial \boldsymbol{\sigma}} + \dfrac{\partial \Phi}{\partial R}\dfrac{\partial^2 \psi}{\partial \alpha \partial \alpha}\dfrac{\partial \Phi}{\partial R} + \dfrac{\partial \Phi}{\partial \boldsymbol{B}} : \dfrac{\partial^2 \psi}{\partial \boldsymbol{\beta} \otimes \partial \boldsymbol{\beta}} : \dfrac{\partial \Phi}{\partial \boldsymbol{B}}}
$$

with

$$\begin{aligned}
\frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon^e} \otimes \partial \boldsymbol{\varepsilon^e}} &= \mathbb{C} \\
\frac{\partial^2 \psi}{\partial \alpha \partial \alpha} &= -kr \\
\frac{\partial^2 \psi}{\partial \boldsymbol{\beta} \otimes \partial \boldsymbol{\beta}} &= -k\,[1-r]\,\mathbb{I}^{sym} \\
\frac{\partial \Phi}{\partial \boldsymbol{\sigma}} &= \frac{\partial \|\boldsymbol{\sigma}^{dev} + \boldsymbol{B}\|}{\partial\left(\boldsymbol{\sigma}^{dev} + \boldsymbol{B}\right)} : \frac{\partial\left(\boldsymbol{\sigma}^{dev} + \boldsymbol{B}\right)}{\partial \boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}^{dev} + \boldsymbol{B}}{\|\boldsymbol{\sigma}^{dev} + \boldsymbol{B}\|} : \mathbb{I}^{sym} = \boldsymbol{n} \\
\frac{\partial \Phi}{\partial R} &= \sqrt{\frac{2}{3}} \\
\frac{\partial \Phi}{\partial \boldsymbol{B}} &= \boldsymbol{n}
\end{aligned}$$

Then

$$\begin{aligned}
\Rightarrow \quad \dot{\lambda} &= \frac{\boldsymbol{n} : \mathbb{C} : \dot{\boldsymbol{\varepsilon}}}{\boldsymbol{n} : \mathbb{C} : \boldsymbol{n} + \sqrt{\frac{2}{3}}\,[-kr]\,\sqrt{\frac{2}{3}} + \boldsymbol{n} : \left[-k\,[1-r]\,\mathbb{I}^{sym}\right] : \boldsymbol{n}} \\
&= \frac{\boldsymbol{n} : \mathbb{C} : \dot{\boldsymbol{\varepsilon}}}{\boldsymbol{n} : \mathbb{C} : \boldsymbol{n} - \frac{2}{3}kr - k\,[1-r]} \\
&= \frac{\boldsymbol{n} : \mathbb{C} : \dot{\boldsymbol{\varepsilon}}}{\boldsymbol{n} : \mathbb{C} : \boldsymbol{n} + k\left[\frac{1}{3}r - 1\right]}
\end{aligned}$$

4.2 Elasto-plastic tangent modulus (continuous setting)

- Goal

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{C}}^{ep} : \dot{\boldsymbol{\varepsilon}}$$

- Stress rate

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}^e}$$

$$\Rightarrow \quad \dot{\boldsymbol{\sigma}} = \frac{\partial^2 \Psi}{\partial \boldsymbol{\varepsilon}^e \otimes \partial \boldsymbol{\varepsilon}^e} : \dot{\boldsymbol{\varepsilon}}^e = \boldsymbol{\mathcal{C}} : [\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^p] = \boldsymbol{\mathcal{C}} : \left[\dot{\boldsymbol{\varepsilon}} - \dot{\lambda}\frac{\partial \Phi}{\partial \boldsymbol{\sigma}}\right]$$

- Remember: Plastic potential $\Phi\left(\boldsymbol{\sigma}, \boldsymbol{S}\right)$ with evolution laws $\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial \Phi}{\partial \boldsymbol{\sigma}}$, $\dot{\boldsymbol{d}} = \dot{\lambda}\frac{\partial \Phi}{\partial \boldsymbol{S}}$
  - Definition and derivatives of stress-like terms $\boldsymbol{S} = [R, \boldsymbol{B}]$ with internal vari-

ables $\boldsymbol{d} = [\alpha, \boldsymbol{\beta}]$

$$\boldsymbol{S} = -\frac{\partial \Psi}{\partial \boldsymbol{d}}$$
$$\dot{\boldsymbol{S}} = -\frac{\partial^2 \Psi}{\partial \boldsymbol{d} \otimes \partial \boldsymbol{d}} \circ \dot{\boldsymbol{d}} = -\frac{\partial^2 \Psi}{\partial \boldsymbol{d} \otimes \partial \boldsymbol{d}} \circ \dot{\lambda} \frac{\partial \Phi}{\partial \boldsymbol{S}}$$

