On a Nonlocal Finite Element Model for Mode-III Brittle Fracture with Surface-Tension Excess Property

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Overview

Notation and preliminaries

- Let $\boldsymbol{X}$ be any arbitrary point in the reference configuration and $\mathbf{x}$ denotes the corresponding point in the deformed configuration.

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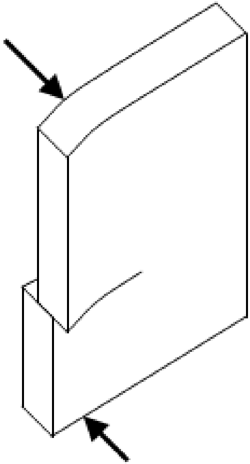
- $\epsilon = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)$
- If we linearize using the assumption of displacement gradients are small, we can approximate $\mathbf{E}$ and ϵ. Then there is no distinction between reference and deformed configuration.

Mode-III Fracture

The Mode-III fracture (or anti-plane shear fracture):

- The fracture surfaces slide relative to each other skew-symmetrically with shear stress acting as shown in the figure.
- Displacement:
  - $u_1 = 0$ and $u_2 = 0$
  - $u_3 = u_3(x_1, x_2)$



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- It predict singular crack-tip strains and stresses.
- Also it predicts an elliptical crack-surface opening displacement with a blunt crack-tip.

Thus, several remedies have been attempted: appealing to a non-linear theory of elasticity, the introduction of a cohesive zone around the crack tip and non-local theories.

Mode-III Brittle Fracture Problem Formulation

The problem studied here is the straight, static, anti-plane shear crack, lying on $|x_1| < a$, $x_2 = 0$ in an infinite, isotropic, linear elastic body subjected to uniform far-field anti-plane shear loading (σ_{23}^{∞}). The stress-strain relations are:

$$\tau_{23} = \mu \frac{\partial u_3}{\partial x_2} \quad \text{and} \quad \tau_{13} = \mu \frac{\partial u_3}{\partial x_1},$$

where τ_{23} and τ_{13} are the relevant stress components, and u_3 denotes the $z-$displacement.

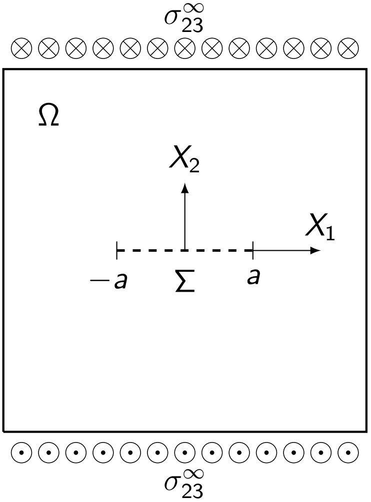

Figure: Physical description of the problem.

Mode-III fracture problem formulation

To derive the governing equations for this problem, we follow the study of Sendova and Walton[1].

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Then we consider a surface tension model which depend on (linearized) curvature of the out-of-plane displacement by:

$$\gamma = \gamma_0 + \gamma_1 u_{3,11}(x_1, 0),$$

where γ_0 and γ_1 are surface tension parameters.

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$$\gamma = \gamma_0 + \gamma_1 u_{3,11}(x_1, 0),$$

where γ_0 and γ_1 are surface tension parameters. Then the resulting boundary condition on the upper crack-surface is give by:

$$u_{3,2}(x_1, 0) = -\gamma_1 u_{3,111}(x_1, 0)$$

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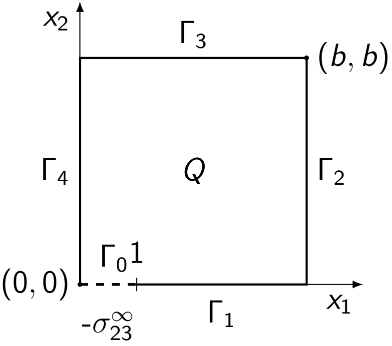

Figure: Finite computational domain Q.

