```
> read `npolyio.txt`:
  read `npolyops.txt`:
  read `npolysubs.txt`:
  read `A22-rules.txt`:
  read `misc.txt`:
  read `A22-L4-iniRules.txt`:
```

```
> w := exp(Pi*I/3);
  w0 := exp(Pi*I/6);
```

$$w := \frac{1}{2} + \frac{1}{2}\,\mathrm{I}\sqrt{3}$$

$$w0 := \frac{1}{2}\sqrt{3} + \frac{1}{2}\,\mathrm{I} \qquad \textbf{(1)}$$

For the (4,0)-module

We have, $f_1 v_0 = 0$.

```
> p := f[1].v[0];
  P := parsePoly(p);
  is0inV(P, "40");
```

$$p := f_1.v_0$$

$$P := [[1, [f_1, v_0]]]$$

true **(1.1)**

```
> Q := rewritePoly(P, Risom union Rvop);
```

$$Q := \left[\left[-\frac{4}{\sqrt{3}+\mathrm{I}}, [X_{-1}, v_0]\right], \left[\frac{1}{3}\sqrt{3}, [a_{-1}, v_0]\right]\right] \qquad \textbf{(1.2)}$$

```
> # Normalize
  Q0 := sMulPoly((sqrt(3)+I)/4, Q):
  writePoly(Q0);
```

$$-X_{-1}.v_0 + \frac{1}{12}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\,\left(a_{-1}.v_0\right) \qquad \textbf{(1.3)}$$

The above relation shows that (1) is reducible.

For the (2,1)-module

We have, $f_1^2 v_0 = 0$.

```
> p := (f[1]^2).v[0];
  P := parsePoly(p);
  is0inV(P, "21");
```

$$p := \left(f_1^2\right).v_0$$

$$P := [[1, [f_1, f_1, v_0]]]$$

true **(2.1)**

```
> Q := rewritePoly(P, Risom union Rvop);
```

$$Q := \left[\left[\frac{16}{\left(\sqrt{3}+\mathrm{I}\right)^2}, \left[X_{-1}, X_{-1}, v_0\right]\right], \left[-\frac{8}{3}\frac{\sqrt{3}}{\sqrt{3}+\mathrm{I}}, \left[a_{-1}, X_{-1}, v_0\right]\right], \left[\frac{1}{3}, \left[a_{-1}, a_{-1}, v_0\right]\right], \left[\frac{4}{3}\frac{\sqrt{3}}{\sqrt{3}+\mathrm{I}}, \left[X_{-2}, v_0\right]\right]\right] \quad (2.2)$$

```
> # Normalize
  Q0 := sMulPoly(-(sqrt(3)+I)^2/16, Q):
  writePoly(Q0);
```

$$-\left(X_{-1}^2\right).v_0 + \frac{1}{6}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(a_{-1}.X_{-1}.v_0\right) - \frac{1}{48}\left(\sqrt{3}+\mathrm{I}\right)^2\left(\left(a_{-1}^2\right).v_0\right) - \frac{1}{12}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(X_{-2}.v_0\right) \quad (2.3)$$

The above relation shows that (1,1) is reducible. We have added the above rule as 'rini21a' in A22-L4-iniRules.txt.

We have, $f_0^3 v_0 = 0$.
We will add rini21a in our substitution rules to reduce the above.

```
> p := (f[0]^3).v[0];
  P := parsePoly(p);
  is0inV(P, "21");
```

$$p := \left(f_0^3\right).v_0$$

$$P := \left[\left[1, \left[f_0, f_0, f_0, v_0\right]\right]\right]$$

true **(2.4)**

```
> Q := rewritePoly(P, Risom union Rvop union {rini21a});
```

$$Q := \left[\left[\frac{72\sqrt{2}}{\sqrt{3}+\mathrm{I}}, \left[a_{-1}, a_{-1}, X_{-1}, v_0\right]\right], \left[-\frac{96\sqrt{2}\sqrt{3}}{\left(\sqrt{3}+\mathrm{I}\right)^2}, \left[X_{-2}, X_{-1}, v_0\right]\right], \left[-\frac{48\sqrt{2}}{\sqrt{3}+\mathrm{I}}, \left[a_{-1}, X_{-2}, v_0\right]\right]\right] \quad (2.5)$$

```
> # Normalize
  Q0 := sMulPoly((sqrt(3)+I)^2/(96*sqrt(6)), Q):
  writePoly(Q0);
```

$$\frac{1}{4}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(\left(a_{-1}^2\right).X_{-1}.v_0\right) - X_{-2}.X_{-1}.v_0 - \frac{1}{6}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(a_{-1}.X_{-2}.v_0\right) \quad (2.6)$$

This gives no new relations. We have added the above rule as 'rini21b' in A22-L4-iniRules.txt.