- Use consistency condition to compute $\dot{\lambda}$:

$$\dot{\lambda}\dot{\Phi} = 0 \quad \Rightarrow \quad \dot{\Phi} = 0 \quad \text{for plastic loading}$$

$$\begin{aligned}
\dot{\Phi} &= \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \frac{\partial \Phi}{\partial \boldsymbol{S}} \circ \dot{\boldsymbol{S}} \\
&= \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \boldsymbol{\mathcal{C}} : \left[\dot{\boldsymbol{\varepsilon}} - \dot{\lambda} \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} \right] - \frac{\partial \Phi}{\partial \boldsymbol{S}} \circ \frac{\partial^2 \Psi}{\partial \boldsymbol{d} \otimes \partial \boldsymbol{d}} \circ \dot{\lambda} \frac{\partial \Phi}{\partial \boldsymbol{S}} \\
&= 0 \\
\Rightarrow \quad \dot{\lambda} &= \frac{1}{D} \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \boldsymbol{\mathcal{C}} : \dot{\boldsymbol{\varepsilon}} \quad \text{with} \\
D &= \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \boldsymbol{\mathcal{C}} : \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} + \frac{\partial \Phi}{\partial \boldsymbol{S}} \circ \frac{\partial^2 \Psi}{\partial \boldsymbol{d} \otimes \partial \boldsymbol{d}} \circ \frac{\partial \Phi}{\partial \boldsymbol{S}}
\end{aligned}$$

- Elasto-plastic tangent

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{C}}^{ep} : \dot{\boldsymbol{\varepsilon}} = \underbrace{\left[\boldsymbol{\mathcal{C}} - \frac{1}{D} \left[\boldsymbol{\mathcal{C}} : \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} \right] \otimes \left[\frac{\partial \Phi}{\partial \boldsymbol{\sigma}} : \boldsymbol{\mathcal{C}} \right] \right]}_{\boldsymbol{\mathcal{C}}^{ep} \text{ (symmetric)}} : \dot{\boldsymbol{\varepsilon}}$$

- Result for isotropic hardening

$$\boldsymbol{\mathcal{C}}^{ep,dev} = 2\mu \boldsymbol{\mathcal{I}}^{dev} - \frac{[2\mu]^2}{2\mu + \frac{2}{3}k} \boldsymbol{n} \otimes \boldsymbol{n} \tag{16a}$$
$$\boldsymbol{\mathcal{C}}^{ep,vol} = \kappa \mathbf{1} \otimes \mathbf{1} \tag{16b}$$

with

$$\boldsymbol{n} = \frac{\boldsymbol{\sigma}^{dev}}{\|\boldsymbol{\sigma}^{dev}\|}$$

5 Integration algorithms for elasto-plasticity

5.1 Radial return algorithm for von Mises plasticity with linear isotropic hardening

- Constitutive setting

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{e} + \boldsymbol{\varepsilon}^{p} = \boldsymbol{\varepsilon}^{vol} + \boldsymbol{\varepsilon}^{dev}$$

$$\psi = \frac{1}{2}\kappa\left[\mathrm{tr}\left(\boldsymbol{\varepsilon}^{vol}\right)\right]^2 + \mu\mathrm{tr}\left(\left[\boldsymbol{\varepsilon}^{e,dev}\right]^2\right) + \frac{1}{2}k\alpha^2$$

$$\Phi\left(\boldsymbol{\sigma}, R\right) = \|\boldsymbol{\sigma}^{dev}\| - \sqrt{\frac{2}{3}}\left[\sigma_y - R\right] \leq 0$$

$$\dot{\boldsymbol{\varepsilon}}^{p} = \dot{\lambda}\frac{\partial\Phi}{\partial\boldsymbol{\sigma}} = \dot{\lambda}\boldsymbol{n} \quad , \quad \boldsymbol{n} = \frac{\boldsymbol{\sigma}^{dev}}{\|\boldsymbol{\sigma}^{dev}\|}$$

$$\dot{\alpha} = \dot{\lambda}\frac{\partial\Phi}{\partial R} = \sqrt{\frac{2}{3}}\dot{\lambda}$$