$$-\Delta\, u_3(x_1, x_2) = 0, \quad \text{in} \quad Q$$

Boundary conditions:

$$\begin{aligned}
&\text{on} \quad \Gamma_0 \quad u_{3,2}(x_1, 0) = -\sigma_{23}^{\infty} - \gamma_1\, u_{3,111},\\
&\text{on} \quad \Gamma_1 \quad u_3 = 0,\\
&\text{on} \quad \Gamma_2 \quad \vec{n} \cdot \nabla u_3 = 0,\\
&\text{on} \quad \Gamma_3 \quad \vec{n} \cdot \nabla u_3 = 0,\\
&\text{on} \quad \Gamma_4 \quad \vec{n} \cdot \nabla u_3 = 0.
\end{aligned}$$

Weak Formulation and Numerical Strategy

Appealing to the BVP, the weak formulation for the problem on hand is found by integrating the PDE against a test function v over Ω. This yields

$$\int_Q \nabla v \cdot \nabla u_3 \, dQ - \int_{\partial Q} v \left(\vec{n} \cdot \nabla u_3\right) d\partial Q = 0.$$

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$$\int_Q \nabla u_3 \cdot \nabla v \, dQ - \int_{\Gamma_0} v \; u_{3,2}(x_1, 0) \, dx_1 = 0 \; ,$$

Reformulation of the Crack-Face Boundary Condition

We consider the crack-surface boundary condition and rearrange the equation to obtain

$$-u_{3,111}(x_1, 0) = \frac{1}{\gamma_1}\left[u_{3,2}(x_1, 0) + \sigma_{23}^{\infty}\right] \quad \text{on} \quad \Gamma_0 \; .$$

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- $u_3(x_1, x_2)$ is an odd function in x_2, therefore $u_{3,1}(0, 0) = 0$.
- Also regularization on $\Gamma_0 \cup \Gamma_1$ requires $u_{3,1}(1, 0) = 0$.

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Then the solution to the two point boundary value problem is given by

$$u_{3,1}(x,0) = \mathcal{G}\{f\}(x) := \int_0^1 G(x,q) f(q) dq.$$

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$$\begin{aligned} u_{3,1}(x,0) &= \frac{1}{\gamma_1} \int_0^1 G(x,q) [u_{3,2}(q,0) + \sigma_{23}^{\infty}]\, dq \\ &= \frac{1}{\gamma_1} \int_0^1 G(x,q)\, u_{3,2}(q,0)\, dq - \frac{\sigma_{23}^{\infty}}{2\gamma_1}\, x(1-x). \end{aligned}$$

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Now, we know that the Hilbert transform gives the Dirichlet-to-Neumann map, ie

$$\begin{aligned} u_{3,2}(x,0^+) &= \mathcal{H}\{u_{3,1}\} \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} u_{3,1}(q,0^+) \, \frac{dq}{q-x} \\ &= \frac{1}{\pi} \int_0^1 u_{3,1}(q,0^+) \, \frac{2q}{q^2 - x^2} \, dq, \end{aligned}$$

Reformulation of the Crack-Face Boundary Condition

$$u_{3,2}(x,0) = \frac{1}{\pi\gamma_1}\int_0^1 k(x,q)\, u_{3,2}(q,0)\, dq - \frac{\sigma_{23}^{\infty}}{2\pi\gamma_1}\, g(x), \quad \text{on} \quad \Gamma_0,$$

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where $k(x,q)$ and $g(x)$ are given by:

$$\begin{aligned} k(x,q) =& (q+x)\ln(q+x) + (q-x)\ln|q-x| \\ & - q(1+x)\ln(1+x) - q(1-x)\ln|1-x| \end{aligned}$$

$$g(x) = 1 - x(1+x)\ln\left(\frac{1+x}{x}\right) + x(1-x)\ln\left|\frac{1-x}{x}\right|$$

Reformulation of the Crack-Face Boundary Condition

Applying this result to the earlier *Weak-Form* yields the final weak form

$$\int_Q \nabla u_3 \cdot \nabla v + \frac{1}{\pi\gamma_1}\int_0^1 v(x,0)\int_0^1 k(x,q)\, u_{3,2}(q,0)\, dq\, dx$$
$$= \frac{\sigma_{23}^{\infty}}{2\pi\gamma_1}\int_0^1 v(x,0)\, g(x)\, dx.$$

Note that this weak form has no higher-order derivatives, thus the standard FEM can now be applied.