We have, $f_0^4 v_0 = 0$.
We will add rini21b in our substitution rules toreduce the above.

```
> p := (f[0]^4).v[0];
  P := parsePoly(p);
  is0inV(P, "21");
```

$$p := \left(f_0^4\right).v_0$$

$$P := [[1, [f_0, f_0, f_0, f_0, v_0]]]$$

true **(2.7)**

```
> Q := rewritePoly(P, Risom union Rvop
    union {rini21a, rini21b});
```

$$Q := \left[\left[-\frac{384\sqrt{3}}{\sqrt{3}+\mathrm{I}}, [a_{-1}, a_{-1}, a_{-1}, X_{-1}, v_0]\right], [-12, [a_{-1}, a_{-1}, a_{-1}, a_{-1}, v_0]], \left[\frac{288\sqrt{3}}{\sqrt{3}+\mathrm{I}}, [a_{-1}, a_{-1}, X_{-2}, v_0]\right], \left[\frac{768}{(\sqrt{3}+\mathrm{I})^2}, [X_{-3}, X_{-1}, v_0]\right], \left[\frac{576}{(\sqrt{3}+\mathrm{I})^2}, [X_{-2}, X_{-2}, v_0]\right], \left[-\frac{64\sqrt{3}}{\sqrt{3}+\mathrm{I}}, [a_{-1}, X_{-3}, v_0]\right], \left[-4\,\mathrm{I}\sqrt{3}\left(\mathrm{I}\sqrt{3}+1\right), [X_{-4}, v_0]\right]\right] \quad \textbf{(2.8)}$$

```
> # Normalize
  Q0 := sMulPoly(-(sqrt(3)+I)^2/576, Q):
  writePoly(Q0);
```

$$\frac{2}{3}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(\left(a_{-1}^3\right).X_{-1}.v_0\right) + \frac{1}{48}\left(\sqrt{3}+\mathrm{I}\right)^2\left(\left(a_{-1}^4\right).v_0\right) - \frac{1}{2}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(\left(a_{-1}^2\right).X_{-2}.v_0\right) - \frac{4}{3}X_{-3}.X_{-1}.v_0 - \left(X_{-2}^2\right).v_0 + \frac{1}{9}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(a_{-1}.X_{-3}.v_0\right) + \frac{1}{144}\mathrm{I}\left(\sqrt{3}+\mathrm{I}\right)^2\sqrt{3}\left(\mathrm{I}\sqrt{3}+1\right)\left(X_{-4}.v_0\right) \quad \textbf{(2.9)}$$

The above relation shows that (2,2) is reducible.
We have added the above rule as 'rini21c' in A22-L4-iniRules.txt.

We have $f_0^5 v_0 = 0$.
We will add rini21c in our substitution rules to reduce the above.

```
> p := (f[0]^5).v[0];
  P := parsePoly(p);
  is0inV(P, "21");
```

$$p := \left(f_0^5\right).v_0$$

$$P := [[1, [f_0, f_0, f_0, f_0, f_0, v_0]]]$$

true **(2.10)**

```
> Q := rewritePoly(P, Risom union Rvop
    union {rini21a, rini21b, rini21c});
```

$$Q := \left[\left[-\frac{1440\sqrt{2}}{\sqrt{3}+\mathrm{I}}, [a_{-1}, a_{-1}, a_{-1}, a_{-1}, X_{-1}, v_0]\right], \left[-24\sqrt{2}\sqrt{3}, [a_{-1}, a_{-1}, a_{-1}, a_{-1}, a_{-1}, v_0]\right], \left[\frac{1440\sqrt{2}}{\sqrt{3}+\mathrm{I}}, [a_{-1}, a_{-1}, a_{-1}, X_{-2}, v_0]\right], \left[-\frac{1920\sqrt{2}\sqrt{3}}{(\sqrt{3}+\mathrm{I})^2}, [a_{-1}, X_{-3}, X_{-1}, v_0]\right], \left[-\frac{960\sqrt{2}}{\sqrt{3}+\mathrm{I}}, [a_{-1}, a_{-1}, X_{-3}, v_0]\right], \left[\frac{1920\sqrt{2}\sqrt{3}}{(\sqrt{3}+\mathrm{I})^2}, [X_{-4}, X_{-1}, v_0]\right], \left[\frac{1920\sqrt{2}\sqrt{3}}{(\sqrt{3}+\mathrm{I})^2},\right. \quad \textbf{(2.11)}$$