- Apply backward Euler method to evolution equations

$$\boldsymbol{\varepsilon}^{p}_{n+1} = \boldsymbol{\varepsilon}^{p}_{n} + \Delta t\,\dot{\boldsymbol{\varepsilon}}^{p}_{n+1} = \boldsymbol{\varepsilon}^{p}_{n} + \Delta t\,\dot{\lambda}_{n+1}\boldsymbol{n}_{n+1}$$

$$\alpha_{n+1} = \alpha_n + \Delta t\,\dot{\alpha}_{n+1} = \alpha_n + \Delta t\,\dot{\lambda}_{n+1}\sqrt{\frac{2}{3}}$$

- Deviatoric stress

$$\begin{aligned}
\boldsymbol{\sigma}^{dev}_{n+1} &= 2\mu\boldsymbol{\varepsilon}^{e,dev}_{n+1} = 2\mu\left[\boldsymbol{\varepsilon}^{dev}_{n+1} - \boldsymbol{\varepsilon}^{p,dev}_{n+1}\right] \\
&= 2\mu\boldsymbol{\varepsilon}^{dev}_{n+1} - 2\mu\left[\boldsymbol{\varepsilon}^{p}_{n} + \Delta t\,\dot{\lambda}_{n+1}\boldsymbol{n}_{n+1}\right] \\
&= \underbrace{2\mu\left[\boldsymbol{\varepsilon}^{dev}_{n+1} - \boldsymbol{\varepsilon}^{p}_{n}\right]}_{\boldsymbol{\sigma}^{dev,trial}_{n+1}} - 2\mu\Delta\lambda_{n+1}\frac{\boldsymbol{\sigma}^{dev}_{n+1}}{\|\boldsymbol{\sigma}^{dev}_{n+1}\|} \quad \text{(Note: } \Delta t\,\dot{\lambda}_{n+1} = \Delta\lambda_{n+1}\text{)}
\end{aligned}$$

$$\Rightarrow \quad \boldsymbol{\sigma}^{dev}_{n+1} = \left[1 + \frac{2\mu\Delta\lambda_{n+1}}{\|\boldsymbol{\sigma}^{dev}_{n+1}\|}\right]^{-1}\boldsymbol{\sigma}^{dev,trial}_{n+1}$$

* The final stress state is obtained by projecting the trial stress state onto the current yield surface.

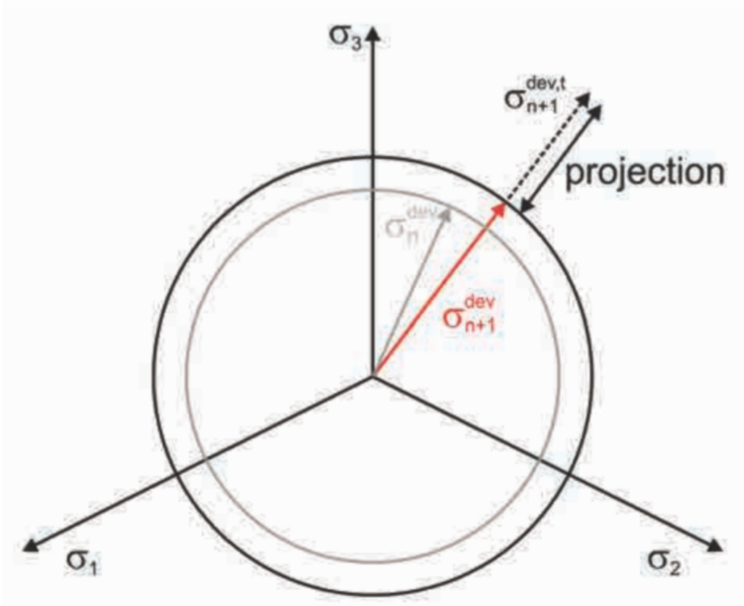


Figure 1: Radial projection [1]