Parameter Determination

Theorem

The Fredholm integral equation

$$\gamma_1 u(x) - \mathcal{K}[u](x) = -\frac{\sigma_{23}^{\infty}}{2\pi} g(x), \quad \textit{for} \quad 0 \leq x \leq 1,$$

where $\mathcal{K}$ *is the integral operator*

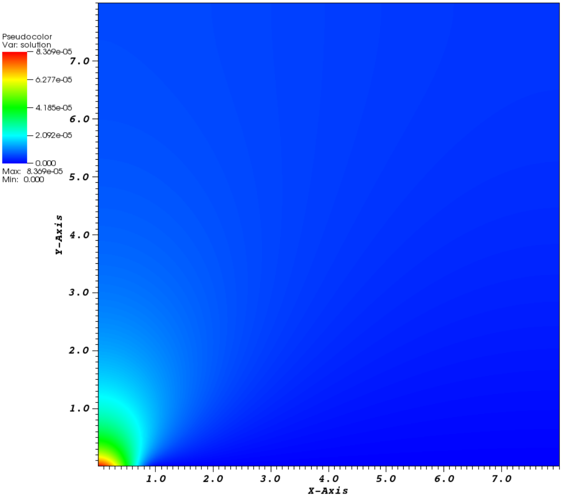
$$\mathcal{K}[\psi](x) = \frac{1}{\pi} \int_0^1 k(x, q)\, \psi(q)\, dq,$$

has a unique, continuous solution for all but countably many values of γ_1.

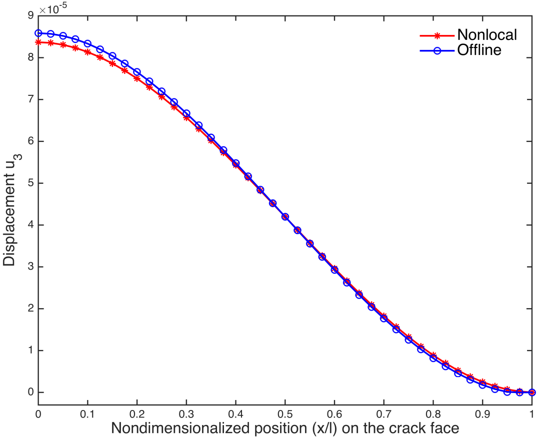
Where $k(x, q)$ is given by:

$$\begin{aligned} k(x, q) = & (q + x) \ln (q + x) + (q - x) \ln |q - x| \\ & - q(1 + x) \ln (1 + x) - q(1 - x) \ln |1 - x| \end{aligned}$$

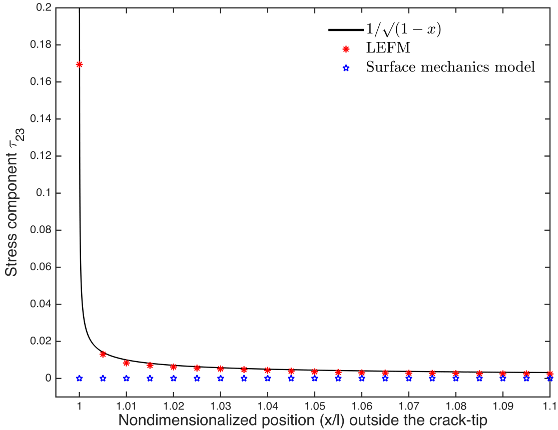
Numerical Results: Displacement



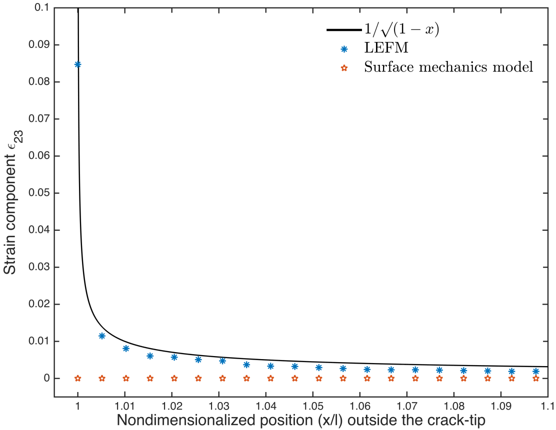
Numerical Results: Crack-Face Displacement



