PARTITION IDENTITIES ARISING FROM THE STANDARD $A_2^{(2)}$-MODULES OF LEVEL 4

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ABSTRACT OF THE DISSERTATION

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In this dissertation, we propose a set of new partition identities, arising from a twisted vertex operator construction of the level 4 standard modules for the affine Kac-Moody algebra of type $A_2^{(2)}$. These identities have an interesting new feature, absent from previously known examples of this type.

This work is a continuation of a long line of research of constructing standard modules for affine Kac-Moody algebras via vertex operators, and the associated combinatorial identities. The interplay between representation theory and combinatorial identities was exemplified by the vertex-algebraic proof of the famous Rogers-Ramanujan-type identities using standard $A_1^{(1)}$-modules by J. Lepowsky and R. Wilson. In his Ph.D. thesis, S. Capparelli proposed new combinatorial identities using a twisted vertex operator construction of the standard $A_2^{(2)}$-modules of level 3, which were later proved independently by G. Andrews, S. Capparelli, and M. Tamba-C. Xie.

We begin with an obvious spanning set for each of the level 4 standard modules for $A_2^{(2)}$, and reduce this spanning set using various relations. Most of these relations come from certain product generating function identities which are valid for all the level 4 modules. There are also other ad-hoc relations specific to a particular module of level 4. In this way, we reduce our spanning sets to match with the graded dimensions of the said modules as closely as possible. We conjecture and present strong evidence for three partition identities based on the spanning sets for the three standard $A_2^{(2)}$-modules of level 4.

One surprising result of our work is the discovery of relations of arbitrary length. Consequently, the partitions corresponding to these spanning sets cannot be described by difference conditions of finite length.

The spanning set result proves one inequality of the proposed identities. There is strong evidence for the validity of the conjecture (i.e., the opposite inequality), since it has been verified to hold for partitions of $n \leq 170$, and $n = 180, 190$ and 200.

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Chapter 1

Introduction

In this work, we conjecture and present strong evidence for possible partition identities arising from the standard modules of level 4 for the affine Lie algebra $\mathrm{A}_2^{(2)}$ using a twisted vertex operator construction. Historically, the discovery of vertex operator constructions of representations of affine Lie algebras was motivated by a conjectured interplay between classical partition identities and standard modules for affine Kac-Moody Lie algebras.

The first famous example of such interplay arises from the Rogers-Ramanujan identities, which may be stated as follows:

(i) *The number of partitions of a nonnegative integer n in which the difference between any two successive parts is at least 2 is the same as the number of partitions of n into parts congruent to 1 or 4 modulo 5.*

(ii) *The number of partitions of a nonnegative integer n in which the difference between any two successive parts is at least 2 and such that the smallest part is at least 2 is the same as the number of partitions of n into parts congruent to 2 or 3 modulo 5.*

A connection between the congruence conditions and standard modules for $\mathrm{A}_1^{(1)}$ was discovered by J. Lepowsky and S. Milne [LM78]. A vertex operator theoretic interpretation and proof of the Rogers-Ramanujan identities, "explaining" the difference conditions, was given by Lepowsky and R. Wilson in [LW82, LW84]. They used monomials, acting on a highest weight vector, in certain new operators whose indices reflected the difference conditions. They extended their work to all the standard $\mathrm{A}_1^{(1)}$-modules in [LW82, LW84, LW85], giving a vertex-algebraic interpretation of a family of Rogers-Ramanujan-type identities, discovered by B. Gordon, G. Andrews and D. Bressoud. The case for the level 2 standard $A_1^{(1)}$-modules was described by certain "difference-one" conditions (where adjacent parts have difference at least one). For level 3 it was described by "difference-two" conditions (where adjacent parts have difference at least two). For levels greater than 3, the description changed into "difference-two-at-a-distance" and parity conditions, reflecting the sum sides of the Gordon-Andrews-Bressoud identities.

The linear independence of the relevant monomials (applied to a highest weight vector) for standard $A_1^{(1)}$-modules of level greater than 3 was not proved in the sequel [LW82, LW84, LW85]. This problem was solved by A. Meurman and M. Primc [MP87], providing a vertex-algebraic proof of the Gordon-Andrews-Bressoud identities beyond the case of Rogers-Ramanujan identities.

