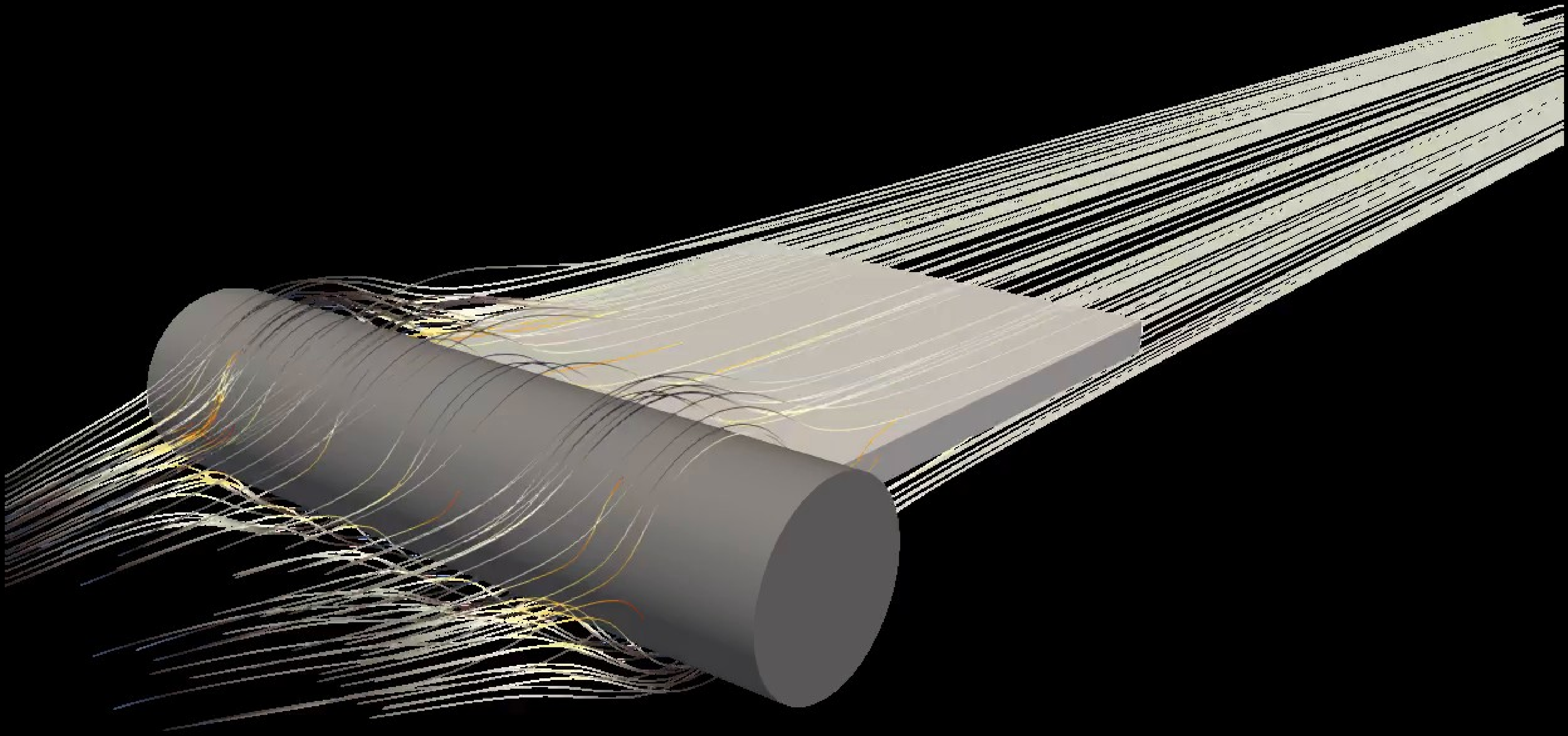
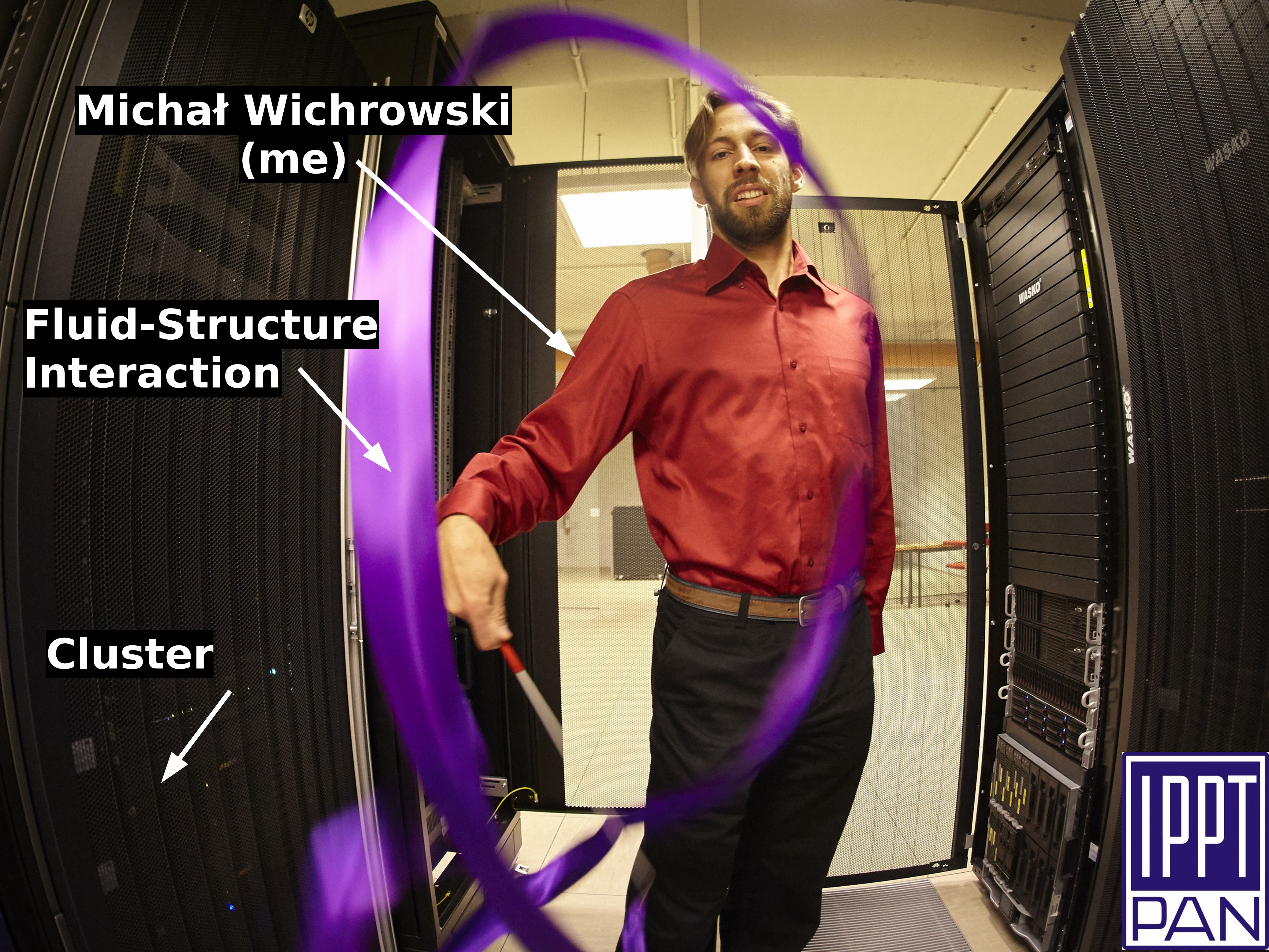


