Application of the Fictitious Domain method to flow problems with complex geometries

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OUTLINE

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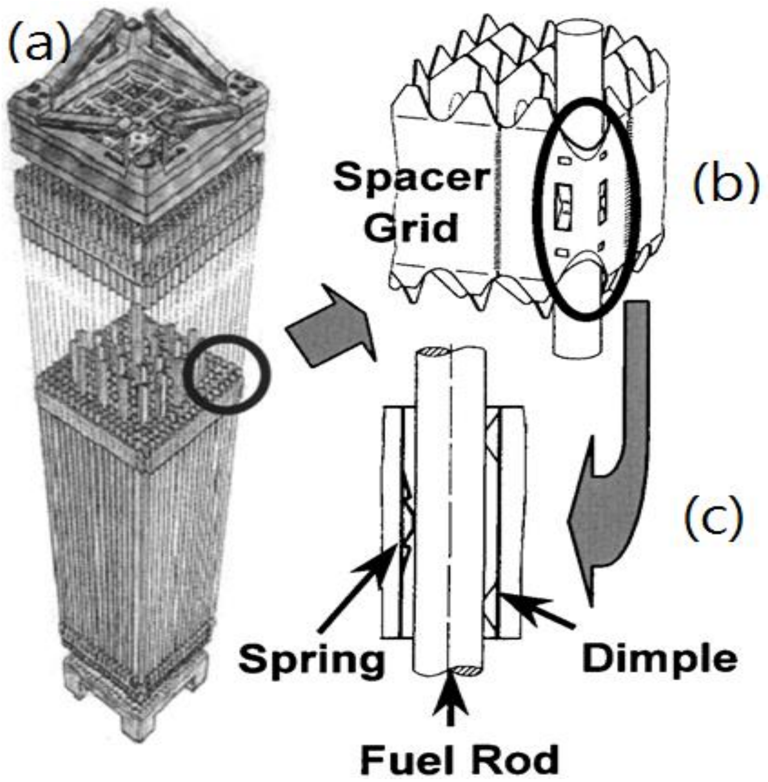
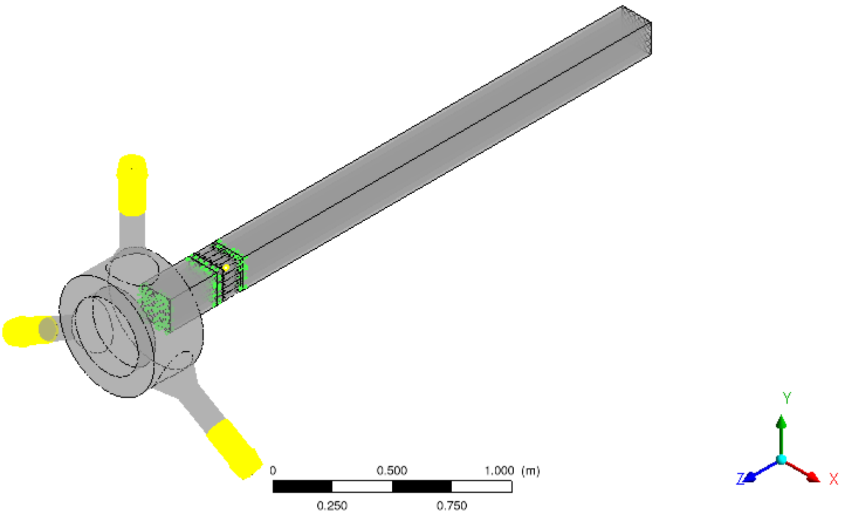
Introduction



PWR fuel assembly and spacer grid spring
(Kim et al. 2001)

- Flow problems with complex geometries are widespread

- CFD simulation for one smaller sized assembly,
  68 million cells (Wei 2013,)

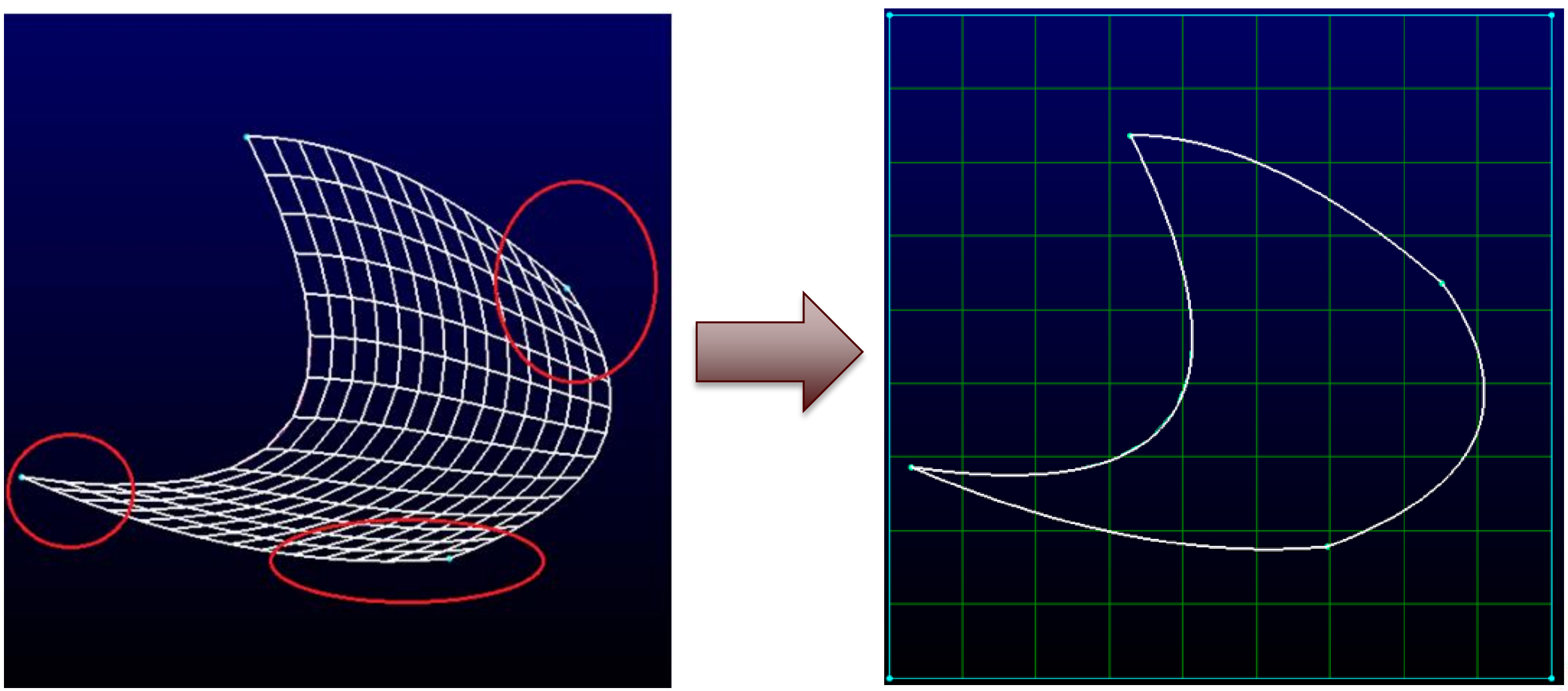


- Body-fitted mesh is not suitable

- Our choice: Fictitious Domain method (FD method)