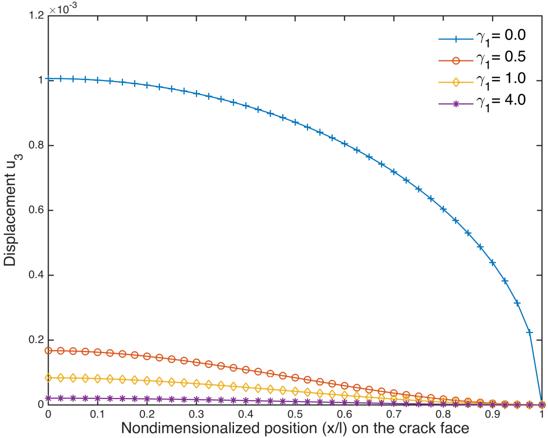
Numerical Results: Near-Tip Stress τ_{23}



Numerical Results: Near-Tip Strain ϵ_{23}



Numerical Results: Crack-Face Displacement



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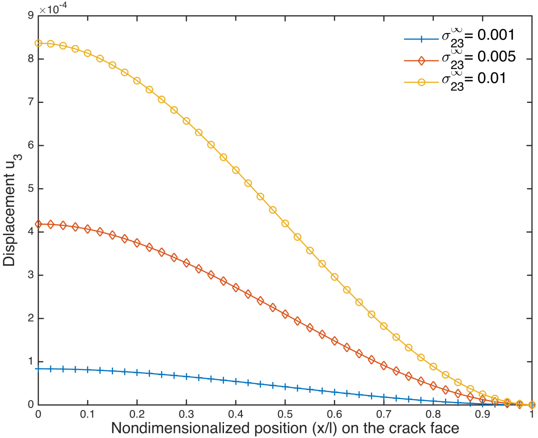

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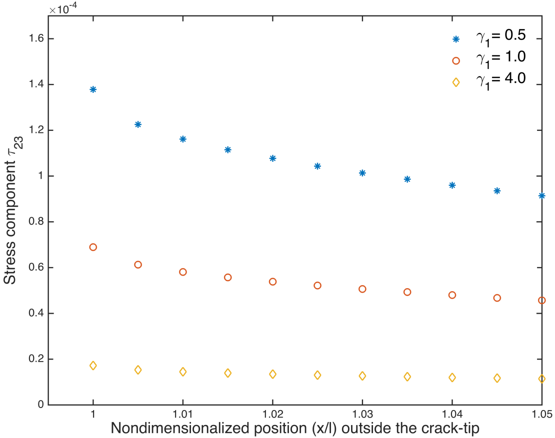

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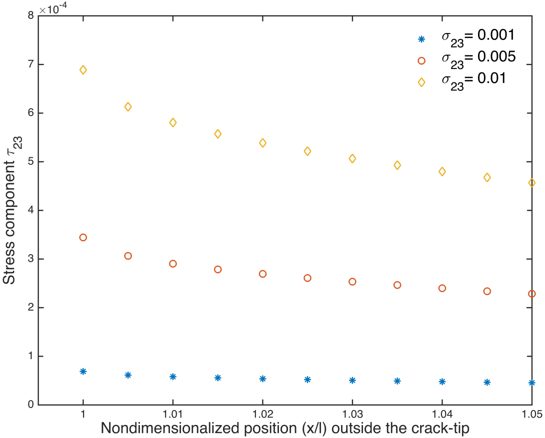

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- We showed that the two FEM implementations agree well with each other. In particular, the model predicts bounded crack-tip stresses (also strains) and a cusp-like crack opening profile with a sharp crack-tip.
- We are currently developing a corresponding implementation of both pure mode-I and mixed-mode (combination of mode-I and mode-II) fracture.

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THANK YOU