Fluid-structure interaction

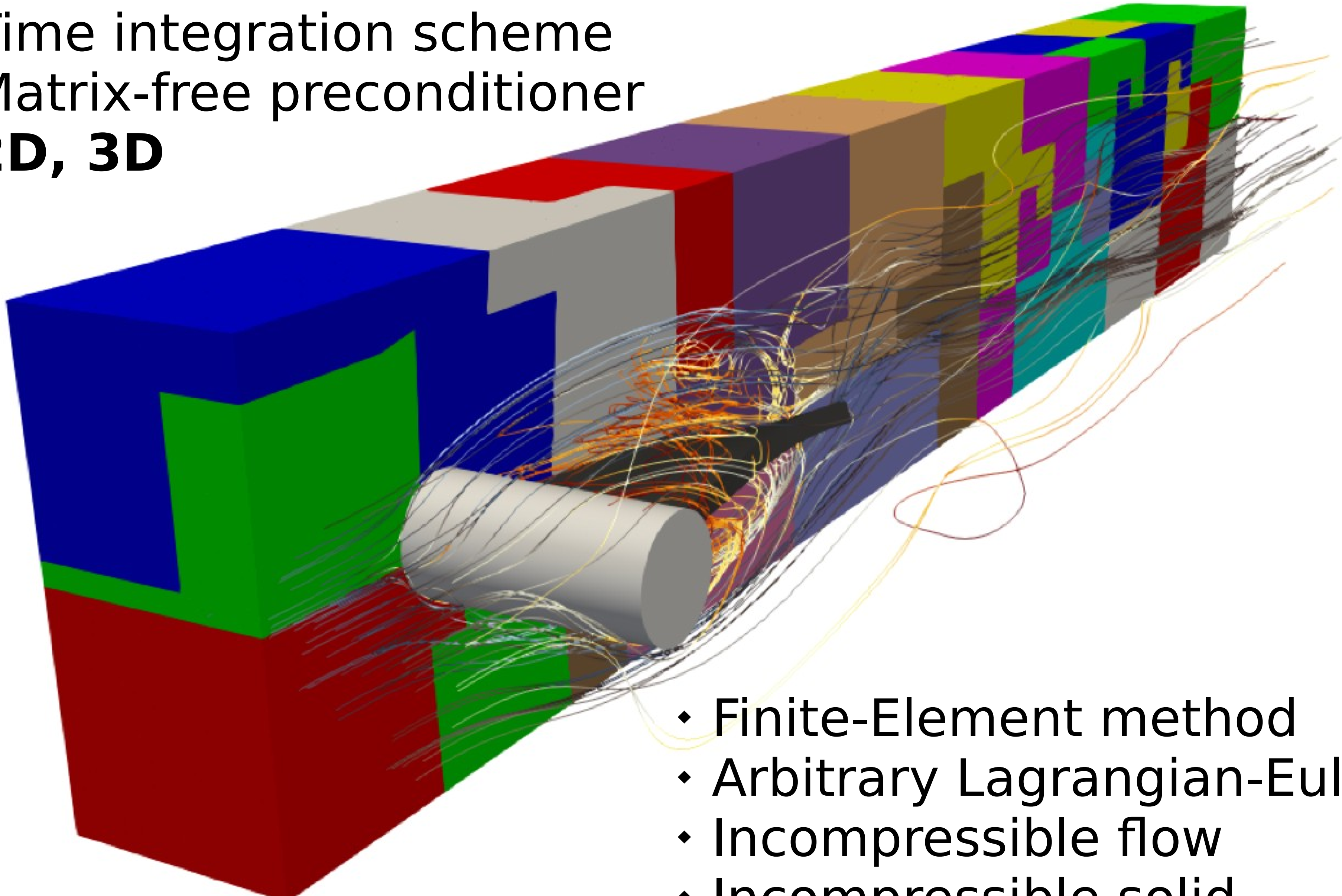
monolithic, matrix-free

Michał Wichrowski
S. Stupkiewicz P. Krzyżanowski L. Heltai


Michał Wichrowski
(me)
Fluid-Structure
Interaction
Cluster
WASKO
IPPT
PAN

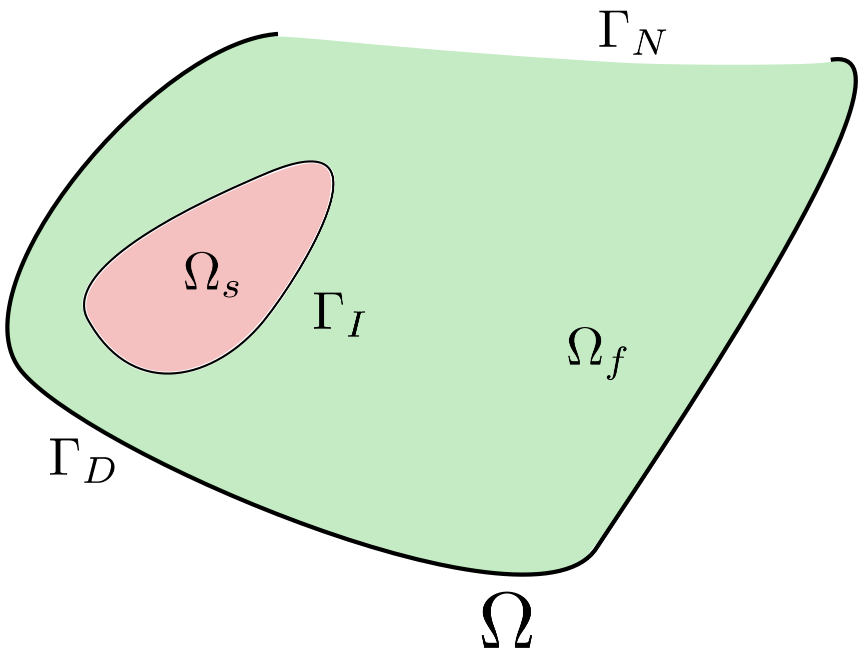
Goal: parallel monolithic matrix-free solver for FSI problems

- Time integration scheme
- Matrix-free preconditioner
- **2D, 3D**



- Finite-Element method
- Arbitrary Lagrangian-Eulerian
- Incompressible flow
- Incompressible solid

Problem formulation



Interface conditions

$$\left.\begin{aligned}\sigma_f n_f + \sigma_s n_s &= 0\\ v_f &= v_s\end{aligned}\right\}\ \text{on } \Gamma_I$$