$$\left[X_{-3}, X_{-2}, v_0\right]\Big], \left[\frac{480\sqrt{2}}{\sqrt{3}+\mathrm{I}}, \left[a_{-1}, X_{-4}, v_0\right]\right], \left[-16\sqrt{2}\sqrt{3}, \left[a_{-5}, v_0\right]\right]\Big]$$

```
> # Normalize
  Q0 := sMulPoly(-(sqrt(3)+I)^2/(1920*sqrt(6)), Q):
  writePoly(Q0);
```

$$\frac{1}{4}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(\left(a_{-1}^4\right).X_{-1}.v_0\right)+\frac{1}{80}\left(\sqrt{3}+\mathrm{I}\right)^2\left(\left(a_{-1}^5\right).v_0\right)-\frac{1}{4}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(\left(a_{-1}^3\right).X_{-2}.v_0\right)+a_{-1}.X_{-3}.X_{-1}.v_0+\frac{1}{6}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(\left(a_{-1}^2\right).X_{-3}.v_0\right)-X_{-4}.X_{-1}.v_0-X_{-3}.X_{-2}.v_0-\frac{1}{12}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(a_{-1}.X_{-4}.v_0\right)+\frac{1}{120}\left(\sqrt{3}+\mathrm{I}\right)^2\left(a_{-5}.v_0\right) \quad \textbf{(2.12)}$$

We have added the above rule as 'rini21d' in A22-L4-iniRules.txt.

We have $f_0^6 v_0 = 0$.
We will add rini21d in our substitution rules to reduce the above.

```
> p := (f[0]^6).v[0];
  P := parsePoly(p);
  is0inV(P, "21");
```

$$p := \left(f_0^6\right).v_0$$

$$P := \left[\left[1, \left[f_0, f_0, f_0, f_0, f_0, f_0, v_0\right]\right]\right]$$

$$\mathit{true} \quad \textbf{(2.13)}$$

```
> Q := rewritePoly(P, Risom union Rvop
    union {rini21a, rini21b, rini21c, rini21d});
```

$$Q := \Big[\left[-\frac{1152\sqrt{3}}{\sqrt{3}+\mathrm{I}}, \left[a_{-1}, a_{-1}, a_{-1}, a_{-1}, a_{-1}, X_{-1}, v_0\right]\right], \left[\frac{23040}{\left(\sqrt{3}+\mathrm{I}\right)^2}, \left[a_{-1}, a_{-1}, X_{-3}, X_{-1}, v_0\right]\right], \left[\frac{1920\sqrt{3}}{\sqrt{3}+\mathrm{I}}, \left[a_{-1}, a_{-1}, a_{-1}, X_{-3}, v_0\right]\right], \left[-\frac{23040}{\left(\sqrt{3}+\mathrm{I}\right)^2}, \left[a_{-1}, X_{-4}, X_{-1}, v_0\right]\right], \left[-\frac{1920\sqrt{3}}{\sqrt{3}+\mathrm{I}}, \left[a_{-1}, a_{-1}, X_{-4}, v_0\right]\right], \left[\frac{7680}{\left(\sqrt{3}+\mathrm{I}\right)^2}, \left[X_{-5}, X_{-1}, v_0\right]\right], \left[-\frac{3840}{\left(\sqrt{3}+\mathrm{I}\right)^2}, \left[X_{-3}, X_{-3}, v_0\right]\right], \left[32\,\mathrm{I}\sqrt{3}\left(\mathrm{I}\sqrt{3}+1\right), \left[a_{-5}, X_{-1}, v_0\right]\right], \left[-160, \left[a_{-5}, a_{-1}, v_0\right]\right], \left[\frac{1280\sqrt{3}}{\sqrt{3}+\mathrm{I}}, \left[a_{-1}, X_{-5}, v_0\right]\right], \left[-\frac{320\sqrt{3}}{\sqrt{3}+\mathrm{I}}, \left[X_{-6}, v_0\right]\right]\Big] \quad \textbf{(2.14)}$$

```
> # Normalize
  Q0 := sMulPoly((sqrt(3)+I)^2/3840, Q):
  writePoly(Q0);
```

$$-\frac{3}{10}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(\left(a_{-1}^5\right).X_{-1}.v_0\right)+6\left(\left(a_{-1}^2\right).X_{-3}.X_{-1}.v_0\right)+\frac{1}{2}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(\left(a_{-1}^3\right).X_{-3}.v_0\right)-6\left(a_{-1}.X_{-4}.X_{-1}.v_0\right)-\frac{1}{2}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(\left(a_{-1}^2\right).X_{-4}.v_0\right)+2\left(X_{-5}.X_{-1}.v_0\right) \quad \textbf{(2.15)}$$

$$-\left(X_{-3}^2\right).v_0+\frac{1}{120}\,\mathrm{I}\left(\sqrt{3}+\mathrm{I}\right)^2\sqrt{3}\left(\mathrm{I}\sqrt{3}+1\right)\left(a_{-5}.X_{-1}.v_0\right)-\frac{1}{24}\left(\sqrt{3}+\mathrm{I}\right)^2\left(a_{-5}.a_{-1}.v_0\right)+\frac{1}{3}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(a_{-1}.X_{-5}.v_0\right)-\frac{1}{12}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(X_{-6}.v_0\right)$$

The above relation shows that (3,3) is reducible.