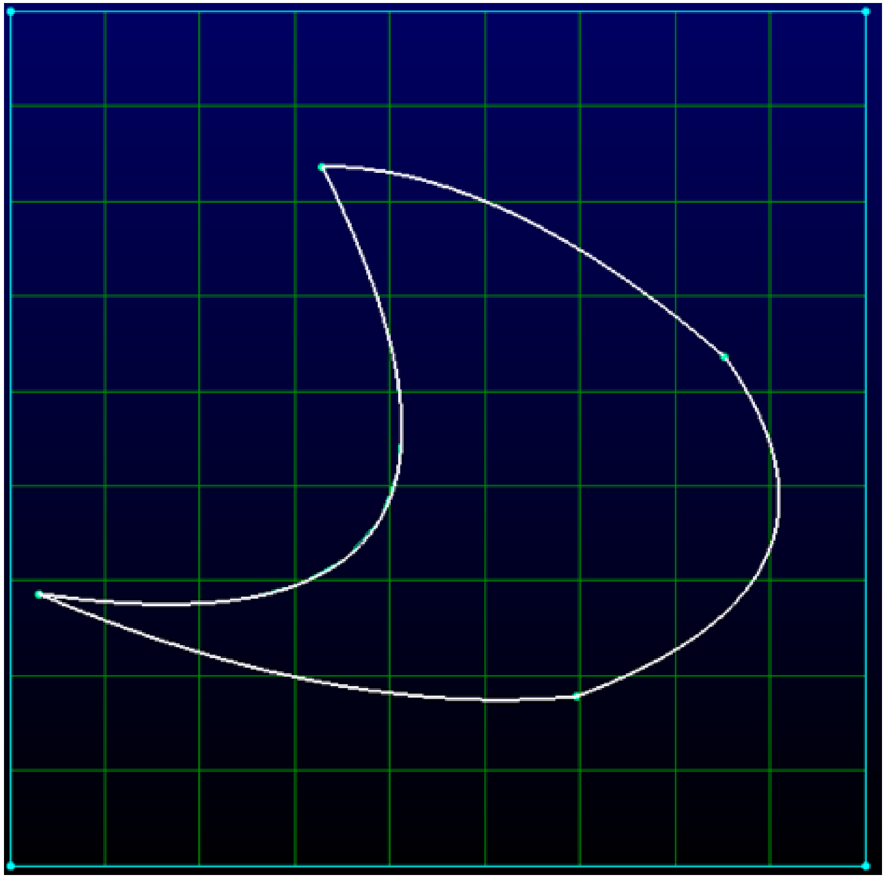
- **Basic concept regarding the FD method**:

Whenever a problem needs to be solved on a domain with an irregular boundary, it may be useful to embed it into a larger domain of a simpler shape (Quarteroni and Valli).



Critical point regarding the FD method implementation:

How to include the influence from the “immersed” BC



Strategies regarding this immersed boundary issue:

1. Penalty function method (Ramière et al. 2005, Zhou and Saito 2014, Saito and Zhou 2014)

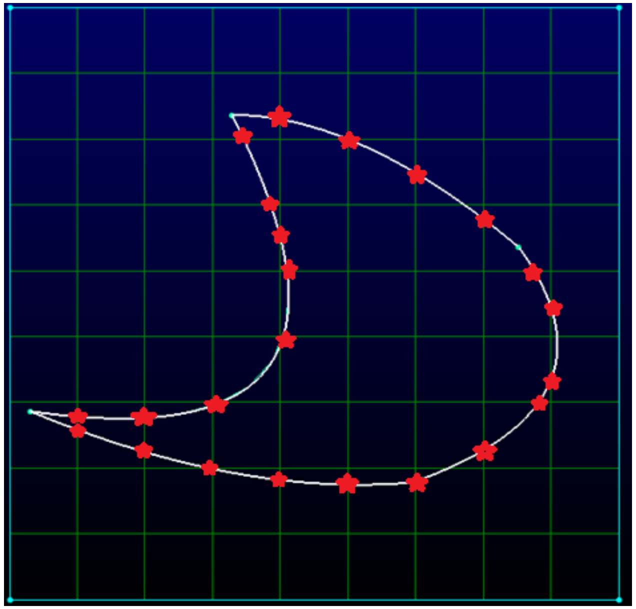
$$\frac{1}{\varepsilon}\int_{\Omega\backslash\omega}(v\cdot\tilde{u})dx$$

=> limitation exists ($e \sim C(\sqrt{\varepsilon}+h)$)

Strategies regarding this immersed boundary issue:

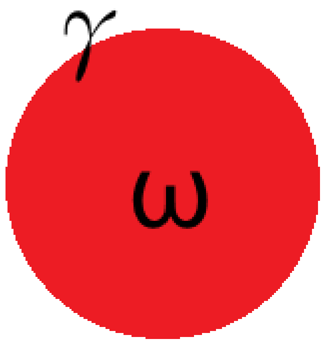
2. Lagrange multiplier method (Glowinski et al. 1994, Glowinski et al. 1995, Glowinski et al. 1998)

$$\int_{\gamma} \lambda \cdot (u - g)dx$$



⇒ Powerful, but may be not suitable for problems with complex 3d geometries

Case 1: Poisson problem



Governing equation

$$-\Delta u = f \qquad in\ \omega,$$

$$u = g \qquad on\ \gamma,$$

where $f \in H^{-1}(\omega)$,
$g \in H^{\frac{1}{2}}(\gamma)$, which is Dirichlet BC

Weak formulation

$$a_{\omega}(v,u) = \langle v,f\rangle \quad \forall v \in H_0^1(\omega),$$

where

$$a_{\omega}(v,u) = \int_{\omega} (\nabla v \cdot \nabla u)dx \qquad\qquad \forall u,v \in H_0^1(\omega),$$

$$\langle v,f\rangle = \int_{\omega} (v \cdot f)dx.$$

Here, V_g is the solution space and its Definition is:

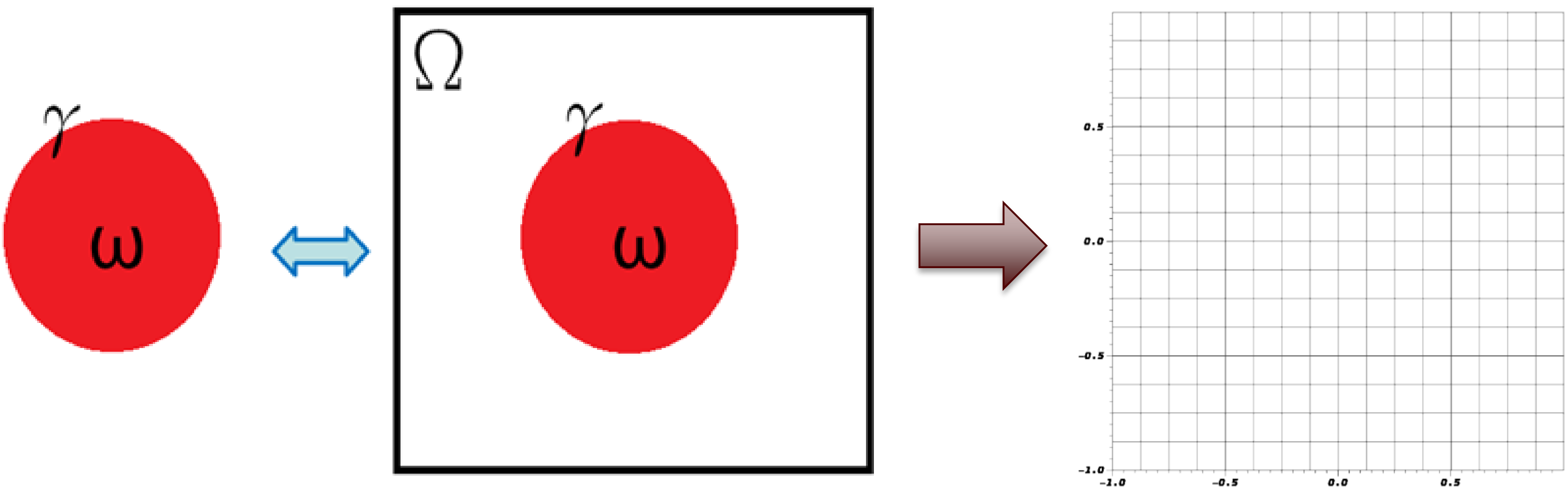
$$V_g = \{v \mid v \in H^1(\omega), v = g \text{ on } \gamma\}.$$