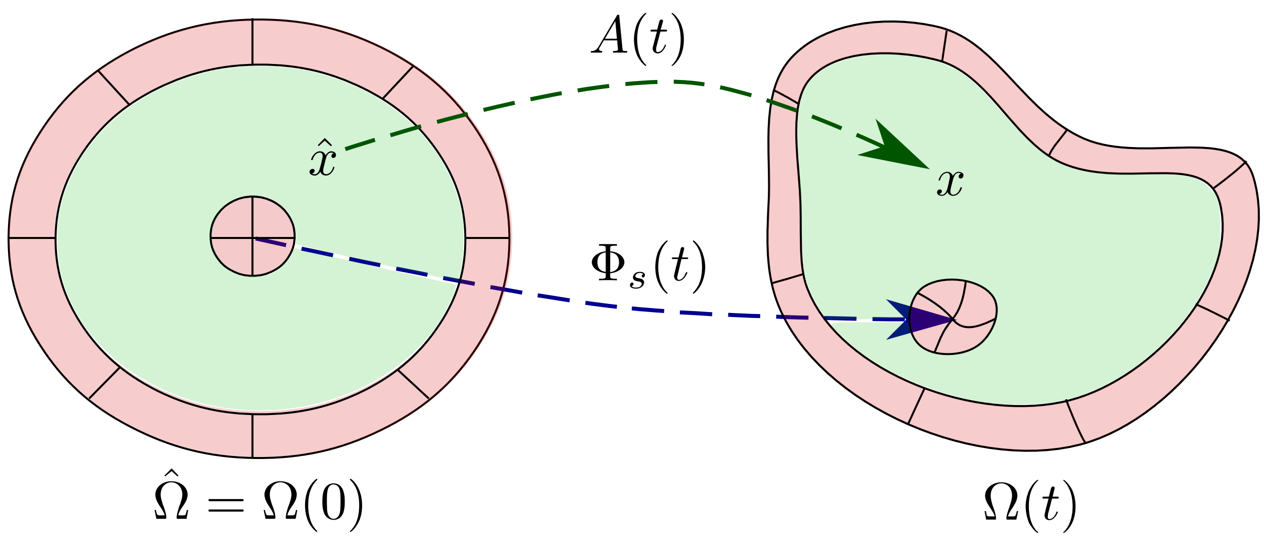
$$v(x) = \begin{cases} v_s(x) & x \in \Omega_s \\ v_f(x) & x \in \Omega_f. \end{cases}$$

$$\int_{\Omega_f} \rho_f \left(\frac{\partial v}{\partial t} + \nabla v\, v\right) \cdot \phi\, \partial x + \int_{\Omega_f} \sigma_f : \nabla \phi\, \partial x - \int_{(\partial\Omega_f \cap \Gamma_N) \cup \Gamma_i} \sigma_f n_f \cdot \phi\, \partial s = \int_{\Omega_f} g \cdot \phi\, \partial x \qquad \forall \phi \in H^1_D(\Omega),$$

$$\int_{\Omega_s} \rho_s \left(\frac{\partial v}{\partial t} + \nabla v\, v\right) \cdot \phi\, \partial x + \int_{\Omega_s} \sigma_s : \nabla \phi\, \partial x - \int_{(\partial\Omega_s \cap \Gamma_N) \cup \Gamma_i} \sigma_s n_s \cdot \phi\, ds = \int_{\Omega_s} g \cdot \phi\, \partial x \qquad \forall \phi \in H^1_D(\Omega),$$

Momentum balance (weak form)

Arbitrary Lagrangian-Eulerian



$$A(t;\hat{x}) = \hat{u}_A(t,\hat{x}) + \hat{x}$$
$$\hat{u}_A = \mathrm{Ext}(\hat{u}_s)$$

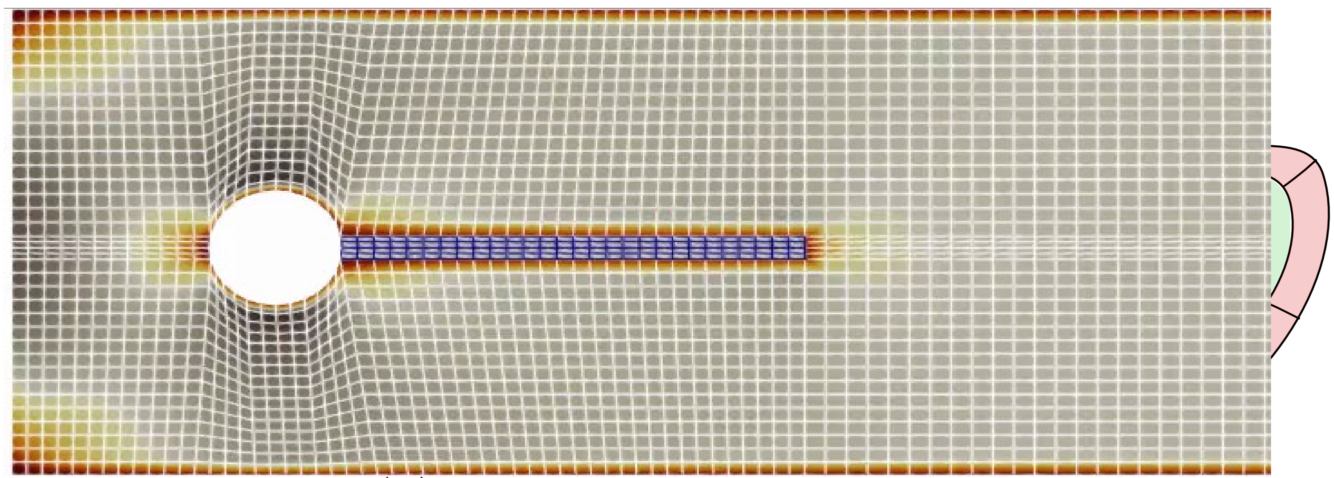
$$\hat{v}(t,\hat{x}) = v(t, A(t;\ \hat{x}))$$
$$v(t,x) = \hat{v}(t, A^{-1}(t;\ x))$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} v(t, x(t,\hat{x})) &= \frac{\partial v}{\partial t}(t, x(t,\hat{x})) + \nabla v(t, x(t,\hat{x}))\ \frac{\partial x}{\partial t}(t,\hat{x}) \\ &= \frac{\partial v}{\partial t}(t, x(t,\hat{x})) + \nabla v(t, x(t,\hat{x}))\ \hat{v}_A(t,\hat{x}). \end{aligned}$$

$$\int_\Omega \rho \left(\frac{\mathrm{d}v}{\mathrm{d}t} + \nabla v\ (v - v_A) \right) \cdot \phi\ \mathrm{d}x + \int_{\Omega_f} \sigma_f : \nabla\phi\ \mathrm{d}x + \int_{\Omega_s} \sigma_s : \nabla\phi\ \mathrm{d}x = \int_\Omega g \cdot \phi\ dx + \int_{\Gamma_N} \tau^* \cdot \phi\ \mathrm{d}s \qquad \forall \phi \in H^1_D(\Omega).$$

$\hat{v}_A = \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_A$ **Momentum balance (weak form), ALE**