In his Ph.D. thesis [Cap88], S. Capparelli proposed a pair of combinatorial identities based on the standard $A_2^{(2)}$-modules of level 3. He also demonstrated that the construction of the level 2 standard modules for $A_2^{(2)}$ in this way gives rise to another vertex operator theoretic interpretation of the classical Rogers-Ramanujan identities (see also [Cap92, Cap93]). It was believed that once a few low level cases for standard $A_2^{(2)}$-modules had been successfully analyzed in this way, a general construction for all levels would emerge. However, the cases for $A_2^{(2)}$ turned out to be much harder and subtler than those for $A_1^{(1)}$ which had been extensively studied. One of Capparelli's identities, arising from the level 3 standard $A_2^{(2)}$-modules, may be stated as follows:

> *The number of partitions of a nonnegative integer n into parts different from 1 and such that the difference of two successive parts is at least 2, and is exactly 2 or 3 only if their sum is a multiple of 3, is the same as the number of partitions of n into parts congruent to $\pm 2, \pm 3$ modulo 12.*

A q-series proof of this identity was given by G. Andrews [And94], proving Capparelli's conjecture. Capparelli also provided a direct vertex operator theoretic proof of his identities by proving the linear independence of his spanning sets in [Cap96]. M. Tamba and C. Xie [TX95] independently gave another vertex operator theoretic proof of Capparelli's identities. See [Lep07] for more details.

In this work, we give combinatorial interpretations of the graded dimensions of the three inequivalent standard $A_2^{(2)}$-modules of level 4. The level 4 case turns out to be much more difficult and subtle even compared to the level 3 case, showing even more surprising results.

A partition can be thought of as a non-increasing sequence of positive integers. A partition $(m_1, \ldots, m_s)$ is said to satisfy a difference condition $[d_1, \ldots, d_{s-1}]$ if $m_i - m_{i+1} = d_i$ for all $1 \leq i \leq s-1$. The partition identities we propose, based on the three inequivalent standard $A_2^{(2)}$-modules of level 4, may be stated as follows:

(i) *The number of partitions of a nonnegative integer n into parts different from 1 and such that there is no sub-partition satisfying the difference conditions $[1]$, $[0,0]$, $[0,2]$, $[2,0]$ or $[0,3]$, and such that there is no sub-partition with an odd sum of parts satisfying the difference conditions $[3,0]$, $[0,4]$, $[4,0]$ or $[3,2^*,3,0]$ (where 2^* indicates zero or more occurrence of 2), is the same as the number of partitions of n into parts congruent to $\pm 2, \pm 3$ or ± 4 modulo 14.*

(ii) *The number of partitions of a nonnegative integer n such that 1, 2 and 3 may occur at most once as a part, and such that there is no sub-partition satisfying the difference conditions $[1]$, $[0,0]$, $[0,2]$, $[2,0]$ or $[0,3]$, and such that there is no sub-partition with an odd sum of parts satisfying the difference conditions $[3,0]$, $[0,4]$, $[4,0]$ or $[3,2^*,3,0]$ (where 2^* indicates zero or more occurrence of 2), is the same as the number of partitions of n into parts congruent to $\pm 1, \pm 4$ or ± 6 modulo 14.*

(iii) *The number of partitions of a nonnegative integer n into parts different from 1 and 3, such that 2 may occur at most once as a part, and such that there is no sub-partition satisfying the difference condition $[3,2^*]$ (where 2^* denotes zero or more occurrence of 2) ending with a 2, and such that there is no sub-partition satisfying the difference conditions $[1]$, $[0,0]$, $[0,2]$, $[2,0]$ or $[0,3]$, and such that there is no sub-partition with an odd sum of parts satisfying the difference conditions $[3,0]$, $[0,4]$, $[4,0]$ or $[3,2^*,3,0]$, is the same as the number of partitions of n into parts congruent to $\pm 2, \pm 5$ or ± 6*

modulo 14.

Each of the above statements corresponds to computing the graded dimension of a level 4 standard $A_2^{(2)}$-module in two ways—from the principal specialization of the Weyl-Kac character formula given by the numerator formula (see [LM78, Lep78]) (describing the "congruence conditions") and an explicit construction of a graded basis for the module (describing the "difference conditions" and "initial conditions").