Lagrangian functional:

$$L(u,\lambda) = \frac{1}{2}\int_{\omega} (\nabla u \cdot \nabla u)dx - \int_{\omega} (f \cdot u)dx + \int_{\gamma} \lambda \cdot (u-g)dx \qquad in\ \omega$$

FD method implementation

1. Embed the original domain



FD method implementation

2. Define Lagrangian functional for the fictitious domain Ω:

$$L(\tilde{u}, \lambda) = \frac{1}{2}\int_{\Omega} (\nabla\tilde{u} \cdot \nabla\tilde{u})dx - \int_{\Omega} \left(\tilde{f} \cdot \tilde{u}\right)dx + \int_{\gamma} \lambda \cdot (\tilde{u} - \tilde{g})dx \qquad in\, \Omega\,.$$

Solution $\tilde{u}$ is valid in the whole domain Ω and the Lagrange multiplier λ is valid on the immersed boundary γ.
In addition, $\tilde{f}\big|_{\omega} = f$ and $\tilde{g}|_{\gamma} = g$.

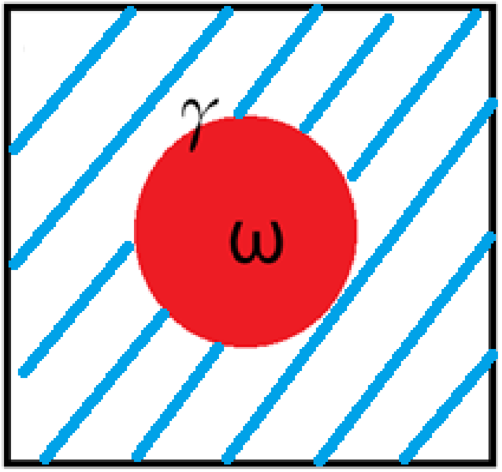
Corresponding weak formulation for Ω:

$$\int_\Omega (\nabla v \cdot \nabla \tilde{u})dx - \int_\Omega (v \cdot \tilde{f})dx + \int_\gamma (v \cdot \lambda)d\gamma = 0,$$

$$\int_\gamma \mu \cdot (\tilde{u} - \tilde{g})d\gamma = 0.$$

Our strategies

1. Approach 1: penalty lives everywhere



e.g.

$$\int_\gamma (v \cdot \lambda)d\gamma \Rightarrow \int_{\Omega\backslash\omega}(v \cdot \lambda)dx$$

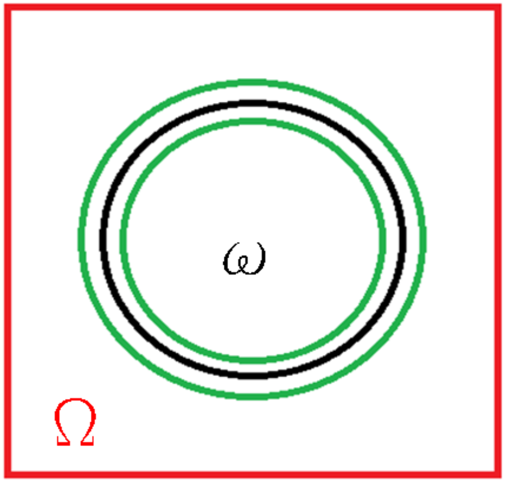
$$\Rightarrow \int_\Omega (w(x) \cdot v \cdot \lambda)\, dx$$

where

$$w(x) = \begin{cases} 1, & x \in \Omega\backslash\omega \\ 0, & x \in \omega \end{cases}$$

- **Our strategies**

2. Approach 2:



$\bar{\omega}$: $d(x) < r_a - \theta,$

$\bar{\gamma}$: $r_a - \theta \le d(x) \le r_a + \theta,$

$\Omega \backslash \bar{\omega} \backslash \bar{\gamma}$: $r_a + \theta \le d(x).$

$$\theta = c \times h$$

e.g.

$$\int_\gamma (v \cdot \lambda) d\gamma \Rightarrow \int_\Omega (w(x) \cdot \boldsymbol{k}(\boldsymbol{x}) \cdot v \cdot \lambda)\, dx$$

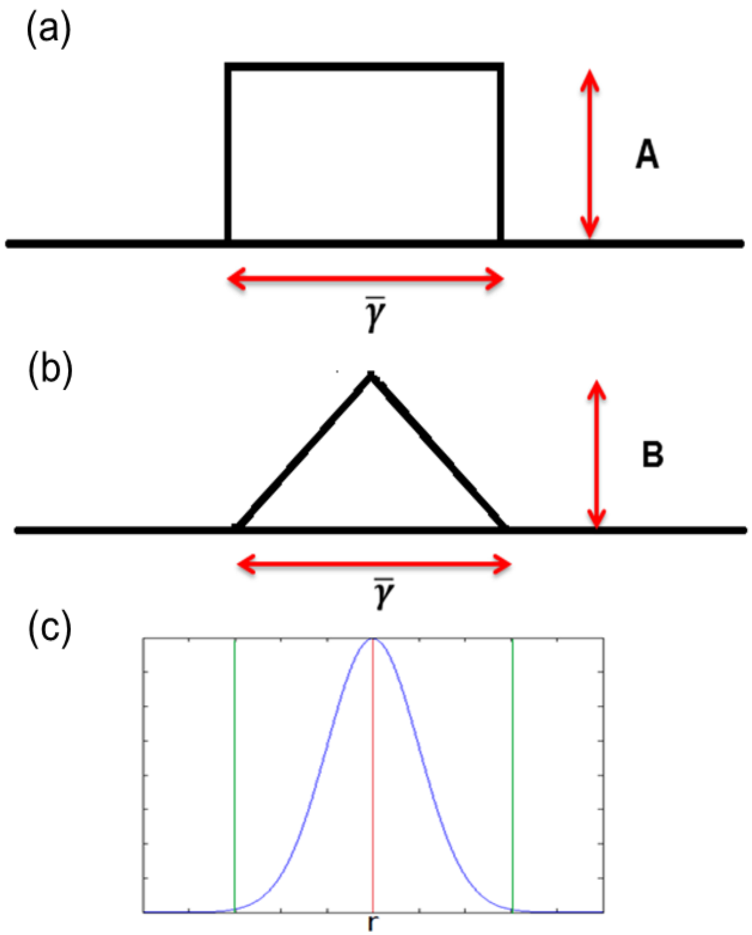
w(x) is equal to one in $\bar{\gamma}$ and equal to zero in the rest areas.