– Still need to determine $\|\boldsymbol{\sigma}_{n+1}^{dev}\|$. Use coaxiality of the stresses:

$$\frac{\boldsymbol{\sigma}_{n+1}^{dev,trial}}{\|\boldsymbol{\sigma}_{n+1}^{dev,trial}\|} = \frac{\boldsymbol{\sigma}_{n+1}^{dev}}{\|\boldsymbol{\sigma}_{n+1}^{dev}\|} = \boldsymbol{n}_{n+1}$$

* Contract both sides of previous equation by $\boldsymbol{n}_{n+1}$ and use definition of tensor norm $\|\boldsymbol{\sigma}\|$:

$$\boldsymbol{\sigma}_{n+1}^{dev} : \frac{\boldsymbol{\sigma}_{n+1}^{dev}}{\|\boldsymbol{\sigma}_{n+1}^{dev}\|} = \boldsymbol{\sigma}_{n+1}^{dev,trial} : \frac{\boldsymbol{\sigma}_{n+1}^{dev,trial}}{\|\boldsymbol{\sigma}_{n+1}^{dev,trial}\|} - 2\mu\Delta\lambda_{n+1} \frac{\boldsymbol{\sigma}_{n+1}^{dev}}{\|\boldsymbol{\sigma}_{n+1}^{dev}\|} : \frac{\boldsymbol{\sigma}_{n+1}^{dev}}{\|\boldsymbol{\sigma}_{n+1}^{dev}\|}$$
$$\Rightarrow \quad \|\boldsymbol{\sigma}_{n+1}^{dev}\| = \|\boldsymbol{\sigma}_{n+1}^{dev,trial}\| - 2\mu\Delta\lambda_{n+1}$$

– Yield function (plastic flow: $\Phi = 0$)

$$\Phi_{n+1} = \|\boldsymbol{\sigma}_{n+1}^{dev}\| - \sqrt{\frac{2}{3}}\left[\sigma_y + \underbrace{k\alpha_{n+1}}_{-R_{n+1}}\right] = 0$$
$$\Rightarrow \quad \|\boldsymbol{\sigma}_{n+1}^{dev}\| = \sqrt{\frac{2}{3}}\left[\sigma_y + k\left[\alpha_n + \Delta\lambda_{n+1}\sqrt{\frac{2}{3}}\right]\right]$$

– Plastic multiplier update computed by equating the two above equations

$$\|\boldsymbol{\sigma}_{n+1}^{dev}\| = \|\boldsymbol{\sigma}_{n+1}^{dev,trial}\| - 2\mu\Delta\lambda_{n+1} = \sqrt{\frac{2}{3}}\left[\sigma_y + k\left[\alpha_n + \Delta\lambda_{n+1}\sqrt{\frac{2}{3}}\right]\right]$$

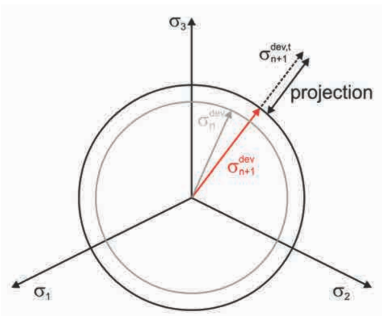
$$\Rightarrow \quad \underbrace{\|\boldsymbol{\sigma}_{n+1}^{dev,trial}\| - \sqrt{\frac{2}{3}}\left[\sigma_y + k\alpha_n\right]}_{\Phi_{n+1}^{trial}} = \Delta\lambda_{n+1}\left[2\mu + \frac{2}{3}k\right]$$

$$\Rightarrow \quad \Delta\lambda_{n+1} = \frac{\Phi_{n+1}^{trial}}{2\mu + \frac{2}{3}k}$$

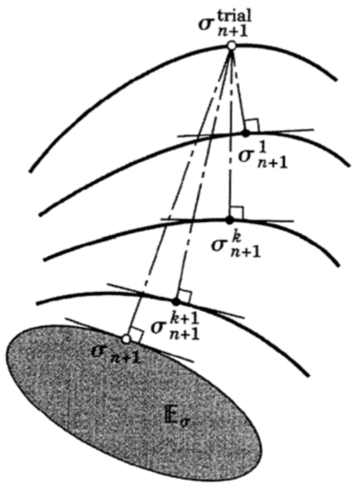
- Remarks
  - Within ideal von Mises plasticity or with linear hardening, $\dot{\lambda}_{n+1}$ can be computed directly. Otherwise, Φ_{n+1} has to be solver iteratively for $\dot{\lambda}_{n+1}$ (e.g. using Newton's method)
  - For general yield functions, the closest point projection is the extension of the radial return algorithm.