The graded dimension, given by the principal specialization of the Weyl-Kac character formula and the numerator formula of [LM78, Lep78], can be factored as $\chi(q) = H(q)F(q)$, as a formal power series in q, where $F(q)$ is the series that counts the partitions with the "congruence conditions." The extra factor $H(q)$ is similar to the "fudge factor" in [LM78, LW82, LW84, LW85]. In their works, Lepowsky and Wilson used a certain "vacuum space" and certain "Z-operators" to cancel out the "fudge factor." We show an equivalent cancellation without using such a "vacuum space."

In this dissertation, we prove the appropriate "spanning set" result. The starting point is a certain obvious spanning set, parametrized by two sets of partitions. The elements of this spanning set can be described as products of two types of operators—the "negative Heisenberg operators" and the "$X(\bullet)$ operators"—acting on a highest weight vector v_0. The partitions describe the degrees and the order of these operators applied to v_0. We show that no relations among the "negative Heisenberg operators" exist and that these operators are accounted for by the "fudge factor" $H(q)$. The only relations, therefore, come from the relations among the $X(\bullet)$ operators acting on v_0.

We eliminate extraneous elements from this spanning set based on these relations. The resulting pruned spanning set can be described as parametrized by the set of all partitions which do not contain certain "forbidden" sub-partitions. The most surprising result in our work was the discovery of forbidden sub-partitions of arbitrary lengths. These forbidden partitions can be described by the "difference conditions" mentioned above. In all previously known analogous situations arising from representations of affine Kac-Moody algebras, the forbidden partitions could be described by difference conditions of bounded length. For example, in the Rogers-Ramanujan identities and Capparelli's identities, the difference conditions are of length one (reflecting the difference between adjacent parts). In our case, there are forbidden partitions satisfying arbitrarily long difference conditions. These difference conditions are the same for the all standard $A_2^{(2)}$-modules of level 4. The differentiating factors are then the "initial conditions" associated with the three inequivalent level 4 standard modules.

If the resulting spanning set is linearly independent, then the product side given by $F(q)$ may be expressed as $\sum_{n\geq 0} A_n q^n$, where A_n is the number of partitions of n not containing any forbidden sub-partitions. We call these partitions "allowed" partitions.

Our spanning set result states that in each of the above cases, the number of partitions of n described by various initial and difference conditions is greater than or equal to the number of partitions of n into parts satisfying the corresponding modulo 14 conditions. Experimental evidence shows that the equality holds for $n \leq 170$, as well as for $n = 180, 190$ and 200.

It is interesting to note how we discovered the family of "exceptional" forbidden partitions of arbitrary lengths (i.e., partitions of an odd number satisfying the difference conditions $[4, 0]$, $[3, 2^*, 3, 0]$). We set out to compare the graded dimension of the $(4, 0)$-module (one of the level 4 standard modules), with the spanning set we got after eliminating partitions into parts different from 1 (the initial condition for this module), and the other partitions containing forbidden sub-partitions, using relations similar to what Capparelli used in [Cap88, Cap93]. We found the first discrepancy at $n = 13$, and the next one at $n = 19$. In each case, there was an extra partition in our pruned spanning set compared to what the corresponding graded dimension would suggest. From certain "periodicity properties" of our relations, we could infer that we must have missed a forbidden triplet (partition into 3 parts). The smallest such triplet surviving in our spanning set (for $n = 13$) was $(7, 3, 3)$. We then eliminated $(7, 3, 3)$ and all its 2-translates (i.e., partitions of the form $(7 + 2k, 3 + 2k, 3 + 2k)$, $k \geq 0$). We compared our resulting spanning set with the graded dimension again, and noticed that the next two discrepancies were at $n = 21$ and $n = 29$, and in each case there was one extra partition in our spanning set. Once again, the "periodicity properties" suggested that we must have missed a forbidden quadruplet (i.e., a partition into 4 parts). Eliminating the smallest surviving quadruplet and its 2-translates gave us a contradiction, i.e., we got a smaller number of partitions in the spanning set than required by the graded dimension, for n sufficiently large. Therefore, we proceeded to eliminate the second quadruplet, which was $(9, 6, 3, 3)$, and its 2-translates. Proceeding in similar fashion, a clear pattern emerged for the family of forbidden partitions of arbitrary lengths.

The task of proving that these partitions are indeed forbidden turned out to be very subtle. Unlike in [Cap88, Cap93], we needed to keep track of terms containing "positive Heisenberg elements" in the relations that we used. In the case of the level 3 standard modules, the forbidden partitions arose directly from certain generating function identities. In our case, we obtain "longer" relations by multiplying similar generating function identities by suitable operators. The "exceptional" forbidden partitions of arbitrary length arise from these relations. Also, the initial conditions are significantly more difficult for level 4 than for level 3.