- **Our strategies**

2. Approach 2:

   about function *k(x)*

$$\int_{\gamma} s(x)\,dx \approx \int_{\bar{\gamma}} k(x)\cdot s(x)dx$$



Three *k(x)* functions

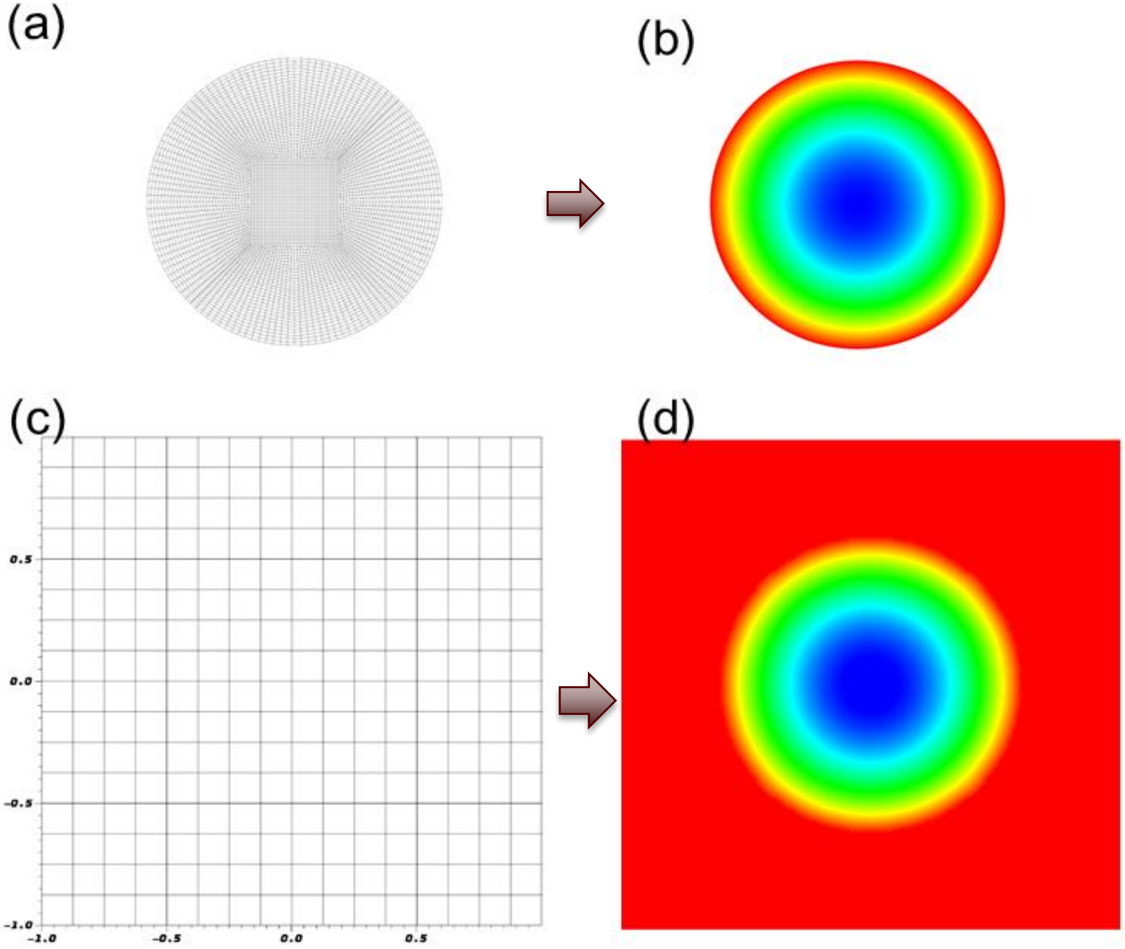
Numerical details

1. FEM, $u_h,\ \lambda_h \in Q_1(K)$

2. Gaussian Quadrature for computing the integration

3. Direct solver UMFPACK for solving the resulting linear system

4. L-2 error norm for the error analysis

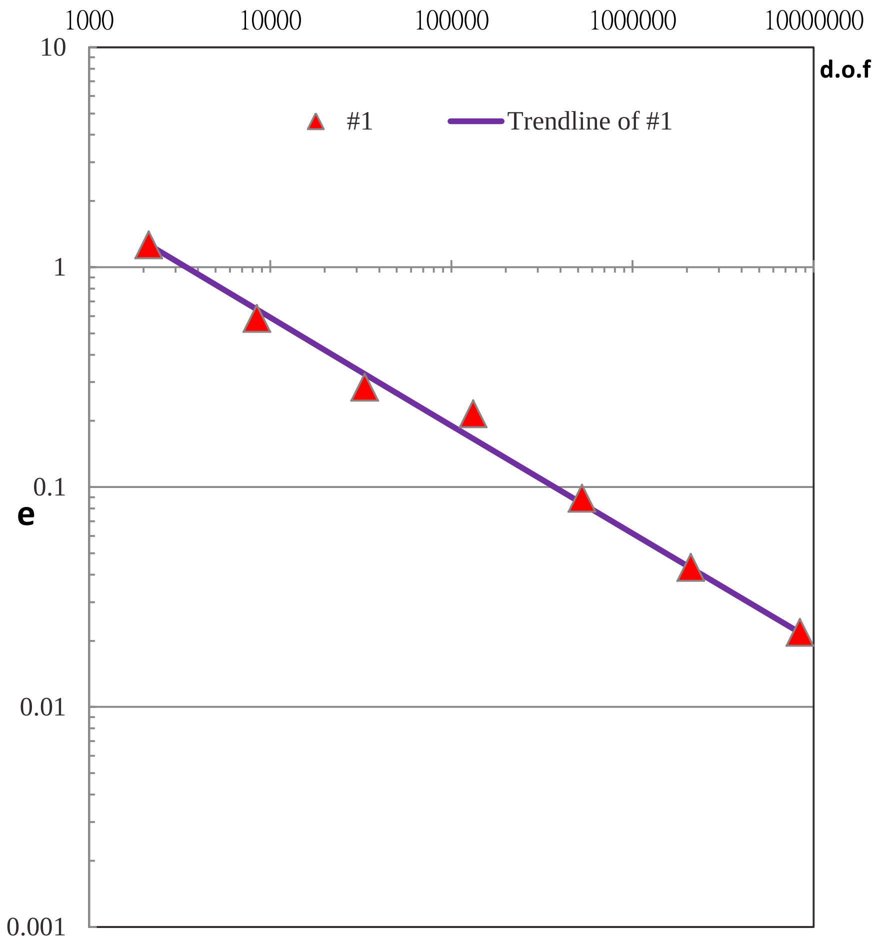


(a) and (b) are direct result, computed from step-6
(c) and (d) are results from the FD method