5.2 General closest-point projection (for general plasticity and hardening laws)

- Difference between radial and closest point projection



(a) Radial projection [1]



(b) Closest-point projection [2]

5.3 Consistent elastoplastic tangent modulus for von Mises plasticity with only linear isotropic hardening (time discrete setting)

- Note: The definition of the consistent elastoplastic tangent depends on the algorithms used to update stresses, internal variables
- Goal: Tangent to compute

$$\boldsymbol{\mathcal{C}}^{ep} = \frac{\partial \boldsymbol{\sigma}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \boldsymbol{\mathcal{C}}^{ep,vol} + \boldsymbol{\mathcal{C}}^{ep,dev} \tag{17}$$

- Stress tensors

$$\begin{aligned}
\boldsymbol{\sigma}_{n+1} &= \boldsymbol{\sigma}^{vol}_{n+1} + \boldsymbol{\sigma}^{dev}_{n+1} \\
\boldsymbol{\sigma}^{vol}_{n+1} &= \kappa \boldsymbol{\varepsilon}^{vol}_{n+1} \\
\boldsymbol{\sigma}^{dev}_{n+1} &= \boldsymbol{\sigma}^{dev,trial}_{n+1} - 2\mu \Delta\lambda_{n+1} \boldsymbol{n}_{n+1} \\
\boldsymbol{\sigma}^{dev,trial}_{n+1} &= 2\mu \left[\boldsymbol{\varepsilon}^{dev}_{n+1} - \boldsymbol{\varepsilon}^{p}_{n} \right]
\end{aligned}$$

- Derivatives (for tangent stiffness contributions for trial solution)

$$\begin{aligned}
\boldsymbol{\mathcal{C}}^{ep,vol} &= \frac{\partial \boldsymbol{\sigma}^{vol}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \kappa \mathbf{1} \otimes \mathbf{1} \\
\boldsymbol{\mathcal{C}}^{ep,dev,trial} &= \frac{\partial \boldsymbol{\sigma}^{dev,trial}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = 2\mu \boldsymbol{\mathcal{I}}^{dev}
\end{aligned}$$

- Derivatives (for tangent stiffness contributions when plastic flow)
  - Remember:

$$\Delta\lambda_{n+1} = \frac{\Phi^{trial}_{n+1}}{2\mu + \frac{2}{3}k} = \frac{\|\boldsymbol{\sigma}^{dev,trial}_{n+1}\| - \sqrt{\frac{2}{3}}\left[\sigma_y - R_n\right]}{2\mu + \frac{2}{3}k} \quad \text{(Note: } R_n \text{ is fixed)}$$

$$\boldsymbol{n}_{n+1} = \frac{\boldsymbol{\sigma}^{dev,trial}_{n+1}}{\|\boldsymbol{\sigma}^{dev,trial}_{n+1}\|}$$

  - Therefore:

$$\frac{\partial \|\boldsymbol{\sigma}^{dev,trial}_{n+1}\|}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial \|\boldsymbol{\sigma}^{dev,trial}_{n+1}\|}{\partial \boldsymbol{\sigma}^{dev,trial}_{n+1}} : \frac{\partial \boldsymbol{\sigma}^{dev,trial}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \underbrace{\frac{\boldsymbol{\sigma}^{dev,trial}_{n+1}}{\|\boldsymbol{\sigma}^{dev,trial}_{n+1}\|}}_{\boldsymbol{n}_{n+1}} : 2\mu \boldsymbol{\mathcal{I}}^{dev} = 2\mu \boldsymbol{n}_{n+1}$$

$$\frac{\partial \Delta\lambda_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{1}{2\mu + \frac{2}{3}k} \frac{\partial \|\boldsymbol{\sigma}^{dev,trial}_{n+1}\|}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{2\mu}{2\mu + \frac{2}{3}k} \boldsymbol{n}_{n+1}$$