For the (0,2)-module

We have $f_0 v_0 = 0$.

```
> p := f[0].v[0];
  P := parsePoly(p);
  is0inV(P, "02");
```

$$p := f_0.v_0$$

$$P := [[1, [f_0, v_0]]]$$

$$\mathit{true} \tag{3.1}$$

```
> Q := rewritePoly(P, Risom union Rvop);
```

$$Q := \left[\left[\frac{4\sqrt{2}}{\sqrt{3}+\mathrm{I}}, [X_{-1}, v_0]\right], \left[\frac{2}{3}\sqrt{2}\sqrt{3}, [a_{-1}, v_0]\right]\right] \tag{3.2}$$

```
> # Normalize
  Q0 := sMulPoly(-(sqrt(3)+I)/(4*sqrt(2)), Q):
  writePoly(Q0);
```

$$-X_{-1}.v_0-\frac{1}{6}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(a_{-1}.v_0\right) \tag{3.3}$$

The above relation shows that (1) is reducible.
We have added the above rule as 'rini02a' in A22-L4-iniRules.txt.

We have $f_1^3 v_0 = 0$.
We will add rini02a in our substitution rules to reduce the above.

```
> p := (f[1]^3).v[0];
  P := parsePoly(p);
  is0inV(P, "02");
```

$$p := \left(f_1^3\right).v_0$$

$$P := [[1, [f_1, f_1, f_1, v_0]]]$$

$$\mathit{true} \tag{3.4}$$

```
> Q := rewritePoly(P, Risom union Rvop union {rini02a});
```

$$Q := \left[\left[3\sqrt{3}, [a_{-1}, a_{-1}, a_{-1}, v_0]\right], \left[\frac{36}{\sqrt{3}+\mathrm{I}}, [a_{-1}, X_{-2}, v_0]\right], \left[-\frac{24}{\sqrt{3}+\mathrm{I}}, [X_{-3}, v_0]\right]\right] \tag{3.5}$$

```
> # Normalize
  Q0 := sMulPoly((sqrt(3)+I)/24, Q):
  writePoly(Q0);
```

(3.6)

$$\frac{1}{8}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(\left(a_{-1}^{3}\right).v_{0}\right)+\frac{3}{2}\,a_{-1}.X_{-2}.v_{0}-X_{-3}.v_{0} \tag{3.6}$$

The above relation shows that (3) is reducible.
We have added the above rule as 'rini02b' in A22-L4-iniRules.txt.

We have $f_1^4 v_0 = 0$.
We will add rini02b in our substitution rules to reduce the above.

```
> p := (f[1]^4).v[0];
  P := parsePoly(p);
  is0inV(P, "02");
```

$$p := \left(f_1^4\right).v_0$$

$$P := \left[\left[1,\ \left[f_1, f_1, f_1, f_1,\ v_0\right]\right]\right]$$

$$\mathit{true} \tag{3.7}$$

```
> Q := rewritePoly(P, Risom union Rvop union
    {rini02a, rini02b});
```

$$Q := \left[\left[-27,\ \left[a_{-1}, a_{-1}, a_{-1}, a_{-1}, v_0\right]\right], \left[-\frac{72\sqrt{3}}{\sqrt{3}+\mathrm{I}},\ \left[a_{-1}, a_{-1}, X_{-2}, v_0\right]\right], \left[\frac{144}{\left(\sqrt{3}+\mathrm{I}\right)^2},\ \left[X_{-2}, X_{-2}, v_0\right]\right], \left[\frac{36\sqrt{3}}{\sqrt{3}+\mathrm{I}},\ \left[X_{-4}, v_0\right]\right]\right] \tag{3.8}$$

```
> # Normalize
  Q0 := sMulPoly(-(sqrt(3)+I)^2/120, Q):
  writePoly(Q0);
```

$$\frac{9}{40}\left(\sqrt{3}+\mathrm{I}\right)^2\left(\left(a_{-1}^{4}\right).v_{0}\right)+\frac{3}{5}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(\left(a_{-1}^{2}\right).X_{-2}.v_{0}\right)-\frac{6}{5}\left(X_{-2}^{2}\right).v_{0}-\frac{3}{10}\left(\sqrt{3}+\mathrm{I}\right)\sqrt{3}\left(X_{-4}.v_{0}\right) \tag{3.9}$$

The above relation shows that (2,2) is reducible.