- **Solution profile**

1. In quality, these two results are similar

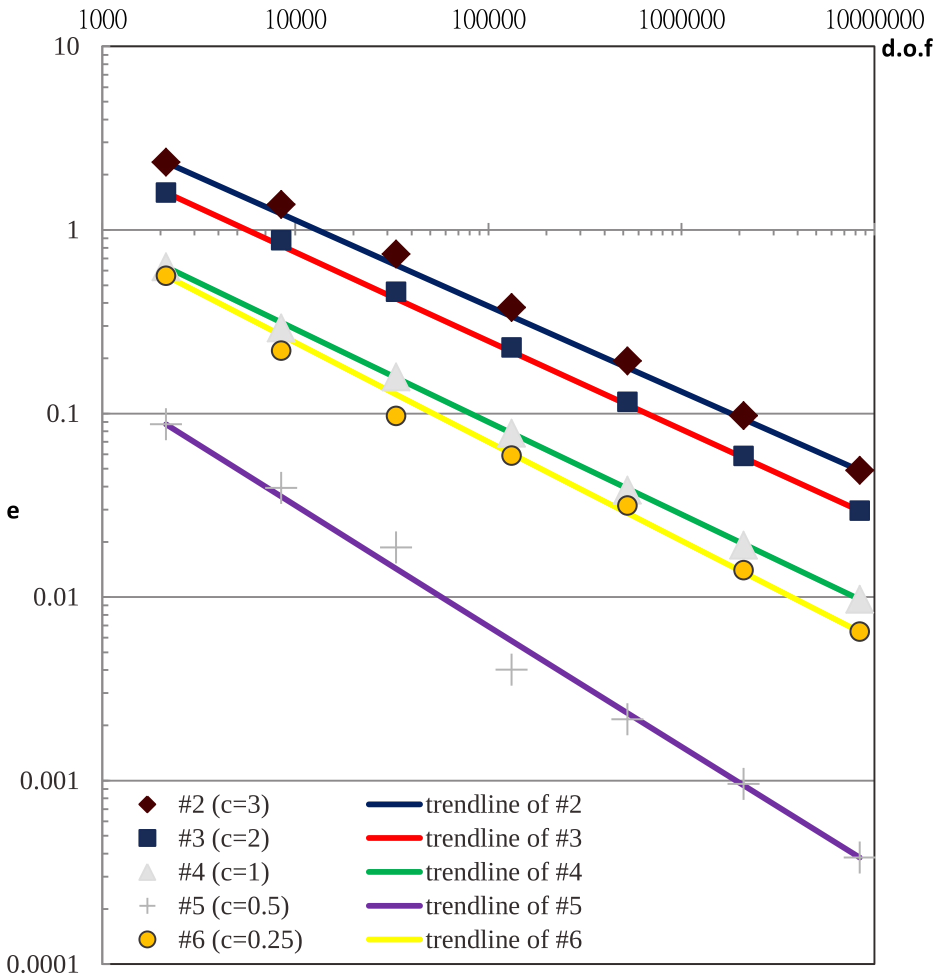
- **Error analysis: Approach 1**



1. Oscillation appears in coarse mesh

2. $e \sim O(h^{0.98})$, comparable to the penalty function method (Ramière et al. 2005)

- **Error analysis: Approach 2, with constant function**



1. When θ is small (i.e., c is small), the result is not stable $\left(\begin{matrix}\theta = c \times h, \\ c = 0.25, 0.5, 1.0, 2.0 \; and \; 3.0\end{matrix}\right)$

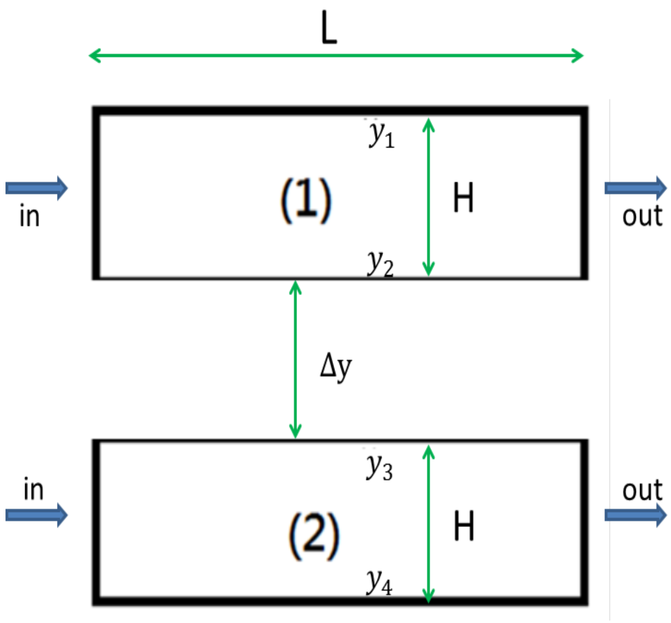
2. $e \sim O(h^1)$, for c=1

   $e \sim O(h^{1.3})$, for c=0.5

Summary 1

1. Both of our two approaches can get reasonable results

Case 2: Stokes flow problem



Two parallel, fully developed, Poiseuille flow with very low velocity

Governing equation

$$-\mathrm{div}[2\eta \cdot \varepsilon(u)] + \nabla p = 0 \quad in\ \omega,$$

$$-\mathrm{div}\, u = 0 \quad in\ \omega,$$

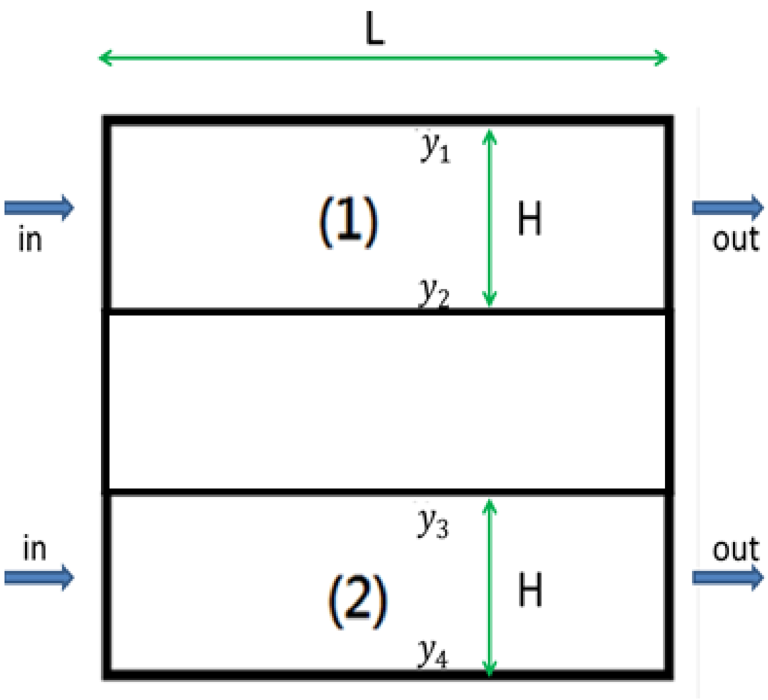
where

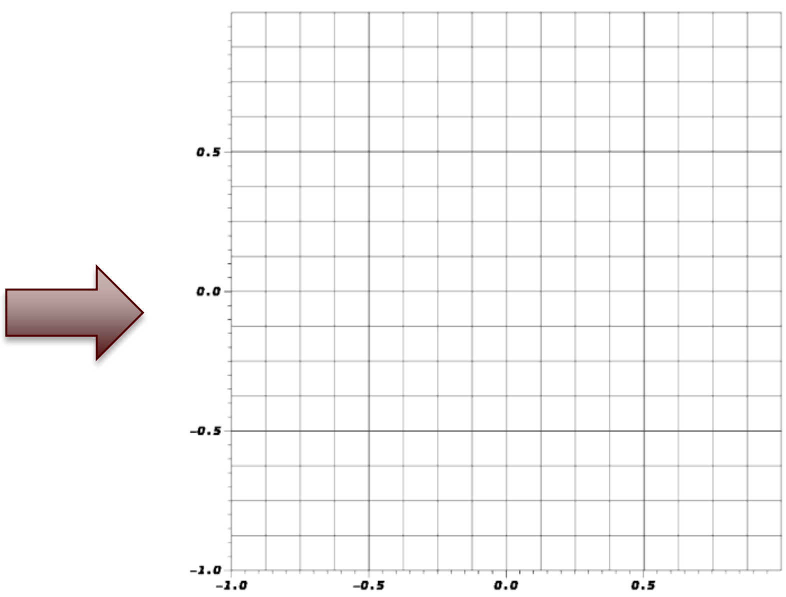
$$\varepsilon(u) = \frac{1}{2}[(\nabla u) + (\nabla u)^T],$$

$$u = 0.0 \text{ at } \Gamma.$$

FD method implementation

1. Embed the original domain





FD method implementation

2. Define Lagrangian functional for the fictitious domain Ω:

$$L(\tilde{u}, \tilde{p}, \lambda) = \quad 2\eta \frac{1}{2} \int_{\Omega} |\nabla \tilde{\varepsilon}|^2 \, dx - \int_{\Omega} p \nabla \tilde{u} \, dx$$

$$- \int_{\Omega} \left(\tilde{F} \cdot \tilde{u}\right) dx + \int_{\gamma} \lambda \cdot (\tilde{u} - \tilde{g}) dx \;\; in \, \Omega,$$