Arbitrary Lagrangian-Eulerian



$$A(t;\hat{x}) = \hat{u}_A(t,\hat{x}) + \hat{x}$$
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$$\int_\Omega \rho\left(\frac{\mathrm{d}v}{\mathrm{d}t} + \nabla v\ (v - v_A)\right)\cdot\phi\ \mathrm{d}x + \int_{\Omega_f}\sigma_f : \nabla\phi\ \mathrm{d}x + \int_{\Omega_s}\sigma_s : \nabla\phi\ \mathrm{d}x = \int_\Omega g\cdot\phi\ dx + \int_{\Gamma_N}\tau^*\cdot\phi\ \mathrm{d}s \qquad \forall\phi\in H^1_D(\Omega).$$

$\hat{v}_A = \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_A$ **Momentum balance (weak form), ALE**

Material models

Newtonian fluid

$$\sigma_f = 2\mu_f \epsilon(v_f) + p_f I$$

$$\nabla \cdot (\rho_s v_s) = 0$$

Symmetric gradient

$$\epsilon(v) = \tfrac{1}{2}(\nabla v + \nabla v^T)$$

Mooney-Rivlin solid

$$\hat{F} = \hat{\nabla}\Phi_s = I + \hat{\nabla}\hat{u}_s$$

$$F = \hat{F} \circ \Phi_s^{-1}$$

$$B = FF^T$$

$$p_s = p_s^* - \mu_1 + \mu_2$$

$$\sigma = \cancel{\mu_1 B} - \mu_2 B^{-1} + p_s^* I$$

$$\boxed{\sigma = \mu_1 FF^T + \mu_2 \big(2\epsilon(u_s) - (\nabla u_s)^T \nabla u_s - I\big) + p_s^* I}$$

Challenge: incompressibility

$$\nabla \cdot (\rho_s v_s) = -\frac{\partial \rho_s}{\partial t} \Rightarrow \boxed{\nabla \cdot (\rho_s v_s) = -\frac{1}{\eta_V}(\det(\hat{F}) - 1)}$$

$$\frac{\partial \rho_s}{\partial t} = -\frac{1}{\eta_V}\left(\frac{\rho_s}{\rho_{s\,0}} - 1\right)$$

$$\frac{\rho_s}{\rho_{s\,0}} = \det(\hat{F})$$

Time integration scheme

Generalization of Geometry Convective-Explicit Scheme [Xu & Yang 2014] [Murea & Soyibou 2017]

- Implicit
- Explicit
- Semi-implicit
- Known

$$\begin{cases} a_i(v^n,\phi) + b(\phi,p) &= g(\phi) \quad \forall \phi \in H^1_D, \\ b(v^n,q) &= -\left(\frac{1}{\eta_V}(\det(\hat{F})-1),\hat{q}\right)_{\hat{\Omega}_s} \quad \forall q \in L^2(\Omega), \\ \delta_k \hat{u}^n_s &= \hat{v}^n_s, \\ \hat{u}^n_A &= \mathrm{Ext}(\hat{u}^n_s), \end{cases}$$

Convection: semi-implicit

$$v^\circ = v^n - v^n_A, \qquad v^\star = v^n.$$

BDF

$$\begin{aligned} a_i(v^n,\phi) &= (\rho\,\delta_k v^n + \rho\,\nabla v^\star\, v^\circ, \phi)_{\Omega^n} + (2\mu_f \epsilon(v^n) : \epsilon(\phi))_{\Omega^n_f} \\ &\quad + (2\mu_s \epsilon(u^n_s), \epsilon(\phi))_{\Omega^n_s} - \left(\mu_s (\nabla u^n_s)^T \nabla u^n_s, \epsilon(\phi)\right)_{\Omega^n_S}, \\ b(v^n,q) &= (\nabla \cdot v^n, q)_{\Omega^n}, \end{aligned}$$

"Velocity formulation":

$$\hat{u}^n_s = \gamma \Delta t \hat{v}^n_s - \sum_{i=1}^{k} \alpha_i \hat{u}^{n-i}_s$$

BDF, <u>Second-order</u>

(improvement)

Fixed-point method

Data: $\hat{u}_A^{n-1}, v^{n-1}, \hat{v}_{\text{Ext}}^{n-1}, \hat{u}^{n-1}$
Result: $\hat{u}_A^{n}, v^{n}, \hat{v}_{\text{Ext}}^{n}, \hat{u}^{n}$
begin

$\hat{v}_{\text{Ext}}^{\square} := \hat{v}_{\text{Ext}}^{n-1}$ **for** $j = 1$ **to** 2 **do**

$\hat{u}_A^{\#} := \gamma \Delta t \hat{v}_{\text{Ext}}^{n} - \sum_{i=1}^{k} \alpha_i \hat{u}^{n-k}$ ◁ `Explicit step`

$\hat{v}_A^{\square} := \hat{v}_{\text{Ext}}^{\square}$ $A^{\#} = \text{Id} + \hat{u}_A^{\#}, \qquad \Omega^{\#} = A^{\#}(\hat{\Omega})$ ◁ `New geometry`

Find $v^{\square} \in H^1(\Omega^{\#})$ and $p^n \in L_2(\Omega)$ ◁ `Implicit step`

$$\begin{cases} a_v(v^{\square}, \phi) + b(\phi, p^n) & = g_v(\phi) \quad \forall \phi \in H_D^1(\Omega^{\#}), \\ b(v^{\square}, q) & = g_p(q) \quad \forall q \in L^2(\Omega^{\#}). \end{cases}$$

$\hat{v}_{\text{Ext}}^{\square} := \text{Ext}(\hat{v}^{\square})$ ◁ `Extension`

end

$v^n := v^{\square} \qquad \hat{v}_{\text{Ext}}^{n} := \hat{v}_{\text{Ext}}^{\square}$

$\hat{u}^n := \gamma \Delta t \hat{v}_{\text{Ext}}^{n} - \sum_{i=1}^{k} \alpha_i \hat{u}^{n-k}$ ◁ `Recover displacement`

end

Predictor-corrector scheme

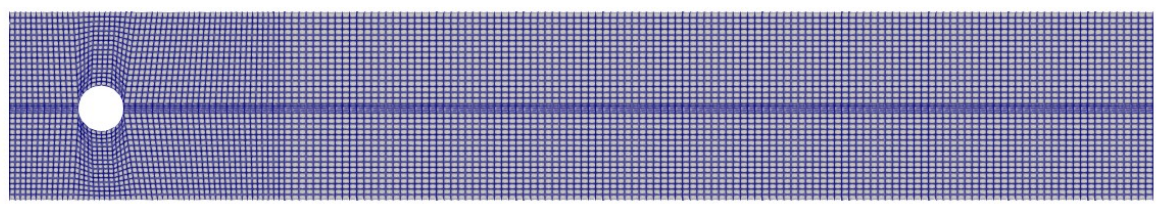
Spatial discretization

Finite-Element discretization Q2-Q1, SUPG-like stabilization

$$\begin{cases} a(v,\phi) + b(\phi,p) = g_v(\phi) \\ b(v,q) = g_p(q) \end{cases} \quad \Rightarrow \quad \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