$$
\begin{aligned}
\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} &\equiv \|\boldsymbol{\sigma}_{n+1}^{dev,trial}\|^{-1}\frac{\partial \boldsymbol{\sigma}_{n+1}^{dev,trial}}{\partial \boldsymbol{\varepsilon}_{n+1}} + \boldsymbol{\sigma}_{n+1}^{dev,trial} \otimes \frac{\partial \|\boldsymbol{\sigma}_{n+1}^{dev,trial}\|^{-1}}{\partial \boldsymbol{\varepsilon}_{n+1}} \\
&= \|\boldsymbol{\sigma}_{n+1}^{dev,trial}\|^{-1} 2\mu \boldsymbol{\mathcal{I}}^{dev} - \|\boldsymbol{\sigma}_{n+1}^{dev,trial}\|^{-2} \boldsymbol{\sigma}_{n+1}^{dev,trial} \otimes \frac{\partial \|\boldsymbol{\sigma}_{n+1}^{dev,trial}\|}{\partial \boldsymbol{\varepsilon}_{n+1}} \\
&= \|\boldsymbol{\sigma}_{n+1}^{dev,trial}\|^{-1}\left[2\mu \boldsymbol{\mathcal{I}}^{dev} - \frac{\boldsymbol{\sigma}_{n+1}^{dev,trial}}{\|\boldsymbol{\sigma}_{n+1}^{dev,trial}\|} \otimes 2\mu \boldsymbol{n}_{n+1}\right] \\
&= \frac{2\mu}{\|\boldsymbol{\sigma}_{n+1}^{dev,trial}\|}\left[\boldsymbol{\mathcal{I}}^{dev} - \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\right]
\end{aligned}
$$

– Resulting deviatoric part of the elasto-plastic tangent

$$
\boldsymbol{\mathcal{C}}^{ep,dev} = \frac{\partial \boldsymbol{\sigma}_{n+1}^{dev}}{\partial \boldsymbol{\varepsilon}_{n+1}} = \frac{\partial \boldsymbol{\sigma}_{n+1}^{dev,trial}}{\partial \boldsymbol{\varepsilon}_{n+1}} - 2\mu\left[\boldsymbol{n}_{n+1} \otimes \frac{\partial \Delta\lambda_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}} + \Delta\lambda_{n+1}\frac{\partial \boldsymbol{n}_{n+1}}{\partial \boldsymbol{\varepsilon}_{n+1}}\right] \tag{18}
$$

$$
= 2\mu \boldsymbol{\mathcal{I}}^{dev} - 2\mu\left[\boldsymbol{n}_{n+1} \otimes \left[\frac{2\mu}{2\mu + \frac{2}{3}k}\boldsymbol{n}_{n+1}\right]\right. \tag{19}
$$

$$
\left. + \Delta\lambda_{n+1}\left[\frac{2\mu}{\|\boldsymbol{\sigma}_{n+1}^{dev,trial}\|}\left[\boldsymbol{\mathcal{I}}^{dev} - \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\right]\right]\right] \tag{20}
$$

$$
= 2\mu \boldsymbol{\mathcal{I}}^{dev} - \frac{[2\mu]^2}{2\mu + \frac{2}{3}k}\boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1} \tag{21}
$$

$$
+ \left[\Delta\lambda_{n+1}\frac{[2\mu]^2}{\|\boldsymbol{\sigma}_{n+1}^{dev,trial}\|}\left[\boldsymbol{\mathcal{I}}_{n+1}^{dev} - \boldsymbol{n}_{n+1} \otimes \boldsymbol{n}_{n+1}\right]\right] \tag{22}
$$

– Compared to continuous case, the tangents are not equal but they converge to the same expression when $\Delta\lambda_{n+1} \to 0$

6 Finite element discretisation

For clarity, it is worth mentioning up front that in this section we will denote the general elasto-plastic tangent by $\mathbb{C}^{ep}$; so, in the elastic region $\mathbb{C}^{ep} \equiv \mathbb{C}$ while in the plastic region $\mathbb{C}^{ep}$ represents the consistent tangent.

Combining eqs. (5) and (17) (specifically, eqs. (16b) and (22)) renders the complete expression of the balance of linear momentum, namely

$$\int\limits_{\Omega} \frac{\partial \delta v_i}{\partial x_j} \, [\mathbb{C}^{ep}]_{ijkl} \, \varepsilon_{kl} \, dv = \int\limits_{\Omega} \delta v_i \, b_i \, dv + \int\limits_{\partial\Omega} \delta v_i \, \underbrace{\sigma_{ij} \, n_j}_{\bar{t}_i} \, da \quad , \tag{23}$$

We discretise the trial solution and test function using vector-valued finite element shape functions (ansatz)

$$\mathbf{u}(\mathbf{x}) \approx \sum_I \mathbf{N}^I(\mathbf{x}) \, u^I \quad , \quad \mathbf{v}(\mathbf{x}) \approx \sum_I \mathbf{N}^I(\mathbf{x}) \, v^I \tag{24}$$

where $\mathbf{N}^I(\mathbf{x})$ is the (position-dependent) vector-valued finite element shape function corresponding to the I^{th} degree-of-freedom, and u^I, v^I are coefficients of the solution and trial function.