The solutions $\tilde{u}$ and $\tilde{p}$ are valid in the whole domain Ω. In addition, $\tilde{F}\big|_{\omega} = F = 0.0$ and $\tilde{g}|_{\gamma} = g = 0.0$.

For simplicity, we set $\tilde{F}\big|_{\Omega \setminus \omega} = 0.0$.

Corresponding weak formulation for Ω:

$$2\nu\big(\varepsilon(v),\varepsilon(\tilde{u})\big)_\Omega - (\nabla v,\tilde{p})_\Omega = \big(v,\tilde{F}\big)_\Omega - (v,\lambda)_\gamma,$$

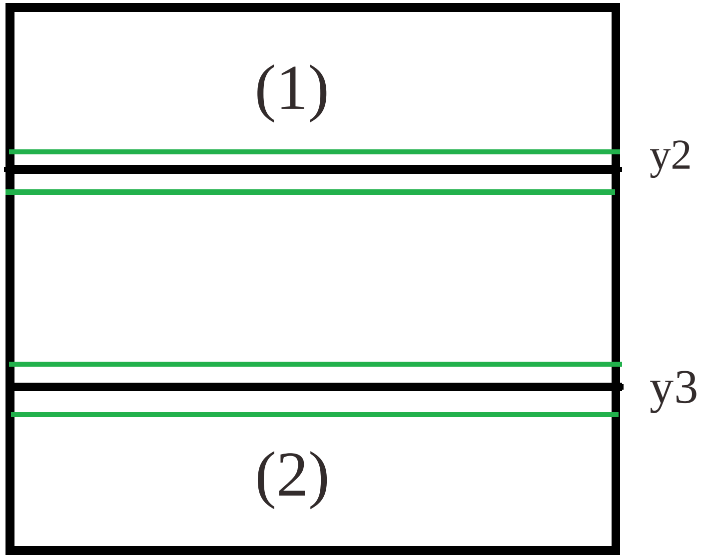
$$-(q,\nabla\tilde{u})_\Omega = 0,$$

$$(\mu,\tilde{u})_\gamma = (\mu,g)_\gamma.$$

Our strategy for the boundary related term

Approach 2, boundary region $\bar{\gamma}$ is used here

$$\bar{\gamma}: \quad y_{ib} - hf \le y_{qp} \le y_{ib} + hf.$$



- **Numerical details**

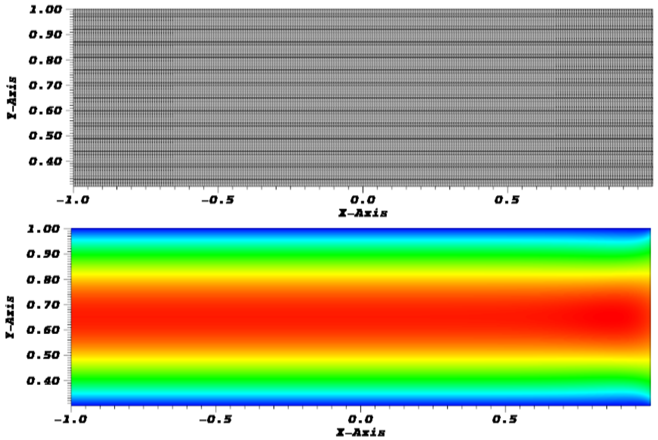
1. FEM, $u_h,\ \lambda_h\ \in Q_2(K), p_h\ \in Q_1(K)$

2. Gaussian Quadrature for computing the integration

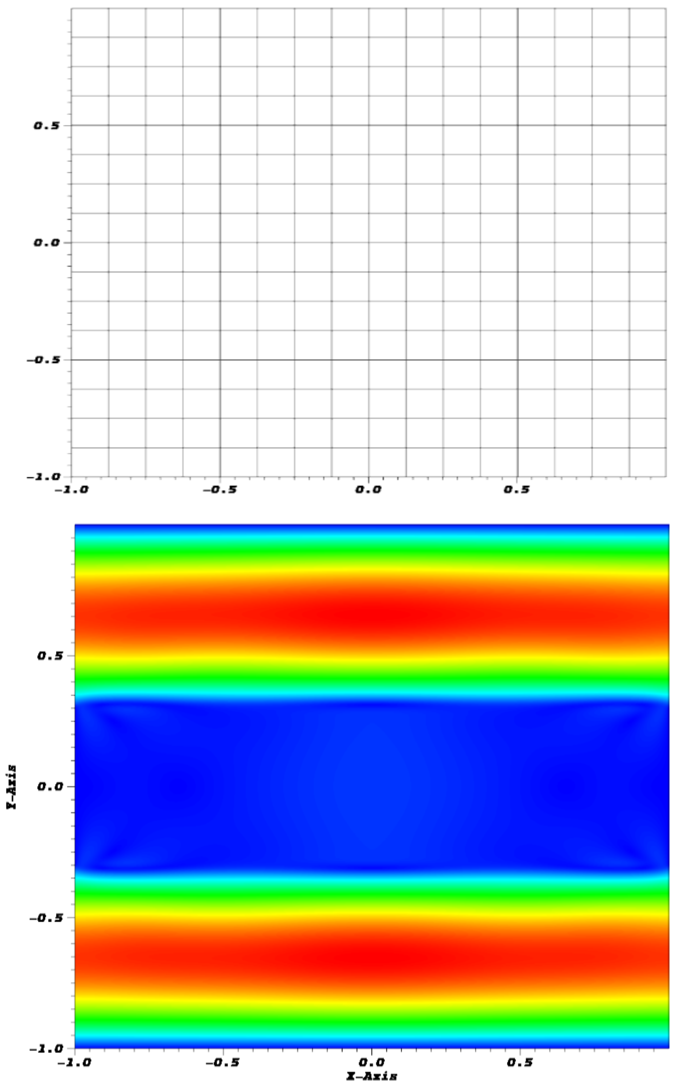
3. The resulting linear system is solved iteratively (modified from Glowinski et al. 1995, a variation of the Uzawa algorithm).