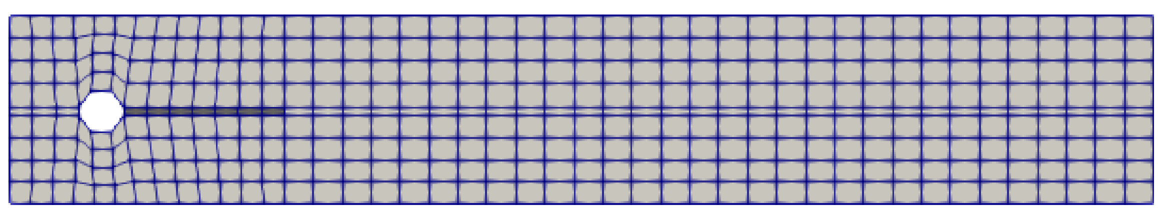
Linear equations

$$\mathcal{M} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$$



$\mathcal{T}$

$$\mathcal{M}_j = \begin{bmatrix} A_j & B_j^T \\ B_j & 0 \end{bmatrix}$$



$\mathcal{T}_j$

Linear solver – idea

Theory

[Braess, Sarazin, 1999]
[Zulehner 2000]

[Xu &Zhu 2008]

Chebyshev smoothers

Mutlilevel method

Implementation

[Kronbichler Kormann, 2012]

deal.II

Parallel, matrix-free

Proposed method:
MG+GMRes

Multigrid

Function $y = \mathrm{MGM}(\mathcal{M}_j, F_j, \mathcal{K}_j, m, x, j)$

  if $j = 0$ **then**

    Solve $\mathcal{M}_0 y = F_0$ ◁ `Direct solve on the coarsest grid` $\mathcal{T}_0$

    return y

  end

  $x^0 = x$ **for** $s = 1$ **to** m **do**

    $x^s = x^{s-1} + \mathcal{K}_j(F_j - \mathcal{M}_j x^{s-1})$ ◁ `pre-smoothing`

  end

  $r_{j-1} = R_j\left(F_j - \mathcal{M}_j x^m\right)$ ◁ `restriction to the coarser grid`

  $e_{j-1} = \mathrm{MGM}(\mathcal{M}_{j-1}, r_{j-1}, \mathcal{K}_{j-1}, m, 0, j-1)$ ◁ `coarse correction; recursive call`

  $e_j = R_{j-1}^T e_{j-1}$ ◁ `prolongation from the coarser grid`

  $y^0 = x^m + e_j$ **for** $s = 1$ **to** m **do**

    $y^s = y^{s-1} + \mathcal{K}_j(F_j - \mathcal{M}_j y^{s-1})$ ◁ `post-smoothing`

  end

  return y^m

end

Smoother

Inverse of block "A"

$$\hat{A}$$

$$\hat{A}^{-1} = \mathrm{Cheb}(A, \mathrm{diag} A, n_A)$$

$$\mathcal{K}_j = \begin{bmatrix} \hat{A}_j & B_j^T \\ B_j & B_j \hat{A}_j^{-1} B_j^T - \hat{S}_j \end{bmatrix}^{-1}$$

[Zulehner 2000]

Challenging!

Inverse of block "S"

$$\tilde{S}_{n_s}^{-1} = \mathrm{Cheb}(S, \mathrm{diag} S, n_S)$$

$$\mathrm{MGCheb}(N, n_S) = \mathrm{MG}_N(S, \tilde{S}_{n_S}, m_S, j)$$

1. $\hat{S}^{-1} = \mathrm{MGCheb}(N_S, n_S)$
2. $\hat{S}^{-1} = \mathrm{MGCG}(N_S, n_S)$
3. $\hat{S}^{-1} = \mathrm{CGCheb}(N_S, n_S)$

Simple approximation

$$\hat{A}_j \approx A_j$$

$$\hat{S}_j \approx S_j = B_j \hat{A}_j^{-1} B_j^T$$

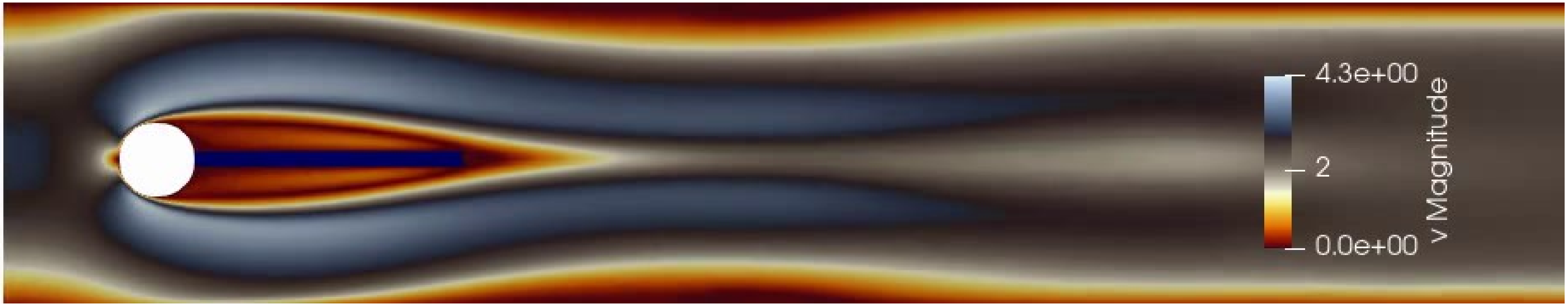
Challenging!

Results

grafia: https://habrastorage.org/getpro/geektimes/post_images/3e2/163/1cc/3e22631cc2bd3eb93fe8c8934d4570eb.png