We now use these shape functions to discretise the weak expression for the balance of linear momentum. Starting on the right-hand side of eq. (23), the body force and traction contributions are computed by

$$\int\limits_{\Omega} \delta v_i \, b_i \, dv = \int\limits_{\Omega} \left[\sum_I \mathbf{N}^I(\mathbf{x}) \, \delta v^I \right]_i b_i \, dv = \sum_I \delta v^I \int\limits_{\Omega} \mathbf{N}^I_i \, b_i \, dv \tag{25}$$

$$\int\limits_{\Omega} \delta v_i \, t_i \, dv = \sum_I \delta v^I \int\limits_{\Omega} \mathbf{N}^I_i \, t_i \, dv \quad . \tag{26}$$

The last component of eq. (23) that we wish to express in discrete form is the left-hand side of the equation. Before we do, we observe that using the minor symmetry of the material stiffness tensor we can re-express the contraction of it and the small strain tensor as

$$\mathbb{C} : \boldsymbol{\varepsilon} = \mathbb{C} : \frac{1}{2} \left[\nabla \mathbf{u} + [\nabla \mathbf{u}]^T \right] \equiv \mathbb{C} : \nabla \mathbf{u} \tag{27}$$

from which we can similarly deduce that

$$\nabla \mathbf{u} : \mathbb{C} \equiv \boldsymbol{\varepsilon} : \mathbb{C} \tag{28}$$

Therefore, this contribution written in discrete form is

$$
\begin{aligned}
\int\limits_{\Omega} \frac{\partial \delta v_i}{\partial x_j} \, [\mathbb{C}^{ep}]_{ijkl} \, \varepsilon_{kl} \, dv &\equiv \int\limits_{\Omega} \frac{\partial \delta v_i}{\partial x_j} \, \mathbb{C}^{ep}_{ijkl} \, \frac{\partial \delta u_k}{\partial x_l} \, dv \\
&= \int\limits_{\Omega} \frac{\partial}{\partial x_j} \left[\sum_I \mathbf{N}^I (\mathbf{x}) \, \delta v^I \right]_i \, \mathbb{C}^{ep}_{ijkl} \, \frac{\partial}{\partial x_l} \left[\sum_J \mathbf{N}^J (\mathbf{x}) \, \delta u^J \right]_k \, dv \\
&= \sum_{I,J} \delta v^I \left[\int\limits_{\Omega} \frac{\partial}{\partial x_j} \left[\mathbf{N}^I (\mathbf{x}) \right]_i \, \mathbb{C}^{ep}_{ijkl} \, \frac{\partial}{\partial x_l} \left[\mathbf{N}^J (\mathbf{x}) \right]_k \, dv \right] \delta u^J \\
&= \sum_{I,J} \delta v^I \left[\int\limits_{\Omega} \left[\mathbf{N}^I (\mathbf{x}) \right]_{i,j} \, \mathbb{C}^{ep}_{ijkl} \, \left[\mathbf{N}^J (\mathbf{x}) \right]_{k,l} \, dv \right] \delta u^J \\
&\equiv \sum_{I,J} \delta v^I \left[\int\limits_{\Omega} \left[\left[\mathbf{N}^I (\mathbf{x}) \right]_{i,j} \right]^S \, \mathbb{C}^{ep}_{ijkl} \, \left[\left[\mathbf{N}^J (\mathbf{x}) \right]_{k,l} \right]^S \, dv \right] \delta u^J \quad . \qquad (29)
\end{aligned}
$$

Equations (25), (26) and (29) are collectively used to develop the system of linear equations that are solved at each time step.

References

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[2] J. C. Simo and Thomas J. R. Hughes. *Computational Inelasticity*, volume 7 of *Interdisciplinary Applied Mathematics*. Springer Science & Business Media, 2000.