4. L-2 error norm for the error analysis

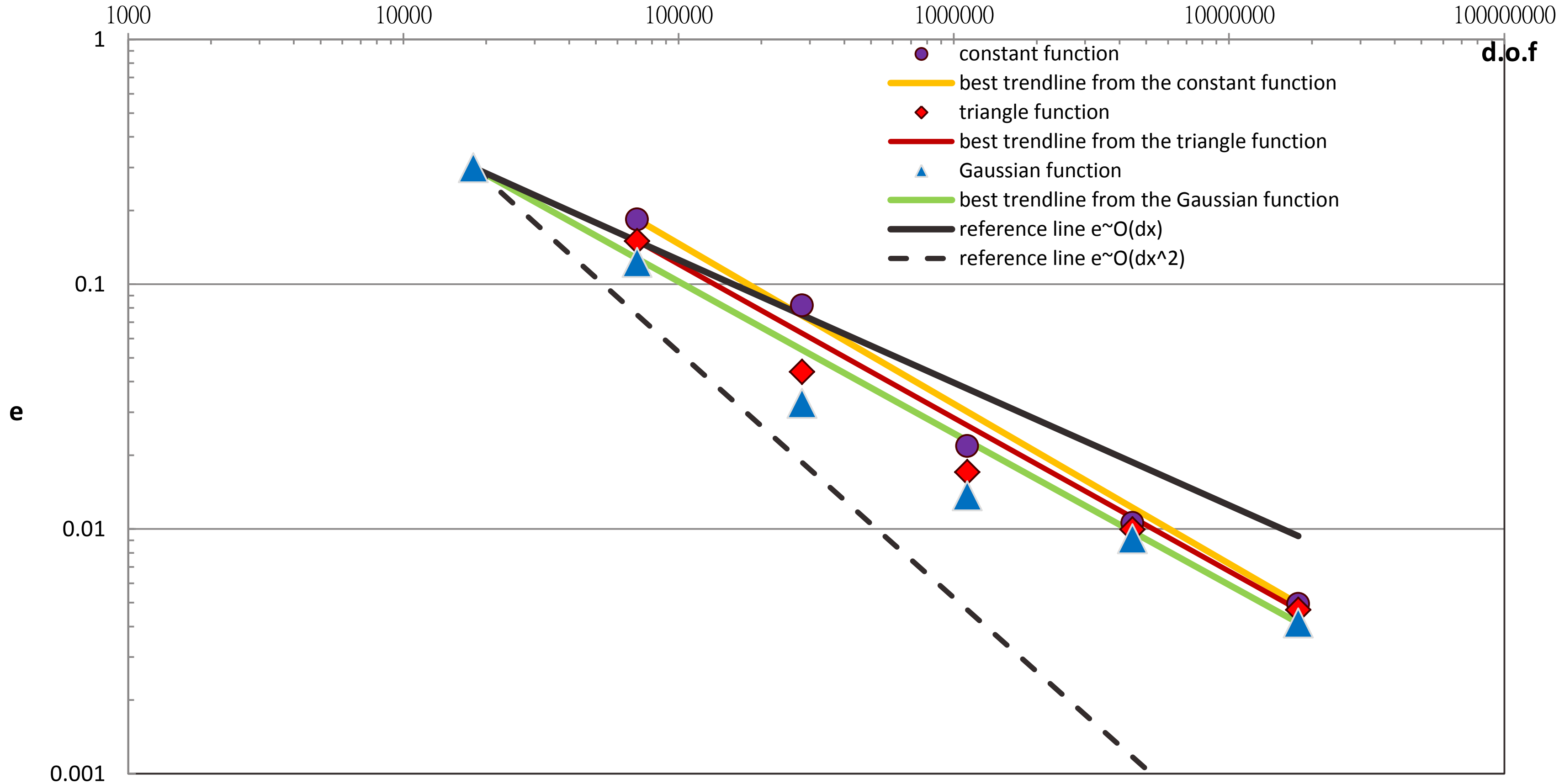
Direct simulation's velocity profile(step-22)



FD method's velocity profile



- Error analysis



- Among the three k(x) functions, Gaussian function may be the best, whose error can be expressed as:

$$e \sim h^{1.236}$$

Summary 2

1. The modified iterative algorithm used here can be parallelized

2. What we have from the Case 2 can be the basis for us to solve the Navier-Stokes problem

Future work

1. Parallelize the code for the Stokes flow problem

2. Solve the Navier-Stokes problem (Operator splitting method)

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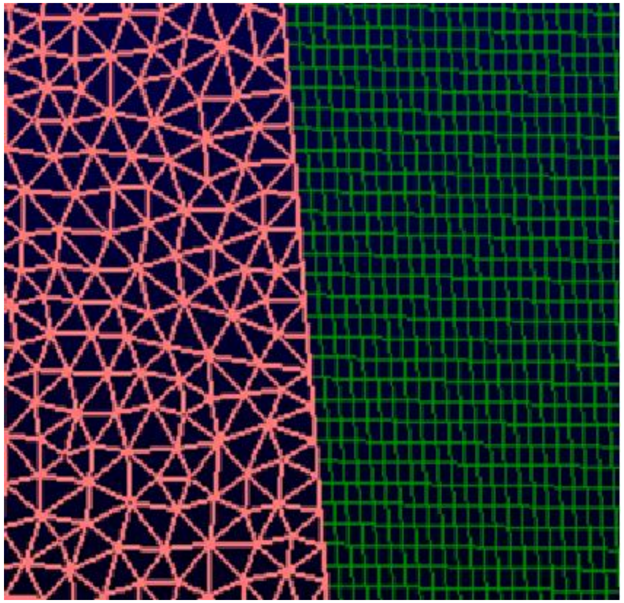
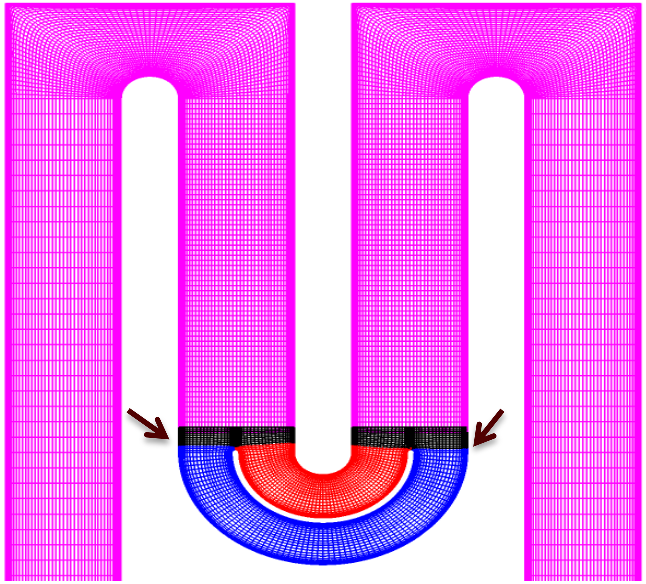
Questions?