SuperComputing Applications and Innovation

Turek benchmark



Turek benchmark FSI3

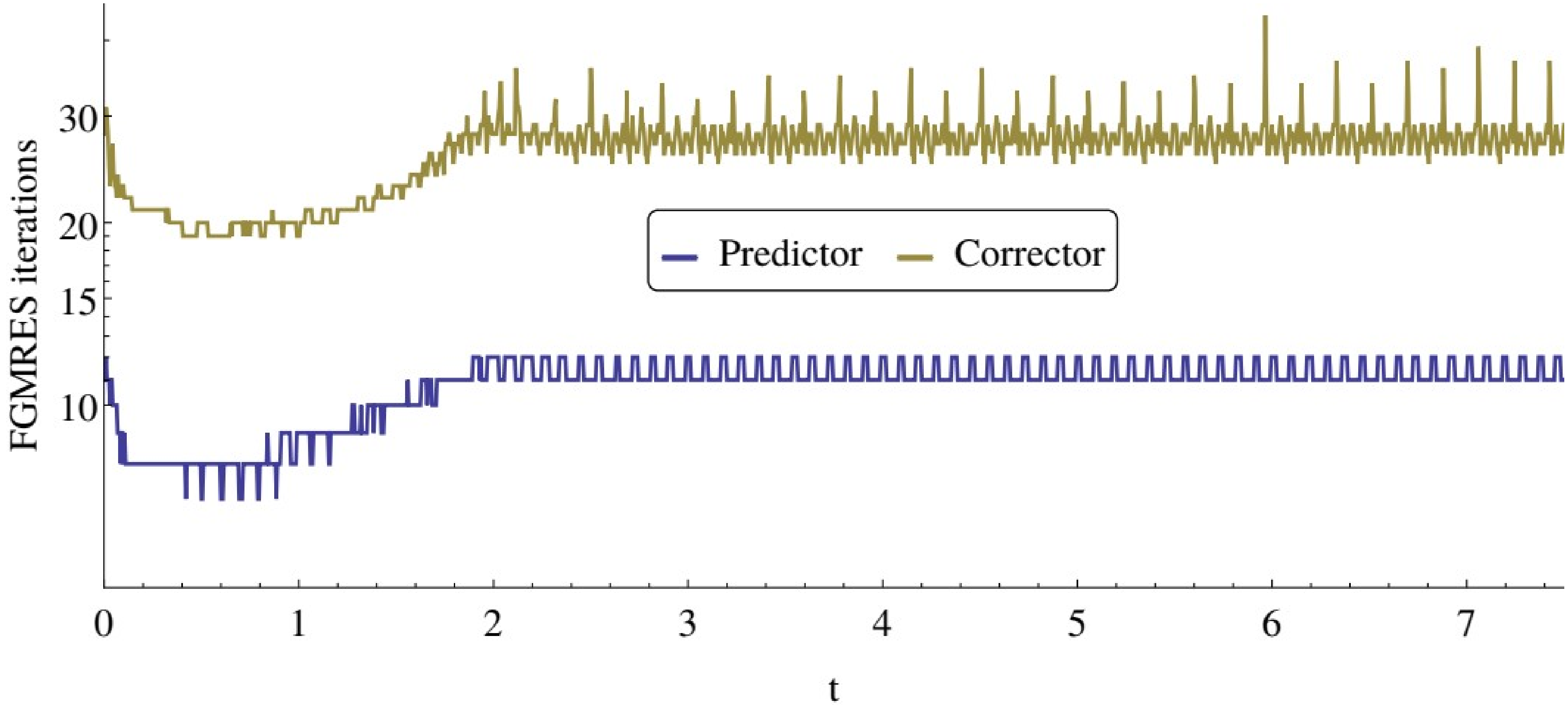
FSI2

	$u_x(\mathsf{A}) \times 10^{-3}$	$u_y(\mathsf{A}) \times 10^{-3}$	Frequency
$J = 5$, $\Delta t = 0.005$,	-14.85 ± 12.89	1.25 ± 80.8	2.00
$J = 5$, $\Delta t = 0.001$	-14.34 ± 12.19	1.10 ± 78.5	1.99
Reference	-14.58 ± 12.44	1.23 ± 80.6	2.0

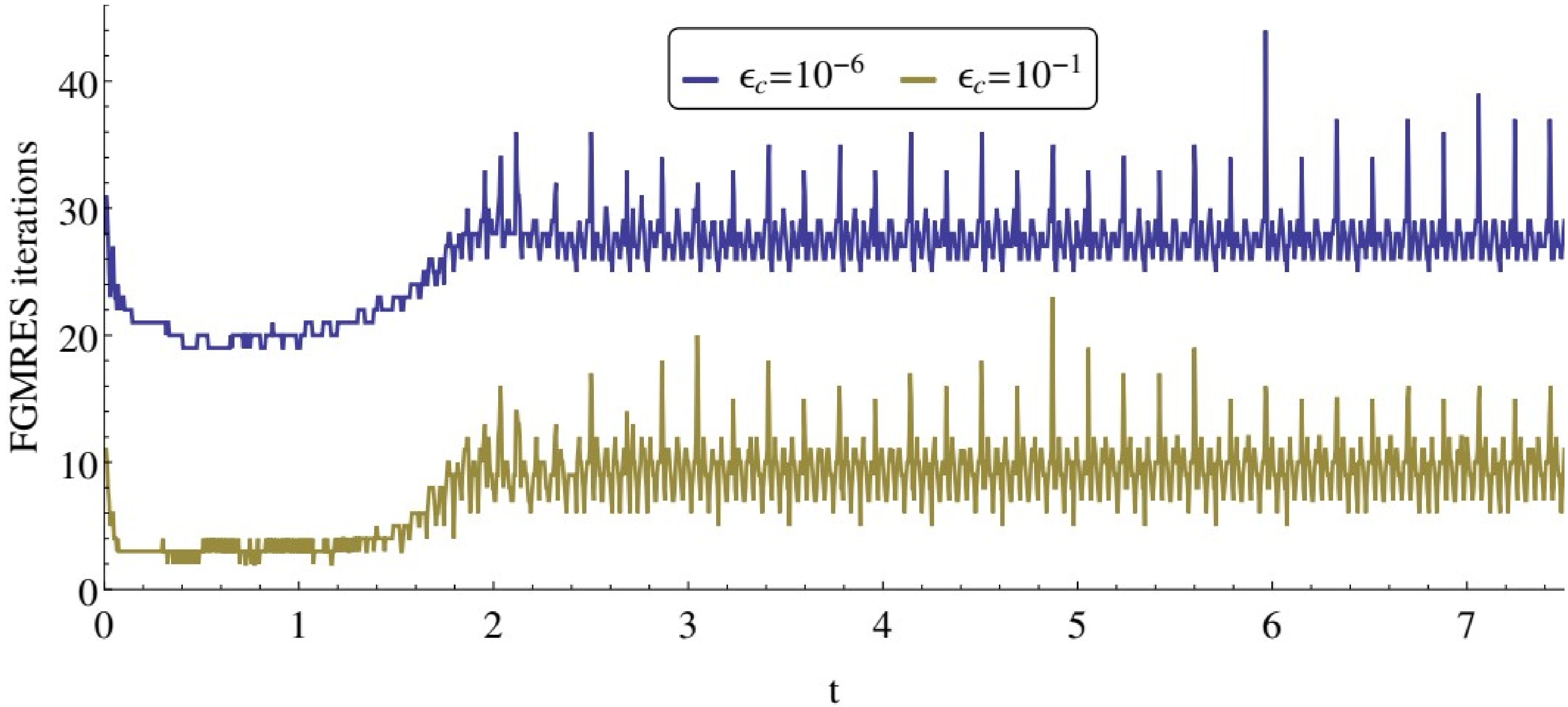
FSI3

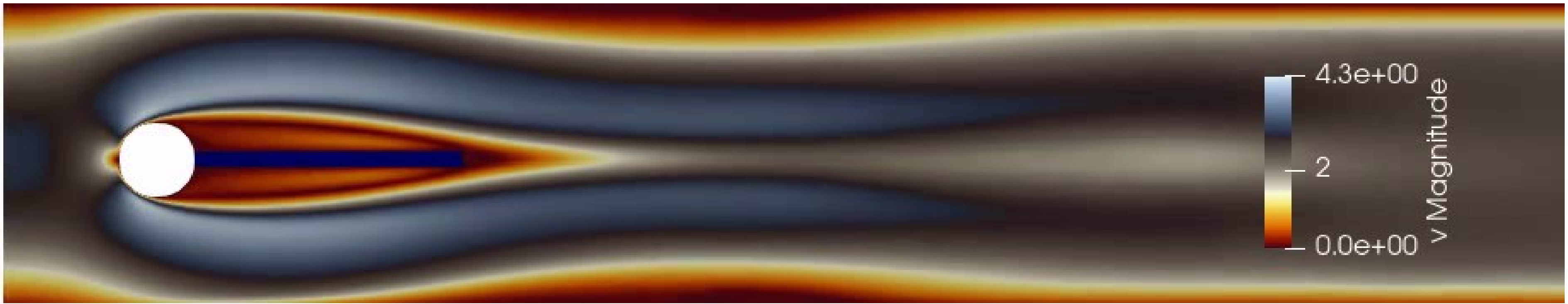
	$u_x(\mathsf{A}) \times 10^{-3}$	$u_y(\mathsf{A}) \times 10^{-3}$	Frequency
$J = 5$, $\Delta t = 0.005$	-2.79 ± 2.46	1.47 ± 34.38	5.40
$J = 5$, $\Delta t = 0.001$	-2.73 ± 2.57	1.55 ± 34.68	5.52
Reference	-2.69 ± 2.53	1.48 ± 34.38	5.3