As illustrated by all of these phenomena, the level 4 theory for $\mathrm{A}_2^{(2)}$ is much more complex than the level 3 theory.

Now we give a brief overview of this dissertation.

In Chapter 2, we recall the basic definitions and results to describe the twisted vertex operator construction of the principally graded realization of the algebra $\mathrm{A}_2^{(2)}$. This is a simplification of the general case, based on vertex operator calculus, described in [Lep85, Fig87, Cap92, Cap93, FLM87, FLM88, DL96], specialized to our specific case of $\mathrm{A}_2^{(2)}$.

In Chapter 3, we recall the basic notions about standard modules for an affine Lie algebra and show that any level 4 standard module can be thought of as embedded in the tensor product of 4 copies of the basic module. We also recall the graded dimensions of these modules given by the principal specialization of the Weyl-Kac character formula and the numerator formula (see [Lep78, LM78] for more details).

In Chapter 4, we present the framework—some notations, definitions and results—on which the rest of the dissertation depends. First, we present a few definitions, notations and results related to partitions and generalized partitions (i.e., any sequence of integers, not necessarily positive, in non-increasing order). Then we describe certain standard monomials—parametrized by these partitions and generalized partitions—in certain operators and the structure of the standard modules in terms of the action of these monomials on a highest weight vector. We also present a number of substantial tools and techniques that we use repeatedly in the later chapters.

In Chapter 5, we present the "product generating function" identities that hold in any level 4 standard module (more generally, on the tensor product of 4 copies of the basic module). These identities are analogous to those used in [Cap88, Cap92, Cap93] for the standard $\mathrm{A}_2^{(2)}$-modules of level 3. We also present the coefficients of the standard monomials that appear in these "product generating function" identities.

Chapter 6 is devoted to finding forbidden partitions using the product generating function identities mentioned above. There are two types of forbidden partitions. Those that follow directly from the product generating function identities, similar to those in the level 3 case in [Cap88, Cap92, Cap93], are called "regular" forbidden partitions. Interestingly, there are other forbidden partitions of arbitrary length (starting from length 3) satisfying a simple pattern of difference conditions. There are no analogues of this type of forbidden partitions in any of the previous cases. We call them "exceptional" forbidden partitions. These exceptional forbidden partitions follow from new relations obtained by multiplying the product generating function identities by suitable operators.

In Chapter 7, we describe the "initial conditions" for each of the three inequivalent level 4 standard modules for $\mathrm{A}_2^{(2)}$. These come from certain ad-hoc relations specific to each of the particular standard $\mathrm{A}_2^{(2)}$-modules of level 4, needed to match the graded dimensions of "low degrees."

Finally, in Chapter 8, we summarize our main results and our three (conjectured) partition identities arising from the three level 4 standard $\mathrm{A}_2^{(2)}$-modules.

Some of the computations used in the proofs were performed using computer programs in Maple. We also wrote a C (standard C99) program to verify the validity of our partition identities. We have collected all the programs that we used in the appendices.

In Appendix A, we present the Maple worksheet and the Maple source files that we used (mainly in Chapter 6 and Chapter 7) for the computations of the relations.

In Appendix B, we present our Maple source files for computations in noncommutative algebras. We also present two Maple worksheets showing the computations used in some of the proofs (notably, in Chapter 7). The Maple programs implementing the operations (addition, multiplication, etc..) in noncommutative algebras were based on the NCFPS (noncommutative formal power series) package of D. Zeilberger (see [Zei12, BRRZ12]). The algorithm to apply substitution rules to straighten out an out-of-order monomial is based on the Maple codes of M. Russell (see [Rus13]). His program was for a finite number of substitution rules over a finite alphabet. We modified his program to work with an infinite number of substitution rules (based on finitely many patterns) over an infinite indexed alphabet.

In Appendix C, we present our C program (written in C99 standard) to verify our partition identities up to $n \leq 200$. (Note that we have done the verification only for $n \leq 170$ and for $n = 180, 190$ and 200. It may take more than 24 hours to complete the computation for $n = 200$.) We used the "accelerated ascending rule" algorithm of J. Kelleher (see [Kel06]) to generate all partitions of a nonnegative integer n.

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