Supplement - introduction

- Several techniques are invented, such as multi-block or overset mesh, but they all cause extra burdens to the user or computers



- Interface between two blocks with different meshes (Multi-block)



- Interpolation area between two blocks (Overset mesh)

Weak formulation

Find $u \in H^1(\omega),\ p \in L^2(\omega)$

$$2\eta\big(\varepsilon(v), \varepsilon(u)\big)_\omega - (\nabla v, p)_\omega = \langle v, 0\rangle_\omega \quad \forall v \in H_0^1(\omega),$$

$$-(q, \nabla u)_\omega = \langle v, 0\rangle_\omega \quad \forall q \in L_0^2(\omega),$$

where

$$\big(\varepsilon(v), \varepsilon(u)\big)_\omega = 2\eta \int_\omega \varepsilon(v) \cdot \varepsilon(\mathrm{u})\, dx,$$

$$(\nabla v, p)_\omega = \int_\omega \nabla v \cdot p\, dx, \quad \langle v, 0\rangle_\omega = \int_\omega v \cdot 0\, dx,$$

$$(q, \nabla u)_\omega = \int_\omega q \cdot \nabla u\, dx, \quad \langle v, G\rangle_\omega = \int_\omega q \cdot 0\, dx,$$

Lagrangian functional

$$L(u,p,\lambda) = \quad 2\eta\frac{1}{2}\int_{\omega}|\nabla\varepsilon|^2\,dx - \int_{\omega}\, p\nabla u\,dx$$

$$-\int_{\omega}(F\cdot u)dx + \int_{\gamma}\ \lambda\cdot(u-g)dx\ \ in\ \omega.$$

Supplement Case 2 Approach

- In Glowinski et al. 1995

Concerning the multiplier λ, its interpretation is very simple since it is equal to the jump of

$$\nu \frac{\partial u}{\partial n} - np$$

Supplement- iterative algorithm 1

Iterative algorithm in Glowinski et al. 1994	Iterative algorithm used for solving our designed problem
Initial guess part	
1. Guess λ^0	
2. Find the initial velocity U^0 and the initial pressure P^0 by solving the Stokes problem $$\nu\int_\Omega \nabla U^0\cdot\nabla \mathrm{v}\,dx-\int_\Omega P^0\nabla\cdot\mathrm{v}\,dx$$ $$=\int_\Omega f\cdot\mathrm{v}\,dx+\int_\gamma \lambda^0\cdot\mathrm{v}\,d\gamma,$$ **and** $$\int_\Omega \mathrm{q}\nabla\cdot U^0\,\mathrm{dx}=0.$$ **In addition, impose:** $$\mathrm{U}^0=g_1\quad on\ \Gamma,$$ **where Γ is the boundary condition for the fictitious domain Ω.**	2. solve $$2\nu\big(\varepsilon(v),\varepsilon(\tilde{u}^0)\big)_\Omega-(\nabla v,\tilde{p}^0)_\Omega$$ $$=\big(v,\tilde{F}\big)_\Omega-(w(x)\cdot k(x)\cdot v,\lambda^0)_\Omega,$$ and $$-(q,\nabla\tilde{u}^0)_\Omega=0.$$ In addition, impose $$\mathrm{U}^0=g_1\quad on\ \Gamma,$$ where $$\tilde{F}\big\vert_\omega=0.0,\ \tilde{F}\big\vert_{\Omega\backslash\omega}=0.0.$$ $$w(x)=\begin{cases}0, & in\ \Omega\backslash\bar{\gamma}\\ 1, & in\ \bar{\gamma}\end{cases}.$$ There are three setting regarding the weight function $k(x)$, when we use Approach 2
3. Compute g^0: $$\int_\gamma g^0\cdot\mu\,d\gamma=\int_\gamma (U^0-g_2)\cdot\mu\,d\gamma$$	3. Compute g^0 by solving $$(w(x)\cdot k(x)\cdot\mu,g^0)_\Omega$$ $$+([1-\mathrm{w}(x)]\cdot\mu,g^0)_\Omega$$ $$=\big(w(x)\cdot k(x)\cdot\mathrm{\mu},(\tilde{u}^0-g)\big)_\Omega$$ $$+(\,[1-w(x)]\cdot\mu,0)_\Omega$$ $g=0.0$ in this designed problem (i.e., no-slip boundary condtion

Supplement- iterative algorithm 2

4. Set $W^0 = g^0$

Iteration loop starts from step-5

5. Find $\bar{U}^n$ and $\bar{P}^n$ which satisfy $$\nu\int_\Omega \nabla\bar{U}^n\cdot\nabla \mathrm{v}\,dx - \int_\Omega \bar{P}^n\nabla\cdot \mathrm{v}\,dx = \int_\gamma W^n\cdot \mathrm{v}\,d\gamma$$ **and** $$\int_\Omega q\nabla\cdot U^0\,\mathrm{dx} = 0.$$	5. Solve $$2\nu\big(\varepsilon(v),\varepsilon(\bar{U}^n)\big)_\Omega - (\nabla v,\bar{P}^n)_\Omega$$ $$= (w(x)\cdot k(x)\cdot \mathrm{v}, W^n)_\Omega,$$ and $$-(q,\nabla\bar{U}^n)_\Omega = 0.$$
6. Compute ρ_n by using: $$\rho_n = \frac{\int_\gamma \lvert g^n\rvert^2\,d\gamma}{\int_\gamma \bar{U}^n\cdot W^n\,d\gamma}$$	6. compute $$\rho_n = \frac{\int_\Omega w(x)\cdot k(x)\cdot \lvert g^n\rvert^2\ dx}{\int_\Omega w(x)\cdot k(x)\cdot \bar{u}^n\cdot W^n\ dx}$$
7. Get new λ, U and P: $$\lambda^{n+1} = \lambda^n - \rho_n W^n$$ $$U^{n+1} = U^n - \rho_n\bar{U}^n$$ $$P^{n+1} = P^n - \rho_n\bar{P}^n$$	7. Get new λ, $\tilde{u}$ and $\tilde{p}$: $$\lambda^{n+1} = \lambda^n - \rho_n W^n$$ $$\tilde{u}^{n+1} = \tilde{u}^n - \rho_n\bar{U}^n$$ $$\tilde{p}^{n+1} = \tilde{p}^n - \rho_n\bar{P}^n$$

Supplement- iterative algorithm 3

8. Renew g^n by $$\int_\gamma g^{n+1}\cdot\mu\, d\gamma = \int_\gamma g^n\cdot\mu\, d\gamma - \rho^n \int_\gamma \bar{U}^n\cdot\mu\, d\gamma$$	**8. Renew g^n** $$(w(x)\cdot k(x)\cdot\mu, g^{n+1})_\Omega + ([1-\mathrm{w}(x)]\cdot\mu, g^{n+1})_\Omega = (w(\mathrm{x})\cdot k(x)\cdot\mu, g^n)_\Omega - \rho^n(w(x)\cdot k(x)\cdot\mu, \bar{U}^n) + ([1-w(x)]\cdot\mu, g^{n+1})_\Omega$$
9. If it reaches the stop criteria: $$\frac{\int_\gamma \lvert g^{n+1}\rvert^2\, d\gamma}{\int_\gamma \lvert g^0\rvert^2\, d\gamma}\le\varepsilon$$ **solutions are λ^{n+1}, U^{n+1} and P^{n+1}.**	9. Compute the iteration error: $$\frac{\int_\Omega w(x)\cdot k(x)\cdot\lvert g^{n+1}\rvert^2\, dx}{\int_\Omega \mathrm{w(x)}\cdot k(x)\cdot\lvert g^0\rvert^2 dx}$$ If it is equal or smaller than the stop criterion ε, then, the solutions are λ^{n+1}, $\tilde{u}^{n+1}$ and $\tilde{p}^{n+1}$
10. If it hasn't reached the stop criteria, compute: $$r_n = \frac{\int_\gamma \lvert g^{n+1}\rvert^2\, d\gamma}{\int_\gamma \lvert g^n\rvert^2\, d\gamma},$$ **and get** $$W^{n+1} = g^{n+1} + r_n W^n.$$ **Restart to compute the step-5 by using this new W^{n+1}**	10. If it hasn't reached the stop criteria, compute: $$r_n = \frac{\int_\Omega w(x)\cdot k(x)\cdot\lvert g^{n+1}\rvert^2\, dx}{\int_\Omega w(x)\cdot k(x)\cdot\lvert g^n\rvert^2\, dx},$$ and get $$W^{n+1} = g^{n+1} + r_n W^n.$$ Restart to compute the step-5 by using this new W^{n+1}