FGMRES iterations - FSI3

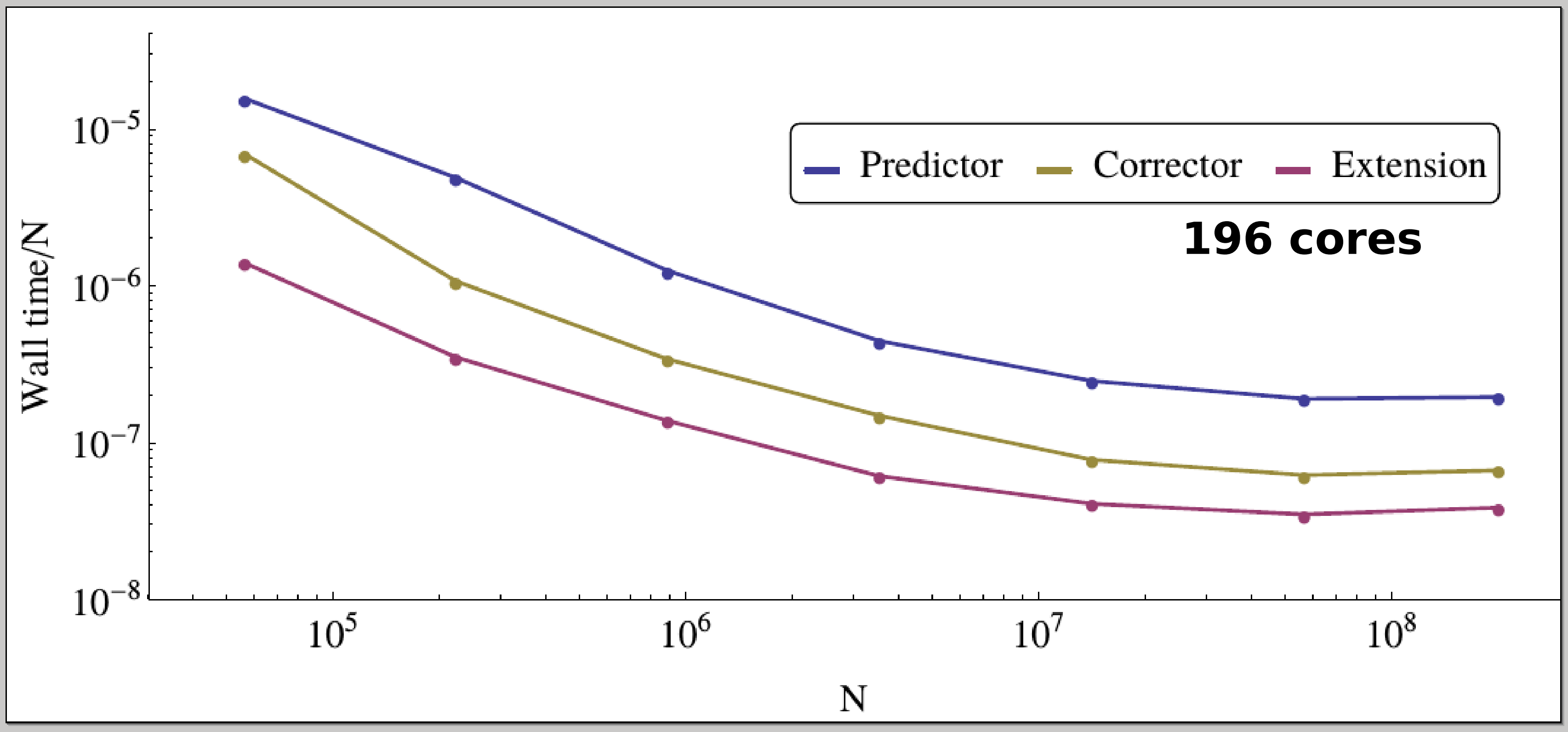


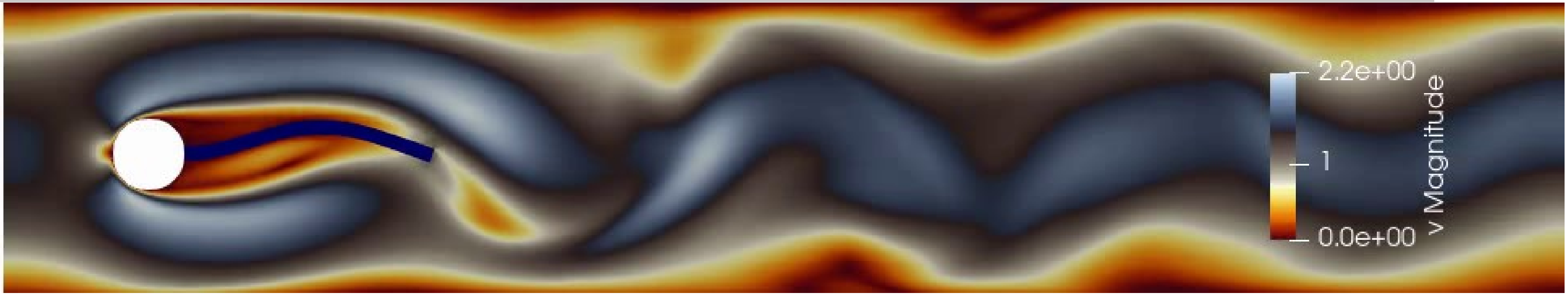
FGMRES iterations - FSI3



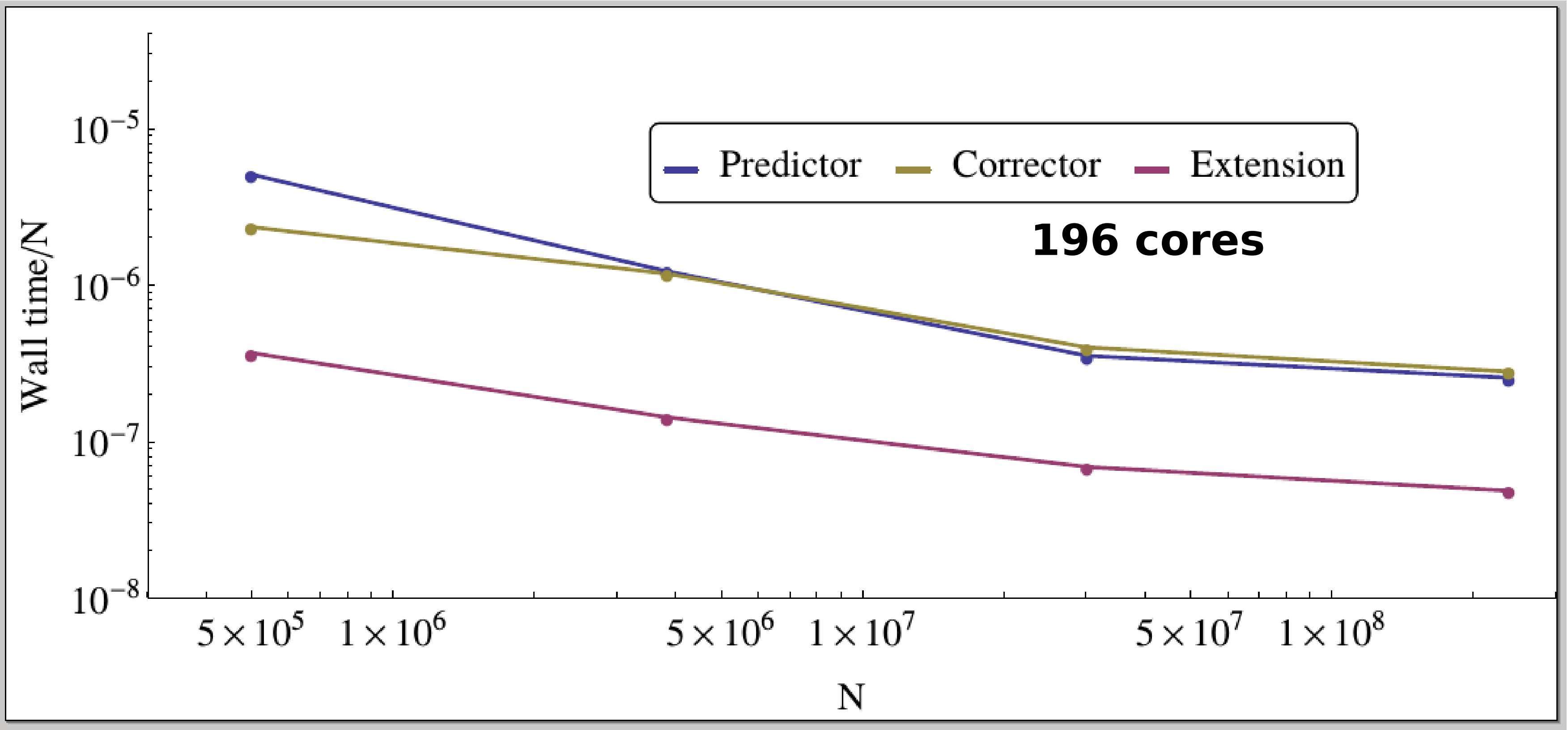


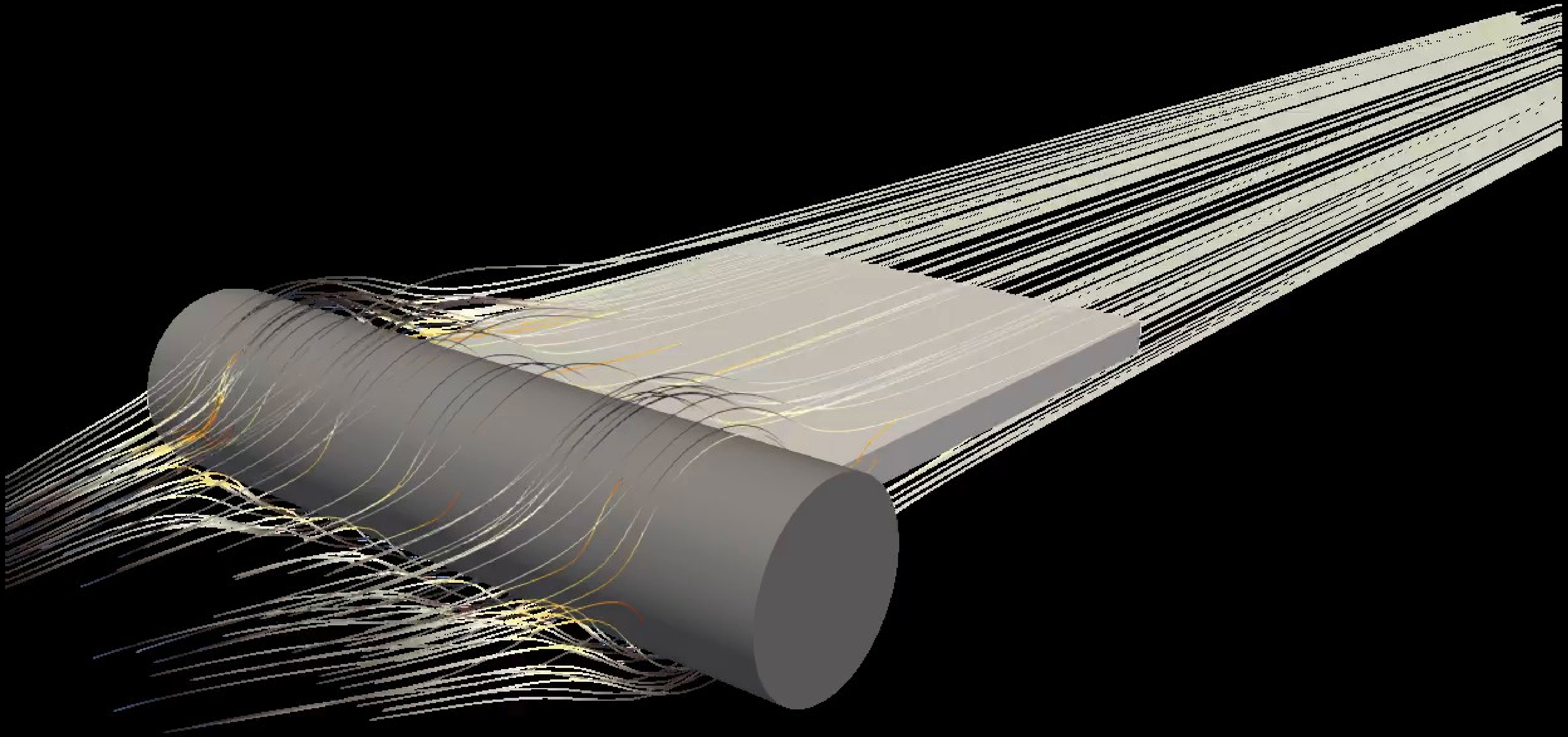
FGMRES performance – 2D





FGMRES performance – 3D





Re = 2000

Time-step size: 0.01

30M DoF

FGMRES iterations:

Predictor: 8

Corrector: 3-10

Original contributions:

- Generalization of GCE scheme: hyperelastic solid, 2^{nd} order
- New matrix-free preconditioner for the generalized Stokes with strongly variable coefficients
- Volume correction method for the solid
- Matrix-free implementation

FSI = Stokes problem with extra steps